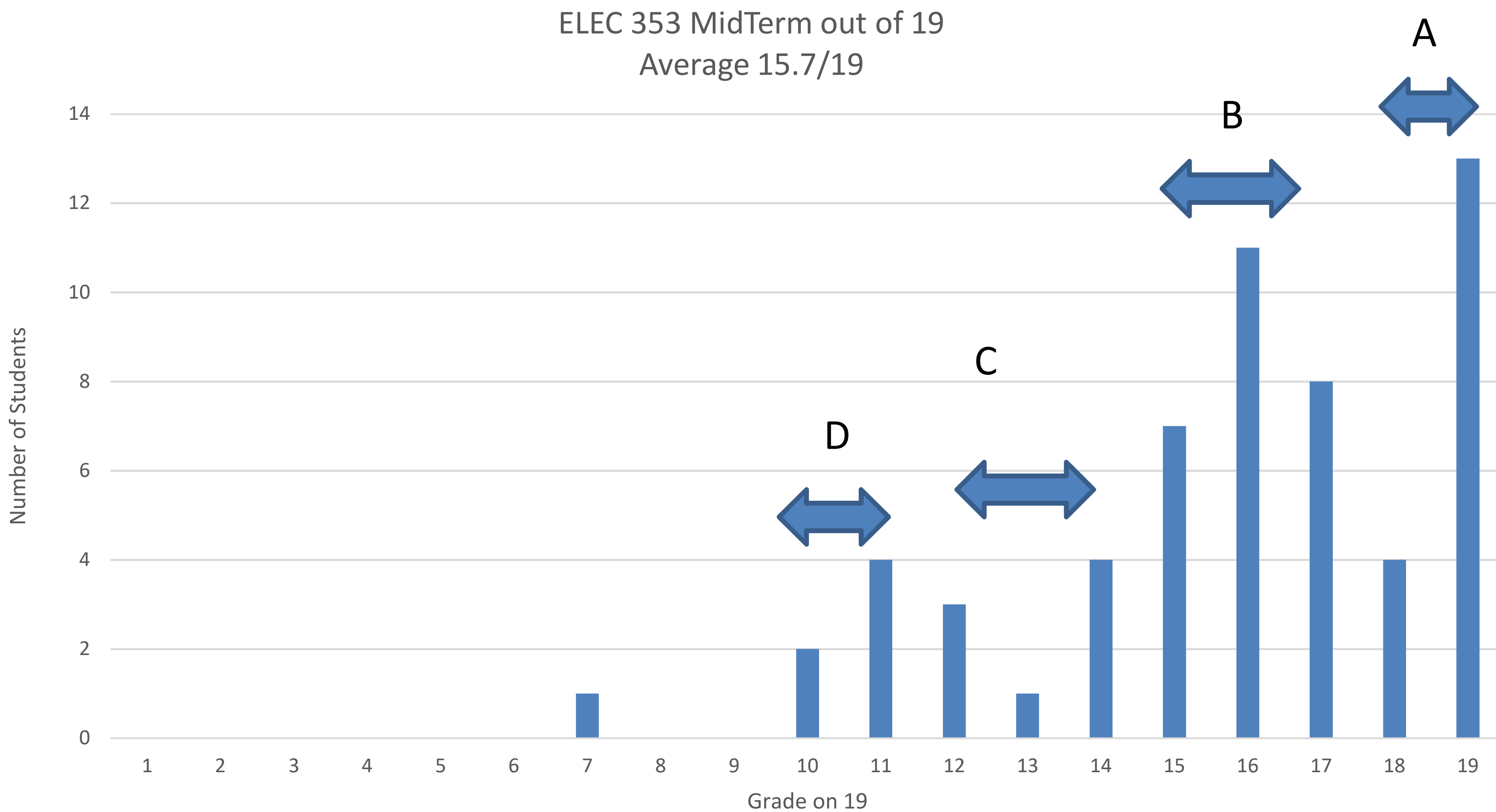
# ELEC353 Lecture Notes Set 10

The homework assignments are posted on the course web site.
http://users.encs.concordia.ca/~trueman/web_page_353.htm

Homework #6: Do homework #6 by February 22, 2019.
Homework #7: Do homework #7 by March 8, 2019.
Homework #8: Do homework #8 by March 15, 2019.

Tentative final exam date: Tuesday April 23, 2019, 9:00 to 12:00.

# Mid Term Grades 2019



# Topics to be Covered

Transmission Lines (TLs)

- Wave Equation and Solution
- Solving a TL Circuit
- Standing Wave Patterns
- Impedance Matching
- Bandwidth of Digital Signal

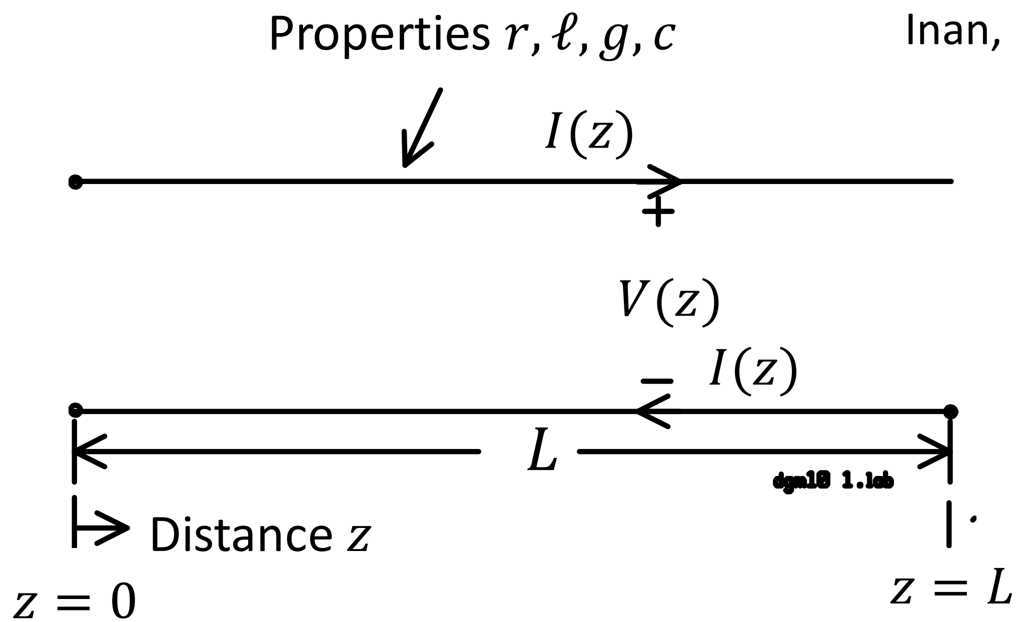
Plane Waves

- Maxwell's Equations and the Wave Equation
- Plane waves
- Material Boundaries
- Transmission Through a Wall

Antennas

# Transmission Line Circuits in the Sinusoidal Steady State

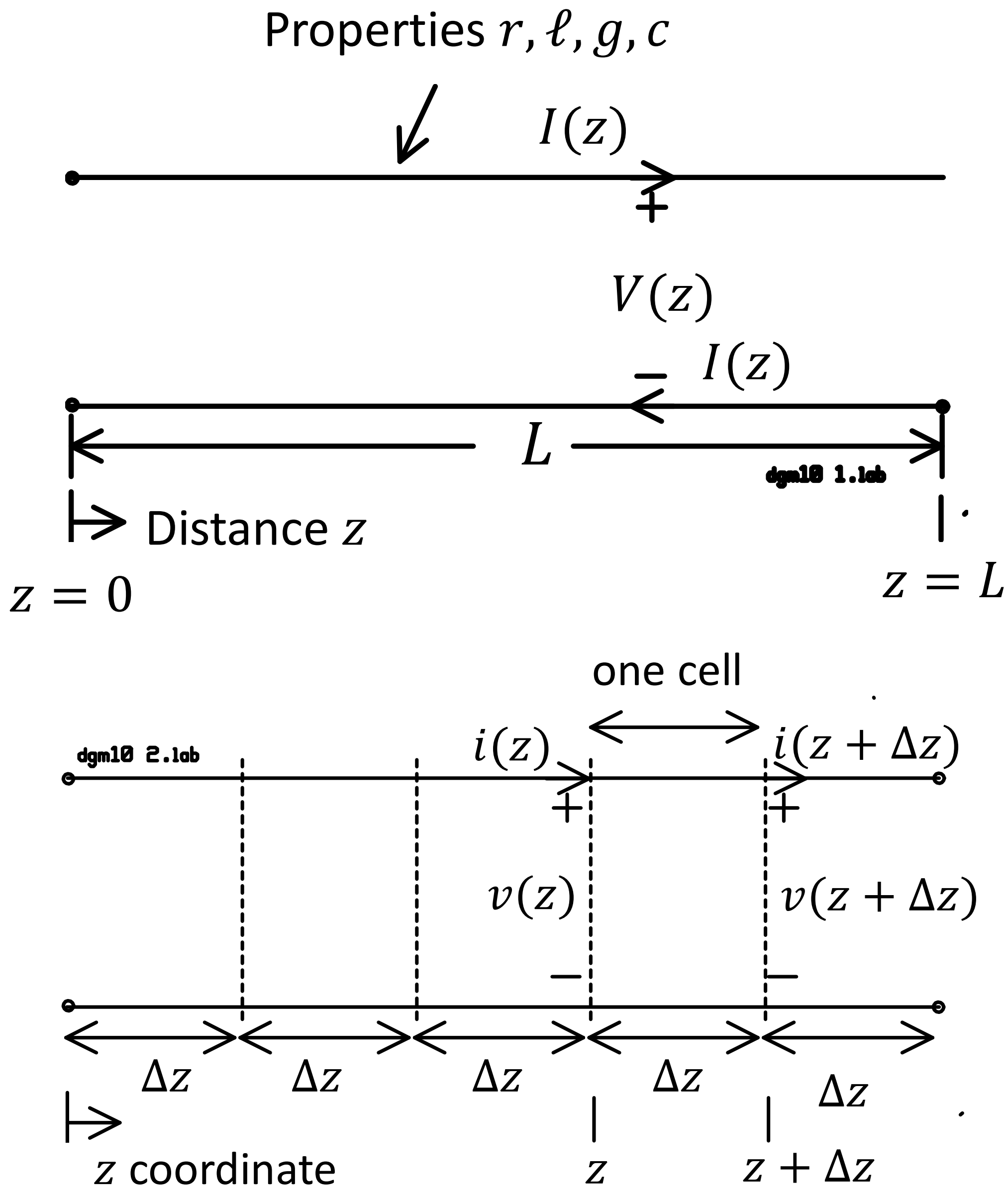
Inan, Inan and Said Section 2.2

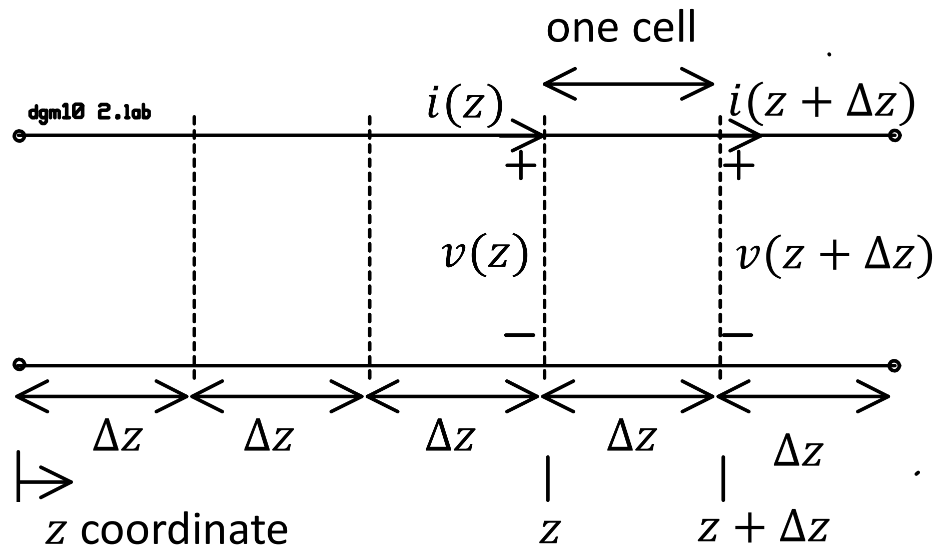



The transmission line parameters are:

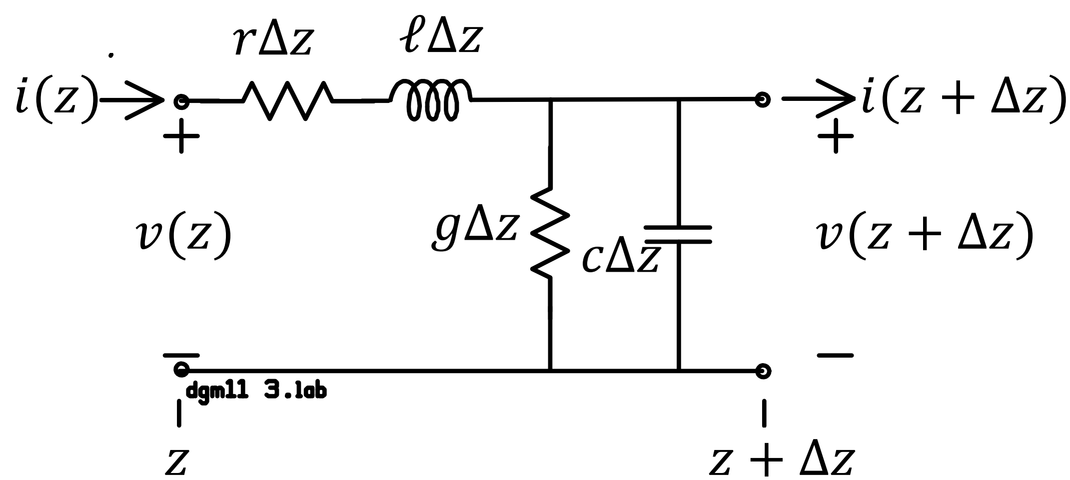
- $\ell$ H/m = inductance per unit length
- $c$ F/m = capacitance per unit length
- $r$ ohms/meter = series resistance per unit length
  - series resistance arises because of the currents flowing in the metal of the conductors
- $g$ Siemens/meter = shunt conductance per unit length

# Lossy Transmission Line Equations

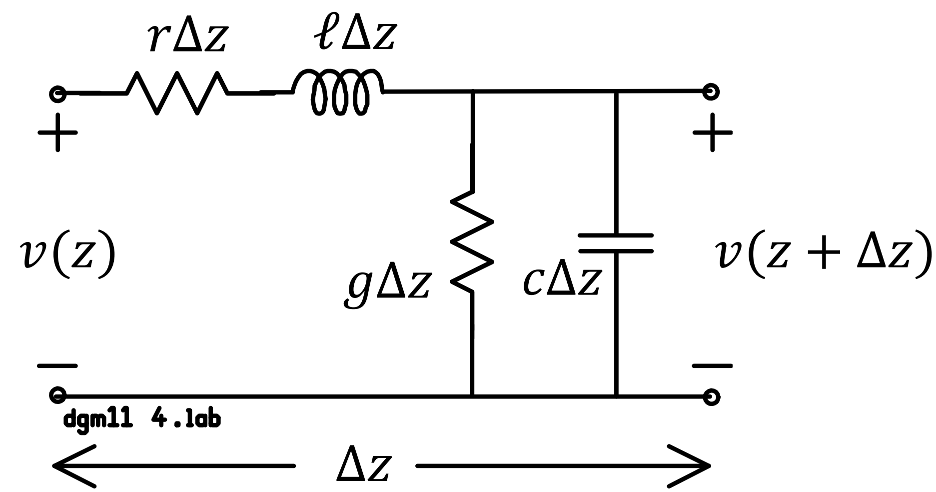





Examine one individual cell.



# KVL Equation




KVL for the cell states:

$$v(z) - r\Delta z i - \ell\Delta z \frac{\partial i}{\partial t} - v(z+\Delta z) = 0$$

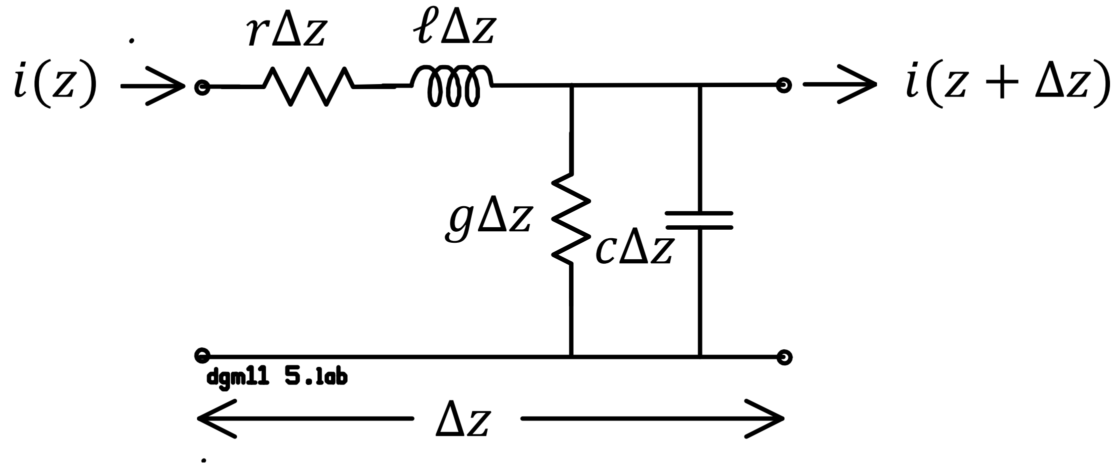
Rearrange the KVL equation:

$$v(z+\Delta z) - v(z) = -r\Delta z i - \ell\Delta z \frac{\partial i}{\partial t}$$

$$\lim_{\Delta z \to 0} \frac{v(z+\Delta z) - v(z)}{\Delta z} = -ri - \ell \frac{\partial i}{\partial t}$$

$$\frac{\partial v}{\partial z} = -ri - \ell \frac{\partial i}{\partial t}$$

# KCL Equation




KCL for the cell states:

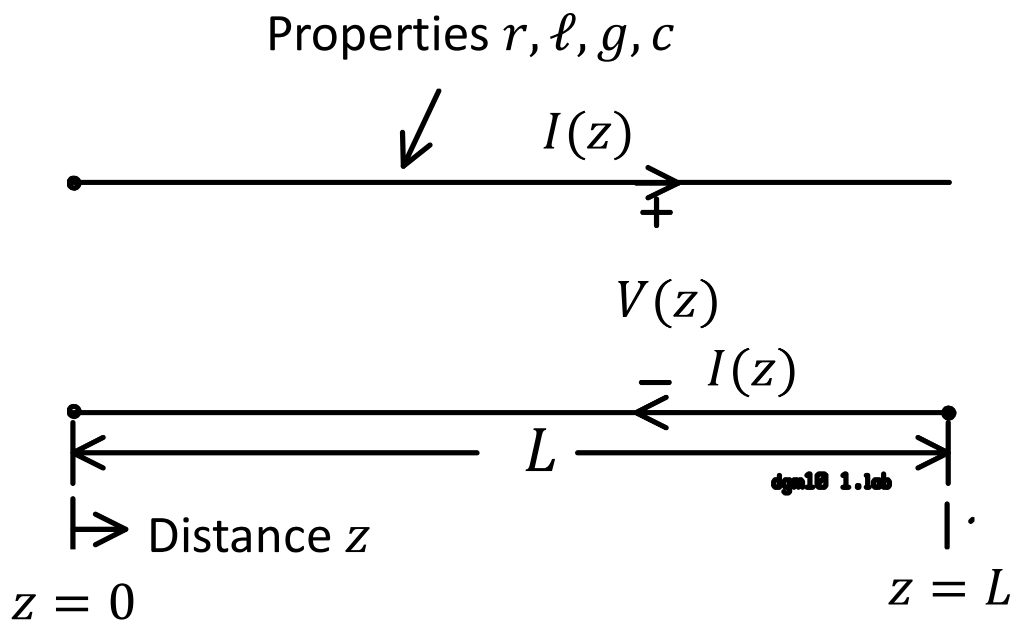
$$i(z) - g\Delta z v - c\Delta z \frac{\partial v}{\partial t} - i(z+\Delta z) = 0$$

Rearrange the KCL equation:

$$i(z+\Delta z) - i(z) = -g\Delta z v - c\Delta z \frac{\partial v}{\partial t}$$

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z) - i(z)}{\Delta z} = -gv - c\frac{\partial v}{\partial t}$$

$$\frac{\partial i}{\partial z} = -gv - c\frac{\partial v}{\partial t}$$




We have shown that:

$$\frac{\partial v}{\partial z} = -ri - \ell \frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial z} = -gv - c\frac{\partial v}{\partial t}$$

These equations are called "lossy" transmission line equations or "lossy" Telegrapher's Equations.

# Transmission-Line Equations for the Phasor Voltage and Current

$$v(z,t) = A(z)\cos(\omega t + \theta(z)) \longleftrightarrow V(z) = A(z)e^{j\theta(z)}$$

$$i(z,t) = B(z)\cos(\omega t + \phi(z)) \longleftrightarrow I(z) = B(z)e^{j\phi(z)}$$

The **magnitude** of the phasor is the **amplitude** of the cosine.
The **angle** of the phasor is the **phase angle** of the cosine.

time differentiation $\frac{\partial}{\partial t}$ is equivalent to multiplication by $j\omega$

Homework:
Prove this!

$$\frac{\partial v}{\partial z} = -ri - \ell\frac{\partial i}{\partial t} \quad \Leftrightarrow \quad \frac{dV}{dz} = -(r + j\omega\ell)I$$

$$\frac{\partial i}{\partial z} = -gv - c\frac{\partial v}{\partial t} \quad \Leftrightarrow \quad \frac{dI}{dz} = -(g + j\omega c)V$$

These are the transmission line equations in the frequency domain.

# Wave Equation for Phasors

Inan, Inan and Said, Section 3.1

$$\frac{dV}{dz} = -(r + j\omega\ell)I \qquad \frac{dI}{dz} = -(g + j\omega c)V$$

$$\frac{d^2V}{dz^2} = -(r + j\omega\ell)\frac{dI}{dz}$$

$$\frac{d^2V}{dz^2} = -(r + j\omega\ell)\left[-(g + j\omega c)V\right]$$

$$\frac{d^2V}{dz^2} = (r + j\omega\ell)(g + j\omega c)V$$

Define the "propagation constant" as $\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)}$

$$\frac{d^2V}{dz^2} = \gamma^2 V$$

# Solution to the Wave Equation

$$\frac{d^2V}{dz^2} = \gamma^2 V$$

$$V(z) = V^+e^{-\gamma z} + V^-e^{\gamma z}$$

Prove that $V(z) = V^+e^{-\gamma z}$ satisfies the wave equation:

$$\frac{dV}{dz} = \frac{d}{dz}(V^+e^{-\gamma z}) = -\gamma V^+e^{-\gamma z}$$

$$\frac{d^2V}{dz^2} = \frac{d}{dz}(-\gamma V^+e^{-\gamma z}) = (-\gamma)(-\gamma)V^+e^{-\gamma z} = \gamma^2V^+e^{-\gamma z} = \gamma^2 V$$

Hence $\frac{d^2V}{dz^2} = \gamma^2 V$ and $V = V^+e^{-\gamma z}$ does indeed satisfy the wave equation.

Homework: prove that $V = V^-e^{\gamma z}$ satisfies the wave equation.

# Propagation Constant

$$\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)}$$

$$\gamma = \alpha + j\beta$$

- $\alpha$ is the "attenuation constant" in "Nepers/meter"
  - "Neper" is a "unitless" unit like "radian".
- $\beta$ is the "phase constant" in radians/meter.

# Current on the Transmission Line

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$\frac{dV}{dz} = -(r + j\omega\ell)I$$

$$I(z) = \frac{-1}{(r + j\omega\ell)} \frac{dV}{dz}$$

$$I(z) = \frac{-1}{(r + j\omega\ell)} \frac{d}{dz}\left(V^+ e^{-\gamma z} + V^- e^{\gamma z}\right)$$

$$I(z) = \frac{-1}{(r + j\omega\ell)}\left(-\gamma V^+ e^{-\gamma z} + \gamma V^- e^{\gamma z}\right)$$

$$I(z) = \frac{\gamma}{(r + j\omega\ell)}\left(V^+ e^{-\gamma z} - V^- e^{\gamma z}\right)$$

# Characteristic Impedance

$$V = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I(z) = \frac{\gamma}{(r + j\omega\ell)}\left(V^+ e^{-\gamma z} - V^- e^{\gamma z}\right)$$

- The "characteristic resistance" $R_c$ was defined earlier in the course as

  $R_c = \sqrt{\frac{\ell}{c}}$ ohms

- For a "lossy" transmission line, define the "characteristic impedance" $Z_c$ as

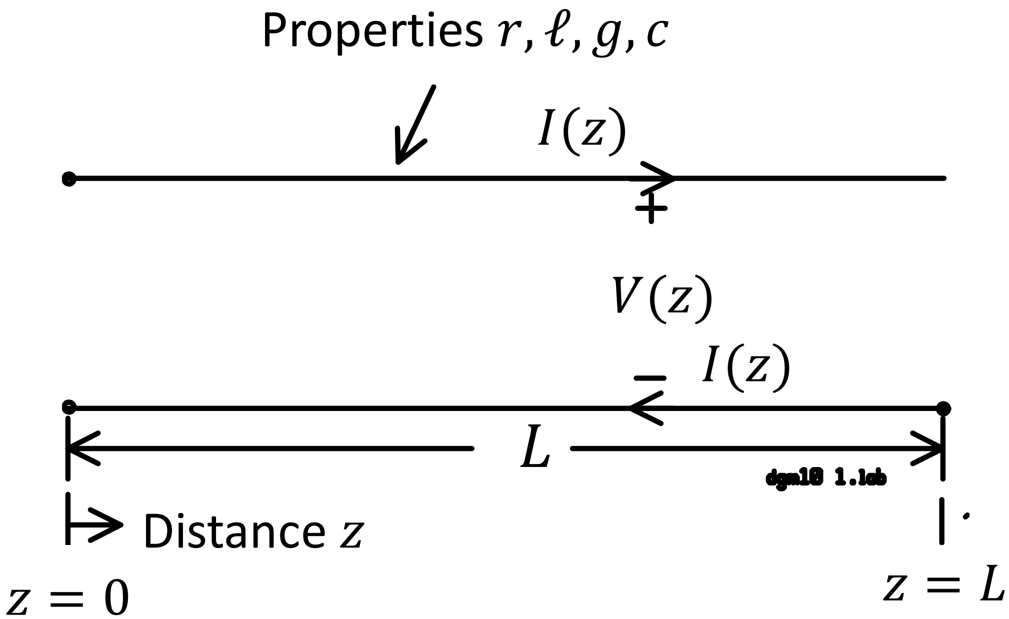
  $$Z_c = \frac{r + j\omega\ell}{\gamma} = \frac{r + j\omega\ell}{\sqrt{(r + j\omega\ell)(g + j\omega c)}} = \sqrt{\frac{(r + j\omega\ell)^2}{(r + j\omega\ell)(g + j\omega c)}}$$

  $Z_c = \sqrt{\frac{r + j\omega\ell}{g + j\omega c}}$ ohms

- Then the current on the transmission line is

  $$I(z) = \left(\frac{V^+}{Z_c} e^{-\gamma z} - \frac{V^-}{Z_c} e^{\gamma z}\right)$$

# Summary-Sinusoidal Steady State




$$\frac{dV}{dz} = -(r + j\omega\ell)I$$

$$\frac{dI}{dz} = -(g + j\omega c)V$$

$$\frac{d^2V}{dz^2} = \gamma^2 V$$

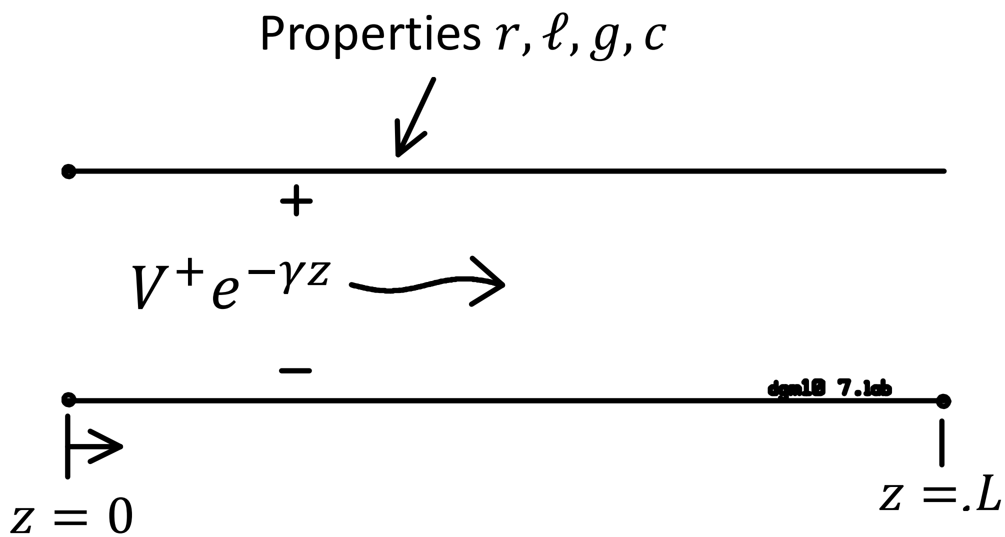
$$\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)}$$

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

$$I(z) = \frac{V^+}{Z_c} e^{-\gamma z} - \frac{V^-}{Z_c} e^{\gamma z}$$

$$Z_c = \sqrt{\frac{r + j\omega\ell}{g + j\omega c}}$$

# Attenuation with Distance Travelled




$V^+ = Ae^{j\theta}$
$\gamma = \alpha + j\beta$

$V(z) = V^+e^{-\gamma z}$
$V(z) = Ae^{j\theta}e^{-\gamma z}$
$V(z) = Ae^{j\theta}e^{-(\alpha+j\beta)z}$
$V(z) = Ae^{-\alpha z}e^{j(-\beta z+\theta)}$

In the time domain:

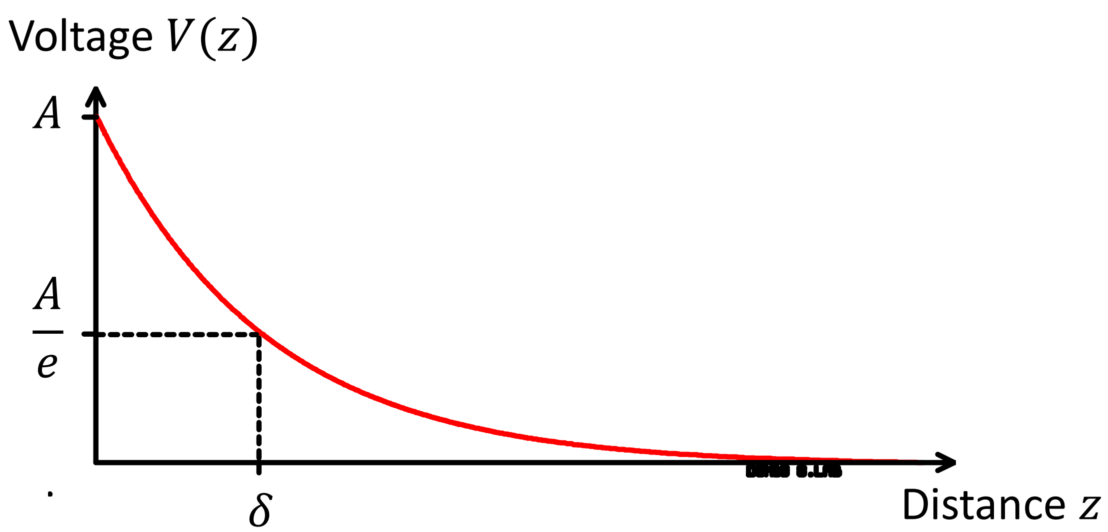
$v(z,t) = Ae^{-\alpha z}\cos(\omega t - \beta z + \theta)$

The amplitude is $Ae^{-\alpha z}$

# Exponential Attenuation

$$v(z,t) = Ae^{-\alpha z}\cos(\omega t - \beta z + \theta)$$

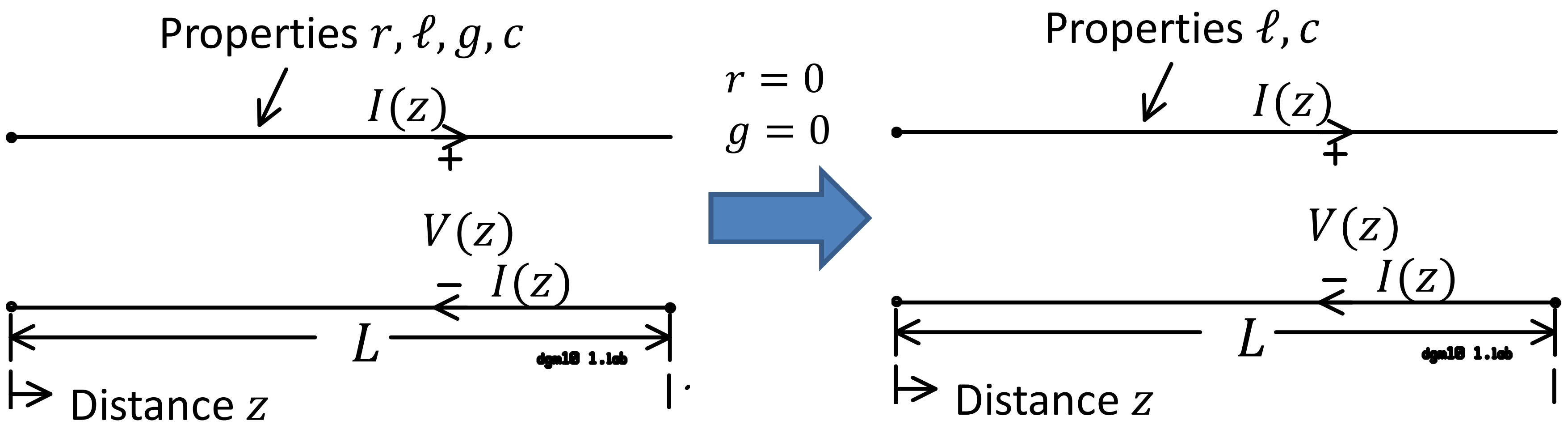
The amplitude of $v(z,t)$ is $Ae^{-\alpha z}$




- The distance $\delta$ required for the amplitude to decrease by a factor of $1/e$ is called the "penetration depth", or for metals the "skin depth".

$$Ae^{-\alpha\delta} = \frac{A}{e} \quad \text{so} \quad e^{-\alpha\delta} = \frac{1}{e} \quad \text{so} \quad \delta = \frac{1}{\alpha}$$

# Lossless Transmission Lines




On a "lossless" transmission line, the series-resistance-per-unit-length is zero ( $r = 0$ ) and the shunt-conductance-per-unit-length is zero ( $g = 0$ ).

$$\frac{dV}{dz} = -(r + j\omega\ell)I \quad \Rightarrow \quad \frac{dV}{dz} = -j\omega\ell I$$

$$\frac{dI}{dz} = -(g + j\omega c)V \quad \Rightarrow \quad \frac{dI}{dz} = -j\omega c V$$

# Lossless Wave Equation

$$\frac{dV}{dz} = -j\omega\ell I$$

$$\frac{dI}{dz} = -j\omega c V$$

$$\frac{d^2V}{dz^2} = -j\omega\ell\frac{dI}{dz} = -j\omega\ell(-j\omega cV) = -\omega^2\ell cV$$

$$\frac{d^2V}{dz^2} = -\omega^2\ell cV$$

$$\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)} = \sqrt{(j\omega\ell)(j\omega c)} = j\omega\sqrt{\ell c}$$

$\gamma = \alpha + j\beta$ So for "lossless" lines:

- $\alpha = 0$ so $\gamma = j\beta$
- $\beta = \omega\sqrt{\ell c}$

$\beta^2 = \left(\omega\sqrt{\ell c}\right)^2 = \omega^2\ell c$

$$\frac{d^2V}{dz^2} = -\beta^2 V$$

# Solution to the Lossless Wave Equation

$$\frac{d^2V}{dz^2} = -\beta^2 V$$

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

(Homework: prove this is a solution to $\frac{d^2V}{dz^2} = -\beta^2 V$ by direct substitution.)

$$V^+ = \left|V^+\right| e^{j\theta^+} = C^+ e^{j\theta^+}$$
$$V^- = \left|V^-\right| e^{j\theta^-} = C^- e^{j\theta^-}$$

$$V(z) = C^+ e^{j\theta^+} e^{-j\beta z} + C^- e^{j\theta^-} e^{j\beta z}$$

$$V(z) = C^+ e^{j(-\beta z + \theta^+)} + C^- e^{j(\beta z + \theta^-)}$$

In the time domain: $v(z,t) = C^+ \cos\left(\omega t - \beta z + \theta^+\right) + C^- \cos\left(\omega t + \beta z + \theta^-\right)$

# Find the Current on the Lossless Transmission Line

$$\frac{dV}{dz} = -j\omega \ell I \qquad \beta = \omega\sqrt{\ell c}$$

$$\frac{dI}{dz} = -j\omega c V \qquad V(z) = V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}$$

$$I = \frac{-1}{j\omega\ell}\frac{dV}{dz} = \frac{-1}{j\omega\ell}\frac{d}{dz}\left(V^{+}e^{-j\beta z} + V^{-}e^{j\beta z}\right)$$

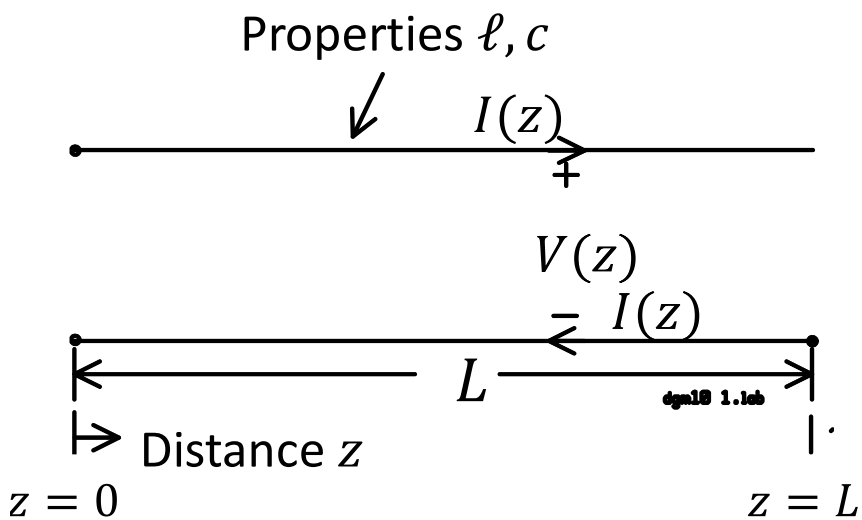
$$I = \frac{-1}{j\omega\ell}\frac{dV}{dz} = \left(\frac{-1}{j\omega\ell}\right)\left(-j\beta V^{+}e^{-j\beta z} + j\beta V^{-}e^{j\beta z}\right)$$

$$I = \frac{\beta}{\omega\ell}V^{+}e^{-j\beta z} - \frac{\beta}{\omega\ell}V^{-}e^{j\beta z}$$

$$R_c = \frac{\omega\ell}{\beta} = \frac{\omega\ell}{\omega\sqrt{\ell c}} = \sqrt{\frac{\ell}{c}} \text{ ohms}$$

$$I(z) = \frac{V^{+}}{R_c}e^{-j\beta z} - \frac{V^{-}}{R_c}e^{j\beta z}$$

# Summary – Lossless Transmission Lines




Lossless transmission-line equations:

$$\frac{dV}{dz} = -j\omega\ell I$$

$$\frac{dI}{dz} = -j\omega c V$$

Lossless wave equation:

$$\frac{d^2V}{dz^2} = -\beta^2 V$$

where $\beta = \omega\sqrt{\ell c}$ is the phase constant.

Characteristic Resistance

$$R_c = \sqrt{\frac{\ell}{c}} \text{ ohms}$$

Voltage and Current

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

$$I(z) = \frac{V^+}{R_c} e^{-j\beta z} - \frac{V^-}{R_c} e^{j\beta z}$$

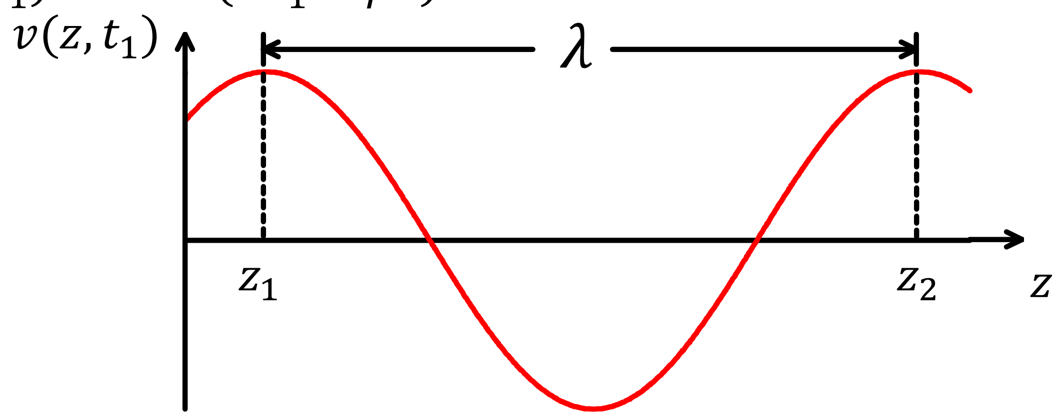
# Find the Wavelength

$$\bar{V} = Ae^{-j\beta z} \Rightarrow v(z,t) = \mathrm{Re}[\bar{V}e^{j\omega t}] = \mathrm{Re}[Ae^{-j\beta z}e^{j\omega t}] = \mathrm{Re}[Ae^{j(\omega t - \beta z)}]$$

$$v(z,t) = \mathrm{Re}[A\cos(\omega t - \beta z) + jA\sin(\omega t - \beta z)]$$

$$v(z,t) = A\cos(\omega t - \beta z)$$

At $t = t_1$, graph $v(z, t_1) = Acos(\omega t_1 - \beta z)$:




$\cos(\omega t - \beta z)$ is a maximum when $(\omega t - \beta z) = \ldots, -2\pi, 0, 2\pi, \ldots$

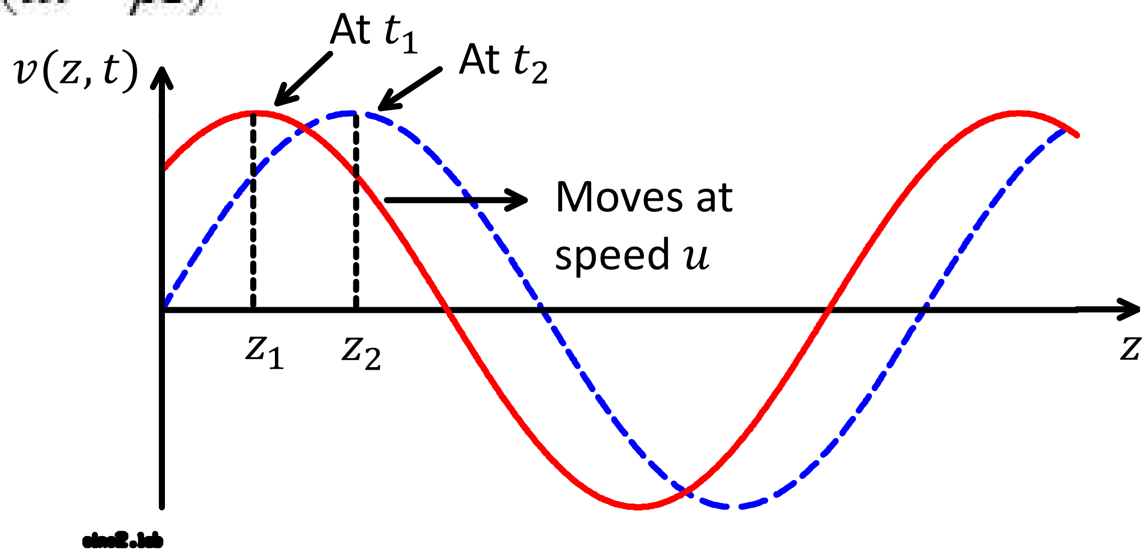
$z_1 = ?$ $\quad \omega t_1 - \beta z_1 = 0$ so $z_1 = \frac{\omega t_1}{\beta}$

$z_2 = ?$ $\quad z_2 > z_1$ and as $z$ increases, $(\omega t - \beta z)$ becomes more negative, so
$\omega t_1 - \beta z_2 = -2\pi$ so $z_2 = \frac{\omega t_1 + 2\pi}{\beta}$

$\lambda = ?$ $\quad \lambda = z_2 - z_1 = \frac{\omega t_1 + 2\pi}{\beta} - \frac{\omega t_1}{\beta} = \frac{2\pi}{\beta}$

# Find the Speed of Travel

$v(z,t) = A\cos(\omega t - \beta z)$




How far does the wave travel between $t = t_1$ and $t = t_2$ ?

At $t = t_1$ find $z_1 = ?$ $\quad \omega t_1 - \beta z_1 = 0$ so $z_1 = \dfrac{\omega t_1}{\beta}$

At $t = t_2$ find $z_2 = ?$ $\quad \omega t_2 - \beta z_2 = 0$ so $z_2 = \dfrac{\omega t_2}{\beta}$

The speed of travel or "phase velocity is $u$= distance / time = ?

$$u = \frac{z_2 - z_1}{t_2 - t_1} = \frac{\frac{\omega t_2}{\beta} - \frac{\omega t_1}{\beta}}{t_2 - t_1} = \frac{\frac{\omega}{\beta}(t_2 - t_1)}{t_2 - t_1} = \frac{\omega}{\beta}$$

# Wavelength and Speed of Travel

In general the wavelength is $\lambda = \dfrac{2\pi}{\beta}$

In general the speed of travel is $u = \dfrac{\omega}{\beta}$

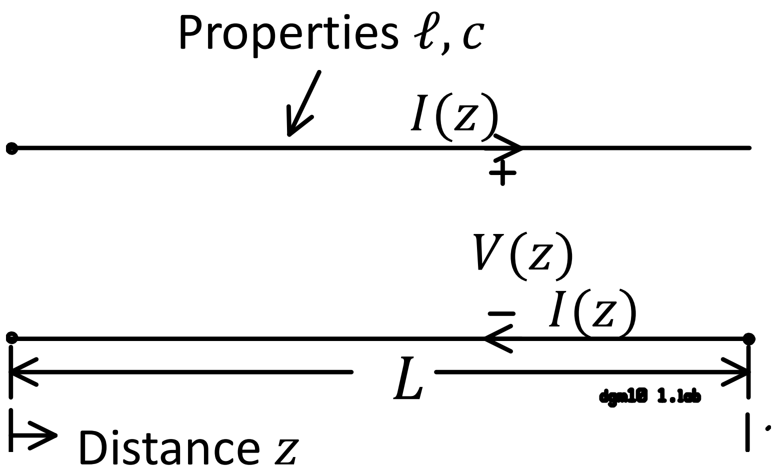
For a lossless transmission line $\beta = \omega\sqrt{\ell c}$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\ell c}} = \frac{1}{\sqrt{\ell c}} \qquad u = \frac{1}{\sqrt{\ell c}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega/u} = \frac{2\pi u}{2\pi f} = \frac{u}{f} \qquad \lambda = \frac{u}{f}$$

$$\beta = \omega\sqrt{\ell c} = \frac{\omega}{1/\sqrt{\ell c}} = \frac{\omega}{u}$$

# Summary – Lossless Transmission Lines




Lossless transmission line equations:

$$\frac{dV}{dz} = -j\omega\ell I$$

$$\frac{dI}{dz} = -j\omega c V$$

Lossless wave equation:

$$\frac{d^2V}{dt^2} = -\beta^2 V$$

where $\beta = \omega\sqrt{\ell c}$ is the phase constant.

Voltage and current:

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

$$I(z) = \frac{V^+}{R_c} e^{-j\beta z} - \frac{V^-}{R_c} e^{j\beta z}$$

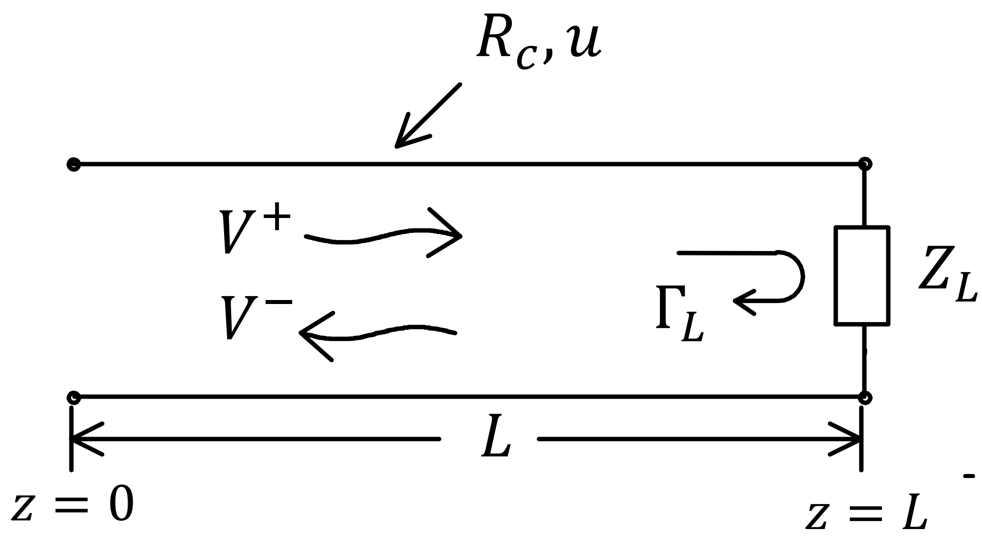
Characteristic resistance:

$$R_c = \sqrt{\frac{\ell}{c}}$$

| In general: | Lossless case: |
|---|---|
| $u = \dfrac{\omega}{\beta}$ | $u = \dfrac{1}{\sqrt{\ell c}}$ |
| $\lambda = \dfrac{2\pi}{\beta}$ | $\lambda = \dfrac{u}{f}$ |
| | $\beta = \omega\sqrt{\ell c} = \dfrac{\omega}{u}$ |

# Transmission Line Terminated with a Load

Inan and Inan Sections 3.2 and 3.3




Voltage and current:

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

$$I(z) = \frac{V^+}{R_c} e^{-j\beta z} - \frac{V^-}{R_c} e^{j\beta z}$$

If we know $V^+$, can we find $V^-$?

At the load we must satisfy $V(L) = Z_L I(L)$

$$V(L) = V^+ e^{-j\beta L} + V^- e^{j\beta L}$$

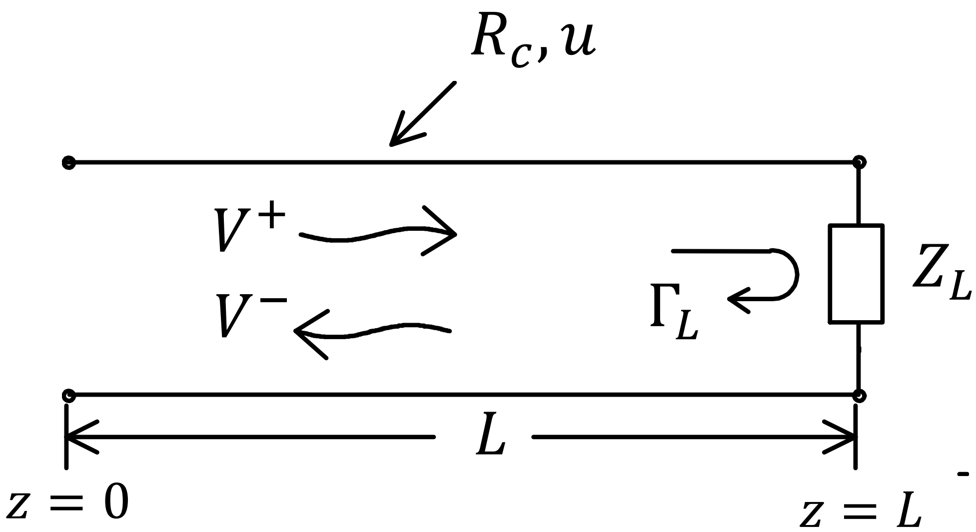
$$I(L) = \frac{V^+}{R_c} e^{-j\beta L} - \frac{V^-}{R_c} e^{j\beta}$$

$$V(L) = Z_L I(L)$$

$$V^+ e^{-j\beta L} + V^- e^{j\beta} = Z_L \left( \frac{V^+}{R_c} e^{-j\beta L} - \frac{V^-}{R_c} e^{j\beta L} \right)$$

$$V^- = \frac{Z_L - R_c}{Z_L + R_c} e^{-j2\beta} \; V^+$$

# Reflection Coefficient at the Load




Definition:

$$\Gamma_L = \frac{V^- e^{j\beta L}}{V^+ e^{-j\beta L}}$$

From the previous page:

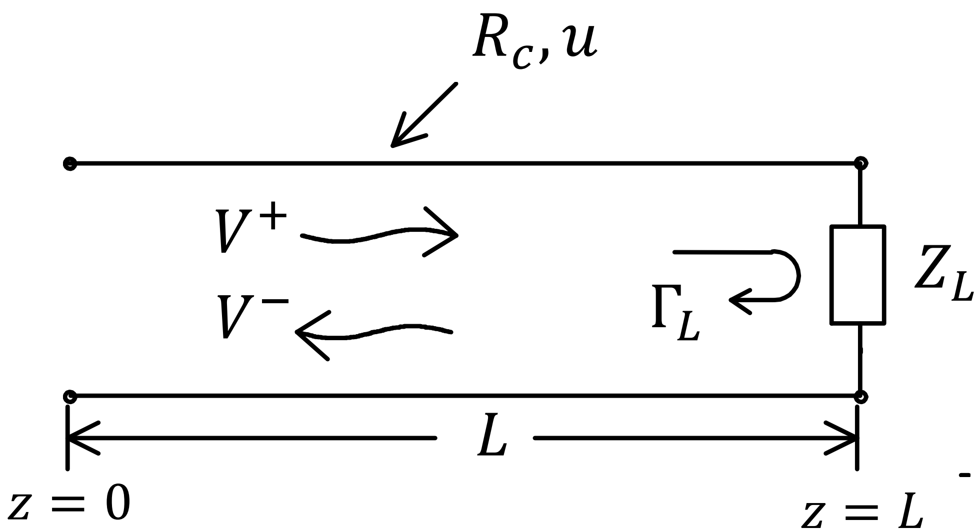
$$V^- = V^+ e^{-2j\beta L} \frac{Z_L - R_c}{Z_L + R_c}$$

$$\Gamma_L = \frac{V^+ e^{-2j\beta L} \dfrac{Z_L - R_c}{Z_L + R_c} e^{j\beta L}}{V^+ e^{-j\beta L}}$$

$$\Gamma_L = \frac{Z_L - R_c}{Z_L + R_c}$$

$$V^- = \Gamma_L e^{-j2\beta L} V^+$$

# Matched Load




$$\Gamma_L = \frac{Z_L - R_c}{Z_L + R_c}$$

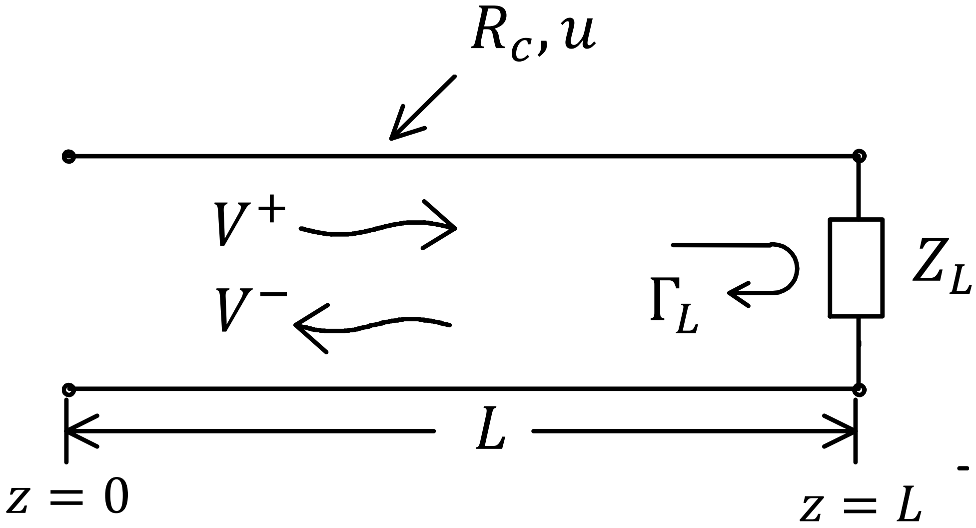
Matched load $Z_L = R_c$:

$$\Gamma_L = \frac{Z_L - R_c}{Z_L + R_c} = \frac{R_c - R_c}{R_c + R_c} = 0$$

The reflected voltage is

$V^- = \Gamma_L e^{-j2\beta L} V^+ = 0$

# Voltage and Current using $\Gamma_L$




Voltage and Current

$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z}$$

$$I(z) = \frac{V^+}{R_c} e^{-j\beta z} - \frac{V^-}{R_c} e^{j\beta z}$$

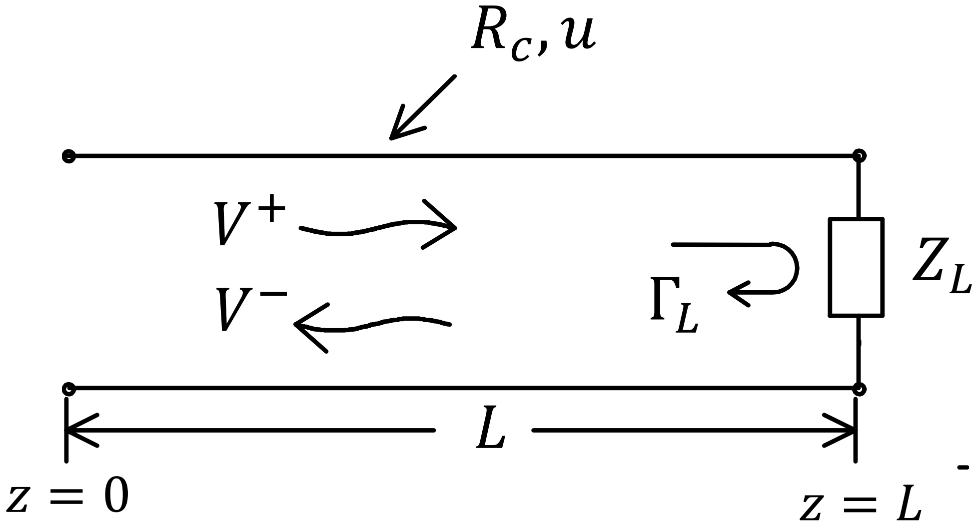
$$V^- = \Gamma_L e^{-j2\beta L} V^+$$

$$\begin{aligned} V(z) &= V^+ e^{-j\beta z} + V^- e^{j\beta z} \\ &= V^+ e^{-j\beta z} + \Gamma_L e^{-j2\beta L} V^+ e^{j\beta z} \\ &= V^+ \left( e^{-j\beta z} + \Gamma_L e^{-j2\beta L} e^{j\beta z} \right) \end{aligned}$$

$$V(z) = V^+ e^{-j\beta L} \left( e^{-j\beta(z-L)} + \Gamma_L e^{j\beta(z-L)} \right)$$

$$I(z) = V^+ e^{-j\beta L} \left( \frac{1}{R_c} e^{-j\beta(z-L)} - \frac{1}{R_c} \Gamma_L e^{j\beta(z-L)} \right)$$

# Impedance as a function of position




$$V(z) = V^+ e^{-j\beta L}\left(e^{-j\beta(z-L)} + \Gamma_L e^{j\beta(z-L)}\right)$$

$$I(z) = V^+ e^{-j\beta L}\left(\frac{1}{R_c} e^{-j\beta(z-L)} - \frac{1}{R_c}\Gamma_L e^{j\beta(z-L)}\right)$$

At any location $z$ on the transmission line, define the impedance $Z(z)$ as

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V^+ e^{-j\beta L}\left(e^{-j\beta(z-L)} + \Gamma_L e^{j\beta(z-L)}\right)}{V^+ e^{-j\beta L}\left(\frac{1}{R_c} e^{-j\beta(z-L)} - \frac{1}{R_c}\Gamma_L e^{j\beta(z-L)}\right)}$$

# Impedance $Z(z)$

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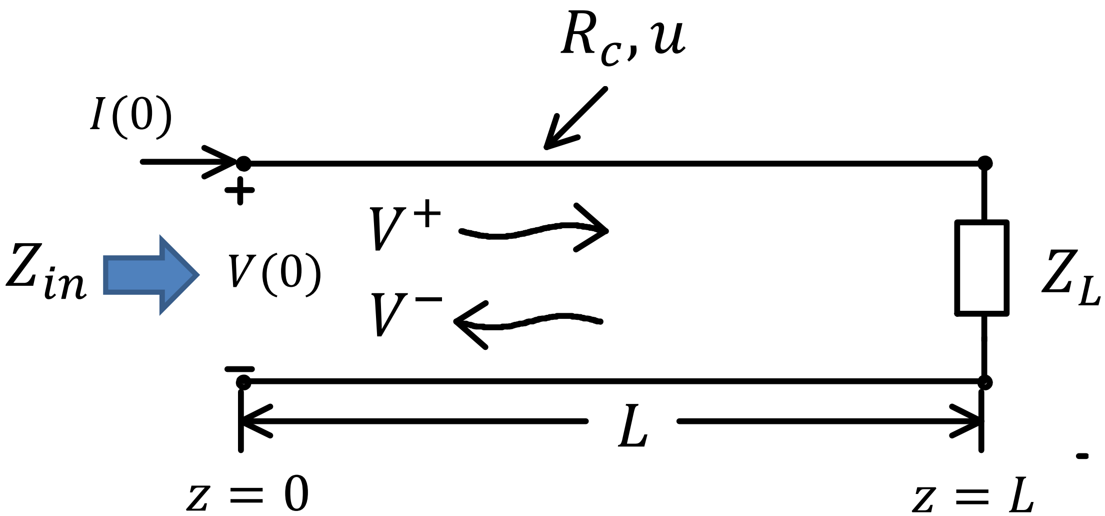
$$Z(z) = R_c\frac{\left(e^{-j\beta(z-L)} + \Gamma_L e^{j\beta(z-L)}\right)}{\left(e^{-j\beta(L-z)} - \Gamma_L e^{j\beta(L-z)}\right)}$$

$$Z(z) = R_c\frac{e^{-j\beta(z-L)}\left(1 + \Gamma_L e^{j2\beta(z-L)}\right)}{e^{-j\beta(z-L)}\left(1 - \Gamma_L e^{j2\beta(z-L)}\right)}$$

$$Z(z) = R_c\frac{\left(1 + \Gamma_L e^{j2\beta(z-L)}\right)}{\left(1 - \Gamma_L e^{j2\beta(z-L)}\right)}$$

We will use this formula later!

# The Input Impedance




$$V(z) = V^+ e^{-j\beta L}\left(e^{-j\beta(z-L)} + \Gamma_L e^{j\beta(z-L)}\right)$$

$$I(z) = V^+ e^{-j\beta L}\left(\frac{1}{R_c} e^{-j\beta(z-L)} - \frac{1}{R_c}\Gamma_L e^{j\beta(z-L)}\right)$$

$$\Gamma_L = \frac{Z_L - R_c}{Z_L + R_c}$$

$$Z_{in} = \frac{V(0)}{I(0)}$$

$$V(0) = V^+ e^{-j\beta L}\left(e^{-j\beta(0-L)} + \Gamma_L e^{j\beta(0-L)}\right)$$

$$V(0) = V^+ e^{-j\beta L}\left(e^{j\beta L} + \Gamma_L e^{-j\beta L}\right)$$

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$$I(0) = V^+ e^{-j\beta L}\left(\frac{1}{R_c} e^{j\beta L} - \frac{1}{R_c}\Gamma_L e^{-j\beta L}\right)$$

$$Z_{in} = \frac{V(z=0)}{I(z=0)} = \frac{V^+ e^{-j\beta L}\left(e^{j\beta L} + \Gamma_L e^{-j\beta L}\right)}{V^+ e^{-j\beta L}\left(\frac{1}{R_c} e^{j\beta L} - \frac{1}{R_c}\Gamma_L e^{-j\beta L}\right)}$$

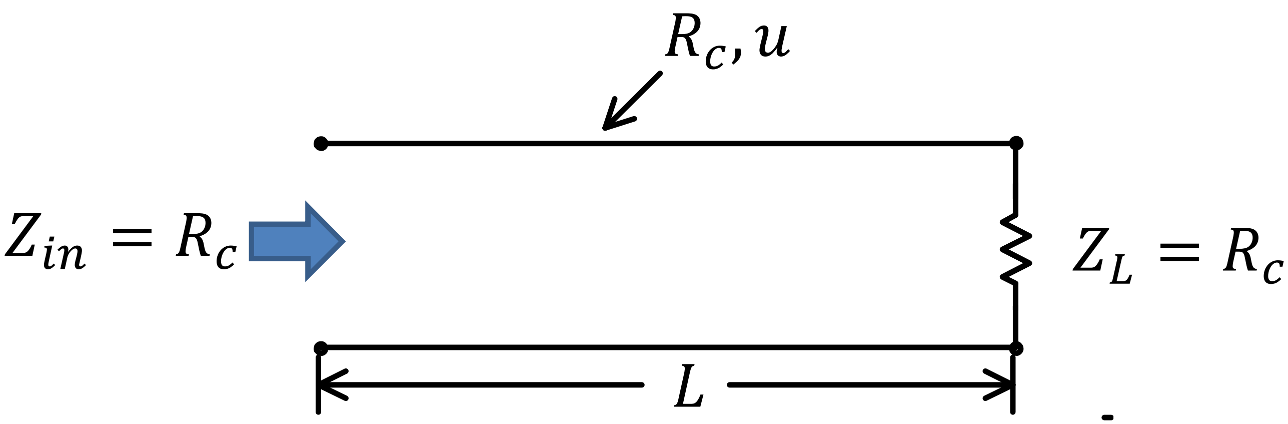
$$Z_{in} = R_c \frac{Z_L + jR_c \tan \beta L}{R_c + jZ_L \tan \beta L}$$

Homework: do the missing algebra to derive this formula.

# Important Special Cases

## Matched Load

- What is the input impedance of a transmission line terminated with a matched load?



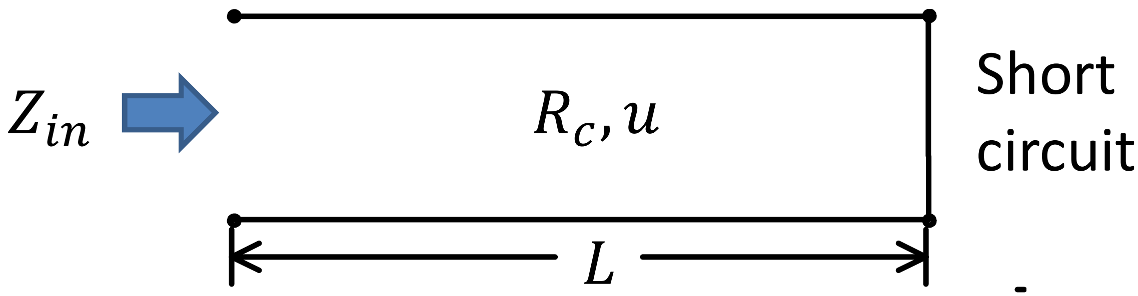

- If $Z_L = R_c$, then the input impedance is

$$Z_{in} = R_c \frac{Z_L + jR_c \tan \beta L}{R_c + jZ_L \tan \beta L} = R_c \frac{R_c + jR_c \tan \beta L}{R_c + jR_c \tan \beta L} = R_c$$

- So the input impedance of a transmission line terminated with a matched load is $Z_{in} = R_c$.

Short-Circuited Transmission Line

- What is the input impedance of a transmission line terminated with a short circuit?

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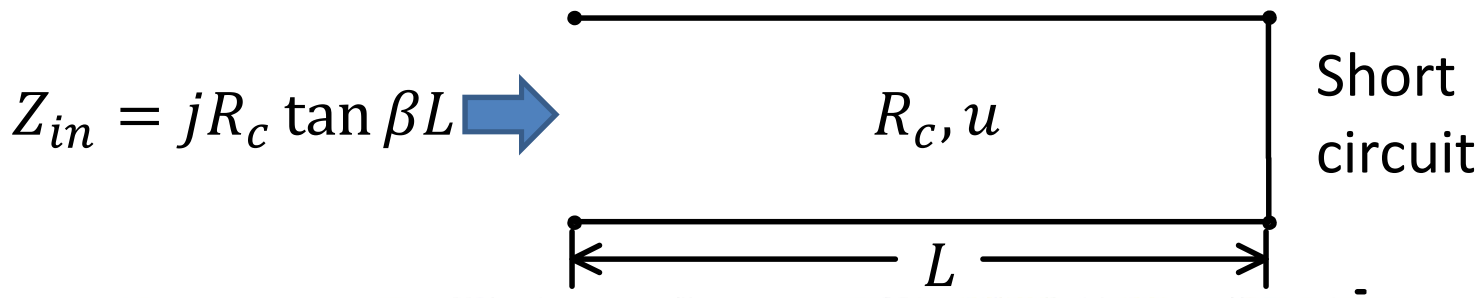
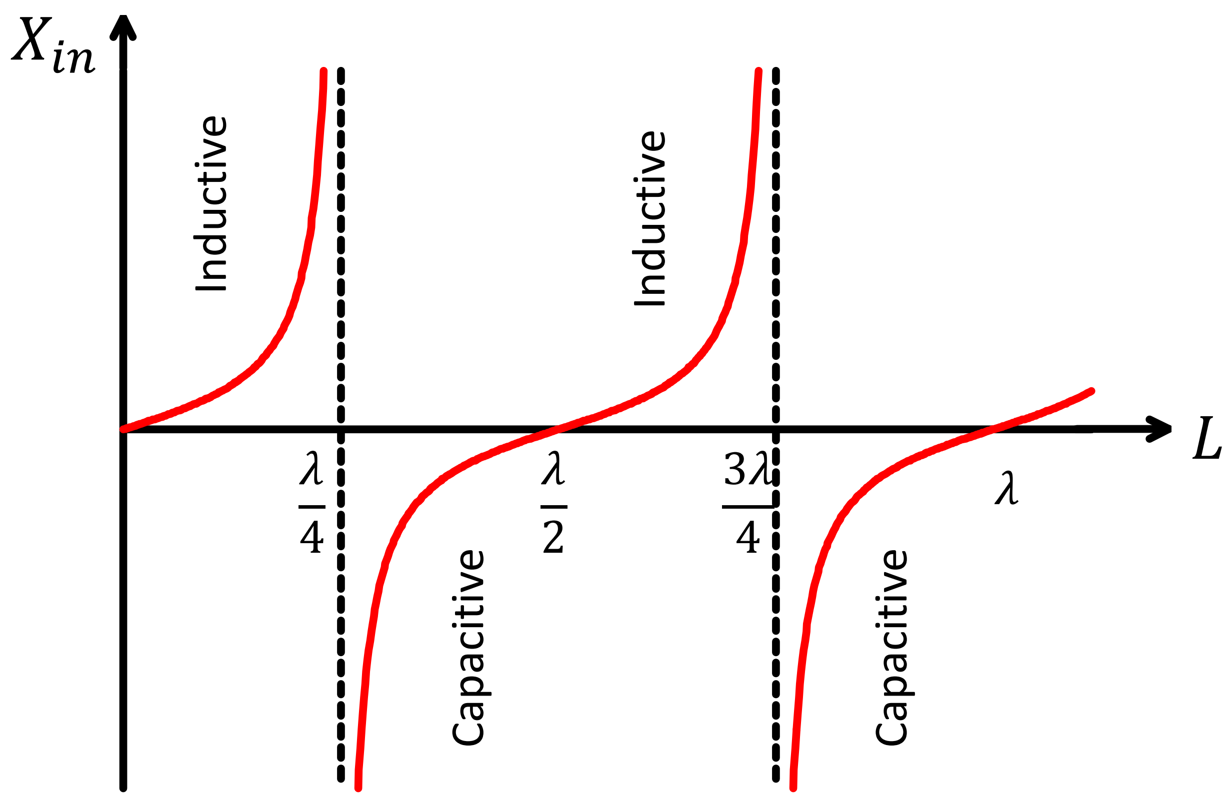
- A short length of transmission line terminated with a short circuit is called a "stub" and can be used for impedance matching. (We will do this later in the course.)
- What is the input impedance of a transmission line terminated with a short circuit, $Z_L = 0$?

$$Z_{in} = R_c \frac{Z_L + jR_c \tan \beta L}{R_c + jZ_L \tan \beta L} = R_c \frac{0 + jR_c \tan \beta L}{R_c + j0 \tan \beta L} = jR_c \tan \beta L$$




- $Z_{in} = jR_c \tan \beta L$ is "reactive", meaning that it is pure imaginary.
- We can write $Z_{in} = jX_{in}$ where the reactance is $X_{in} = R_c \tan \beta L$




When $\beta L \to \frac{\pi}{2}$, $\tan \beta L \to +\infty$.

$$L = \frac{\pi}{2\beta} = \frac{\pi}{2\frac{2\pi}{\lambda}} = \frac{\lambda}{4}$$

## Open-Circuited Transmission Line

- What is the input impedance of a transmission line terminated with an open circuit, $Z_L \to \infty$?

$$Z_{in} = \lim_{Z_L \to \infty} R_c \frac{Z_L + jR_c \tan \beta L}{R_c + jZ_L \tan \beta L}$$

$$Z_{in} = R_c \frac{Z_L}{jZ_L \tan \beta L} = -jR_c \cot \beta L$$

- $Z_{in} = -jR_c \cot \beta L$ is "reactive" meaning pure imaginary.
- We can write $Z_{in} = jX_{in}$ where the reactance is $X_{in} = -R_c \cot \beta L$