ELEC353 Lecture Notes Set 3

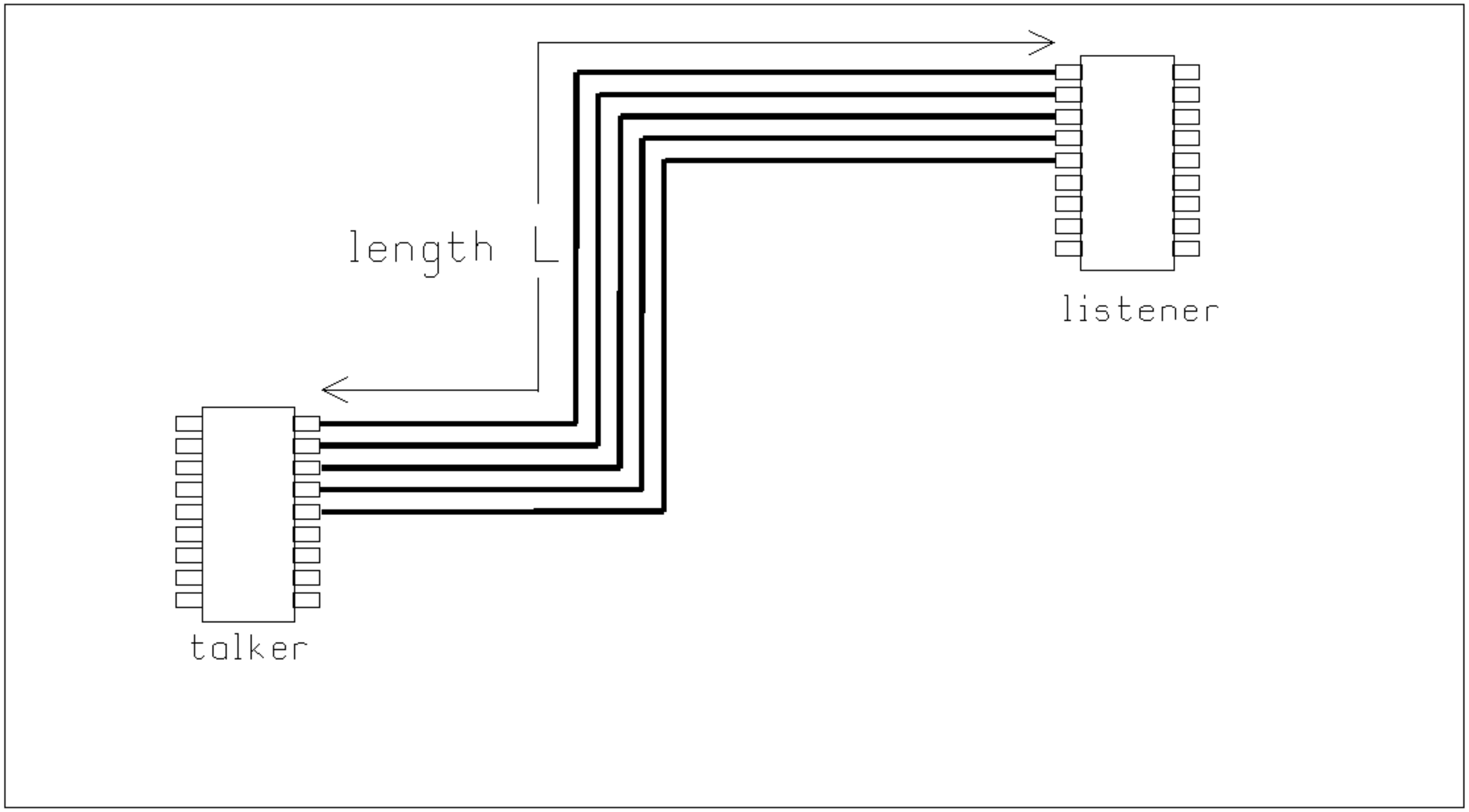
The course web site is:
www.ece.concordia.ca/~trueman/web_page_353.htm

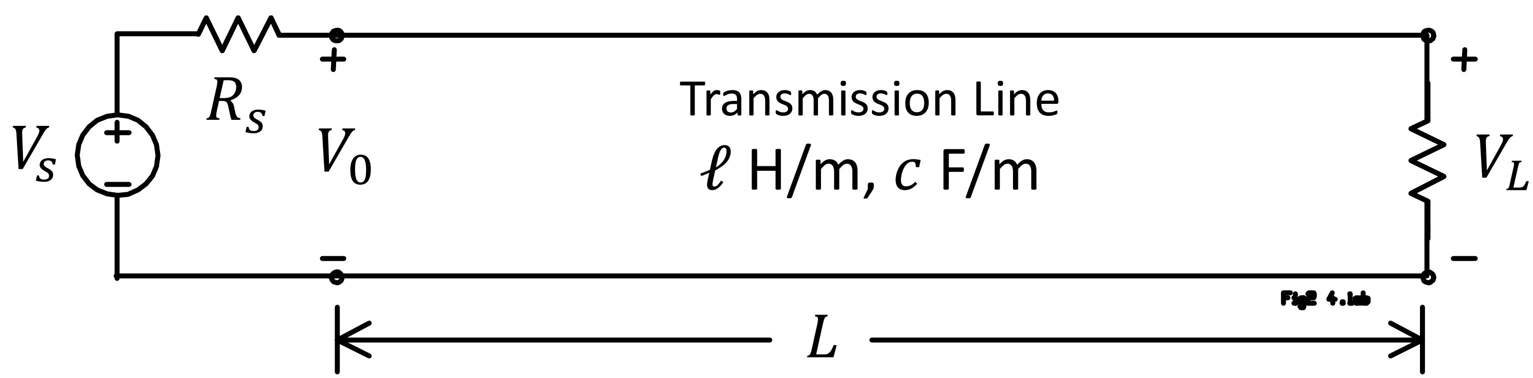
- The course outline
- The lecture notes
- The homework assignment each week
- A set of practice problems with solutions
- Software: BOUNCE, TRLINE, WAVES

The homework assignments are posted on the course web site.

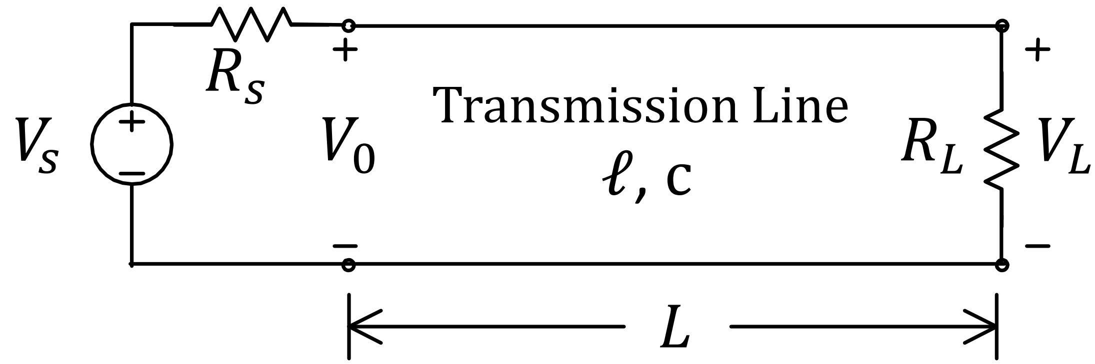
Homework #1: Do this assignment by January 18th, 2019.
Homework #2: Do this assignment by January 25nd.

Equivalent Circuit for an Interconnection

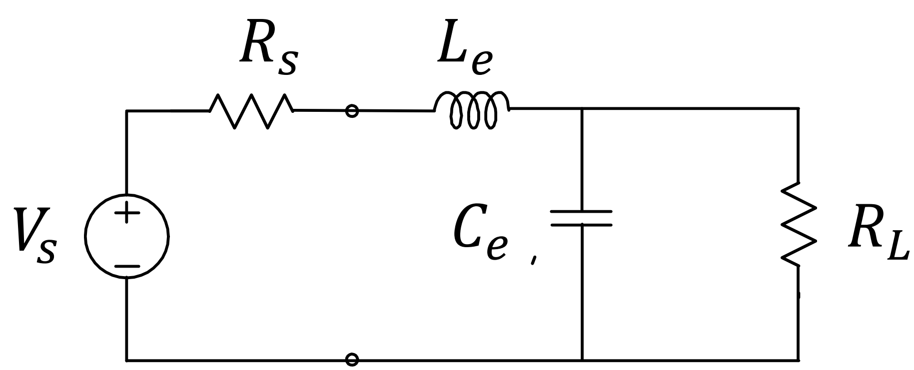




Modelling the Transmission Line

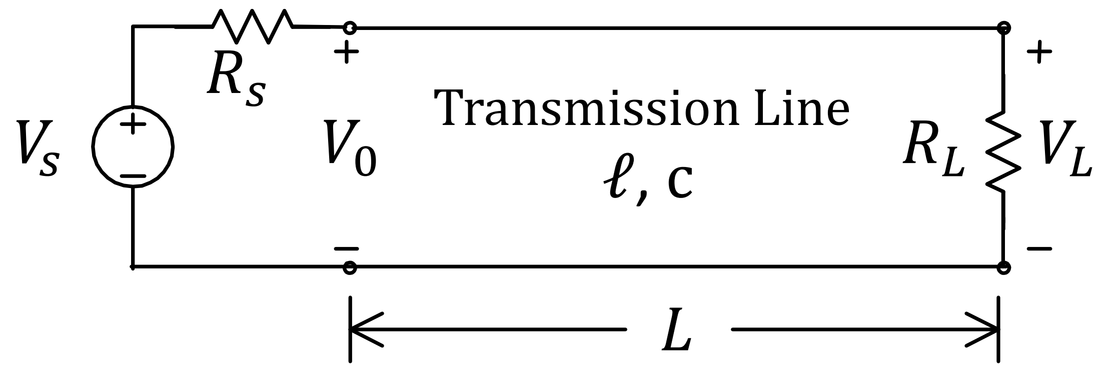


ℓ =Inductance-per-unit-length
c =Capacitance-per-unit-length

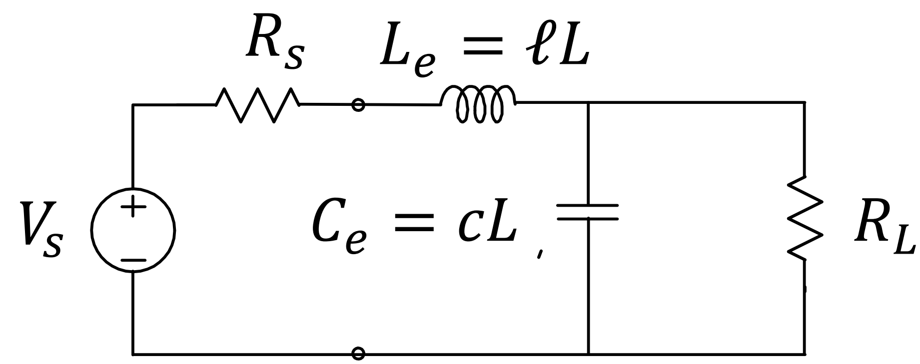


$L_e = \ell L$= Inductance-per-unit-length x Length
$C_e = cL$= Capacitance-per-unit-length x Length

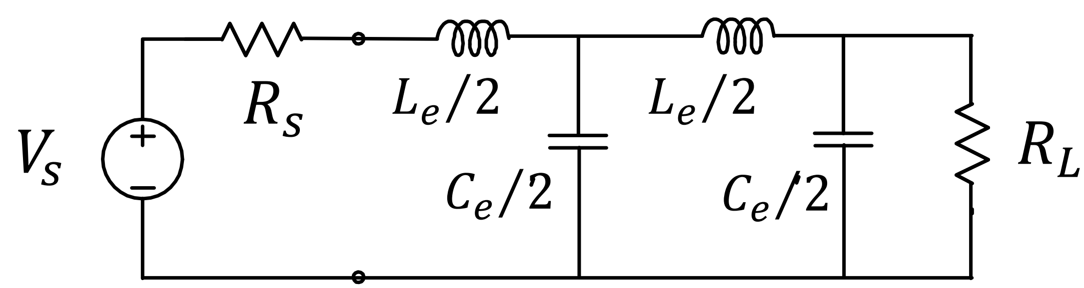
Ladder Network Approximations



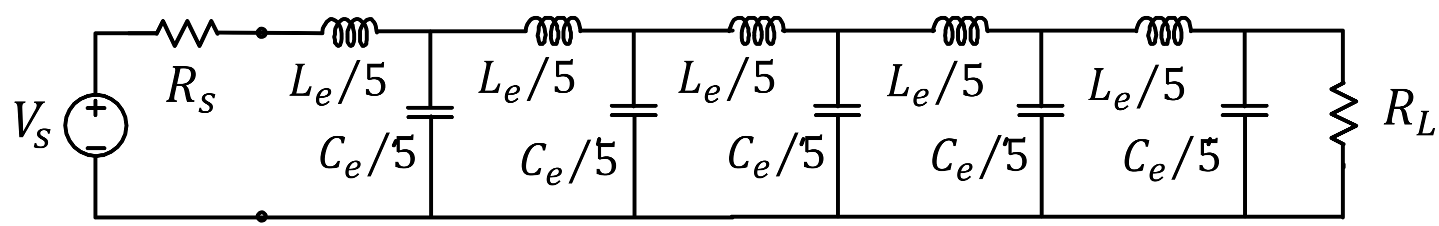
One cell:



Two cells:

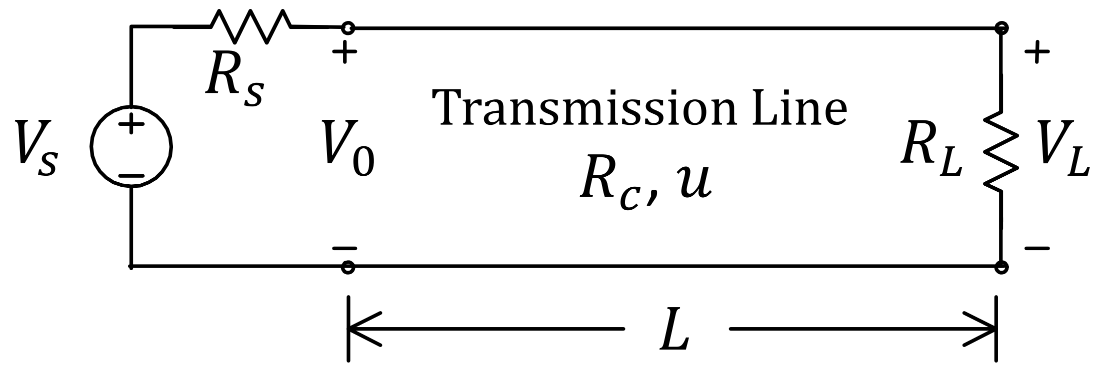


Five cells:



Step Response of a Transmission Line

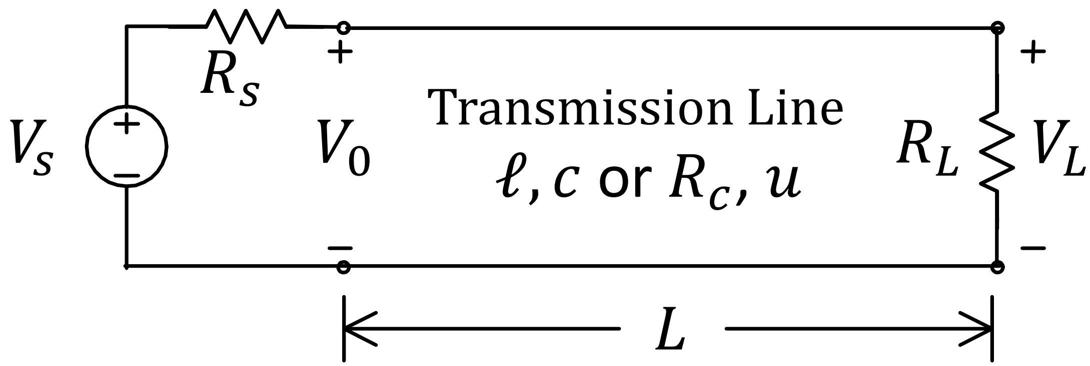
We will use this problem to test ladder networks.



A logic chip has internal resistance $R_s = 15\Omega$ and generates a step-function voltage, that steps up from $V_s = 0$ volts to $V_s = 10$ volt at $t = 0$. The rise time of the step function is $T_r = 0.1$ ns. The output of the chip is connected to the input of another logic chip by a circuit path that is $L = 25$ cm in length. The interconnect has inductance-per-unit-length $\ell = 0.320$ microHenries/meter and capacitance-per-unit=length $c = 50.0$ picoFarads/meter. The load resistor has value $R_L = 400\ \Omega$.

Find the voltage at the load, V_L, as a function of time.

Homework: Solve this problem with the BOUNCE program.



- In a later lecture, we will show that $R_c = \sqrt{\frac{\ell}{c}}$ and $u = \frac{1}{\sqrt{\ell c}}$.
- Given: $\ell = 0.320$ microHenries/meter, $c = 50$ picoFarads/meter

Analysis:

The characteristic resistance of the transmission line is $R_c = \sqrt{\frac{\ell}{c}}$ =80 ohms.

The speed of travel of waves on the transmission line is $u = \frac{1}{\sqrt{\ell c}} = 25$ cm/ns.

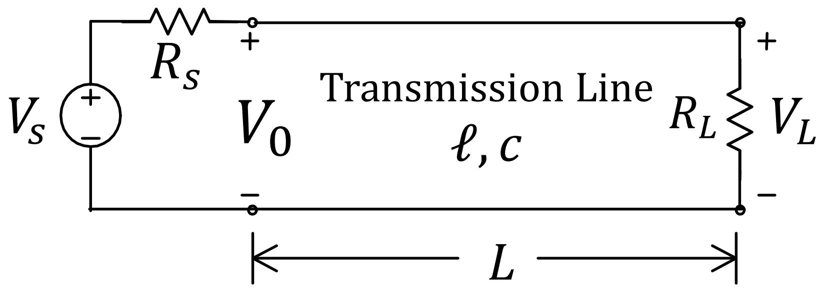
Hence the time delay is $T_d = \frac{L}{u} = \frac{25}{25} = 1$ ns.

The "rule of thumb" states that "distributed" circuit analysis must be used if $\frac{T_r}{T_d} < 2.5$.

T_r=0.1 ns

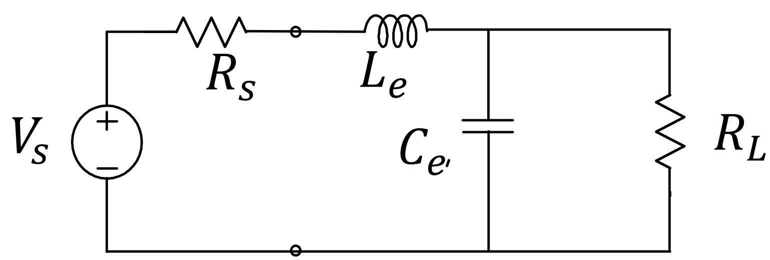
For this circuit, $\frac{T_r}{T_d} = \frac{0.1}{1} = 0.1$ so we need to use distributed circuit analysis.

Cell Model Approximations: Ladder Networks

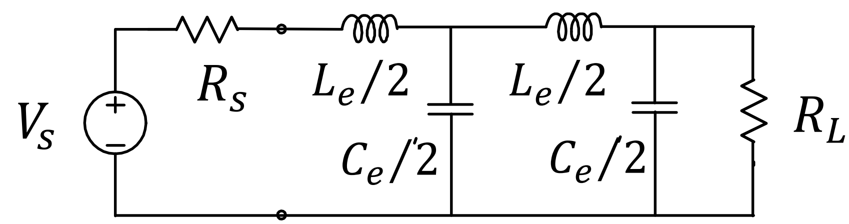


Line length L= 25 cm
Inductance per unit length: $\ell = 0.320$ microH/m
Capacitance per unit length: $c = 50$ picoF/m
Inductance $L_e = \ell L = 0.320x0.25 = 80$ nH
Capacitance $C_e = cL = 50x0.25 = 12.5$ pF

One cell model:



Two cell model:

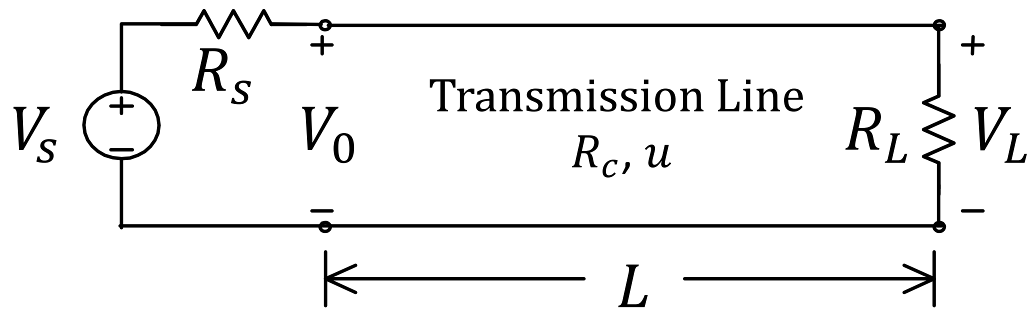


Let's try simulating the transmission line with ladder networks of 1, 2, 3, 5 and 10 cells, and compare with the true response using distributed circuit analysis.

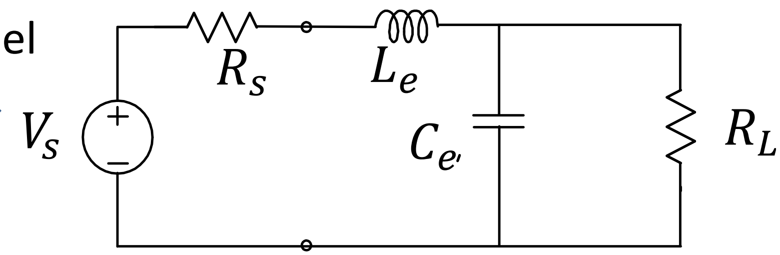
Model	Cell length	Inductance	Capacitance
1 cell	25 cm	80 nH	12.5 pF
2 cells	12.5	40	6.25
3 cells	8.333	26.67	4.167
5 cells	5	16	2.5
10 cells	2.5	8	1.25

One-Cell Ladder Network

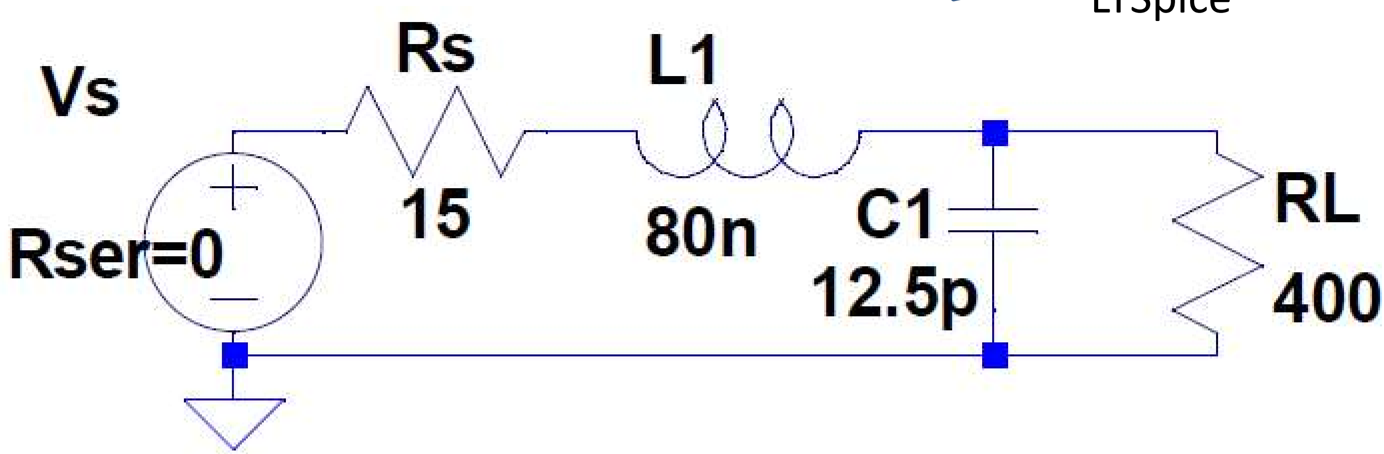
I used “LTSpice IV” for these simulations.
LTSpice is a free download from http:www.linear.com/software



model

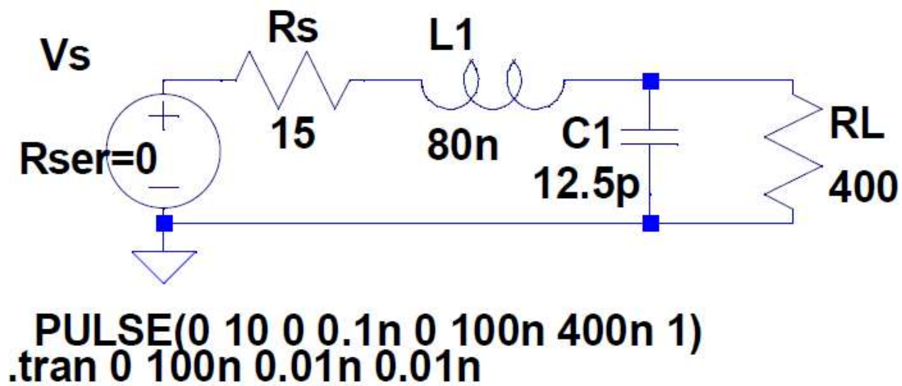


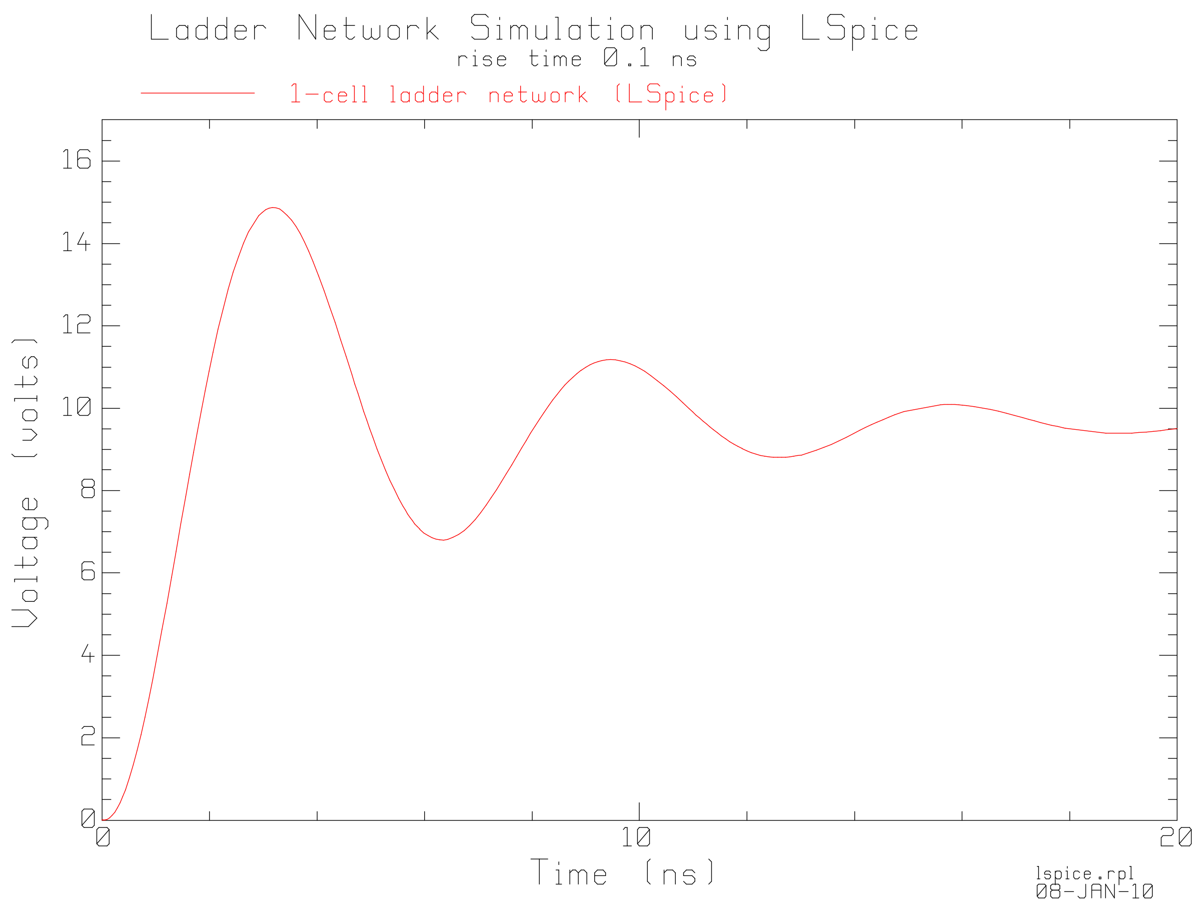
Implementation in
LTSpice



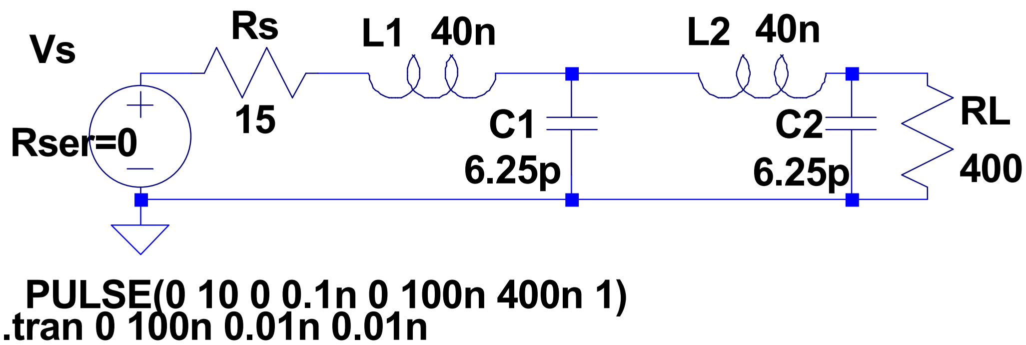
PULSE(0 10 0 0.1n 0 100n 400n 1)
.tran 0 100n 0.01n 0.01n

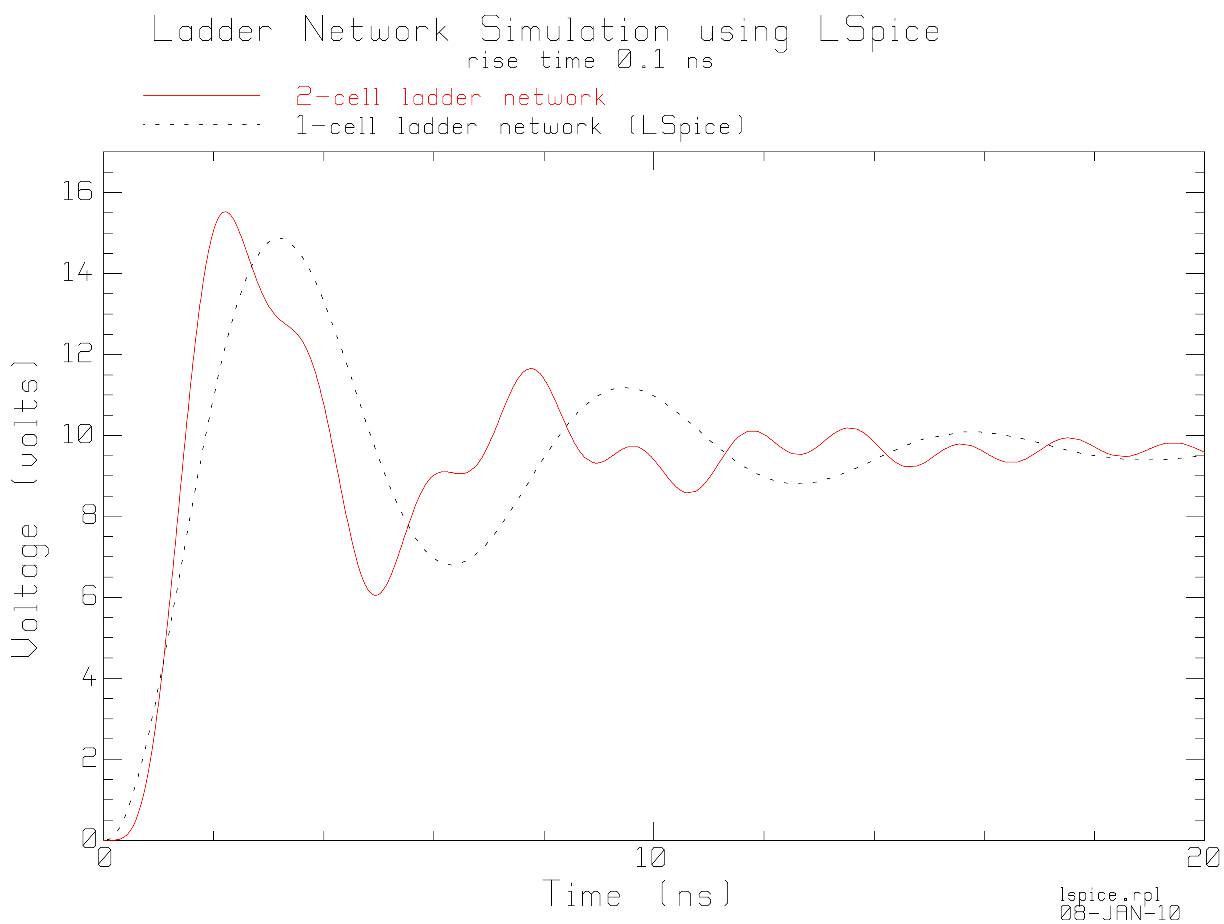
Response of the one-cell ladder network:



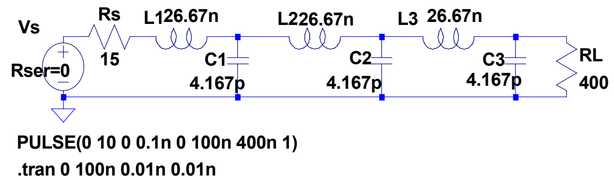


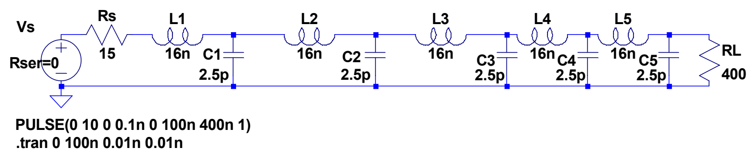
Two-Cell Ladder Network

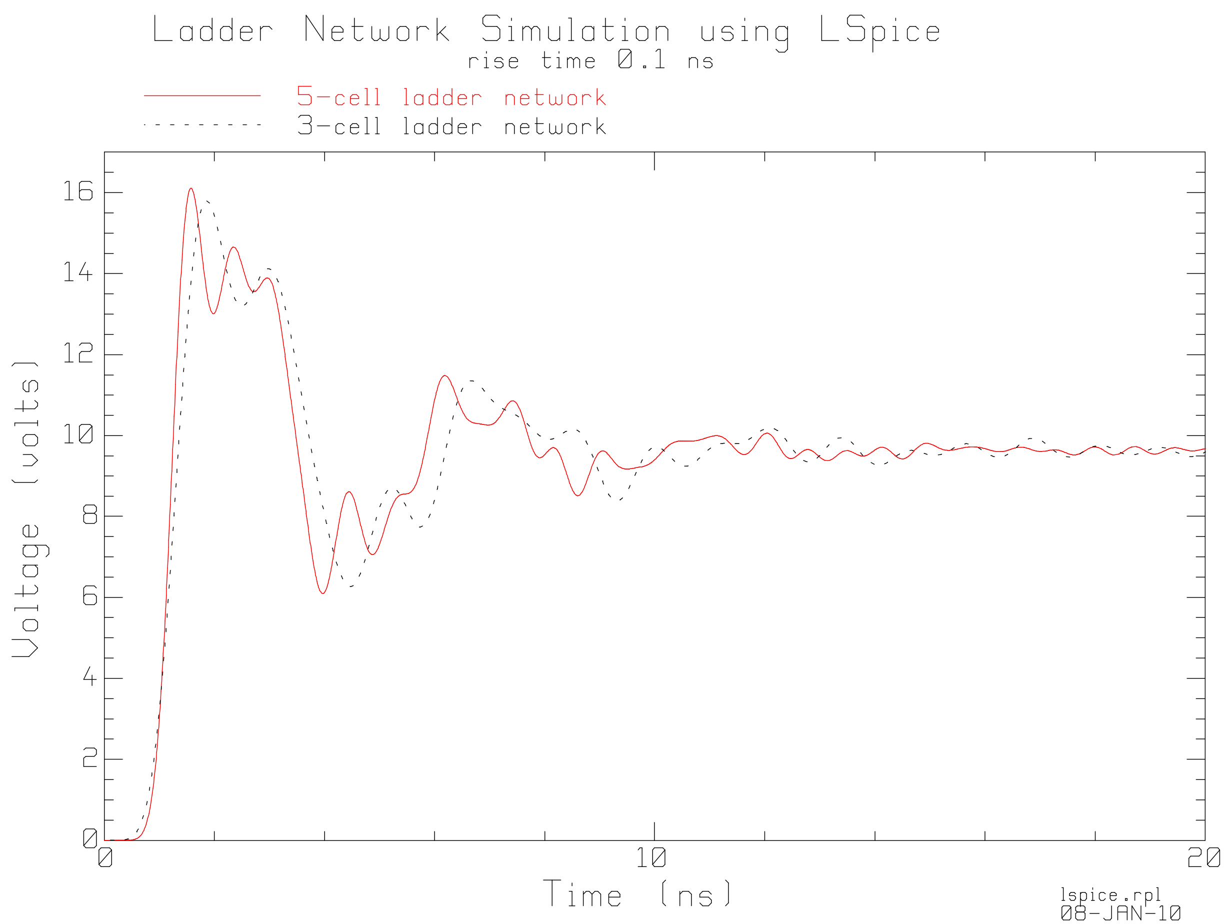




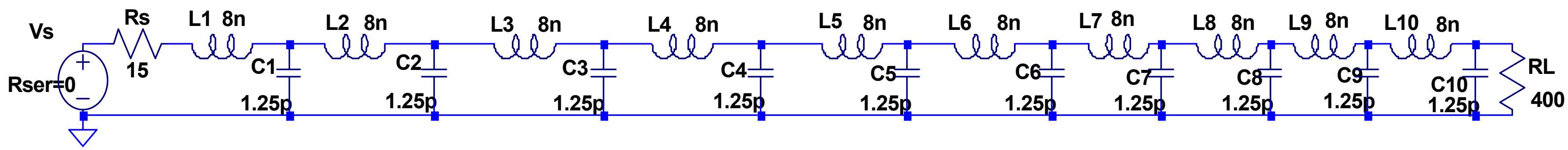
Three-Cell and Five-Cell Ladder Network





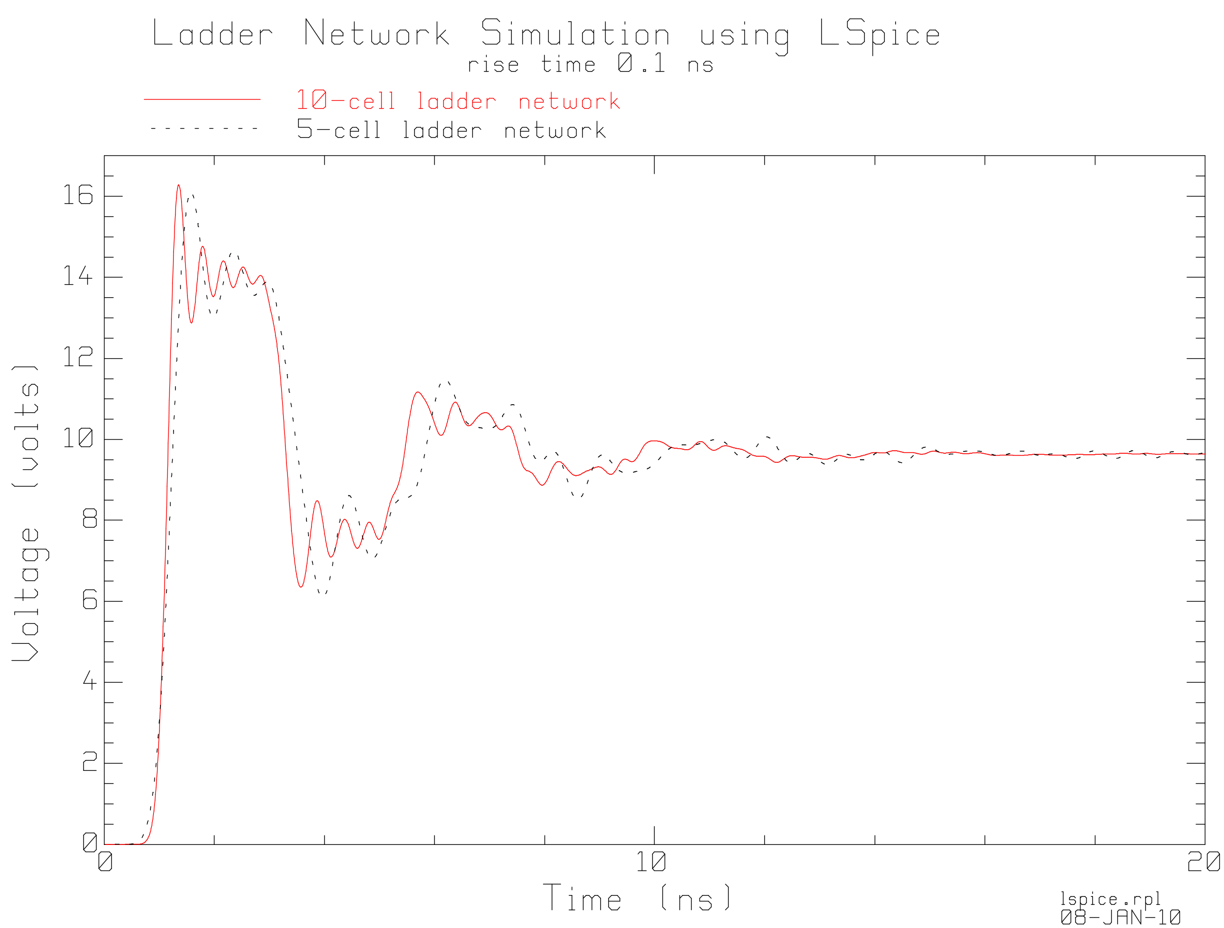


Ten-Cell Ladder Network

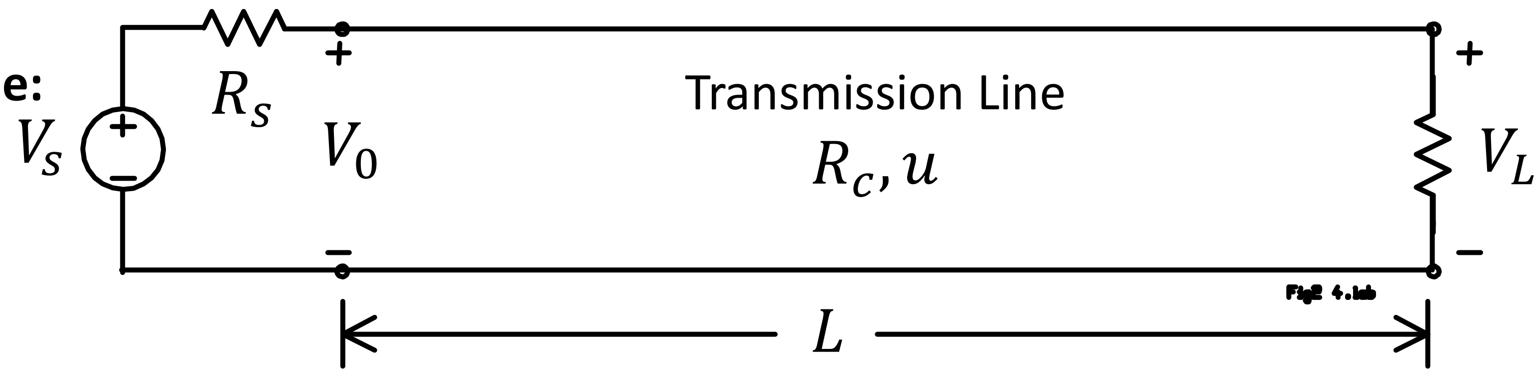


PULSE(0 10 0 0.1n 0 100n 400n 1)

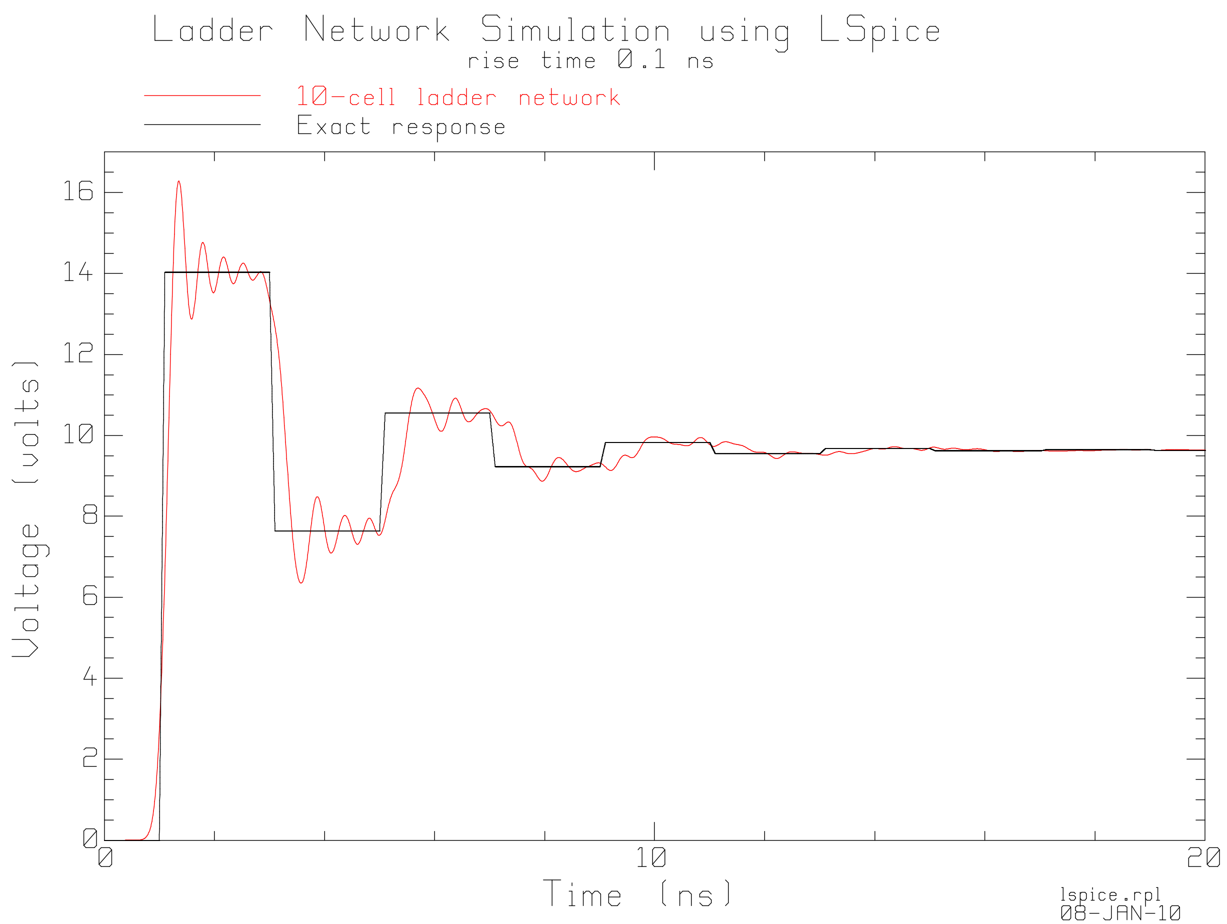
.tran 0 100n 0.01n 0.01n



Exact Response:



The time delay is $T_d = \frac{L}{u} = \frac{25}{25} = 1$ ns



Conclusion:

- The two-cell model is better than the one-cell.
- The more cells we use, the better the approximation.

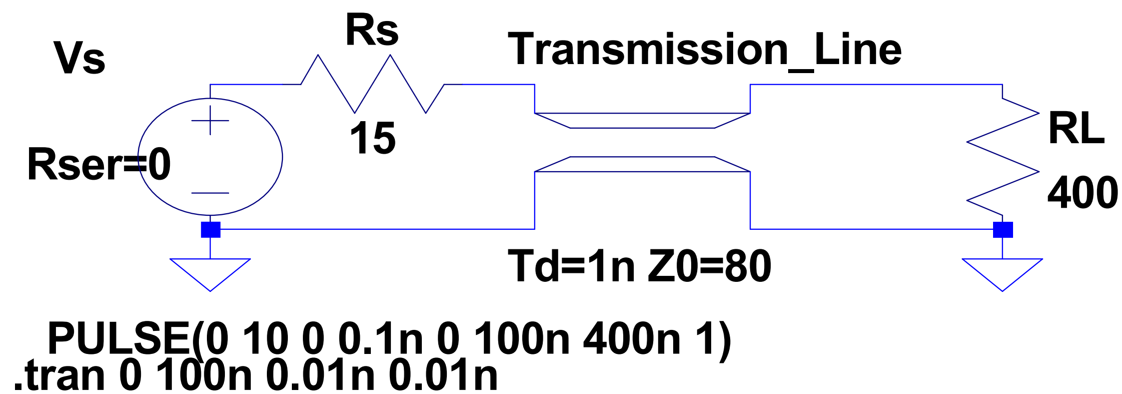
- The ten-cell model has “ringing” compared to the exact response.

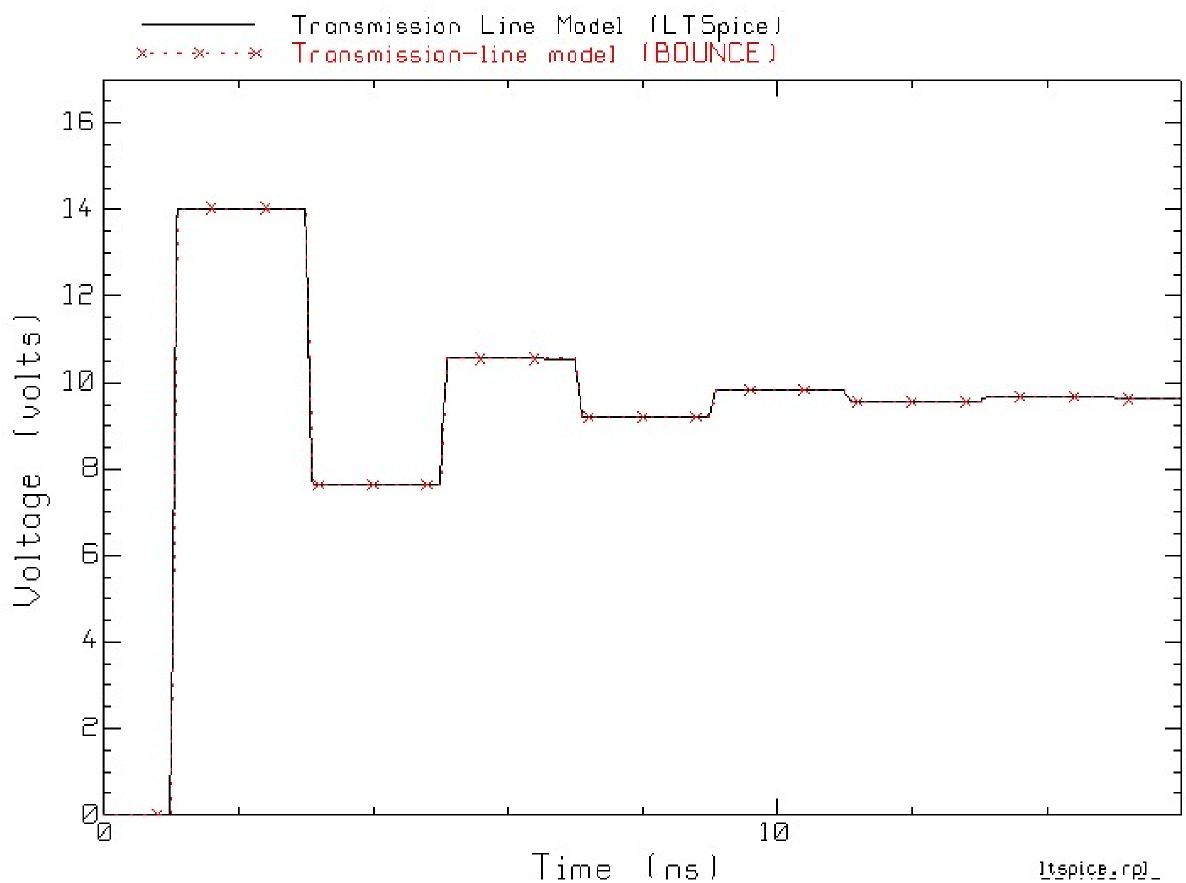
- Isn’t there a better way to simulate the transmission line?

YES

We can do exact analysis using calculus.

Remark: LTSpice includes a Transmission Line Model

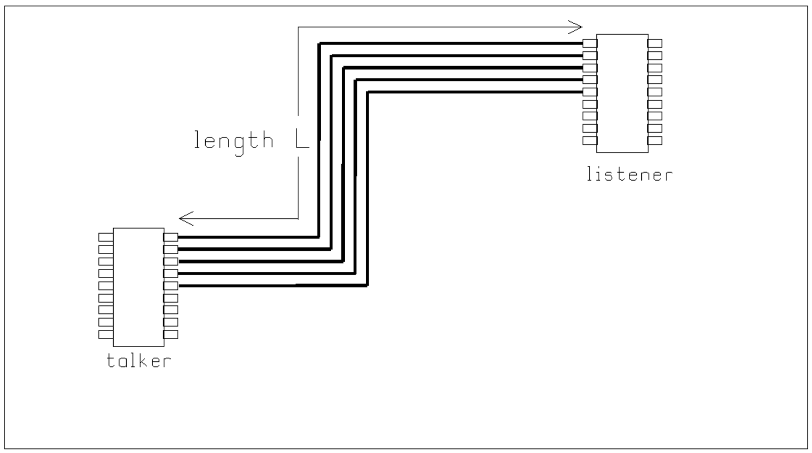


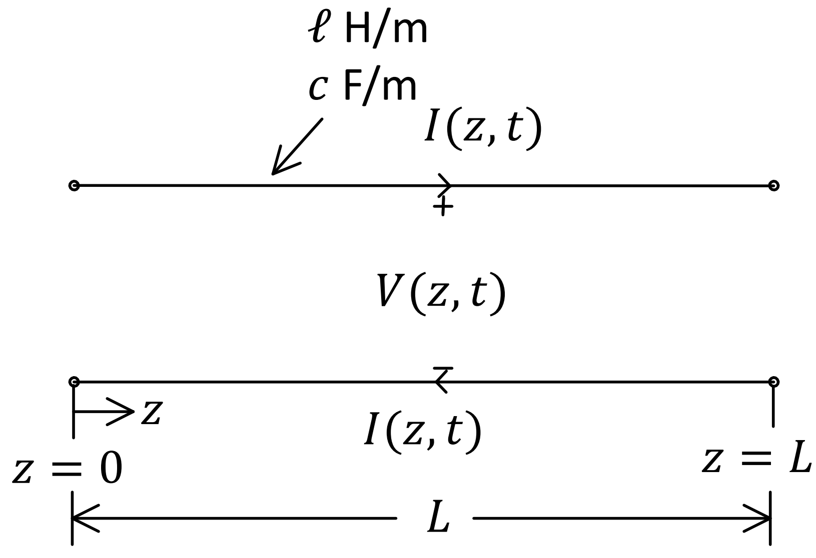


LTSpice and BOUNCE calculate exactly the same waveform for this problem.

Exact Analysis of Transmission Lines

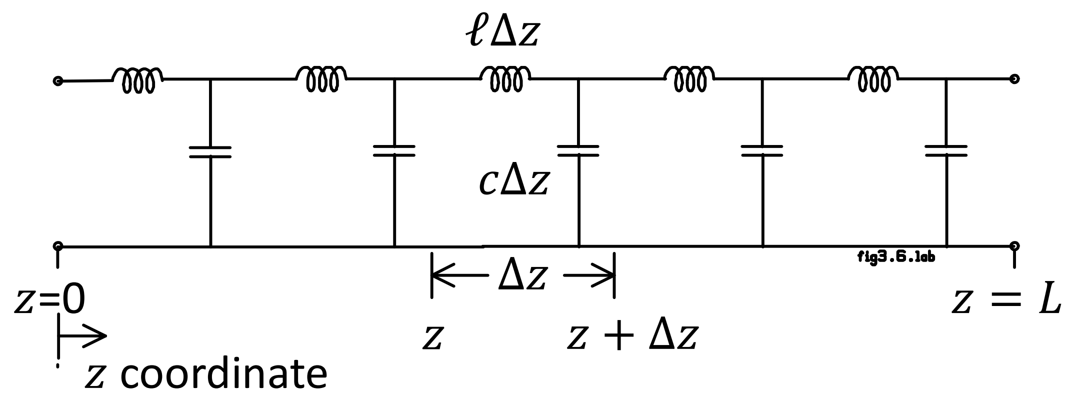
Inan, Inan and Said Section 2.2





- The circuit schematic for a "transmission line" is two parallel straight lines, labelled with:
  - The length of the line L
  - The inductance per unit length ℓ H/m
  - The capacitance per unit length c F/m
  - Or equivalently, the characteristic resistance R_c ohms and the speed of travel u m/s.
- We need a distance coordinate z to measure distance along the transmission line from the start of the transmission line at $z=0$ to the end of the transmission line at $z=L$.

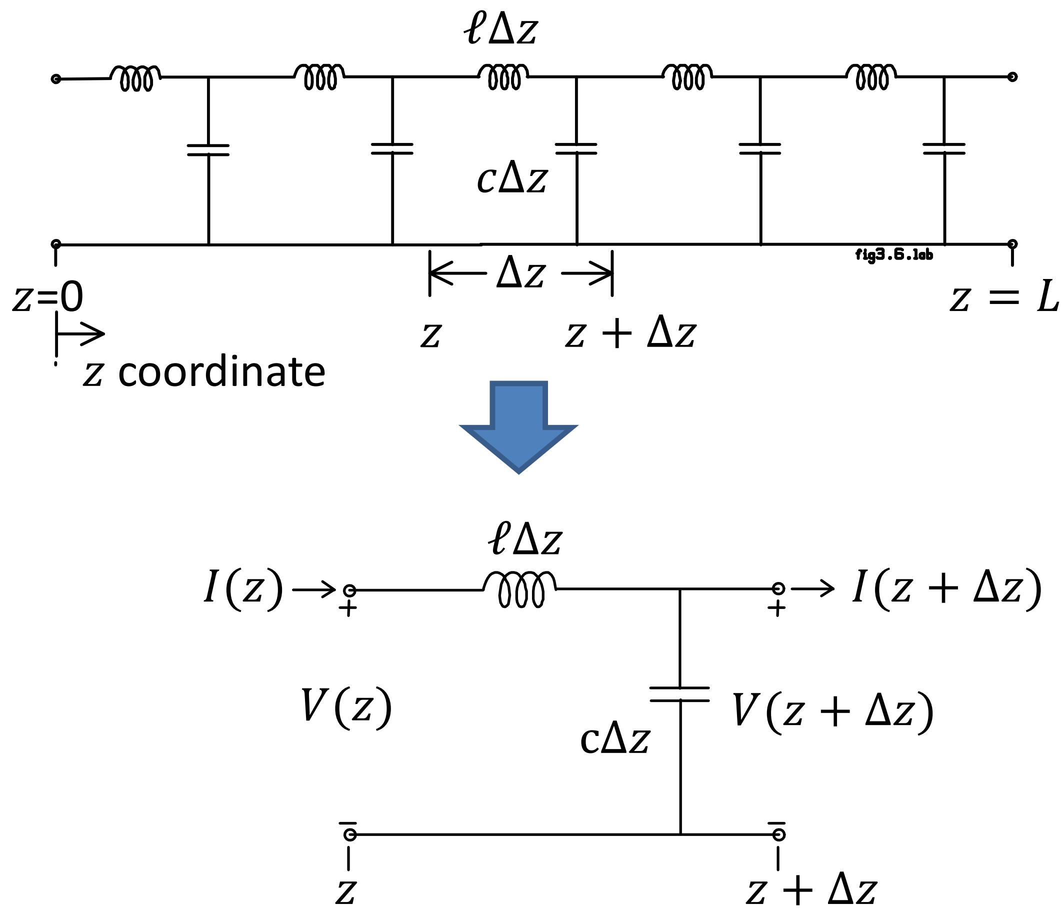
Circuit Analysis for the Cell Model



- Divide the transmission line into N cells, each of length $\Delta z = \frac{L}{N}$
- Each cell has inductance $\frac{\ell L}{N} = \ell\Delta z$ and capacitance $\frac{cL}{N} = c\Delta z$

Consider one typical cell:

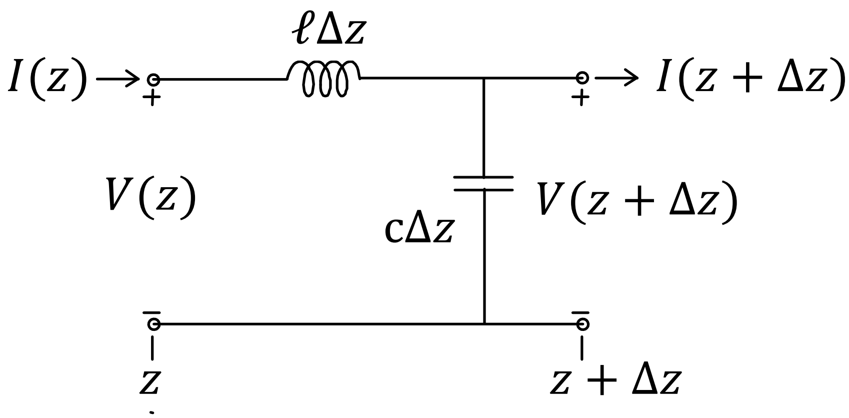
- The inductance of the cell is $\ell\Delta z$
- The capacitance is $c\Delta z$
- The coordinate at the "input" of the cell is z
- The coordinate at the "output" is $z + \Delta z$



Focus on this one cell:

- The input voltage is $V(z)$
- The input current is $I(z)$
- The output voltage is $V(z+\Delta z)$
- The output current is $I(z+\Delta z)$

Write KVL and KCL for One Cell



Inductance:

$$v = L\frac{di}{dt}$$

- $L = \ell\Delta z$
- $I(z,t)$ is a function of two variables so use the partial derivative:

$$v = \ell\Delta z\frac{\partial I}{\partial t}$$

Kirchhoff's Voltage Law:

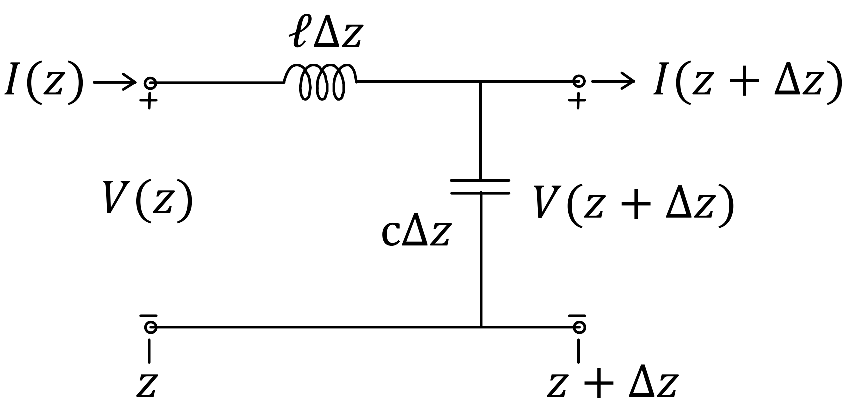
$$V(z) - (\ell\Delta z)\frac{\partial I}{\partial t} - V(z+\Delta z) = 0$$

$$\frac{V(z+\Delta z) - V(z)}{\Delta z} = -\ell\frac{\partial I}{\partial t}$$

$$\lim_{\Delta z \to 0}\frac{V(z+\Delta z) - V(z)}{\Delta z} = -\ell\frac{\partial I}{\partial t}$$

$$\frac{\partial V}{\partial z} = -\ell\frac{\partial I}{\partial t}$$

Kirchhoff's Current Law



$$I(z)-(c\Delta z)\frac{\partial V}{\partial t}-I(z+\Delta z)=0$$

$$\frac{I(z+\Delta z)-I(z)}{\Delta z}=-c\frac{\partial V}{\partial t}$$

$$\lim_{\Delta z\rightarrow 0}\frac{I(z+\Delta z)-I(z)}{\Delta z}=-c\frac{\partial V}{\partial t}$$

$$\frac{\partial I}{\partial z}=-c\frac{\partial V}{\partial t}$$

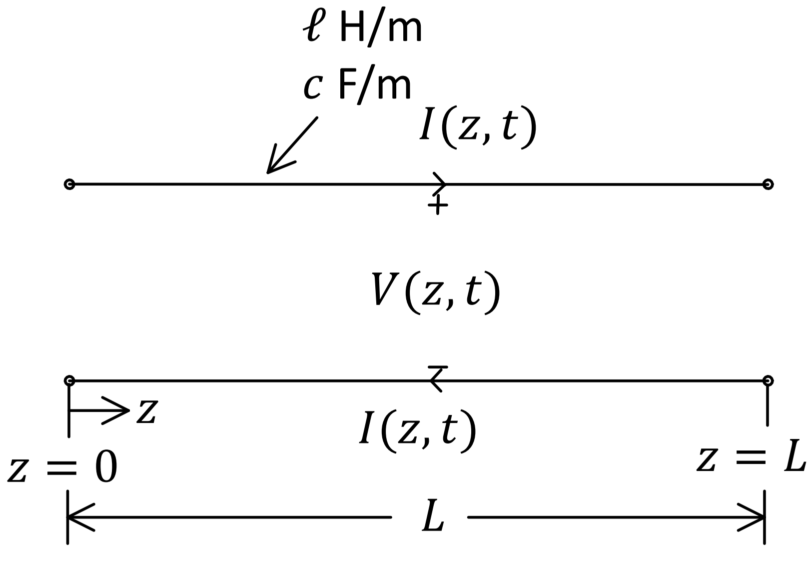
Capacitance:

$$i=C\frac{dv}{dt}$$

- $C=c\Delta z$
- $V(z,t)$ is a function of two variables so use the partial derivative:

$$i=c\Delta z\frac{\partial V}{\partial t}$$

Summary-Transmission Line Equations



$$\frac{\partial V}{\partial z} = -\ell \frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -c \frac{\partial V}{\partial t}$$

These equations are called "Transmission Line Equations".
They are also sometimes called "Telegrapher's Equations" because they were first derived about 120 years ago in the design of telegraph systems.

Solving the Transmission Line Equations

Inan, Inan and Said Section 2.2.2

$$\frac{\partial V}{\partial z} = -\ell \frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -c \frac{\partial V}{\partial t}$$

Eliminate the current I(z):

$$\frac{\partial V}{\partial z} = -\ell \frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -c \frac{\partial V}{\partial t}$$

$$\frac{\partial^2 V}{\partial z^2} = -\ell \frac{\partial}{\partial z} \frac{\partial I}{\partial t}$$

$$\frac{\partial^2 V}{\partial z^2} = -\ell \frac{\partial}{\partial t} \left(-c \frac{\partial V}{\partial t} \right)$$

$$\frac{\partial^2 V}{\partial z^2} = \ell c \frac{\partial^2 V}{\partial t^2}$$

Similarly we may show that:

$$\frac{\partial^2 I}{\partial z^2} = \ell c \frac{\partial^2 I}{\partial t^2}$$

These two equations are called "**wave equations**".

Phase Velocity:

$$u = \frac{1}{\sqrt{\ell c}} \text{ m/s}$$

Rewrite the wave equations as:

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 I}{\partial t^2}$$

General Solution to the Wave Equation

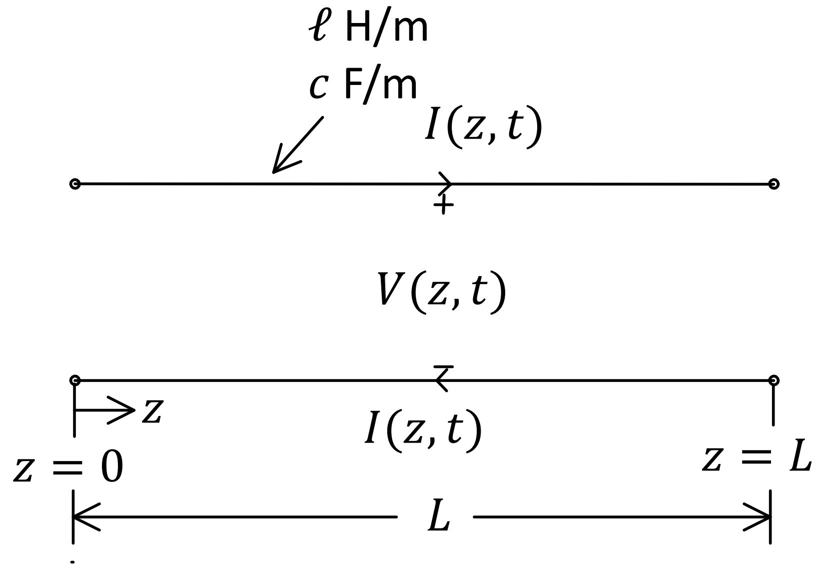
$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}$$

$$V(z,t) = f^+\left(t - \frac{z}{u}\right) + f^-\left(t + \frac{z}{u}\right)$$

where f^+ is *any function*, and f^- is *any function*.

- This says that any function of the variable $t - \frac{z}{u}$ is a solution to the wave equation, and also any function of the variable $t + \frac{z}{u}$ is a solution to the wave equation.
- In practice f^+ and f^- are determined by the wave shape of the generator: step function, pulse function, sinusoid, …

Interpretation of the Solution:



The voltage is $V(z) = f^+\left(t - \frac{z}{u}\right) + f^-\left(t + \frac{z}{u}\right)$

$V(z) = f^+\left(t - \frac{z}{u}\right)$ is voltage "wave" that travels in the positive z direction, from the source towards the load.

$V(z) = f^-\left(t + \frac{z}{u}\right)$ is a voltage "wave" that travels in the negative z direction, from the load towards the source.

Prove that $V(z,t) = f^{+}\left(t - \frac{z}{u}\right)$ Satisfies the Wave Equation $\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}$

Calculate

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}\left[f^{+}\left(t - \frac{z}{u}\right)\right]$$

We can do this using the "Chain Rule". Let $s = t - \frac{z}{u}$ and calculate

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}\left[f^{+}(s)\right] = \frac{\partial f^{+}}{\partial s}\frac{\partial s}{\partial z}$$

where $\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(t - \frac{z}{u}\right) = -\frac{1}{u}$

so

$$\frac{\partial V}{\partial z} = \frac{\partial f^{+}}{\partial s}\frac{\partial s}{\partial z} = -\frac{1}{u}\frac{\partial f^{+}}{\partial s}$$

and since $f^{+}(s)$ is a function of only one variable, s, we can write

$$\frac{\partial V}{\partial z} = -\frac{1}{u}\frac{df^{+}}{ds}$$

Then calculate

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}\left(-\frac{1}{u}\frac{df^+}{ds}\right)$$

by the Chain Rule as

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial s}\left(-\frac{1}{u}\frac{df^+}{ds}\right)\frac{\partial s}{\partial z} = \left(-\frac{1}{u}\frac{d^2 f^+}{ds^2}\right)\left(-\frac{1}{u}\right) = \frac{1}{u^2}\frac{d^2 f^+}{ds^2}$$

Thus we have evaluated the left-hand side of the wave equation as

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{d^2 f^+}{ds^2}$$

Wave Equation:

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}$$

We have evaluated the L.H.S. as:

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{d^2 f^+}{ds^2}$$

Evaluate the R.H.S. of the wave equation: $$\frac{\partial^2 V}{\partial z^2}=\frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial V}{\partial t}=\frac{\partial}{\partial t}\left[f^+(s)\right]=\frac{\partial f^+}{\partial s}\frac{\partial s}{\partial t}=\frac{df^+}{ds}\frac{\partial s}{\partial t}$$

where $s=t-\frac{z}{u}$ so $\frac{\partial s}{\partial t}=\frac{\partial}{\partial t}\left(t-\frac{z}{u}\right)=1$

So

$$\frac{\partial V}{\partial t}=\frac{df^+}{ds}\frac{\partial s}{\partial t}=\frac{df^+}{ds}(1)=\frac{df^+}{ds}$$

$$\frac{\partial^2 V}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{df^+}{ds}\right)=\frac{d}{ds}\left(\frac{df^+}{ds}\right)\frac{\partial s}{\partial t}=\frac{d^2 f^+}{ds^2}(1)=\frac{d^2 f^+}{ds^2}$$

So the R.H.S. of the wave equation is:

$$\frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}=\frac{1}{u^2}\frac{d^2 f^+}{ds^2}$$

Does $V(z,t) = f^{+}\left(t - \frac{z}{u}\right)$ satisfy the wave equation $\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}$?

	$\frac{\partial^2 V}{\partial z^2}$	$=$	$\left(\frac{1}{u^2}\right)\frac{\partial^2 V}{\partial t^2}$
	$\frac{1}{u^2}\frac{d^2 f^{+}}{ds^2}$	$=$	$\left(\frac{1}{u^2}\right)\frac{d^2 f^{+}}{ds^2}$

Yes, the wave equation is satisfied by $V(z,t) = f^{+}\left(t - \frac{z}{u}\right)$

Homework:

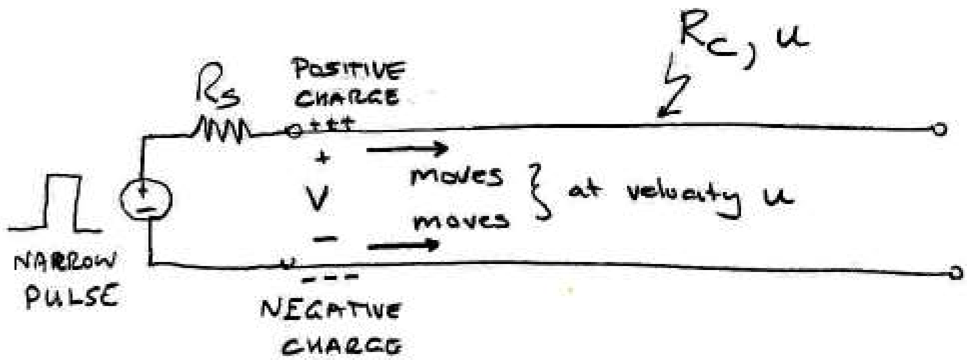
Recall that the general solution to the wave equation is

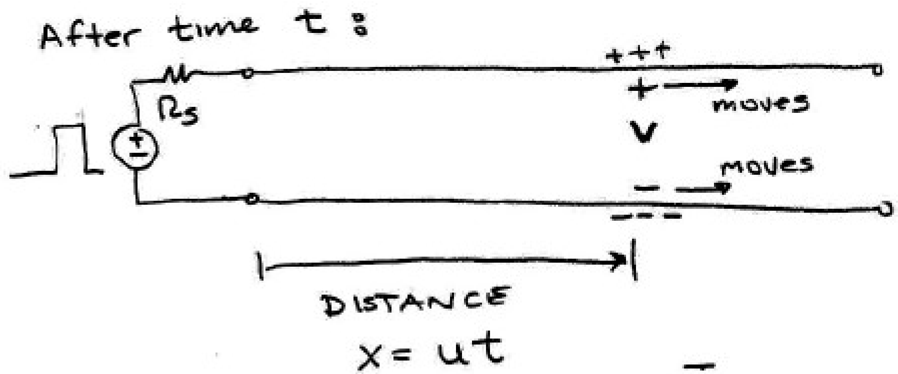
$$V(z,t) = f^{+}\left(t - \frac{z}{u}\right) + f^{-}\left(t + \frac{z}{u}\right)$$

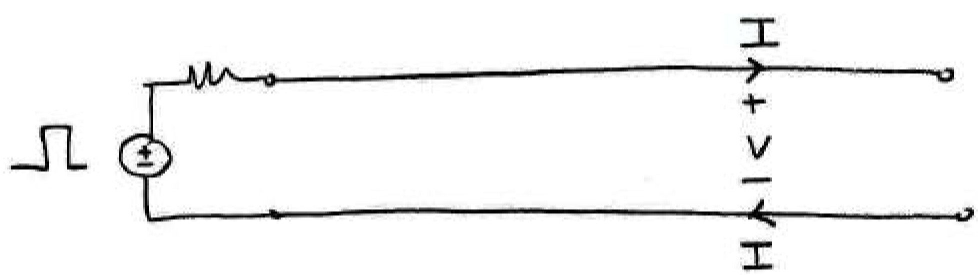
Prove that $V(z,t) = f^{-}\left(t + \frac{z}{u}\right)$ satisfies the wave equation.

Hint: Let $s = t + \frac{z}{u}$ and follow the same steps as we have just done.

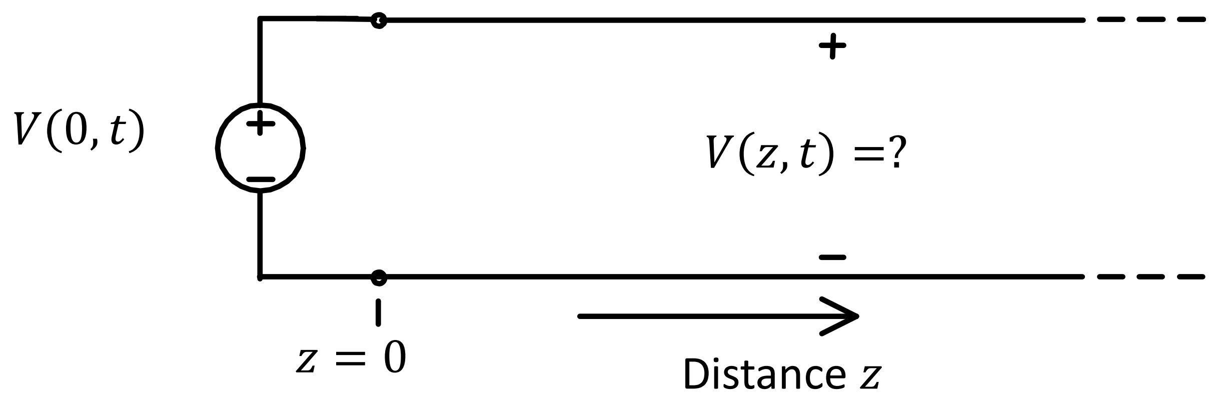
How Transmission Lines Work







Interpretation of $V(z,t) = f^+\left(t - \frac{z}{u}\right)$ as a "Traveling Wave"



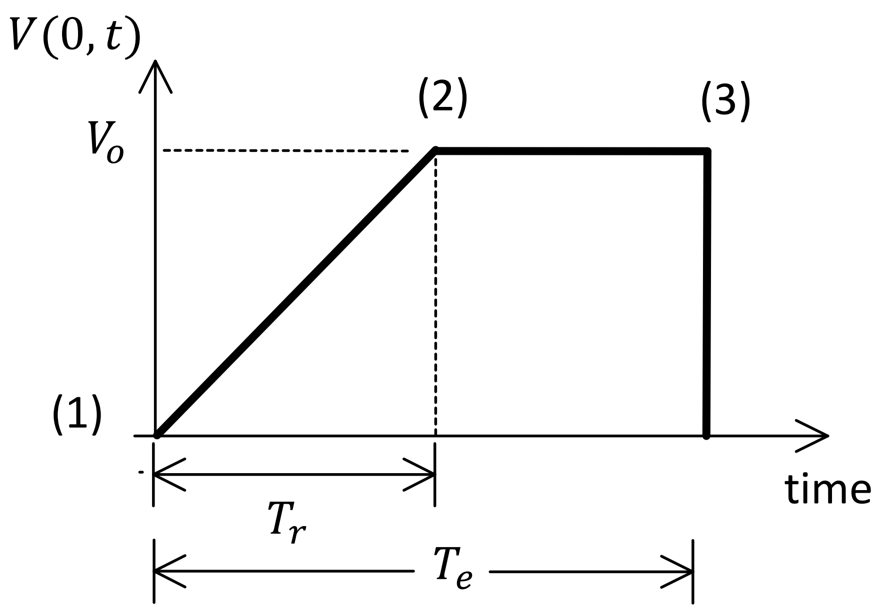
What is f^+?

In general,

$$V(z,t) = f^+\left(t - \frac{z}{u}\right)$$

at $z = 0$ we have

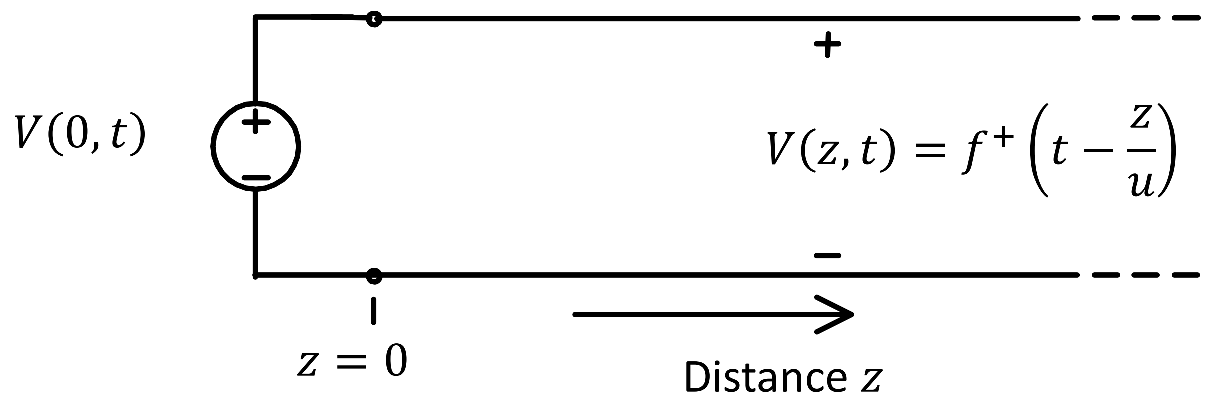
$$V(0,t) = f^+(t - \frac{0}{u}) = f^+(t)$$



So at $z = 0$:

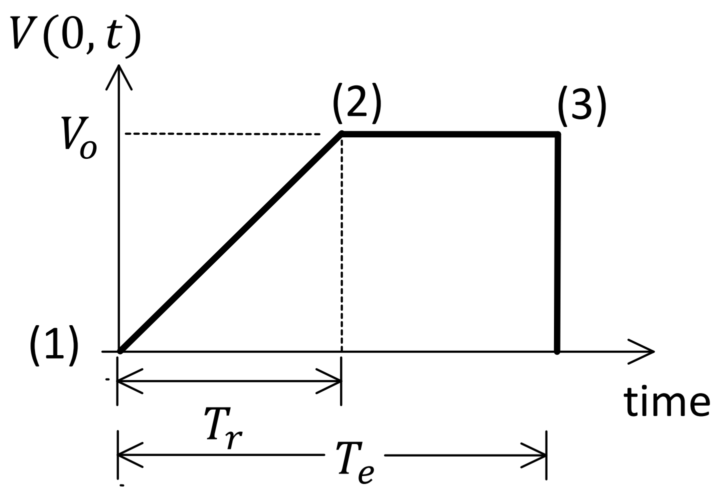
$$f^+(t) = V(0,t)$$

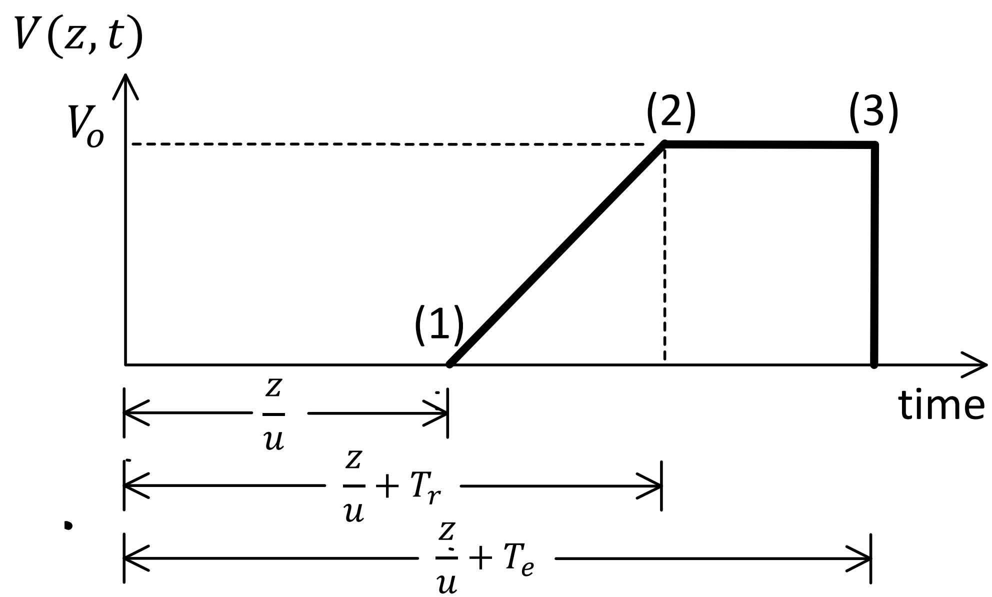
Find the voltage as a function of time at location *z*:



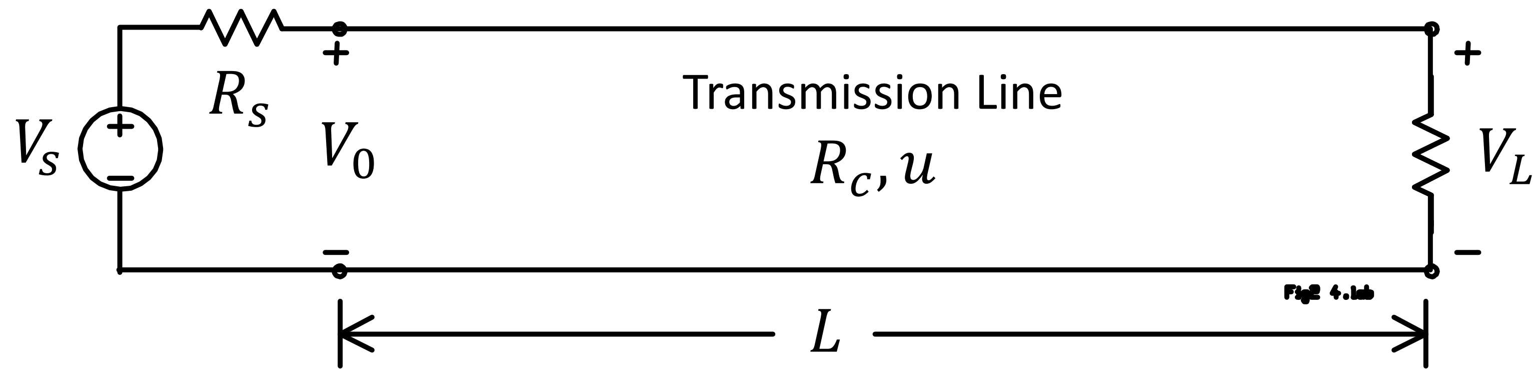
Each point on the waveform has a time delay of $\dfrac{z}{u}$

At z=0: $f^+(t) = V(0,t)$ At z>0: $V(z,t) = f^+\left(t - \dfrac{z}{u}\right)$





Voltage as a function of distance z



$V_s =$ Pulse of amplitude 5 v, rise time 1 ns and overall length 2 ns

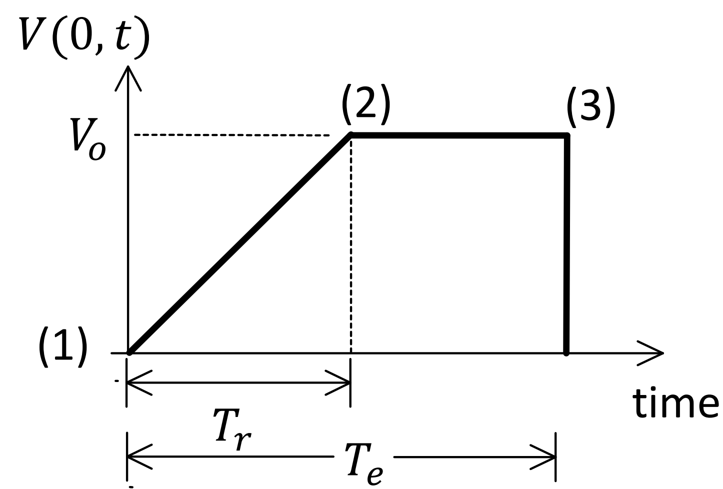
$R_s = 0\Omega$

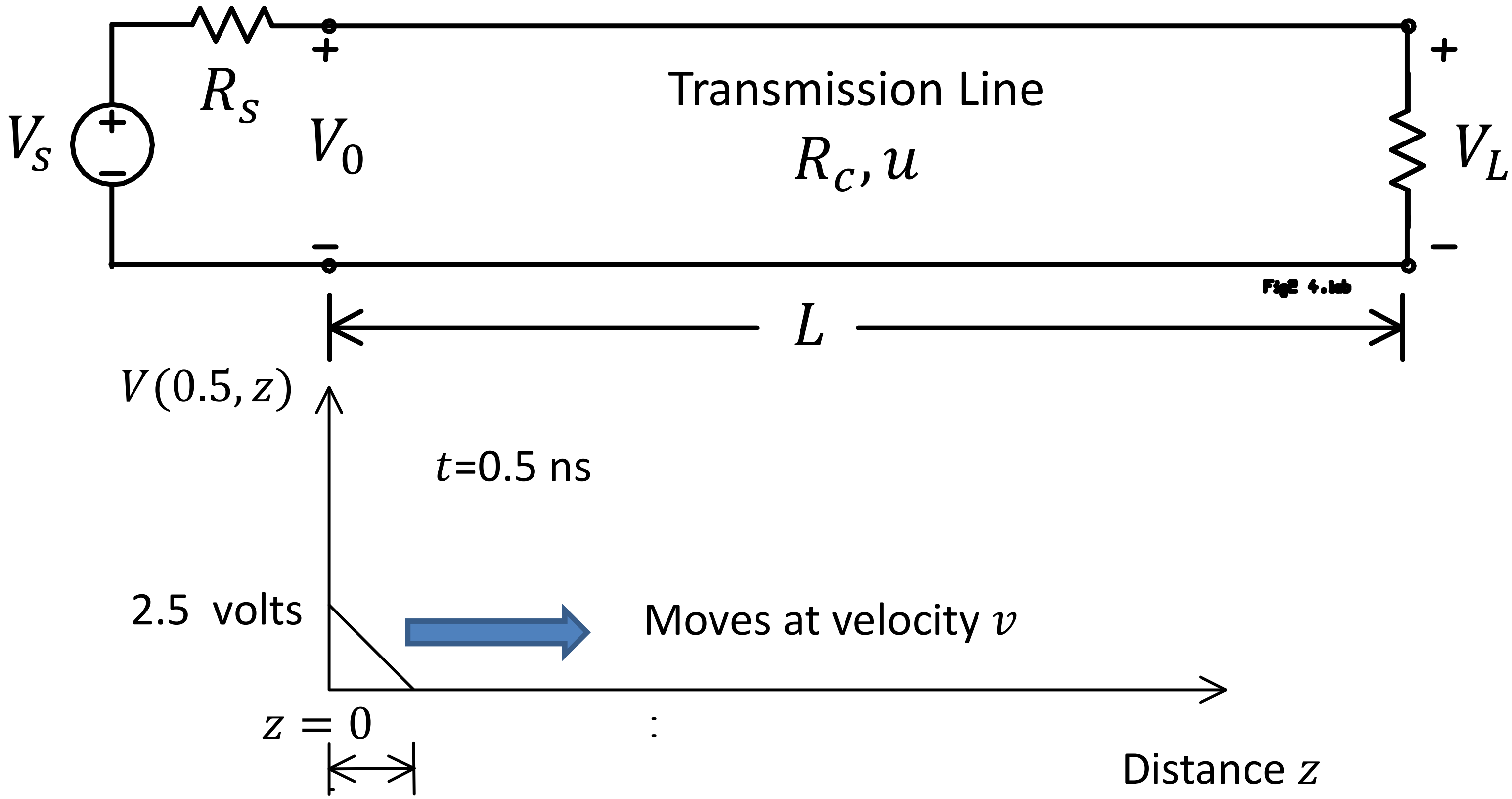
$R_c = 50\Omega$

$u = 20$ cm/ns

$L = 100$ cm

$R_L = 50\Omega$

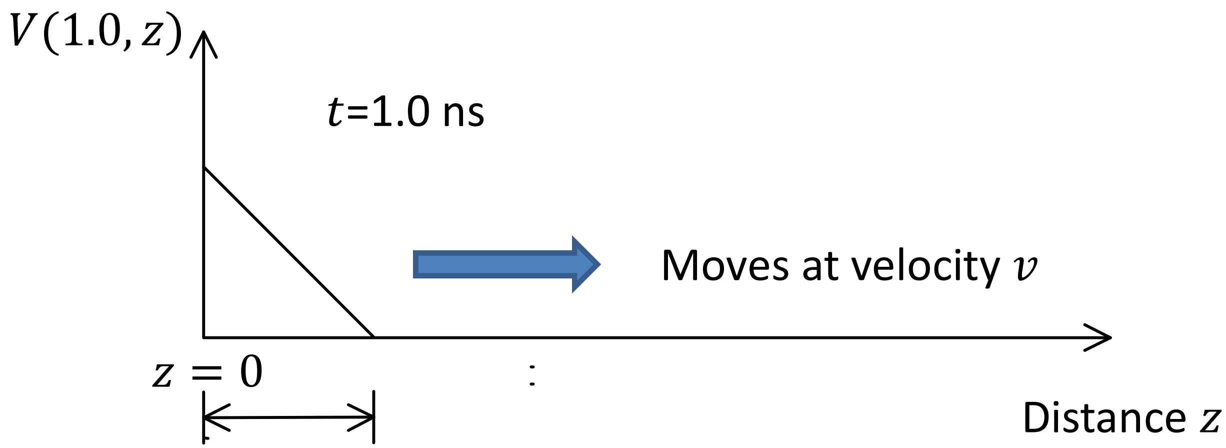




- ut = 20 cm/ns x 0.5 ns = 10 cm

- At $t = 0$, the voltage at the generator terminals begins to rise linearly from 0 volts towards 5 volts.
- As the voltage rises, the *leading edge* travels out onto the transmission line at a speed of $u = 20$ cm/ns.
- After 0.5 ns, the voltage at the generator terminals has risen to 2.5 volts, and the leading edge has travelled 10 cm.

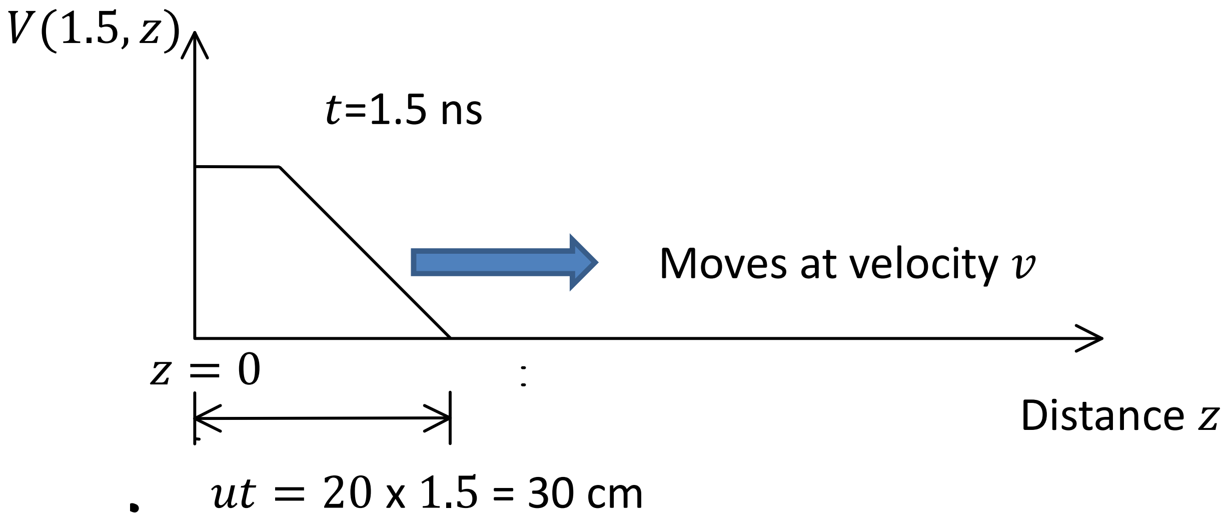
At t=1 ns:



$ut = 20$ cm/ns x 1.0 ns = 20 cm

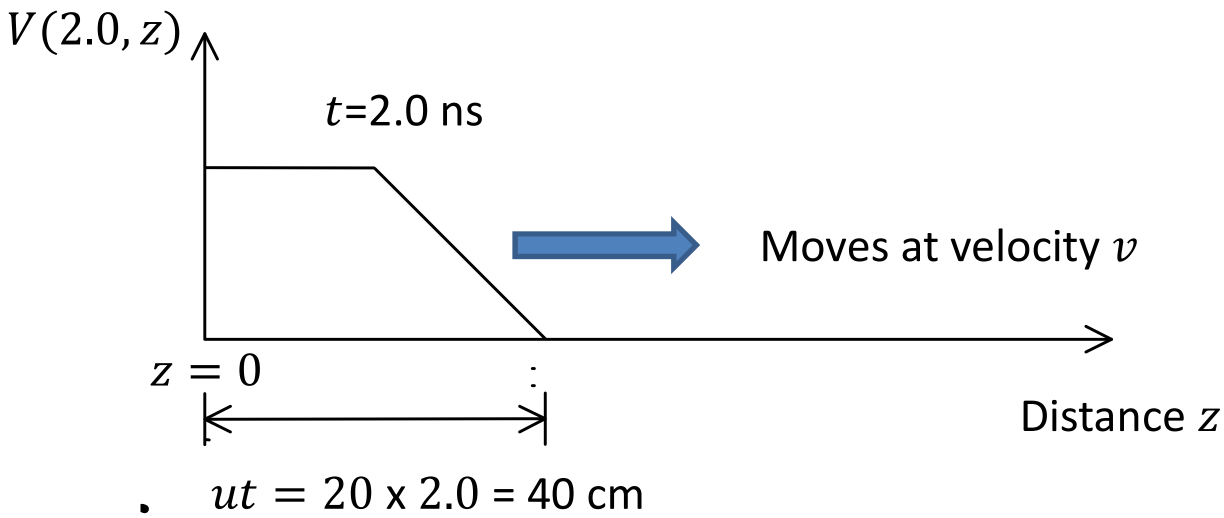
- As more time goes by, the voltage at the generator terminals rises towards 5 volts, and the leading edge continues to travel out onto the transmission line at a speed of $u = 20$ cm/ns.
- After 1 ns has gone by, the leading edge has travelled 20 cm and the voltage at the generator terminals has risen to 5 volts.

At $t = 1.5$ ns:



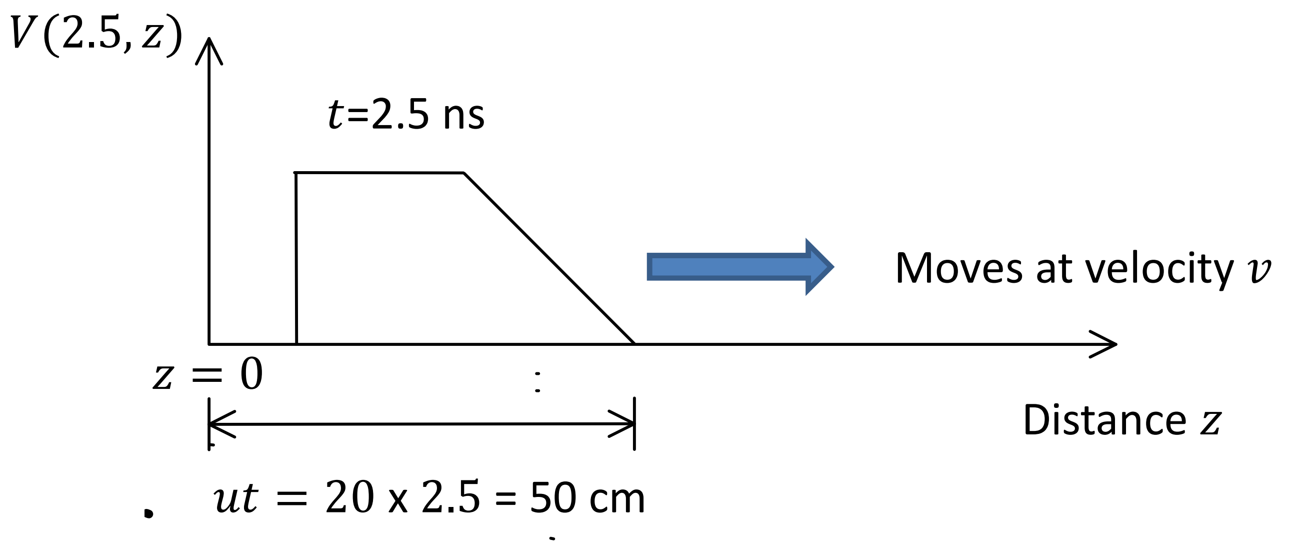
- As time advances beyond 1 ns, the voltage at the generator terminals stays constant at 5 volts.
- The leading edge continues to travel out onto the transmisson line, and has travelled a distance of 30 cm at t=1.5 ns.

At t=2.0 ns:



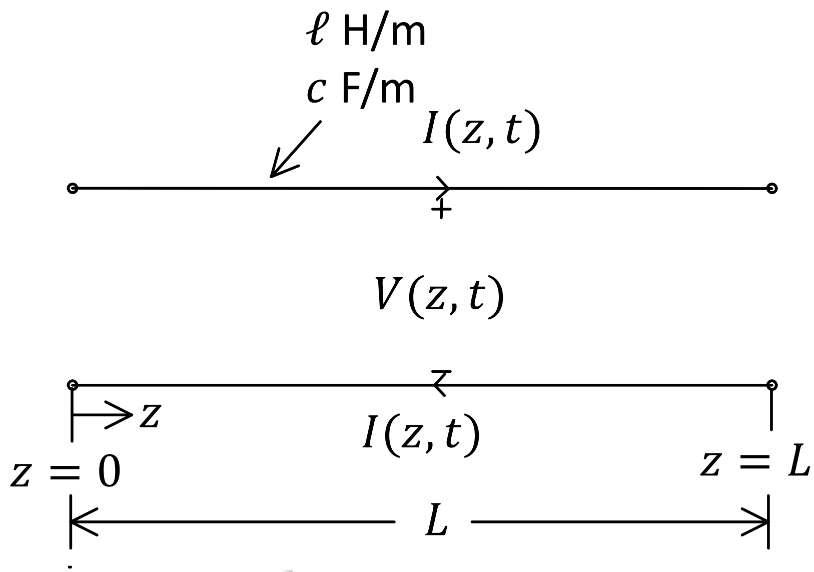
- At 2 ns, the voltage at the generator terminals is still 5 volts and the leading edge has travelled 40 cm.
- But just after 2 ns, the voltage at the generator terminals falls quickly to zero volts.

At t=2.5 ns:



- After 2 ns, the voltage at the generator terminals is zero.
- The leading edge continues to travel out onto the transmission line at 20 cm/ns, and has reached 50 cm at 2.5 ns.
- The “shape” of the voltage wave as a function of distance is the mirror image of the “shape” as a function of time.

What we have done so far:



$$\frac{\partial V}{\partial z} = -\ell \frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -c \frac{\partial V}{\partial t}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 V}{\partial t^2} \qquad u = \frac{1}{\sqrt{\ell c}} \text{ m/s}$$

$$V(z,t) = f^+\left(t - \frac{z}{u}\right) + f^-\left(t + \frac{z}{u}\right)$$

- $f^+\left(t - \frac{z}{u}\right)$ is a voltage *wave* that travels in the +z direction
- $f^-\left(t + \frac{z}{u}\right)$ is a voltage *wave* that travels in the -z direction

Find the Current Wave:

If the voltage "wave" is $V(z,t) = f^{+}\left(t - \frac{z}{u}\right)$.

Then what is the current "wave"? $I(z,t) = ?$

We can find the current by evaluating $\frac{\partial V}{\partial z} = -\ell \frac{\partial I}{\partial t}$

Left side:

- If $V(z,t) = f^{+}\left(t - \frac{z}{u}\right)$, we have shown that

$$\frac{\partial V}{\partial z} = -\frac{1}{u}\frac{df^{+}}{ds} \qquad \text{where} \qquad s = t - \frac{z}{u}$$

Right side:

- Calculate

$$\frac{\partial I}{\partial t} = \frac{dI}{ds}\frac{\partial s}{\partial t} \qquad \text{where } \frac{\partial s}{\partial t} = \frac{\partial}{\partial t}\left(t - \frac{z}{u}\right) = 1$$

so

$$\frac{\partial I}{\partial t} = \frac{dI}{ds}\frac{\partial s}{\partial t} = \frac{dI}{ds}$$

$$\frac{\partial V}{\partial z} = -\ell \frac{\partial I}{\partial t}$$

$$-\frac{1}{u}\frac{df^{+}}{ds} = -\ell \frac{dI}{ds}$$

so

$$\frac{dI}{ds} = \frac{1}{\ell u}\frac{df^{+}}{ds}$$

and integrating with respect to s we have

$$I(s) = \frac{1}{\ell u} f^{+}(s)$$

so with $s = t - \frac{z}{u}$

$$I(z,t) = \frac{1}{\ell u} f^{+}\left(t - \frac{z}{u}\right)$$

Characteristic Resistance:

$$V(z,t) = f^+\left(t - \frac{z}{u}\right)$$

$$I(z,t) = \frac{1}{\ell u} f^+\left(t - \frac{z}{u}\right)$$

$$R_c = \frac{\text{positive - going voltage wave}}{\text{positive - going current wave}}$$

$$R_c = \frac{f^+}{\frac{1}{\ell u} f^+} = \ell u = \ell \frac{1}{\sqrt{\ell c}} = \sqrt{\frac{\ell}{c}} \quad \text{ohms}$$

$$R_c = \sqrt{\frac{\ell}{c}}$$

$$I(z,t) = \frac{1}{R_c} f^+\left(t - \frac{z}{u}\right)$$

$$V(z,t) = f^+\left(t - \frac{z}{u}\right)$$

$$I(z,t) = \frac{V(z,t)}{R_c}$$

- Inan, Inan and Said page 34 use Z_0 and use the term "characteristic impedance".
- I often use $R_c = Z_0$ and use the term "characteristic resistance".
- Some textbooks use Z_c and the term "characteristic impedance".

Homework:

If the voltage is $V(z,t) = f^{-}\left(t+\frac{z}{u}\right)$, then show that the current is given by

$$I(z,t) = -\frac{1}{R_c} f^{-}\left(t+\frac{z}{u}\right)$$

Note that leading MINUS sign!

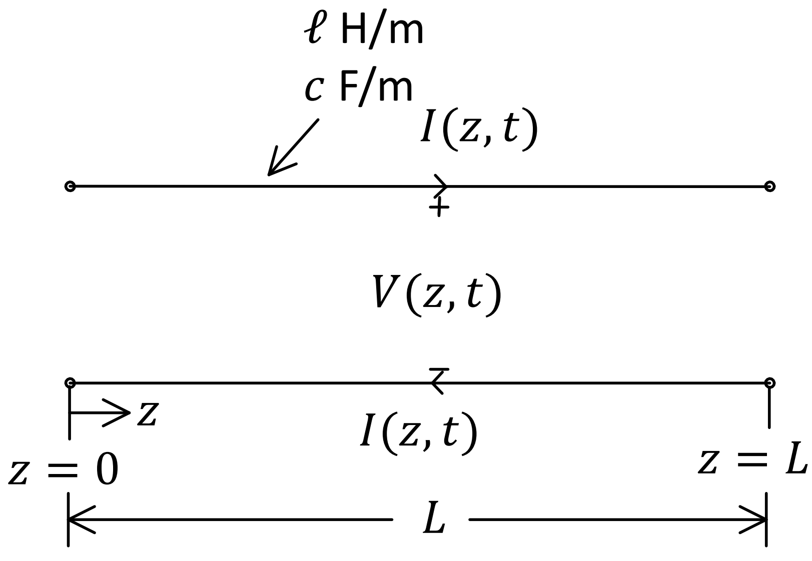
$$V(z,t) = f^{+}\left(t-\frac{z}{u}\right) + f^{-}\left(t+\frac{z}{u}\right)$$

Inan, Inan and Said page 34

$$R_c = \sqrt{\frac{\ell}{c}}$$

$$I(z,t) = \frac{1}{R_c} f^{+}\left(t-\frac{z}{u}\right) - \frac{1}{R_c} f^{-}\left(t+\frac{z}{u}\right)$$

Summary



$$\frac{\partial V}{\partial z} = -\ell \frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -c \frac{\partial V}{\partial t}$$

$$u = \frac{1}{\sqrt{\ell c}} \text{ m/s}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2} \frac{\partial^2 V}{\partial t^2}$$

$$V(z,t) = f^+\left(t - \frac{z}{u}\right) + f^-\left(t + \frac{z}{u}\right)$$

$$R_c = \sqrt{\frac{\ell}{c}}$$

$$I(z,t) = \frac{1}{R_c} f^+\left(t - \frac{z}{u}\right) - \frac{1}{R_c} f^-\left(t + \frac{z}{u}\right)$$