ELEC353 Lecture Notes Set 16

The homework assignments are posted on the course web site.
http://users.encs.concordia.ca/~trueman/web_page_353.htm

Homework #10: Do homework #10 by March 29, 2019.
Homework #11: Do homework #11 by April 5, 2019.
Homework #12: Do homework #12 by April 12, 2019.

Tutorial Workshop #11: Friday March 29, 2019.
Tutorial Workshop #12: Friday April 5, 2019.
Tutorial Workshop #13: Friday April 12, 2019

Last Day of Classes: April 13, 2019.

Final exam date: Tuesday April 23, 2019, 9:00 to 12:00, in H531.

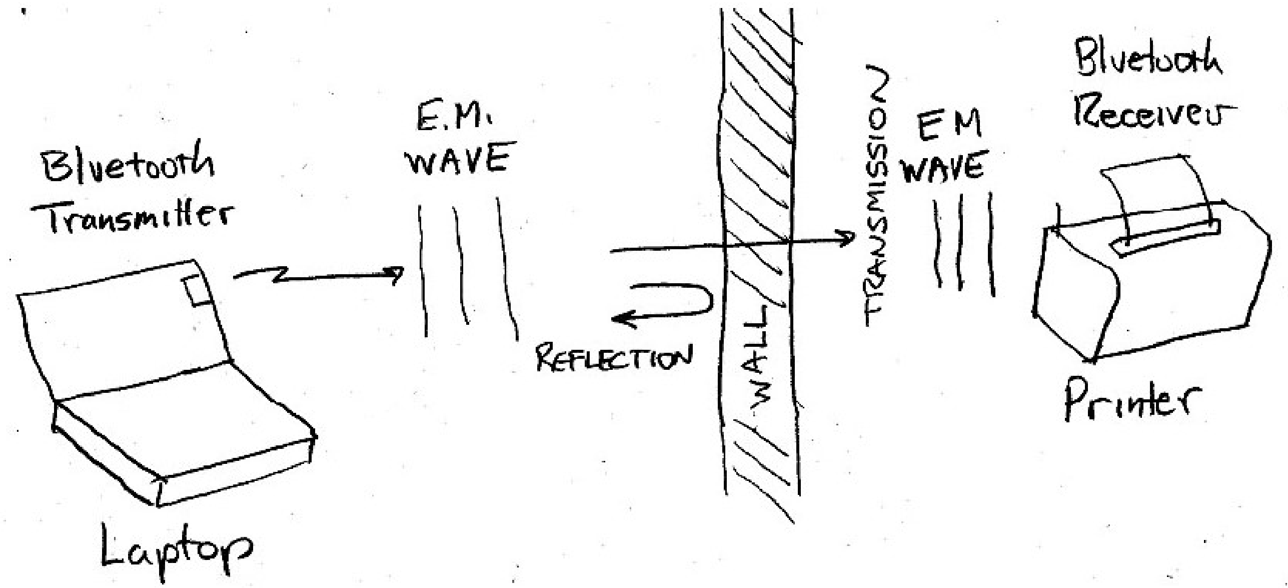
Topics to be Covered

Plane Waves

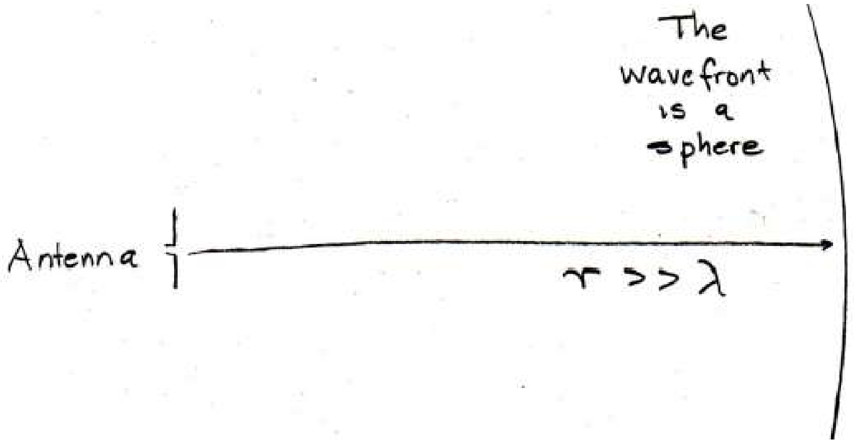
- Maxwell’s Equations and the Wave Equation – done
- Plane waves - today
- Material Boundaries
- Transmission Through a Wall

Antennas

Plane Waves in Space



Spherical Waves and Plane Waves



Spherical wave:

$$E = \frac{A}{r} e^{-j\beta r}$$

Approximate the spherical wave with a plane wave:

For a small region of space at a distance of r_0

$$E \approx \frac{A}{r_0} e^{-j\beta r}$$

$$E_0 = \frac{A}{r_0} \qquad E \approx E_0 e^{-j\beta r}$$

In rectangular coordinates assuming that *z* and *r* point in the same direction:

$$E \approx E_0 e^{-j\beta z}$$ plane wave

Review: Wave Equation

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \qquad\qquad \nabla \times \overline{H} = (\sigma + j\omega\varepsilon)\overline{E}$$

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu\nabla \times \overline{H}$$

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu\left[(\sigma + j\omega\varepsilon)\overline{E}\right]$$

Propagation constant:

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\nabla \times \nabla \times \overline{E} = -\gamma^2\overline{E}$$

Vector identity: $\nabla \times \nabla \times \overline{E} = \nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2\overline{E}$

$$\nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2\overline{E} = -\gamma^2\overline{E}$$

$\nabla^2\overline{E}$ is the “vector Laplacian” of the electric field

Review: Wave equation, continued

$$\nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2 \overline{E} = -\gamma^2 \overline{E}$$

Gauss' Law: $\nabla \cdot \overline{E} = \dfrac{\rho_v}{\varepsilon}$

Source-free region: $\rho_v = 0$ so $\nabla \cdot \overline{E} = 0$

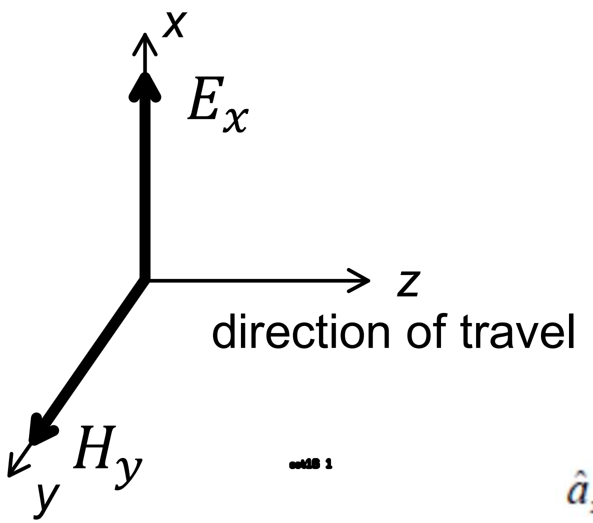
$$-\nabla^2 \overline{E} = -\gamma^2 \overline{E}$$

$$\nabla^2 \overline{E} = \gamma^2 \overline{E}$$

- Vector wave equation
- Vector Helmholtz equation

Plane Waves

Inan and Inan Section 8.1.3



$$\overline{E} = \hat{a}_x E_x(z)$$

E_x is a function of z only

$$\nabla^2 \overline{E} = \gamma^2 \overline{E}$$

$$\hat{a}_x\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\right) + \hat{a}_y\left(\frac{\partial^2 E_y}{\partial x^2}+\frac{\partial^2 E_y}{\partial y^2}+\frac{\partial^2 E_y}{\partial z^2}\right) + \hat{a}_z\left(\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+\frac{\partial^2 E_z}{\partial z^2}\right) = \gamma^2\left(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z\right)$$

$$\overline{E} = \hat{a}_x E_x(z)$$

$$\hat{a}_x\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\right) = \gamma^2 \hat{a}_x E_x$$

$$\hat{a}_x\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}+\frac{\partial^2 E_x}{\partial z^2}\right)=\gamma^2\hat{a}_x E_x$$

E_x is a function only of z

$$\frac{\partial^2 E_x}{\partial z^2}=\gamma^2 E_x$$

$$\frac{d^2 E_x}{dz^2}=\gamma^2 E_x$$

exactly the same as $\frac{d^2V}{dz^2}=\gamma^2 V$

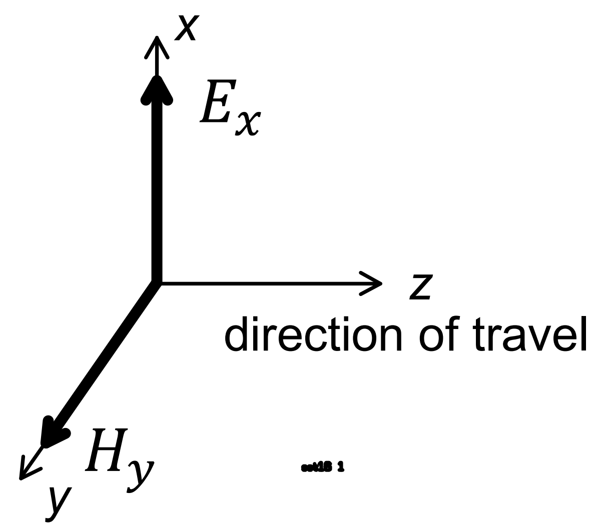
$$V=V^+e^{-\gamma z}+V^-e^{\gamma z}$$

$$E_x(z)=E_x^+e^{-\gamma z}+E_x^-e^{\gamma z}$$

propagation constant $\gamma=\alpha+j\beta$

wavelength $\lambda=\dfrac{2\pi}{\beta}$

speed of travel $u=\dfrac{\omega}{\beta}$



Does $E_x(z) = E_x^+ e^{-\gamma z}$ satisfy the wave equation?

$$\frac{d^2 E_x}{dz^2} = \gamma^2 E_x$$

$$\frac{dE_x}{dz} = \frac{d}{dz}\left(E_x^+ e^{-\gamma z}\right) = -\gamma E_x^+ e^{-\gamma z}$$

$$\frac{d^2 E_x}{dz^2} = \frac{d}{dz}\left(-\gamma E_x^+ e^{-\gamma z}\right) = (-\gamma)(-\gamma) E_x^+ e^{-\gamma z} = \gamma^2 E_x$$

So $E_x^+ e^{-\gamma z}$ does satisfy the wave equation.

Find the Magnetic Field

$$E_x(z) = E_x^+ e^{-\gamma z}$$

Faraday's Law: $\nabla \times \overline{E} = -j\omega\mu\overline{H}$

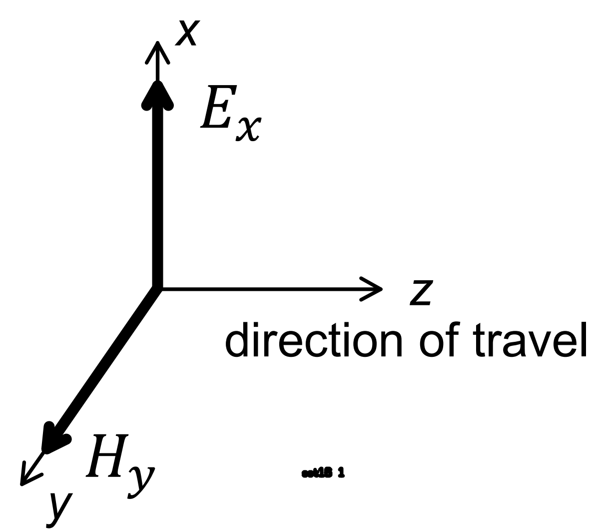
$$\overline{H} = \frac{-1}{j\omega\mu}\nabla\times\overline{E} = \frac{-1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x^+ e^{-\gamma z} & 0 & 0 \end{vmatrix}$$

$$\overline{H} = \frac{-1}{j\omega\mu}\left[-\hat{a}_y\left(-\frac{\partial}{\partial z}\left(E_x^+ e^{-\gamma z}\right)\right) + \hat{a}_z\left(-\frac{\partial}{\partial y}\left(E_x^+ e^{-\gamma z}\right)\right)\right]$$

$$\overline{H} = \frac{-1}{j\omega\mu}\hat{a}_y\left(-\gamma E_x^+ e^{-\gamma z}\right)$$

$$\overline{H} = \frac{\gamma}{j\omega\mu}E_x^+ e^{-\gamma z}\hat{a}_y$$

$$H_y = \frac{\gamma}{j\omega\mu}E_x^+ e^{-\gamma z}$$



Intrinsic Impedance

$$E_x(z) = E_x^+ e^{-\gamma z}$$

"characteristic impedance" $Z_0 = \dfrac{\text{voltage wave}}{\text{current wave}}$

$$H_y = \frac{\gamma}{j\omega\mu} E_x^+ e^{-\gamma z}$$

"intrinsic impedance" $\eta = \dfrac{\text{E wave}}{\text{H wave}}$

$$\eta = \frac{E_x}{H_y} = \frac{E_x^+ e^{-\gamma z}}{\frac{\gamma}{j\omega\mu} E_x^+ e^{-\gamma z}}$$

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{(j\omega\mu)^2}{j\omega\mu(\sigma + j\omega\varepsilon)}}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \text{ ohms}$$

$$H_y = \frac{1}{\eta} E_x^+ e^{-\gamma z}$$

Remarks: $$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \text{ ohms}$$

(1) In general the intrinsic impedance η is a complex number.

(2) For lossless media, $\sigma = 0$ so

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{0 + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \quad \text{ohms}$$

(3) For free space, we use the symbol η_0 for the "intrinsic impedance of free space".
(4) The value is

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\pi x 10^{-7}}{8.854 x 10^{-12}}} = 376.73 \quad \text{ohms}$$

(5) Sometimes people use the approximation

$$\eta_0 \approx 120\pi = 376.99 \quad \text{ohms}$$

The error is about 0.7%

Loss Tangent

Ampere's Law: $\nabla \times \overline{H} = (\sigma + j\omega\varepsilon)\overline{E}$

$$\nabla \times \overline{H} = \sigma\overline{E} + j\omega\varepsilon\overline{E}$$

Conduction current density: $\overline{J}_c = \sigma\overline{E}$

Displacement current density: $\overline{J}_d = j\omega\varepsilon\overline{E}$

Total current density: $\overline{J} = \overline{J}_c + \overline{J}_d \qquad \nabla \times \overline{H} = \overline{J}$

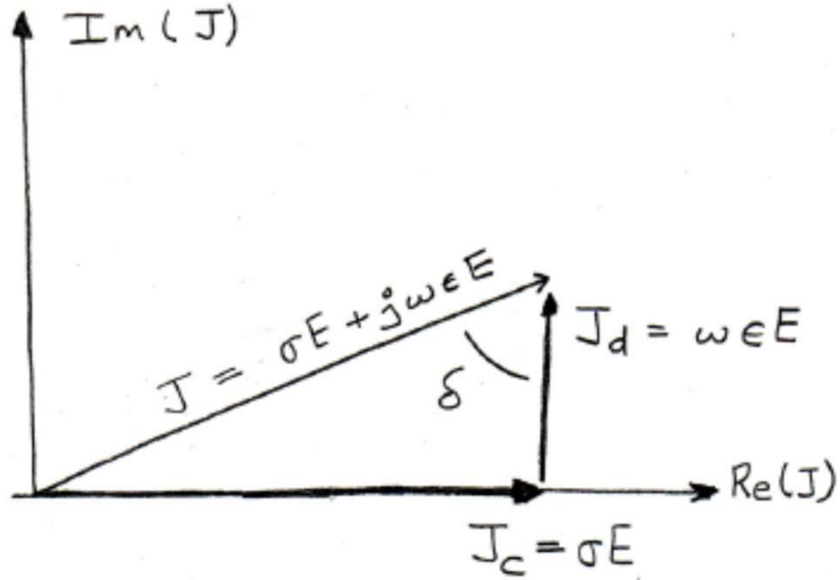
In a "**good conductor**" the conduction current density is much greater than the "displacement" current density:

$$J_c >> J_d$$

But in a "**good insulator**" the displacement current density is much greater than the conduction current density:

$$J_d >> J_c$$

$$\bar{J} = \bar{J}_c + \bar{J}_d = \sigma\bar{E} + j\omega\varepsilon\bar{E}$$



Loss Tangent

$$\tan\delta = \frac{J_c}{J_d} = \frac{\sigma}{\omega\varepsilon}$$

If the *loss tangent is large*, then the conduction current density is large compared to the displacement current density and the material is a *good conductor*.

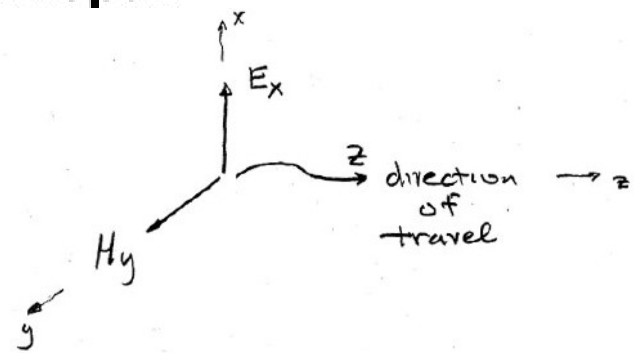
If *the loss tangent is small*, then the displacement current is large compared to the conduction current, and the material is a *good insulator*.

Loss tangent is usually used to describe the behavior of insulating materials such as circuit board materials. These materials should be perfect insulators but in reality are "lossy" and absorb some energy.

Example: FR4 Epoxy circuit board material: $\epsilon_r = 4.4$, $\tan\delta = 0.02$

Attenuation with Distance: Penetration Depth

Inan and Inan page 671



$$E_x(z) = E_x^+ e^{-\gamma z}$$

$$\gamma = \alpha + j\beta$$

$$E_x(z) = E_x^+ e^{-\alpha z} e^{-j\beta z}$$

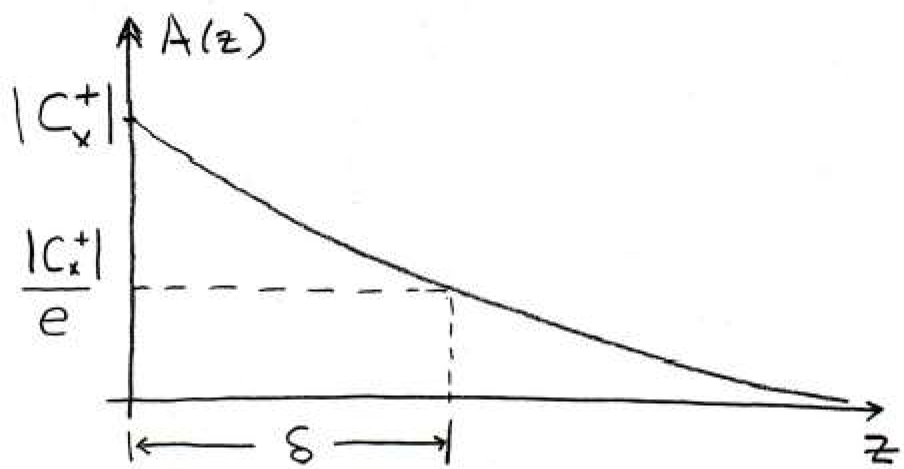
$$E_x^+ = C_x^+ e^{j\theta^+} \qquad E_x(z) = C_x^+ e^{j\theta^+} e^{-\alpha z} e^{-j\beta z}$$

Phasor $$E_x(z) = C_x^+ e^{-\alpha z} e^{j(\theta^+ - \beta z)}$$

Time function $$E_x(z,t) = C_x^+ e^{-\alpha z} \cos(\omega t + \theta^+ - \beta z)$$

The amplitude is

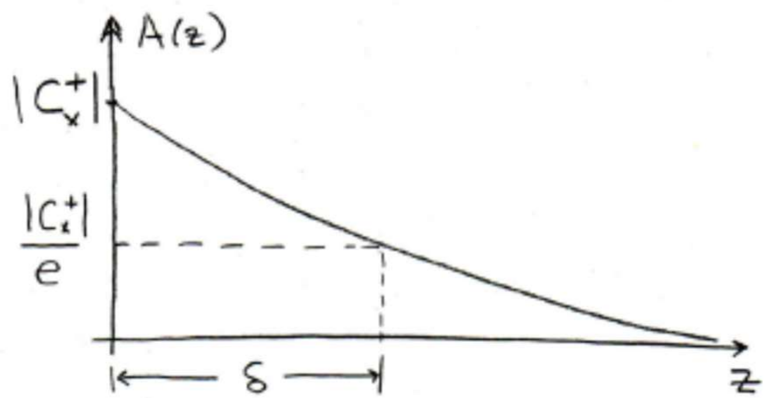
$$A(z) = C_x^+ e^{-\alpha z}$$



Penetration Depth

$$E_x(z,t) = C_x^+ e^{-\alpha z}\cos\left(\omega t + \theta^+ - \beta z\right)$$

Amplitude: $$A(z) = C_x^+ e^{-\alpha z}$$



Penetration depth: the distance that the wave travels for the amplitude to decrease by a factor of 1/e.

At $z = 0$: $A(0) = C^+$

At $z = \delta$: $A(\delta) = C^+ e^{-\alpha\delta} = \dfrac{C^+}{e}$

Solve: $$\delta = \frac{1}{\alpha}$$

For good conductors, the penetration depth is very small and is usually called the “skin depth”.

Example: Metal at 2.45 GHz.

Find the "skin depth" in copper at 2.45 GHz. The conductivity is $\sigma = 5.8 \text{ x } 10^7$ S/m, the relative permittivity is $\varepsilon_r = 1$ and the material is "non-magnetic".

Solution

- Evaluate the radian frequency: $\omega = 2\pi f = 2\pi \cdot 2.45x10^9 = 1.539x10^{10}$ rad/sec.
- Evaluate the propagation constant:

  $$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

  $$\gamma = \sqrt{\left(j \cdot 1.539x10^{10} \cdot 4\pi x10^{-7}\right)\left(5.8x10^7 + j \cdot 1.539x10^{10} \cdot 8.854x10^{-12}\right)}$$

  $$\gamma = \sqrt{(19340\angle 90°)(5.8x10^7 + j \cdot 0.13636)}$$

- We can neglect 0.13636 compared to $5.8\text{x}10^7$, so

  $$\gamma = \sqrt{(19340\angle 90°)(5.8x10^7)} = \sqrt{1.122x10^{12}\angle 90°}$$

- To take the square root of a complex number, take the square root of the magnitude and half the angle:

  $$\gamma = \sqrt{1.122x10^{12} e^{j90°}} = \sqrt{1.122x10^{12}}\sqrt{e^{j90°}} = 1.059x10^6 e^{j45°}$$

  $$\gamma = 1.059x10^6 \angle 45° = 749{,}000 + j749{,}000$$

- Since $\gamma = \alpha + j\beta$, we have $\alpha = 749000$ Np/m and $\beta = 749000$ rad/meter.

$$\gamma = 1.059x10^6 \angle 45° = 749{,}000 + j749{,}000$$

$$\gamma = \alpha + j\beta ;$$

$\alpha = 749000$ Np/m and $\beta = 749000$ rad/meter.

- The skin depth is $\delta = \dfrac{1}{\alpha} = \dfrac{1}{749000} = 1.335x10^{-6}$ m.

- This is typical of metals or "good conductors". The skin depth is very, very small and the current flows in a very thin layer right at the surface, of depth about 4 times the skin depth.
- Loss tangent:
  - $\tan\delta = \dfrac{\sigma}{\omega\varepsilon} = \dfrac{5.8x10^7}{0.13636} = 4.25x10^8 >> 1$
  - for a good conductor we find that $\tan\delta >> 1$ as expected.

Example: Muscle at 915 MHz

Find the penetration depth in muscle tissue at 915 MHz. The "electrical properties" of muscle at 915 MHz are: $\varepsilon_r = 51$, $\sigma = 1.6$ S/m, $\mu = \mu_0$ ("non-magnetic").

Solution

- Evaluate the radian frequency: $\omega = 2\pi f = 2\pi \cdot 915x10^6 = 5.749x10^9$ rad/sec.
- Evaluate the propagation constant:

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\gamma = \sqrt{(j \cdot 5.749x10^9 \cdot 4\pi x10^{-7})(1.6 + j \cdot 5.749x10^9 \cdot 51 \cdot 8.854x10^{-12})}$$

$$\gamma = \sqrt{(7224.6\angle 90°)(1.6 + j \cdot 2.596)}$$

- Since the conductivity $\sigma = 1.6$ is about equal to $\omega\varepsilon = 2.596$, muscle at 915 MHz cannot classified as a "good conductor" or as a "good dielectric".
- Loss tangent: $\tan\delta = \dfrac{\sigma}{\omega\varepsilon} = \dfrac{1.6}{2.596} = 0.6163$
- Finish evaluating the propagation constant:

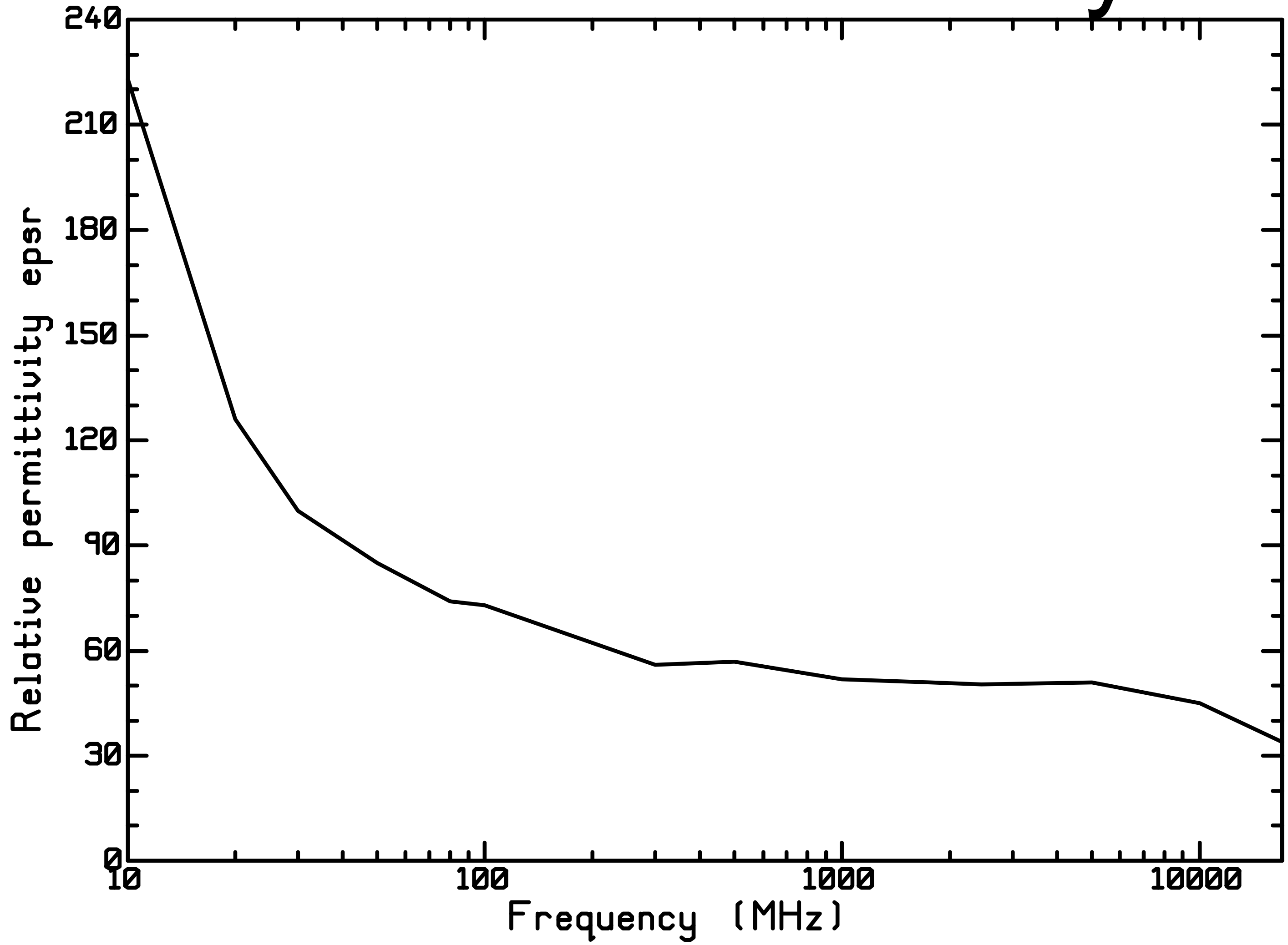
$$\gamma = \sqrt{(7224.6\angle 90°)(3.049\angle 58.4°)} = \sqrt{22031\angle 148.4°}$$

- To take the square root of a complex number, take the square root of the magnitude and half the angle:

$$\gamma = 148.4\angle 74.2° = 40.41 + j142.8$$

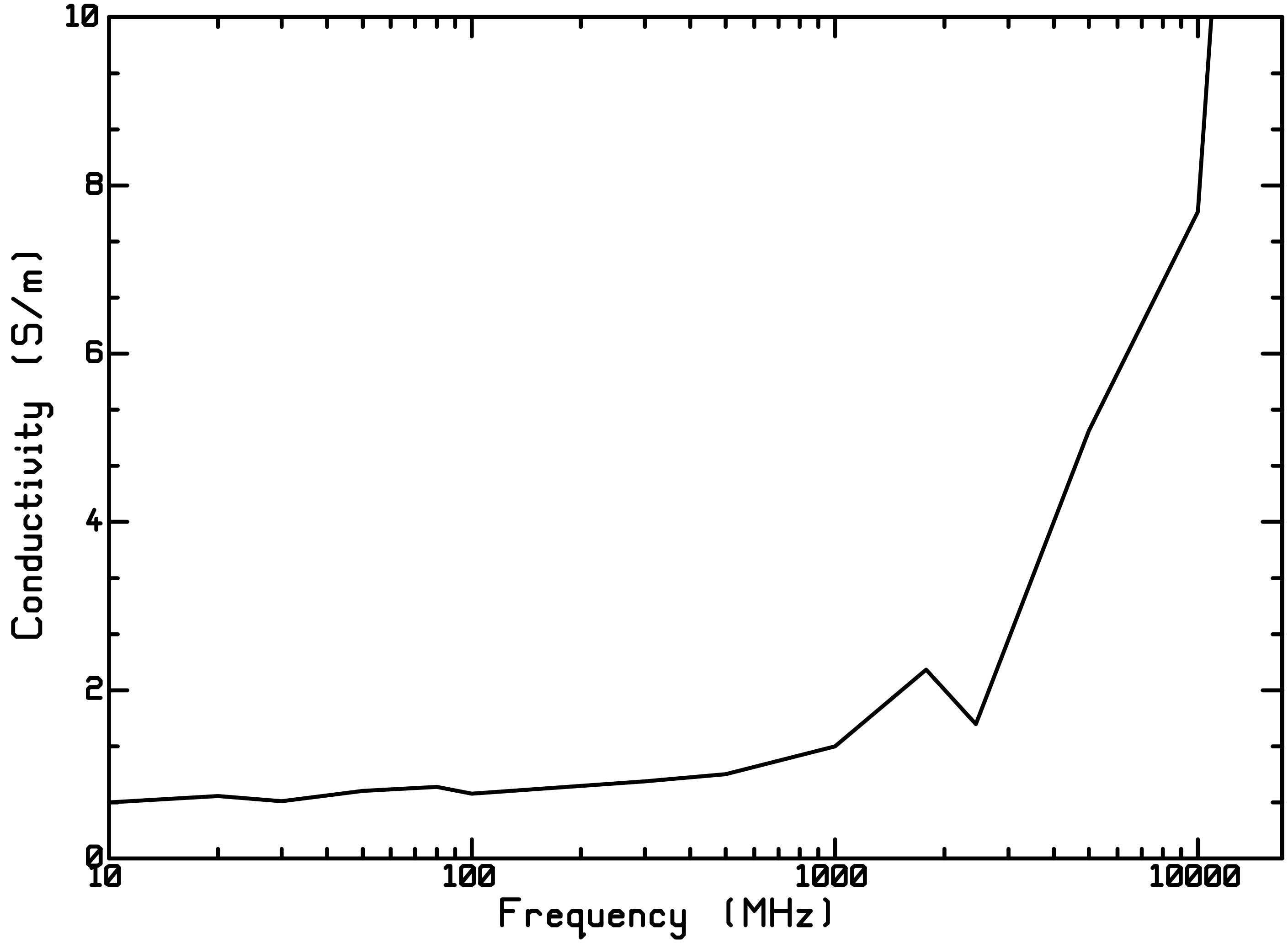
- Since $\gamma = \alpha + j\beta$, we have $\alpha = 40.41$ Np/m and $\beta = 142.8$ rad/meter.
- The penetration depth is $\delta = \dfrac{1}{\alpha} = \dfrac{1}{40.41} = 2.47$ cm.

Electrical Parameters of Muscle
Relative Permittivity



Data from: W.D. Hurt, “Multiterm Debye Dispersion Relations for Permittivity of Muscle”, IEEE Transactions on Biomedical Engineering, Vol. BME-32, No. 1, pp.60-64, January 1985.

Conductivity of Muscle



Data from : W.D. Hurt, “Multiterm Debye Dispersion Relations for Permittivity of Muscle”, IEEE Transactions on Biomedical Engineering, Vol. BME-32, No. 1, pp.60-64, January 1985.

Complex Permittivity

- Ampere's Law reads $\nabla \times \overline{H} = (\sigma + j\omega\varepsilon)\overline{E}$
- Define the complex permittivity ε_c in such a way that Ampere's Law reads

$$\nabla \times \overline{H} = j\omega\varepsilon_c \overline{E}$$

- Then we must have

$$j\omega\varepsilon_c = \sigma + j\omega\varepsilon$$

so

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}$$

- It is customary to decompose the complex permittivity into real and imaginary parts as

$$\varepsilon_c = \varepsilon' - j\varepsilon''$$

so $\varepsilon' = \varepsilon$

and $\varepsilon'' = \dfrac{\sigma}{\omega}$

- The loss tangent is

$$\tan\delta = \frac{\sigma}{\omega\varepsilon} = \frac{\omega\varepsilon''}{\omega\varepsilon'} = \frac{\varepsilon''}{\varepsilon'}$$

- Which "system" to use? ε and σ? Or ε' and ε''?
  - Answer: you will find both systems in use in textbooks and journals.
  - So you need to be familiar with both systems.

Plane Waves in Lossless Media

- A "lossy medium" has non-zero conductivity σ and absorbs energy.
- A "lossless" medium has $\sigma = 0$ and does not absorb any energy.
- Propagation constant

  $$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{j\omega\mu(j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon} = 0 + j\beta$$

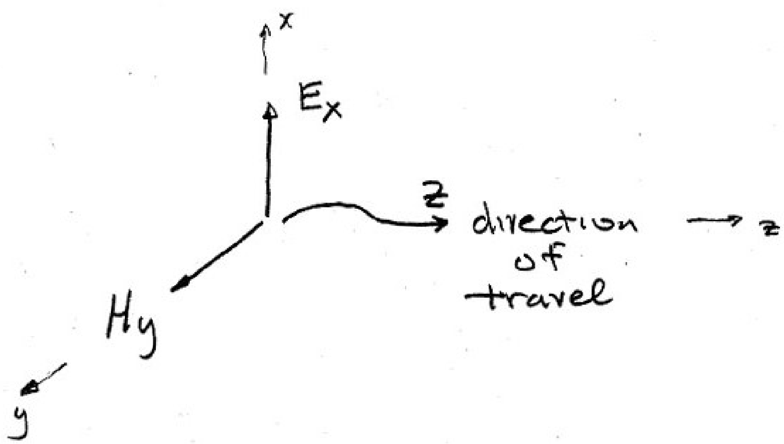
  - $\alpha = \mathrm{Re}(\gamma)$=0 in lossless materials
  - $\beta = \mathrm{Im}(\gamma) = \omega\sqrt{\mu\varepsilon}$ in lossless materials
- Intrinsic impedance

  $$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \text{ ohms}$$

  - For free space, $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.734 \approx 377 \text{ ohms}$
- Speed of travel $u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \text{ m/s}$
  - For free space, $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 2.997929458x10^8 \approx 3x10^8 \text{ m/s}$
  - $c = 300$ meters per microsecond = 30 cm/ns

- Wavelength $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{1/\sqrt{\mu\varepsilon}}{\omega/(2\pi)} = \frac{u}{f}$ meters
- In lossless media our plane waves look much simpler!
- The wave equation $\nabla^2\overline{E} = \gamma^2\overline{E}$ has $\gamma = j\beta$ so $\gamma^2 = -\beta^2$ and the wave equation in lossless media reads

$$\nabla^2\overline{E} = -\beta^2\overline{E}$$

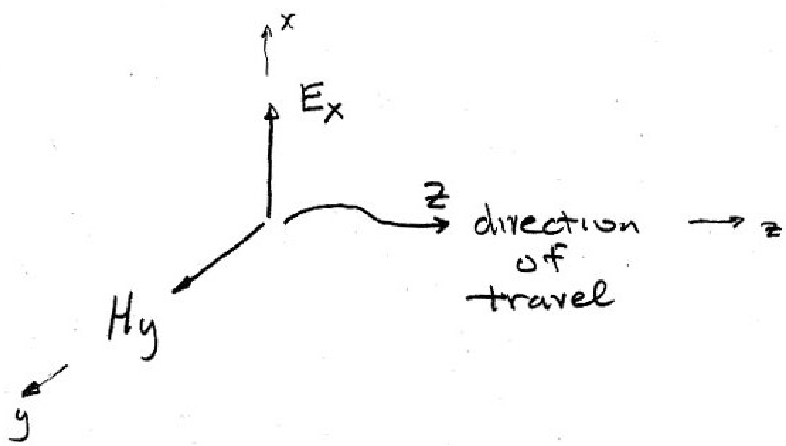


- For a one-dimensional plane wave $\overline{E} = \hat{a}_x E_x$, we have

$$\frac{d^2E_x}{dz^2} = -\beta^2 E_x$$

- For lossless media, $\gamma = j\beta$ and the solution to the wave equation is

$$E_x(z) = E_x^+ e^{-j\beta z} + E_x^- e^{j\beta z}$$



- The corresponding magnetic field is

$$H_y(z) = \frac{1}{\eta} E_x^+ e^{-j\beta z} - \frac{1}{\eta} E_x^- e^{j\beta z}$$

where:

  - The speed of travel is $u = \dfrac{1}{\sqrt{\mu\varepsilon}}$ meters/second
  - The wavelength is m $\lambda = \dfrac{u}{f}$ meters
  - The phase constant is $\beta = \dfrac{2\pi}{\lambda}$ radians/meter
  - The intrinsic impedance is $\eta = \sqrt{\dfrac{\mu}{\varepsilon}}$ ohms

- The term $E_x^+ e^{-j\beta z}$ is a traveling wave in the $+z$ direction.
- The term $E_x^- e^{j\beta z}$ is a traveling wave in the $-z$ direction.
- The mathematics of plane waves is *exactly the same* as that of voltage waves on transmission lines.

Power Flow in the Plane Wave

- In circuits, the "average power" is $P_{av} = \frac{1}{2}\mathrm{Re}\left(VI^*\right)$ watts, where V and I are phasors written relative to the amplitude of the cosine voltage.
- So we would expect that in dealing with fields, power should be the "voltage" quantity E times the "current" quantity H.
- The product EH^* has the units of volts/meter times amps/meter or watts per square meter.
- This is called "power flow *density*" because it is power spread over area.
- So in electromagnetics the "power flow density" vector is given by

$$\overline{S}_{av} = \frac{1}{2}\mathrm{Re}\left(\overline{E} \times \overline{H}^*\right) \quad \text{watts per square meter}$$

- This is also called the "Poynting Vector".
- The magnitude of S_{av} tells us the amount of power flow.
- The direction of $\overline{S}_{av}$ tells us the direction in which the power is flowing.

- For the plane wave in a lossless medium we have

$$\overline{E} = \hat{a}_x E_x^+ e^{-j\beta z}$$

$$\overline{H} = \hat{a}_y \frac{1}{\eta} E_x^+ e^{-j\beta z}$$

and the power flow density or Poynting Vector is

$$\overline{S}_{av} = \frac{1}{2}\mathrm{Re}\left(\overline{E} \times \overline{H}^*\right) = \frac{1}{2}\mathrm{Re}\left(\hat{a}_x E_x^+ e^{-j\beta z} \times \hat{a}_y \frac{E_x^{+*}}{\eta} e^{+j\beta z}\right)$$

where we must take the complex conjugate of E_x^+,

$$\overline{S}_{av} = \frac{1}{2}\mathrm{Re}\left(E_x^+ E_x^{+*} \frac{1}{\eta}\hat{a}_z\right)$$

and since E_x^+ times its complex conjugate E_x^{+*} equals the magnitude squared $\left|E_x^+\right|^2$, we have

$$\overline{S}_{av} = \frac{1}{2}\mathrm{Re}\left(\frac{\left|E_x^+\right|^2}{\eta}\hat{a}_z\right)$$

$$\overline{S}_{av} = \frac{\left|E_x^+\right|^2}{2\eta}\hat{a}_z \quad \text{watts/meter}^2$$

- So the magnitude of the power flow density is $\dfrac{\left|E_x^+\right|^2}{2\eta}$ watts/meter^2 and the direction of power flow is the direction of travel of the plane wave, $\hat{a}_z$