ELEC353 Lecture Notes Set 15

The homework assignments are posted on the course web site.
http://users.encs.concordia.ca/~trueman/web_page_353.htm

Homework #9: Do homework #9 by March 22, 2019.
Homework #10: Do homework #10 by March 29, 2019.
Homework #11: Do homework #11 by April 5, 2019.
Homework #12: Do homework #12 by April 12, 2019.

Tutorial Workshop #10: Friday March 22, 2019.
Tutorial Workshop #11: Friday March 29, 2019.
Tutorial Workshop #12: Friday April 5, 2019.
Tutorial Workshop #13: Friday April 12, 2019

Last Day of Classes: April 13, 2019.

Final exam date: Tuesday April 23, 2019, 9:00 to 12:00, in H531.

Topics to be Covered

Transmission Lines (TLs)

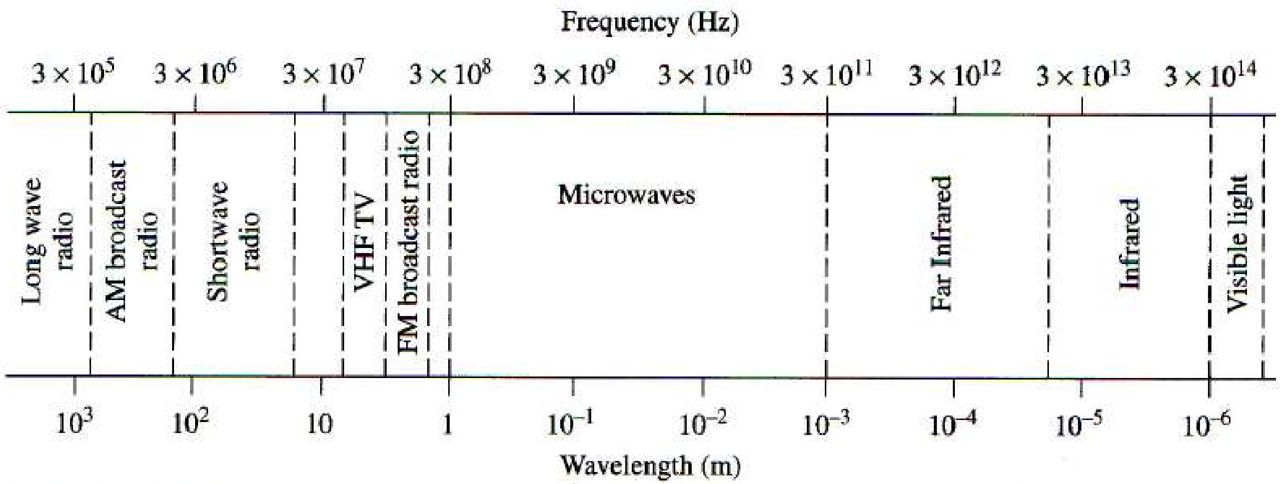
- Wave Equation and Solution - done
- Solving a TL Circuit - done
- Standing Wave Patterns - done
- Impedance Matching – done
- Bandwidth of Digital Signal - done

Plane Waves

- Maxwell’s Equations and the Wave Equation (today)
- Plane waves
- Material Boundaries
- Transmission Through a Wall

Antennas

The Frequency Spectrum



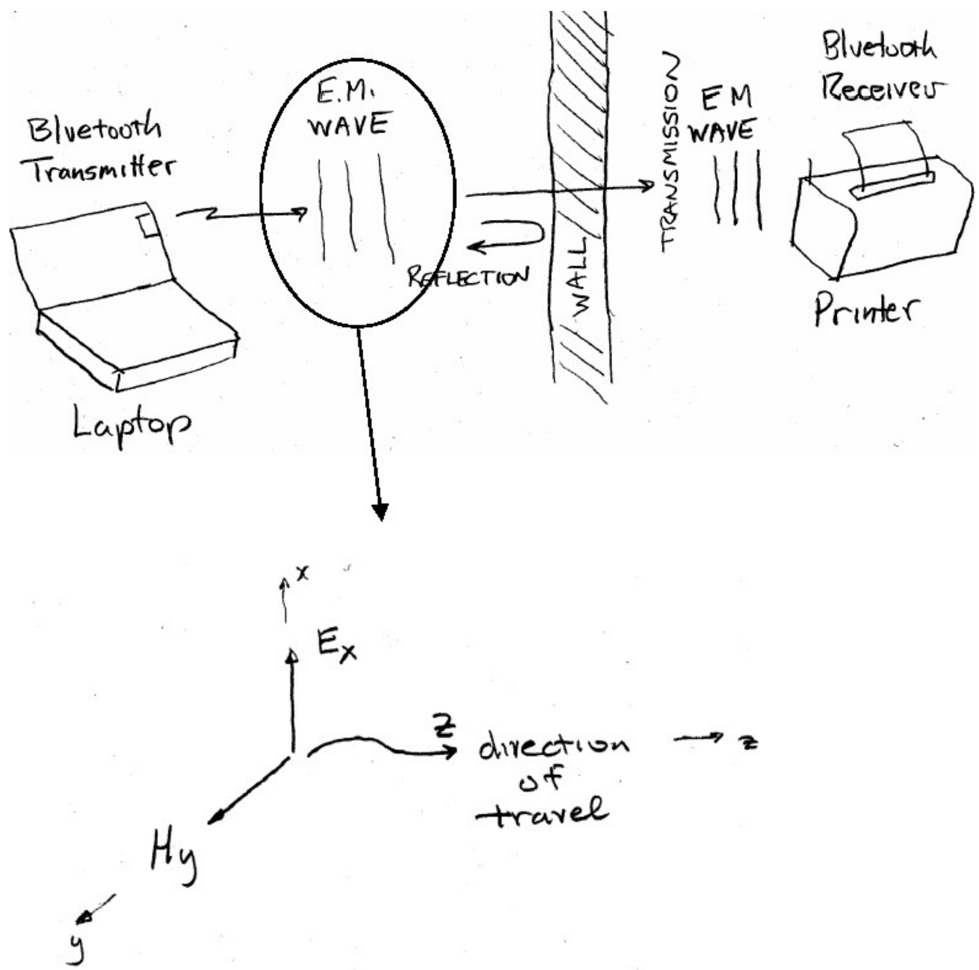
Typical Frequencies	
AM broadcast band	535–1605 kHz
Short wave radio band	3–30 MHz
FM broadcast band	88–108 MHz
VHF TV (2–4)	54–72 MHz
VHF TV (5–6)	76–88 MHz
UHF TV (7–13)	174–216 MHz
UHF TV (14–83)	470–890 MHz
US cellular telephone	824–849 MHz
	869–894 MHz
European GSM cellular	880–915 MHz
	925–960 MHz
GPS	1575.42 MHz
	1227.60 MHz
Microwave ovens	2.45 GHz
US DBS	11.7–12.5 GHz
US ISM bands	902–928 MHz
	2.400–2.484 GHz
	5.725–5.850 GHz
US UWB radio	3.1–10.6 GHz

Approximate Band Designations	
Medium frequency	300 kHz to 3 MHz
High frequency (HF)	3 MHz to 30 MHz
Very high frequency (VHF)	30 MHz to 300 MHz
Ultra high frequency (UHF)	300 MHz to 3 GHz
L band	1–2 GHz
S band	2–4 GHz
C band	4–8 GHz
X band	8–12 GHz
Ku band	12–18 GHz
K band	18–26 GHz
Ka band	26–40 GHz
U band	40–60 GHz
V band	50-75 GHz
E band	60-90 GHz
W band	75-110 GHz
F band	90-140 GHz

Pozar, “Microwave Engineering”, 3rd edition, Wiley, 2005.

Waves in Space

The objective of the *last segment* of ELEC353 is to study a "wireless" communication link such as those used by Bluetooth or IEEE 802.11b or e at 2450 MHz:



Fields and Maxwell's Equations

- An "antenna" changes a voltage on a transmission line into an "electric field" E traveling through space.
- There is no "circuit" for a wave traveling in space so it is not sensible to talk about voltage and current.
- Instead we deal with:
    - "voltage spread across space", called the "electric field" E in volts per meter.
    - "current spread across space", called the "magnetic field" H in amps per meter.

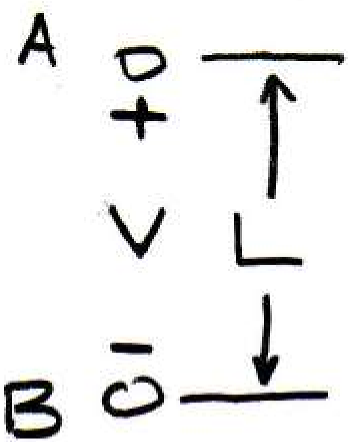
$$\mathrm{V} \Longrightarrow \mathrm{E}$$

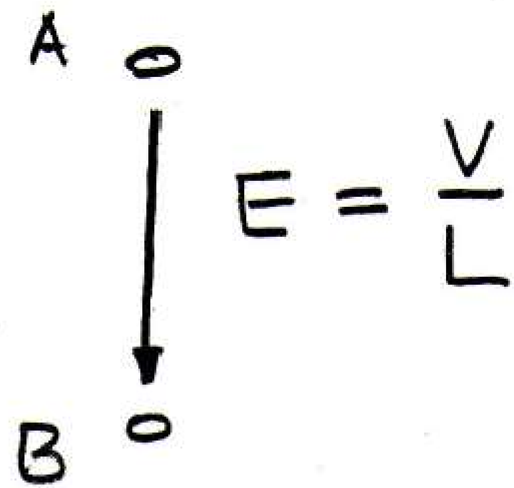
$$\mathrm{I} \Longrightarrow \mathrm{H}$$

Voltage and Electric Field

Inan and Inan Section 4.3 and 4.4 (review of CEGEP Electricity and Magnetism)

- What is the relationship between voltage and electric field?

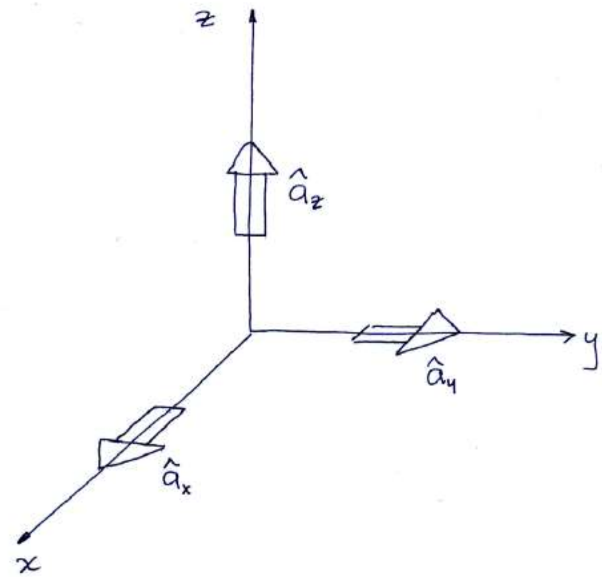




- Suppose there is a voltage of V volts measured at point A relative to point B, a distance of L meters.
- The "electric field" is the voltage spread across the space between the two points:

$$\text{average electric field } E = \frac{V}{L} \text{ volts per meter}$$

- Think of electric field as "voltage per unit distance".
- The electric field is a *vector* quantity which has a *direction* as well as a *magnitude*.
- The direction of the electric field is that it points from the positive terminal A towards the negative terminal B.

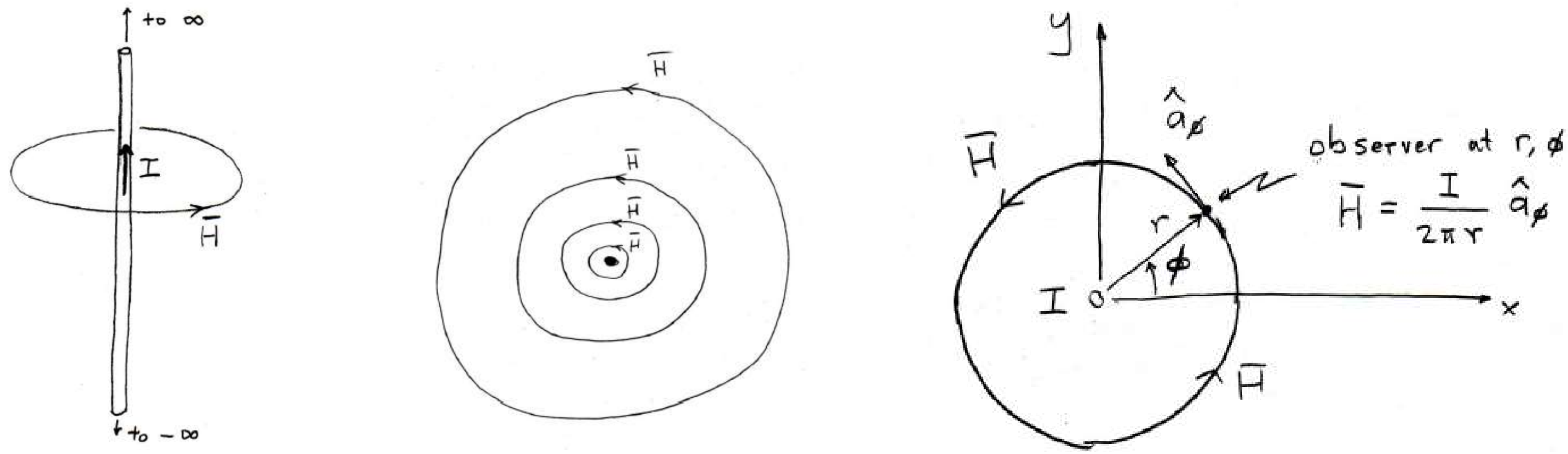


- In general the electric field is a vector quantity with three vector components:
$$\overline{E} = E_x(x,y,z)\hat{a}_x + E_y(x,y,z)\hat{a}_y + E_z(x,y,z)\hat{a}_z$$
where the unit vectors along the coordinate axes are $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$.
- Each vector component E_x, E_y and E_z can, in general, be a function of all three coordinates x, y, and z.
- But in ELEC353, we will usually deal with electric fields that are a function of only one space coordinate and only have one vector component.

Current and Magnetic Field

Inan and Inan Section 6.2

- What is the relationship between current and magnetic field?

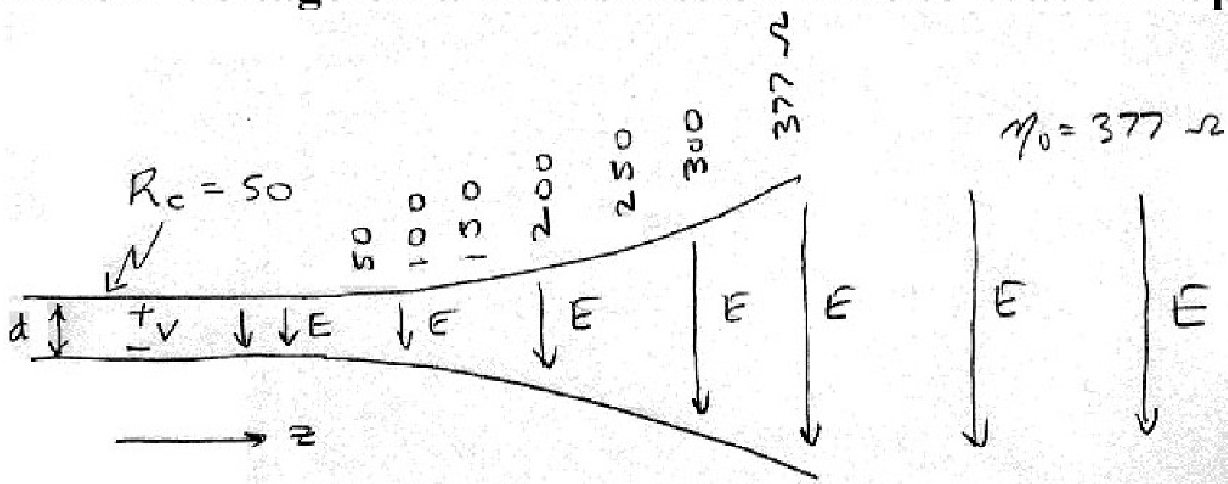


- Suppose we have an infinitely long, straight wire oriented along the z axis, carrying current I.
- Then the magnetic field forms closed circles centered on the wire, and can be written as (Inan and Inan page 455)

$$\overline{H} = \frac{I}{2\pi r}\hat{a}_\phi \text{ amps per meter}$$

- Distance r is the distance from the center of the wire to the observer.
- For an observer at location (x, y), the distance r is the distance to the z axis, $r = \sqrt{x^2 + y^2}$

Transition from Voltage-on-a-Transmission-Line to Wave-in-Space



- Consider a two-wire transmission line with the wires separated by distance d and with characteristic impedance $R_c = 50$ ohms
- The voltage on the transmission line is a traveling wave in the $+z$ direction
$$V(z) = V^+ e^{-j\beta z}$$
- Electric field is voltage divided by distance, so the electric field between the wires is (approximately)
$$E(z) = \frac{V(z)}{d} = \frac{V^+}{d} e^{-j\beta z} = E^+ e^{-j\beta z}$$
- So we expect the electric field to be a traveling wave in the $+z$ direction.

Maxwell's Equations

Inan and Inan Section 7.4.2 and Section 7.5, and Section 8.1

Gauss' Law for $\overline{D}$: $$\nabla \cdot \overline{D} = \rho_v$$	Ampere's Law: $$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$
Gauss' Law for $\overline{B}$: $$\nabla \cdot \overline{B} = 0$$	Faraday's Law: $$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

Electric Flux Density $\overline{D}$ and Electric Field $\overline{E}$

Inan and Inan Section 4.5

- Vector $\overline{D}$ is called the "electric flux density" and has the units of coulombs per square meter.
- Vector $\overline{E}$ is the "electric field" and has the units of volts per meter.
- Vectors $\overline{D}$ and $\overline{E}$ are related by the properties of the material in which the fields exist.
- Most materials are:
    - "linear": if you double the strength of the sources then you double the strength of the fields
    - "isotropic": the behavior of the material is the same in all directions in the material.
    - "homogeneous": the behavior of the material does not change as we move around from one location to another inside the material.

- For linear, isotropic, homogeneous materials:

$$\overline{D} = \varepsilon \overline{E}$$

  - Vectors $\overline{D}$ and $\overline{E}$ both point in the same direction
  - The magnitude of $\overline{D}$ is proportional to the magnitude of $\overline{E}$.
  - The proportionality constant ε is called the "permittivity" of the material, and has the units "farads per meter", which is "capacitance spread over space".
  - Empty space is called "free space" and the value of the permittivity of free space is

$$\varepsilon_0 = 8.854x10^{-12} \text{ Farads/meter}$$

- The "relative permittivity" ε_r is the ratio of the permittivity of the material to the permittivity of free space;

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$$

- Relative permittivity for certain materials: (Inan and Inan Table 4.1)

Material	Relative Permittivity
Free Space	1
Air	≈ 1
Styrofoam	1.03
Glass	4 to 9
Polyethylene	2.26
Teflon	2.1
Water (distilled)	81
Sea Water	72
Metals (high conductivity)	1

- Note that the permittivity is a function of frequency so we need to know the permittivity value at the given frequency of operation.
- Permittivity ε has the units of "Farads per meter" and measures the ability of the material to store energy in the electric field.
- The stored energy density in the electric field is

$$w_e = \frac{1}{2}\varepsilon E^2 \text{ Joules per cubic meter}$$

Magnetic Flux Density $\overline{B}$ and Magnetic Field $\overline{H}$

Inan and Inan Section 6.1

- Vector $\overline{B}$ is called the "magnetic flux density" and has the units of "Webers per square meter" (older textbooks) or "Tesla" (newer textbooks).
- Vector $\overline{H}$ is the "magnetic field" and has the units of "Amps per meter".
- For magnetic fields, most materials are linear, isotropic and homogeneous.
- For linear, isotropic, homogeneous materials:

$$\overline{B} = \mu\overline{H}$$

  - The proportionality constant μ is called the "permeability" of the material, and has the units "Henrys per meter", or inductance spread across space.
  - The permeability has the units of "inductance spread across space" and measures the ability of the material to store energy in the magnetic field.
  - The stored energy in a magnetic field H is

$$w_m = \frac{1}{2}\mu H^2 \text{ Joules per cubic meter}$$

  - Empty space or "free space" has permeability

$$\mu_0 = 4\pi x 10^{-7} \text{ Henries/meter}$$

- The "relative permeability" μ_r is the ratio of the permeability of the material to the permeability of free space;

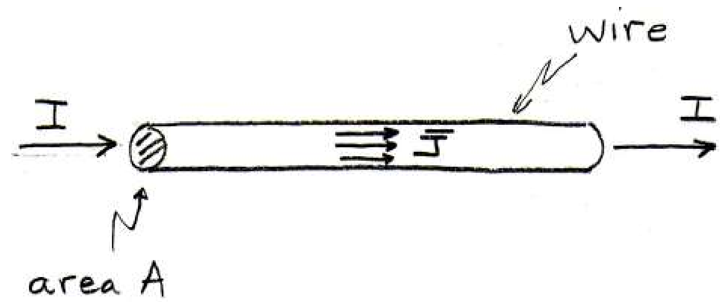
$$\mu_r = \frac{\mu}{\mu_0}$$

- Most materials are "non-magnetic", meaning that $\mu_r \approx 1$ (See Table 6.2 in Inan and Inan, page 531)

- An important class of materials is called "ferromagnetic" materials:
    - They are used to make magnetic memory devices like floppy discs or hard discs. (Inan and Inan Section 6.8)
    - Ferromagnetic materials are anisotropic and non-linear
    - Ferromagnetic materials have "memory": if you magnetize a ferromagnetic material in a certain direction, it stays magnetized in that direction (like a permanent magnet) and you can "read back" the direction of magnetization.
    - "Core" memory works by using small toroids of ferromagnetic material. A "1" bit is represented by magnetization in the "right hand rule" sense; a "0" bit by magnetization in the "left hand rule" sense. In 1970 magnetic core memory was the ONLY practical form of computer memory, aside from individual two-transistor flip-flops! 16 kilobytes of memory occupied about 20 circuit boards!
    - A hard disc works by assigning a tiny amount of area on the surface of the disc to each binary 'bit' to be stored: if the area is magnetized in one direction the bit is "zero"; if it is magnetized in the other direction the bit is "one".
- In ELEC353, we will always deal with linear, isotropic, homogeneous magnetic materials, so $\overline{B} = \mu\overline{H}$ and we can replace $\overline{B}$ in Maxwell's Equations with $\mu\overline{H}$.
- Also our materials will be "non-magnetic" meaning that $\mu_r = 1$.

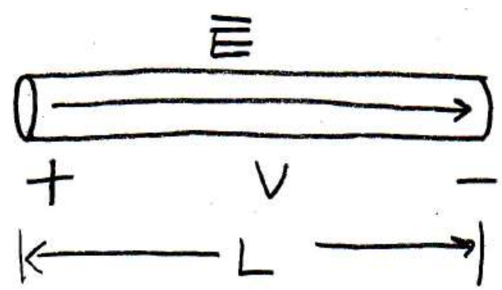
Current, Conductivity and Current Density

Inan and Inan Section 5.2

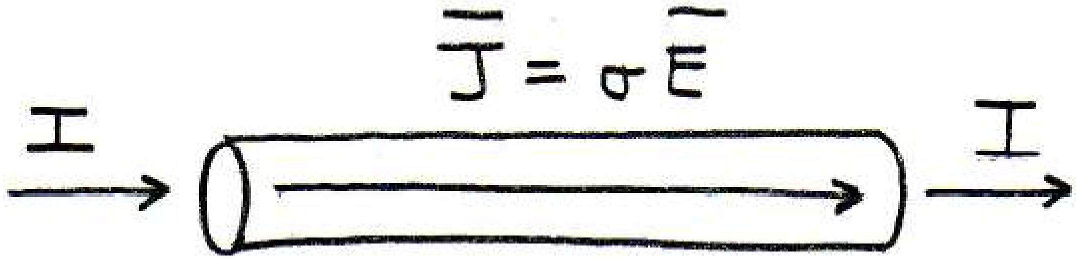


- Suppose that a wire has cross-sectional area A and carries current I.
- If the current is uniformly spread over the area A, then the current density is

$$J = \frac{I}{A} \text{ amps per square meter}$$



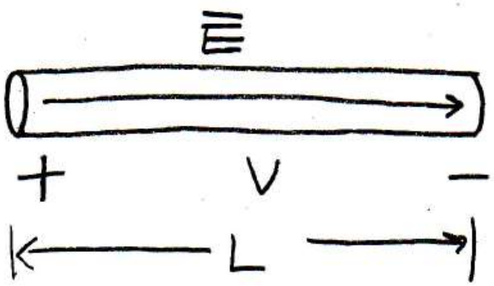
- If we apply a voltage V across the ends of a uniform wire of length L, then the electric field in the wire is $E = \frac{V}{L}$ volts per meter



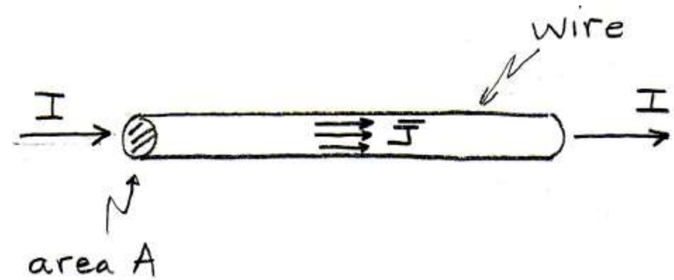
Ohm's Law

$$\bar{J} = \sigma \bar{E} \qquad \sigma \quad \text{Conductivity in Siemens/meter}$$

$$\bar{E} = \rho \bar{J} \qquad \rho = \frac{1}{\sigma} \quad \text{Resistivity in Ohm-meters}$$



$$E = \frac{V}{L}$$



$$J = \frac{I}{A}$$

$$E = \rho J$$

Ohm's Law

$$\frac{V}{L} = \rho \frac{I}{A} \qquad V = \frac{\rho L}{A} I \qquad R = \frac{\rho L}{A} \qquad V = RI$$

- Conductivity of Materials: (Inan and Inan Table 5.1)

Material	Conductivity (S/m)
Silver	6.17×10^{7}
Copper	5.8×10^{7}
Brass	2.56×10^{7}
Gold	4.1×10^{7}
Aluminum	3.82×10^{7}
Iron	1.03×10^{7}
Seawater	4
Marshy Soil	10^{-2}
Dry sandy soil	10^{-3}
Water (distilled and de-ionized)	10^{-4}
Mica	10^{-15}

- Metals are “good conductors” and have a very high conductivity.
- Conversely, “insulators” such as mica have very low conductivity.

Maxwell's Equations

Inan and Inan Section 7.4.2 and Section 7.5, and Section 8.1

Gauss' Law for $\overline{D}$: $$\nabla \cdot \overline{D} = \rho_v$$	Ampere's Law: $$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$$
Gauss' Law for $\overline{B}$: $$\nabla \cdot \overline{B} = 0$$	Faraday's Law: $$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

Maxwell's Equations for Linear, Isotropic, Homogeneous Material

- If the material is linear, isotropic and homogeneous we can simplify Maxwell's Equations:
    - $\overline{D} = \varepsilon \overline{E}$, so we can eliminate $\overline{D}$
    - $\overline{B} = \mu \overline{H}$, so we can eliminate $\overline{B}$
    - $\overline{J} = \sigma \overline{E}$, so we can eliminate $\overline{J}$
- These three equations are called the "constitutive equations" for the material.
- The parameters ε, μ, σ are called the "electrical properties" of the material.

Maxwell's Equations for Linear, Isotropic, Homogeneous Media

General Maxwell's Equations:	Specialized to linear, isotropic, homogeneous materials: $\overline{D} = \varepsilon \overline{E}$, $\overline{B} = \mu \overline{H}$, $\overline{J} = \sigma \overline{E}$
Gauss' Law for $\overline{D}$: $\nabla \cdot \overline{D} = \rho_v$	Gauss' Law for the electric field: $\nabla \cdot \overline{E} = \dfrac{\rho_v}{\varepsilon}$
Gauss' Law for $\overline{B}$: $\nabla \cdot \overline{B} = 0$	Gauss' Law for the magnetic field: $\nabla \cdot \overline{H} = 0$
Ampere's Law: $\nabla \times \overline{H} = \overline{J} + \dfrac{\partial \overline{D}}{\partial t}$	Ampere's Law: $\nabla \times \overline{H} = \sigma \overline{E} + \varepsilon \dfrac{\partial \overline{E}}{\partial t}$
Faraday's Law: $\nabla \times \overline{E} = -\dfrac{\partial \overline{B}}{\partial t}$	Faraday's Law: $\nabla \times \overline{E} = -\mu \dfrac{\partial \overline{H}}{\partial t}$

- It is useful to think of the charge density ρ_v as the "generator" for the problem.
- In A.C. problems the charge density varies sinusoidally with time, of the form $\rho_v(t) = A\cos(\omega t)$ at some frequency $f = \dfrac{\omega}{2\pi}$ Hz.

$$\nabla \cdot \overline{E} = \frac{\rho_v}{\varepsilon}$$

- **Gauss' Law** for $\overline{E}$ relates the space derivatives of the electric field to the charge density at each point in space:

$$\nabla \cdot \overline{E} = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \right) \cdot \left(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \right)$$

so Gauss' Law reads:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho_v}{\varepsilon}$$

$$\nabla \cdot \overline{H} = 0$$

- **Gauss' Law** for $\overline{H}$ states that there are no magnetic charges so the space derivatives of $\overline{H}$ must add up to zero at every point:

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

Ampere's Law

$$\nabla \times \overline{H} = \sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t}$$

$$\nabla \times \overline{H} = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \right) \times \left(H_x \hat{a}_x + H_y \hat{a}_y + H_z \hat{a}_z \right) = \sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t}$$

$$\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \hat{a}_x \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) - \hat{a}_y \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) + \hat{a}_z \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)$$

$$\hat{a}_x \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) - \hat{a}_y \left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) + \hat{a}_z \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)$$

$$= \sigma \left(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \right) + \varepsilon \left(\frac{\partial E_x}{\partial t} \hat{a}_x + \frac{\partial E_y}{\partial t} \hat{a}_y + \frac{\partial E_z}{\partial t} \hat{a}_z \right)$$

Faraday's Law

$$\nabla \times \overline{E} = -\mu \frac{\partial \overline{H}}{\partial t}$$

$$\hat{a}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\mu \frac{\partial \overline{H}}{\partial t}$$

$$\hat{a}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$= -\mu\left(\frac{\partial H_x}{\partial t}\hat{a}_x + \frac{\partial H_y}{\partial t}\hat{a}_y + \frac{\partial H_z}{\partial t}\hat{a}_z\right)$$

Review: Maxwell's Equations

General Maxwell's Equations:	Specialized to linear, isotropic, homogeneous materials: $\overline{D} = \varepsilon\overline{E}$, $\overline{B} = \mu\overline{H}$, $\overline{J} = \sigma\overline{E}$
Gauss' Law for $\overline{D}$: $\nabla \cdot \overline{D} = \rho_v$	Gauss' Law for the electric field: $\nabla \cdot \overline{E} = \frac{\rho_v}{\varepsilon}$
Gauss' Law for $\overline{B}$: $\nabla \cdot \overline{B} = 0$	Gauss' Law for the magnetic field: $\nabla \cdot \overline{H} = 0$
Ampere's Law: $\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$	Ampere's Law: $\nabla \times \overline{H} = \sigma\overline{E} + \varepsilon\frac{\partial \overline{E}}{\partial t}$
Faraday's Law: $\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$	Faraday's Law: $\nabla \times \overline{E} = -\mu\frac{\partial \overline{H}}{\partial t}$

- It is useful to think of the charge density ρ_v as the "generator" for the problem.
- In A.C. problems the charge density varies sinusoidally with time, of the form $\rho_v(t) = A\cos(\omega t)$ at some frequency $f = \frac{\omega}{2\pi}$ Hz.

Time-Harmonic Fields

- Suppose the generator in the problem is "time harmonic", meaning that it varies with time as $\cos(\omega t)$.
- Maxwell's equations for linear materials describe a "linear, time-invariant system" or LTI system, so we expect that in the sinusoidal steady state, each component of the electric field will also be proportional to $\cos(\omega t)$.
- So we can write the electric field as
$$\overline{E}(t) = A_x \cos(\omega t + \theta_x)\hat{a}_x + A_y \cos(\omega t + \theta_y)\hat{a}_y + A_z \cos(\omega t + \theta_z)\hat{a}_z$$
where A_x is the amplitude of the x component of the electric field and θ_x is the phase of the x component of the electric field, and so forth.
- It is always convenient to deal with A.C. quantities with phasors, so we can "code" the amplitude and phase of each component into a phasor:

$$A_x \cos(\omega t + \theta_x) \Rightarrow E_x = A_x e^{j\theta_x}$$
$$A_y \cos(\omega t + \theta_y) \Rightarrow E_y = A_y e^{j\theta_y}$$
$$A_z \cos(\omega t + \theta_z) \Rightarrow E_z = A_z e^{j\theta_z}$$

$$\overline{E}(t) = A_x \cos(\omega t + \theta_x)\hat{a}_x + A_y \cos(\omega t + \theta_y)\hat{a}_y + A_z \cos(\omega t + \theta_z)\hat{a}_z$$

$$A_x \cos(\omega t + \theta_x) \Rightarrow E_x = A_x e^{j\theta_x}$$
$$A_y \cos(\omega t + \theta_y) \Rightarrow E_y = A_y e^{j\theta_y}$$
$$A_z \cos(\omega t + \theta_z) \Rightarrow E_z = A_z e^{j\theta_z}$$

- Then we can assemble the three components of the field into a vector that is also a phasor, sometimes called a "**vector-phasor**":
$$\overline{E} = A_x e^{j\theta_x}\hat{a}_x + A_y e^{j\theta_y}\hat{a}_y + A_z e^{j\theta_z}\hat{a}_z$$
- We can write this more compactly as
$$\overline{E} = E_x\hat{a}_x + E_y\hat{a}_y + E_z\hat{a}_z$$
where we understand that E_x, E_y and E_z are complex numbers called "phasors" which represent the A.C. field components.

Time-Harmonic Maxwell's Equations

Inan and Inan Section 7.4

- Recall that if V is a phasor representing an A.C. voltage $v(t)$, then the time derivative $\frac{dv}{dt}$ is represented by the phasor $j\omega V$.
- For A.C. fields, we represent the field with a "vector-phasor" $\overline{E}$.
- Then the time derivative of the field, $\frac{\partial \overline{E}(t)}{\partial t}$, is represented by the vector-phasor $j\omega\overline{E}$.

Equation	Time Domain	Frequency Domain
Gauss' Law for the electric field	$\nabla \cdot \overline{E} = \frac{\rho_v}{\varepsilon}$	$\nabla \cdot \overline{E} = \frac{\rho_v}{\varepsilon}$
Gauss' Law for the magnetic field	$\nabla \cdot \overline{H} = 0$	$\nabla \cdot \overline{H} = 0$
Ampere's Law	$\nabla \times \overline{H} = \sigma\overline{E} + \varepsilon\frac{\partial \overline{E}}{\partial t}$	$\nabla \times \overline{H} = (\sigma + j\omega\varepsilon)\overline{E}$
Faraday's Law	$\nabla \times \overline{E} = -\mu\frac{\partial \overline{H}}{\partial t}$	$\nabla \times \overline{E} = -j\omega\mu\overline{H}$

Wave Equation

Review: Transmission Lines

$$\frac{dV}{dz} = -(r + j\omega\ell)I \qquad \frac{dI}{dz} = -(g + j\omega c)V$$

$$\frac{d^2V}{dz^2} = -(r + j\omega\ell)\frac{dI}{dz}$$

$$\frac{d^2V}{dz^2} = -(r + j\omega\ell)\left[-(g + j\omega c)V\right]$$

Propagation constant:

$$\gamma = \sqrt{(r + j\omega\ell)(g + j\omega c)}$$

$$\frac{d^2V}{dz^2} = \gamma^2 V \quad \Longrightarrow$$

Solution:

$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z}$$

Wave Equation for Waves in Space

$$\nabla \times \overline{E} = -j\omega\mu\overline{H} \qquad\qquad \nabla \times \overline{H} = (\sigma + j\omega\varepsilon)\overline{E}$$

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu\nabla \times \overline{H}$$

$$\nabla \times \nabla \times \overline{E} = -j\omega\mu\left[(\sigma + j\omega\varepsilon)\overline{E}\right]$$

Propagation constant:

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\nabla \times \nabla \times \overline{E} = -\gamma^2\overline{E}$$

Vector identity: $\nabla \times \nabla \times \overline{E} = \nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2\overline{E}$

$$\nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2\overline{E} = -\gamma^2\overline{E}$$

$\nabla^2\overline{E}$ is the “vector Laplacian” of the electric field; explained below.

Wave equation, continued:

$$\nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2 \overline{E} = -\gamma^2 \overline{E}$$

Gauss' Law: $\nabla \cdot \overline{E} = \dfrac{\rho_v}{\varepsilon}$

Source-free region: $\rho_v = 0$ so $\nabla \cdot \overline{E} = 0$

$$-\nabla^2 \overline{E} = -\gamma^2 \overline{E}$$

$$\nabla^2 \overline{E} = \gamma^2 \overline{E}$$

- Vector wave equation
- Vector Helmholtz equation

$$\nabla^2 \overline{E} = \gamma^2 \overline{E}$$

Propagation Constant: (Inan and Inan page 659)

- Define the "propagation constant" $\gamma = \alpha + j\beta$ as

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

- $\alpha = \mathrm{Re}(\gamma)$ is the "attenuation constant" in Nepers/meter
- $\beta = \mathrm{Im}(\gamma)$ is the "phase constant" in radians/meter
- Lossless Materials: If $\sigma = 0$, then

$$\gamma = \sqrt{j\omega\mu(j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon} = 0 + j\beta$$

- $\alpha = \mathrm{Re}(\gamma) = 0$ in lossless materials
- $\beta = \mathrm{Im}(\gamma) = \omega\sqrt{\mu\varepsilon}$ in lossless materials

How the Scalar Laplacian Arises

Problem: given some electric charge density as ‘sources’, find the electric field.

Solution: The electric field is related to the charge density by Gauss’ Law:

$$\nabla \cdot \overline{E} = \frac{\rho_v}{\varepsilon}$$

Use the voltage or “scalar potential” *V(x,y,z)* as an intermediate step:

$$\overline{E} = -\nabla V$$

$$\nabla \cdot \overline{E} = -\nabla \cdot \left(\nabla V\right)$$

$$\nabla \cdot \left(\nabla V\right) = -\frac{\rho_v}{\varepsilon}$$

Define the “scalar Laplacian”:

$$\nabla^2 V = \nabla \cdot \left(\nabla V\right)$$

$$\Longrightarrow$$

Poisson’s equation:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

Formula for the Scalar Laplacian

$$\nabla^2 V = \nabla \cdot \left(\nabla V \right)$$

Gradient: $$\nabla V = \hat{a}_x \frac{\partial V}{\partial x} + \hat{a}_y \frac{\partial V}{\partial y} + \hat{a}_z \frac{\partial V}{\partial z}$$

Divergence: $$\nabla \cdot \overline{E} = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \right) \cdot \left(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \right)$$

$$\nabla \cdot \overline{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)$$

Scalar Laplacian = divergence of the gradient:

$$\nabla^2 V = \nabla \cdot \left(\nabla V \right) = \left(\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z} \right) \cdot \left(\hat{a}_x \frac{\partial V}{\partial x} + \hat{a}_y \frac{\partial V}{\partial y} + \hat{a}_z \frac{\partial V}{\partial z} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Vector Laplacian

$$\overline{E} = E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z$$

$$\nabla^2 \overline{E} = \gamma^2 \overline{E}$$

Use the scalar Laplacian on each of the three vector components:

$$\nabla^2 \overline{E} = \hat{a}_x \nabla^2 E_x + \hat{a}_y \nabla^2 E_y + \hat{a}_z \nabla^2 E_z$$

$$\nabla^2 \overline{E} = \hat{a}_x \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) + \hat{a}_y \left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right) + \hat{a}_z \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right)$$

$$\hat{a}_x \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) + \hat{a}_y \left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right)$$

$$+ \hat{a}_z \left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) = \gamma^2 \left(E_x \hat{a}_x + E_y \hat{a}_y + E_z \hat{a}_z \right)$$