ELEC353 Lecture Notes Set 6

The homework assignments are posted on the course web site.
http://users.encs.concordia.ca/~trueman/web_page_353.htm

Homework #3: Do homework #3 by February 1, 2019.
Homework #4: Do homework #4 by February 8, 2019.
Homework #5: Do homework #5 by February 14, 2019.

Mid-term test: Thursday February 14, 2019.

- Includes Homework #5!

- See the course web site for sample mid-term tests with solutions.
- Study tip:
    - Download the question paper for a mid-term from a previous year.
    - Spend one hour 15 minutes solving the test with your calculator and the formula sheet, but no textbook or notes.
    - Grade your answer against the solution to the test!

Tentative final exam date: Tuesday April 23, 2019, 9:00 to 12:00.

Mid-term Test: Thursday February 14, 2019

What is covered on the mid-term?
Everything done in class up to February 12.

Homework assignments #1 to 5
#1 = lumped and distributed circuit analysis, RC model of a TL
#2 = introductory TL questions, bounce diagram
#3 = TL in series, TL with shunt load
#4 = branching TL, RL load, TDR
#5 = RL load, transmission lines in series, pulse generator, TDR

Transmission Line Topics:

1. Junctions Between Transmission Lines Done!
2. Transmission Line with Shunt Load Done!
3. Branching Transmission Lines Done!
4. Inductive and Capacitive Terminations
5. Time Domain Reflectometry

(Class Test)

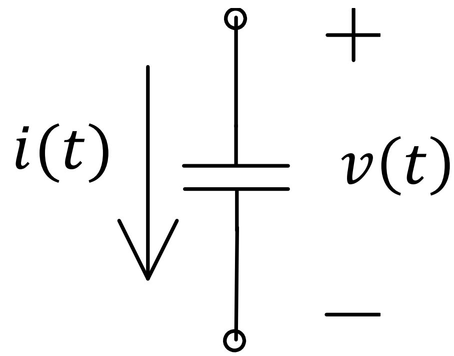
6. The Sinusoidal Steady State

Inductive and Capacitive Terminations

Inan, Inan and Said, page 60

1. Review RC circuits.
2. Solve transmission line circuits with RC terminations.
3. Review RL circuits.
4. Solve transmission line circuits with RL terminations.

Capacitor



- The "terminal relationship" for a capacitor is

$$i = C\frac{dv}{dt}$$

- Hence the voltage across a capacitor can be calculated as

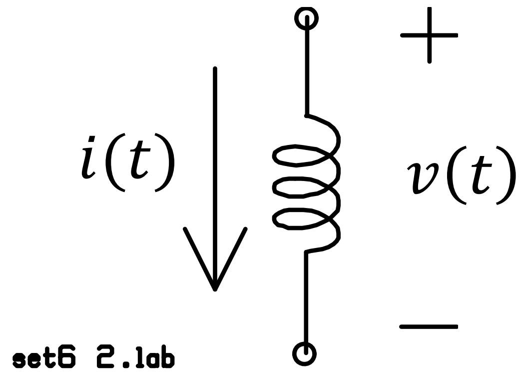
$$dv = \frac{1}{C}i(t)dt$$

$$v(t) = \frac{1}{C}\int_0^t i(t')dt' + v(0)$$

where $v(0)$ is the voltage across the capacitor at $t = 0$, and is called the "initial condition" for the capacitor.

- In most of our problems the capacitor is "initially uncharged":
  - This means that $v(0)$=0.
  - Since the voltage across the capacitor is zero, at $t = 0$ we can replace it with a short circuit to calculate the "initial" values of the voltages and currents in the circuit.

Inductor



- The "terminal relationship" for and inductor is

$$v = L\frac{di}{dt}$$

- Hence the current flowing throw the inductance can be calculated as

$$di = \frac{1}{L} v dt$$

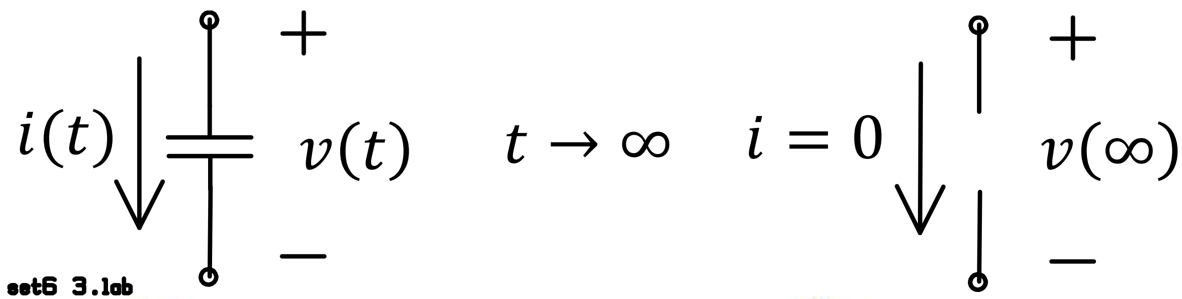
$$i(t) = \frac{1}{L}\int_0^t v(t')dt' + i(0)$$

where $i(0)$ is the current through the inductor at $t = 0$, and is called the "initial condition" for the inductor.

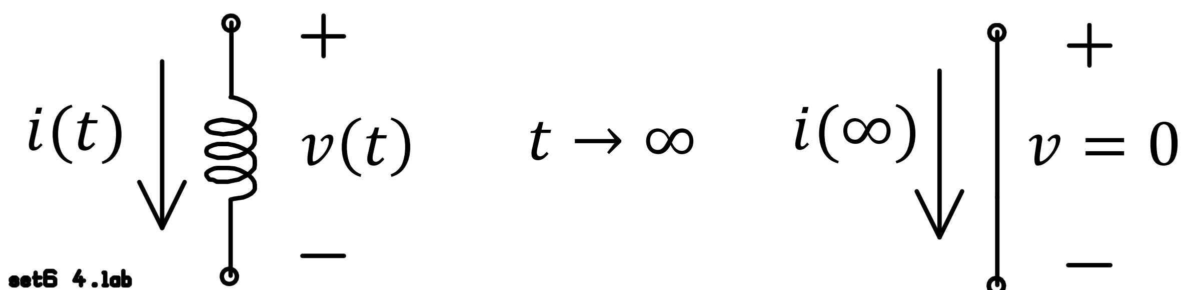
- In most of our problems the inductor is "initially uncharged":
  - This means at $t = 0$ there is no current flowing through the inductor, so $i(0)=0$.
  - Since the current through the inductor is zero, at $t = 0$ we can replace it with an open circuit to calculate the "initial" values of the voltages and currents in the circuit.

Final Conditions

- If a circuit has a "D.C." source that does NOT vary with time as $t \to \infty$, then all the voltages and currents in the circuit become CONSTANT with time as $t \to \infty$.

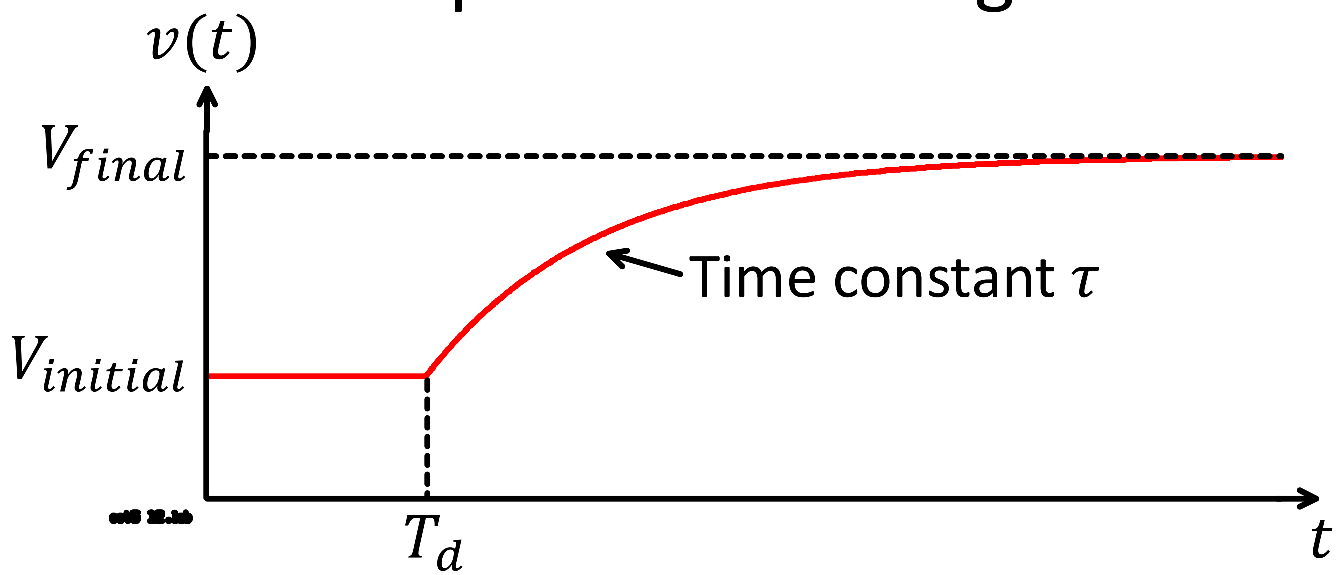


- Since $i = C\dfrac{dv}{dt}$, when v becomes constant then $\dfrac{dv}{dt} \to 0$ so the current in the capacitor becomes zero. The **capacitor** becomes an **open circuit** as $t \to \infty$.



- Since $v = L\dfrac{di}{dt}$, when i becomes constant then $\dfrac{di}{dt} \to 0$ so the voltage across the inductor becomes zero. The **inductor** becomes a **short circuit** as $t \to \infty$.
- We can calculate the voltages and currents in the circuit as $t \to \infty$ by replacing the capacitors by open circuits and the inductors by short circuits.

Exponential Changes



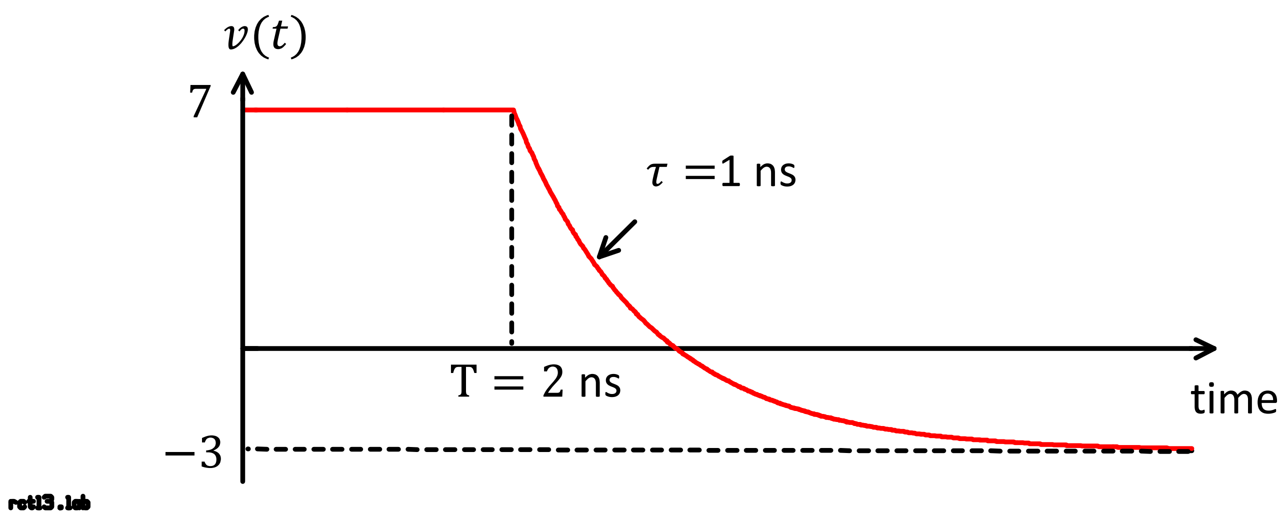
A voltage changes exponentially from $V_{initial}$ to V_{final}. The change starts at time T_d and has time constant τ.
For $t > T_d$

$$v(t) = V_{final} + \left(V_{initial} - V_{final}\right)e^{-\frac{t-T_d}{\tau}}$$

- When does the exponential charging start? At $t = T_d$.
- What is the initial value $V_{initial}$?
- What is the final value V_{final}?
- What is the time constant?

Example

A capacitor has initial voltage 7 volts. At $t = 2$ ns, the capacitor begins to "charge" from 7 volts to -3 volts, with a time constant of $\tau = 1$ ns. Write an equation for the capacitor voltage.



Solution

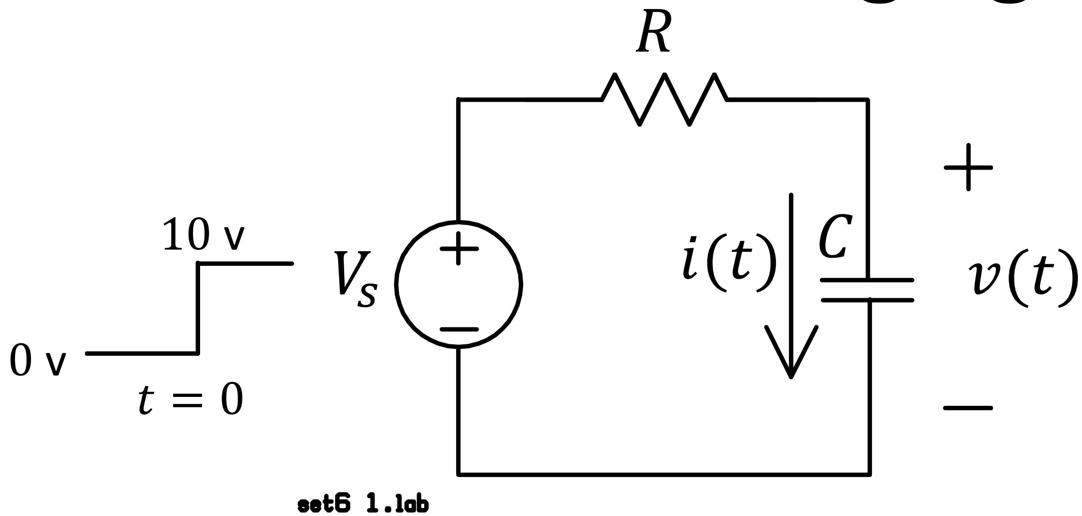
$$v(t) = V_{final} + (V_{initial} - V_{final})e^{-\frac{t-T_d}{\tau}} \qquad \text{for } t > T_d$$

The initial value is $V_{initial} = 7$ volts. The final value is $V_{final} = -3$ volts. The change starts at $T = 2$ ns, and the time constant is $\tau = 1$ ns. So we can write the voltage as

$$v(t) = -3 + (7 - (-3))e^{-\frac{t-2}{1}} = -3 + 10e^{-(t-2))}$$

for $t > 2$ ns.

Review: Charging Capacitor



The capacitor is initially uncharged so $v(0) = 0$.
The generator V_s is a step function of height 10 volts that starts at $t = 0$.

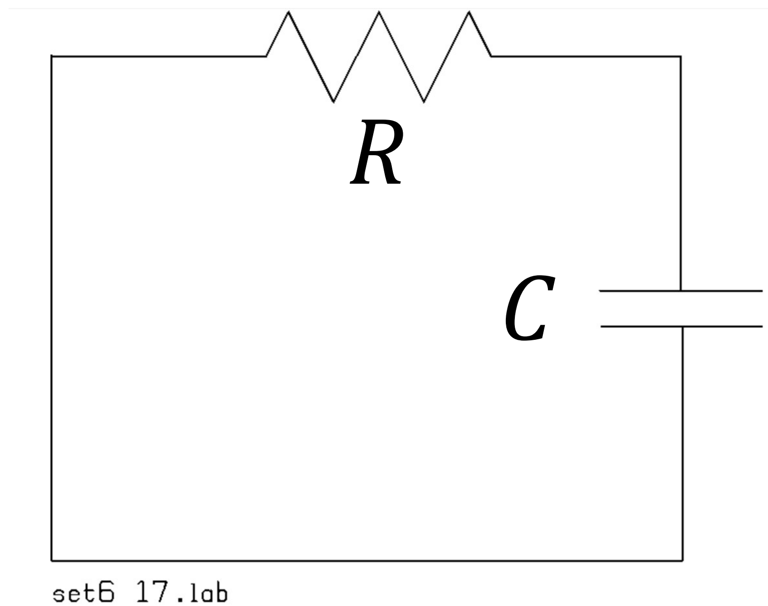
When does the exponential charging start?
The charging starts at $t = 0$ so $T_d = 0$.

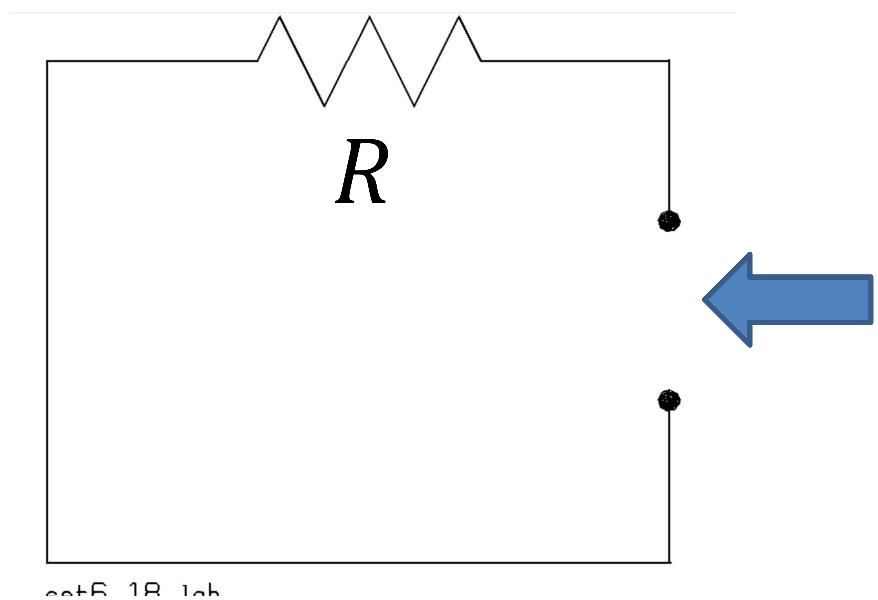
What is the initial value $V_{initial}$?
At $t = 0$, the capacitor is uncharged so $V_{initial} = 0$.

What is the final value V_{final}?
As $t \to \infty$, the capacitor becomes an open circuit and so $V_{final} = V_s$.

What is the time constant?
The time constant is $\tau = R_e C$, where R_e is the resistance of the circuit seen from the capacitor terminals.
For this circuit $R_e = R$, so $\tau = RC$.

How to find the time constant: $\tau = R_e C$





$$R_e = R$$

$$\tau = RC$$

When does the exponential charging start? At $T_d = 0$.

What is the initial value $V_{initial}$? The initial value is $V_{initial} = 0$.

What is the final value V_{final}? The final value is $V_{final} = V_s$.

What is the time constant? The time constant is $\tau = RC$.

$$v(t) = V_{final} + \left(V_{initial} - V_{final}\right)e^{-\frac{t-T_d}{\tau}}$$

$$v(t) = V_s + (0 - V_s)e^{-\frac{t-0}{RC}}$$

We can write this more simply as

$$v(t) = V_s - V_s e^{-\frac{t}{\tau}}$$

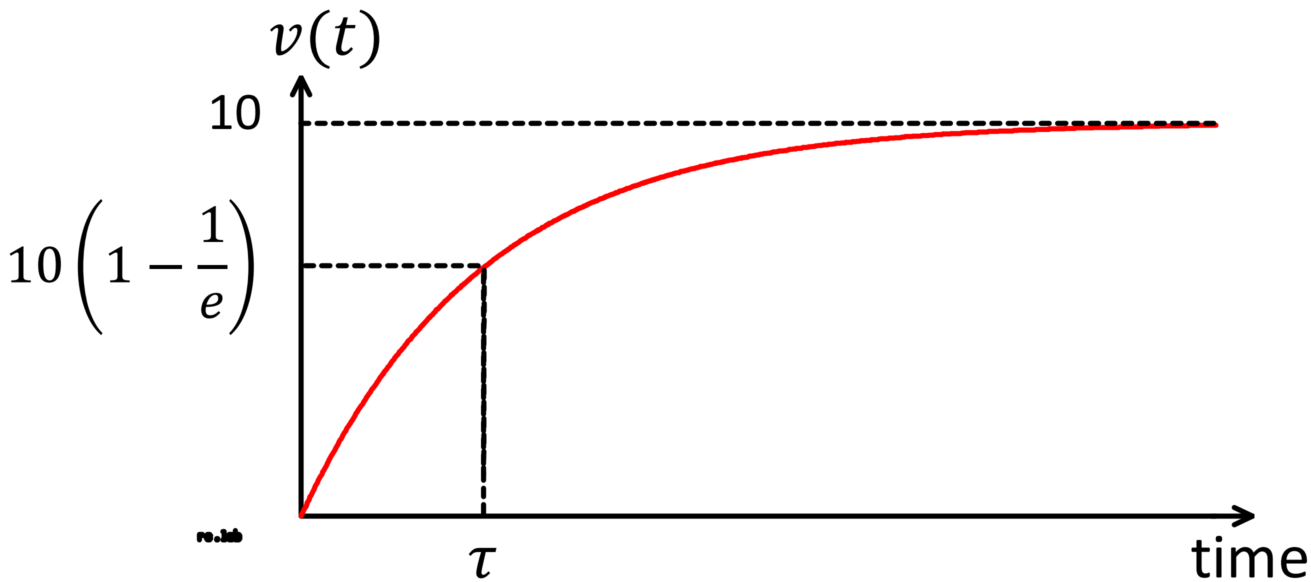
or

$$v(t) = V_s\left(1 - e^{-\frac{t}{\tau}}\right)$$

Draw a graph of the response:

With $V_s = 10$ volts,

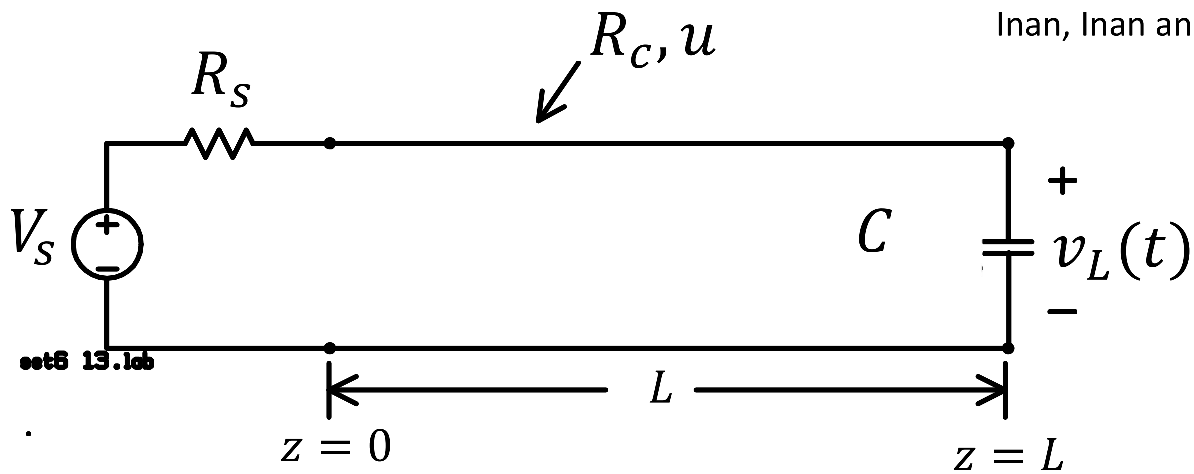
$$v(t) = 10\left(1 - e^{-t/\tau}\right)$$



After one time constant, at $t = \tau$, the voltage is $v(t=\tau) = 10\left(1 - e^{-1}\right) = 10\left(1 - \frac{1}{e}\right)$

Transmission Line Terminated with a Capacitor

Inan, Inan and Said Section 2.5



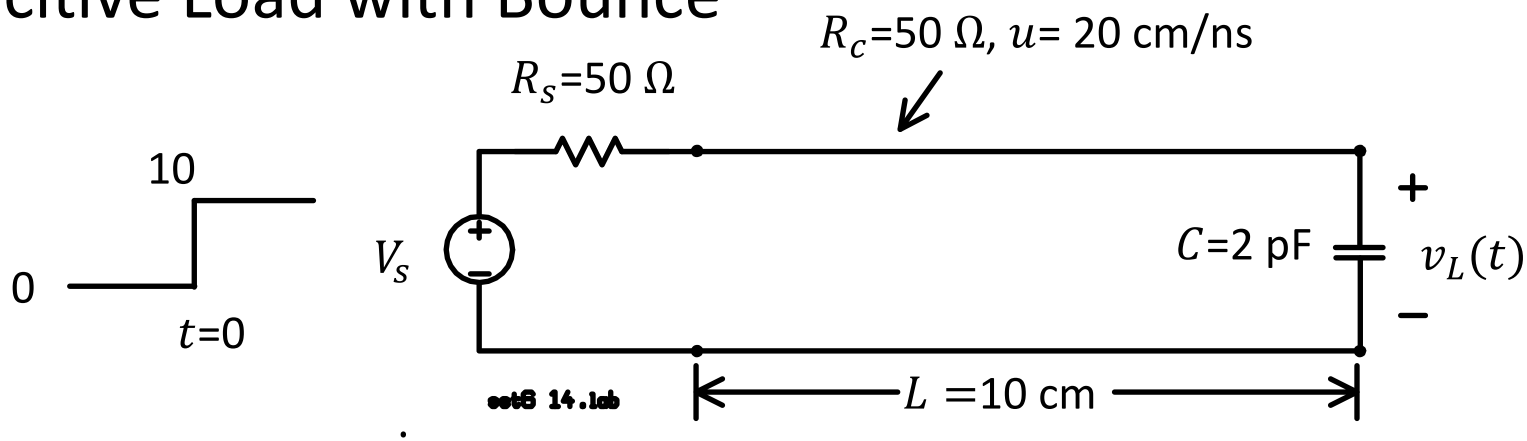
The capacitor is initially uncharged, so at $t = 0$, the load voltage is $v_L(0) = 0$

Find the load voltage as a function of time.

The source is matched, so $R_s = R_c = 50\Omega$

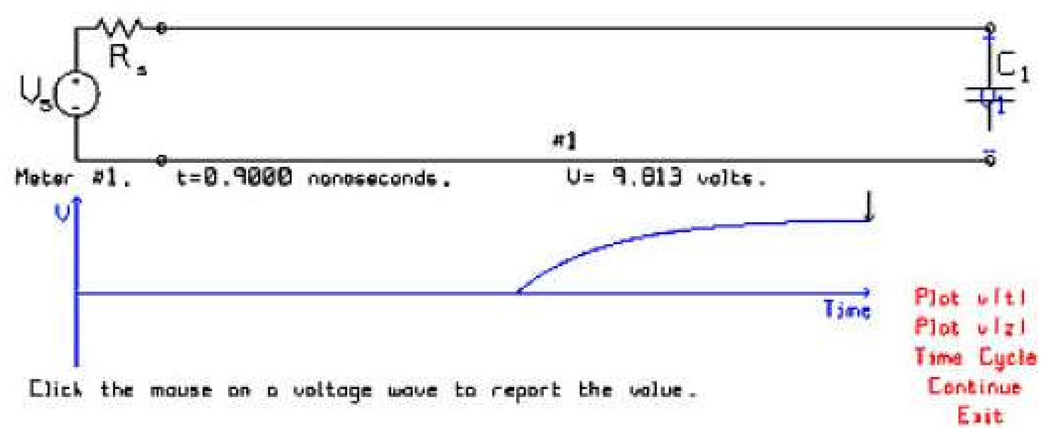
$$V(z=0) = \frac{R_c}{R_s + R_c} V_s = 5$$

Capacitive Load with Bounce

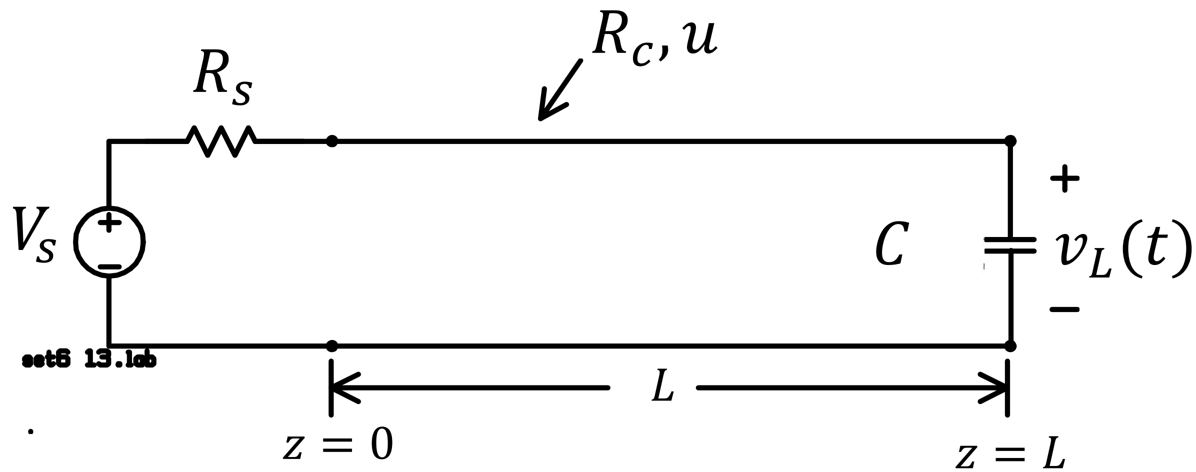


- the generator is a 10 volt step at t=0, with $R_s = 50$ ohms.
- the line has $R_c = 50$ ohms, $u = 20$ cm/ns, and length 10 cm.
- the one-way delay time is $\frac{L}{u} = \frac{10}{20} = 0.5$ ns
- the load is a 2 pF capacitor.

$$V(z=0) = \frac{R_c}{R_s + R_c} V_s = 5$$



Find the voltage across the capacitor:



Inan, Inan and Said Section 2.5

$V_s = 10$ volts
$R_s = 50$ ohms
$R_c = 50$ ohms
The capacitor is initially uncharged, so the load voltage is zero:

$$v_L(0) = 0$$

The load voltage is given by

$$v(t) = V_{final} + \left(V_{initial} - V_{final}\right)e^{-\frac{t-T_d}{\tau}}$$

The step arriving from the source is

$$V^+ = \frac{R_c V_s}{R_c + R_s} = \frac{50x10}{50+50} = 5$$

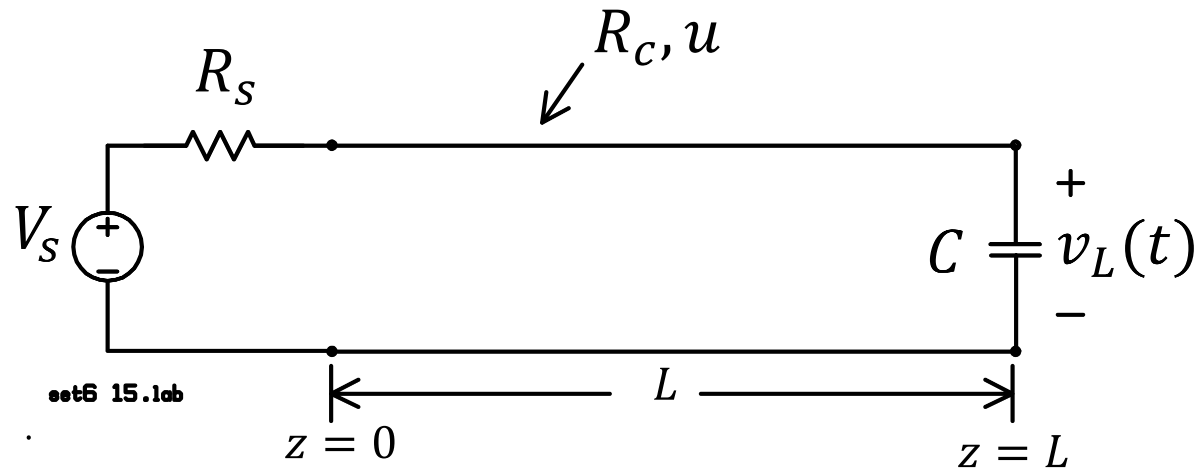
The step arrives at $t = T_d = \frac{L}{u}$ and then the load voltage starts to increase exponentially.

The load voltage is given by

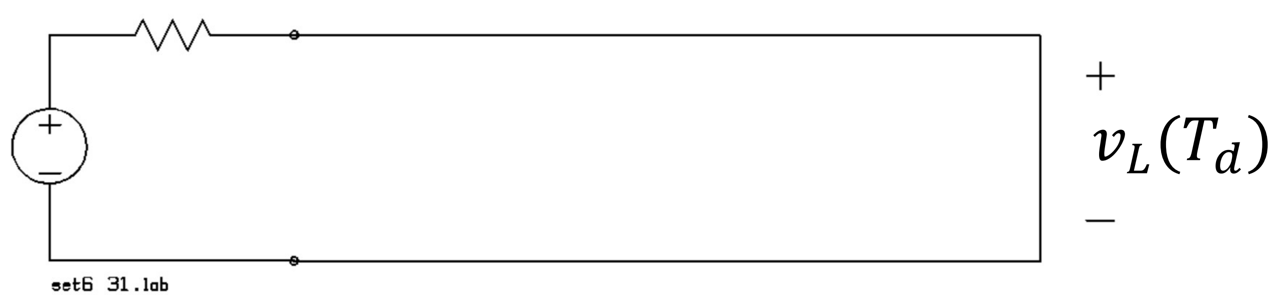
$$v_L(\mathrm{t}) = V^+(L,t) + V^-(L,t)$$

What is the initial value of V^- at $t = T_d$ when the step from the generator arrives?
What is the final value of V^- as $t \to \infty$?
What is the time constant τ?

Initial Value of V^- using the Reflection Coefficient



The capacitor is initially uncharged, so v_L=0 until the V^+ step function arrives at $t = T_d$.
An uncharged capacitor with v_L=0 looks like a short circuit:

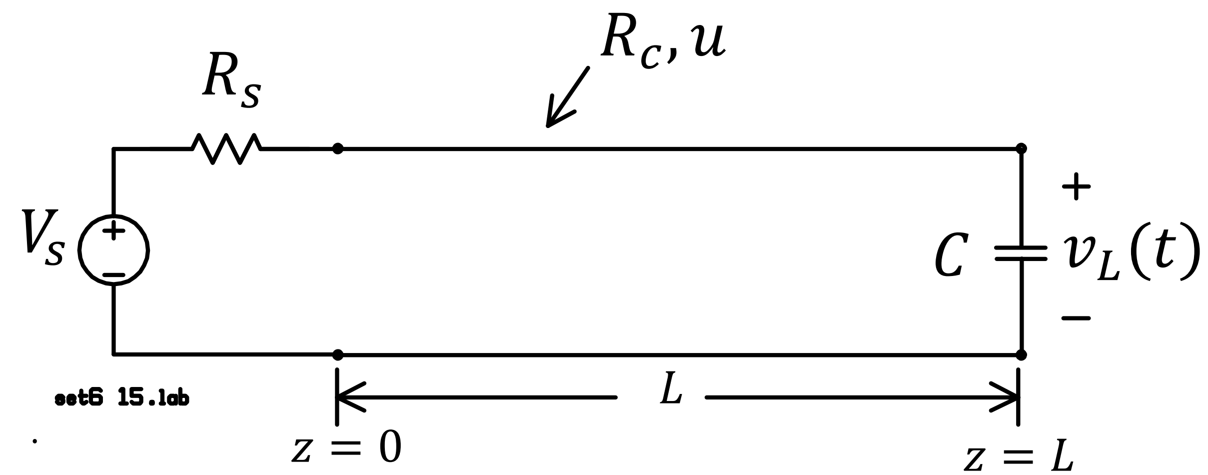


A short circuit has reflection coefficient $\Gamma_L = -1$.

Hence we expect $V^- = \Gamma_L V^+ = -1x5 = -5$ volts.

$V_{initial} = V^+ + V^- = 5 - 5 = 0$ volts

Final Value of V^- using the Reflection Coefficient



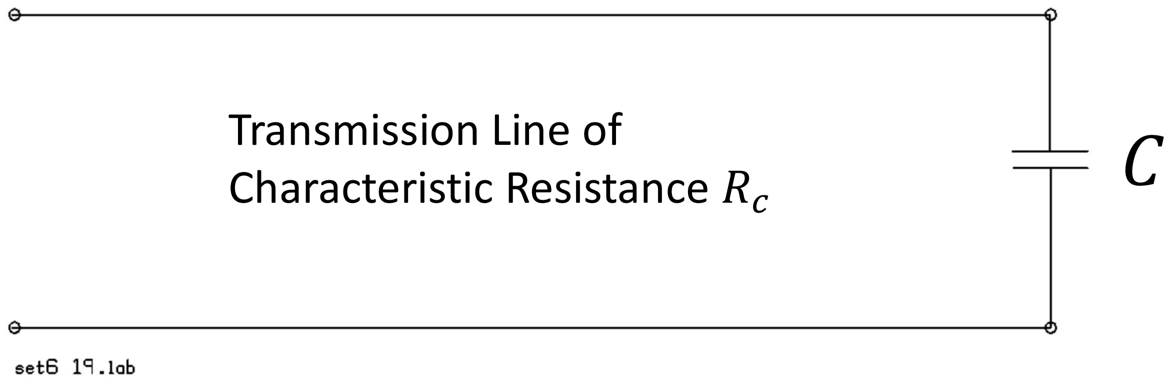
As $t \to \infty$, the capacitor becomes fully charged and the voltage becomes constant. Then $i = C\frac{dv}{dt} = 0$ and the capacitor behaves like an open circuit. A fully charged capacitor with i=0 looks like an open circuit.

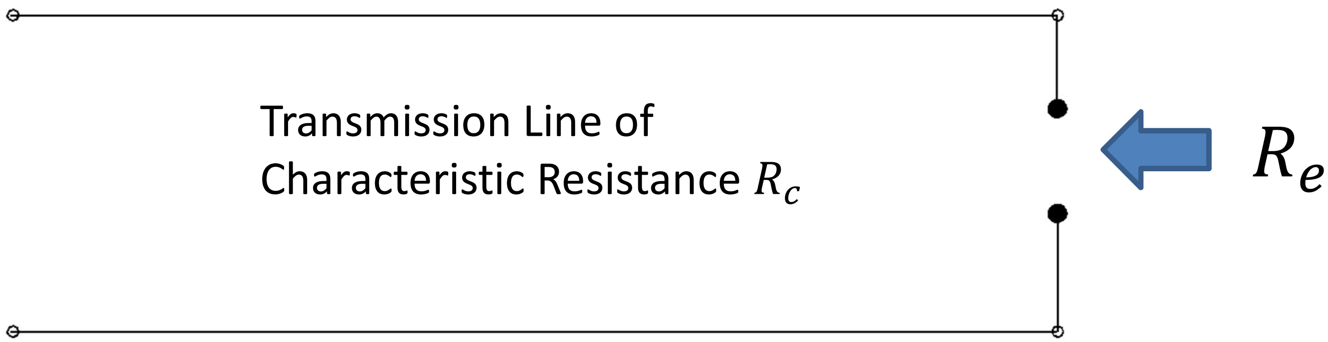
The reflection coefficient of an open circuit is $\Gamma_L = +1$.

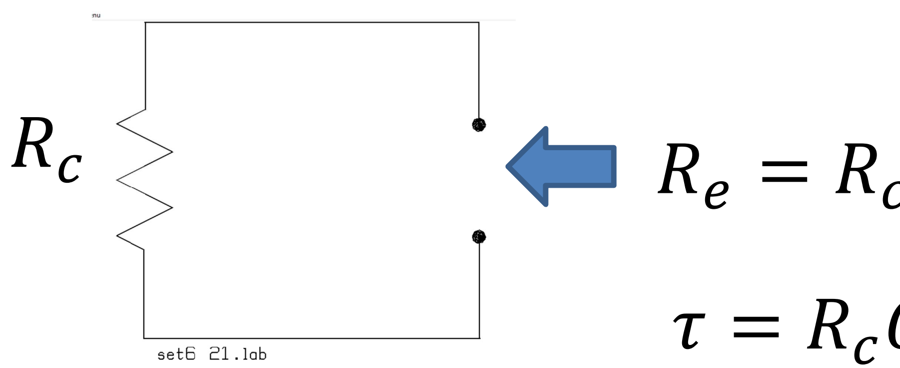
Hence we expect $V^- = \Gamma_L V^+ = +1x5 = +5$ volts.

$V_{final} = V^+ + V^- = 5 + 5 = 10$ volts

Find the time constant $\tau = R_e C$:







$$\tau = R_c C$$

$$v(t) = V_{final} + \left(V_{initial} - V_{final}\right)e^{-\frac{t-T_d}{\tau}}$$

The step arrives at $T_d = \frac{L}{u}$.

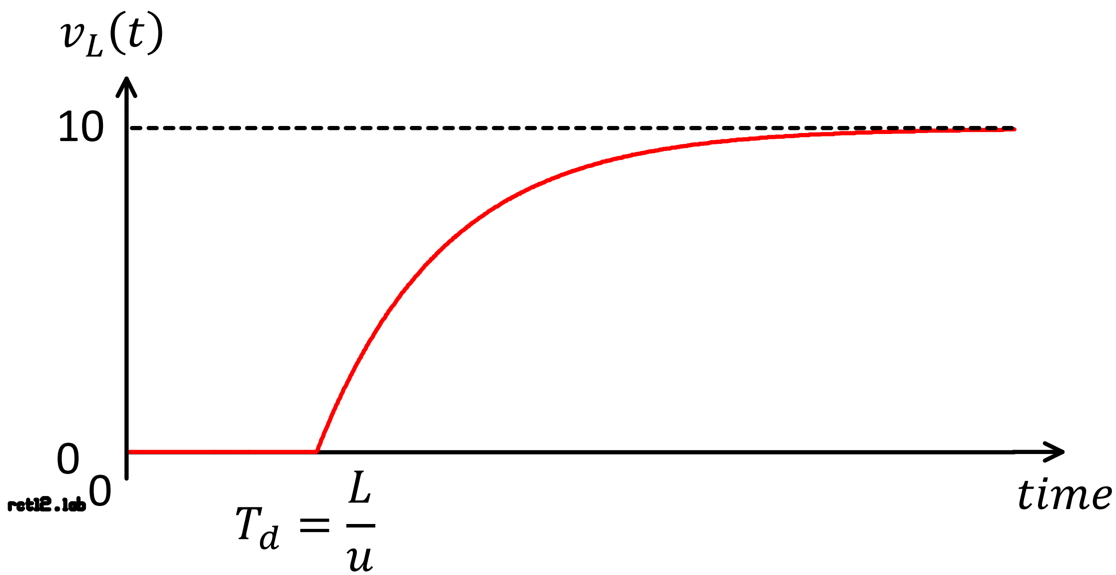
The time constant is $\tau = R_c C$

The initial value is $V_{initial} = 0$ volts.

The final value is $V_{initial} = 10$ volts.

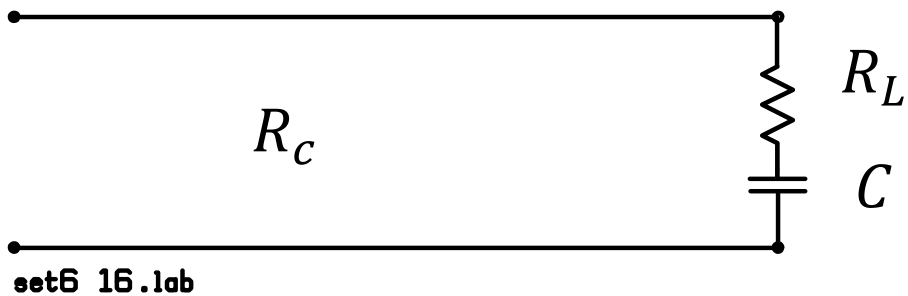
$$v(t) = 10 + (0 - 10)e^{-\frac{t-T_d}{\tau}}$$

$$v(t) = 10\left(1 - e^{-\frac{t-T_d}{\tau}}\right)$$

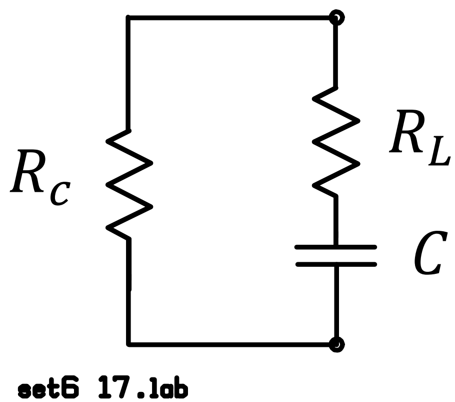


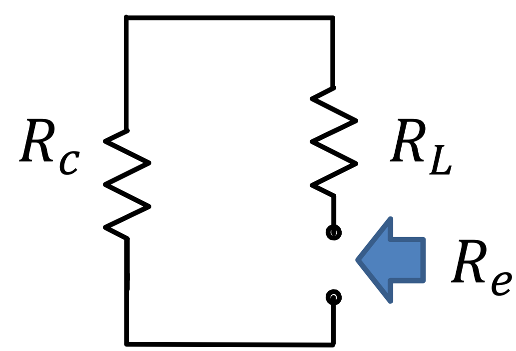
Finding the Time Constant

Find the time constant for a series RC load:



The transmission line behaves as a resistor of value equal to the characteristic resistance R_c:





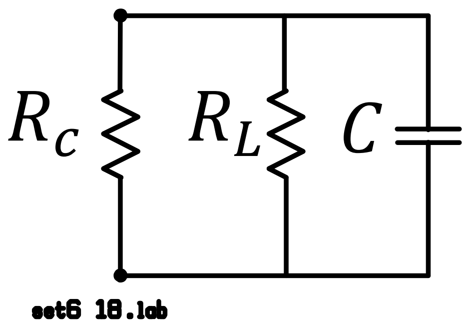
The resistance seen from the capacitor terminals is

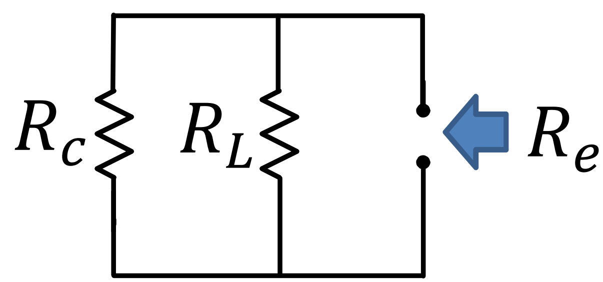
$$R_e = R_L + R_c$$

so the time constant is $\tau = R_e C = (R_L + R_c)C$

Find the time constant for a parallel RC load:

The transmission line behaves as a resistor of value equal to the characteristic resistance R_c:





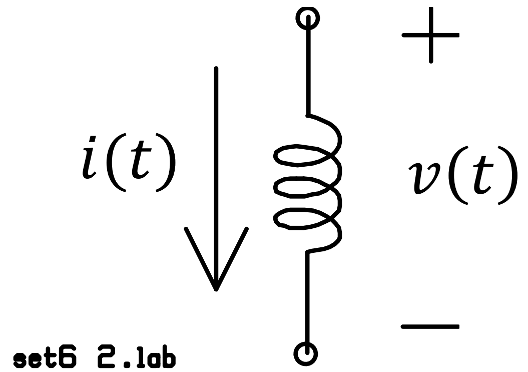
The resistance seen from the capacitor terminals is R_c in parallel with R_L is

$$R_e = \frac{R_c R_L}{R_c + R_L}$$

So the time constant is $\tau = R_e C = \frac{R_c R_L}{R_c + R_L} C$

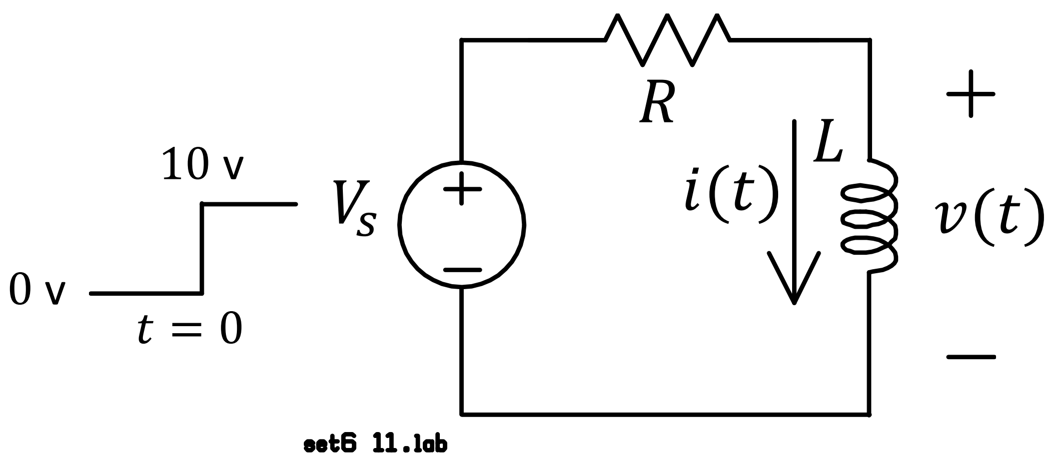
Inductive Load

1. Examine a "charging" RL circuit.
2. Look at a transmission line terminated with an inductance.



$$v = L\frac{di}{dt}$$

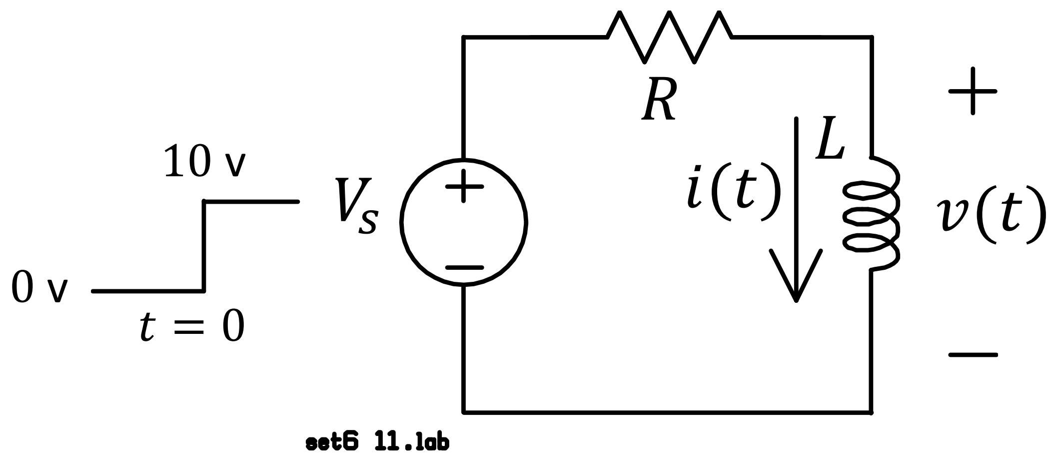
Charging RL Circuit



The initial current in the inductor is zero:
$i_L(0) = 0$

Find the voltage across the inductor $v_L(t)$ as a function of time.

Exponential Charging:

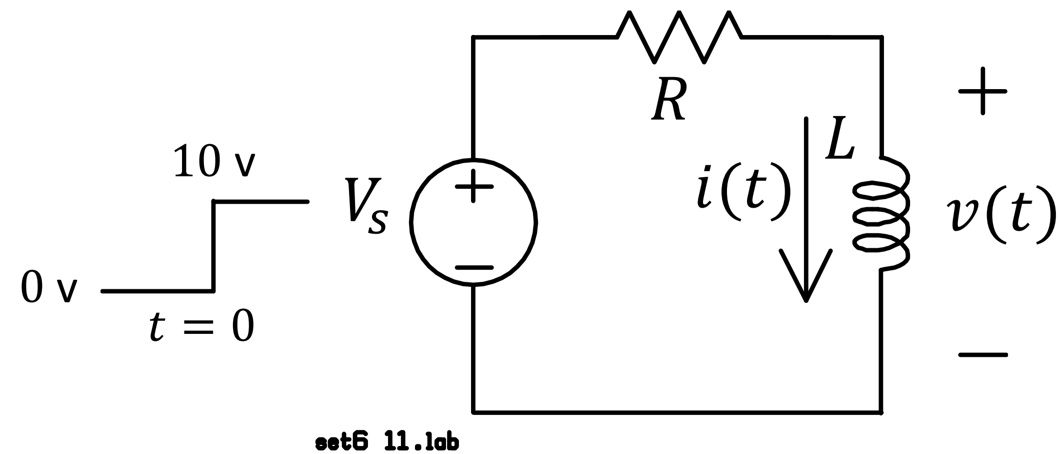


For $t > T_d$

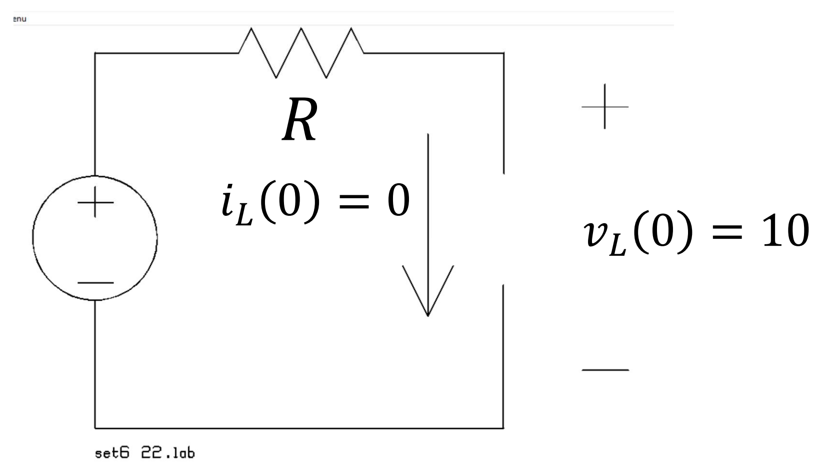
$$v(t) = V_{final} + \left(V_{initial} - V_{final}\right)e^{-\frac{t-T_d}{\tau}}$$

- When does the exponential charging start? At $t = 0$ when the generator turns on.
- What is the initial value $V_{initial}$?
- What is the final value V_{final}?
- What is the time constant?

What is the initial value of v_L?

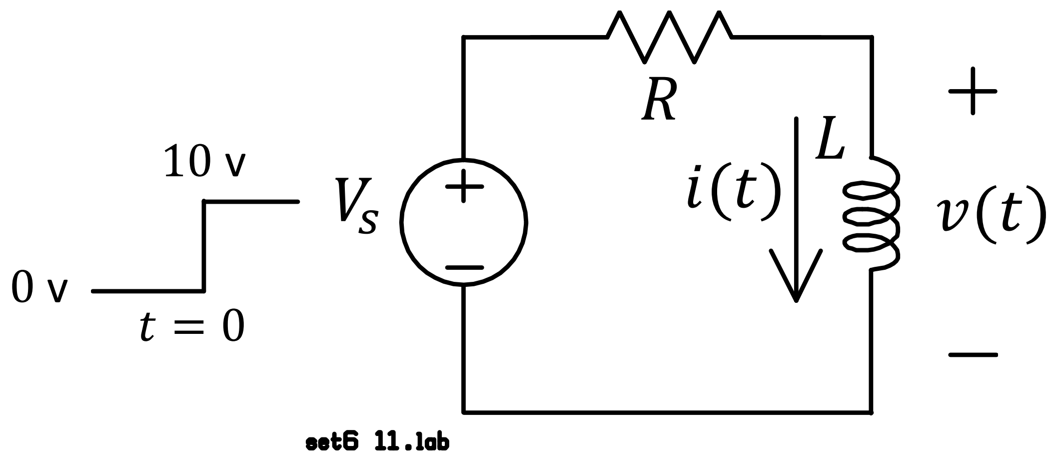


At t=0:

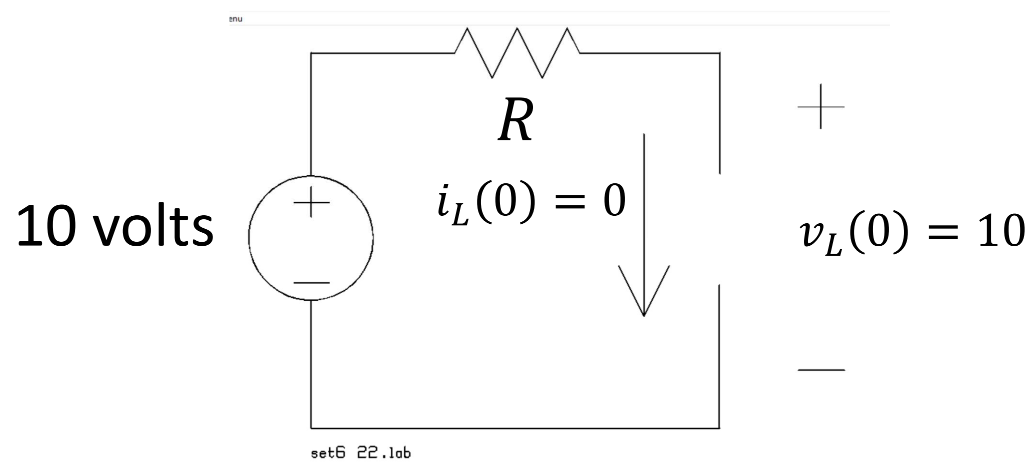


$$V_{initial} = 10 \text{ volts}$$

What is the initial value of v_L?

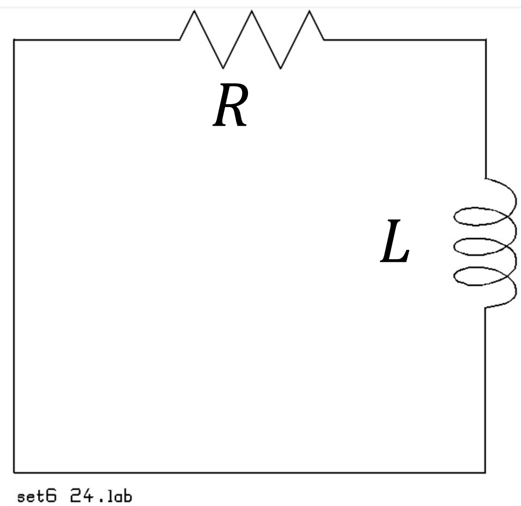


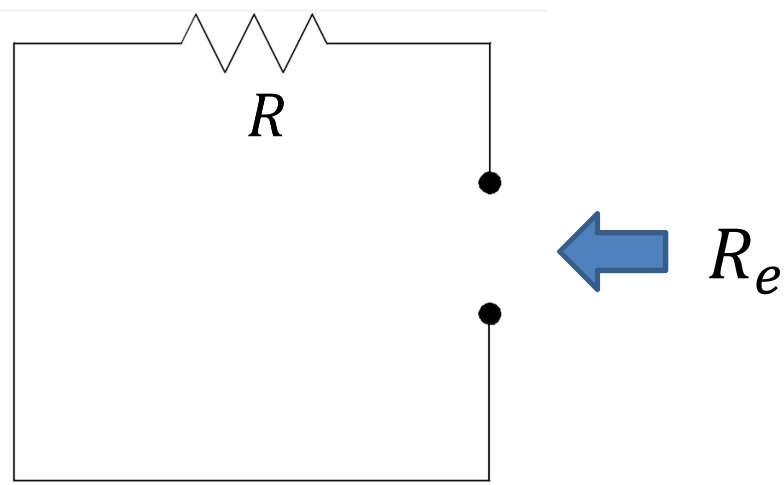
The initial current in the inductor is zero: $i_L(0) = 0$



$$V_{initial} = 10 \text{ volts}$$

Find the time constant $\tau = \frac{L}{R_e}$:





$$R_e = R$$

$$\tau = \frac{L}{R}$$

$$v(t) = V_{final} + \left(V_{initial} - V_{final}\right)e^{-\frac{t-T_d}{\tau}}$$

Delay: $T_d = 0$

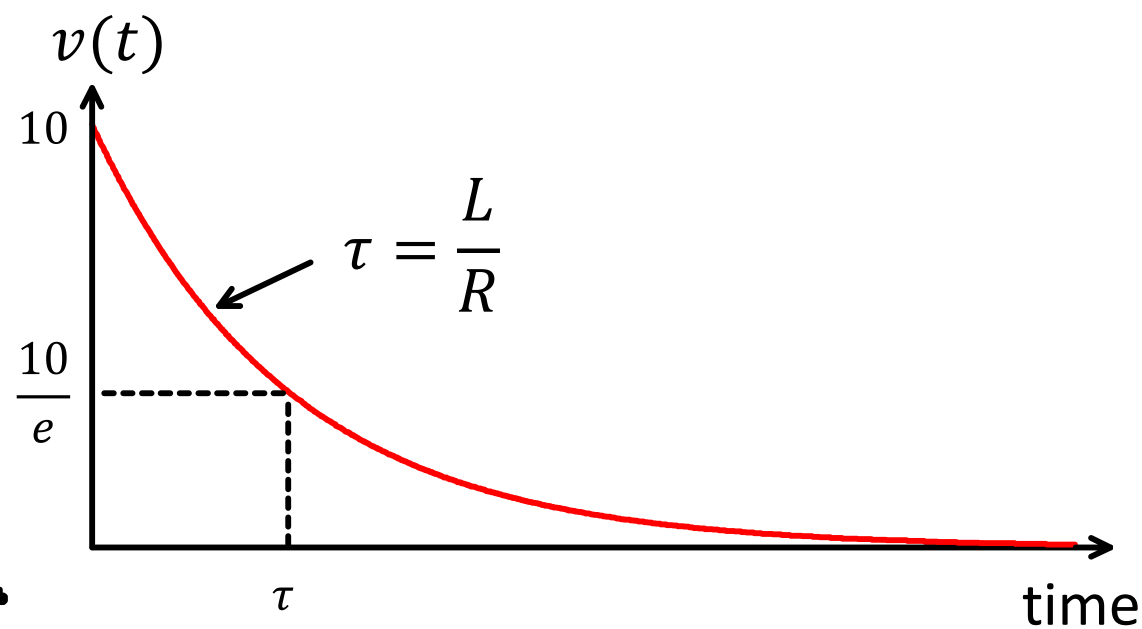
Time Constant: $\tau = \frac{L}{R}$

$$\tau = \frac{L}{R}$$

$V_{initial} = 10$

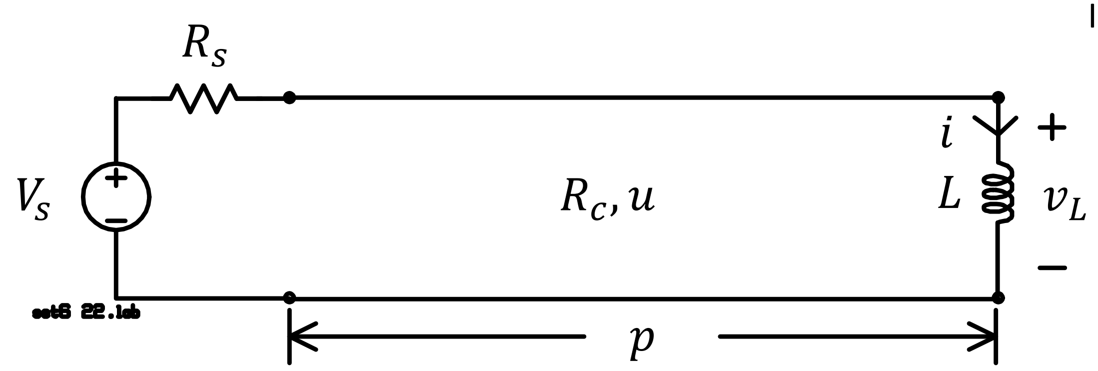
$V_{final} = 0$

$$v(t) = 10e^{-t/\tau}$$



Transmission Line Terminated with an Inductor

Inan, Inan and Said Section 2.5



The inductor is initially "uncharged", meaning that the current in the inductor is initially equal to zero.

Find the load voltage $v_L(t)$ as a function of time.

The generator is a step function starting at *t*=0, of height $V_s = 10$ volts.
The generator is "matched" with $R_s = R_c$.

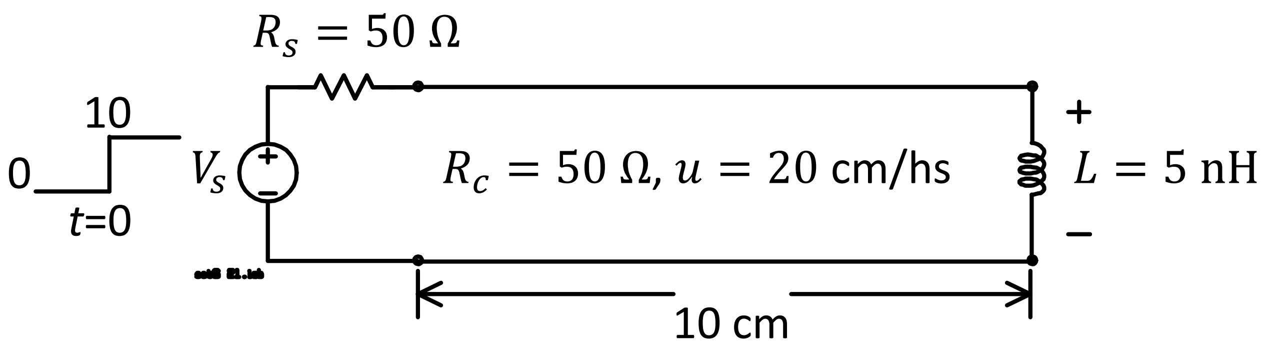
The initial step launched onto the transmission line has value

$$V_0 = V(0) = \frac{R_c}{R_s + R_c} V_s \text{ volts}$$

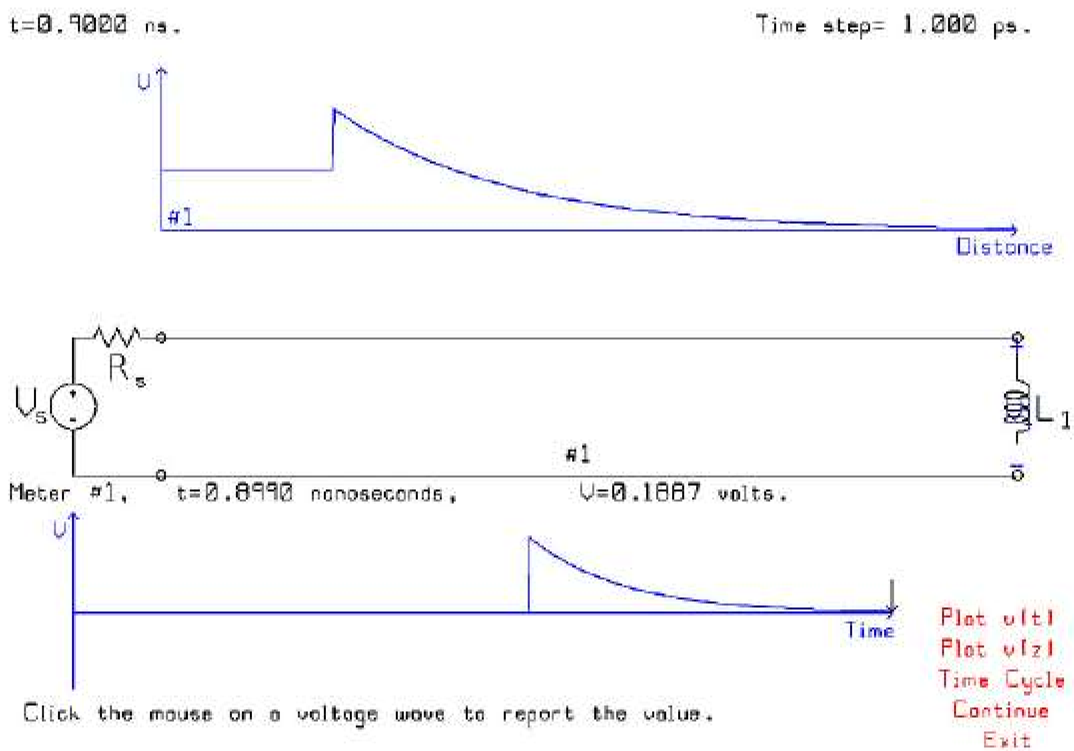
$$T_d = \frac{p}{u}$$

where p is the length of the transmission line.

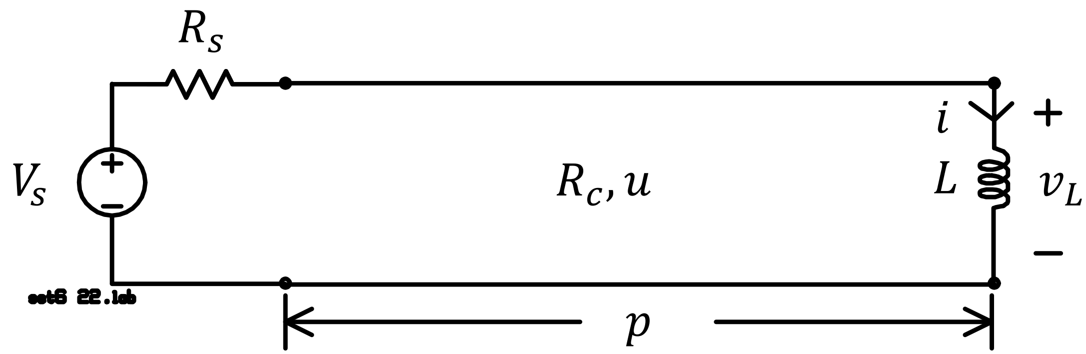
Inductive Load with BOUNCE



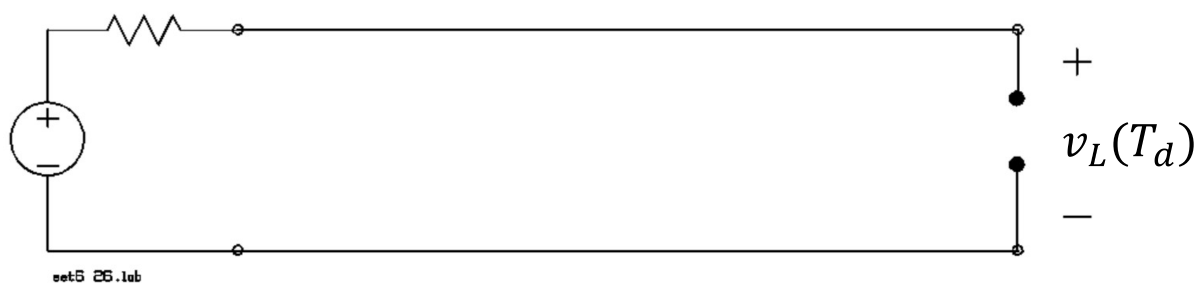
A source generates a 10 volt step function and has a 50 ohm internal resistance. Find the load voltage as a function of time.



Initial Value of V^- using the Reflection Coefficient



Initially, the inductor current is zero, so the inductor behaves like an open circuit.



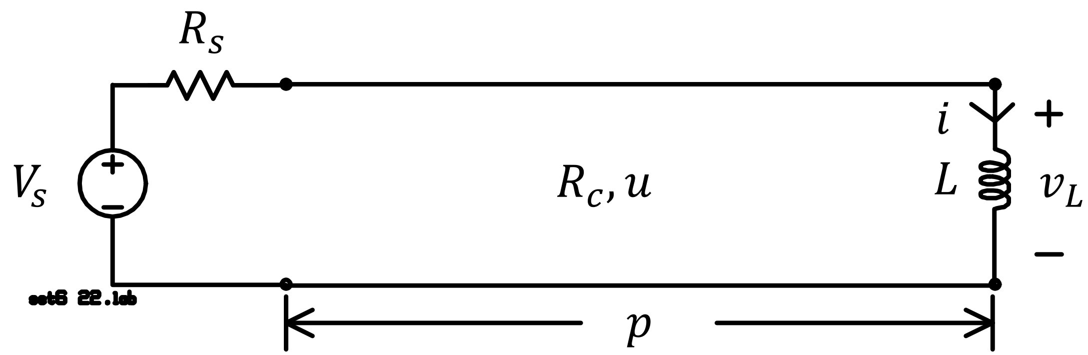
The initial reflection coefficient is $\Gamma_L = +1$.

When the $V^+ = V_0$ step from the generator arrives at $t = T_d$, the reflected voltage is

$V^- = \Gamma_L V^+ = +1xV_0 = V_0$.

$V_{initial} = V^+ + V^- = V_0 + V_0 = 2V_0$

Final Value of V^- as $t \to \infty$ using the Reflection Coefficient



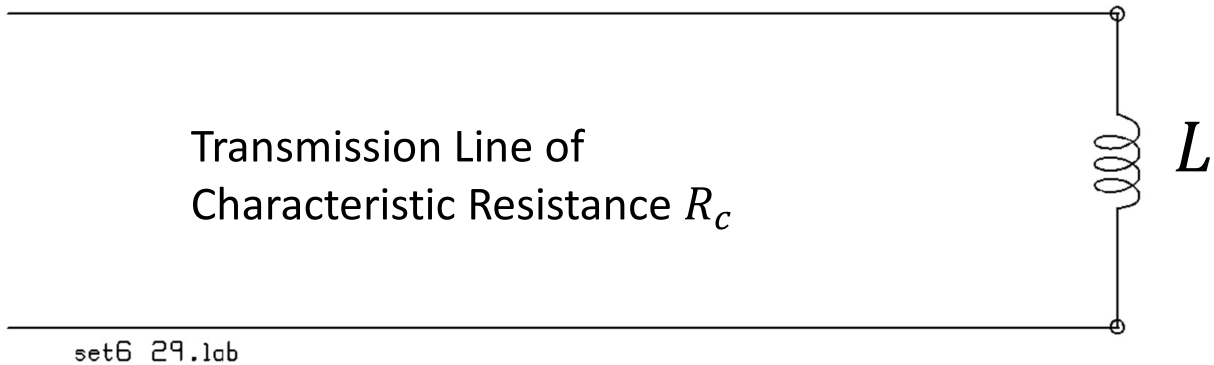
As $t \to \infty$, the current through the inductor becomes constant with time so $\frac{di}{dt}$=0, and so $v_L = L\frac{di}{dt} = 0$. The inductor behaves like a short circuit.

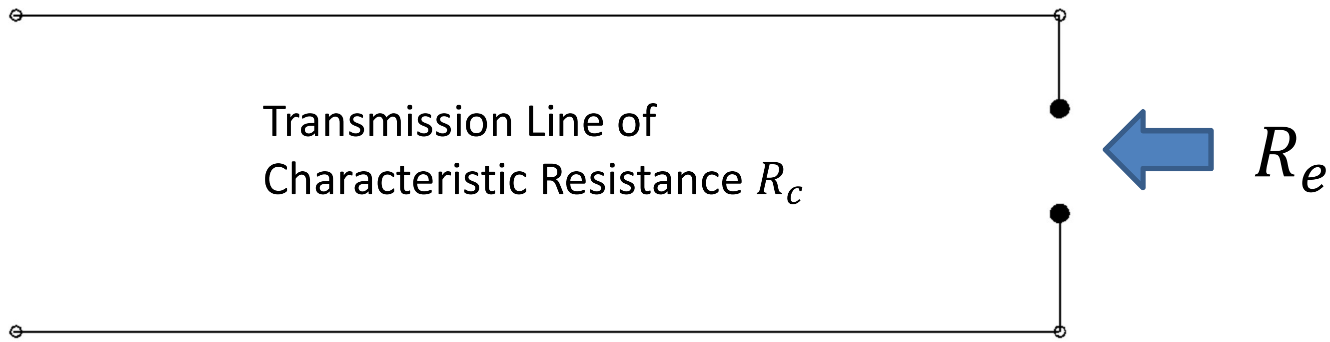
The final value of the reflection coefficient is $\Gamma_L = -1$.

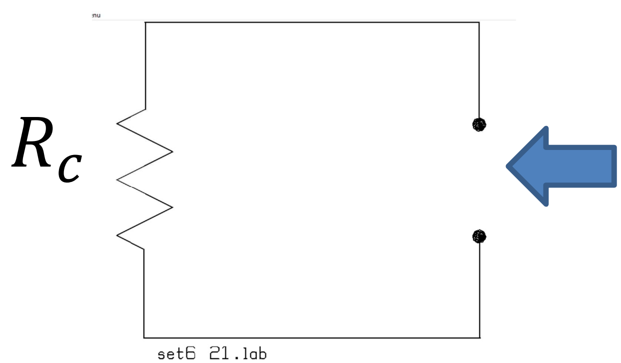
So as $t \to \infty$, the reflected voltage is $V^- = \Gamma_L V^+ = -1xV_0 = -V_0$.

$V_{final} = V^+ + V^- = V_0 - V_0 = 0$

Find the time constant $\tau = \frac{L}{R_e}$:







$$R_e = R_c$$

$$\tau = \frac{L}{R_c}$$

$v_L(t) = V_{final} + \left(V_{initial} - V_{final}\right)e^{-\frac{t-T_d}{\tau}}$

Delay: $T_d = \frac{p}{u}$ where p is the length of the transmission line.

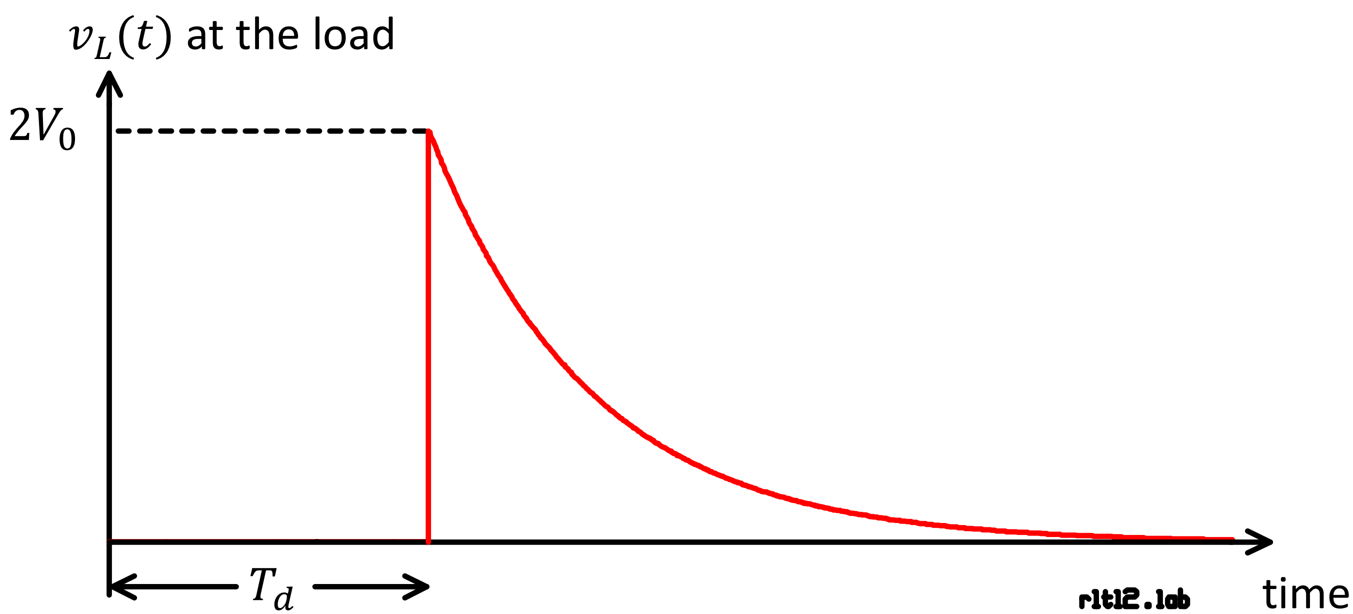
Time Constant: $\tau = \frac{L}{R}$

$V_{initial} = 2V_o$

$V_{final} = 0$

$v_L(t) = 0 + (2V_o - 0)e^{-\frac{t-T_d}{\tau}}$

$$v_L(t) = 2V_0 e^{-(t-T_d)/\tau}$$



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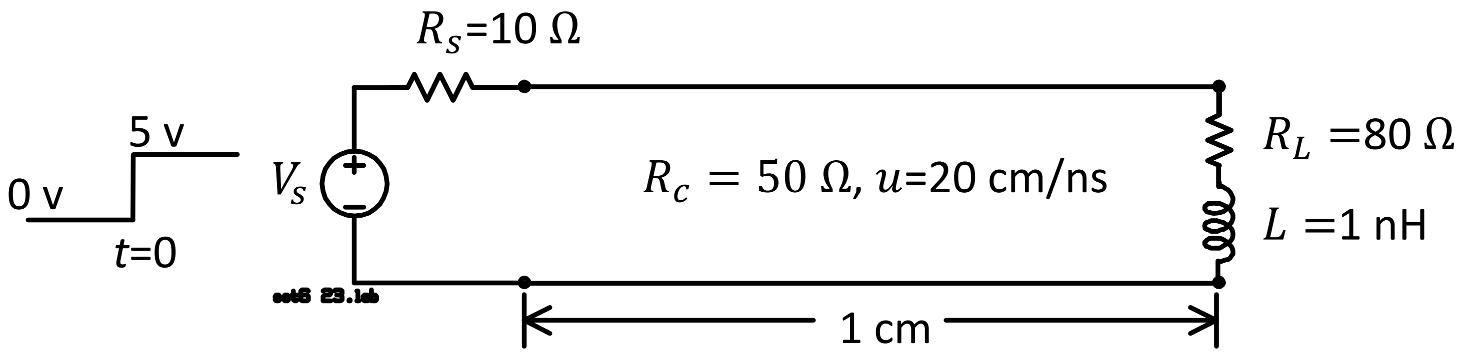
Another Example: RL load, unmatched source.



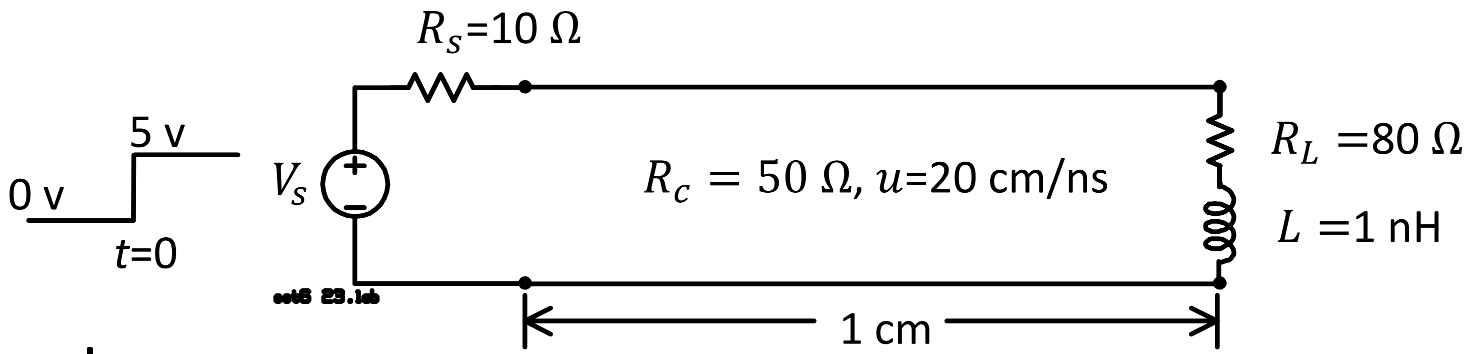
The generator steps up from 0 volts to 5 volts at t=0.
The inductance is initially "uncharged", meaning that the current is zero at t=0.
Find the voltage across the load as a function of time.

Solution:

Initially the generator launches a step of height

$$V(0) = \frac{R_c V_s}{R_c + R_s} = \frac{50x5}{50+10} = 4.167 \text{ volts}$$

This step arrives at the load after a delay time of $T_d = \frac{1}{20} = 0.05$ ns



Initial value:

The inductance is an open circuit so

$$\Gamma = +1$$

$$V^- = \Gamma V^+ = +1x4.167 = 4.167$$

$$V_{initial} = V^+ + V^- = 4.167 + 4.167 = 8.334$$

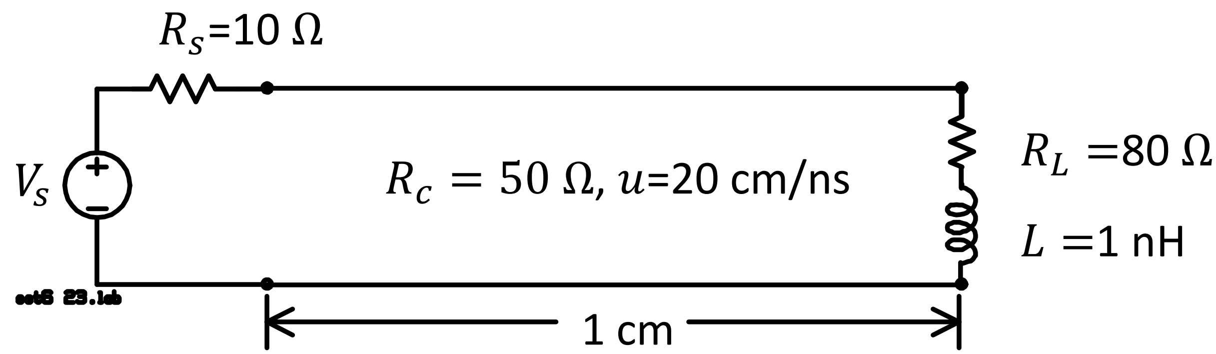
Final value:

The inductance is a short circuit so

$$\Gamma = \frac{R_L - R_c}{R_L + R_c} = \frac{80-50}{80+50} = \frac{30}{130} = 0.2308$$

$$V^- = \Gamma V^+ = 0.2308x4.167 = 0.9616$$

$$V_{final} = V^+ + V^- = 4.167 + 0.9616 = 5.129$$



Time Constant:

$$\tau = \frac{L}{R_{eq}} \quad \text{where} \quad R_{eq} = R_L + R_c = 80 + 50 = 130 \quad \text{ohms}$$

$$\tau = \frac{L}{R_{eq}} = \frac{1}{130} = 0.00769 \quad \text{ns}$$

Complete solution:

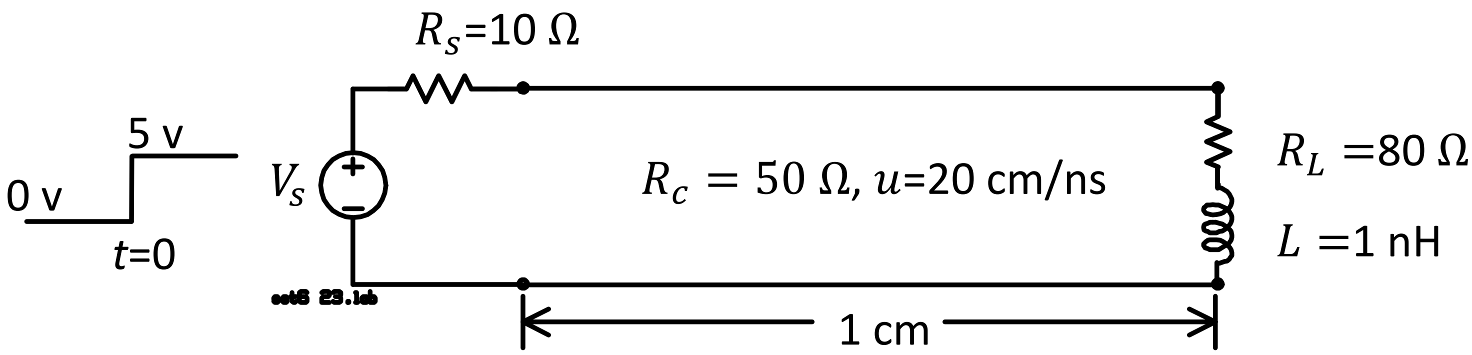
$$v(t) = V_{final} + \left(V_{initial} - V_{final}\right) e^{-(t-T_d)/\tau}$$

$V_{initial} = 8.334$ $\quad V_{final} = 5.129$ $\quad T_d = 0.05$ $\quad \tau = 0.00769$

$$v(t) = 5.129 + (8.334 - 5.129) e^{-(t-0.05)/0.00769}$$

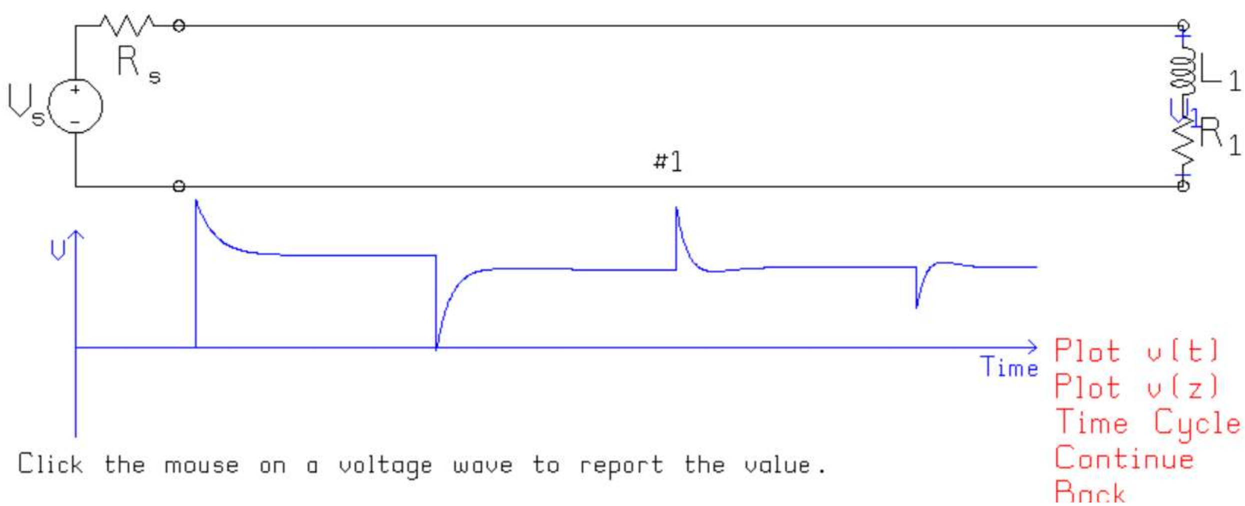
$$v(t) = 5.129 + 3.205 e^{-(t-0.05)/0.00769}$$

Unmatched Source

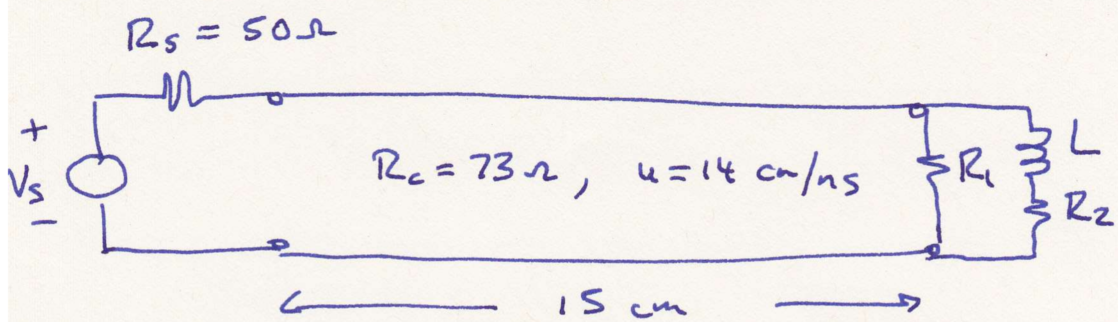


In this problem, the source is not matched. So the exponential waveform that is reflected from the load will be re-reflected from the source.

We can simulate this with BOUNCE.



Another Example



$R_1 = 150\Omega$

$R_2 = 10\Omega$

$L = 5$ nH

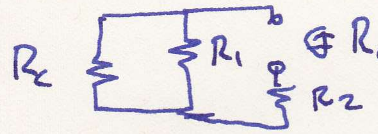
$V_S = 0$ for $t < 0$

$V_S = 10$ for $t > 0$

The generator is a step function of height 10 volts at t=0.
The current in the inductor is zero at t=0.
Find the voltage across the load as a function of time.

DELAY $T = \frac{L}{u} = \frac{15}{14} = 1.071$ ns

Time Constant



$R_e = R_2 + R_1 \| R_c$

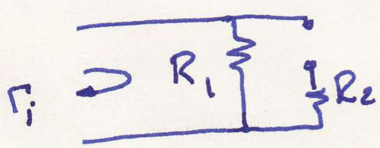
$= 10 + \frac{150 \times 73}{150 + 73}$

$=59.103$

$$\tau = \frac{L}{R_e} = \frac{5}{59.103} = 0.0846 \text{ ns}$$

Initial Step $V_o = \frac{R_c V_s}{R_s + R_c} = \frac{73 \times 10}{73 + 50} = 5.935$ volts

At $t = T$, L is an open circuit



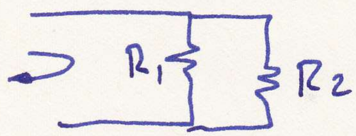
$\Gamma_i = \frac{R_1 - R_c}{R_1 + R_c} = \frac{150 - 73}{150 + 73} = 0.345$

$V_L = V^+ + V^- = V_o + \Gamma_i V_o = 5.935(1 + 0.345)$

$V_{initial} = V_L = 7.984$

After a "long time", L is a short circuit

Γ_f



$$R_L = R_1 \| R_2 = \frac{10 \times 150}{10 + 150} = 9.375\ \Omega$$

$$\Gamma_f = \frac{R_L - R_c}{R_L + R_c} = \frac{9.375 - 73}{9.375 + 73} = -0.7724$$

$$V_L = V^+ + V^- = 5.935\,(1 + \Gamma_f)$$

$$= 5.935\,(1 - 0.7724)$$

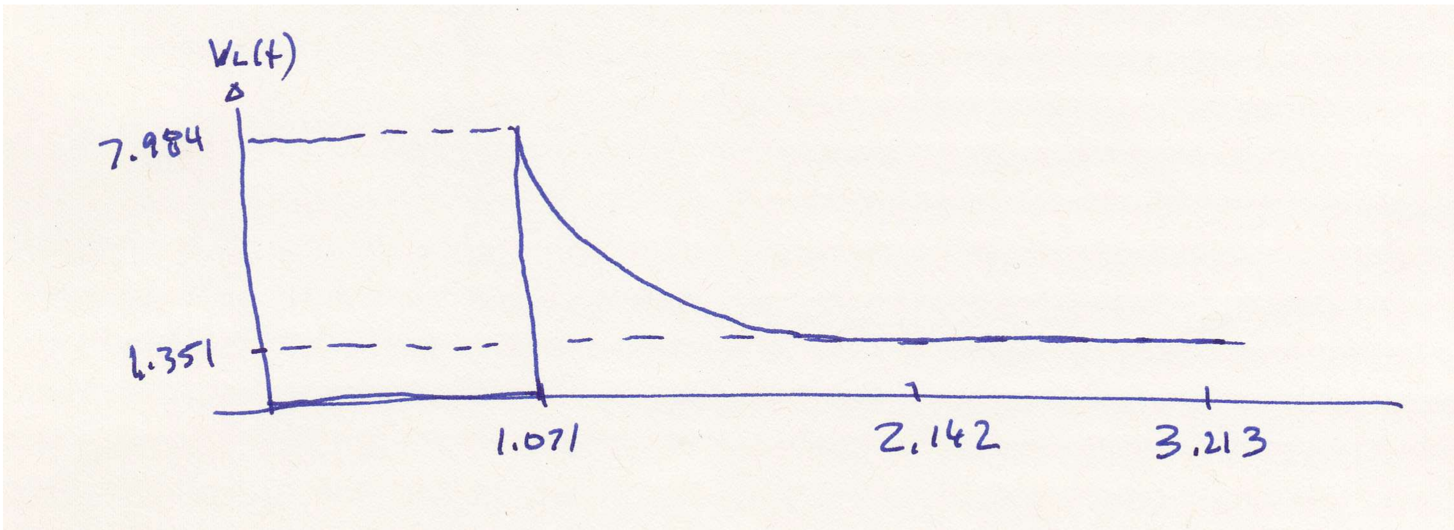
$$= 1.3509$$

$V_{final} = 1.3509$ volts

$$V_L(t) = V_{final} + (V_{initial} - V_{final})\, e^{-\frac{(t-T)}{\tau}}$$

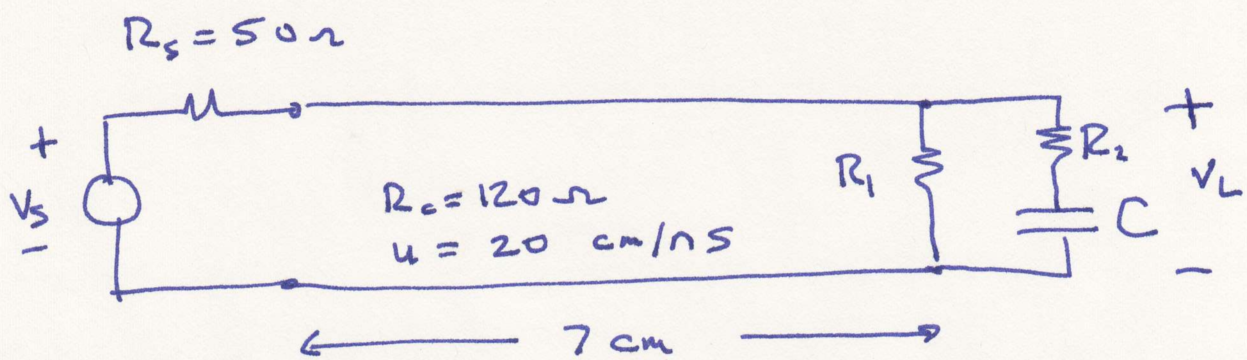
$$v_L(t) = 1.351 + (7.984 - 1.351)e^{-\frac{t-1.071}{0.0846}}$$

$$v_L(t) = 1.351 + (7.984 - 1.351)e^{-\frac{t-1.071}{0.0846}}$$



Note that $R_s = 50$ and $R_c = 73$, so the source is not matched and there will be further reflections back and forth before the circuit “settles”.

And Another Example



$R_1 = 15\,\Omega$

$R_2 = 25\,\Omega$

$C = 15$ pF

$V_s = 0 \quad t < 0$

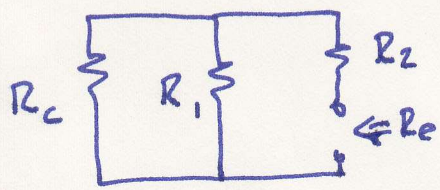
$V_s = 10 \quad t > 0$

The capacitor is uncharged at $t = 0$.
The generator steps up from 0 volts to 10 volts at $t = 0$.

Find the voltage across the load as a function of time.

Time Delay $T = \frac{L}{u} = \frac{7}{20} = 0.35 \text{ ns}$

Time Constant



$$R_e = R_2 + R_1 \| R_c$$

$$= 25 + \frac{15 \times 120}{15 + 120}$$

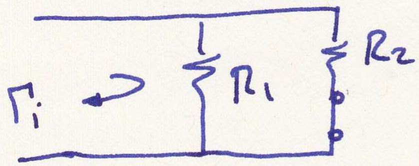
$$= 25 + 13.33$$

$$= 38.33$$

$$\tau = R_e C = 38.33 \times 15 = 0.575 \text{ ns}$$

Initial Step $V_0 = \frac{R_c V_s}{R_c + R_s} = \frac{120 \times 10}{120 + 50} = 7.059$ volts

At $t = T$, C is a short circuit

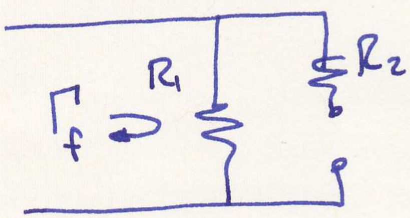


$$R_e = R_1 \| R_2 = \frac{15 \times 25}{15+25} = 9.375\ \Omega.$$

$$\Gamma_i = \frac{R_e - R_c}{R_e + R_c} = \frac{9.375 - 120}{9.375 + 120} = -0.8526$$

$$V_{initial} = V_0(1+\Gamma_i) = 7.059\,(1 \cancel{+} - 0.8526) = \cancel{13.08}\ 1.0405 \text{ volts}$$

After a long time, C is an open circuit



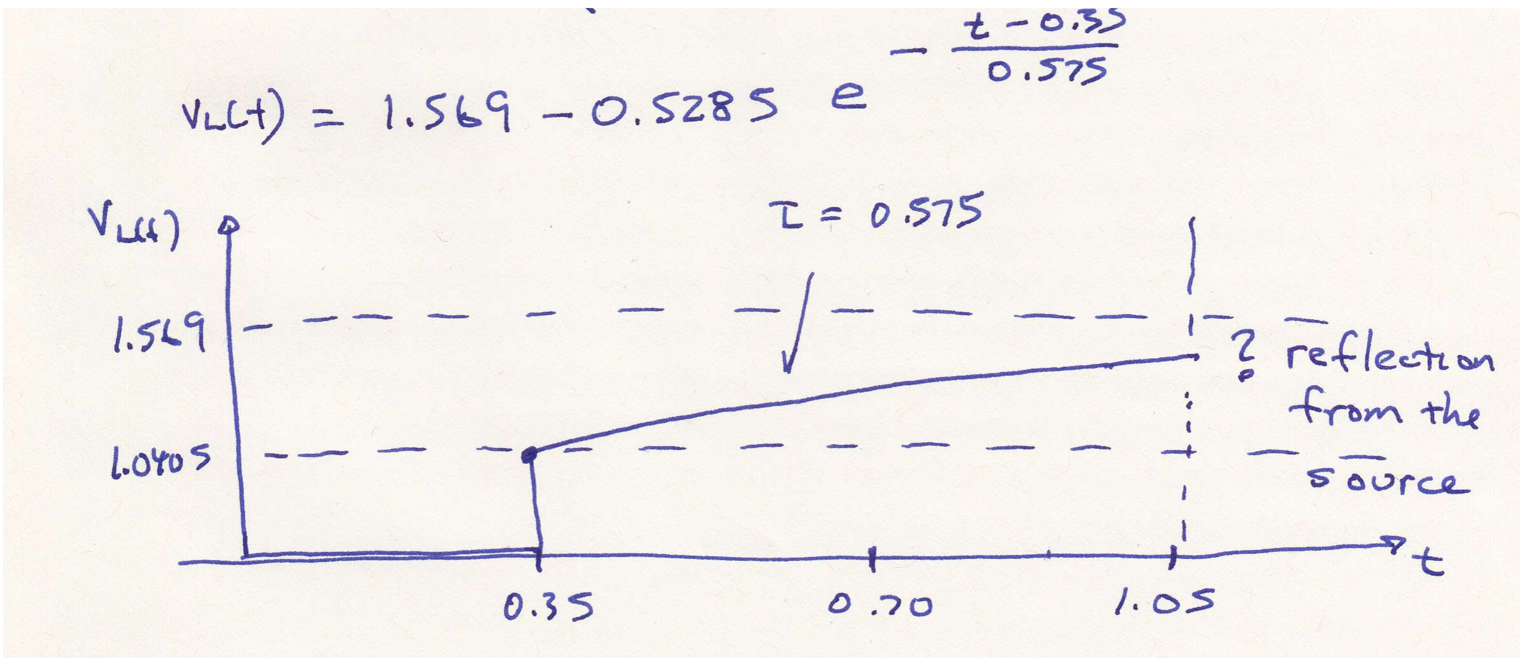
$$\Gamma_f = \frac{R_1 - R_c}{R_1 + R_c} = \frac{15 - 120}{15 + 120} = -0.7778 \ \cancel{\text{volts}}$$

$$V_{final} = V_0(1+\Gamma_f) = 7.059\,(1 - 0.7778) = 1.569 \text{ volts}$$

$$V_L(t) = V_{final} + (V_{initial} - V_{final})\, e^{-\frac{t-T}{\tau}}$$

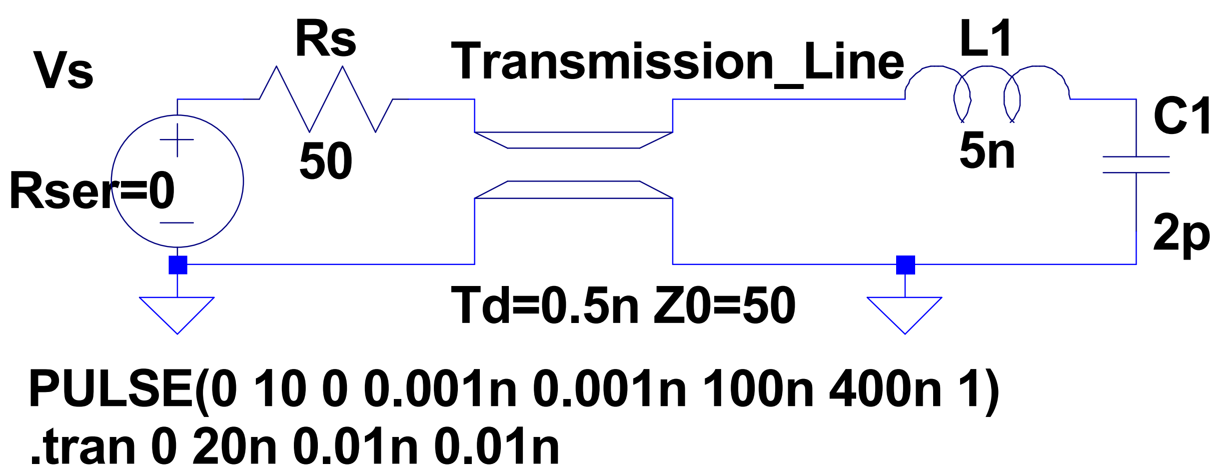
$$= 1.569 + (1.0405 - 1.569)\, e^{-\frac{t-0.35}{0.575}}$$

$$= 1.569 - 0.5285\, e^{-\frac{t-0.35}{0.575}}$$



The source is not matched ($R_s = 50$ and $R_c = 120$) and there will be further reflections back and forth to reach the final value as $t \to \infty$.

Pin Inductance in series with Gate Capacitance



LTSpice Simulation:
Here we have a pin inductance of 5 nH in series with a gate capacitance of 2 pF.

What is the voltage across the load?

Computed with LTSpice…