6.3.4

$\dot{x} = y + x - x^3$

$\dot{y} = -y$

Steady states $\dot{y} = 0 \Rightarrow y = 0$

$\dot{x} = 0 \Rightarrow 0 = y + x - x^3 \Rightarrow 0 = x - x^3$

$0 = x(1 - x^2)$

$\Rightarrow x = -1, 0, 1.$

So three steady states $(x, y) = (-1, 0), (0, 0) \& (1, 0)$.

3/10

Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial}{\partial x}\dot{x} & \frac{\partial}{\partial y}\dot{x} \\ \frac{\partial}{\partial x}\dot{y} & \frac{\partial}{\partial y}\dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(y + x - x^3) & \frac{\partial}{\partial y}(y + x - x^3) \\ \frac{\partial}{\partial x}(-y) & \frac{\partial}{\partial y}(-y) \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{pmatrix}$$

So $J(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $\Rightarrow tr(J) = 0$

$det(J) = -1 - 0 = -1$

$det(J) < 0$ so this is a saddle point.

2/10

Deal with $(\pm 1, 0)$ together:

$$J(\pm 1, 0) = \begin{pmatrix} 1 - 3 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$$

$tr(J) = -3$

$det(J) = 2$

$$\lambda^2 - tr(J)\lambda + det(J) = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda+2)(\lambda+1) = 0$$

$$\Rightarrow \lambda = -2 \ \& \ \lambda = -1.$$

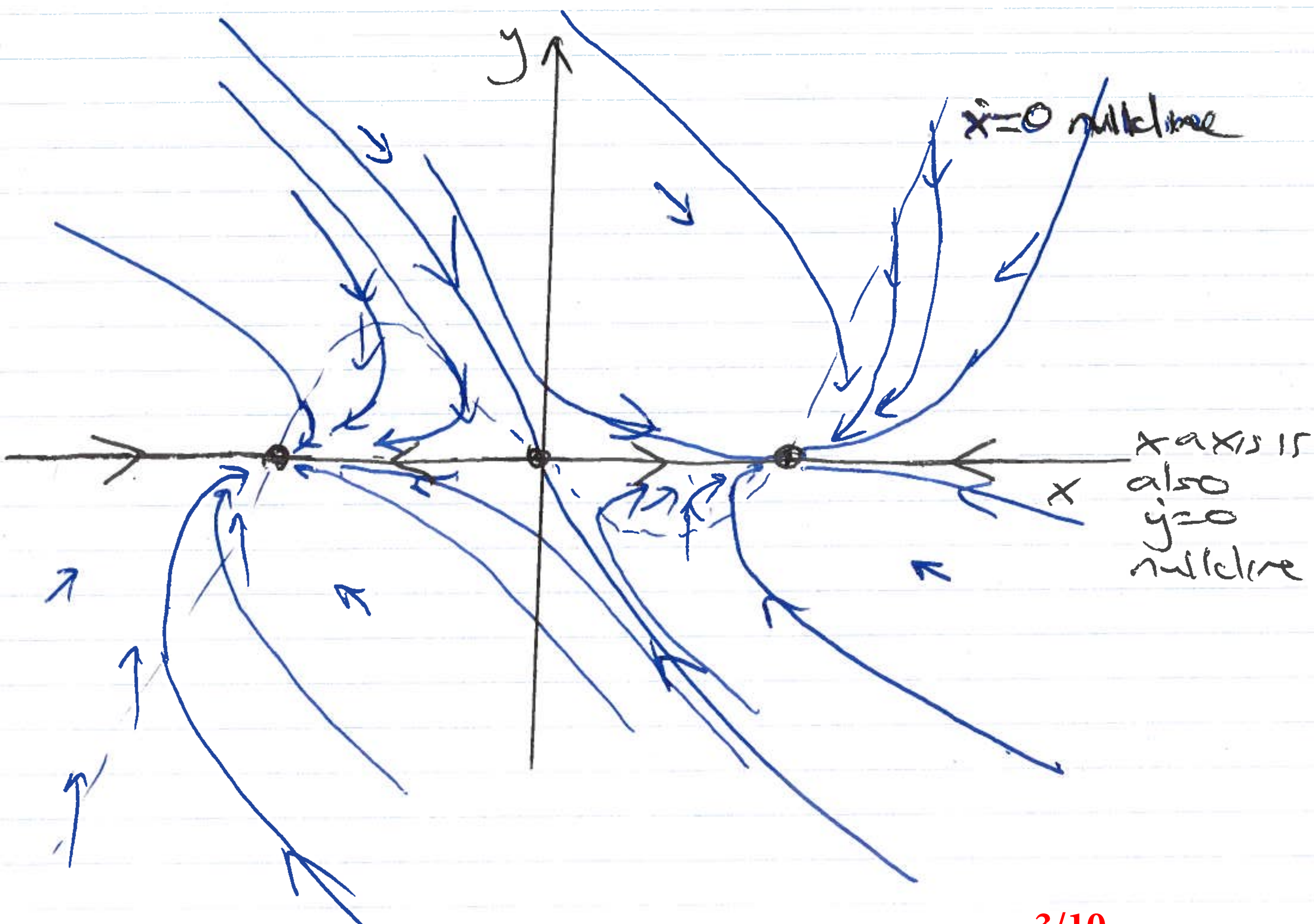
There are both stable nodes.

2/10

To plot dynamics lets also find nullclines

$\dot{y} = 0 \Rightarrow y = 0 \Rightarrow$ This implies x-axis is invariant. If $y(0) = 0$ $y(t) = 0$ $\forall t$.

$\dot{x} = 0 \Rightarrow y = -x + x^3$



3/10

6.3.10

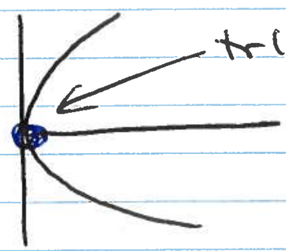
$\dot{x} = xy$

$\dot{y} = x^2 - y$

$(x,y) = (0,0)$ is a steady state

$$J = \begin{pmatrix} y & x \\ 2x & -y \end{pmatrix} \Rightarrow J(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So a double zero eigenvalue $\lambda_1 = \lambda_2 = 0$



$tr(J) = det(J) = 0.$

linear system $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

would have (x,y) a steady state for all $(x,y) \in \mathbb{R}^2$!

3/10

b) Lets solve for all steady states.

$\dot{y} = 0 \Rightarrow y = x^2$

then $\dot{x} = 0 \Rightarrow 0 = xy = x(x^2) = x^3$

So need $x = 0$ & $y = x^2 = 0$ & $(x,y) = (0,0)$ is only steady state.

3/10

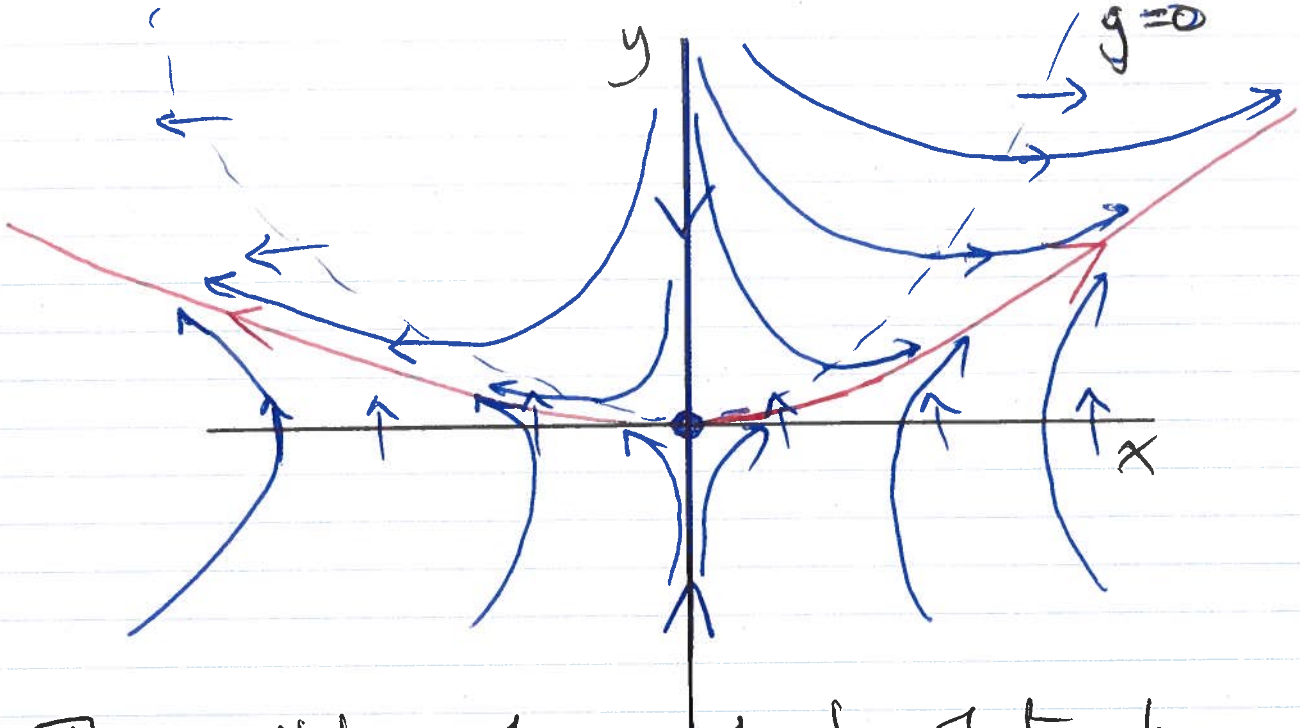
c) Notice if $x = 0$ then $\dot{x} = 0$ so y-axis is invariant.

On y-axis $\dot{y} = -y$ so $(0,0)$ stable in this direction

So steady state is either stable or a saddle.

Nullclines $\dot{x} = 0 \Rightarrow x = 0$ or $y = 0$

$\dot{y} = 0 \Rightarrow y = x^2.$



The nullclines show steady state has saddle stability. Unstable manifold is shown in red

4/10

6.4-2

$$\dot{x} = x(3-2x-y) \qquad x,y \geq 0$$
$$\dot{y} = y(2-x-y)$$

Fixed points $\dot{x} = \dot{y} = 0 \Rightarrow \begin{cases} 0 = x(3-2x-y) \\ 0 = y(2-x-y) \end{cases}$

$\Rightarrow \begin{cases} x = 0 \\ \& \ 0 = y(2-x-y) \end{cases}$ or $\begin{cases} 3-2x-y=0 \\ \& \ 0 = y(2-x-y \end{cases}$

Hence 4 fixed points $\underline{(0,0), (0,2), (3/2, 0)}$

or $0 = 3-2x-y = 2-x-y \Rightarrow (x,y) = \underline{(1,1)}$. **3/10**

Jacobian $J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{pmatrix}$

<u>At (0,0)</u>

$J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = 3, \lambda_2 = 2 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Fixed Point is Unstable node (v_2 is dominant direction $0 < \underline{\lambda_2} < \lambda_1$)

<u>At (0,2)</u> $J = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = -2$ (triangular matrix)

$\Rightarrow (0,2)$ is a saddle point.

<u>At (3/2, 0)</u> $J = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix} \Rightarrow \lambda_1 = -3, \lambda_2 = 1/2$

$\Rightarrow (3/2, 0)$ is also a saddle point

At (1,1) $J = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$ $\det(J) = 1, \ tr(J) = -3$
$tr(J)^2 - 4\det(J) = 5 > 0$

Thus (1,1) is a stable node **4/10**

For eigenvalues **(Okay if you skipped following steps...**

$$\lambda^2 - tr(J)\lambda + det(J) = 0 \Rightarrow \lambda^2 + 3\lambda + 1 = 0$$

$$\Rightarrow \lambda_\pm = \frac{-3 \pm \sqrt{9-4}}{2} = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$$

~~Let~~ Note $\lambda_- < -\frac{3}{2} < \lambda_+ < 0$.

For eigenvectors $Av = \lambda v \Rightarrow (A - \lambda I)v = 0$

$$\Rightarrow \begin{pmatrix} -2-\lambda & -1 \\ -1 & -1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

First equation is

$$-(2+\lambda)x - y = 0 \Rightarrow y = -(2+\lambda)x$$

Let $x = 1 \Rightarrow y = -(2+\lambda)$

Thus $\lambda_- = -\frac{3}{2} - \frac{1}{2}\sqrt{5} \Rightarrow V_- = \begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{1}{2}\sqrt{5} \end{pmatrix}$

$\lambda_+ = -\frac{3}{2} + \frac{1}{2}\sqrt{5} \Rightarrow V_+ = \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{1}{2}\sqrt{5} \end{pmatrix}$

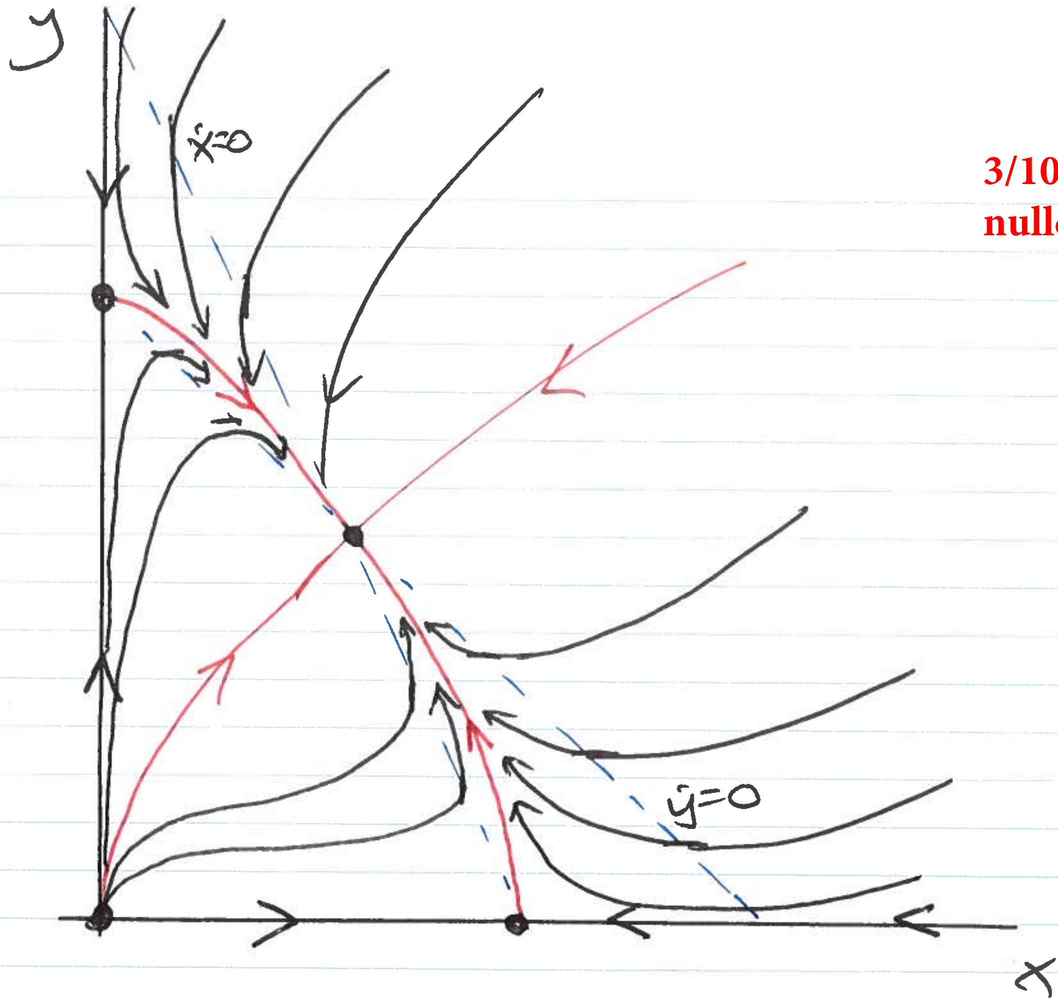
...up to here)

Nullclines $\dot{x} = 0 \Rightarrow x = 0$ (so y axis invariant)

or $y = 3 - 2x$

$\dot{y} = 0 \Rightarrow y = 0$ (so x axis invariant)

or $y = 2 - x$



3/10 (must include nullclines)

$\dot{x}=0$ on $y=3-2x$

on this curve

$$\dot{y} = y(2-x-y) = (3-2x)(2-x-3+2x)$$
$$= (3-2x)(x-1)$$
$$<0 \quad x\in(0,1)$$
$$>0 \quad x\in(1,3/2)$$

$\dot{y}=0$ on $y=2-x$

on this curve

$$\dot{x} = x(3-2x-y) = x(3-2x-2+x)$$
$$= x(1-x)$$
$$>0 \quad x\in(0,1)$$
$$<0 \quad x>1.$$

6.5-6 $\dot{x} = -kxy$ $\qquad x, y \geq 0$

$\dot{y} = kxy - ly$ $\qquad k, l > 0$

a) Fixed Points $\dot{x} = \dot{y} = 0$

Clearly $(x, 0)$ is fixed point $\forall x \geq 0$.

If $y \neq 0$ then $\dot{x} = 0 \Rightarrow x = 0$, then $\dot{y} = 0 \Rightarrow y = 0$, so no other fixed points.

The entire positive x-axis is a line of fixed points.

Jacobian $J = \begin{pmatrix} -ky & -kx \\ ky & kx - l \end{pmatrix}$

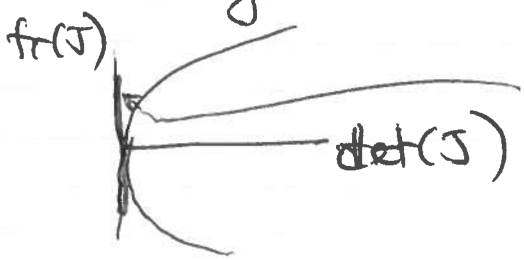
$$J(x, 0) = \begin{pmatrix} 0 & -kx \\ 0 & kx - l \end{pmatrix}$$

$\Rightarrow$ Eigenvalues are $\lambda_1 = 0$ & $\lambda_2 = kx - l$

So $\lambda_2 < 0$ if $x < \frac{k}{l}$ & $\lambda_2 > 0$ if $x > \frac{k}{l}$

Clearly $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Have a line of nonisolated fixed points with one zero eigenvalue & other eigenvalue of either sign, so fixed points not hyperbolic; we are on boundary case between saddle points & all other cases [I did not state a name for this type of fixed point]



3/10

b) Nullclines $\dot{x}=0 \Rightarrow kxy=0 \Rightarrow x=0$ or $y=0$

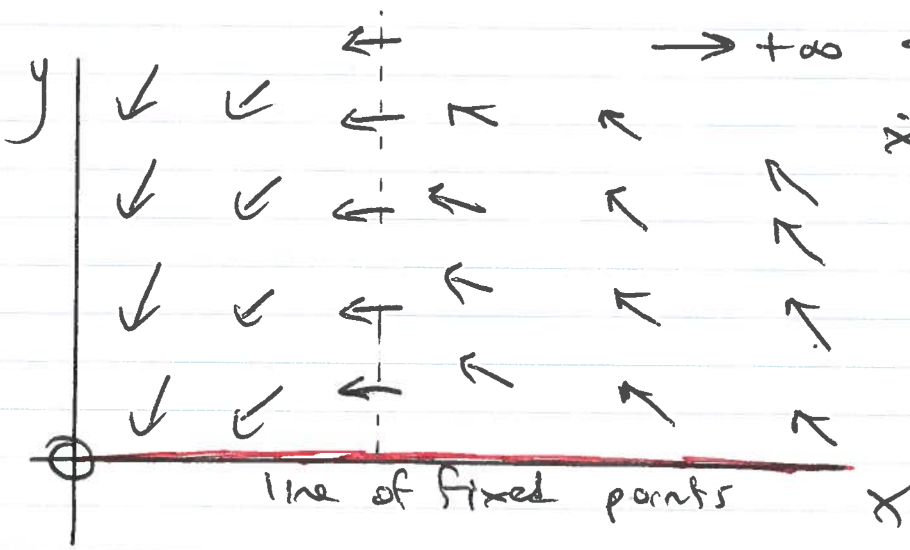
$\dot{y}=0 \Rightarrow (kx-l)y=0 \Rightarrow y=0$ or $x=k/l$

$$\frac{dy}{dx} = \frac{(kx-l)y}{-kxy} = -\frac{(kx-l)}{kx} \quad \text{function of } x \text{ only}$$

$\rightarrow -1$ as $x \rightarrow +\infty$

$\rightarrow +\infty$ as $x \rightarrow 0$

$\dot{x}<0 \ \forall x>0, y>0$



2/10

c) $$\frac{dy}{dx} = -\frac{(kx-l)}{kx} = -1 + \frac{l}{kx}$$

$$\Rightarrow \int dy = \int -1 + \frac{l}{kx}\, dx$$

$$\Rightarrow y = -x + \frac{l}{k}\ln x + c \qquad (x>0)$$

$\Rightarrow c = x + y - \frac{l}{k}\ln x$ is conserved quantity.

2/10

d) Solutions follow vector field from (b) on curves defined by

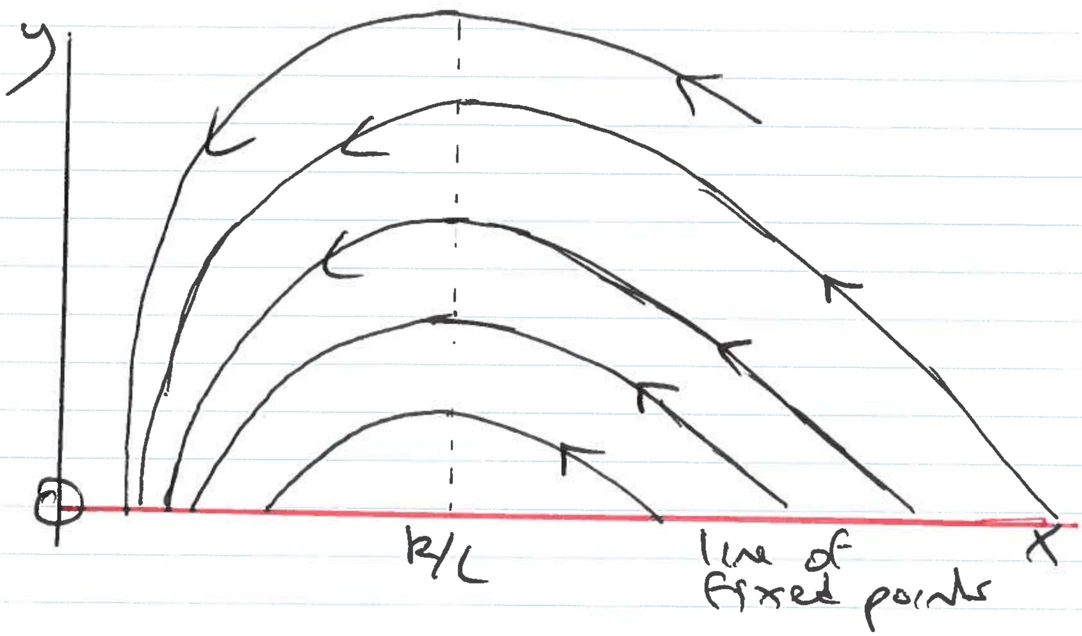
$$c = x + y - \frac{L}{k}\ln x.$$

Clearly as $t \to \infty$ $y \to 0$ which implies

$x(t) \longrightarrow x^*$ where $c = x^* - \frac{L}{k}\ln x^*$

& value of c defined by initial condition $(x(0), y(0))$

$c = x(0) + y(0) - \frac{L}{k}\ln x(0).$



3/10

6.5.20

a) Paper beats rock, rock beats scissors, scissors beats paper

So growth terms in P, R, S include R, S, P term respectively.
The decay terms $-S, -P, -R$ come from what paper, rock and scissors lose too.

For biological assumption just look at first equation

$\dot{P} = P(R-S)$ not $\dot{P} = R-S$ nor $\dot{P} = k(R-S)$.
The factor P in PR & $-PS$ essential says chances of two species encountering each other is proportional to the product of their populations.

2/10

b) $\dot{P} + \dot{R} + \dot{S} = P(R-S) + R(S-P) + S(P-R) = 0$

so $\frac{d}{dt}(P+R+S) = \dot{P} + \dot{R} + \dot{S} = 0$

so $P+R+S$ is a conserved quantity.

2/10

c) $\frac{d}{dt}(PRS) = \dot{P}RS + \dot{R}PS + \dot{S}PR$

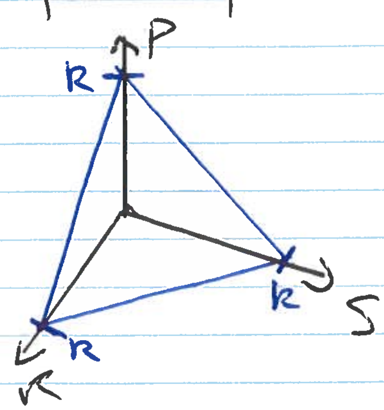
$= PRS(R-S) + PRS(S-P) + PRS(P-R)$

$= 0$

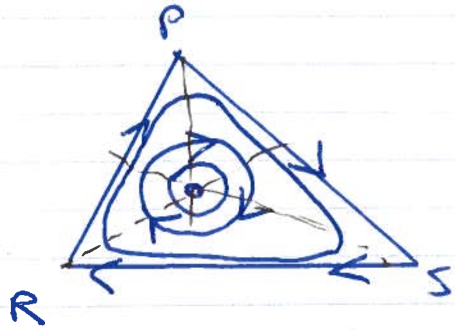
So PRS also conserved

2/10

d) Consider positive quadrant where P, R, S all non negative.



Let $P+R+S = k > 0$.
This defines surface shown.
On this surface $PRS = 0$ on boundaries where at least one of P, R, S is zero & is positive inside the plane with a maximum in the centre by symmetry.



Resulting dynamics as shown.
For direction of rotation notice
when $P=0$
$\dot{R} = RS > 0$
$\dot{S} = -RS < 0$
so arrow on bottom axis
from right to left. Similar
for other cases.

4/10

Using constrained optimization it would be possible to prove rigorously that the function PRS subject to constraints $P \geq 0, R \geq 0, S \geq 0$ & $P+R+S = K > 0$ has a global maximum & locate it, but that is beyond the technique of this course.

6.7.2 $\ddot{\theta} + \sin\theta = \gamma$ 2 points for each of a-e

Let $x = \theta$, $y = \dot{\theta}$ $\Rightarrow$ $\dot{x} = y$, $\dot{y} = \gamma - \sin x$

a) Steady states $\dot{x} = 0 \Rightarrow y = 0$
$\dot{y} = 0 \Rightarrow \sin x = \gamma$

Steady states are $(\theta, \dot{\theta}) = (x, y) = (\sin^{-1}(\gamma), 0)$.

So
i) if $|\gamma| > 1$ there are no steady states.

ii) if $\gamma = 1$ steady states are $(\theta, \dot{\theta}) = (\frac{\pi}{2} + 2n\pi, 0)$ $n \in \mathbb{N}$

iii) For $\gamma \in (-1, 1)$ there are infinitely many steady states u_n with one steady state in each interval $(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi)$ such that $u_n = (x_n, 0) = (\sin^{-1}(\gamma), 0)$

If $\gamma = -1$ steady states are $(\theta, \dot{\theta}) = (\frac{3\pi}{2} + 2n\pi, 0)$.

To classify the steady states for $\gamma \in (-1, 1)$

$\dot{x} = y$, $\dot{y} = \gamma - \sin x$ $\Rightarrow$ Jacobian $J = \begin{pmatrix} \frac{\partial}{\partial x}\dot{x} & \frac{\partial}{\partial y}\dot{x} \\ \frac{\partial}{\partial x}\dot{y} & \frac{\partial}{\partial y}\dot{y} \end{pmatrix}$

$\Rightarrow J = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$ so $\mathrm{tr}(J) = 0$ & if $\det(J) < 0$ steady state is a saddle & if $\mathrm{tr}(J) > 0$ steady state is a linear centre.

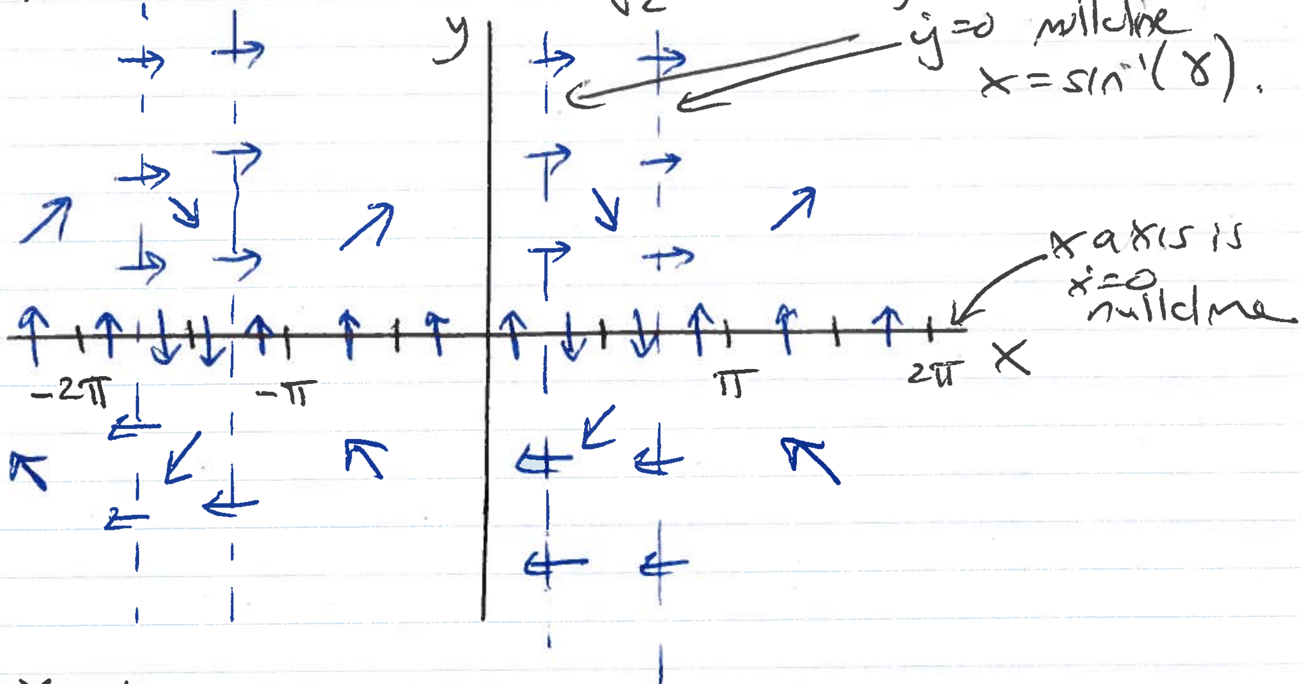
At steady states $(x_{2n}, 0)$ where $x_{2n} \in (-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi)$
$\cos x > 0 \Rightarrow \det(J) > 0 \Rightarrow$ linear centre.

At steady states $(x_{2n+1}, 0)$ with $x_{2n+1} \in (-\frac{\pi}{2} + (2n+1)\pi, \frac{\pi}{2} + (2n+1)\pi)$
$\cos x < 0 \Rightarrow \det(J) < 0 \Rightarrow$ saddle point.

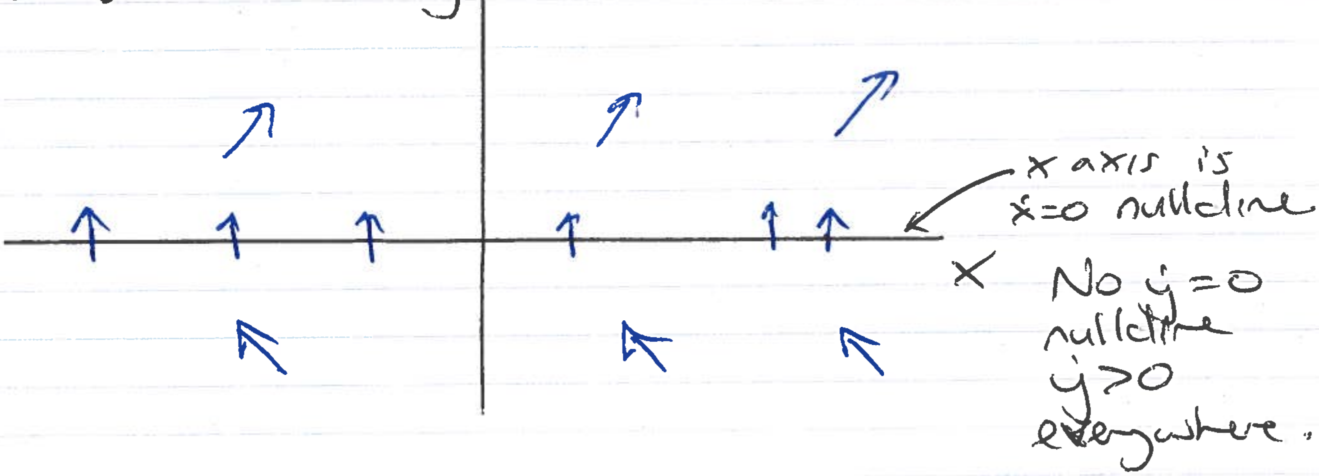
b) Nullclines $\dot{x}=0 \Rightarrow y=0$

$\dot{y}=0 \Rightarrow \sin x = \gamma$

Lets sketch nullclines for $\gamma = \frac{1}{\sqrt{2}} \in (-1,1)$



And for $\gamma > 1$



c) Yes, system is conservative. One way to see this is to consider

$$\ddot{\theta} + \sin\theta = \gamma$$

$$\Rightarrow \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta = \gamma\dot{\theta}$$

$$\Rightarrow \int \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta \, dt = E + \int \gamma\dot{\theta} \, dt \qquad (*)$$

where E is a constant of integration.

But $\int \ddot{\theta}\dot{\theta} \, dt = \frac{1}{2}\dot{\theta}^2$

$$\int \dot{\theta}\sin\theta \, dt = -\cos\theta$$

$$\int \gamma\dot{\theta} \, dt = \gamma\theta$$

Therefore $(*)$ implies

$$\frac{1}{2}\dot{\theta}^2 - \cos\theta = E + \gamma\theta$$

$$\Rightarrow E = \frac{1}{2}\dot{\theta}^2 - \cos\theta - \gamma\theta$$

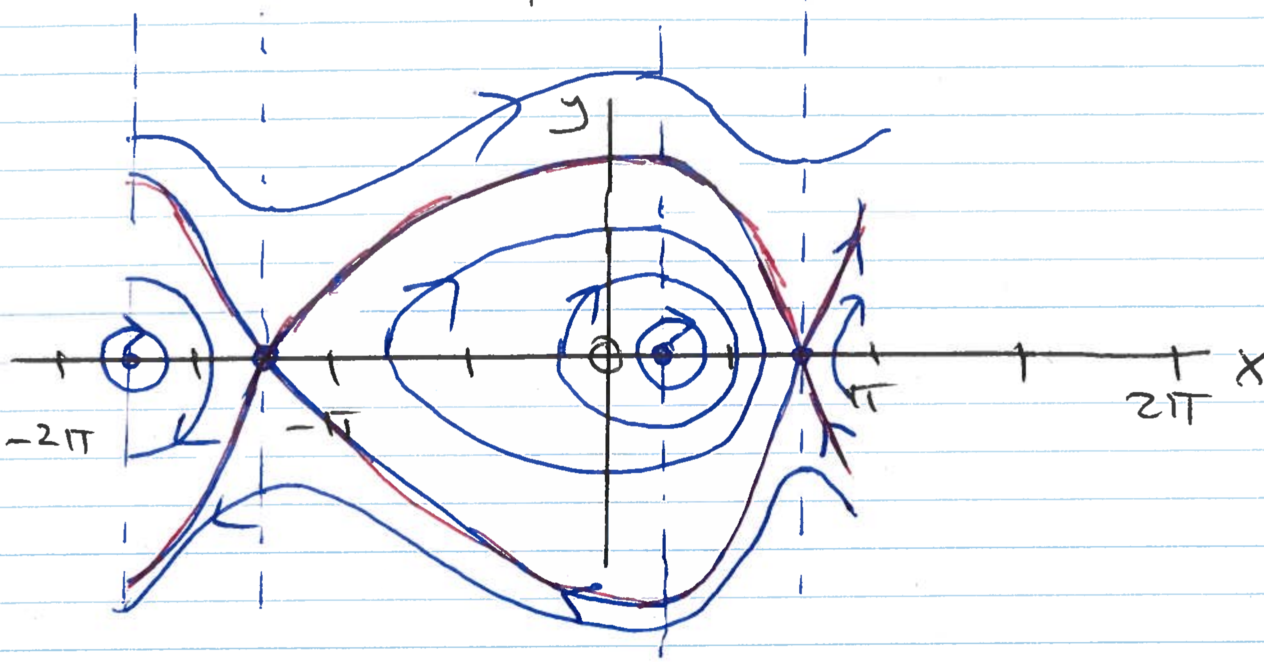
is a conserved quantity.

Writing $\dot{x} = y = f(x,y)$
$\dot{y} = \gamma - \sin x = g(x,y)$

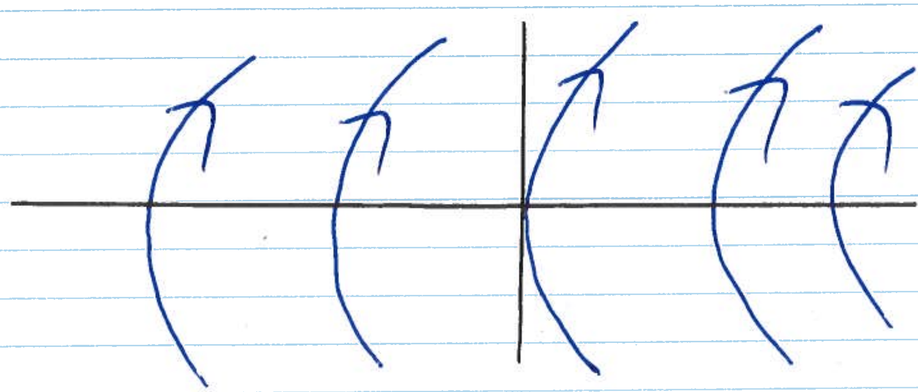
we see f is odd in y & g is even in y so system is also reversible.

$*$ Since system conservative ~~[illegible]~~ all of the linear centres are nonlinear centres. Reversibility also implies linear centres are nonlinear centres since they are all on the x-axis.

d) $\gamma \in (-1,1)$ (we'll take $\gamma = \frac{1}{\sqrt{2}}$ again, other values qualitatively similar).



$\gamma > 1$



e) At centre

$$\lambda^2 - tr(J)\lambda + \det J = 0$$

but $tr(J) = 0 \Rightarrow \lambda^2 = -\det J \Rightarrow \lambda = \pm i\omega$

where $\omega = \sqrt{\det J} = \sqrt{\cos x} = \sqrt{\cos(\sin^{-1}\gamma)}$

For $\gamma \approx 0$ $\sin\gamma \approx \gamma \Rightarrow \gamma \approx \sin^{-1}\gamma$

$$\Rightarrow \omega \approx \sqrt{\cos\gamma} \approx \sqrt{1 - \frac{\gamma^2}{2} + O(\gamma^4)} \approx \left(1 - \frac{\gamma^2}{2}\right)^{1/2}$$

$$\approx 1 - \frac{\gamma^2}{4} \quad \text{so} \quad \omega \approx 1 \;\&\; T \approx \frac{2\pi}{\omega} \approx \underline{\underline{2\pi}}$$