SYMMETRY IN FRACTALS

Symmetry—that topic you probably encountered in primary school maths and then never again—is far more interesting and applicable than it seems at a glance.

Why, for instance, does everything from a penguin to a literal sea cucumber have bilateral symmetry? How does the sonic symmetry of a rhyme scheme enhance our enjoyment of music? What makes a symmetrical face attractive?

This program will answer exactly none of those questions.

What it will do, though, is give you a better understanding of the different types of symmetry out there so that you may (try to) answer those questions yourself.

THE TYPES OF SYMMETRY

Broadly speaking, there are three types of symmetries. You have the fairly common mirror and rotational symmetries. You can reflect an object or rotate the object, but it still looks the same as it used to. Shocker, I know.

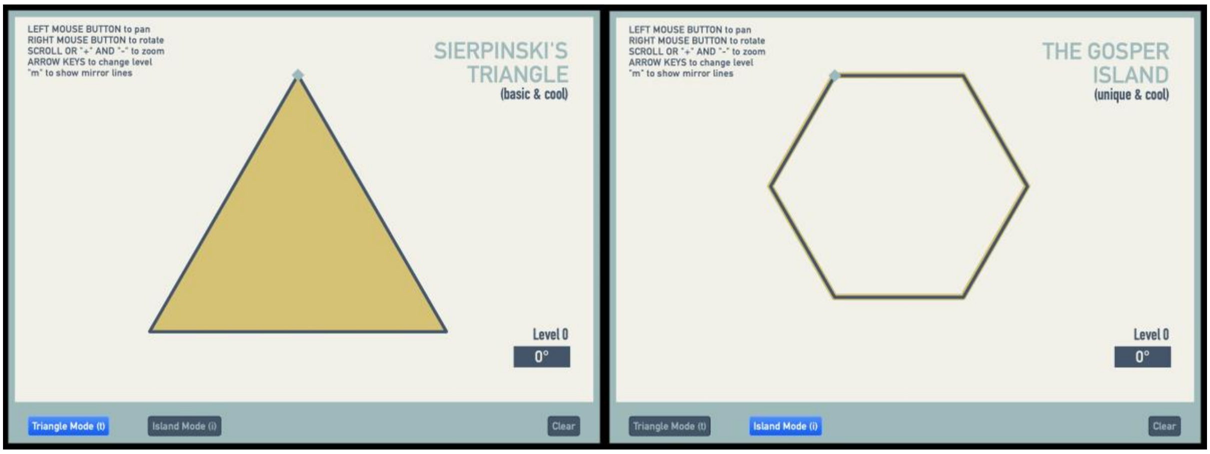
The much less common and far more interesting type of symmetry is **scale symmetry**. This is when you can zoom in or out on object and still have it look exactly the same. Objects that have this property are special types of **fractals**.

THE FRACTALS

A fractal is any shape whose macroscopic and microscopic structures look similar. Think about the coastline of your favourite land mass (mine is, of course, the entirety of Afro-Eurasia). On Google Maps' highest zoom level, the coastline is slightly jagged and slightly smooth. Zoom in, and you'll see that the coastline is still slightly jagged and slightly smooth, even if it looks a little different. Keep zooming, and you'll notice that same pattern no matter what zoom level you are at. That's a fractal.

Before we get started with the actual exploration, you'll need to download the desktop app first. Thankfully, the files are less than 20 MB. Ok, so go on and download the zip file. Then unpackage it (this is the original "virtual unboxing"). Inside you will find two files named "symmetry_and_fractals". Both contain the same information, but they are for different computers. Find one that works for you! If you encounter an error that says "symmetry_and_fractals.app" can't be opened because Apple cannot check it for malicious software", then that means that I am not a big time app developer (yet!). Try right-clicking and selecting "Open" from there. That should fix things!

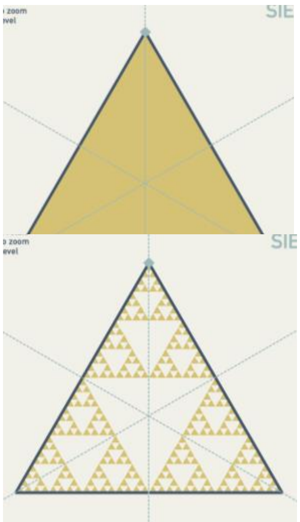
Now that you have the app up and running, we can actually look at these fractals. The two fractals you can look at in this program are known as Sierpinski's Triangle (click "Create Triangle") and the Gosper Island (click "Create Island"). After you play around with these, you can brag to your friends[1] that you know one cool and overused fractal (Sierpinski's Triangle) and one cool and unique one (the Gosper Island).



What you should see when you click "Create Triangle (t)" (left) and "Create Triangle (i)" (right). At first, there is only the very basic shape.

Both fractals have scale symmetry. However, where Sierpinksi's Triangle has all the symmetries, the Gosper Island does not.

THE FRACTALS II: REFLECTION VS ROTATION



After you "Create Triangle", you should see a yellow triangle. If you then hit "m" on your keyboard, you can see how this basic-bitch shape has three mirror lines. You can also use your right mouse button to drag the shape around or your left mouse button to rotate it. Now, if you go down a few levels using your arrow keys, you can start seeing how the actual fractal is made. Lo, behold! No matter what level you go down to, the triangle should

Sierpinski's Triangle at level 0 (top) and level 5 (bottom). Despite the much higher complexity of the level 5 Triangle, the mirror lines are in the exact same place!

---

[1] This brag will only work if your friends are nerds. Trust me. I have tried.

always have three mirror lines and should always fit in the original outline at 120°, 240° and 0°.

Meanwhile, if you do the same for the Island option, you will first see a black hexagon. Quite boring, lights up at 60°, 120°, 180°, 240°, 300°, and 0°, has six mirror lines. But, once you start descending into the depths of this fractal, you may notice that its reflective symmetry disappears[2].

That's actually a pretty key observation. An object can have rotational symmetry without reflective symmetry[3]. The converse is very much not true, though. In fact, notice how the Triangle's three mirror lines correspond to three rotational symmetries. Even the six short-lived mirror lines of the hexagon correspond to six rotational symmetries. Whenever an object has reflective symmetries, the number of mirror lines you can construct is the same as the number of symmetric rotations.

This is also the reason we distinguish between something called "gyration point" and point of rotational symmetry. The former is a point where the only way to get an object to look like itself is through rotation. The latter means objects that look the same purely through rotation are just as good as objects that look the same through reflection and rotation.

Okay, so those are some of the things we observe about reflective and rotational symmetry, but now it's time for

THE FRACTALS III: INCEPTION

So, I said before that both fractals have scale symmetry. But you've probably noticed that this scale symmetry doesn't look quite the same for the Triangle and the Island. Where you can zoom in on the triangle itself and have it look the same, for the Island, you can only really see this happening on the outline. That, I'd say is not particularly interesting.

[2] The reason that the Island loses its reflective symmetry is because the way it's made involves iterating through a shape that itself does not have reflective symmetry . On the other hand, the Triangle is made by iterating shapes that do have reflective symmetry (aka, more equilateral triangles) from points on the mirror lines. Honestly, I am not certain how true this is in general, but it is what is happening in these two specific cases.

[3] All the details can be found in this book, if you are interested.

However, and here's the plot twist, everything changes when tile these objects. For the Triangle, since its outline is just a, well, triangle, it can definitely tile a plane. Hit the "Tile Me" button to see one version of what this looks like. Notice, though, that once the Triangle has tiled the plane, its scale symmetry breaks apart[4]. Zoom out on the triangle, and that gap in the centre fills in. Still, the symmetries of the original object are retained. Not only that, but those symmetries are repeated infinitely across the plane. This pattern is unique and has a signature of *632[5].

What you should see when you click "Tile!!" for a level 1 Sierpinski's Triangle.

What you should see when you click "Tile!!" for a level 1 Island, and then go to level 2.

If you [do the same for the Island](), you get a very different result. First, the fact that this weird shape can tile is already pretty cool[6]. Further, if you enlarge the island, you will see that the six Gosper Islands surrounding your original island have an outline that is exactly the same as your original Island, but bigger. As with the Triangle, the Island does retain its rotational symmetries, forming a plane pattern with signature of 333.

THE FRACTALS IV: THE FINAL ACT

So, we have now seen how fractals are formed and how they can exhibit the three types of symmetry. In spite of how weird our fractals ended up being at the higher levels, they were still governed by some simple rules. In each iteration, the same action was applied to the fractals to make it more complex.

---

[4] Something to note here is that I have very purposefully only shown you a single tiling of the Triangle. In that sense, I have misled you. The way that I have defined "tiling" is taking a single instance of the objects and then tessellating them. You could just as easily have zoomed in on the Triangle until you couldn't see the original edges and then called *th* symmetry of the Triangle easily. You would have to make your definitions a lot more finicky if you wanted to do the same for the Island. For a really interesting but unrelated way to tile a plane with triangles, check this out. (I am also sorry for being that person and sticking everything into a footnote. Oops.)

[5] For more Information, you can visit www2.clarku.edu/faculty/djoyce/wallpaper/seventeen.html, which is a very 90s website with some good information.

[6] This is in part because the Gosper Island was first created as the outline of a space-filling curve—a single line that passes through every point of a given space.

Moreover, the rotational and reflective symmetries of the objects did not change as we got to the more complex iterations.

Have a play around with this program. Hopefully you'll get a feel for how these symmetries work intuitively. And then once you find the answers to those three questions, please tell me because I am also dying to know.