CIS 581 Homework 1
Due: Tuesday September 15th

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Convolution

$$I = \begin{pmatrix} 0 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0.5 & 0.5 & 0 & 0.5 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0.5 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad (1)$$

For the following filtering operation, use $a = 0.4$(Gaussian). Let $J1 = (I \otimes Gx) \otimes Gy$ Let$Gxy = Gx \otimes Gy$, and $J2 = I \otimes Gxy$. Compute J1 and J2 using the image I and verify they are the same. How many operations(addition and multiplication) are required for computing J1 and J2?

Solution To demonstrate convolution I picked $I(0,0)$, $I(7,11)$, $I(7,5)$, and $I(5,5)$. for$a = .4$

$$G_x = \begin{vmatrix} .05 & .25 & .4 & .25 & .05 \end{vmatrix}$$

Where the index starts from -2 and goes to 2 for J_1 First we convolve G_x with the image I

$$\big(\sum_k I(i-k, j-l)G(k)\big)$$

Example calculation for $I(0,0)$, $I(7,11)$, $I(7,5)$, and $I(5,5)$. First pad the Image in both x and y directions with column indices from -2 to 12, and row indices from -2 to 10.

$$I_{pad} = \begin{pmatrix} 0.5 & 1 & 1 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0.5 & 1 & 0 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0.5 & 1 & 0 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0.5 & 1 & 0 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0.5 & 1 & 1 & 1 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0.5 & 1 & 1 & 1 & 0.5 & 0.5 & 0 & 0.5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0.5 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0.5 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0.5 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0.5 & 1 & 1 & 1 & 0.5 & 0.5 & 0 & 0.5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
&I_{new}(0,0)\\
&\quad= \Big(\sum_k I_{pad}(i-k,j)G_x(k)\Big) = I_{pad}(-2,0)G_x(2)\\
&\quad+ I_{pad}(-1,0)G_x(1) + I_{pad}(0,0)G_x(0) + I_{pad}(2,0)G_x(-2) + I_{pad}(1,0)G_x(-1) = .55
\end{aligned}
$$

$$
\begin{aligned}
&J1(0,0)\\
&\quad= \Big(\sum_l I(i,j-l)G_y(l)\Big) = I_{new}(0,-2)G_y(2) + I_{new}(0,-1)G_y(1)\\
&\quad+ I_{new}(0,0)G_y(0) + I_{new}(0,1)G_y(-1) + I_{new}(0,2)G_y(-2)\\
&\quad= .95*.05 + .55*.25 + .55*.4 + .55*.25 + .95*.05 = .59
\end{aligned}
$$

$$
\begin{aligned}
&I_{new}(6,10)\\
&\quad= \Big(\sum_k I_{pad}(i-k,j)G_x(k)\Big) = I_{pad}(4,10)G_x(2)\\
&\quad+ I_{pad}(5,10)G_x(1) + I_{pad}(6,10)G_x(0) + I_{pad}(7,10)G_x(-1) + I_{pad}(8,10)G_x(-2)\\
&\quad= .1000
\end{aligned}
$$

$$
\begin{aligned}
&J1(6,10)\\
&\quad= \Big(\sum_l I(i,j-l)G_y(l)\Big) = I_{new}(6,8)G_y(2) + I_{new}(6,9)G_y(1)\\
&\quad+ I_{new}(6,10)G_y(0) + I_{new}(6,11)G_y(-1) + I_{new}(6,12)G_y(-2)\\
&\quad= .1*.05 + .4*.25 + .1*.4 + .4*.25 + .1*.05 = .25
\end{aligned}
$$

$$
\begin{aligned}
&I_{new}(5,5)\\
&\quad= \Big(\sum_k I(i-k,j)G_x(k)\Big) = I_{pad}(3,5)G_x(2) + I_{pad}(4,5)G_x(1)\\
&\quad+ I_{pad}(5,5)G_x(0) + I_{pad}(6,5)G_x(-1) + I_{pad}(7,5)G_x(-2) = .3250
\end{aligned}
$$

$$
\begin{aligned}
&J1(5,5)\\
&\quad= \Big(\sum_l I(i,j-l)G_y(l)\Big) = I_{new}(5,3)G_y(2) + I_{new}(5,4)G_y(1)\\
&\quad+ I_{new}(5,5)G_y(0) + I_{new}(5,6)G_y(-1) + I_{new}(5,7)G_y(-2) = .4312
\end{aligned}
$$

$$
\begin{aligned}
&I_{new}(6,0)\\
&\quad= \Big(\sum_k I(i-k,j)G_x(k)\Big) = I_{pad}(4,0)G_x(2) + I_{pad}(5,0)G_x(1)\\
&\quad+ I_{pad}(6,0)G_x(0) + I_{pad}(7,0)G_x(-1) + I_{pad}(8,0)G_x(-2) = .0500
\end{aligned}
$$

$$
\begin{aligned}
&J1(7,0)\\
&\quad = \big(\sum_l I(i,j-l)G_y(l)\big) = I_{new}(6,-2)G_y(2) + I_{new}(6,-1)G_y(1)\\
&\quad + I_{new}(6,0)G_y(0) + I_{new}(6,1)G_y(-1) + I_{new}(7,2)G_y(-2) = .1400
\end{aligned}
$$

For J_2 we neex to calculate G_{xy}

$$
G_{xy} = \begin{pmatrix} 0.0025 & 0.0125 & 0.02 & 0.0125 & 0.0025 \\ 0.0125 & 0.0625 & 0.1 & 0.0625 & 0.0125 \\ 0.02 & 0.1 & 0.16 & 0.1 & 0.02 \\ 0.0125 & 0.0625 & 0.1 & 0.0625 & 0.0125 \\ 0.0025 & 0.0125 & 0.02 & 0.0125 & 0.0025 \end{pmatrix}
$$

$$
\begin{aligned}
&J_2(0,0)\\
&\quad = I_{pad}(-2,-2) * G_{xy}(-2,-2) + I_{pad}(-2,-1) * G_{xy}(-2,-1) + I_{pad}(-2,0) * G_{xy}(-2,0)\\
&\quad + I_{pad}(-2,1) * G_{xy}(-2,1) + I_{pad}(-2,2) * G_{xy}(-2,2)\\
&\quad + I_{pad}(-1,-2) * G_{xy}(-1,-2) + I_{pad}(-1,-1) * G_{xy}(-1,-1) + I_{pad}(-1,0) * G_{xy}(-1,0)\\
&\quad + I_{pad}(-1,1) * G_{xy}(-1,1) + I_{pad}(-1,2) * G_{xy}(-1,2)\\
&\quad + I_{pad}(0,-2) * G_{xy}(0,-2) + I_{pad}(0,-1) * G_{xy}(0,-1) + I_{pad}(0,0) * G_{xy}(0,0)\\
&\quad + I_{pad}(0,1) * G_{xy}(0,1) + I_{pad}(0,2) * G_{xy}(0,2)\\
&\quad + I_{pad}(1,-2) * G_{xy}(1,-2) + I_{pad}(1,-1) * G_{xy}(1,-1) + I_{pad}(1,0) * G_{xy}(1,0)\\
&\quad + I_{pad}(1,1) * G_{xy}(1,1) + I_{pad}(1,2) * G_{xy}(1,2)\\
&\quad + I_{pad}(1,-2) * G_{xy}(2,-2) + I_{pad}(2,-1) * G_{xy}(2,-1) + I_{pad}(2,0) * G_{xy}(2,0)\\
&\quad + I_{pad}(1,1) * G_{xy}(2,1) + I_{pad}(1,2) * G_{xy}(2,2) = .59
\end{aligned}
$$

$$
\begin{aligned}
&J_2(6,10)\\
&\quad = I_{pad}(4,8) * G_{xy}(-2,-2) + I_{pad}(4,9) * G_{xy}(-2,-1) + I_{pad}(4,10) * G_{xy}(-2,10)\\
&\quad + I_{pad}(4,11) * G_{xy}(-2,1) + I_{pad}(4,12) * G_{xy}(-2,2)\\
&\quad + I_{pad}(5,8) * G_{xy}(-1,-2) + I_{pad}(5,9) * G_{xy}(-1,-1) + I_{pad}(5,10) * G_{xy}(-1,0)\\
&\quad + I_{pad}(5,11) * G_{xy}(-1,1) + I_{pad}(5,12) * G_{xy}(-1,2)\\
&\quad + I_{pad}(6,8) * G_{xy}(0,-2) + I_{pad}(6,9) * G_{xy}(0,-1) + I_{pad}(6,10) * G_{xy}(0,0)\\
&\quad + I_{pad}(6,11) * G_{xy}(0,1) + I_{pad}(6,12) * G_{xy}(0,2)\\
&\quad + I_{pad}(7,8) * G_{xy}(1,-2) + I_{pad}(7,9) * G_{xy}(1,-1) + I_{pad}(7,10) * G_{xy}(1,0)\\
&\quad + I_{pad}(7,11) * G_{xy}(1,1) + I_{pad}(7,12) * G_{xy}(1,2)\\
&\quad + I_{pad}(8,8) * G_{xy}(2,-2) + I_{pad}(8,9) * G_{xy}(2,-1) + I_{pad}(8,10) * G_{xy}(2,0)\\
&\quad + I_{pad}(8,11) * G_{xy}(2,1) + I_{pad}(8,12) * G_{xy}(2,2) = .25
\end{aligned}
$$

$J_2(5,5)$

$$
\begin{aligned}
&= I_{pad}(3,3)*G_{xy}(-2,-2)+I_{pad}(3,4)*G_{xy}(-2,-1)+I_{pad}(3,5)*G_{xy}(-2,10)\\
&+I_{pad}(3,6)*G_{xy}(-2,1)+I_{pad}(3,7)*G_{xy}(-2,2)\\
&+I_{pad}(4,3)*G_{xy}(-1,-2)+I_{pad}(4,4)*G_{xy}(-1,-1)+I_{pad}4,5)*G_{xy}(-1,0)\\
&+I_{pad}(4,6)*G_{xy}(-1,1)+I_{pad}(4,7)*G_{xy}(-1,2)\\
&+I_{pad}(5,3)*G_{xy}(0,-2)+I_{pad}(5,4)*G_{xy}(0,-1)+I_{pad}(5,5)*G_{xy}(0,0)\\
&+I_{pad}(5,6)*G_{xy}(0,1)+I_{pad}(5,7)*G_{xy}(0,2)\\
&+I_{pad}(6,3)*G_{xy}(1,-2)+I_{pad}(6,4)*G_{xy}(1,-1)+I_{pad}(6,5)*G_{xy}(1,0)\\
&+I_{pad}(6,6)*G_{xy}(1,1)+I_{pad}(6,7)*G_{xy}(1,2)\\
&+I_{pad}(7,3)*G_{xy}(2,-2)+I_{pad}(7,4)*G_{xy}(2,-1)+I_{pad}(7,5)*G_{xy}(2,0)\\
&+I_{pad}(7,6)*G_{xy}(2,1)+I_{pad}(7,7)*G_{xy}(2,2)=.1400
\end{aligned}
$$

Operations for convolutions For J_1 for each point there are 4 addition and 5 multiplication operations per element per convoloution, which makes a total of $2*(4*77)=616$ addition operations and $2*(5*77)=770$ multiplication operations. For J_2 For the convolution of G_{xy} it takes 25 multiplications, additionally there are there are 20 addition operations and 25 multiplication operations for one element. Therefore for all 77 elements there would be $20*77=1540$ addition operations and $25+25*77=1950$ multiplication operations.

Image Gradient

Compute the image gradient $\Delta I = [I_x, I_y]$ where $I_x = I \otimes G_x \otimes \delta_x \otimes G_y$
$I_y = I \otimes G_y \otimes \delta_y \otimes G_x$

Since the convolutions are associative we can say that where J1 has 0 padding $I_x = I \otimes G_x \otimes G_y \otimes \delta_x = J_1\delta_x$
$I_y = I \otimes G_x \otimes G_y otimes \delta_y = J_1\delta_y$

where $\delta_x = [1 - 1]$ and $\delta_y = delta_x^T$ For

$$
\begin{aligned}
I_x(0,0) &= J_1(0,0)*\delta_x(1)+J_1(1,0)*\delta_x(0)=0.5900*-1+0.6250*1=.035\\
I_y(0,0) &= J_1(0,0)*\delta_y(1)+J_1(0,1)*\delta_y(0)=0.5900*-1+0.66750*1=.0775\\
I_x(6,10) &= J_1(6,10)*\delta_x(1)+J_1(7,10)*\delta_x(0)=.25*-1+0*1=-.25\\
I_y(6,10) &= J_1(6,10)*\delta_y(1)+J_1(7,10)*\delta_y(0)=.25*-1+0*1=-.25\\
I_x(5,5) &= J_1(5,5)*\delta_x(1)+J_1(6,5)*\delta_x(0)=.6287*-1+.8225=.1938\\
I_y(5,5) &= J_1(5,5)*\delta_y(1)+J_1(5,6)*\delta_y(0)=-0.6287+.7050=.0763
\end{aligned}
$$

Compute the magnitude

$$\|\Delta I_{0,0}\| = \sqrt[I_x(0,0)^2+I_y(0,0)^2]{2} = \sqrt[0.35^2+0.0775^2]{2} = 0.0850$$
$$\|\Delta I_{6,10}\| = \sqrt[I_x(6,10)^2+I_y(6,10)^2]{2} = \sqrt[-0.25^2+-0.25^2]{2} = 0.3536$$
$$\|\Delta I_{5,5}\| = \sqrt[I_x(5,5)^2+I_y(5,5)^2]{2} = \sqrt[0.1938^2+0.0763^2]{2} = 0.2082$$

How edges move after image smoothing

$$(a) I = \begin{vmatrix}1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\end{vmatrix}, G1 = G.$$
$$(b) I = \begin{vmatrix}0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\end{vmatrix}, G2 = G.$$
$$(c) I = \begin{vmatrix}0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2\end{vmatrix}, G3 = G \otimes G$$

solution: First we convolve the Image with with δ_x since they are all row vectors. This convolution (with 0 padding) will give us the edges for the "Image". Then after applying the filter we will see the edges will remain at the same point in the row vector but the magnitude spread out over several pixels.
a)

$$I \otimes \delta_x = \begin{vmatrix}0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\end{vmatrix}$$

We see that the edge is at $I(4)$, so when we apply the Gaussian kernel we have

$$I \otimes \delta_x \otimes G = \begin{vmatrix}0 & 0 & -0.0500 & -0.2500 & -0.4000 & -0.2500 & -0.0500 & 0 & 0 & 0\end{vmatrix}$$

We see that the greatest magnitude still occurs at I(4), but is blurred across 5 pixels.
b)

$$I \otimes \delta_x = \begin{vmatrix}0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0\end{vmatrix}$$

We see that the edge is at $I(3) and I(6)$, so when we apply the Gaussian kernel we have

$$I \otimes \delta_x \otimes G = \begin{vmatrix}0 & 0.050 & 0.25 & 0.40 & 0.20 & -0.20 & -0.40 & -0.20 & -0.050 & 0\end{vmatrix}$$

We see that the greatest magnitude still occurs at I(3) and I(6) , but is blurred across the image.
c)

$$I \otimes \delta_x = \begin{vmatrix}0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -2\end{vmatrix}$$

We see that the edge is at $I(2), I(5), and I(9)$, so when we apply the Gaussian kernel we have

$$I \otimes \delta_x \otimes G \otimes G = \begin{vmatrix}0.1025 & 0.2250 & 0.2900 & 0.3275 & 0.3275 & 0.2900 & 0.2250 & -0.1025 - 0.4500 & -0.5800\end{vmatrix}$$

We see that with this new filter that we have effectively blurred most of the image leaving the only change in magnitude between I(6) and I(7).