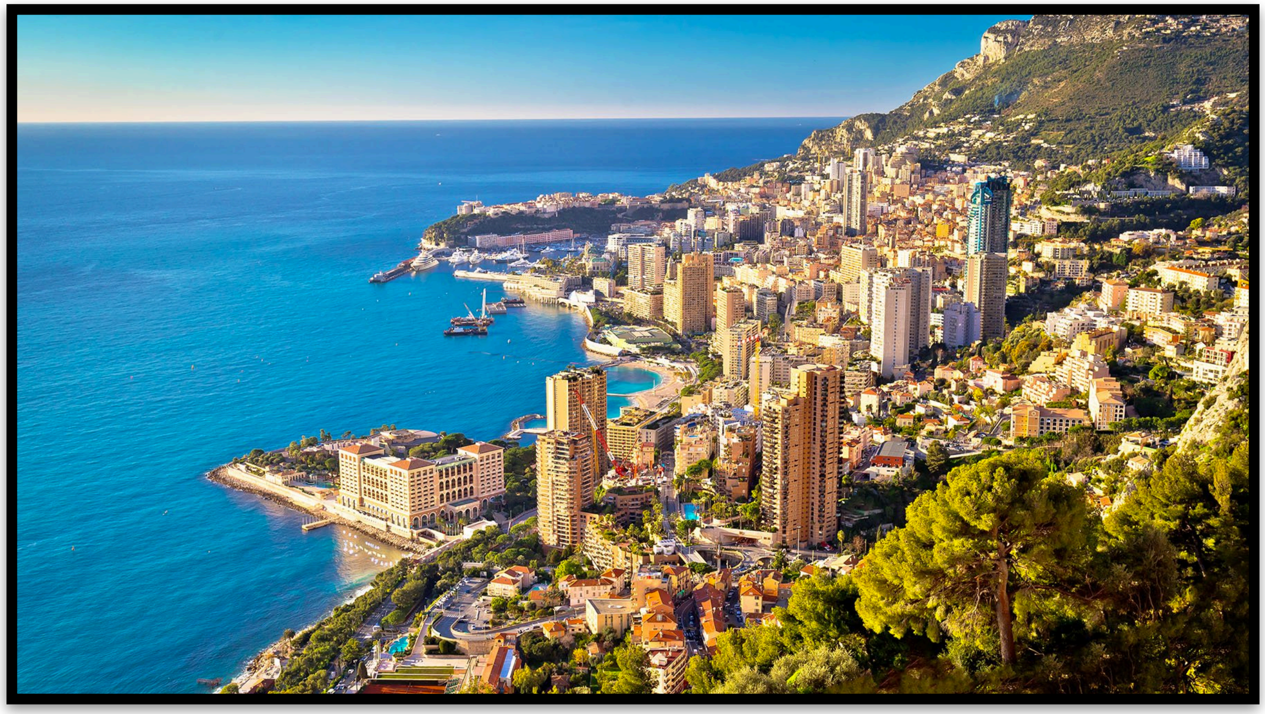
Counterparty Credit Risk

From Statistics

Monte Carlo, Monaco



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Introduction

<u>Counterparty credit risk</u>

How much money would we lose in the future if a counterparty defaults

Can we derive OSFI prescribed formulas from statistics?

Can we account for correlations?

Introduction

We will derive

- **Part 1: ECL Formula**
- **Part 2: Alpha Factor in EAD**
- **Part 3: PD term in RWA formula**

Notation

- T : future reference time
- $D = \{0,1\}$: default at $t \in (0,T]$
- $E \in [0,\infty)$: how much they owe us at t
- $L \in [0,1]$: loss incurred at t as a % of E
  - $L = 0$ when $D = 0$

- N_D defaults between $(0,T]$
- N_{ND} non-defaults between $(0,T]$
- $N = N_N + N_{ND}$

- $\bar{D} \equiv PD = \sum D_i/N = N_D/N$: Probability Default
- LGD $= \sum_{\text{defaulted cpty}} L_i/N_D$
- $\bar{L} = \sum L_i/N =$ LGD $\cdot\, N_D/N =$ LGD $\cdot$ PD

Part 1
Expected Credit Loss

Expected Credit Loss

$ECL = EAD \times LGD \times PD$

$L \in [0,1]$ **: loss incurred at** t **as a % of exposure** E

$D = \{0,1\}$**: default at** $t \in (0,T]$

- $ECL \equiv \cdot \mathbb{E}(L \cdot D)$
- Assume for simplicity that $EAD = 1$ is a constant (fixed loan)

What is meant by this $\mathbb{E}$ average?

<u>Frequentist approach</u>: draw many samples of L and D for the *same* counterparty and take the average. Impossible!

Workarounds:

1) Do Monte Carlo Simulation (e.g Merton model)
2) Consider *similar* counterparties (e.g. industry, geography) as realizations of a single hypothetical counterparty.

Expected Credit Loss

$ECL = EAD \times LGD \times PD$

First Principles (easy method)

Using law of total expectation $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X \mid Y))$, we can write:

$$\begin{aligned}\mathbb{E}(L \cdot D) &= \mathbb{E}(L \cdot D \mid \text{default})P(\text{default}) + \mathbb{E}(L \cdot D \mid \text{non-default})P(\text{non-default}) \\ &= \mathbb{E}(L \mid \text{default})P(\text{default}) \\ &= LGD \cdot PD\end{aligned}$$

$L = D = 0$ **when no default**

Too easy…

Did we *really* account for correlations between L and D?

Expected Credit Loss

$ECL = EAD \times LGD \times PD$

First Principles (hard method)

Using $\mathbb{E}(L \cdot D) = cov(L, D) + \mathbb{E}(L) \cdot \mathbb{E}(D)$ we can write:

$$cov(L, D) = \frac{1}{N} \sum^{N} (L_i - \bar{L})(D_i - \bar{D})$$

$$= \frac{1}{N} \sum_{\text{defaulted cpty}}^{N_D} (L_i - \bar{L})(\underbrace{D_i}_{1} - \bar{D}) + \frac{1}{N} \sum_{\text{non-defaulted cpty}}^{N_{ND}} (\underbrace{L_i}_{0} - \bar{L})(\underbrace{D_i}_{0} - \bar{D})$$

$$= \frac{N_D N_{\mathrm{ND}}}{N^2} \cdot (1 - \mathrm{PD}) \cdot \mathrm{LGD} + \frac{N_D N_{ND}}{N^2} \cdot \underbrace{\mathrm{PD}}_{\frac{N_D}{N}} \cdot \underbrace{\mathrm{LGD}}_{\sum_i^{N_D} \frac{L_i}{N_D}}$$

$$= \mathrm{LGD} \frac{N_D N_{\mathrm{ND}}}{N^2}$$

$$= \mathrm{LGD} \cdot \mathrm{PD} \cdot (1 - \mathrm{PD})$$

$$\left(= \mathrm{LGD} \cdot var(D) \right)$$

Expected Credit Loss

$ECL = EAD \times LGD \times PD$

First Principles (hard method)

Using $\mathbb{E}(L \cdot D) = cov(L, D) + \mathbb{E}(L) \cdot \mathbb{E}(D)$ we can write:

$$\begin{aligned} \mathbb{E}(L) \cdot \mathbb{E}(D) &\equiv \bar{L} \cdot \bar{D} \\ &= (\text{LGD PD}) \cdot \text{PD} \\ &= \text{LGD} \cdot \text{PD}^2 \end{aligned}$$

Expected Credit Loss

$ECL = EAD \times LGD \times PD$

First Principles (hard method)

Using $\mathbb{E}(L \cdot D) = cov(L, D) + \mathbb{E}(L) \cdot \mathbb{E}(D)$ we can write:

$$\begin{aligned} \mathbb{E}(L \cdot D) &= cov(L, D) + \mathbb{E}(L) \cdot \mathbb{E}(D) \\ &= \mathsf{LGD} \cdot \mathsf{PD}(1 - \mathsf{PD}) + \mathsf{LGD} \cdot \mathsf{PD}^2 \\ &= \mathsf{LGD} \cdot \mathsf{PD} \end{aligned}$$

Expected Credit Loss

$ECL = EAD \times LGD \times PD$

Insights

- Without correlations between L and D, we would have $ECL \propto PD^2$
- Correlations correct the PD^2 scaling to a PD scaling

Part 2
Alpha Factor

Alpha Factor

Motivation
In many formulas from Basel, we often see a mysterious factor α, which we are told to set to 1.4

$$EAD_{\text{SACCR}} = \alpha \cdot (RC + PFE)$$

$$EAD_{\text{IMM}} = \alpha \cdot \text{Effective EPE}$$

Why do we need α?

Alpha Factor

First Principles (1 counterparty)

$D = \{0,1\}$: **default at** $t \in (0,T]$

$E \in [0,\infty)$: **how much money they owe us at time** t

$$\mathbb{E}(D \cdot E) \equiv \sum d \cdot eP(d,e)$$

- Sum over all possible exposures and default states at future time t, weighted by the joint probability distribution $P(d,e)$
- Take $L = 1$ as constant for simplicity

Alpha Factor

First Principles Derivation (1 counterparty)

Assumption

- Let's *assume* that the exposure dependence in the joint probability distribution $P(e, d)$ can be captured by a constant fudge factor α.

This can be loosely motivated by using Bayes' theorem:

$$P(d \mid e) = \frac{P(e \mid d)}{P(e)} P(d)$$
$$\approx \alpha P(d)$$

Fine Print: This actually violates many theorems in probability, but let's call this a minor inconvenience and keep going

Alpha Factor

First Principles Derivation (1 counterparty)

Using this assumption, we can write the following:

$$
\begin{aligned}
\mathbb{E}[D \cdot E] &= \sum_{i,k} e_i d_k P(d_k, e_i) \\
&= \sum_{i,k} e_i d_k P(d_k \,|\, e_i) P(e_i) \\
&\approx \sum_{i,k} e_i d_k \alpha P(d_k) P(e_i) \\
&= \left[\alpha \sum_i e_i P(e_i) \right] \left[\sum_k d_k P(d_k) \right] \\
&\equiv \text{EAD} \cdot \text{PD}
\end{aligned}
$$

Alpha Factor

First Principles Derivation (1 counterparty)

Re-arranging the previous calculation, we obtain:

$$\alpha = \frac{\sum_{i,k} e_i d_k P(d_k \,|\, e_i) P(e_i)}{\left[\sum_i e_i P(e_i)\right]\left[\sum_i d_k P(d_k)\right]}$$

Insights

- α accounts for the correlation between default probabilities and exposures in the regulatory capital calculation

Alpha Factor

First Principles (many counterparties)

Let's assume $N = 2$ (general case follows).
We now want to calculate:

$$\mathbb{E}\left[(D \cdot E)_{\text{cpty A}} + (D \cdot E)'_{\text{cpty B}}\right]$$
$$= \sum (ed + e'd')P(d, d', e, e')$$

Alpha Factor

First Principles Derivation (many counterparties)

Assumptions

- Exposures of counterparty A and B are independent:
  $P(e, e') = P(e)P(e')$

- Defaults of A and B are independent conditional on exposures:
  $P(d, d' | e, e') = P(d | e, e')P(d' | e, e')$

- α_A correlates defaults of A on exposures of A and B:
  $P(d | e, e') \approx \alpha_A P(d)$

- α is universal:
  $\alpha_A = \alpha_B = \alpha$

Alpha Factor

First Principles Derivation (many counterparties)

Using these assumptions and a lot of algebra, we obtain:

$$
\begin{aligned}
&\mathbb{E}[(D \cdot E)_{\text{cpty A}} + (D \cdot E)^{'}_{\text{cpty B}}] \\
&= \sum_{i,j,k,l} (e_i d_k + e_j' d_l') P(d_k, d_l', e_i, e_j') \\
&= \sum_{i,j,k,l} e_i d_k P(d_k, d_l' \,|\, e_i, e_j') P(e_i, e_j') + \sum_{i,j,k,l} e_j' d_l' P(d_k, d_l' \,|\, e_i, e_j') P(e_i, e_j') \\
&\approx \sum_{i,k} \alpha e_i P(e_i) d_k P(d_k) + \sum_{j,l} \alpha e_j' P(e_j') d_l' P(d_l') \\
&= (EAD \cdot PD)_{\text{cpty A}} + (EAD \cdot PD)'_{\text{cpty B}}
\end{aligned}
$$

Alpha Factor

First Principles Derivation (many counterparties)

Re-arranging the previous calculation, we obtain more generally

$$\alpha = \frac{\sum_{i,j,k,l} (e_i d_k + e'_j d'_l) P(d_k, d'_l, e_i, e'_j)}{\sum_{i,k} e_i P(e_i) d_k P(d_k) + \sum_{j,l} e'_j P(e'_j) d'_l P(d'_l)}$$

Insights

- α can be generalized to many counterparties

Part 3
RWA Default Probabilities

Risk Weighted Assets

Motivation

Can we write down an explicit model to correlate D and E?

Yes - will show how it leads to Worst Case Default Rate (WCDR) in the prescribed OSFI formula:

$$RWA_{AIRB} = 12.5 \times EAD \times (WCDR - PD) \times LGD \times MA$$

Risk Weighted Assets

Consider a 1-factor model. For counterparty i, let's define:

$M \sim N(0,1)$ (systemic factor)

$Z_i \sim N(0,1)$ (idiosyncratic factor)

$X_i = \sqrt{\rho}M + \sqrt{1-\rho}Z_i$ (e.g. asset returns)

X_i was chosen as such to have the following properties:

$$
\left.
\begin{aligned}
\mathbb{E}(X_i) &= \mathbb{E}(\sqrt{\rho}M + \sqrt{1-\rho}Z_i) \\
&= \sqrt{\rho}\mathbb{E}(M) + \sqrt{1-\rho}\mathbb{E}(Z_i) \\
&= 0 \\
Var(X_i) &= \mathbb{E}((X_i - \mathbb{E}(X_i))(X_i - \mathbb{E}(X_i))) \\
&= \mathbb{E}(\rho M^2 + 2\sqrt{\rho(1-\rho)}MZ_i + (1-\rho)Z_i^2) \\
&= \rho 1 + 0 + (1-\rho)1 = 1 \\
cov(X_i, X_j) &= \rho
\end{aligned}
\right\}
\quad \boxed{X_i \sim N(0,1)}
$$

Risk Weighted Assets

Define defaults as when $X_i < \bar{x}_i$, for some threshold $\bar{x}_i$

$$
\begin{aligned}
PD &= P(X_i < \bar{x}_i) \\
&\equiv \int_{-\infty}^{\infty} 1(X_i < \bar{x}_i) d\phi(x) \\
&= (\phi(x_1) - \phi(-\infty)) + (\phi(x_2) - \phi(x_1)) + \cdots + (\phi(\bar{x}_i) - \phi(x_l)) \\
&= \phi_{\text{normal}}(\bar{x}_i)
\end{aligned}
$$

Rieman-Stieltjes integral: like normal integral, but with $d\Phi_{CDF}$ instead of dx

Now fix $M = m$. $X_i \sim N(0,1) \rightarrow N\left(\sqrt{\rho}m, 1 - \rho\right)$

$$
\begin{aligned}
\textit{Conditional PD} &= P\left(\sqrt{\rho}m + \sqrt{1-\rho}Z_i < \bar{x}_i \,|\, M = m\right) \\
&= P\left(Z_i < \frac{\bar{x}_i - \sqrt{\rho}m}{\sqrt{1-\rho}} \,|\, M = m\right) \\
&= \phi_{\text{normal}}\left(\frac{\bar{x}_i - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)
\end{aligned}
$$

Risk Weighted Assets

Average $P\left(X_i < \bar{x}_i \,|\, M = m\right)$ over all values of m.

$$\mathbb{E}_m\left(P(X_i < \bar{x}_i \,|\, M = m)\right) = \mathbb{E}_m\left(\phi\left(\frac{\bar{x}_i - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)\right)$$

$$= \int_{-\infty}^{\infty} \phi\left(\frac{\bar{x}_i - \sqrt{\rho}m}{\sqrt{1-\rho}}\right)\phi'(m)dm$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{\bar{x}_i - \sqrt{\rho}m}{\sqrt{1-\rho}}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\frac{e^{-m^2/2}}{\sqrt{2\pi}}dydm$$

$$= \phi(\bar{x}_i)$$

$$= PD$$

Risk Weighted Assets

Let's define

$$g(m) = P(X_i < \bar{x}_i | M = m)) - E_m(P(X_i < \bar{x}_i | M = m))$$
$$= \phi\left(\frac{\bar{x}_i - \sqrt{\rho}m}{\sqrt{1-\rho}}\right) - \phi(\bar{x}_i)$$

Since $M \sim N(0,1)$, it is true that in 99% of scenarios, $M > \phi^{-1}(0.01)$.
Since ϕ is monotonic, we have in 99% of scenarios:

$$g(m) < g(\phi^{-1}(0.01))$$
$$\equiv WCDR - \phi(\bar{x}_i)$$
$$\equiv WCDR - PD$$

Risk Weighted Assets

$$RWA = 12.5 \times EAD \times (WCDR - PD) \times LGD \times MA$$

Insights

- The statement "RWA is the unexpected loss" comes from the $WCDR$ term.
- $IMM\ EAD$ and PD are averaged over *all* scenarios, not just tail scenarios

Bonus!
Generalizing Results

Generalizations

Generalization #1

- Extend PD to full joint probability of X_i across all counterparties: $P_\phi(X_1 < \phi^{-1}(u_1), \ldots, X_n < \phi_n^{-1}(u_n))$

Full joint probability is complicated! However conditional on $M = m$, the various $X_i(m)$ are now independent. Hence, we can write:

$$
\begin{aligned}
&P_\phi(X_1 < \phi^{-1}(u_1), \ldots, X_n < \phi_n^{-1}(u_n)) \\
&= \int_{-\infty}^{\infty} P_\phi(X_1 < \phi^{-1}(u_1), \ldots, X_n < \phi_n^{-1}(u_n) \,|\, M = m) d\phi(m) \\
&= \int_{-\infty}^{\infty} \prod_i^n P_{\phi_i}(X_i < \phi^{-1}(u_i) \,|\, M = m)) d\phi(m)
\end{aligned}
$$

Generalizations

Generalization #2

- Relate joint CDF of X_i to the joint CDF of t_i, where t_i is the time (random variable) until counterparty i defaults.

Introduce copulas (Sklar's theorem): Given a d-dimensional continuous CDF F_n with marginals $F_1, \cdots, F_n$, there exists a unique copula function C, such that $F_N(t_1, \ldots, t_n) = C(F_1(t_1), \ldots, F_n(t_n))$.

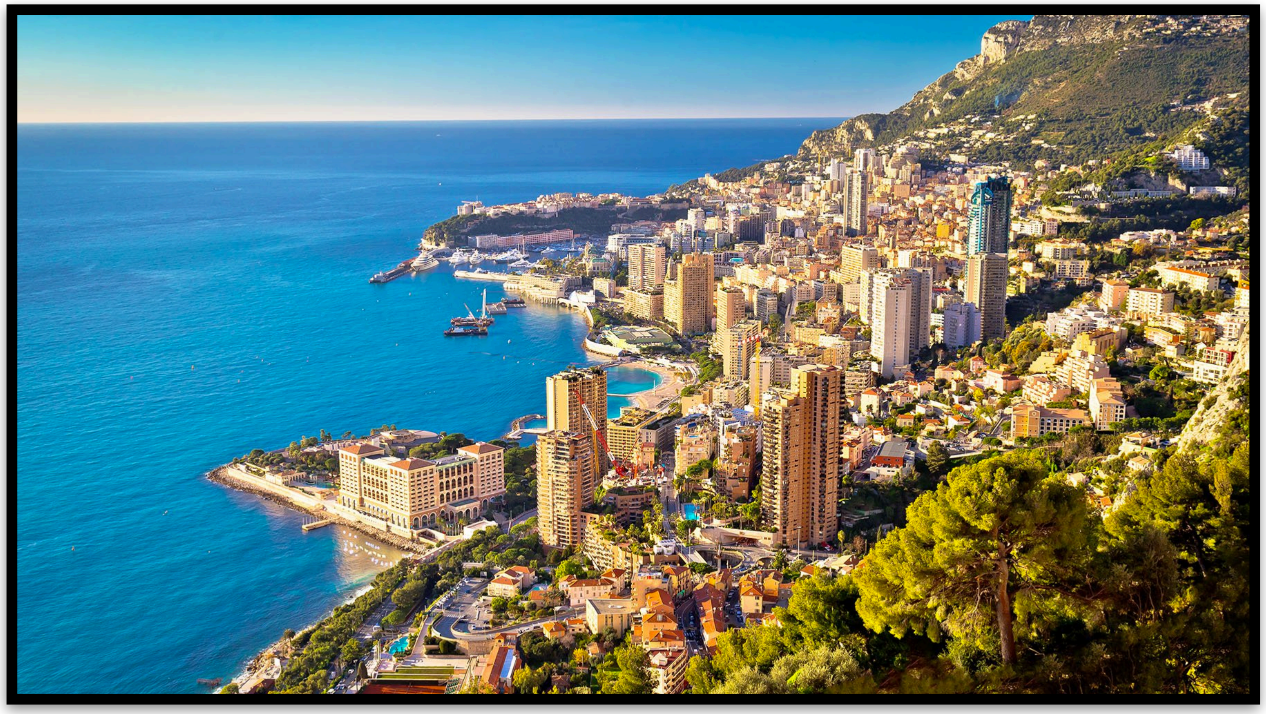
Generalizations

Generalization #2

- Relate joint CDF of X_i to the joint CDF of t_i, where t_i is the time (random variable) until counterparty i defaults.

Assume t_i copula are equal to X_i copula:

$$\begin{aligned}
P_F(T_1 < t_1, \ldots, T_n < t_n) &\equiv F_N(t_1, \ldots, t_n) \\
&= F_N(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)) \\
&= C_F(u_1, \ldots, u_n) \\
&= C_\phi(u_1, \ldots, u_n) \\
&= \phi_N(\phi_1^{-1}(u_1), \ldots, \phi_n^{-1}(u_n)) \\
&\equiv P_\phi(X_1 < \phi^{-1}(u_1), \ldots, X_n < \phi_n^{-1}(u_n))
\end{aligned}$$



Thank you!

Questions?