Analytic Results for Expectation of Loss and Default

Gabriel Magill

2020-12-08

1 Motivation

We want to explore the ideas behind asserting that risk averaged losses when a counterparty owes you money is

$$ECL = EAD \cdot LGD \cdot PD. \tag{1}$$

In this note, we consider the situation when the money that is owed is a constant, such as an fixed outstanding loan. In this case, PD and LGD are random variables, but EAD is not. So in the rest of this document, we set EAD=1.

This note will answer the following questions:

- How can we obtain this formula from a statistical framework?
- Did we account for the correlations between losses and default in the ECL formula?

2 Correlating Defaults and Loss

2.1 Setup

Let $D = \{0, 1\}$ be an indicator function denoting whether a counterparty has defaulted between now and future time T. Let $0 \leq L \leq 1$ be the loss incurred during the counterparty default in terms of our exposure at default [1]. By convention, for a non-defaulted counterparty, $L \equiv 0$. Both L and D are random variables. Given a fixed T, we are interested in calculating ECL $\equiv E(L \cdot D)$ (EAD=1 and a constant).

[1] It is actually possible that $L > 1$, meaning we lose more money than what the defaulting counterparty owes us (lawyer fees, etc).

Question: What is meant by $E(L \cdot D)$?

Before continuing, we need to be precise about what is meant by this expectation. L and D apply to a single counterparty. The frequentist approach would have us draw many samples of L and D. But how can we randomly sample the *same* counterparty defaulting? Here are three ways to think about this:

1. Philosophical Definition: Consider a multiverse where there are various copies of planet Earth. Measure L and D for the same counterparty in each version of Earth. Take the average.
2. Simulated Definition: Run a Monte Carlo simulation to generate multiple future scenarios of the same counterparty. For example one could simulate fluctuations in asset prices and apply the Merton Model.
3. Historical Cluster Definition: Consider a collection of distinct historical counterparties within a similar segment (geography, industry, etc). Within this segment, we assume that these different counterparties are similar enough that they can be thought of as various realizations of a single hypothetical counterparty. Measure L and D for each counterparty over a historical period of time.

Let us denote

- N_D as the number of defaulted counterparties
- N_{ND} as the number of non-defaulted counterparties
- $N = N_D + N_{ND}$ as the total number of counterparties
- $\bar{D} \equiv \text{PD} = \sum D_i/N = N_D/N$ as the probability of default between time 0 and T
- $\text{LGD} = \sum_{\text{defaulted cpty}} L_i/N_D$ as the mean of L over only the defaulted counterparties
- $\bar{L} = \sum L_i/N$ as the mean of L over all counterparties. Since $L = 0$ when $D = 0$, we can write $\bar{L} = \sum_{\text{defaulted}} L_i/N = \text{LGD} \cdot N_D/N = \text{LGD} \cdot \text{PD}$.

2.2 The Easy Way

Using the law of total expectation $E(X) = E(E(X|Y))$, we can write:

$$\begin{aligned} E(L \cdot D) &= E(L \cdot D|\text{default})P(\text{default}) + E(L \cdot D|\text{non-default})P(\text{non-default}) \\ &= E(L|\text{default})P(\text{default}) \\ &= LGD \cdot PD \end{aligned} \tag{2}$$

We obtain the same factors that go into the ECL formula.

Let's define an event H as $Y = y$. Conditional expectations of X given an event H is the average of X over only the scenarios s in which we had H. Namely,

$$E(X|H) = \frac{\sum_{s\in H} X(s)}{N_{s\in H}}. \tag{3}$$

2.3 The Hard Way

Let's try to evaluate $E(L \cdot D)$ through the lens of the correlation between L and D. Using the definition of covariance, we can generally write:

$$E(L \cdot D) = cov(L, D) + E(L) \cdot E(D) \tag{4}$$

The (biased) covariance is given as follows:

$$\begin{aligned}
cov(L, D) &= \frac{1}{N} \sum (L_i - \bar{L})(D_i - \bar{D}) \\
&= \frac{1}{N} \sum_{\text{defaulted cpty}} (L_i - \bar{L})(D_i - \bar{D}) + \frac{1}{N} \sum_{\text{non-defaulted cpty}} (L_i - \bar{L})(D_i - \bar{D}) \\
&= \frac{1}{N} \sum_{\text{defaulted cpty}} (L_i - \bar{L})(1 - \bar{D}) + \frac{1}{N} \sum_{\text{non-defaulted cpty}} (0 - \bar{L})(0 - \bar{D}) \\
&= (1 - \bar{D}) \cdot \frac{N_D}{N}(\text{LGD} - \bar{L}) + \bar{L} \cdot \bar{D} \cdot \frac{N_{ND}}{N} \\
&= \frac{N_D}{N} \cdot (1 - \text{PD}) \cdot (\text{LGD} - \bar{L}) + \frac{N_{ND}}{N} \cdot \text{PD} \cdot \bar{L} \\
&= \frac{N_D N_{\text{ND}}}{N^2} \cdot (1 - \text{PD}) \cdot \text{LGD} + \frac{N_D N_{ND}}{N^2} \cdot \text{PD} \cdot \text{LGD} \\
&= \text{LGD} \frac{N_D N_{\text{ND}}}{N^2} \\
&= \text{LGD} \cdot \text{PD} \cdot (1 - \text{PD}) \\
&= \text{LGD} \cdot var(D)
\end{aligned} \tag{5}$$

In the last line, we used the fact that $D^2 = D$ to write $var(D) = E(D) - E(D)^2$. Using eq. 5, we can now write our original expression as:

$$\begin{aligned} E(L \cdot D) &= cov(L, D) + E(L) \cdot E(D) \\ &= cov(L, D) + \bar{L} \cdot \bar{D} \\ &= \mathrm{LGD} \cdot \mathrm{PD}(1 - \mathrm{PD}) + \mathrm{LGD} \cdot \mathrm{PD}^2 \\ &= \mathrm{LGD} \cdot \mathrm{PD} \end{aligned} \tag{6}$$

We obtain the same result, as we should.

3 Conclusion

Given that the two derivations obtained the same result, we conclude that writing

$$ECL = EAD \cdot LGD \cdot PD \tag{7}$$

is actually capturing the correlations between L and D!

Although it is tempting to think of ECL as calculating $E(L) \cdot E(D)$ without correlations, this is wrong since $E(L) \cdot E(D) \propto \mathrm{PD}^2$. The effect of including correlations in the ECL calculation effectively corrects the PD^2 scaling to a PD scaling.