PROBABILITY

GABRIEL MOISE

PROBLEM SHEET 6

1\. (a) Let X be a constant random variable, say $\mathbb{P}(X=a)=1$ for some $a \in \mathbb{N}$.

Then, $G_X(s) = \sum_{k=0}^{\infty} s^k \mathbb{P}(X=k) \Rightarrow \boxed{G_X(s) = s^a}$

(b) Let $G_Y(s)$ be the p.g.f of Y, and $m, n \in \mathbb{N}_+$. Let $Z = mY + n$, then

$$\underline{G_Z(s)} = \mathbb{E}(s^Z) = \mathbb{E}(s^{mY+n}) = \mathbb{E}((s^m)^Y \cdot s^n) = s^n \cdot \mathbb{E}((s^m)^Y) = \underline{s^n \cdot G_Y(s^m)}.$$

2\. (a) We perform a sequence of independent trials, each of which has probability p of success. Y denotes the number of trials until we get the m^{th} success, where $m \geq 1$ fixed.

Now, let X_i denote the number of trials until we get a success, where $i \in \{1,2,\dots,m\}$. Therefore, $X_1, X_2, \dots, X_m$ are independent and identically distributed random variables with $X_i \sim \text{Geom}(p)$, for all $i \in \{1,2,\dots,m\}$. It follows that $Y = X_1 + X_2 + \dots + X_m$.

So, we have:

$\mathbb{P}(Y=k) = \mathbb{P}(X_1 + X_2 + \dots + X_m = k) = \mathbb{P}(\{X_1 = k_1\} \cap \{X_2 = k_2\} \cap \dots \cap \{X_m = k_m\})$, with $k_1 + k_2 + \dots + k_m = k$, where $k_1, k_2, \dots, k_m \geq 1$ (a geometric distribution always starts from $k=1,2,\dots$). Now, by fixing the $k_1, k_2, \dots, k_m$ we see that $\{X_1 = k_1\} \cap \{X_2 = k_2\} \cap \dots \cap \{X_m = k_m\}$ is independent of $\{k_1 + k_2 + \dots + k_m = k\}$,

so $\mathbb{P}(Y=k) = \mathbb{P}(\{X_1=k_1\} \cap \{X_2=k_2\} \cap \dots \cap \{X_m=k_m\}) \underbrace{N_0}_{\text{number of ways of obtaining this equality}}(k_1+k_2+\dots+k_m=k)$ (*)

- First, $\mathbb{P}(\{X_1=k_1\} \cap \{X_2=k_2\} \cap \dots \cap \{X_m=k_m\}) = \mathbb{P}(X_1=k_1)\mathbb{P}(X_2=k_2) \cdot \ldots \cdot \mathbb{P}(X_m=k_m)$ (we already said that $X_1, X_2, \dots, X_m$ are independent) $= p(1-p)^{k_1-1} \cdot p(1-p)^{k_2-1} \cdot \ldots \cdot p(1-p)^{k_m-1} = p^m (1-p)^{k_1+k_2+\dots+k_m-m}$,

so $\underline{\mathbb{P}(\{X_1=k_1\} \cap \{X_2=k_2\} \cap \dots \cap \{X_m=k_m\}) = p^m(1-p)^{k-m}}$. (1)

- Second, $N_0(k_1+k_2+\dots+k_m=k)$. We have m boxes and k balls, each of them has at least 1 ball, so by putting one in each, we are left with $k-m$ balls. The ways of doing that is $\binom{m-1}{k-m+m-1} = \binom{m-1}{k-1}$ (same reasoning with $k-m$ balls and $m-1$ bars between them). Therefore,

$\underline{N_0(k_1+k_2+\dots+k_m=k) = \binom{m-1}{k-1}}$ (2)

From (1) and (2) we conclude that $\underline{\mathbb{P}(Y=k) = \binom{m-1}{k-1} p^m (1-p)^{k-m}}$, obviously for $k \geq m$ (for each X_i we need at least $k_i \geq 1$), which proves (*)

(This is called the "negative binomial" distribution)

(b) As we said at (a), $Y = X_1 + X_2 + \dots + X_m$, where $X_1, X_2, \dots, X_m$ are i.i.d. r.vs. with $X_i \sim \text{Geom}(p)$ for all $i \in \{1,2,\dots,m\} \Rightarrow G_{X_i}(s) = \frac{ps}{1-(1-p)s}$

$G_Y(s) = \mathbb{E}(s^Y) = \mathbb{E}(s^{X_1+X_2+\dots+X_m}) = \mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2}) \cdot \ldots \cdot \mathbb{E}(s^{X_m}) = (\mathbb{E}(s^{X_1}))^m = (G_{X_1}(s))^m$, so

$$G_Y(s) = \left(\frac{ps}{1-s+ps}\right)^m.$$

3. Let $X_1, X_2, \dots$ be a sequence of i.i.d. non-negative integer valued r.vs., and let N be a non-negative integer valued r.v., which is independent of the sequence $X_1, X_2, \dots$

Let $Z = X_1 + X_2 + \dots + X_N$ (if $N=0$, then $Z=0$)

(a) We want to show that

$$\mathbb{E}(Z) = \mathbb{E}(N)\,\mathbb{E}(X_1)$$

$$\underline{\mathbb{E}(Z)} = \mathbb{E}\left(X_1+X_2+\dots+X_N\right) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\,\mathbb{E}\left(X_1+X_2+\dots+X_N \mid N=n\right) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\,\mathbb{E}\left(X_1+\dots+X_n\right) =$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n} \mathbb{E}(X_i) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\, n\, \mathbb{E}(X_1) = \mathbb{E}(X_1) \sum_{n=0}^{\infty} n\,\mathbb{P}(N=n) = \underline{\mathbb{E}(X_1)\,\mathbb{E}(N)}$$

TO USE:
independence of the sequence ①
identically distributed r.vs. ②

Now we want to show that

$$\mathrm{Var}(Z) = \mathrm{var}(N)\left(\mathbb{E}(X_1)\right)^2 + \mathbb{E}(N)\,\mathrm{Var}(X_1)$$

$$\mathrm{Var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}^2(Z)$$

From above, we know that $\mathbb{E}(Z) = \mathbb{E}(N)\,\mathbb{E}(X_1) \Rightarrow \underline{\mathbb{E}^2(Z) = \mathbb{E}^2(N)\,\mathbb{E}^2(X_1)}$ (A)

Now, we'll calculate $\mathbb{E}(Z^2)$:

$$\underline{\mathbb{E}(Z^2)} = \mathbb{E}\left(\left(X_1+X_2+\dots+X_N\right)^2\right) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\,\mathbb{E}\left(\left(X_1+X_2+\dots+X_N\right)^2 \mid N=n\right) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\,\mathbb{E}\left(\left(X_1+X_2+\dots+X_n\right)^2\right) =$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(N=n)\,\mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right) = \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}(X_i X_j) = \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n} \mathbb{E}(X_i^2) +$$

$$+ \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \mathbb{E}(X_i X_j) \overset{①}{=} \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \mathbb{E}(X_i)\,\mathbb{E}(X_j) \overset{②}{=}$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n} \mathbb{E}(X_1^2) + \sum_{n=0}^{\infty} \mathbb{P}(N=n) \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \mathbb{E}^2(X_1) = \sum_{n=0}^{\infty} \mathbb{P}(N=n)\cdot n\,\mathbb{E}(X_1^2) +$$

$$+ \sum_{n=0}^{\infty} \mathbb{P}(N=n)\cdot n(n-1)\,\mathbb{E}^2(X_1) = \sum_{n=0}^{\infty} n\,\mathbb{P}(N=n)\,\mathbb{E}(X_1^2) + \sum_{n=0}^{\infty} n^2\,\mathbb{P}(N=n)\,\mathbb{E}^2(X_1) - \sum_{n=0}^{\infty} n\,\mathbb{P}(N=n)\,\mathbb{E}^2(X_1) =$$

$$= \underline{\mathbb{E}(X_1^2)\,\mathbb{E}(N) + \mathbb{E}(N^2)\,\mathbb{E}^2(X_1) - \mathbb{E}(N)\,\mathbb{E}^2(X_1)} \quad (B)$$

From (A) and (B) we obtain:

$$\underline{\mathrm{Var}(Z)} = \mathbb{E}(Z^2) - \mathbb{E}^2(Z) = \mathbb{E}(X_1^2)\,\mathbb{E}(N) + \mathbb{E}(N^2)\,\mathbb{E}^2(X_1) - \mathbb{E}(N)\,\mathbb{E}^2(X_1) - \mathbb{E}^2(N)\,\mathbb{E}^2(X_1) =$$

$$= \mathbb{E}^2(X_1)\left(\mathbb{E}(N^2) - \mathbb{E}^2(N)\right) + \mathbb{E}(N)\left(\mathbb{E}(X_1^2) - \mathbb{E}^2(X_1)\right) = \underline{\mathbb{E}^2(X_1)\,\mathrm{Var}(N) + \mathbb{E}(N)\,\mathrm{Var}(X_1)} \quad \square.$$

(b) We know that $N \sim Po(\lambda)$ and $X_1 \sim Bin(p) \Rightarrow \mathbb{E}(X_1) = p$, $\mathrm{Var}(X_1) = p(1-p)$

$\mathbb{E}(N) = \lambda$, $\mathrm{Var}(N) = \lambda \quad \Big| \Rightarrow$

$$\Rightarrow \mathrm{Var}(Z) = \mathbb{E}^2(X_1)\,\mathrm{Var}(N) + \mathbb{E}(N)\,\mathrm{Var}(X_1) = p^2\lambda + \lambda p(1-p) = \cancel{p^2\lambda} + \lambda p - \cancel{\lambda p^2} \Rightarrow$$

$$\Rightarrow \boxed{\mathrm{Var}(Z) = \lambda p}$$

(c) Now we remove the condition that N is independent of the sequence (X_i).

Let $X_i \sim Ber(p)$ and $N = X_1 + X_2$. Then, $Z = X_1 + X_2 + \dots + X_{X_1+X_2}$

As $X_1, X_2 \sim Ber(p)$ and they are independent and identically distributed, we have

$\mathbb{E}(N) = \mathbb{E}(X_1+X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 2p$

$\mathbb{P}(X_1+X_2=0) = (1-p)^2$

$\mathbb{P}(X_1+X_2=1) = 2p(1-p)$ and $\mathbb{P}(X_1+X_2=m) = 0$ for all $m>2$.

$\mathbb{P}(X_1+X_2=2) = p^2$

Now, $\mathbb{E}(Z) = \mathbb{E}(X_1+X_2+\dots+X_{X_1+X_2}) = \sum_{n=0}^{\infty} \mathbb{P}(X_1+X_2=m)\, \mathbb{E}\left(X_1+X_2+\dots+X_{X_1+X_2} \mid X_1+X_2=m\right)$

$= (1-p)^2 \cdot 0 + \underline{2p(1-p)\,\mathbb{E}(X_1 \mid X_1+X_2=1)} + p^2 \cdot 2$

$\mathbb{E}(X_1) = \mathbb{E}(X_1 \mid X_1+X_2=0)\mathbb{P}(X_1+X_2=0) + \mathbb{E}(X_1 \mid X_1+X_2=1)\mathbb{P}(X_1+X_2=1) + \mathbb{E}(X_1 \mid X_1+X_2=2)\mathbb{P}(X_1+X_2=2) \Rightarrow$

$\Rightarrow p = 0\cdot(1-p)^2 + \underline{\mathbb{E}(X_1 \mid X_1+X_2=1)\cdot 2p(1-p)} + p^2 \Rightarrow \mathbb{E}(X_1 \mid X_1+X_2=1)\,2p(1-p) = p-p^2$, so

$\mathbb{E}(Z) = p-p^2+2p^2 \Rightarrow \mathbb{E}(Z) = p+p^2$

$\mathbb{E}(N)\mathbb{E}(X_1) = 2p\cdot p \Rightarrow \mathbb{E}(N)\mathbb{E}(X_1) = 2p^2$

Supposing that $\mathbb{E}(Z) = \mathbb{E}(N)\mathbb{E}(X_1)$

$\Rightarrow p+p^2 = 2p^2$

$p = p^2 \Rightarrow p(p-1) = 0 \Rightarrow p=0$ or $p=1$, which is not always true (we could have $p=\frac{1}{3}$ for example). Therefore, N needs to be independent of the sequence (X_i) for the $\mathbb{E}(Z) = \mathbb{E}(X_1)\mathbb{E}(N)$ to be true.

4. $G_X(s) = \sum_{k=0}^{\infty} s^k p_k$, where $p_k = \mathbb{P}(X=k)$.

We want to calculate $\mathbb{P}(X \text{ even}) = \sum_{k=0}^{\infty} \mathbb{P}(X=2k) = p_0+p_2+p_4+\dots$

$G_X(1) = \sum_{k=0}^{\infty} p_k$

$G_X(-1) = \sum_{k=0}^{\infty} (-1)^k p_k$

$\Rightarrow G_X(1) + G_X(-1) = \sum_{k=0}^{\infty} p_k + (-1)^k p_k$

$G_X(1) + G_X(-1) = \sum_{a=0}^{\infty} p_{2a} + (-1)^{2a} p_{2a} + p_{2a+1} + (-1)^{2a+1} p_{2a+1}$

$G_X(1) + G_X(-1) = \sum_{a=0}^{\infty} 2p_{2a}$

$G_X(1) + G_X(-1) = 2\mathbb{P}(X \text{ is even}) \Rightarrow \boxed{\mathbb{P}(X \text{ is even}) = \frac{G_X(-1)+G_X(1)}{2}}$

<u>Possible extension:</u>

$G_X(1) = \sum_{k=0}^{\infty} p_k = \sum_{a=0}^{\infty} p_{4a} + p_{4a+1} + p_{4a+2} + p_{4a+3}$

$G_X(-1) = \sum_{k=0}^{\infty} (-1)^k p_k = \sum_{a=0}^{\infty} p_{4a} - p_{4a+1} + p_{4a+2} - p_{4a+3}$

Now, we also have

$$G_X(i) = \sum_{k=0}^{\infty} i^k p_k = \sum_{a=0}^{\infty} p_{4a} + i p_{4a+1} - p_{4a+2} - i p_{4a+3}$$

$$G_X(-i) = \sum_{k=0}^{\infty} (-i)^k p_k = \sum_{a=0}^{\infty} p_{4a} - i p_{4a+1} - p_{4a+2} + i p_{4a+3}$$

By adding them, we get:

$$G_X(1) + G_X(-1) + G_X(i) + G_X(-i) = \sum_{a=0}^{\infty} 4 p_{4a} \Rightarrow \boxed{P(X \text{ is divisible by } 4) = \frac{G_X(1) + G_X(-1) + G_X(i) + G_X(-i)}{4}}$$

5. $P(\text{success}) = \frac{1}{4} \Rightarrow 2$ cells

$P(\text{death}) = \frac{1}{12} \Rightarrow 0$ cells (each minute)

$P(\text{nothing}) = \frac{2}{3} \Rightarrow 1$ cell

We begin with a single cell $\Rightarrow X_0 = 1$

After 1 minute, we have a p.g.f $G_1(s) = G(G_0(s))$, where $G_0(s) = s$, so

$$G_1(s) = G(s) = \sum_{i=0}^{\infty} p(i) s^i = \frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2. \text{ Therefore,}$$

$$G_2(s) = G(G(s)) = \sum_{i=0}^{\infty} p(i)\left(\frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2\right)^i = \frac{1}{12} + \frac{2}{3}\left(\frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2\right) + \frac{1}{4}\left(\frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2\right)^2$$

$$G_2(s) = \frac{1}{12} + \frac{1}{18} + \frac{4}{9}s + \frac{1}{6}s^2 + \frac{1}{4}\left(\frac{1}{144} + \frac{4}{9}s^2 + \frac{1}{16}s^4 + \frac{1}{9}s + \frac{1}{24}s^2 + \frac{1}{3}s^3\right)$$

$$G_2(s) = \left(\frac{1}{12} + \frac{1}{18} + \frac{1}{576}\right) + \left(\frac{4}{9} + \frac{1}{36}\right)s + \left(\frac{1}{6} + \frac{1}{9} + \frac{1}{96}\right)s^2 + \frac{1}{12}s^3 + \frac{1}{64}s^4$$

$$G_2(s) = \frac{81}{576} + \frac{17}{36}s + \frac{83}{288}s^2 + \frac{1}{12}s^3 + \frac{1}{64}s^4$$

$$P(X_2 = 0) = G_2(0) = \frac{81}{576} = \frac{9}{64}$$

6. $p(2) = p$

$p(0) = 1-p$

X_n = the size of the n^{th} generation

$X_0 = 1$

(a) $$\mu = \sum_{i=0}^{\infty} i p(i) = 0p(0) + 2p(2) \Rightarrow \boxed{\mu = 2p}$$

$$G(s) = \sum_{i=0}^{\infty} s^i p(i) = (1-p)s^0 + 0 \cdot s^1 + p \cdot s^2 \Rightarrow \boxed{G(s) = (1-p) + ps^2}$$

(b) From Theorem 4.14., the extinction probability q is the smallest non-negative solution of $x = G(x)$, which is $x = (1-p) + px^2 \Rightarrow px^2 - x + (1-p) = 0$

$$\Delta = 1 - 4p(1-p) = 1 - 4p + 4p^2 = (2p-1)^2 \Rightarrow x_{1/2} = \frac{1 \pm 2p - 1}{2p}$$

So, $x_1 = 1$ and $x_2 = \frac{1-p}{p}$

if $1 \leq \frac{1-p}{p}$, or $p \leq 1-p$
$2p \leq 1$
$\mu \leq 1$, then $\boxed{q=1}$, so we conclude that the population will surely die out.

if $1 > \frac{1-p}{p}$, or $p > 1-p$
$2p > 1$
$\mu > 1$, then $\boxed{q = \frac{1-p}{p}}$, so the probability that the process survives

for ever is positive ($1-q>0$, as $q<1$) if and only if $\mu>1$. So, $q = \min\left\{1, \frac{1-p}{p}\right\}$

(c) Let $\beta_n = \mathbb{P}(X_n>0)$, the probability that the process survives for at least n generations.

For $p=\frac{1}{2}$, we have

$G(s) = \left(1-\frac{1}{2}\right) + \frac{1}{2}s^2 = \frac{1}{2} + \frac{1}{2}s^2$

We have $\beta_n = \mathbb{P}(X_n>0) \Rightarrow 1-\beta_n = \mathbb{P}(X_n=0) \Rightarrow G_n(0) = 1-\beta_n \Rightarrow \boxed{\beta_n = 1 - G_n(0)}$

$\beta_n = 1 - G_n(0) \overset{\text{Th. 4.11}}{=} 1 - G(G_{n-1}(0)) = 1 - G(1-\beta_{n-1}) = 1 - \frac{1}{2} - \frac{1}{2}(1-\beta_{n-1})^2$

$\beta_n = \frac{1}{2} - \frac{1}{2}(1-2\beta_{n-1}+\beta_{n-1}^2) = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \beta_{n-1} - \frac{1}{2}\beta_{n-1}^2$

$\boxed{\beta_n = \beta_{n-1} - \frac{1}{2}\beta_{n-1}^2}$

Now, we want to show that $\frac{1}{n+1} \leq \beta_n \leq \frac{2}{n+2}$, for all n. We will do that by induction on n.

Base case: $S(1)$: $\frac{1}{2} \leq \beta_1 \leq \frac{2}{3}$, with $\beta_1 = \frac{1}{2}$, which is true.

Inductive step

Inductive Hypothesis: We know that $S(n)$ is true, and we'll show that $S(n+1)$ is also true.

First, $\beta_{n+1} \overset{2}{=} \beta_n - \frac{\beta_n^2}{2} = \frac{2\beta_n - \beta_n^2}{2} = \frac{1-1+2\beta_n-\beta_n^2}{2} = \frac{1-(1-\beta_n)^2}{2}$ (*)

Now, from IH:

$$\frac{1}{n+1} \leq \beta_n \leq \frac{2}{n+2} \quad \Big| \cdot (-1)$$

$$-\frac{2}{n+2} \leq -\beta_n \leq -\frac{1}{n+1} \quad \Big| +1$$

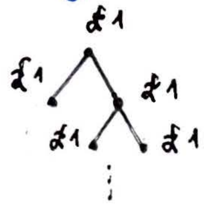
$$1-\frac{2}{n+2} \leq 1-\beta_n \leq 1-\frac{1}{n+1} \quad \Big| ()^2$$

$$\left(1-\frac{2}{n+2}\right)^2 \leq (1-\beta_n)^2 \leq \left(1-\frac{1}{n+1}\right)^2$$

(Continuing on page 6., at the end).

(d) In the gambler's ruin model, we start from £1. Then, after each step we can go up with £1 or down £1. Let's link that to ~~our~~ branching process.

By starting with 1, we can go and have 2 children (so £2) or 0 children (so £0), then we lose as the population has died out. By going to 2, we now have two games of gambler's ruin, each starting from £1, and we need to know at what generation they are dead. Therefore, as $q = \begin{cases} 1, & \text{if } p \le \frac{1}{2} \\ \frac{1-p}{p}, & \text{if } p > \frac{1}{2} \end{cases}$, then, by linking the gambler's ruin model to it, the probability that we ever hit 0 is the same.



(visualizing the branching process, but with money instead of population)

(c) $\left(1-\frac{2}{n+2}\right)^2 \le (1-\beta_n)^2 \le \left(1-\frac{1}{n+1}\right)^2 \quad \Big| \cdot (-1)$

$-\left(1-\frac{1}{n+1}\right)^2 \le -(1-\beta_n)^2 \le -\left(1-\frac{2}{n+2}\right)^2 \quad \Big| +1$

$1-\left(1-\frac{1}{n+1}\right)^2 \le 1-(1-\beta_n)^2 \le 1-\left(1-\frac{2}{n+2}\right)^2 \quad \Big| \cdot \frac{1}{2}$

$\frac{1-\left(1-\frac{1}{n+1}\right)^2}{2} \le \frac{1-(1-\beta_n)^2}{2} \le \frac{1-\left(1-\frac{2}{n+2}\right)^2}{2}$ (now we use (*))

$\frac{1-\left(1-\frac{1}{n+1}\right)^2}{2} \le \beta_{n+1} \le \frac{1-\left(1-\frac{2}{n+2}\right)^2}{2}$

CLAIM 1: $\frac{1-\left(1-\frac{1}{n+1}\right)^2}{2} \geq \frac{1}{n+2}$, for all $n \in \mathbb{N}$

PROOF 1: $1-\left(1-\frac{2}{n+1}+\frac{1}{(n+1)^2}\right) \geq \frac{2}{n+2}$

$\frac{2(n+1)-1}{(n+1)^2} \geq \frac{2}{n+2}$

$(2n+1)(n+2) \geq 2(n+1)^2 = 2(n^2+2n+1)$

$\cancel{2n^2}+5n+\cancel{2} \geq \cancel{2n^2}+4n+\cancel{2}$

$5n \geq 4n$

$5 \geq 4$ YES

CLAIM 2: $\frac{1-\left(1-\frac{2}{n+2}\right)^2}{2} \le \frac{2}{n+3}$

PROOF 2:

$1-\left(1-\frac{4}{n+2}+\frac{4}{(n+2)^2}\right) \le \frac{4}{n+3}$

$\frac{4(n+2)-4}{(n+2)^2} \le \frac{4}{n+3} \quad \Big| :4$

$$\frac{n+2-1}{(n+2)^2} \leq \frac{1}{n+3}$$

$$(n+1)(n+3) \leq (n+2)^2$$

$$\cancel{n^2} + \cancel{4n} + 3 \leq \cancel{n^2} + \cancel{4n} + 4$$

$3 \leq 4$ YES

Coming back to $\dfrac{1-\left(1-\frac{1}{n+1}\right)^2}{2} \leq \beta_{n+1} \leq \dfrac{1-\left(1-\frac{2}{n+2}\right)^2}{2}$ and using CLAIM 1 and CLAIM 2, we get $\dfrac{1}{n+2} \leq \dfrac{1-\left(1-\frac{1}{n+1}\right)^2}{2} \leq \beta_{n+1} \leq \dfrac{1-\left(1-\frac{2}{n+2}\right)^2}{2} \leq \dfrac{2}{n+3}$, so

$\dfrac{1}{n+2} \leq \beta_{n+1} \leq \dfrac{2}{n+3}$, which is exactly $S(n+1)$.

Therefore, the induction is complete, so we can confirm that

$\dfrac{1}{n+1} \leq \beta_n \leq \dfrac{2}{n+2}$, for all $n \geq 1$.