LINEAR ALGEBRA MT2018

GABRIEL MOISE

WEEK 7

1. Let λ be an eigenvalue of A.

(i) Let $k \geq 2$. We'll prove that λ^k is an eigenvalue of A^k.

λ eigenvalue of $A \Rightarrow (\exists)\, u$ a non-zero vector such that $Au = \lambda u$

$$Au = \lambda u \quad | \cdot \lambda$$

$$\lambda A u = \lambda^2 u$$

$$A(\lambda u) = \lambda^2 u$$

$$A \cdot A \cdot u = \lambda^2 u$$

$$A^2 u = \lambda^2 u$$

By using induction:

<u>Base case</u>: $S(2): A^2 u = \lambda^2 u$

<u>Inductive step</u>: $S(k) \Rightarrow S(k+1)$

$$A^k u = \lambda^k u \Rightarrow A^k \lambda u = \lambda^{k+1} u \Rightarrow A^{k+1} u = \lambda^{k+1} u \Rightarrow \text{OK.}$$

(ii) A is invertible.

$$A^{-1} \cdot | \; Au = \lambda u$$

$$u = A^{-1} \lambda u \quad | : \lambda$$

$$\frac{1}{\lambda} u = A^{-1} u \Rightarrow \frac{1}{\lambda} \text{ is an eigenvalue of } A^{-1}$$

(iii) $(\forall)\, \alpha$ scalar, $\lambda + \alpha$ is an eigenvalue of $A + \alpha I$

$$(A + \alpha I) u = (\lambda + \alpha) u$$

$$Au + \cancel{\alpha I u} = \lambda u + \cancel{\alpha u}$$

$$Au = \lambda u \quad \text{OK.}$$

2. $B = A^T A$

(a) $B = A^T A \Rightarrow B^T = (A^T A)^T = A^T (A^T)^T = A^T A \Rightarrow B^T = B$

Suppose λ and μ are (possibly complex) eigenvalues of B. We then have

$$Bv = \lambda v \text{ and } Bw = \mu w \text{ for eigenvectors } v \text{ and } w.$$

We may then write

$$\mu v^T w = v^T(\mu w) = v^T B w = v^T B^T w = (Bv)^T w = \lambda v^T w \Rightarrow$$

$$\Rightarrow (\mu - \lambda) v^T w = 0$$

$$(\mu - \lambda) v \cdot w = 0$$

Now, suppose λ is complex. We then have

eigenvector $\Rightarrow$ non-zero

$$Bv = \lambda v, \; B\bar{v} = \bar{\lambda}\bar{v} \Rightarrow (\bar{\lambda} - \lambda) v \cdot \bar{v} = 0 \Rightarrow (\bar{\lambda} - \lambda) \|v\|^2 = 0 \Rightarrow \bar{\lambda} = \lambda \Rightarrow \lambda \in \mathbb{R}.$$

(b) Let $v_1, v_2, \dots, v_m$ be the eigenvectors of B and $\lambda_1, \lambda_2, \dots, \lambda_n$ the corresponding eigenvalues.

Then, we have $\|Av_i\|^2 = (Av_i)^T \cdot (Av_i) = v_i^T A^T A v_i = v_i^T (A^T A) v_i = v_i^T B v_i$, $i \in \{1, 2, \dots, m\}$

v_i eigenvector $\Rightarrow Bv_i = \lambda_i v_i \Rightarrow \|Av_i\|^2 = v_i^T \lambda_i v_i = \lambda_i \|v_i\|^2 \Rightarrow \lambda_i = \dfrac{\|Av_i\|^2}{\|v_i\|^2} \geq 0 \Rightarrow$

$\Rightarrow \lambda_i \geq 0 \Rightarrow$ all eigenvalues are non-negative.

3. $B = A^T A$, where A is non-singular

(a) Then, from 2. b) we know that all eigenvalues of B are non-negative $\Rightarrow \lambda_i \geq 0$ $(\forall)$ $i \in \{1, 2, \dots, n\}$.

Let's suppose there is an eigenvalue which is 0. $\Rightarrow$

$\Rightarrow Av_i = 0 v_i \Rightarrow Av_i = 0 \Rightarrow A^{-1} A v_i = 0 \Rightarrow v_i = 0$ (but the eigenvectors are non-zero). So, all $\lambda_i > 0$.

(b) We know that there exist D, P matrices such that

$D = P^T B P$, where $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $P^T P = I$, $P = (v_1\ v_2\ \dots\ v_n)$, where $v_1, v_2, \dots, v_n$ are the orthogonal eigenvectors of B. $\Rightarrow \{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{R}^n$.

Let $v \in \mathbb{R}^n \Rightarrow v = \sum_{i=1}^{m} \alpha_i v_i$.

$\|v\|^2 = v^T \cdot v = \sum_{i=1}^{n} \alpha_i v_i^T \cdot \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i^2$ $(v_i \cdot v_j = 0$ for $i \neq j)$

$v^T B v = \sum_{i=1}^{n} \alpha_i v_i^T \cdot \sum_{i=1}^{n} \alpha_i B v_i = \sum_{i=1}^{n} \alpha_i v_i^T \cdot \sum_{i=1}^{n} \alpha_i \lambda_i v_i = \sum_{i=1}^{n} \alpha_i^2 \lambda_i \|v_i\|^2$

Now, we want to prove that:

$\lambda_{min} \|v\|^2 \leq v^T B v \leq \lambda_{max} \|v\|^2$

$\lambda_{min} \sum_{i=1}^{n} \alpha_i^2 \|v_i\|^2 \leq \sum_{i=1}^{n} \lambda_i \alpha_i^2 \|v_i\|^2 \leq \lambda_{max} \sum_{i=1}^{n} \alpha_i^2 \|v_i\|^2$

As $\lambda_{min} \leq \lambda_i \leq \lambda_{max}$ for all $i \in \{1, 2, \dots, n\}$

$\Rightarrow$ this is true

c) $w^T B w = \sum_{i=1}^{n} \lambda_i \alpha_i^2 \|v_i\|^2$ where $w = \sum_{i=1}^{n} \alpha_i v_i$

$\|w\| = \sum_{i=1}^{n} \alpha_i^2 \|v_i\|^2$

$\Rightarrow \lambda_{max} \|w\|^2 - w^T B w = \sum_{i=1}^{n} \alpha_i^2 (\lambda_{max} - \lambda_i) \|v_i\|^2 = 0$

As $\lambda_{max} - \lambda_i \geq 0$ (there can be values which are > 0), therefore $\alpha_i = 0$ $(\forall)$ $i \in \{1, 2, \dots, n\} \setminus \{j\}$, where j is the value where $\lambda_j = \lambda_{max} \Rightarrow \boxed{w = c\, v_{max}}$, where c is a constant and v_{max} is the corresponding eigenvector for λ_{max}

The same thing applies for $\boxed{y = d\, v_{min}}$.

4. (a) $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 & 2 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} = (2-\lambda)\lambda^2 - (2-\lambda) = (2-\lambda)(\lambda^2-1) = (\lambda-1)(\lambda+1)(2-\lambda) \Rightarrow$$

$\Rightarrow \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$

$$Av_1 = \lambda_1 v_1 \Rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Rightarrow \left.\begin{matrix} 2p+q+2r = p \\ r = q \\ q = r \end{matrix}\right| \Rightarrow p+q+2q = 0 \Rightarrow p = -3q \Rightarrow$$

$\Rightarrow v_1 = q\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$ we take $v_1 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

$$Av_2 = \lambda_2 v_2 \Rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -p \\ -q \\ -r \end{pmatrix} \Rightarrow \left.\begin{matrix} 2p+q+2r = -p \\ r = -q \\ q = -r \end{matrix}\right| \Rightarrow 3p = q \Rightarrow p = \frac{1}{3}q \Rightarrow v_2 = q\begin{pmatrix} \frac{1}{3} \\ 1 \\ -1 \end{pmatrix} \Rightarrow$$ we

take $v_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \\ -1 \end{pmatrix}$

$$Av_3 = \lambda_3 v_3 \Rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2p \\ 2q \\ 2r \end{pmatrix} \Rightarrow \left.\begin{matrix} 2p+q+2r = 2p \\ r = 2q \\ q = 2r \end{matrix}\right| \Rightarrow q = r = 0 \;\Big|\; \Rightarrow v_3 = p\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$ we take

$v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Then, $S = \begin{pmatrix} -3 & \frac{1}{3} & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$ and $S^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{4}{3} & \frac{5}{3} \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and $D = S^{-1}AS$

(b) $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}$

$$\det(A-\lambda I) = 0 \Rightarrow \det\begin{pmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 4-\lambda & -1-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)(-1-\lambda) + (-1-\lambda) = 0$$

$$(-\lambda-1)(3-3\lambda-\lambda+\lambda^2+1) = 0$$

$$(\lambda+1)(\lambda^2-4\lambda+4) = 0$$

$$(\lambda+1)(\lambda-2)^2 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = \lambda_3 = 2$$

$$Av_1 = \lambda_1 v_1 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -p \\ -q \\ -r \end{pmatrix} \Rightarrow \left.\begin{matrix} p+q = -p & \Rightarrow & 2p+q = 0 \\ -p+3q = -q & \Rightarrow & -p+4q = 0 \\ -p+4q-r = -r & \Rightarrow & -p+4q = 0 \end{matrix}\right| \Rightarrow p = 0, q = 0 \Rightarrow$$

$\Rightarrow v_1 = r\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$ we take $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$Av_2 = \lambda_2 v_2 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 2p \\ 2q \\ 2r \end{pmatrix} \Rightarrow \left.\begin{matrix} p+q = 2p & \Rightarrow & q = p \\ -p+3q = 2q & \Rightarrow & q = p \\ -p+4q-r = 2r & \Rightarrow & 3r = -p+4q \end{matrix}\right| \Rightarrow p = q = r \Rightarrow$$ we

only get one vector, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, so we can't create D such that $D = S^{-1}AS$, where D is diagonal.

(c) $A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$

$\det(A-\lambda I)=0 \Rightarrow \det\begin{pmatrix}2-\lambda&1&1\\1&2-\lambda&1\\1&1&2-\lambda\end{pmatrix}=0 \Rightarrow \lambda^3-6\lambda^2+9\lambda-4=0$

$(\lambda-1)^2(\lambda-4)=0 \Rightarrow \lambda_1=\lambda_2=1,\ \lambda_3=4$

$Av_1=\lambda_1 v_1 \Rightarrow \begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}p\\q\\r\end{pmatrix} \Rightarrow \left.\begin{array}{l}2p+q+r=p\\p+2q+r=q\\p+q+2r=r\end{array}\right| \Rightarrow p+q+r=0 \Rightarrow$ we choose two independent vectors $v_1=\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $v_2=\begin{pmatrix}1\\1\\-2\end{pmatrix}$.

$Av_3=\lambda_3 v_3 \Rightarrow \begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}4p\\4q\\4r\end{pmatrix} \Rightarrow \left.\begin{array}{l}2p+q+r=4p\\p+2q+r=4q\\p+q+2r=4r\end{array}\right| \Rightarrow \left.\begin{array}{l}q+r=-2p\\p+r=-2q\\p+q=-2r\end{array}\right| \Rightarrow p=q=r \Rightarrow$ we choose $v_3=\begin{pmatrix}1\\1\\1\end{pmatrix}$

Then, $S=\begin{pmatrix}1&1&1\\-1&1&1\\0&-2&1\end{pmatrix} \Rightarrow S^{-1}=\begin{pmatrix}\frac12&-\frac12&0\\\frac16&\frac16&-\frac13\\\frac13&\frac13&\frac13\end{pmatrix}$

$D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix}$, and $D=S^{-1}AS$.

5. $A=\begin{pmatrix}-\frac34&\frac14\\\frac14&-\frac34\end{pmatrix}=\frac14\begin{pmatrix}-3&1\\1&-3\end{pmatrix}$

(a) $\frac{1}{16}\det\begin{pmatrix}-3-\lambda&1\\1&-3-\lambda\end{pmatrix}=0 \Rightarrow (\lambda+3)^2-1=0$

$\lambda^2+6\lambda+8=0$

$(\lambda+2)(\lambda+4)=0 \Rightarrow \lambda_1=-2,\ \lambda_2=-4$

$Av_1=\lambda_1 v_1 \Rightarrow \begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow -x+y=0 \Rightarrow v_1=\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$

$Av_2=\lambda_2 v_2 \Rightarrow \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x+y=0 \Rightarrow v_2=\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$

$P=\frac{1}{\sqrt2}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \Rightarrow P^T=\frac{1}{\sqrt2}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \Rightarrow P\cdot P^T=I$

$D=\begin{pmatrix}-\frac12&0\\0&-1\end{pmatrix}$ with $D=P^TAP \Rightarrow A=PDP^T=PDP^{-1}$

(b) $A^n=(PDP^T)^n=PD^nP^T=\frac12\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}(-\frac12)^n&0\\0&(-1)^n\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\frac12\begin{pmatrix}(-\frac12)^n&(-1)^n\\(-\frac12)^n&(-1)^{n+1}\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=$

$=\frac{(-1)^n}{2}\begin{pmatrix}\frac{1}{2^n}&1\\\frac{1}{2^n}&-1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\frac{(-1)^n}{2}\begin{pmatrix}\frac{1}{2^n}+1&\frac{1}{2^n}-1\\\frac{1}{2^n}-1&\frac{1}{2^n}+1\end{pmatrix}$

(c) As n goes to infinity, we have:

n odd $\Rightarrow A^n \to \frac{-1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}=\begin{pmatrix}-\frac12&\frac12\\\frac12&-\frac12\end{pmatrix}$

n even $\Rightarrow A^n \to \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(d) Let λ be the eigenvalue with the smallest modulus. u is the corresponding eigenvector.
Then $\lambda = -\frac{1}{2} \Rightarrow u = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A^n u = \alpha\frac{(-1)^n}{2}\begin{pmatrix} \frac{1}{2^n}+1 & \frac{1}{2^n}-1 \\ \frac{1}{2^n}-1 & \frac{1}{2^n}+1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha\frac{(-1)^n}{2}\begin{pmatrix} \frac{1}{2^n}\cancel{+1}+\frac{1}{2^n}\cancel{-1} \\ \frac{1}{2^n}\cancel{-1}+\frac{1}{2^n}\cancel{+1} \end{pmatrix} =$$

$$= \alpha\frac{(-1)^n}{2}\begin{pmatrix} \frac{1}{2^{n-1}} \\ \frac{1}{2^{n-1}} \end{pmatrix} = \alpha\left(-\frac{1}{2}\right)^n\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{n\to\infty} \alpha\cdot 0\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(e) $$\frac{dy_1}{dt} = -\frac{3}{4}y_1 + \frac{1}{4}y_2$$

$$\frac{dy_2}{dt} = \frac{1}{4}y_1 - \frac{3}{4}y_2$$

Writing this in matrix form:

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad ; \quad A = PDP^T$$

We may then write the system of differential equations as:

$$\frac{dy}{dt} = PDP^T y$$

As the entries of P are constant this may be written

$$\frac{d}{dt}(P^T y) = (DP^T)y$$

Setting $z = P^T y$, we obtain

$$\frac{dz}{dt} = Dz$$

which may be written

$$\frac{dz_1}{dt} = z_1, \quad \frac{dz_2}{dt} = z_2$$

We then have, for arbitrary constants A, B:

$$z_1 = Ae^{-\frac{1}{2}}, \quad z_2 = Be^{-1}$$

Finally,

$$y = Pz = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} Ae^{-\frac{1}{2}t} \\ Be^{-t} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}Ae^{-\frac{t}{2}} + \frac{1}{\sqrt{2}}Be^{-t} \\ \frac{1}{\sqrt{2}}Ae^{-\frac{t}{2}} - \frac{1}{\sqrt{2}}Be^{-t} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$