CONTINUOUS MATHS HT 2019: PROBLEM SHEET 0

GABRIEL MO

Basic differentiation: reminders and exercises

0.1.

(i) $y = 1$

$$\frac{dy}{dx} = 0$$

(ii) $y = 2x^2 + 3x^3$

$$\frac{dy}{dx} = 4x + 9x^2$$

(iii) $y = \frac{1}{x} - \frac{3}{x^2} + \sin x + e^x$

$$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{6}{x^3} + \cos x + e^x$$

(iv) $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}$$

(v) $y = \log_2 x$

$$\frac{dy}{dx} = \frac{1}{x \ln 2}$$

(vi) $y = \sum_{i=1}^{n} \frac{x^i}{i}$

$$\frac{dy}{dx} = \sum_{i=1}^{n} x^{i-1} = \frac{1-x^n}{1-x}$$

0.2 The Chain Rule: $\frac{d(g \circ f)}{dx} = \left(\frac{dg}{dx} \circ f\right) \frac{df}{dx}$

(i) $y = e^{\sin x}$

$g = e^x$

$f = \sin x$

$$\frac{dy}{dx} = e^{\sin x} \cdot \cos x$$

(ii) $y = \ln(1 - x^a)$

$g = \ln x$

$f = 1 - x^a$

$$\frac{dy}{dx} = \frac{1}{1-x^a} \cdot \left(-ax^{a-1}\right) = \frac{ax^{a-1}}{x^a - 1}$$

(iii) $y = \sin(\cos x)$

$g = \sin x$

$f = \cos x$

$$\frac{dy}{dx} = \cos(\cos x) \cdot (-\sin x) = -\sin x \cdot \cos(\cos x)$$

(iv) $y = \sqrt{\cos x^2}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\cos x^2}} \cdot (-\sin x^2) \cdot (2x) = \frac{-x \sin x^2}{\sqrt{\cos x^2}}$$

(v) $y = \exp(\exp(x^{-1}))$

$$\frac{dy}{dx} = \exp(\exp(x^{-1})) \cdot \exp(x^{-1}) \cdot \left(-\frac{1}{x^2}\right) = e^{e^{\frac{1}{x}}} \cdot e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)$$

(vi) $y = \sin^2\left(\frac{e^{-x}}{1+e^{-x}}\right) + \cos^2\left(\frac{1}{1+e^x}\right)$

$$\frac{dy}{dx} = 2\sin\left(\frac{e^{-x}}{1+e^{-x}}\right) \cdot \cos\left(\frac{e^{-x}}{1+e^{-x}}\right) \cdot \frac{(-e^{-x}) \cdot (1+e^{-x}) - e^{-x} \cdot (-e^{-x})}{(1+e^{-x})^2} +$$

$$+ 2\cos\left(\frac{1}{1+e^x}\right) \cdot \left(-\sin\left(\frac{1}{1+e^x}\right)\right) \cdot \frac{-e^x}{(1+e^x)^2}$$

$$\frac{dy}{dx} = \sin \frac{2e^{-x}}{1+e^{-x}} \cdot \frac{-e^{-x}}{(1+e^{-x})^2} + \sin\left(\frac{2}{1+e^x}\right) \cdot \frac{e^x}{(1+e^x)^2}$$

$$\frac{dy}{dx} = \sin\left(\frac{2}{1+e^x}\right) \cdot \left(\frac{-e^{-x}}{1+2e^{-x}+e^{-2x}} + \frac{e^x}{1+2e^x+e^{2x}}\right)$$

$$\frac{dy}{dx} = \sin\left(\frac{2}{1+e^x}\right) \cdot \left(\frac{-1}{e^x+2+e^{-x}} + \frac{1}{e^{-x}+2+e^x}\right)$$

$$\frac{dy}{dx} = 0$$

$\boxed{0.3}$ The Product Rule: $\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$

(i) $x = \sin t \cos t$

$$\frac{dx}{dt} = -\sin^2 t + \cos^2 t = \cos(2t)$$

(ii) $x = (1-2t+3t^2)e^{4t}$

$$\frac{dx}{dt} = (-2+6t)e^{4t} + 4(1-2t+3t^2)e^{4t} = e^{4t}(-2+6t+4-8t+12t^2) = e^{4t}(12t^2-2t+2)$$

$$\frac{dx}{dt} = 2e^{4t}(6t^2-t+1)$$

(iii) $x = t(\sin t)e^t$

$$\frac{dx}{dt} = (\sin t)e^t + t(\cos t)e^t + t(\sin t)e^t = e^t(\sin t + t\cos t + t\sin t)$$

$$\frac{dx}{dt} = e^t\left((1+t)\sin t + t\cos t\right)$$

$\boxed{0.4}$

$$\frac{d\left(\frac{f}{g}\right)}{dx} = \frac{d\left(f\cdot\frac{1}{g}\right)}{dx} = f\frac{d\left(\frac{1}{g}\right)}{dx} + \frac{1}{g}\frac{df}{dx} = f\cdot\frac{-1}{g^2}\cdot\frac{dg}{dx} + \frac{1}{g}\frac{df}{dx} \Rightarrow$$

$$\Rightarrow \frac{d\left(\frac{f}{g}\right)}{dx} = \frac{-f\frac{dg}{dx} + g\frac{df}{dx}}{g^2} = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

(i) $l = \frac{z+1}{z-1}$

$$\frac{dl}{dz} = \frac{(z-1)-(z+1)}{(z-1)^2} = \frac{-2}{(z-1)^2}$$

(ii) $l = \frac{\sin(\pi z)}{\pi z}$

$$\frac{dl}{dz} = \frac{\cos(\pi z)\cdot\pi^2 z - \pi\sin(\pi z)}{\pi^2 z^2} = \frac{\pi z\cos(\pi z) - \sin(\pi z)}{\pi z^2}$$

(iii) $l = \frac{e^z}{\ln z}$

$$\frac{dl}{dz} = \frac{e^z\ln z - e^z\cdot\frac{1}{z}}{\ln^2 z} = \frac{e^z\left(\ln z - \frac{1}{z}\right)}{\ln^2 z}$$

$\boxed{0.5}$ (i) $f = \exp(x\ln x) = e^{x\ln x}$

$$\frac{df}{dx} = e^{x\ln x}(1+\ln x)$$

(ii) $f = e^{\sin^2(\ln x)} e^{\cos^2(\ln x)}$

$$\frac{df}{dx} = e^{\sin^2(\ln x)} \cdot e^{\cos^2(\ln x)} \cdot 2\cos(\ln x) \cdot (-\sin(\ln x)) \cdot \frac{1}{x} + e^{\sin^2(\ln x)} \cdot 2\sin(\ln x) \cdot \cos(\ln x) \cdot \frac{1}{x} \cdot e^{\cos^2(\ln x)}$$

$$\frac{df}{dx} = e^{\sin^2(\ln x)} \cdot e^{\cos^2(\ln x)} \cdot \frac{2}{x}\left(\cos(\ln x)\cdot(-\sin(\ln x)) + \sin(\ln x)\cdot\cos(\ln x)\right)$$

$$\frac{df}{dx} = 0$$

(iii) $f = (x^4-1)^3(x^3+1)^4$

$$\frac{df}{dx} = 3(x^4-1)^2 \cdot 4x^3 \cdot (x^3+1)^4 + (x^4-1)^3 \cdot 4(x^3+1)^3 \cdot 3x^2$$

$$\frac{df}{dx} = (x^4-1)^2(x^3+1)^3\left(3\cdot 4x^3\cdot(x^3+1) + (x^4-1)\cdot 4\cdot 3x^2\right)$$

$$\frac{df}{dx} = (x^4-1)^2(x^3+1)^3\left(12x^6+12x^3+12x^6-12x^2\right) = 12(x^4-1)^2(x^3+1)^3(2x^6+x^3-x^2)$$

$$\frac{df}{dx} = 12x^2(x^4-1)^2(x^3+1)^3(2x^4+x-1)$$

(iv) $f = \log\left(\frac{1+x^2}{1-x^2}\right)$

$$\frac{df}{dx} = \frac{1-x^2}{1+x^2} \cdot \frac{2x(1-x^2)+(1+x^2)\cdot 2x}{(1-x^2)^2}$$

$$\frac{df}{dx} = \frac{2x}{1-x^4}$$

(v) $f = \sum_{i=1}^{n} \ln(1+e^{ix})$

$$\frac{df}{dx} = \sum_{i=1}^{n} \frac{1}{1+e^{ix}} \cdot ie^{ix} = \sum_{i=1}^{n} \frac{ie^{ix}}{1+e^{ix}}$$

(vi) $f = -x\log_2 x - (1-x)\log_2(1-x)$

$$\frac{df}{dx} = -\log_2 x - x\cdot\frac{1}{x\ln 2} + \log_2(1-x) + (1-x)\cdot\frac{1}{(1-x)\ln 2}$$

$\boxed{0.6}$ $y = \text{cst.}$

(i) $f = x^2+y^2+2xy$

$$\frac{df}{dx} = 2x+2y = 2(x+y)$$

(ii) $f = 2x\cos y + 3y\sin x$

$$\frac{df}{dx} = 2\cos y + 3y\cos x$$

(iii) $f = \sqrt{4x+4y}$

$x = \text{cst.}$

(i) $f = x^2+y^2+2xy$

$$\frac{df}{dy} = 2y+2x = 2(x+y)$$

(ii) $f = 2x\cos y + 3y\sin x$

$$\frac{df}{dy} = -2x\sin y + 3\sin x$$

(iii) $f = \sqrt{4x+4y}$

$$\frac{df}{dx} = \frac{1}{2\sqrt{4x+4y}} \cdot 4 = \frac{1}{\sqrt{x+y}}$$

$$\frac{df}{dy} = \frac{1}{2\sqrt{4x+4y}} \cdot 4 = \frac{1}{\sqrt{x+y}}$$

(iv) $f = \frac{x-y}{x+y}$

$$\frac{df}{dx} = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{df}{dy} = \frac{-(x+y)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

(v) $f = \frac{e^{xy}}{xy}$

$$\frac{df}{dx} = \frac{ye^{xy} \cdot xy - ye^{xy}}{(xy)^2} = \frac{ye^{xy}(xy-1)}{x^2y^2}$$

$$\frac{df}{dx} = \frac{e^{xy}(xy-1)}{x^2y}$$

$$\frac{df}{dy} = \frac{xe^{xy} \cdot xy - xe^{xy}}{(xy)^2} = \frac{xe^{xy}(xy-1)}{x^2y^2}$$

$$\frac{df}{dy} = \frac{e^{xy}(xy-1)}{xy^2}$$

(vi) $f = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = (x+y)^n$

$$\frac{df}{dx} = n(x+y)^{n-1}$$

$$\frac{df}{dy} = n(x+y)^{n-1}$$

$\boxed{0.7}$ (i) $f = \sum_{i=1}^{n} x_i^2$

$$\frac{\partial f}{\partial x_j} = 2x_j$$

(ii) $f = \left(\sum_{i=1}^{n} x_i\right)^2$

$$\frac{\partial f}{\partial x_j} = 2\left(\sum_{i=1}^{n} x_i\right) = 2\sum_{i=1}^{n} x_i$$

(iii) $f = \frac{\sum_{i=1}^{n} x_i}{\sqrt{\sum_{i=1}^{n} x_i^2}}$

$$\frac{\partial f}{\partial x_j} = \frac{\sqrt{\sum_{i=1}^{n} x_i^2} - \sum_{i=1}^{n} x_i \cdot \frac{1}{\cancel{2}\sqrt{\sum_{i=1}^{n} x_i^2}} \cdot \cancel{2} x_j}{\sum_{i=1}^{n} x_i^2}$$

$$\frac{\partial f}{\partial x_j} = \frac{\sqrt{\sum_{i=1}^{n} x_i^2} - \sum_{i=1}^{n} x_i \cdot \frac{x_j}{\sqrt{\sum_{i=1}^{n} x_i^2}}}{\sum_{i=1}^{n} x_i^2}$$

$$\frac{\partial f}{\partial x_j} = \frac{\sum_{i=1}^{n} x_i^2 - x_j \cdot \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2 \cdot \sqrt{\sum_{i=1}^{n} x_i^2}}.$$