Determinants

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Permutations

Let $S = \{1, 2, \ldots, N\}$ be the set of integers from 1 to N, arranged in ascending order

A re-arragement of the elements of S is known as a permutation of S

For example, if S is the set $S = \{1, 2, 3\}$ then there are $3! = 6$ distinct permutations of S (why?)

These 6 permutations are $123, 132, 213, 231, 312, 321$

Let $S = \{1, 2, \ldots, N\}$

A permutation $j_1 j_2 \ldots j_N$ of S is said to have an inversion if a larger integer lies on the left of a smaller one

A permutation is said to be even if the total number of inversions is even, and odd if the total number of inversions is odd

For example, if S is the set $S = \{1, 2, 3, 4\}$, in the permutation 4312: 4 precedes 3; 4 precedes 1; 4 precedes 2; 3 precedes 1; and 3 precedes 2

There are 5 inversions, and so 4312 is an odd permutation of S

Determinants

We will now give two definitions of the determinant of a square matrix

We will claim that these definitions are equivalent, and will verify this for the special case of 2×2 and 3×3 matrices

First definition of a determinant

Let A be a $N \times N$ matrix

The determinant of A is defined by

$$\det(A) = \sum \pm A_{1,j_1} A_{2,j_2} \dots A_{N,j_N}$$

where

- The sum is taken over all permutations $j_1 j_2 \dots j_N$ of the set $S = \{1, 2, \dots, N\}$
- The sign of each component of the sum is positive if the permutation is an even permutation, and negative if it is an odd permutation

Observe that each component in the sum on the previous slide is of the form

$$\pm A_{1,j_1} A_{2,j_2} \dots A_{N,j_N}$$

Each term in the sum contains one entry from each row

As $j_1 j_2 \dots j_N$ is a permutation of the set $S = \{1, 2, \dots, N\}$ it contains exactly one instance of all integers between 1 and N inclusive

Each term in the sum therefore contains one entry from each column

The determinant of a 2×2 matrix

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

The set $S = \{1, 2\}$ has 2 permutations — the even permutation 12, and the odd permutation 21

The determinant of A is then given by

$$\det(A) = A_{11}A_{22} - A_{12}A_{21}$$

The determinant of a 3×3 matrix

Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

The set $S = \{1, 2, 3\}$ has 6 permutations — the even permutations 123, 231, 312, and the odd permutations 213, 321, 132

The determinant of A is then given by

$$\det(A) = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32}$$

Second definition of a determinant

Let A be a $N \times N$ matrix

If A is of size 1×1 then $\det(A) = A_{11}$

Let D_{ij} be the $(N-1) \times (N-1)$ matrix obtained from A by deleting row i and column j

Number the rows and columns so that the first row is row 1, and the first column is column 1 — note that C, C++, Java, etc. start the indexing of arrays from 0

Set $\Delta_{ij} = (-1)^{i+j} \det(D_{ij})$

Δ_{ij} is sometimes called the cofactor of A_{ij}

Then, for any fixed i:

$$\det(A) = \sum_{j=1}^{N} A_{ij} \Delta_{ij}$$

Alternatively, for any fixed j:

$$\det(A) = \sum_{i=1}^{N} A_{ij} \Delta_{ij}$$

This method of calculating a determinant is known as the cofactor expansion

Note that this is a recursive definition of a matrix

We first define the determinant of a $N \times N$ matrix to be the sum of determinants of $(N-1) \times (N-1)$ matrices

Using the definition of the determinant of a $(N-1) \times (N-1)$ matrix we may write this as a sum of determinants of $(N-2) \times (N-2)$ matrices

Eventually we will write the determinant of a $N \times N$ matrix to be the sum of determinants of 1×1 matrices

The determinant of a 2×2 matrix

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

Then, setting $i = 1$ in the definition of $\det(A)$ on the previous slide,

$$\begin{aligned} \det(A) &= A_{11} \left((-1)^{1+1} \det(A_{22})\right) + A_{12} \left((-1)^{1+2} \det(A_{21})\right) \\ &= A_{11} A_{22} - A_{12} A_{21} \end{aligned}$$

The determinant of a 3×3 matrix

Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

Setting $i = 1$ in the definition of a determinant gives

$$\det(A) = A_{11}\left((-1)^{1+1}\det\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}\right) + A_{12}\left((-1)^{1+2}\det\begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{pmatrix}\right) + A_{13}\left((-1)^{1+3}\det\begin{pmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix}\right)$$

$$= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32}$$

Properties of determinants

We now give some useful properties of determinants

We give the properties in terms of rows — the properties are equally valid for the columns of the matrix

We will give sketch proofs of these properties, usually illustrating why these properties hold for 2×2 matrices (although you may assume that they are true for all square matrices)

These properties are useful for evaluating determinants

Property 1

Let $\mathcal{I}_N$ be the identity matrix of size $N \times N$. We then have

$$\det \mathcal{I}_N = 1$$

Proof:

Using the first definition of the determinant, we have a sum over all permutations.

Each term includes the product of exactly one entry from each row, and exactly one entry from each column.

The only permulation with a non-zero contribution to the sum is the permutation $\{1, 2, \ldots, N\}$, which has no inversions, and is therefore an even permutation.

The determinant is then the product of entries on the diagonal, which is equal to 1

Property 2

The determinant changes sign when two rows are interchanged.

Sketch proof:

$$\begin{aligned}\det\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= ad - bc \\ &= -(bc - ad) \\ &= -\det\begin{pmatrix} c & d \\ a & b\end{pmatrix}\end{aligned}$$

Property 3

Let A be a $N \times N$ matrix

Let $B = A$, and then multiply row k of B by λ so that

$$B_{ij} = \begin{cases} A_{ij} & i \neq k \\ \lambda A_{ij} & i = k \end{cases}$$

We then have $\det(B) = \lambda \det(A)$

Sketch proof:

Let

$$A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \qquad B = \begin{pmatrix} a & b \\ \lambda c & \lambda d\end{pmatrix}$$

Then

$$\begin{aligned}\det(B) &= \det\begin{pmatrix} a & b \\ \lambda c & \lambda d\end{pmatrix} \\ &= \lambda(ad - bc) \\ &= \lambda \det(A)\end{aligned}$$

Property 4

Let A be a $N \times N$ matrix

Let $B = A$, and then add the vector $\mathbf{v}^\top$ to row k of B so that

$$B_{ij} = \begin{cases} A_{ij} & i \neq k \\ A_{ij} + v_j & i = k \end{cases}$$

Let $C = A$, and then replace row k of C by the vector $\mathbf{v}^\top$ so that

$$C_{ij} = \begin{cases} A_{ij} & i \neq k \\ v_j & i = k \end{cases}$$

We then have $\det(B) = \det(A) + \det(C)$

Sketch proof:

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} a+e & b+f \\ c & d \end{pmatrix}, \qquad C = \begin{pmatrix} e & f \\ c & d \end{pmatrix}, \qquad \text{i.e. } \mathbf{v}^\top = \begin{pmatrix} e & f \end{pmatrix}$$

Then

$$\begin{aligned} \det(B) &= (a+e)d - (b+f)c \\ &= (ad - bc) + (ed - fc) \\ &= \det(A) + \det(C) \end{aligned}$$

Property 5

If a matrix has a row of zeros then the determinant is zero

Proof:

Recall the first definition of the determinant:

$$\det(A) = \sum \pm A_{1,j_1} A_{2,j_2} \dots A_{N,j_N}$$

where

- The sum is taken over all permutations $j_1 j_2 \dots j_N$ of the set $S = \{1, 2, \dots, N\}$
- The sign of each component of the sum is positive if the permutation is an even permutation, and negative if it is an odd permutation

Each term in the sum is a product that contains an entry from each row

As the matrix contains a row of zeros each term in the sum is a product that includes at least one factor that is zero

We therefore have $\det(A) = 0$

Property 6

The determinant of an upper (or lower) triangular matrix is the product of entries on the diagonal

Proof:

We may write the first definition of the determinant as:

$$\det(A) = A_{11} A_{22} A_{33} \dots A_{NN} + \sum \pm A_{1,j_1} A_{2,j_2} \dots A_{N,j_N}$$

where

- The sum is taken over all permutations $j_1 j_2 \dots j_N$ of the set $S = \{1, 2, \dots, N\}$ apart from the permutation $123 \dots N$
- The sign of each component of the sum is positive if the permutation is an even permutation, and negative if it is an odd permutation

Suppose A is a upper triangular matrix, and so $A_{ij} = 0$ when $i > j$

Let $j_1 j_2 \dots j_N$ be a permutation of the set $S = \{1, 2, \dots, N\}$ of the set $S = \{1, 2, \dots, N\}$ that is not $123 \dots N$

There will be at least one $j_m < m$ in this permutation — we then have $A_{m,j_m} = 0$ as A is lower triangular

All terms in the sum on the previous slide are zero and so $\det(A) = A_{11}A_{22}A_{33} \dots A_{NN}$

Property 7

Let A be a $N \times N$ matrix

Suppose two rows of A are identical. Then $\det(A) = 0$

Proof:

Let B be the matrix obtained by interchanging the two identical rows

By Property 2, $\det(B) = -\det(A)$

But, as the rows are identical, $B = A$ and so $\det(B) = \det(A)$

Therefore $\det(B) = -\det(A) = \det(A)$

This can only be true if $\det(A) = 0$

Property 8

Subtracting a multiple of one row from another row leaves the determinant unchanged

Proof:

$$\begin{aligned}
\det \begin{pmatrix} a - \lambda c & b - \lambda d \\ c & d \end{pmatrix} &= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \det \begin{pmatrix} \lambda c & \lambda d \\ c & d \end{pmatrix} && \text{by property 4} \\
&= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda \det \begin{pmatrix} c & d \\ c & d \end{pmatrix} && \text{by property 3} \\
&= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} && \text{by property 7}
\end{aligned}$$

Property 9

If both A and B are $N \times N$ matrices then

$$\det(AB) = \det(A)\det(B)$$

Sketch proof:

For the 2×2 case, verify by calculating the matrix product AB and calculating the determinant of this matrix

Suppose A, B and C are $N \times N$ matrices

We may extend this property as follows:

$$\begin{aligned}\det(ABC) &= \det(A(BC)) \\ &= \det(A)\det(BC) \\ &= \det(A)\det(B)\det(C)\end{aligned}$$

Similarly, if $A_1, A_2, \ldots, A_M$ are $N \times N$ matrices we have

$$\det(A_1A_2\ldots A_M) = \det(A_1)\det(A_1)\ldots\det(A_M)$$

Property 10

A square matrix is invertible if, and only if, $\det(A) \neq 0$

Proof:

The fundamental theorem of invertible matrices may be used to deduce that

$$A \text{ is invertible} \Leftrightarrow A \text{ is the product of elementary matrices}$$

Three types of elementary matrices are

- The non-zero scaling of a row of the identity matrix
- Interchanging two rows of the identity matrix
- Adding a multiple of one row of the identity matrix to another row

The determinant of an elementary matrix is non-zero (why?)

A is the product of elementary matrices if, and only if,

$$A = E_1E_2\ldots E_M$$

where $E_1, E_2, \ldots, E_M$ are elementary matrices

We then have

$$\det(A) = \det(E_1)\det(E_2)\ldots\det(E_M) \neq 0$$

and so

$$\begin{aligned}A \text{ is invertible} &\Leftrightarrow A \text{ is the product of elementary matrices} \\ &\Leftrightarrow \det(A) \neq 0\end{aligned}$$

Property 11

Let A be a square matrix. Then $\det(A) = \det(A^\top)$

Sketch proof:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Example

Calculate the determinant of the matrix

$$A = \begin{pmatrix} 3 & 4 & 6 & 8 \\ 1 & 2 & 3 & 4 \\ 7 & 2 & 3 & 5 \\ 9 & 1 & 6 & 7 \end{pmatrix}$$

By property 8, the determinant will be unchanged if we: (i) subtract twice row 2 from row 1; and (ii) subtract row 2 from row 3

We then have

$$\det(A) = \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 6 & 0 & 0 & 1 \\ 9 & 1 & 6 & 7 \end{pmatrix}$$

Using the second definition of the determinant we then have

$$\begin{aligned} \det(A) &= \det\begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 1 & 6 & 7 \end{pmatrix} \\ &= -\det\begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix} \\ &= -9 \end{aligned}$$

Practical computation of the determinant

Recall the first definition of the determinant:

Let A be a $N \times N$ matrix

The determinant of A is defined by

$$\det(A) = \sum \pm A_{1,j_1} A_{2,j_2} \dots A_{N,j_N}$$

where

- The sum is taken over all permutations $j_1 j_2 \dots j_N$ of the set $S = \{1, 2, \dots, N\}$
- The sign of each component of the sum is positive if the permutation is an even permutation, and negative if it is an odd permutation

How much work is needed to compute the determinant of a square matrix of size N?

We sum over all permutations of the set $S = \{1, 2, \dots, N\}$ — there are $N!$ such permutations

In each sum we need to:

1. Multiply N numbers
2. Decide whether the permutation is odd or even

Calculating the area of a triangle

The area of the triangle with nodes at the points (x_1, y_1), (x_2, y_2), (x_3, y_3), is given by the magnitude of

$$\frac{1}{2} \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Suppose a line is drawn through the points (x_1, y_1), (x_2, y_2).

A triangle with nodes at the points (x_1, y_1), (x_2, y_2), (x, y) will have zero volume if, and only if, (x, y) lies on that line

The equation of the line is then given by

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x \\ y_1 & y_2 & y \end{pmatrix} = 0$$

Calculating the volume of a tetrahedron

The volume of the tetrahedron with nodes at the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), is given by the magnitude of

$$\frac{1}{6}\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

Suppose a plane is drawn through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)

A tetrahedron with nodes at the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x, y, z) will have zero volume if, and only if, (x, y, z) lies on that plane

The equation of the plane is then given by

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x \\ y_1 & y_2 & y_3 & y \\ z_1 & z_2 & z_3 & z \end{pmatrix} = 0$$

The vector product

Define $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to be unit vectors in the x-, y- and z-directions respectively, so that

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The vector product between two vectors $\mathbf{a}$ and $\mathbf{b}$, written $\mathbf{a} \times \mathbf{b}$, is defined to be the vector

- of magnitude $\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$
- with direction perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in the sense of a right hand screw turned from $\mathbf{a}$ to $\mathbf{b}$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$

This definition implies that

$$\begin{array}{lll} \mathbf{i}\times\mathbf{i} = \mathbf{0}, & \mathbf{i}\times\mathbf{j} = \mathbf{k}, & \mathbf{i}\times\mathbf{k} = -\mathbf{j}, \\ \mathbf{j}\times\mathbf{i} = -\mathbf{k}, & \mathbf{j}\times\mathbf{j} = \mathbf{0}, & \mathbf{j}\times\mathbf{k} = \mathbf{i}, \\ \mathbf{k}\times\mathbf{i} = \mathbf{j}, & \mathbf{k}\times\mathbf{j} = -\mathbf{i}, & \mathbf{k}\times\mathbf{k} = \mathbf{0}, \end{array}$$

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ the vector product is given by

$$\mathbf{a}\times\mathbf{b} = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$