Discrete Mathematics MT16: Problem Sheet 5

Chapters 8 (Orders) and Vacation Revision

5.1 Consider the set $A = \mathcal{P}(\{1,2,3,4,5,6\})$, with the order $\subseteq$. In this order, an example of a chain is $\{\emptyset, \{1,2\}, \{1,2,3,4,5\}\}$ and an example of an antichain is $\{\{1,2\},\{2,3\},\{3,4\}\}$.

(i) Explain why any chain $\{B_1, B_2, \ldots, B_n\}$ cannot have more than seven elements.

(ii) How many different chains are there with exactly seven elements?

(iii) The largest antichain has 20 elements, and there is only one such antichain. Describe it.

5.2 Which of the following relations are preorders, which are partial orders, and which are linear orders? For those which are partial orders, determine whether every pair of elements has a least upper bound: if they do, describe how the lub can be computed; otherwise, give an example of a pair with no lub.

(i) On $S = \{A \mid A \text{ is a finite set of real numbers}\}$, $A \preceq B$ if $\max A \leq \max B$.

(ii) On $S = \{\text{all sequences of natural numbers}\}$, $(x_n) \preceq (y_n)$ if $x_i \leq y_i$ for all $i \in \mathbb{N}$.

(iii) On $S = \{1, 2, 3, \ldots, 20\}$, the "divides" order $\mid$.

(iv) On $(0, \infty)$, $x \preceq y$ if $1/x \leq 1/y$.

5.3 Let $A = \{1, 2, 3, 6\}$ be ordered by $\mid$. Draw the Hasse diagram of this order, and of the derived lexicographic order on $A \times A$.

With respect to this lexicographic order on $A \times A$, compute the glb (a.k.a meet) of each of the pairs:

(i) $(1,2)$ and $(1,3)$;

(ii) $(2,3)$ and $(3,2)$;

(iii) $(6,2)$ and $(3,3)$.

5.4 Let $A, B \subseteq \mathbb{R}$ be nonempty and suppose that $\operatorname{lub} A$ and $\operatorname{lub} B$ exist. Prove that

$$\operatorname{lub}\{a + b \mid a \in A \text{ and } b \in B\} = \operatorname{lub} A + \operatorname{lub} B.$$

[Hint: $\mathbb{R}$ is a linear order and it is best to use the alternative definition of lub found in the lecture notes: $m = \operatorname{lub} S$ if and only if (i) $x \leq m$ for all $x \in S$, and (ii) for all $y \in \mathbb{R}$, if $y < m$ then there exists an element $x \in S$ with $y < x$.]

Revision Questions

5.5 Prove the following: S and T are disjoint if, and only if, $S \oplus T = S \cup T$.

(The definition of $S \oplus T$ is $(S \setminus T) \cup (T \setminus S)$.)

5.6 If $m, n \in \mathbb{N}_+$ find a formula, in terms of m and n, for the number of m-digit positive integers (no leading zeros) which are divisible by n.

How many 4-digit positive integers are divisible by 6 and 15? How many are divisible by 6 or 15? How many by 6, 10, or 15?

5.7 Let $f : \mathbb{Z} \to \mathbb{Z}$ be a polynomial with integral coefficients, i.e.

$$f(n) = a_0 + a_1 n + a_2 n^2 + \ldots + a_k n^k$$

where $a_0, a_1, \ldots, a_k \in \mathbb{Z}$.

(i) Prove that, for any distinct integers m and n, $n - m \,|\, f(n) - f(m)$.

(ii) Prove that, if $f(0) = f(3) = 0$ then $1 \notin \mathrm{Im}(f)$.

Hint: Suppose that $f(0) = f(3) = 0$ and also that $f(n) = 1$ for some n; then use part (i) to find a contradiction.

5.8 The sequence of *Catalan numbers* (C_n) is defined by the recurrence

$$C_0 = 1, \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n.$$

(i) Using the recurrence, compute C_1, C_2, C_3, and C_4.

(ii) Prove, by induction on n, that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(iii) Explain, using the formula in part (ii), why if p is a prime number, $n > 1$, and $p \,|\, C_n$ then $p < 2n$.

(iv) Prove, by induction on n, that $C_n > 2n - 1$ for $n \geq 4$.

(v) Deduce that the only prime Catalan numbers are C_2 and C_3.

(vi) Using the fact that there exist nonzero constants b and c such that

$$bn^{n+\frac{1}{2}} \exp(-n) \leq n! \leq cn^{n+\frac{1}{2}} \exp(-n)$$

for all $n \in \mathbb{N}_+$ (the second of which was proved in Section 7.4 of the lecture notes), prove that

$$C_n = O(4^n n^{-3/2}).$$