Linear Algebra MT18 - Week 1

Chapter 1 (Vectors and Vector Spaces)

1. If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$ and c is a scalar explain why the following make no sense:
   (a) $\mathbf{u} \cdot \mathbf{v} + \mathbf{w}$
   (b) $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$
   (c) $c \cdot (\mathbf{u} + \mathbf{w})$

2. (optional) Under what conditions for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$ is the following true
$$\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|.$$

3. Let $\lambda \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ vectors. Given that
$$0 \leq \|\mathbf{u} - \lambda \mathbf{v}\|^2$$
choose an appropriate choice for λ in order to prove the Cauchy-Schwarz inequality. Hence, prove the triangle inequality.

4. (optional) Any point $\mathbf{x} = [x, y]^T$ lying on the line, $\mathcal{L}$, that is parallel to the vector $\mathbf{d} = [a, b]^T$ and contains the point $\mathbf{p} = [x_0, y_0]^T$ satisfies
$$\mathbf{x} = \mathbf{p} + t\mathbf{d},$$
for some $t \in \mathbb{R}$. We call this the vector form of the equation of a line.

   Given two lines with slopes m_1 and m_2 show that they are perpendicular if and only if $m_1 m_2 = -1$.

5. (optional) Any point $\mathbf{x} = [x, y, z]^T$ lying on the plane, $\mathcal{P}$, that is normal to the vector $\mathbf{n} = [a, b, c]^T$ and contains the point $\mathbf{p} = [x_0, y_0, z_0]^T$ satisfies
$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0.$$
We call this the normal form of the equation of a plane.

   The plane $\mathcal{P}_1$ has the equation
$$4x - y + 5z = 2.$$
Deduce whether, or not, the following planes are parallel, perpendicular, or neither

(a) $2x + 3y - z = 1$

(b) $4x - y + 5z = 0$

(c) $x - y - z = 3$

(d) $4x + 6y - 2z = 0$

6. State, with reason, whether, or not, the following are vector spaces:

   (a) Consider the set $V = \mathbb{R}^2$ with standard addition and the following definition of scalar multiplication

   $$\alpha \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} x \\ \alpha y \end{bmatrix}.$$

   (b) Consider the set $V = \mathbb{R}^2$ with standard scalar multiplication and the following definition of addition

   $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}.$$

   (c) For any $n \geq 1$ and integer, let $\mathcal{P}^n$ denote the set of all polynomials with maximum degree n or less with real coefficients.

   (d) For any $n \geq 1$ and integer, the set of all n degree polynomials with real coefficients.

7. State, with reason, whether, or not, the following are vector subspaces of the given vector space, V:

   (a) $V = \mathcal{P}^n$: The space $\mathcal{P}^{n-1}$.

   (b) $V = \mathbb{R}^2$: The set of all vectors $[x, y]^T$ such that $y = x^2$.

   (c) $V = \mathbb{R}^3$: The set of all vectors $[x, y, z]^T$ such that $x = 3y$ and $z = -2y$.

   (d) $V = \mathbb{R}^3$: The set of all vectors $[x, y, z]^T$ such that $x = 3y + 1$ and $z = -2y$.

Applications

1. A credit card number, $\mathbf{x} \in \mathbb{Z}_{10}^{16}$, consists of 16 digits, e.g.

$$\mathbf{x} = 5412\ 3456\ 7890\ 432d,$$

where d is the check digit and uses the check vector

$$\mathbf{c} = [2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1]^T.$$

Unlike the Bar Code error detection seen in class, an extra check is made in this case. Let h denote the number of digits in odd positions that are greater than 4. It is now required that

$$\mathbf{c} \cdot \mathbf{x} + h = 0 \quad \text{in} \quad \mathbb{Z}_{10}.$$

Find the check digit d. Interchange any two adjacent numbers in $\mathbf{x}$ and see if the error will be detected.

2. The international standard book number (ISBN) takes the form

$$\mathbf{b} = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, X]^T.$$

It has check vector

$$\mathbf{c} = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]^T$$

and it is required that $(\mathbf{c} \cdot \mathbf{b}) = 0$ in $\mathbb{Z}_{11}$. The ISBN for *Linear Algebra and its Applications* by Gilbert Strang is

$$\mathbf{b} = [0, 5, 3, 4, 4, 2, 2, 0, 0, x]^T.$$

Find the check digit x.

Note, since we are using $\mathbb{Z}_{11}$ the number 10 is allowed. To overcome the fact that this has two digits the Roman numeral **X** is used.

For the ISBN

$$\mathbf{b} = [0, 8, 3, 7, 0, 9, 9, 0, 2, 6]^T$$

show that an error has occurred. Given that this error was a transposition of two adjacent numbers, find the correct ISBN.