Linear Algebra - Week 6

1. Suppose U and V are vector spaces. In lectures we said that $T : U \to V$ is a linear transformation if the following conditions are satisfied:

   - if $\mathbf{u} \in U$, then $T(\mathbf{u}) \in V$;
   - if $\mathbf{u_1}, \mathbf{u_2} \in U$, then $T(\mathbf{u_1} + \mathbf{u_2}) = T(\mathbf{u_1}) + T(\mathbf{u_2})$; and
   - if $\mathbf{u} \in U$ and c is any scalar, then $T(c\mathbf{u}) = cT(\mathbf{u})$.

   (a) Prove that the conditions in the last two bullet points are equivalent to

   if $\mathbf{u_1}, \mathbf{u_2} \in U$ and c is a scalar, then $T(\mathbf{u_1} + c\mathbf{u_2}) = T(\mathbf{u_1}) + cT(\mathbf{u_2})$.

   (b) Prove that if: (i) $T : U \to V$ is a linear transformation; (ii) $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \in U$; and (iii) $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars, then

   $$T\left(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i\right) = \sum_{i=1}^{n} \alpha_i T\left(\mathbf{u}_i\right).$$

2. Let U be the set of points in two-dimensional space, i.e. if $\mathbf{u} \in U$ then $\mathbf{u}$ may be written

   $$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

   Which of the following $T : U \to U$ are linear transformations?

   (a) T is defined by

   $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ y-1 \end{pmatrix}.$$

   (b) T is defined by

   $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}.$$

   (c) T is defined by

   $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2x+3y \end{pmatrix}.$$

   (d) T is defined by

   $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}.$$

3. Let F be the set of functions $f(x)$ such that the integral $\int_0^1 f(x)\,\mathrm{d}x$ takes a finite value.

(a) Prove that F is a vector space.

(b) $T : F \to \Re$, is defined by

$$T(f(x)) = \int_0^1 f(x)\ \mathrm{d}x.$$

Evaluate $T(x^2)$ and $T(\sin \pi x)$.

(c) Prove that T is a linear transformation.

4. (a) Write down the matrix representing a rotation of an angle θ about the origin.
   (b) Write down the matrix representing a reflection in the y-axis.
   (c) Write down the matrix representing a reflection in the line $y = x$.
   (d) Write down the product of matrices that represents: a reflection in the line $y = x$, followed by a rotation of an angle θ about the origin, followed by a reflection in the y-axis.

5. In this question we will develop a matrix representation of a rotation anti-clockwise through an angle $\pi/2$ (i.e. 90 degrees) about the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. You will need to use homogeneous coordinates.

   (a) Write down the matrix representing a translation of the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ to the origin.
   (b) Write down the matrix representing (in homogeneous coordinates) a rotation anti-clockwise through an angle $\pi/2$ about the origin.
   (c) Write down the matrix representing a translation of the origin to the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.
   (d) Write down the product of matrices that represents a rotation anti-clockwise through an angle $\pi/2$ about the point $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

6. Suppose a composite linear transformation, $T : U \to V$, is defined, for $\mathbf{u} \in U$, by

$$T(\mathbf{u}) = CBA\mathbf{u},$$

where the dimension of U is n, and A, B, C are matrices of size $n \times n$.

   (a) Explain why implementing an algorithm to evaluate $A\mathbf{u}$ will require n^2 additions and n^2 multiplications.
   (b) If $\mathbf{v} = T(\mathbf{u})$, then we may write, for $i = 1, 2, \ldots, n$:

$$v_i = \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{p=1}^{n} C_{ij} B_{jk} A_{kp} u_p.$$

   The n entries of $\mathbf{v}$ may be evaluated by putting the sum above into nested `for` loops. Explain why this would require n^4 additions and $3n^4$ multiplications.
   (c) By writing

$$T(\mathbf{u}) = C(B(A\mathbf{u})),$$

   explain how $T(\mathbf{u})$ can be evaluated in $3n^2$ additions and $3n^2$ multiplications.

(d) Would you use the method for calculating $T(\mathbf{u})$ from part (b), or the one from part (c)? Why?

7. For T defined in parts (c) and (d) of Question 2:

   (a) Write down the kernel of T.

   (b) Write down the image of T.

   (c) Explain whether T is an onto transformation.

   (d) Explain whether T is a 1-1 transformation.

8. The transformation $T : \Re^5 \to \Re^3$ is defined by

$$T(\mathbf{u}) = A\mathbf{u},$$

where $\mathbf{u} \in \Re^5$, and A is the matrix given by

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{pmatrix}.$$

Find the image of T, and the kernel of T. What is the nullity and rank of T?