# Discrete Mathematics

| | *topic* |
|---|---|
| week 1 | ***Sets*** |
| week 2 | ***Functions*** |
| week 3 | ***Counting*** |
| week 4 | $\boxed{\text{week 4} \quad \textbf{\textit{Relations}}}$ |
| week 5 | ***Sequences*** |
| week 6 | ***Modular Arithmetic*** |
| week 7 | ***Asymptotic Notation*** |
| week 8 | ***Orders*** |


**Jonathan Barrett**

jonathan.barrett@cs.ox.ac.uk

***Material by Andrew Ker***

***University of Oxford***

***Department of Computer Science***

# Discrete Mathematics

**Jonathan Barrett**
jonathan.barrett@cs.ox.ac.uk
*Material by Andrew Ker*
*University of Oxford*
*Department of Computer Science*

# Chapter 4: Relations

# Definition

A **relation** on $A$ is a subset of $A \times A$.

Relations are usually written **infix**: $a \, R \, b$ instead of $(a, b) \in R$

$a \not R \, b$ instead of $(a, b) \notin R$

# Definition

A **relation** on $A$ is a subset of $A \times A$.

Relations are usually written **infix**: $a \, R \, b$ instead of $(a, b) \in R$
$a \not R \, b$ instead of $(a, b) \notin R$

*More generally, we say that a relation from $A$ to $B$ is a subset of $A \times B$.*

*Even more generally, a relation can be a subset on any cartesian product, e.g. a ternary relation between $A$, $B$ and $C$ is a subset of $A \times B \times C$.*

# Properties of Relations

Let $R$ be a relation on $A$.

We say that $R$ is

- **reflexive** if $a \, R \, a$ for all $a \in A$
- **symmetric** if $a \, R \, b \Rightarrow b \, R \, a$ for all $a, b \in A$
- **antisymmetric** if $a \, R \, b$ and $b \, R \, a \Rightarrow a = b$ for all $a, b \in A$
- **transitive** if $a \, R \, b$ and $b \, R \, c \Rightarrow a \, R \, c$ for all $a, b, c \in A$

# Properties of Relations

Let $R$ be a relation on $A$.

We say that $R$ is

- **reflexive** if $a\,R\,a$ for all $a \in A$
- **symmetric** if $a\,R\,b \Rightarrow b\,R\,a$ for all $a, b \in A$
- **antisymmetric** if $a\,R\,b$ and $b\,R\,a \Rightarrow a = b$ for all $a, b \in A$
- **transitive** if $a\,R\,b$ and $b\,R\,c \Rightarrow a\,R\,c$ for all $a, b, c \in A$

- **irreflexive** if $a \not R\, a$ for all $a \in A$
- **serial** if for every $a \in A$ there is some $b \in A$ with $a\,R\,b$

# Properties of Relations

Let $R$ be a relation on $A$.

We say that $R$ is

- **reflexive** if $a\,R\,a$ for all $a \in A$
- **symmetric** if $a\,R\,b \Rightarrow b\,R\,a$ for all $a, b \in A$
- **antisymmetric** if $a\,R\,b$ and $b\,R\,a \Rightarrow a = b$ for all $a, b \in A$
- **transitive** if $a\,R\,b$ and $b\,R\,c \Rightarrow a\,R\,c$ for all $a, b, c \in A$

- **irreflexive** if $a \not R\, a$ for all $a \in A$
- **serial** if for every $a \in A$ there is some $b \in A$ with $a\,R\,b$

*NB: "not reflexive" and "irreflexive" do not mean the same thing.*

*NB: "not symmetric" and "antisymmetric" do not mean the same thing.*

# The Divides Relation

We write

$$m \mid n$$

if $n$ is an integer multiple of $m$. This is a relation on $\mathbb{N}_+$ (can be extended to $\mathbb{Z}$ ).

# Equivalence Relations

An **equivalence relation** on $A$ is a relation which is

**reflexive**, **symmetric**, and **transitive**

$a \, R \, a$

$a \, R \, b \Rightarrow b \, R \, a$

$a \, R \, b$ and $b \, R \, c \Rightarrow a \, R \, c$

If $\sim$ is an equivalence relation on $A$ then for each $a \in A$ we write

$$[a] = \{a' \in A \mid a' \sim a\}$$

This called the **equivalence class** of $a$.

# Partitions

A **partition** of a set $A$ is a collection of subsets $\{B_i \mid i \in I\} \subseteq \mathcal{P}(A)$ satisfying

i. $\displaystyle\bigcup_{i \in I} B_i = A$

ii. $B_i \cap B_j = \emptyset$ for $i \neq j$

iii. $B_i \neq \emptyset$ for any $i$

# Partitions

A **partition** of a set $A$ is a collection of subsets $\{B_i \mid i \in I\} \subseteq \mathcal{P}(A)$ satisfying

i. $\bigcup_{i \in I} B_i = A$ ← ***The $B_i$ cover A***

ii. $B_i \cap B_j = \emptyset$ for $i \neq j$ ← ***The $B_i$ are disjoint***

iii. $B_i \neq \emptyset$ for any $i$ ← ***The $B_i$ are nonempty***

# Partitions

A **partition** of a set $A$ is a collection of subsets $\{B_i \mid i \in I\} \subseteq \mathcal{P}(A)$ satisfying

i. $\displaystyle\bigcup_{i \in I} B_i = A$ ← ***The $B_i$ cover $A$***

ii. $B_i \cap B_j = \emptyset$ for $i \neq j$ ← ***The $B_i$ are disjoint***

iii. $B_i \neq \emptyset$ for any $i$ ← ***The $B_i$ are nonempty***

<u>Claim</u> If $\sim$ is an equivalence relation on $A$ then:

(a) the equivalence classes form a partition of $A$;
(b) any partition of $A$ defines an equivalence relation;
(c) different equivalence relations correspond to different partitions.

# The Divides Relation

We write

$$m \mid n$$

if $n$ is an integer multiple of $m$. This is a relation on $\mathbb{N}_+$ (can be extended to $\mathbb{Z}$ ).

# Modular Congruence

Fix a positive integer $n$. We can define an equivalence relation $\equiv$ on $\mathbb{Z}$ by

$$x \equiv y \pmod{n} \quad \text{if} \quad n \mid (x - y)$$

*$x \equiv y \pmod{n}$ can be understood as*

*"$x$ and $y$ have the same remainder when divided by $n$."*

# Observational Equivalence

Define a relation on **programs** (in some fixed language) by

$$P_1 \approx P_2$$

If, given the same inputs, $P_1$ and $P_2$ always give the same outputs. It is an equivalence relation.

***This is "equality" in functional programs if we take an <u>extensional</u> view.***

# Converse & Composition

If $R$ is a relation on $A$ then we define the **converse** relation by

$$a\,R^{-1}b \quad \text{if} \quad b\,R\,a.$$

If $R$ and $S$ are both relations on $A$ then we define their composition $S \circ R$ by

$$a\,(S \circ R)\,b \text{ if there is some } x \in A \text{ such that } a\,R\,x \text{ and } x\,S\,b.$$

***There are close connections with functional inverse and composition.***

# Transitive Closure

If $R$ is a relation on $A$ then we define the **transitive closure** of $R$, by

$$a\,R^{+}\,b$$

if there is some sequence $x_0, x_1, \ldots, x_n \in A$ with $n \geq 1$ such that

$$a = x_0, \quad x_0\,R\,x_1, \quad x_1\,R\,x_2, \quad \ldots, \quad x_{n-1}\,R\,x_n, \quad x_n = b$$

# Transitive Closure

If $R$ is a relation on $A$ then we define the **transitive closure** of $R$, by

$$a\,R^{+}\,b$$

if there is some sequence $x_0, x_1, \ldots, x_n \in A$ with $n \geq 1$ such that

$$a = x_0, \quad x_0\,R\,x_1, \quad x_1\,R\,x_2, \quad \ldots, \quad x_{n-1}\,R\,x_n, \quad x_n = b$$

If $R$ is a relation on $A$ then we define the **reflexive transitive closure** of $R$, by

$$a\,R^{*}\,b$$

if there is some sequence $x_0, x_1, \ldots, x_n \in A$ with $n \geq 0$ such that

$$a = x_0, \quad x_0\,R\,x_1, \quad x_1\,R\,x_2, \quad \ldots, \quad x_{n-1}\,R\,x_n, \quad x_n = b$$

*$a\,R^{+}\,b$ means that you can get from $a$ to $b$ by "doing" $R$ at least once.*

*$a\,R^{*}\,b$ means that you can get from $a$ to $b$ by "doing" $R$ zero or more times.*

# Directed Graphs

A **directed graph** consists of a set of **nodes** $N$ and a set of **edges** $E \subseteq N \times N$. We say that there is an edge from $n_1$ to $n_2$ if $(n_1, n_2) \in E$.

Digraphs are depicted by drawing the nodes, as labelled points in the plane, and an arrow from $n_1$ to $n_2$ whenever $(n_1, n_2) \in E$.

# Directed Graphs

A **directed graph** consists of a set of **nodes** $N$ and a set of **edges** $E \subseteq N \times N$. We say that there is an edge from $n_1$ to $n_2$ if $(n_1, n_2) \in E$.

Digraphs are depicted by drawing the nodes, as labelled points in the plane, and an arrow from $n_1$ to $n_2$ whenever $(n_1, n_2) \in E$.

***This is the same as a relation on $N$.***

***It is rather common, in mathematics and computer science, to see the same definition given different terminology in different applications.***

# Diversion: Counting Relations

<u>Example</u> If $|A| = n$, how many relations are there, on $A$?

# Diversion: Counting Relations

<u>Example</u> If $|A| = n$, how many relations are there, on $A$?

<u>Answer</u> $2^{n^2}$

# Diversion: Counting Relations

<u>Example</u> If $|A| = n$, how many relations are there, on $A$?

<u>Answer</u> $2^{n^2}$

<u>Example</u> If $|A| = n$, how many **symmetric** relations are there, on $A$?

# Diversion: Counting Relations

Example If $|A| = n$, how many relations are there, on $A$?

Answer $2^{n^2}$

Example If $|A| = n$, how many **symmetric** relations are there, on $A$?

Answer $2^{\frac{n(n+1)}{2}}$

*It is also quite easy to count reflexive relations, antisymmetric relations, and any combinations of these properties.*

*It is much harder to count the number of transitive relations.*

*The number of equivalence relations equals the number of partitions.*

# Discrete Mathematics

**Jonathan Barrett**
jonathan.barrett@cs.ox.ac.uk
***Material by Andrew Ker***
***University of Oxford***
***Department of Computer Science***

# End of Chapter 4