CONTINUOUS MATHEMATICS

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HT 2019

PROBLEM SHEET 2

Optimization, Convexity, Lagrange Multipliers, and Numerical integration

2.1. We want to find and classify the stationary points of:

(a) $g(x,y) = e^{x^3-3x} e^{y^3-3y}$

$$\frac{dg}{d\underline{x}} = \underline{0} \iff \frac{\partial g}{\partial x} = 0 \text{ and } \frac{\partial g}{\partial y} = 0$$

$$\frac{\partial g}{\partial x} = 0 \iff \underbrace{e^{x^3-3x}}_{\neq 0}(3x^2-3)\underbrace{e^{y^3-3y}}_{\neq 0} = 0 \iff 3(x^2-1) = 0 \iff x = \pm 1$$

$$\frac{\partial g}{\partial y} = 0 \iff \underbrace{e^{x^3-3x}}_{\neq 0}\underbrace{e^{y^3-3y}}_{\neq 0}(3y^2-3) = 0 \iff 3(y^2-1) = 0 \iff y = \pm 1$$

We have the following stationary points for g: $(-1,-1)$, $(-1,1)$, $(1,-1)$ and $(1,1)$.
To classify them, we need the Hessian of g

$$\frac{\partial^2 g}{\partial x^2} = e^{y^3-3y}\left(e^{x^3-3x}(3x^2-3)^2 + e^{x^3-3x}(6x)\right)$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x} = e^{x^3-3x}(3x^2-3)\,e^{y^3-3y}(3y^2-3)$$

$$\frac{\partial^2 g}{\partial y^2} = e^{x^3-3x}\left(e^{y^3-3y}(3y^2-3)^2 + e^{y^3-3y}(6y)\right)$$

Then, we have:

$$H(g)(-1,-1) = \begin{pmatrix} -6e^4 & 0 \\ 0 & -6e^4 \end{pmatrix} = -6e^4 I \text{, which is negative definite} \Rightarrow$$

$\Rightarrow (-1,-1)$ is a LOCAL MAXIMUM for g

$$H(g)(-1,1) = \begin{pmatrix} -6 & 0 \\ 0 & 6 \end{pmatrix} \text{, which is indefinite as } \begin{vmatrix} -6 & 0 \\ 0 & 6 \end{vmatrix} = -36 < 0 \Rightarrow$$

$\Rightarrow (-1,1)$ is a SADDLE POINT for g

$$H(g)(1,-1) = \begin{pmatrix} 6 & 0 \\ 0 & -6 \end{pmatrix} \text{, which is indefinite as } \begin{vmatrix} 6 & 0 \\ 0 & -6 \end{vmatrix} = -36 < 0 \Rightarrow$$

$\Rightarrow (1,-1)$ is a SADDLE POINT for g

$$H(g)(1,1) = \begin{pmatrix} 6e^{-4} & 0 \\ 0 & 6e^{-4} \end{pmatrix} = 6e^{-4} I \text{, which is positive definite} \Rightarrow$$

$\Rightarrow (1,1)$ is a LOCAL MINIMUM for g

(b) $h(x,y,z) = \frac{1}{2}x^2 + y^2 + \frac{27}{2}z^2 - xy + xz + 4yz - x - y - 11z$

$$\left.\begin{array}{l} \frac{\partial h}{\partial x} = x - y + z - 1 \\ \frac{\partial h}{\partial y} = -x + 2y + 4z - 1 \\ \frac{\partial h}{\partial z} = x + 4y + 27z - 11 \\ \frac{dh}{d\underline{x}} = \underline{0} \end{array}\right| \Rightarrow \left\{\begin{array}{l} x - y + z - 1 = 0 \Rightarrow x = y - z + 1 \\ -x + 2y + 4z - 1 = 0 \\ x + 4y + 27z - 11 = 0 \end{array}\right| \Rightarrow \left\{\begin{array}{l} y + 5z = 2 \\ 5y - 26z = 10 \end{array}\right| \Rightarrow y = 2,\ z = 0 \Rightarrow x = 3$$

So, the only stationary point that h has is $(3,2,0)$.

Let's find the Hessian:

$$\frac{\partial^2 h}{\partial x^2} = 1 \ ;\ \frac{\partial^2 h}{\partial y^2} = 2;\ \frac{\partial^2 h}{\partial z^2} = 27;\ \frac{\partial^2 h}{\partial x \partial y} = -1\ ;\ \frac{\partial^2 h}{\partial x \partial z} = 1;\ \frac{\partial^2 h}{\partial y \partial z} = 4$$

So, $H(h)(3,2,0) = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 4 \\ 1 & 4 & 27 \end{pmatrix}$

We have the upper-left submatrix $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ with eigenvalues $\left.\begin{array}{l}\lambda_1 = \frac{3-\sqrt{5}}{2} > 0 \\ \lambda_2 = \frac{3+\sqrt{5}}{2} > 0\end{array}\right| \Rightarrow$ it is positive definite $\Rightarrow H(h)(3,2,0)$ is positive definite $\Rightarrow (3,2,0)$ is a local minimum.

$\boxed{2.2.}$ Let $\underline{a} \in \mathbb{R}^n$ with $a_i > 0$ and $\sum_i a_i = 1$

Let Δ_a = diagonal matrix with $\underline{a}$ on the diagonal

We have $\Delta_a - \underline{a}\,\underline{a}^T$ positive semidefinite if $\underline{a}^T \Delta_a^{-1} \underline{a} \leq 1$.

$$\Delta_a = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix} \Rightarrow \Delta_a^{-1} = \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{a_n} \end{pmatrix}$$

$$\underline{a}^T \Delta_{\underline{a}} \underline{a} = (a_1\ a_2 \dots a_n) \begin{pmatrix} \frac{1}{a_1} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{a_n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \mathbb{1}\,\underline{a} = \sum_i a_i = 1 \leq 1 \Rightarrow$$

$\Rightarrow \Delta_a - \underline{a}\,\underline{a}^T$ is positive semidefinite.

We want to prove that l is convex, where

$$l(\underline{x}) = \ln\left(\sum_{i=1}^{n} e^{x_i}\right)$$

$l = f \circ g$, where $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \ln x$, which is concave and strictly increasing

$g: \mathbb{R}^n \to \mathbb{R}$, $g(\underline{x}) = \sum_{i=1}^{n} e^{x_i}$, which is convex, as a sum of convex functions $\Rightarrow$

$\Rightarrow f \circ g$ is convex, therefore l is convex.

<u>Alternative solution</u>

$$\frac{dl}{d\underline{x}} = \frac{1}{\sum_{i=1}^{n} e^{x_i}} \cdot \begin{pmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_n} \end{pmatrix}$$

$$H(l) = J\left(\frac{dl}{d\underline{x}}\right) = J\left(\frac{1}{\sum_{i=1}^{n} e^{x_i}} \cdot \begin{pmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_n} \end{pmatrix}\right) = \begin{pmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_n} \end{pmatrix} \cdot \left(-\frac{1}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2} \cdot \begin{pmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_n} \end{pmatrix}\right)^T +$$

$$+ \frac{1}{\sum_{i=1}^{n} e^{x_i}} \cdot \begin{pmatrix} e^{x_1} & 0 & 0 & \dots & 0 \\ 0 & e^{x_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{x_n} \end{pmatrix} = -\begin{pmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_n} \end{pmatrix} \cdot \left(e^{x_1}\ e^{x_2} \dots e^{x_n}\right) \cdot \frac{1}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2} + \frac{1}{\sum_{i=1}^{n} e^{x_i}} \cdot$$

$$\cdot \begin{pmatrix} e^{x_1} & 0 & 0 & \dots & 0 \\ 0 & e^{x_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{x_n} \end{pmatrix}$$

Let $\underline{a} = \frac{1}{\sum_{i=1}^{n} e^{x_i}} \cdot \begin{pmatrix} e^{x_1} \\ e^{x_2} \\ \vdots \\ e^{x_n} \end{pmatrix} \Rightarrow a_i > 0$ and $\sum_{i=1}^{n} a_i = 1$ and

$H(l) = -\underline{a} \cdot \underline{a}^T + \Delta_a$, which we know it's positive semi-definite, therefore l is convex.

$\boxed{2.3.}$ X, Y, Z with costs p, q, r respectively, $p, q, r > 0$ different, budget $B > 0$.

The value of a product is $g(x, y, z)$.

(a) We want to maximise f on

$$F = \left\{ (x, y, z) \mid x, y, z \geq 0,\ px + qy + rz - B \leq 0 \right\}$$

(b) $g(x, y, z) = xyz$

We have g increasing in all dimensions, so if we want to maximize g with the constraint $px + qy + rz - B \leq 0$, we need to have equality there, so the constraint has to be <u>tight</u>, therefore we can transform it into an equality constraint.

Let $\Lambda(\lambda, \underline{v}) = xyz + (B - px - qy - rz)\lambda$, $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

We need

$$\left.\begin{array}{l} B - px - qy - rz = 0 \\ yz = p\lambda \\ xz = q\lambda \\ xy = r\lambda \end{array}\right| \Rightarrow \left.\begin{array}{l} x = \sqrt{\dfrac{\lambda q r}{p}} \qquad z = \sqrt{\dfrac{\lambda p q}{r}} \\ y = \sqrt{\dfrac{\lambda p r}{q}} \end{array}\right| \Rightarrow B = 3\sqrt{\lambda p q r} \Rightarrow \lambda = \frac{B^2}{9pqr} \Rightarrow$$

$\Rightarrow x = \dfrac{B}{3p}$, $y = \dfrac{B}{3q}$, $z = \dfrac{B}{3r}$ (all positive) $\Rightarrow$ the maximum of g on F is $\dfrac{B^3}{27pqr}$.

(c) We now want to maximise $g(\underline{v}) = e^{x+y+z}$, which is also increasing in every dimension, on F. So, we need: $\left(\Lambda(\lambda, \underline{v}) = e^{x+y+z} + (B - px - qy - rz)\lambda \right)$

$$\left.\begin{array}{l} B - px - qy - rz = 0 \\ e^{x+y+z} = \lambda p \\ e^{x+y+z} = \lambda q \\ e^{x+y+z} = \lambda r \end{array}\right| \Rightarrow \lambda = 0 \Rightarrow e^{x+y+z} = 0, \text{ which is impossible!}$$

Notice that we didn't suceed with set of constraints we have, so we need to try all the possibilities of combinations of tight/slack constraints. We keep $B - px - qy - rz = 0$, as that is obvious from the increasing nature of g in all dimensions. For $x = 0$ and $y, z \neq 0$ we get

$$\left.\begin{array}{l} e^{y+z} = \lambda q \\ e^{y+z} = \lambda r \end{array}\right| \Rightarrow \lambda = 0 \Rightarrow e^{y+z} = 0, \text{ which is impossible!}$$

For $x = y = 0$, we get

$$g(x, y, z) = g(0, 0, z) = e^z$$

$$\Lambda(\lambda, \underline{v}) = e^z + (B - rz)\lambda$$

$$\Lambda(\lambda, v) = e^z + B\lambda - rz\lambda$$

$$\left.\begin{array}{l} B - rz = 0 \Rightarrow z = \dfrac{B}{r} \\ e^z = r\lambda \Rightarrow \lambda = \dfrac{e^z}{r} = \dfrac{e^{\frac{B}{r}}}{r} \end{array}\right| \Rightarrow e^{\frac{B}{r}} \text{ is the maximum}$$

To get the maximum value, we need $\max\left\{\dfrac{B}{p}, \dfrac{B}{q}, \dfrac{B}{r}\right\}$, which is $\dfrac{B}{\min\{p,q,r\} := \min} \Rightarrow$

$\Rightarrow$ the result is $e^{\frac{B}{\min}}$.

$\boxed{2.4.}$ $w_{ij} = w_{ji}$

$$\underset{x_1,\dots,x_n}{\text{minimize}}\ E(\underline{x}) = \sum_i \sum_j w_{ij}(x_i - x_j)^2 \text{ subject to } \sum_i x_i^2 = 1,\ \sum_i x_i = 0.$$

(a) $$E(\underline{x}) = \sum_i \sum_j w_{ij}(x_i - x_j)^2 = \sum_i \sum_j w_{ij}(x_i^2 - 2x_i x_j + x_j^2) =$$

$$= 2\sum_i \sum_j w_{ij} x_i^2 - 2\sum_i \sum_j x_i x_j w_{ij} =$$

$\sum_i \sum_j w_{ij} x_i^2 = \sum_i x_i^2 \sum_j w_{ij} = \sum_i x_i^2 \sum_j w_{ji} = \underline{x}^T D \underline{x}$, where D is the diagonal matrix that satisfies $\underline{1}^T D = \underline{1}^T W$, and W is the symmetric matrix (w_{ij})

$$\sum_i \sum_j x_i x_j W_{ij} = \underline{x}^T W \underline{x}$$

So, $E(\underline{x}) = 2\underline{x}^T D \underline{x} - 2\underline{x}^T W \underline{x} = 2\underline{x}^T(D - W)\underline{x}$

We have $\sum_i x_i^2 = 1 \Rightarrow \underline{x}^T\underline{x} = 1$ and $\sum_i x_i = 0 \Rightarrow \underline{1}^T\underline{x} = 0$

(b) We have $\underline{x}^T(D-W)\underline{x} = \frac{1}{2}\sum_i \sum_j \underbrace{w_{ij}}_{\geq 0}\underbrace{(x_i - x_j)^2}_{\geq 0} \geq 0\ (\forall)\ \underline{x} \in \mathbb{R}^n \Rightarrow$

$\Rightarrow (D-W)$ is positive semidefinite

(c) $\frac{dE}{d\underline{x}} = 2\cdot 2(D-W)\cdot\underline{x} = 4(D-W)\underline{x}$

$\frac{d}{d\underline{x}}\left(\sum_i x_i^2\right) = 2\underline{x}$

$\frac{d}{d\underline{x}}\left(\sum_i x_i\right) = \underline{1}$

$\Rightarrow$ The first-order condition for a stationary point of the Lagrangian is

$$4(D-W)\underline{x} - 2\lambda\underline{x} - \mu\underline{1} = \underline{0} \quad (*)$$

By multiplying $(*)$ with $\underline{1}^T$, we get:

$$4\underbrace{\underline{1}^T(D-W)\underline{x}}_{=0} - 2\lambda\underbrace{\underline{1}^T\underline{x}}_{=0} - \mu\underline{1}^T\underline{1} = 0 \Rightarrow \mu n = 0\ (\forall)\ n \in \mathbb{N}^* \Rightarrow \mu = 0$$

(d) Let $\underline{x} \in \mathbb{R}^n$ be a solution to $(*)$. Then:

$4(D-W)\underline{x} = 2\lambda\underline{x}$

$(D-W)\underline{x} = \frac{\lambda}{2}\underline{x}$, where $\frac{\lambda}{2} \in \mathbb{R} \Rightarrow \underline{x}$ is an eigenvector of $(D-W)$ with the corresponding eigenvalue $\frac{\lambda}{2} = \lambda$

Then, we have

$$E(\underline{x}) = 2\,\underline{x}^T(D-W)\,\underline{x} = 2\,\underline{x}^T \lambda\, \underline{x} = 2\lambda\, \underline{x}^T\underline{x} \overset{\text{as long as the constaints are satisfied}}{=} 2\lambda$$

$$(D-W)\underline{x} = \begin{pmatrix} \sum_{j=1}^{n} w_{j1} - w_{11} & -w_{12} & \cdots & -w_{1n} \\ -w_{21} & \sum_{j=1}^{n} w_{j2} - w_{22} & \cdots & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & \cdots & \sum_{j=1}^{n} w_{jn} - w_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

Let c_j be the sum on the j^{th} column of $D-W$. Then $c_j = 0$ $(\forall)\, j \in \{1, 2, \dots, n\}$

Then the vector from LHS has the sum of the elements 0 (because it is $\sum_j c_j x_j$), so

$$\underline{1}^T(D-W)\,\underline{x} = 0 \Rightarrow \underline{1}^T\lambda\,\underline{x} = 0 \Rightarrow \lambda(\underline{1}^T\underline{x}) = 0$$

if $\lambda = 0$, then $\underline{x}$ can be any vector, so it doesn't necessarily have $\underline{1}^T\underline{x} = 0$.
Unless $\lambda = 0$, we have $\underline{1}^T\underline{x} = 0$.

(e) Let $\underline{x}$ be the solution to the optimization problem. Then we have

$$\sum_i x_i^2 = 1 \Rightarrow \underline{x}^T\underline{x} = 1 \Rightarrow \underline{x} \text{ is a unit vector}$$

From (d), we know that $\underline{x}$ is an eigenvector of $(D-W)$ and its eigenvalue λ is non-zero. As $D-W$ is positive semi-definite, $\lambda > 0$.

We have that $\frac{dE}{d\underline{x}} = 4(D-W)\underline{x}$ and $H(E) = 4(D-W)$, which is positive semi-definite, so E is a convex function. Then, if $\underline{x}$ is a local minimum for E, then it must also be a global minimum.

We know that all the solutions are eigenvectors of $(D-W)$ with non-zero eigenvalues. We will prove that if λ is the smallest eigenvalue of $(D-W)$ (non-zero), then it is the minimum of E:

Let's suppose that $(\exists)\, \underline{v} \in \mathbb{R}^n$ eigenvector of $(D-W)$ with corresponding eigenvalue $\alpha \geq \lambda$. Then, we have

$$\left.\begin{array}{l} E(\underline{x}) = 2\,\underline{x}^T(D-W)\underline{x} = 2\,\underline{x}^T\lambda\,\underline{x} = 2\lambda\,\underline{x}^T\underline{x} = 2\lambda \\ E(\underline{v}) = 2\,\underline{v}^T(D-W)\underline{v} = 2\,\underline{v}^T\alpha\,\underline{v} = 2\alpha\,\underline{v}^T\underline{v} = 2\alpha \geq 2\lambda \end{array}\right| \Rightarrow E(\underline{x}) \leq E(\underline{v}) \Rightarrow \underline{x} \text{ is the}$$

global minimum of E, so it is the solution of the optimization problem. (As λ is the smallest eigenvalue which is non-zero, and because the smallest eigenvalue of $(D-W)$ is 0, then λ is the <u>second</u>-smallest eigenvalue of $(D-W)$).

$\boxed{2.5.}$ Let $f(x) = e^{e^x}$. We will compute the first 4 derivatives of f: (all of them increasing strictly)

$$\frac{df}{dx} = e^{e^x} \cdot e^x$$

$$\frac{d^2f}{dx^2} = e^{e^x} \cdot e^{2x} + e^{e^x} \cdot e^x = e^{e^x+2x} + e^{e^x+x}$$

$$\frac{d^3f}{dx^3} = e^{e^x+2x} \cdot (e^x+2) + e^{e^x+x} \cdot (e^x+1) = e^{e^x+3x} + 3e^{e^x+2x} + e^{e^x+x}$$

$$\frac{d^4f}{dx^4} = e^{e^x+3x} \cdot (e^x+3) + 3e^{e^x+2x} \cdot (e^x+2) + e^{e^x+x}(e^x+1) = e^{e^x+4x} + 6e^{e^x+3x} + 7e^{e^x+2x} + e^{e^x+x}$$

(a) Using the Midpoint rule: (to calculate $\int_0^1 f(x)dx$ with n strips)

$$-\frac{1}{24n^2}\overline{D_2} \leq err(M_n)[f,0,1] \leq -\frac{1}{24n^2}\underline{D_2}$$

$$\overline{D_2} = \max_{x\in(0,1)} \frac{d^2f}{dx^2} = \frac{d^2f}{dx^2}(1) = e^{e+2} + e^{e+1} \approx 153.17$$

$$\underline{D_2} = \min_{x\in(0,1)} \frac{d^2f}{dx^2} = \frac{d^2f}{dx^2}(0) = e^1 + e^1 = 2e \approx 5.44$$

We want $|err(M_n)[f,0,1]| < 10^{-6}$, but $|err(M_n)[f,0,1]| \leq \frac{1}{24n^2}\overline{D_2}$, so we want n such that

$$\frac{1}{24n^2} \cdot 153.17 \leq 10^{-6}$$

$$6.38 \cdot 10^6 \leq n^2$$

$$2.526 \cdot 10^3 \leq n \Rightarrow \boxed{n \geq 2526}$$

(b) Using the Trapezium rule:

$$\frac{1}{12n^2}\underline{D_2} \leq err(T_n)[f,a,b] \leq \frac{1}{12n^2}\overline{D_2}$$

$$\frac{1}{12n^2} \cdot 153.17 \leq 10^{-6}$$

$$12.764 \cdot 10^6 \leq n^2$$

$$3.573 \cdot 10^3 \leq n \Rightarrow \boxed{n \geq 3573}$$

(c) Using Simpson's rule:

$$\frac{1}{180n^4}\underline{D_4} \leq err(S_n)[f,0,1] \leq \frac{1}{180n^4}\overline{D_4}$$

$$\underline{D_4} = \min_{x\in(0,1)} \frac{d^4f}{dx^4} = e^1 + 6e^1 + 7e^1 + e^1 = 15e \approx 40.77$$

$$\overline{D_4} = \max_{x\in(0,1)} \frac{d^4 f}{dx^4} = e^{e+4} + 6e^{e+3} + 7e^{e+2} + e^{e+1} \approx 3478.71$$

$$\frac{1}{180 n^4} \cdot 3478.71 \leq 10^{-6}$$

$$19.326 \cdot 10^6 \leq n^4 \Rightarrow \boxed{n \geq 67}$$

$\boxed{2.6.}$ $$\hat{f}_3(x) = f(\ell) + (x-\ell)\frac{df}{dx}(\ell) + \frac{(x-\ell)^2}{2}\frac{d^2f}{dx^2}(\ell) + \frac{(x-\ell)^3}{6}\frac{d^3 f}{dx^3}(\ell)$$

(a) $$A_1[f,0,2\ell] = \int_0^{2\ell} \hat{f}_3(x)dx = \int_0^{2\ell}\left[f(\ell) + (x-\ell)\frac{df}{dx}(\ell) + \frac{(x-\ell)^2}{2}\frac{d^2f}{dx^2}(\ell) + \frac{(x-\ell)^3}{6}\frac{d^3f}{dx^3}\right]dx$$

$$A_1[f,0,2\ell] = \left[xf(\ell) + \frac{(x-\ell)^2}{2}\frac{df}{dx}(\ell) + \frac{(x-\ell)^3}{6}\frac{d^2f}{dx^2}(\ell) + \frac{(x-\ell)^4}{24}\frac{d^3f}{dx^3}(\ell)\right]_0^{2\ell}$$

$$A_1[f,0,2\ell] = 2\ell f(\ell) + \frac{2\ell^3}{6}\frac{d^2f}{dx^2}(\ell) = 2\ell f(\ell) + \frac{\ell^3}{3}\frac{d^2f}{dx^2}(\ell)$$

(b) $$\underline{D_4} = \min_{\xi\in(0,2l)} \left|\frac{d^4f}{dx^4}(\xi)\right|, \quad \overline{D_4} = \max_{\xi\in(0,2l)}\left|\frac{d^4f}{dx^4}(\xi)\right|$$

$$f(x) = \hat{f}_3(x) + \frac{(x-\ell)^4}{24}\frac{d^4f}{dx^4}(\xi), \text{ with } \xi\in(0,2\ell)$$

$$\hat{f}_3(x) - f(x) = -\frac{(x-\ell)^4}{24}\cdot\frac{d^4f}{dx^4}(\xi)$$

$$\underline{D_4} \leq \frac{d^4f}{dx^4}(\xi) \leq \overline{D_4} \quad \Big| \cdot \left(-\frac{(x-\ell)^4}{24}\right) \leq 0$$

$$-\frac{(x-\ell)^4}{24}\overline{D_4} \leq \hat{f}_3(x) - f(x) \leq -\frac{(x-\ell)^4}{24}\underline{D_4}$$

(c) We have that (from (b))

$$\frac{(x-\ell)^4}{24}\underline{D_4} \leq f(x) - \hat{f}_3(x) \leq \frac{(x-\ell)^4}{24}\overline{D_4}$$

$$\int_0^{2\ell}\frac{(x-\ell)^4}{24}\underline{D_4}\,dx \leq \int_0^{2\ell} f(x)\,dx - \int_0^{2\ell}\hat{f}_3(x)\,dx \leq \int_0^{2\ell}\frac{(x-\ell)^4}{24}\overline{D_4}\,dx$$

$$\frac{\ell^5}{60}\underline{D_4} \leq \int_0^{2\ell} f(x)dx - A_1[f,0,2\ell] \leq \frac{\ell^5}{60}\overline{D_4} \quad \Big| \cdot (-1)$$

$$-\frac{1}{60}\ell^5\overline{D_4} \leq err(A_1)[f,0,2\ell] \leq -\frac{1}{60}\ell^5\underline{D_4}$$

(d) From the subtasks above, we have

$$A_1[f,0,2\ell] = 2\ell f(\ell) + \frac{\ell^3}{3}\frac{d^2 f}{dx^2}(\ell)$$

$$-\frac{1}{60}\ell^5 \overline{D_4} \leq err(A_1)[f,0,2\ell] \leq -\frac{1}{60}\ell^5 \underline{D_4}$$

First, we express a single strip from c to d

$$A_1[f,c,d] = \frac{d-c}{2\ell} A_1\left[x \mapsto f\left(c+\frac{d-c}{2\ell}x\right), 0, 2\ell\right] =$$

$$= \frac{d-c}{2\ell}\left(2\ell f\left(c+\frac{d-c}{2\ell}\ell\right) + \frac{\ell^3}{3}\frac{d^2 f}{dx^2}\left(c+\frac{d-c}{2\ell}\ell\right)\right) =$$

$$= (d-c) f\left(\frac{c+d}{2}\right) + \frac{d-c}{2\ell}\cdot\frac{\ell^3}{3}\cdot\frac{(d-c)^2}{4\ell^2}\cdot\frac{d^2 f}{dx^2}\left(\frac{c+d}{2}\right)$$

$$= (d-c) f\left(\frac{c+d}{2}\right) + \frac{(d-c)^3}{24}\cdot\frac{d^2 f}{dx^2}\left(\frac{c+d}{2}\right) \quad (*)$$

$$err(A_1)[f,c,d] = \frac{d-c}{2\ell} err(A_1)\left[x \mapsto f\left(c+\frac{d-c}{2\ell}x\right), 0, 2\ell\right] \leq$$

$$\leq \left(\frac{d-c}{2\ell}\right)\left(-\frac{1}{60}\ell^5 \min_{x\in(0,2\ell)} \frac{d^4 f}{dx^4}\left(c+\frac{d-c}{2\ell}x\right)\right) =$$

$$= \frac{d-c}{2\ell}\cdot\left(-\frac{1}{60}\ell^5\right)\cdot \min_{x\in(c,d)} \frac{d^4 f}{dx^4}\cdot\frac{(d-c)^4}{(2\ell)^4} =$$

$$= \frac{-(d-c)^5}{1920} \min_{x\in(c,d)} \frac{d^4 f}{dx^4}. \quad (**)$$

Using (*), we can compute $A_n[f,a,b]$:

$$A_n[f,a,b] = \sum_{i=1}^{n} A_1[f, x_{i-1}, x_i] = \frac{b-a}{n}\left(f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \dots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right) +$$

$$+ \frac{(b-a)^3}{24n^3}\left(\frac{d^2 f}{dx^2}\left(\frac{x_0+x_1}{2}\right) + \frac{d^2 f}{dx^2}\left(\frac{x_1+x_2}{2}\right) + \dots + \frac{d^2 f}{dx^2}\left(\frac{x_{n-1}+x_n}{2}\right)\right)$$

$$err(A_1)[f,c,d] = \frac{d-c}{2\ell} err(A_1)\left[x \mapsto f\left(c+\frac{d-c}{2\ell}x\right), 0, 2\ell\right] \geq$$

$$\geq \left(\frac{d-c}{2\ell}\right)\left(-\frac{1}{60}\ell^5 \max_{x\in(0,2\ell)} \frac{d^4 f}{dx^4}\left(c+\frac{d-c}{2\ell}x\right)\right) =$$

$$= \frac{d-c}{2\ell}\cdot\left(-\frac{1}{60}\ell^5\right)\cdot\frac{(d-c)^4}{16\ell^4}\cdot \max_{x\in(c,d)} \frac{d^4 f}{dx^4} =$$

$$= \frac{-(d-c)^5}{1920}\cdot \max_{x\in(c,d)} \frac{d^4 f}{dx^4} \quad (***)$$

From (**) and (***) we get that:

$$-\frac{(d-c)^5}{1920}\overline{\Delta_4} \leqslant -\frac{(d-c)^5}{1920}\max_{x\in(c,d)}\frac{d^4f}{dx^4} \leqslant err(A_1)[f,c,d] \leqslant -\frac{(d-c)^5}{1920}\min_{x\in(c,d)}\frac{d^4f}{dx^4} \leqslant -\frac{(d-c)^5}{1920}\underline{\Delta_4}$$

$$\overline{\Delta_4} \geqslant \max_{x\in(c,d)}\frac{d^4f}{dx^4} \qquad \underline{\Delta_4} \leqslant \min_{x\in(c,d)}\frac{d^4f}{dx^4}$$

$$err(A_n)[f,a,b] = \sum_{i=1}^{n} err(M_1)[f,x_{i-1},x_i]$$

$$-\frac{(b-a)^5}{1920n^5}\overline{\Delta_4} \leqslant err(M_1)[f,x_{i-1},x_i] \leqslant -\frac{(b-a)^5}{1920n^5}\underline{\Delta_4} \quad \Big|\ \sum_{i=1}^{n}()$$

$$-\frac{(b-a)^5}{1920n^4}\overline{\Delta_4} \leqslant err(M_m)[f,a,b] \leqslant -\frac{(b-a)^5}{1920n^4}\underline{\Delta_4}.$$

(e) With Simpson's rule we have:

$$S_n[f,a,b] = \frac{b-a}{3n}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\dots+4f(x_{n-1})+f(x_n)\right)$$

$$\frac{(b-a)^5}{180n^4}\underline{\Delta_4} \leqslant err(S_n)[f,a,b] \leqslant \frac{(b-a)^5}{180n^4}\overline{\Delta_4}$$

<u>Our method</u>:

- $O(n)$ arithmetic operations
- n evaluations of the function f
- n evaluations of the function $\frac{d^2f}{dx^2}$
- $O(n^{-4})$ error with a factor of $\frac{1}{1920}$

<u>Simpson's rule</u>:

- $O(n)$ arithmetic operations
- $(n+1)$ evaluations of the function
- $O(n^{-4})$ error with a factor of $\frac{1}{180}$

<u>Conclusion</u>:

Considering the fact that our method needs to evaluate the second derivative of f n times, Simpson's rule is definitely faster than our method. In terms of accuracy, because of the factor that is bigger at our method ($1920 > 180$), our method is more accurate than Simpson's rule (slightly, as they have the same complexity $O(n^{-4})$).

$\boxed{2.7}$ $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

• Estimating it using the Midpoint rule: $(f: [0,1] \to \mathbb{R},\ f(x) = x^2)$

$$M_2[f,0,1] = \frac{1}{2}\left(f\left(\frac{0+\frac{1}{2}}{2}\right) + f\left(\frac{\frac{1}{2}+1}{2}\right)\right) = \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{2}\left(\frac{1}{16} + \frac{9}{16}\right)$$

$$M_2[f,0,1] = \frac{10}{32} = \frac{5}{16}$$

• Estimating it using the Trapezium rule:

$$T_2[f,0,1] = \frac{1}{4}\left(f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right) = \frac{1}{4}\left(0 + 2\cdot\frac{1}{4} + 1\right) = \frac{1}{4}\cdot\frac{3}{2} = \frac{3}{8}$$

• The error of the Midpoint rule is $\frac{5}{16} - \frac{1}{3} = \frac{15-16}{48} = -\frac{1}{48}$

• The error of the Trapezium rule is $\frac{3}{8} - \frac{1}{3} = \frac{9-8}{24} = \frac{1}{24}$

$\Rightarrow$ the Midpoint rule is two times more accurate!

$$\int_0^1 x^4 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

• Estimating it using Simpson's rule: $(g: [0,1] \to \mathbb{R},\ g(x) = x^4)$

$$S_2[g,0,1] = \frac{1}{6}\left(g(0) + 4g\left(\frac{1}{2}\right) + g(1)\right) = \frac{1}{6}\left(0 + 4\cdot\frac{1}{16} + 1\right) = \frac{1}{6}\cdot\frac{5}{4} = \frac{5}{24}$$

• Estimating it using the method from question 2.6: $\left(\frac{d^2g}{dx^2} = 12x^2\right)$

$$A_2[g,0,1] = \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) + \frac{1}{192}\left(12\cdot\frac{1}{16} + 12\cdot\frac{9}{16}\right)$$

$$A_2[g,0,1] = \frac{1}{2}\left(\frac{1}{256} + \frac{81}{256}\right) + \frac{1}{192}\cdot\frac{120}{16} = \frac{1}{2}\cdot\frac{82}{256} + \frac{15}{384} = \frac{41}{256} + \frac{15}{384}$$

$$A_2[g,0,1] = \frac{153}{768} = \frac{51}{256}$$

• The error of Simpson's rule is $\frac{5}{24} - \frac{1}{5} = \frac{25-24}{120} = \frac{1}{120}$

• The error of our method is $\frac{51}{256} - \frac{1}{5} = \frac{255-256}{1280} = -\frac{1}{1280}$

$\Rightarrow$ Our method is $\frac{32}{3}$ times more accurate!

We have $\frac{d^4g}{dx^4} = 24 \Rightarrow \underline{D_4} = \overline{D_4} = 24$. From the lecture notes, we have

$$24\cdot\frac{1}{180\cdot 16} \leq err(S_2)[g,0,1] \leq 24\cdot\frac{1}{180\cdot 16} \Rightarrow err(S_2)[g,0,1] = \frac{1}{120}$$, as we calculated above.

From question 2.6 we have:

$$-\frac{1}{1920\cdot 16}\cdot 24 \leq err(A_2)[g,0,1] \leq -\frac{1}{1920\cdot 16}\cdot 24 \Rightarrow$$

$\Rightarrow err(A_2)[g,0,1] = -\frac{1}{1280}$, as we calculated above.

2.8. Finally, $\int_0^2 x^{\frac{3}{2}} dx = \left[\frac{2}{5} x^{\frac{5}{2}}\right]_0^2 = \frac{2}{5} \cdot 4\sqrt{2} = \frac{8\sqrt{2}}{5} \simeq 2.2627$

Let's also recall Simpson's rule: $(f: [0,2] \to \mathbb{R},\ f(x) = x^{\frac{3}{2}})$

$$S_n[f,0,2] = \frac{2}{3n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + f(x_n)\right)$$

Now, the program written in Scala:

```
object Simpson
{
  def Simpson (n: int) : Double =     // The function that calculates Sn [f,0,2]
  {
    var sum = 0.00
    sum = sum + Math.pow(0, 1.5)   // f(x0) = f(0)
    for (i <- 1 until n)
    {
      if (i % 2 == 0) sum = sum+2*Math.pow(2.0 * i/n, 1.5)   // 4f(1) + 2f(2) + ... + 4f(n-2) +
                                                                 2f(n-1)
        else sum = sum+4*Math.pow(2.0 * i/n, 1.5)
    }
    sum = sum + Math.pow(2, 1.5)  // f(xn) = f(2)
    sum = (2.0 * sum) / (3.0 * n)  // Sn [f,a,b] = (b-a)/3n Σ ...
    sum   // We return the result
  }
  def main (args: Array [String]) =
  {
    val trials = scala.io.StdIn.readInt   // We try 2^1, 2^2, ..., 2^trials strips
    var N = 1
    val integral = 2.2627   // ∫_0^2 x^(3/2) dx
    for (j <- 1 to trials)
      {
        N = N * 2
        val approx = simpson(N)   // the estimate with N strips
        val error = approx - integral
        println ("Estimate using "+N+" strips is "+approx+" and the error is "+error)
      }
  }
}
```

We observe that the error converges to a value very close to $4.1699 \cdot 10^{-5}$ even after we use 2^{24} strips. This might be because $\overline{D}_4$ in our case is unbounded (close to 0 it goes to $+\infty$).