Linear Algebra — Week 8

1. Suppose A is a non-singular, upper-triangular matrix. Write down an algorithm for solving $A\mathbf{u} = \mathbf{b}$ in a known, finite number of steps without calculating the inverse of A or using Gaussian elimination. How many additions, multiplications, subtractions and divisions does your algorithm take?

2. The Gauss-Seidel method is to be used to solve the linear system $A\mathbf{u} = \mathbf{b}$ iteratively. By decomposing $A = D - L - U$ as in lectures, explain how the iterative step may be written

$$\mathbf{u}_{n+1} = G\mathbf{u}_n + \mathbf{c},$$

   defining G and $\mathbf{c}$ in terms of D, L, U and $\mathbf{b}$.

3. Repeat question 2 for the SOR method.

4. A linear system is given by

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}.$$

   You are given an initial guess to the solution. Determine whether the following methods converge by considering the eigenvalues of an appropriate matrix in each case:

   (a) Jacobi's method;
   (b) the Gauss-Seidel method;
   (c) the SOR method, with $\omega = 0.5$.

5. A non-singular linear system is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

   for given constants $a \neq 0, b, c, d \neq 0, p, q$. You are given an initial guess to the solution. Assume that $bc/ad > 0$

   (a) By considering the eigenvalues of an appropriate matrix, determine a condition on a, b, c, d for Jacobi's method to converge.
   (b) By considering the eigenvalues of an appropriate matrix, determine a condition on a, b, c, d for the Gauss-Seidel method to converge.
   (c) If both Jacobi's method and the Gauss-Seidel method converge, which method is likely to converge faster?

6. How many additions, subtractions, multiplications and divisions are required for each iteration of the SOR method when the linear system is of size N?

7. How many additions, subtractions, multiplications and divisions are required for each iteration of the method of steepest descent when the linear system is of size N?

8. Why won't the steepest descent method work when solving the linear system below?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

9. The matrix A is given by

$$A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

   (a) Calculate $\|A\|$.
   (b) Calculate $\|A^{-1}\|$.
   (c) Calculate the condition number of A.

10. Prelims 2011, "Discrete Mathematics and Linear Algebra", Q8, slightly modified.

   (a) Given a set of linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \Re^n$ define the Gram-Schmidt algorithm for constructing k mutually orthonormal vectors, $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k \in \Re^n$.
   (b) Show that the inverse of a square orthonormal matrix, Q, is $Q^\top$.
   (c) Find the QR factorisation for the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \\ 4 & 8 \end{pmatrix}.$$

   (d) Use the QR factorisation to find the vector such that the least squares function

$$F(\mathbf{x}) = \left| A\mathbf{x} - \begin{pmatrix} 1 \\ 8 \\ 8 \end{pmatrix} \right|^2,$$

   is minimised.