Introduction to Formal Proof
Tutorial Sheet 2*

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Question 1

(a) (Adapted from Huth and Ryan Exercises 2.2.{2,5,6}])

Using the signature

$$\mathcal{C} = \{Mary\}$$

$$\mathcal{P} = \{P(\cdot), S(\cdot), L(\cdot), \cdot \textsf{ attended } \cdot, \cdot \textsf{ admires } \cdot\}$$

translate the following into formal predicate language. The symbols here are intended to have the obvious interpretations on a domain that includes all possible people and lectures.

i. Mary admires every professor.
ii. Some professors admire Mary.
iii. Mary admires herself.
iv. No student attended every lecture.
v. No lecture was attended by every student.
vi. No lecture was attended by any student.

(b) Extending the signature given in the first part with the predicate symbol $\cdot = \cdot$ (intended to mean equality) translate the following into formal predicate language

i. Mary admires one professor.
ii. Mary doesn't admire more than one professor.
iii. There is a student who is also a professor.
iv. There is just one student who is a professor.

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(c) Give an appropriate signature and, after explaining what its constants, function symbols, and predicates are intended to mean, translate the following sentences into formal predicate language:[1]

i. All the red things are in the box.

ii. Only the red things are in the box.

iii. Only red things are in the box.

iv. The box is blue.

v. There's a green thing in the box, and everything else is in the shed.

vi. No mammal is both a horse and a primate.

vii. The prizes were all won by girls.

viii. A girl won all the prizes.

ix. There is nothing in the universe.

x. There are at least two things in the universe.

xi. There are exactly two things in the universe.

Question 2

(a) Prove the following predicate logic conjectures. You may present your proofs in linear or sequent-tree form (I recommend the linear form).

i. $R(j), \forall x \cdot (R(x) \to S(x)) \vdash S(j)$

ii. $\forall x \cdot (R(x) \to S(x)) \vdash (\forall y \cdot R(y)) \to (\forall z \cdot S(z))$

iii. $\forall x \cdot (R(x) \to S(x)), \forall y \cdot (S(y) \to T(y)) \vdash \forall z \cdot (R(z) \to T(z))$

iv. $(\forall x \cdot R(x)) \wedge (\forall y \cdot S(y)) \vdash \forall z \cdot (R(z) \wedge S(z))$

v. $\forall x {\cdot} P(a, x, x), \forall x {\cdot} \forall y {\cdot} \forall z {\cdot} (P(x, y, z) \to P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$
[Hint: to support your intuition in this and the next question, you may find it helpful to think of $P(x, y, z)$ as meaning "adding x to y yields z"; of f as the successor function; and of a as zero.]

vi.

$$\forall x \cdot P(a, x, x), \forall x \cdot \forall y \cdot \forall z \cdot (P(x, y, z) \to P(f(x), y, f(z)))$$
$$\vdash$$
$$\exists z \cdot P(f(a), z, f(f(a)))$$

[Hint: the last line of the proof is likely to be an application of $\exists$-I. Imagine that the term to be substituted for z is T, then see what constraints you can discover on T.

[1] Hint: you will need an equality predicate for one ore more of these.

The best way to go about this is to observe, similarly, that if the quantified implication in the first premiss is ever going to be used you will need to substitute a term for x in an application of $\forall$-E. Likewise, if the second premiss is ever going to be used, you will need to substitute terms for the x, y, z, in three applications of $\forall - E$ – imagine these to be W, X, Y, and Z and see what constraints doing the proof places on the "unknown terms".]

(b) i. $\exists x \cdot R(x) \vdash \neg\forall y \cdot \neg R(y)$

[Hint: consider a proof of $P \vee Q \vdash \neg(\neg P \wedge \neg Q)$]

ii. $\neg\exists x \cdot \neg R(x) \vdash \forall y \cdot R(y)$

[Hint: consider a proof of $\neg(\neg P \vee \neg Q) \vdash P \wedge Q$]

iii. $\neg\forall x \cdot R(x) \vdash \exists y \cdot \neg R(y)$

[Hint: consider a proof of $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$]

(c) (Exercise 2.5.12 Huth and Ryan)

Translate the following argument into a sequent, using a suitable set of predicate symbols.

> "If there are any tax payers, then all politicians are tax payers. If there are any philanthropists, then all taxpayers are philanthropists. So if there are any tax-paying philanththropists, then all politicans are philanthropists."

Now prove the sequent.

Question 3

Prove $\forall x \cdot P(x) \vdash \exists x \cdot P(x)$. Now consider applying this result in a model whose domain of discourse (set of values) is empty, by evaluating the premiss in the model, and evaluating the conclusion in the model.

What does this tell you about the claim that the proof rules given in the lecture notes are sound?