Linear Transformations

Linear Algebra, Michaelmas Term 2018
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Linear Transformations

A linear transformation from a vector space U to a vector space W is a mapping $T : U \to W$ such that, for all $\mathbf{u}, \mathbf{v} \in U$, and for all scalars c,

1. $T(\mathbf{u}) \in V$
2. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
3. $T(c\mathbf{u}) = cT(\mathbf{u})$

The last two conditions are the conditions for T to be linear

A consequence of the definition of a linear transformation is that

$$T(\mathbf{0}) = \mathbf{0}$$

Proof:

The vector $\mathbf{0}$ is a member of any vector space, and so $\mathbf{0} \in U$

Setting $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$, and noting that this implies that $\mathbf{u} + \mathbf{v} = \mathbf{0}$, our second condition on the previous slide gives

$$T(\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$$

and so $T(\mathbf{0}) = \mathbf{0}$ as required.

Example: projection of a point in 3D onto the (x, y) plane

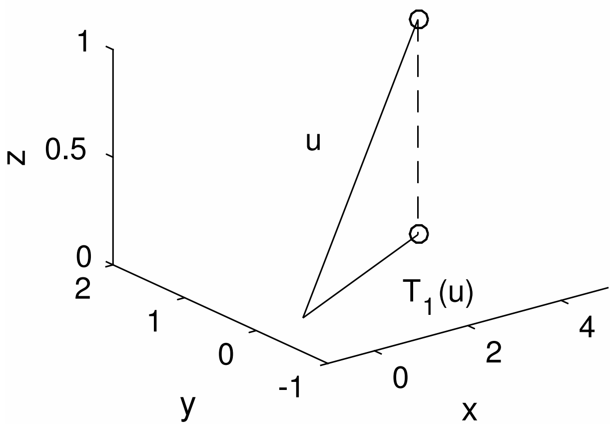
Let $\Re^3$ be the vector space of points in three dimensional space.

If $\mathbf{u} \in \Re^3$ we may write

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Define the linear transformation $T_1 : \Re^3 \to \Re^3$ by, for $\mathbf{u} \in \Re^3$:

$$T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$



We will now confirm that T_1 satisfies the three conditions required to be a linear transformation

From the definition of T_1 we see that this transformation maps a point in 3D space to another point in 3D space and so, for any $\mathbf{u} \in \Re^3$, we have $T_1(\mathbf{u}) \in \Re^3$

We will now show that T_1 is linear

Suppose

$$\mathbf{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Let c be a scalar constant

By definition of T_1 we have

$$T_1(\mathbf{u}) = \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}, \qquad T_1(\mathbf{v}) = \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix}$$

$$\begin{aligned} T_1(\mathbf{u}+\mathbf{v}) &= T_1\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ 0 \end{pmatrix} \\ &= T_1(\mathbf{u}) + T_1(\mathbf{v}) \end{aligned}$$

$$T_1(c\mathbf{u}) = T_1\left(\begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix}\right)$$
$$= \begin{pmatrix} cx_1 \\ cy_1 \\ 0 \end{pmatrix}$$
$$= c\begin{pmatrix} x_1 \\ y_1 \\ 0 \end{pmatrix}$$
$$= cT_1(\mathbf{u})$$

Hence T_1 is a linear transformation

Example: matrix multiplication of a vector

If $\mathbf{u} \in \Re^m$ we may write

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

Let A be a matrix of size $n \times m$, with real entries

Define the linear transformation $T_2 : \Re^m \to \Re^n$ by

$$T_2(\mathbf{u}) = A\mathbf{u}$$

Suppose $\mathbf{u} \in \Re^m$.

As A is a $n \times m$ matrix, and $\mathbf{u}$ is a vector of length m, the matrix-vector product $A\mathbf{u} \in \Re^n$, and so $T_2(\mathbf{u}) \in \Re^n$

Further, if $\mathbf{u}_1, \mathbf{u}_2 \in \Re^m$, then

$$\begin{aligned} T_2(\mathbf{u}_1 + \mathbf{u}_2) &= A(\mathbf{u}_1 + \mathbf{u}_2) \\ &= A\mathbf{u}_1 + A\mathbf{u}_2 \qquad \text{by properties of matrix multiplication} \\ &= T_2(\mathbf{u}_1) + T_2(\mathbf{u}_2) \end{aligned}$$

Also, if $\mathbf{u} \in \Re^m$ and c is a scalar, then

$$\begin{aligned} T_2(c\mathbf{u}) &= A(c\mathbf{u}) \\ &= cA\mathbf{u} \\ &= cT_2(\mathbf{u}) \end{aligned}$$

Hence T_2 is a linear transformation

Example: differentiation of polynomial functions

Let $\mathcal{P}^n$ be the vector space of polynomials of degree n.

That is, if $f(x)$ is in $\mathcal{P}^n$, then we may write

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where $a_0, a_1, a_2, \ldots, a_n$ are given scalars

Define the linear transformation $T_3 : \mathcal{P}^n \to \mathcal{P}^{n-1}$ by, for $f(x) \in \mathcal{P}^n$:

$$T_3(f) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

Suppose

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

where $a_0, a_1, a_2, \ldots, a_n$ are scalars

Then

$$\begin{aligned} T_3(f) &= \frac{\mathrm{d}f}{\mathrm{d}x} \\ &= a_1 + 2a_2 x + \ldots + n a_n x^{n-1} \end{aligned}$$

which is a polynomial of degree $n-1$, and so the transformation maps the vector space $\mathcal{P}^n$ to $\mathcal{P}^{n-1}$, as required

Now need to demonstrate that the map is linear

Suppose

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \\ g(x) &= b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n \end{aligned}$$

where $a_0, a_1, a_2, \ldots, a_n$ and $b_0, b_1, b_2, \ldots, b_n$ are scalars

Clearly,

$$\begin{aligned} T_3(f) &= a_1 + 2a_2 x + \ldots + n a_n x^{n-1} \\ T_3(g) &= b_1 + 2b_2 x + \ldots + n b_n x^{n-1} \end{aligned}$$

Noting that

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n \end{aligned}$$

we see that

$$\begin{aligned} T_3(f+g) &= T_3\left((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n\right) \\ &= (a_1 + b_1) + 2(a_2 + b_2)x + \ldots + n(a_n + b_n)x^{n-1} \\ &= (a_1 + 2a_2 x + \ldots + n a_n x^{n-1}) + (b_1 + 2b_2 x + \ldots + n b_n x^{n-1}) \\ &= T_3(f) + T_3(g) \end{aligned}$$

If c is a scalar then

$$\begin{aligned}(cf)(x) &= cf(x)\\ &= ca_0 + ca_1x + ca_2x^2 + \ldots + ca_nx^n\end{aligned}$$

We then have

$$\begin{aligned}T_3(cf) &= T_3\left(ca_0 + ca_1x + ca_2x^2 + \ldots + ca_nx^n\right)\\ &= ca_1 + 2ca_2x + \ldots + nca_nx^{n-1}\\ &= c(a_1 + 2a_2x + \ldots + na_nx^{n-1})\\ &= cT_3(f)\end{aligned}$$

Hence, T_3 is a linear transformation

Expressing a vector with a given basis

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be a basis for a vector space U

Any vector $\mathbf{u} \in U$ may then be written

$$\begin{aligned}\mathbf{u} &= \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \ldots + \beta_m\mathbf{u}_m\\ &= \sum_{j=1}^{m} \beta_j\mathbf{u}_j\end{aligned}$$

with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$

We then say that $\mathbf{u}$ has coordinates

$$\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{pmatrix}$$

with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$

Example: the space $\Re^3$

A basis for $\Re^3$ is

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

If $\mathbf{u} \in \Re^3$, we can then write

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

or, equivalently,

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Example: the space of polynomial functions of degree n

The vector space $\mathcal{P}^n$ is of dimension $n+1$.

We will use the basis

$$\mathbf{e}_i = x^{i-1}, \qquad i = 1, 2, \ldots, n+1$$

Suppose the polynomial $f(x) \in \mathcal{P}^n$ is given by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

We can then write $f(x)$ as

$$f(x) = a_0\mathbf{e}_1 + a_1\mathbf{e}_2 + a_2\mathbf{e}_3 + \ldots + a_n\mathbf{e}_{n+1}$$

or, with respect to the basis as

$$f(x) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

The matrix of a linear transformation

Suppose U and V are vector spaces, and $T : U \to V$ is a linear transformation

Given bases for U and V there always exists a matrix A such that

$$T(\mathbf{u}) = A\mathbf{u}$$

for all $\mathbf{u} \in U$

We will demonstrate that this is true by constructing the matrix A

Suppose U has dimension m, and V has dimension n

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be a basis for U, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V

As T is a linear transformation, $T(\mathbf{u}_j) \in V$ for $j = 1, 2, \ldots, m$

We can therefore express $T(\mathbf{u}_j)$ as a linear sum of basis vectors for V

$$T(\mathbf{u}_j) = \sum_{i=1}^{n} A_{i,j}\mathbf{v}_i$$

Suppose $\mathbf{u} \in U$ may be written

$$\begin{aligned}\mathbf{u} &= \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \ldots + \beta_m\mathbf{u}_m \\ &= \sum_{j=1}^{m} \beta_j\mathbf{u}_j\end{aligned}$$

with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$

$\mathbf{u}$ then has coordinates

$$\begin{pmatrix}\beta_1 \\ \beta_2 \\ \vdots \\ \beta_m\end{pmatrix}$$

with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$

Let $\mathbf{v} = T(\mathbf{u})$

Suppose $\mathbf{v} \in V$ may be written

$$\begin{aligned}\mathbf{v} &= \gamma_1\mathbf{v}_1 + \gamma_2\mathbf{v}_2 + \ldots + \gamma_n\mathbf{v}_n \\ &= \sum_{j=1}^{n} \gamma_j\mathbf{v}_j\end{aligned}$$

with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

$\mathbf{v}$ then has coordinates

$$\begin{pmatrix}\gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n\end{pmatrix}$$

with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

We may then write

$$\begin{aligned}\mathbf{v} &= \sum_{j=1}^{n} \gamma_j\mathbf{v}_j = T(\mathbf{u}) = T\left(\sum_{j=1}^{m} \beta_j\mathbf{u}_j\right) \\ &= \sum_{j=1}^{m} \beta_j T(\mathbf{u}_j) \qquad \text{This step is on the worksheet for this week} \\ &= \sum_{j=1}^{m} \beta_j \sum_{i=1}^{n} A_{i,j}\mathbf{v}_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{i,j}\beta_j\right)\mathbf{v}_i \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} A_{j,i}\beta_i\right)\mathbf{v}_j\end{aligned}$$

We then have

$$\sum_{j=1}^{n}\left(\gamma_j - \sum_{i=1}^{m} A_{j,i}\beta_i\right)\mathbf{v}_j = 0$$

As $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis for V they are linearly independent

We must then have

$$\gamma_j = \sum_{i=1}^{m} A_{j,i}\beta_i, \qquad j = 1, 2, \ldots, n$$

This may be written as the matrix equation

$$\begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_n\end{pmatrix} = A\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{pmatrix}$$

Hence, the matrix A relates the coordinates of $\mathbf{u} \in U$ and $\mathbf{v} = T(\mathbf{u}) \in V$ under the transformation T with respect to the chosen bases

Example: calculating the matrix for T_1

Recall that $T_1 : \Re^3 \to \Re^3$ is the linear transformation defined by:

$$T_1\left(\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right) = \begin{pmatrix}x\\ y\\ 0\end{pmatrix}$$

We will use the following basis for $\Re^3$:

$$\mathbf{e}_1 = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

We then have

$$T_1(\mathbf{e}_1) = \mathbf{e}_1, \qquad T_1(\mathbf{e}_2) = \mathbf{e}_2, \qquad T_1(\mathbf{e}_3) = \mathbf{0}$$

Remembering that

$$T_1(\mathbf{e}_j) = \sum_{i=1}^{3} A_{i,j}\mathbf{e}_i$$

it follows that

$$T_1(\mathbf{e}_1) = A_{1,1}\mathbf{e}_1 + A_{2,1}\mathbf{e}_2 + A_{3,1}\mathbf{e}_3$$

and so

$$A_{1,1} = 1, \qquad A_{2,1} = 0, \qquad A_{3,1} = 0$$

Similarly,

$$T_1(\mathbf{e}_2) = A_{1,2}\mathbf{e}_1 + A_{2,2}\mathbf{e}_2 + A_{3,2}\mathbf{e}_3$$

and so

$$A_{1,2} = 0, \qquad A_{2,2} = 1, \qquad A_{3,2} = 0$$

and

$$T_1(\mathbf{e}_3) = A_{1,3}\mathbf{e}_1 + A_{2,3}\mathbf{e}_2 + A_{3,3}\mathbf{e}_3$$

and so

$$A_{1,3} = 0, \qquad A_{2,3} = 0, \qquad A_{3,3} = 0$$

The matrix representing T_1 with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is therefore given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example: calculating the matrix for T_3

$T_3 : \mathcal{P}^n \to \mathcal{P}^{n-1}$ is the linear transformation defined by, for $f(x) \in \mathcal{P}^n$:

$$T_3(f) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

The vector space $\mathcal{P}^n$ is of dimension $n+1$. We will use the basis

$$\mathbf{e}_i = x^{i-1}, \qquad i = 1, 2, \ldots, n+1$$

Similarly, the vector space $\mathcal{P}^{n-1}$ is of dimension n. We will use the basis

$$\mathbf{e}_i = x^{i-1}, \qquad i = 1, 2, \ldots, n$$

We then have

$$T_3(\mathbf{e}_1) = T_3(1) = \mathbf{0}$$
$$T_3(\mathbf{e}_j) = T_3(x^{j-1}) = (j-1)x^{j-2} = (j-1)\mathbf{e}_{j-1}, \qquad j = 2, 3, \ldots, n+1$$

We may use a similar method as for the previous example to relate the equations above to the entries of the matrix

The matrix representing T_3 with respect to the bases described above for $\mathcal{P}^n$ and $\mathcal{P}^{n-1}$ is the $n \times (n+1)$ matrix given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & n-1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & n \end{pmatrix}$$

Rotations, reflections, shears

We have shown that a linear transformation can always be written in matrix form when suitable bases are given

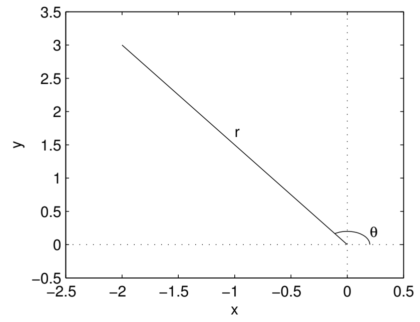
We will now use this observation to write some useful linear transformations in matrix form

Rotations

Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ be a general vector in the xy-plane.

We will now construct a matrix A such that $A\mathbf{u}$ has the effect of rotating $\mathbf{u}$ anti-clockwise about the origin through an angle ϕ.

It is convenient to use polar coordinates



The positive direction for a rotation is anti-clockwise

Writing $\mathbf{u}$ in polar form gives

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Let $\mathbf{v} = A\mathbf{u}$.

If $\mathbf{v} = \begin{pmatrix} x_{\text{rot}} \\ y_{\text{rot}} \end{pmatrix}$ then writing $\mathbf{v}$ in polar form gives

$$x_{\text{rot}} = r\cos(\phi + \theta), \qquad y_{\text{rot}} = r\sin(\phi + \theta)$$

We may then write

$$\begin{aligned} x_{\text{rot}} &= r\cos(\phi + \theta) \\ &= r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ &= x\cos\phi - y\sin\phi \\ y_{\text{rot}} &= r\sin(\phi + \theta) \\ &= r\cos\phi\sin\theta + r\sin\phi\cos\theta \\ &= x\sin\phi + y\cos\phi \end{aligned}$$

In matrix form:

$$\begin{pmatrix} x_{\text{rot}} \\ y_{\text{rot}} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

It is always worth checking a calculation — either analytical or computational one — with suitable test cases

Suitable examples include rotation through $\pi/2$ or π — i.e. 90 degrees or 180 degrees — of vectors such as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

One suitable test — rotation of the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ about the origin by an angle of $\pi/2$. Clearly this point is rotated to the point $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Substituting $\phi = \pi/2$, our rotation matrix becomes

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We then predict that the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is rotated to the point

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

and so our matrix is correct in this case

Reflections

Suppose we want a matrix A so that, for any vector $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, the vector $\mathbf{v}$ given by $\mathbf{v} = A\mathbf{u}$ is the reflection of $\mathbf{u}$ in the x-axis

We will write

$$\mathbf{v} = \begin{pmatrix} x_{\text{ref}} \\ y_{\text{ref}} \end{pmatrix}$$

Clearly,

$$x_{\text{ref}} = x, \qquad y_{\text{ref}} = -y$$

A suitable matrix is then

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exercise: what would be a suitable test that the matrix A is correct?

Example: reflection in a given line passing through the origin

Let $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ be a unit vector, perpendicular to the line we are relecting in

$\mathbf{n}$ is often called a unit normal vector

If the reflection in this line of the a point $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ is the point $\mathbf{u}_{\text{ref}} = \begin{pmatrix} x_{\text{ref}} \\ y_{\text{ref}} \end{pmatrix}$ then

$$\mathbf{u}_{\text{ref}} = \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$$

Writing this in terms of the components of $\mathbf{u}$ and $\mathbf{n}$ gives

$$\begin{aligned}
\mathbf{u}_{\text{ref}} &= \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{n})\mathbf{n} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} - 2(xn_1 + yn_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\
&= \begin{pmatrix} (1 - 2n_1^2)x - 2yn_1n_2 \\ -2xn_1n_2 + (1 - 2n_2^2)y \end{pmatrix} \\
&= \begin{pmatrix} 1 - 2n_1^2 & -2n_1n_2 \\ -2n_1n_2 & 1 - 2n_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

The matrix representing this reflection is

$$\begin{pmatrix} 1 - 2n_1^2 & -2n_1n_2 \\ -2n_1n_2 & 1 - 2n_2^2 \end{pmatrix}$$

Our first example reflection was a reflection in the x-axis, i.e. the line $y = 0$

This line has unit normal $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Substituting $n_1 = 0, n_2 = 1$ into the matrix on the previous slide gives

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which is identical to the matrix generated for our first example

Scalings

Suppose we want to scale a vector $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ by a factor of α in the x-direction, and β in the y-direction

Let

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Then

$$\begin{aligned} A\mathbf{u} &= \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix} \end{aligned}$$

and so A represents this transformation

Suppose now we want to scale a vector by a factor of α in the direction of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and β in the direction of the vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

It is not as straightforward to write down the matrix representing this transformation, as the scalings are not parallel to the coordinate axes

We can get around this by first rotating the axes so that the scaling directions are parallel to these rotated coordinate axes

We then scale the vector, before rotating the axes back again

This is known as a composite transformation

Composite transformations

We will evaluate the scaling of a vector by a factor of α in the direction of the vector $\begin{pmatrix}1\\1\end{pmatrix}$, and β in the direction of the vector $\begin{pmatrix}-1\\1\end{pmatrix}$ by combining transformations with matrices that we know

This is known as a composite transformation

We will start with a vector $\mathbf{u}$ and carry out the following steps:

- We will rotate by an angle of -$\pi/4$ (-45 degrees) so that the vector $\begin{pmatrix}1\\1\end{pmatrix}$ lies along the x-axis, and the vector $\begin{pmatrix}-1\\1\end{pmatrix}$ lies along the y-axis. This rotation will be described by the matrix R_1
- We will scale by a factor of α in the x-direction, and β in the y-direction. This will be described by the matrix S
- Finally, we will rotate by an angle of $\pi/4$ (45 degrees) so that the coordinate axes are returned to their initial locations. This rotation will be described by the matrix R_2

The matrices R_1, S and R_2 are all matrices that are easily calculated using the earlier material

The matrices may be written

$$R_1 = \begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}, \qquad S = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}, \qquad R_2 = \begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}$$

The given scalings acting on the vector $\mathbf{u}$ is given by

$$R_2 S R_1 \mathbf{u}$$

Note the order of the matrices — the first transformation carried out goes on the right of the matrix product

We may confirm this is correct by setting, for example, $\alpha = 4$ and $\beta = 2$

We then expect that:

$$R_2 S R_1 \begin{pmatrix}1\\1\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix}$$
$$R_2 S R_1 \begin{pmatrix}-1\\1\end{pmatrix} = 2\begin{pmatrix}-1\\1\end{pmatrix}$$

An elementary calculation demonstrates that this is true

Homogeneous coordinates and translations

Let $\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$

A translation along the vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the transformation

$T : \Re^2 \to \Re^2$ defined by

$$T(\mathbf{u}) = \mathbf{u} + \mathbf{v} = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$$

Suppose $\mathbf{v} \neq \mathbf{0}$

Then

$$T(\mathbf{0}) = \mathbf{v} \neq \mathbf{0}$$

We proved earlier that, if T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$

Translations are therefore not linear transformations

We can get around this by using homogeneous coordinates — instead of writing

$$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$$

we write

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

To translate along the vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, we use homogeneous coordinates and the matrix

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

We then have

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \\ 1 \end{pmatrix}$$

which describes the translation in homogeneous coordinates

The translation may be described in homogeneous coordinates by a matrix multiplication

We saw earlier that matrix multiplication defined a linear transformation

The translation is a linear transformation with respect to these coordinates

Suppose a linear transformation $T : \Re^2 \to \Re^2$ is represented by a matrix A so that, for all $\mathbf{u} \in \Re^2$ we may write

$$T(\mathbf{u}) = A\mathbf{u}$$

where

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

We may then write the matrix of this transformation with respect to homogeneous coordinates as

$$B = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We know how to write down the matrix representing a rotation about the origin

Suppose we want to calculate a rotation through an angle ϕ about the point $\begin{pmatrix} a \\ b \end{pmatrix}$

This may be done using homogeneous coordinates and composite transformations

- We first translate along the vector $\begin{pmatrix} -a \\ -b \end{pmatrix}$ so that the point $\begin{pmatrix} a \\ b \end{pmatrix}$ lies at the origin. This is represented by the matrix A_1 with respect to homogeneous coordinates
- We rotate about an angle ϕ. This is represented by the matrix R with respect to homogeneous coordinates
- We translate along the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ to restore the origin to the correct position. This is represented by the matrix A_2 with respect to homogeneous coordinates

From earlier we know that

$$A_1 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

The composite transformation is represented by the matrix product A_2RA_1

The kernel and image of a transformation

Suppose U and V are vector spaces, and $T : U \to V$ is a linear transformation

Some definitions:

- The kernel or nullspace of T — often written $\ker(T)$ — is the set of vectors $\mathbf{u} \in U$ such that $T(\mathbf{u}) = \mathbf{0}$
- The range or image of T — often written $\mathrm{Im}(T)$ — is the set of vectors $\mathbf{v} \in V$ such that $\mathbf{v} = T(\mathbf{u})$ for some $\mathbf{u} \in U$

Example: projection of a point in 3D onto the (x, y) plane

Let $\Re^3$ be the vector space of points in three dimensional space.

If $\mathbf{u} \in \Re^3$ we may write

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Define the linear transformation $T_1 : \Re^3 \to \Re^3$ by, for $\mathbf{u} \in \Re^3$:

$$T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

The kernel of T_1 is then vectors of the form $\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$ for all scalars c

The range of T_1 is vectors of the form $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

Both the kernel and the nullspace of a linear transformation $T : U \to V$ are vector spaces

Proof that the kernel is a vector space

Need to show that if $\mathbf{u}_1 \in \ker(T)$ and $\mathbf{u}_2 \in \ker(T)$ implies that $\mathbf{u}_1 + c\mathbf{u}_2 \in \ker(T)$ for all scalars c

If $\mathbf{u}_1 \in \ker(T)$ and $\mathbf{u}_2 \in \ker(T)$ then, by definition,

$$T(\mathbf{u}_1) = \mathbf{0} \qquad \text{and} \qquad T(\mathbf{u}_2) = \mathbf{0}$$

We then have

$$T(\mathbf{u}_1 + c\mathbf{u}_2) = T(\mathbf{u}_1) + cT(\mathbf{u}_2) = \mathbf{0}$$

and so $\mathbf{u}_1 + c\mathbf{u}_2 \in \ker(T)$ as required

Proof that the range is a vector space

Need to show that if $\mathbf{v}_1 \in \mathrm{Im}(T)$ and $\mathbf{v}_2 \in \mathrm{Im}(T)$ implies that $\mathbf{v}_1 + c\mathbf{v}_2 \in \mathrm{Im}(T)$ for all scalars c

As $\mathbf{v}_1, \mathbf{v}_2 \in \mathrm{Im}(T)$ there exist $\mathbf{u}_1, \mathbf{u}_2 \in U$ such that

$$\mathbf{v}_1 = T(\mathbf{u}_1), \qquad \mathbf{v}_2 = T(\mathbf{u}_2)$$

We then have, for all scalars c,

$$\begin{aligned} \mathbf{v}_1 + c\mathbf{v}_2 &= T(\mathbf{u}_1) + cT(\mathbf{u}_2) \\ &= T(\mathbf{u}_1 + c\mathbf{u}_2) \end{aligned}$$

As $\mathbf{u}_1 + c\mathbf{u}_2 \in U$, it then follows that $\mathbf{v}_1 + c\mathbf{v}_2 \in \mathrm{Im}(T)$, as required

As the kernel and the range of a transformation are vector spaces they have a dimension

The dimension of the kernel or nullspace is known as the nullity of a transformation

The dimension of the range or image is known as the rank of a transformation

The kernel, nullity, range and rank may be identified by considering the matrix of the transformation

Theorem

Suppose U and V are vector spaces. Let $T : U \to V$ be a linear transformation.

Then

$$\text{nullity of } T + \text{rank of } T = \text{dimension of } U$$

Proof

We have shown that $\ker(T)$ is a subspace of U

Suppose the nullity of T is m — in other words, the dimension of $\ker(T)$ is m

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be a basis for $\ker(T)$

Let the dimension of U be n

Extend the basis of $\ker(T)$ to a basis for U: $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_n$

Any $\mathbf{u} \in U$ can be written $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$

Any $\mathbf{v} \in \text{range}(T)$ can therefore be written

$$\begin{aligned}\mathbf{v} &= T\left(\sum_{i=1}^{n} \alpha_i \mathbf{u}_i\right) \\ &= \sum_{i=1}^{n} \alpha_i T(\mathbf{u}_i) \\ &= \sum_{i=m+1}^{n} \alpha_i T(\mathbf{u}_i)\end{aligned}$$

as $T(\mathbf{u}_i) = 0$ for $i = 1, 2, \ldots, m$.

We then see that the vectors

$$T(\mathbf{u}_{m+1}), T(\mathbf{u}_{m+2}), \ldots, T(\mathbf{u}_n)$$

span $\text{Im}(T)$

We will now demonstrate that these vectors are linearly independent

We may then deduce that the vectors above form a basis for $\text{Im}(T)$

Suppose

$$\sum_{i=m+1}^{n} \beta_i T(\mathbf{u}_i) = \mathbf{0}$$

We want to show that $\beta_i = 0$ for $i = m+1, m+2, \ldots, n$.

Then

$$T\left(\sum_{i=m+1}^{n} \beta_i \mathbf{u}_i\right) = \mathbf{0}$$

This implies that

$$\sum_{i=m+1}^{n} \beta_i \mathbf{u}_i$$

lies in the kernel of T

The kernel of T is spanned by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$.

This implies that

$$\sum_{i=m+1}^{n} \beta_i \mathbf{u}_i = \sum_{i=1}^{m} \gamma_i \mathbf{u}_i$$

Writing $\gamma_i = -\beta_i$ for $i = 1, 2, \ldots, m$ this becomes

$$\sum_{i=1}^{n} \beta_i \mathbf{u}_i = \mathbf{0}$$

As $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form a basis for U they are linearly independent

This implies that $\beta_i = 0$ for $i = 1, 2, \ldots, n$, as required

As $T(\mathbf{u}_{m+1}), T(\mathbf{u}_{m+2}), \ldots, T(\mathbf{u}_n)$ span $\mathrm{Im}(T)$ and are linearly independent they form a basis for $\mathrm{Im}(T)$

We then have:

- rank = dimension of $\mathrm{Im}(T) = n - m$
- nullity = dimension of $\ker(T) = m$
- dimension of $U = n$

Hence,

nullity of T + rank of T = dimension of U

The rank and nullity theorem has uses

Suppose you identify the kernel of a linear transformation

This then allows you to compute the rank of the transformation relatively easily

"Onto" linear transformations

Suppose U and V are vector spaces

Let $T : U \to V$ be a linear transformation

A linear transformation is said to be onto if the image of T is the vector space V

By definition of a linear transformation, $T(\mathbf{u}) \in V$

The image of T is therefore a sub-space of V

In practice, T is onto if the the dimension of the image of T is the vector space V

Example: projection of a point in 3D onto the (x, y) plane

Define the linear transformation $T_1 : \Re^3 \to \Re^3$ by, for $\mathbf{u} \in \Re^3$:

$$T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Clearly no $\mathbf{u} \in \Re^3$ satisfies $T_1(\mathbf{u}) = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ and so T_1 is not onto

Example: differentiation of polynomial functions

Let $\mathcal{P}^n$ be the vector space of polynomials of degree n.

Define the linear transformation $T_3 : \mathcal{P}^n \to \mathcal{P}^{n-1}$ by, for $f(x) \in \mathcal{P}^n$:

$$T_3(f) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

Suppose $\mathbf{v} \in \mathcal{P}^{n-1}$

Let $\mathbf{u} = \int \mathbf{v}\ \mathrm{d}x$. Then $\mathbf{u} \in \mathcal{P}^n$ and $T_3(\mathbf{u}) = \mathbf{v}$

For any $\mathbf{v} \in \mathcal{P}^{n-1}$ we can find $\mathbf{u} \in \mathcal{P}^n$ such that $T_3(\mathbf{u}) = \mathbf{v}$

T_3 is therefore onto

'1-1' linear transformations

Suppose U and V are vector spaces

Let $T : U \to V$ be a linear transformation

Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the range of T, with $\mathbf{u}_1, \mathbf{u}_2 \in U$ satisfying

$$T(\mathbf{u}_1) = \mathbf{v}_1, \qquad T(\mathbf{u}_2) = \mathbf{v}_2$$

A linear transformation is said to be 1-1 if $\mathbf{v}_1 = \mathbf{v}_2$ implies that $\mathbf{u}_1 = \mathbf{u}_2$

Example: projection of a point in 3D onto the (x, y) plane

Define the linear transformation $T_1 : \Re^3 \to \Re^3$ by, for $\mathbf{u} \in \Re^3$:

$$T_1\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Noting that

$$T_1 \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = T_1 \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

we see that T_1 is not 1-1

Example: multiplication of a vector by a matrix

Define the linear transformation $T_4 : \Re^n \to \Re^n$ by, for $\mathbf{u} \in \Re^n$:

$$T_4(\mathbf{u}) = A\mathbf{u}$$

for a $n \times n$ matrix

Suppose

$$A\mathbf{u}_1 = \mathbf{v}_1, \qquad A\mathbf{u}_2 = \mathbf{v}_1$$

for $\mathbf{u}_1, \mathbf{u}_2 \in \Re^n$

Then $A(\mathbf{u}_1 - \mathbf{u}_2) = 0$

We then have $\mathbf{u}_1 = \mathbf{u}_2$ if, and only if, A is non-singular

Example

The linear transformation $T : \Re^4 \to \Re^3$ is defined by

$$\mathbf{v} = A\mathbf{u}$$

where $\mathbf{u} \in \Re^4$, $\mathbf{v} \in \Re^3$, and A is the matrix given by

$$A = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$$

Find the kernel of T, and the range of T

Is T 1-1?

Is T onto?

Writing

$$\mathbf{u} = \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

we can write the linear transformation as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}$$

This may be written

$$\begin{pmatrix} a \\ b-2a \\ c+a \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}$$

A further row operation gives

$$\begin{pmatrix} a \\ b-2a \\ c+b-a \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}$$

As a consequence $c + b - a = 0$

We may then write any vector $\mathbf{v}$ in the range of T as

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ a-b \end{pmatrix}$$

Suppose $\mathbf{u} \in \ker(T)$

Then

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}$$

and so

$$\begin{aligned} q &= -r - \frac{4}{7}s \\ p &= -4q - 5r - 2s \\ &= -r + \frac{2}{7}s \end{aligned}$$

Hence, if $\mathbf{u} \in \ker(T)$ then

$$\mathbf{u} = \begin{pmatrix} -r + \frac{2}{7}s \\ -r - \frac{4}{7}s \\ r \\ s \end{pmatrix}$$

Note that the nullity and the rank are both 2

Note further that the sum of the rank and the nullity is 4, which is the dimension of $\Re^4$

Is T 1-1?

Setting $s = 0, r = 1$, and $s = 7, r = 0$ in the kernel vectors on the previous slide implies that both

$$\mathbf{u}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} 2 \\ -4 \\ 0 \\ 7 \end{pmatrix}$$

lie in the kernel of T

We then have $T(\mathbf{u}_1) = T(\mathbf{u}_2)$, but $\mathbf{u}_1 \neq \mathbf{u}_2$, and so T is not 1-1

Is T onto?

We proved earlier that any vector $\mathbf{v}$ in the range of T may be written

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ a-b \end{pmatrix}$$

The vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \Re^3$ may not be written in this form, and so T is not onto