FUNCTIONAL PROGRAMMING 2016

GABRIEL MOISE

QUESTION 1

(a) We will start with the definitions for the functions (++), foldl, reverse and foldr as they will come in handy for the equalities we will need to prove:

```
> (++) :: [a] -> [a] -> [a]
> [] ++ ys = ys
> (x:xs) ++ ys = x:(xs ++ ys)
```

```
> foldl :: Foldable t => (b -> a -> b) -> b -> t a -> b
> foldl f e [] = e
> foldl f e (x:xs) = foldl f (f e x) xs
```

```
> foldr :: Foldable t => (a -> b -> b) -> b -> t a -> b
> foldr f e [] = e
> foldr f e (x:xs) = f x (foldr f e xs)
```

```
> reverse :: [a] -> [a]
> reverse [] = []
> reverse (x:xs) = reverse xs ++ [x]
```

Now, we want to prove that

$$\text{foldl } f \ e \ (xs +\!\!+ ys) = \text{foldl } f \ (\text{foldl } f \ e \ xs) \ ys \quad (*)$$

for all finite lists xs and all lists ys.

We will prove this by induction on xs:

I $xs = []$

$$\text{foldl } f \ e \ ([] +\!\!+ ys) = \text{foldl } f \ (\text{foldl } f \ e \ []) \ ys$$

{ definition of (++) and foldl }

$$\text{foldl } f \ e \ ys = \text{foldl } f \ e \ ys \quad \underline{\text{TRUE}}$$

II Now, we assume that (*) holds for xs (iH) and we prove it for (x:xs)

$$\text{foldl } f \ e \ ((x:xs) +\!\!+ ys) = \text{foldl } f \ (\text{foldl } f \ e \ (x:xs)) \ ys$$

{ definition of (++) and foldl }

$$\text{foldl } f \ e \ x:(xs +\!\!+ ys) = \text{foldl } f \ (\text{foldl } f \ (f \ e \ x) \ xs) \ ys$$

{ inductive hypothesis }

$$\text{foldl } f \ e \ x:(xs +\!\!+ ys) = \text{foldl } f \ (f \ e \ x) \ (xs +\!\!+ ys)$$

{ definition of foldl }

$$\text{foldl } f \ (f \ e \ x) \ (xs +\!\!+ ys) = \text{foldl } f \ (f \ e \ x) \ (xs +\!\!+ ys) \quad \underline{\text{TRUE}}$$

From I and II we can conclude that (*) is true for all finite lists xs and all lists ys.

(b) Now, we'll prove that

$$\text{foldr } f\ e\ xs = \text{foldl } (\text{flip } f)\ e\ (\text{reverse } xs) \quad (**)$$

for all finite lists xs, where $\text{flip } f\ x\ y = f\ y\ x$

We will use again induction on xs:

I $\underline{xs = []}$

$$\text{foldr } f\ e\ [] = \text{foldl } (\text{flip } f)\ e\ (\text{reverse } [])$$

{definitions of foldr and reverse}

$$e = \text{foldl } (\text{flip } f)\ e\ []$$

{definition of foldl}

$$e = e \qquad \underline{\text{TRUE}}$$

II Now, we assume that (**) holds for xs (IH) and we'll prove it for (x:xs)

$$\text{foldr } f\ e\ (x:xs) = \text{foldl } (\text{flip } f)\ e\ (\text{reverse } (x:xs))$$

{definitions of foldr and reverse}

$$f\ x\ (\text{foldr } f\ e\ xs) = \text{foldl } (\text{flip } f)\ e\ (\text{reverse } xs \mathbin{+\!\!+} [x])$$

{using (*) from (a), and that [x] is equivalent to x:[]}

$$f\ x\ (\text{foldr } f\ e\ xs) = \text{foldl } (\text{flip } f)\ \big(\text{foldl } (\text{flip } f)\ e\ (\text{reverse } xs)\big)\ (x:[])$$

{inductive hypothesis}

$$f\ x\ (\text{foldr } f\ e\ xs) = \text{foldl } (\text{flip } f)\ (\text{foldr } (\text{flip } (\text{flip } f))\ e\ xs)\ (x:[])$$

{flip (flip f) is equivalent to f}

$$f\ x\ (\text{foldr } f\ e\ xs) = \text{foldl } (\text{flip } f)\ (\text{foldr } f\ e\ xs)\ (x:[])$$

{definition of foldl}

$$f\ x\ (\text{foldr } f\ e\ xs) = \text{foldl } (\text{flip } f)\ \big((\text{flip } f)\ (\text{foldr } f\ e\ xs)\ x\big)\ []$$

{definition of foldl}

$$f\ x\ (\text{foldr } f\ e\ xs) = (\text{flip } f)\ (\text{foldr } f\ e\ xs)\ x$$

{definition of flip}

$$f\ x\ (\text{foldr } f\ e\ xs) = f\ x\ (\text{foldr } f\ e\ xs) \qquad \underline{\text{TRUE}}$$

From I and II we can conclude that (**) is true for all finite lists xs.

(c) First, we have

$$(++) = \text{foldr } (:)$$

Therefore,

$$xs = \text{foldr } (:) \; [\,] \; xs$$

From (**) we get that

$$xs = \text{foldr } (:) \; [\,] \; xs = \text{foldl } (\text{flip } (:)) \; [\,] \; (\text{reverse } xs)$$

which is equivalent to

$$id = (\text{foldl } (\text{flip } (:)) \; [\,]) \, . \, \text{reverse}$$

If we apply . reverse to both, we get

$$\text{reverse} = \text{foldl } (\text{flip } (:)) \; [\,].$$