Introduction to Formal Proof
Tutorial Sheet 1*

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Question 1

(a) In the cases where they are *syntactically correct propositions* show the fully-bracketed form of each of the following:

i. $\neg\neg\neg P$

ii. $\neg\neg P \wedge \neg Q$

iii. $\neg P \neg \vee Q$

iv. $P \leftrightarrow Q \rightarrow Q \leftrightarrow P$

v. $P \rightarrow Q \leftrightarrow Q \rightarrow P$

vi. $(P \rightarrow Q \rightarrow R) \vdash P \wedge Q \rightarrow R$

vii. $(P \rightarrow Q \rightarrow R) \rightarrow P \wedge Q \rightarrow R$

viii. $P \wedge Q \wedge R \rightarrow P \vee Q \vee R$

(b) Show how to represent each of the following statements formally in the language of propositions indicating which elementary statement is represented by each of the propositional letters you have used. Use as few propositional letters as is consistent with avoiding using propositional letters to represent compound statements.

i. The logic lectures have started but the concurrency lectures have not.

ii. If Bernard is at a logic lecture then he is not at a concurrency lecture.

iii. Either Bernard teaches a logic lecture or he stays in bed; he can't do both.

iv. If Dick is moving then Jane is stationary and if Jane is moving then Dick is stationary.

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v. When Dick has drunk he doesn't drive and when he drives he hasn't drunk.

vi. Carol is jealous of Yvonne or not in a good mood.

vii. When the barometer falls it will rain or snow.

viii. If you have been to the lectures and read the first two sections of the notes then you should be able to do the first few tutorial questions; otherwise you will not be able to.

ix. The formula is always true if its subformulae are always true.

x. The formula is always true only if its subformulae are always true.

xi. The formula is always true if and only if its subformulae are always true.

(c) Show how to represent the following conjectures formally.

i. Tomorrow it will either rain or snow. It only snows when it is cold. It will be warm tomorrow. Therefore it will rain tomorrow.

ii. Either the bongo repeller or the footler is broken. There is always a bad smell when the bongo repeller is broken. There isn't a bad smell. Therefore the footler is broken.

iii. Either the UFOs are secret enemy weapons or they are spaceships from an alien world. If they are enemy weapons, then enemy technology is (contrary to current thinking) superior to ours. If they are alien spacecraft then they display a technology more sophisticated than anything we can imagine. In any case, therefore, their builders are technologically more sophisticated than we are.

Question 2

Prove each of the following conjectures using *only* the Natural Deduction rules given in the lecture notes, and (as derived rules) any theorems you need that are proved in the notes or you are able to prove yourself. Present your proofs in "boxed linear form". Except where there is a hint to the contrary, you would be well advised to use the method first discussed in Chapter 1 in the section *Proof rules as "conjecture transformers"*.

(a) i. $P \vdash Q \rightarrow (Q \wedge P)$

ii. $P \vdash Q \rightarrow (P \wedge Q)$

iii. $Q \vdash P \rightarrow (P \wedge Q)$

iv. $P \rightarrow (Q \wedge R) \vdash (P \rightarrow Q) \wedge (P \rightarrow R)$

v. $(P \rightarrow Q) \wedge (P \rightarrow R) \vdash P \rightarrow Q \wedge R$

(b) i. $\vdash (Q \leftrightarrow R) \to (Q \to S) \to (R \to S)$

ii. $\vdash (Q \leftrightarrow R) \to (Q \land S) \to (R \land S)$

(c) (Distributivity results)

i. $(P \land Q) \lor (P \land R) \vdash P \land (Q \lor R)$

ii. $(P \lor Q) \land (P \lor R) \vdash P \lor (Q \land R)$

Finding a proof for this might be harder than you first think. You will have to delay deciding which of the disjuncts of the conclusion to satisfy, and you will find yourself deciding this more than once. A good way to start your search is to "explode" the premiss into its conjuncts using the $\land$-elim rules, then select one of the conjuncts first to use $\lor$-elim on.

iii. $\vdash (P \land Q) \lor (P \land R) \leftrightarrow P \land (Q \lor R)$

Hint: use two already-proven conjectures in this proof.

iv. $\vdash (P \lor Q) \land (P \lor R) \leftrightarrow P \lor (Q \land R)$

Hint: use two already-proven conjectures in this proof.

(d) (Proofs involving negation)

None of the linearly-presented proofs in this part of the question need be longer than about ten lines.

Avoid using RAA if you can; but if you do use it[1] then the best place to use it is at the root of the proof.

i. $\neg P \land \neg Q \vdash \neg(P \lor Q)$

ii. $\neg P \lor \neg Q \vdash \neg(P \land Q)$

iii. $\neg(P \land Q) \vdash \neg P \lor \neg Q$

Hint: You can use RAA, $\neg$-E and the theorem $\neg(E \lor F) \vdash \neg E \land \neg F$ proven in the notes; you will need other rules.

iv. $A, \neg A \vdash B$

v. $T \lor U, T \to D, \neg D \vdash U$

vi. $\vdash \neg(P \land \neg P)$

vii. $(P \to Q) \land (Q \to R), \neg R \vdash \neg P$

viii. $(P \to Q) \land (Q \to \neg P) \vdash \neg P$

ix. $(P \to Q) \vdash \neg P \lor Q$

x. $\neg P \lor Q \vdash P \to Q$

[1] I used it 4 times.

Question 3

In this question we consider some of the steps needed to introduce a new binary logical connective into the language of propositions.

The connective we want to introduce is "exclusive or". The composite proposition P **exclusive or** Q written $P \veebar Q$, is intended to be true when *exactly one* of P, Q are true.

(a) Show how to extend the definition of `eval` (see Chapter 2: "Tautology and Satisfiability Checking") to incorporate a case corresponding to the connective $\veebar$

(b) Design natural deduction introduction and elimination rules for the connective $\veebar$, and explain carefully why you think they are sound.

(c) What conjecture(s) might you attempt to prove to check that the proof rules you gave above are consistent with the prose explanation of $\veebar$ given above. You may give the proofs of this/these if you want.

Question 4

(a) (Goal-directed sequent calculus rules for backward proof discovery)[2]

i. (Optional) Show – by giving a proof of $\neg\varphi \vdash C$ in which its antecedent sequent is used – that the following goal-directed rule can be derived.

$$\frac{\varphi \to \bot \vdash C}{\neg\varphi \vdash C}\ \neg\vdash$$

Hint: the proof is short. If you are working on a linearised form then start with a proof of $\varphi{\to}\bot$ from $\neg\varphi$ that uses $\neg$-elim and $\to$-intro. Once you have that, you can use the antecedent sequent, *viz:* $\varphi \to \bot \vdash C$, "as if it were a theorem" to complete the proof. (If you are working on the sequent tree form of the proof then you will need the CUT rule for this step)

Briefly explain how the structure of your proof could be used to prove the more generally-applicable rule:

$$\frac{\Gamma, \varphi \to \bot \vdash C}{\Gamma, \neg\varphi \vdash C}\ \neg\vdash$$

ii. (Optional) Do likewise for

$$\frac{\vdash \varphi \to \bot}{\vdash \neg\varphi}\ \vdash\neg$$

iii. (Possibly fun) Using the above two rules, together with hyp, $\wedge$-intro where necessary, and the rules $\to$-i (Notes p2.33) and $\to\vdash$ (Notes p 2.35), and applying rules as "sequent transformers", construct proof trees for the sequents: $P \vdash \neg\neg P$, $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$ and $(P \to Q) \wedge (Q \to \neg P) \vdash \neg P$.

The point of this part of the question is to show that with appropriate (single-conclusion) sequent-calculus rules one can work more or less mechanically at discovering proofs of certain kinds of (single-conclusion) sequent.

[2]See part 2 of the notes: in the section "Reformulating ND as a single-conclusion sequent calculus".