Discrete Mathematics

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Chapter 5: Sequences

Sequences

A **sequence** is an ordered list of objects (usually infinite):

$$(x_1, x_2, x_3, \ldots)$$

Alternatively, a sequence is a function whose domain is $\mathbb{N}$ or $\mathbb{N}_+$.

The whole sequence is denoted

$$(x_i)$$

and the i^{th} term

$$x_i$$

The simplest way to define a sequence is to give a formula for its terms:

$$x_n = 2n$$

$$a_i = i^2$$

Recurrence Relations

Sequences can be defined **recursively**, such as

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n$$

(This defines the **Fibonacci sequence**).

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When we are given a recurrence relation, and want to find a nonrecursive formula for the n^{th} term, we speak of **solving** the recurrence.

In this case, $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Proof by Induction

If $S(n)$ is a statement involving a natural number n, and we want to prove $S(n)$ for all n, we often use induction:

The Principle of Induction

If we prove
- $S(0)$
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<u>Claim</u> For the Fibonacci sequence $F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n$, $2 \mid F_{3n}$ for all $n \in \mathbb{N}$.

Variations

The Principle of Induction (different base case)

If we prove
- $\boldsymbol{S(1)}$
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Claim Every positive integer can be written as the sum of distinct Fibonacci numbers.

The Minimal Counterexample

A form of proof by contradiction. If we want to prove *S(n)* for all natural numbers *n*, we suppose that it is not, and define *m* to be:

the smallest natural number for which *S(m)* is false,

and then prove that *S(m')* must also be false for some smaller natural number *m'*.

Claim   The recurrence $x_1 = 1,\ x_2 = 3,\ x_{n+2} = 4x_{n+1} + 3x_n$ generates a sequence of odd numbers.

Sigma Notation

If (a_i) is a sequence, we can write

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdots a_n$$

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<u>Claim</u> For $n \in \mathbb{N}_+$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

A Recurrence for Derangements

We have already derived the number of **derangements** of n objects:

$$d_n = \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!}$$

But we can take another approach using recurrence relations.

<u>Claim</u> The sequence (d_n) satisfies the recurrence

$$d_1 = 0, \quad d_2 = 1, \quad d_n = (n-1)(d_{n-1} + d_{n-2}) \text{ for } n > 2.$$

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Trivia: the same recurrence, but with different boundary conditions, also generates the sequence of factorials.

A Recurrence for Partitions

How many equivalence relations are there on a set of cardinality n?

Equivalently, how many partitions are there of a set of cardinality n?

The number of partitions of a set of cardinality n is written B_n.

The sequence (B_n) is known as the **Bell numbers**. It begins (1, 1, 2, 5, 15, ...)

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It is possible to give a formula for B_n:

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

but this form (Dobinsky, 1877) is not particularly useful for computing B_n.

Diversion: Solving Linear Recurrences

To solve a homogeneous linear recurrence,

$$\lambda_m x_n + \lambda_{m-1} x_{n-1} + \cdots + \lambda_1 x_{n-m+1} + \lambda_0 x_{n-m} = 0 \qquad (1)$$

plus some boundary conditions (usually m of them).

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1. Form the **characteristic polynomial**:

   $$\lambda_m r^m + \lambda_{m-1} r^{m-1} + \cdots + \lambda_1 r + \lambda_0 = 0 \qquad (2)$$

   and solve it for (potentially complex) r.

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2. If the roots of (2) are all different, say $r = r_1,\ r_2,\ \ldots,\ r_m,$
   then the solutions of (1) are

   $$x_n = A_1 r_1^n + A_2 r_2^n + \cdots + A_m r_m^n$$

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3. If some roots are repeated, then duplicate terms must be multiplied by enough powers of n to make them distinct.

Diversion: Solving Linear Recurrences

To solve an inhomogeneous linear recurrence,

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2. Solve the corresponding homogeneous recurrence by deleting $f(n)$, still without regard to boundary conditions.

3. Add up the answers to parts 1 and 2, and finally use the boundary conditions to determine the missing constants.

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End of Chapter 5