Discrete Mathematics

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week 4	***Relations***
week 5	***Sequences***
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week 7	***Asymptotic Notation***
week 8	***Orders***

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Chapter 3: Counting

Laws of Sum & Subtract

Law of Sum: Let P1 and P2 be properties of objects which are **exclusive**. The number of objects with **either** property is the number with property P1 **plus** the number with property P2.

This is true because, if A and B are disjoint, $|A \cup B| = |A| + |B|$

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Law of Subtract: Let P1 and P2 be properties, such that P1 is true at least whenever P2 is. Then the number of objects with property P1 **but not** P2 is the number with property P1 **subtract** the number with property P2.

This is true because, if $B \subseteq A$ $|A \setminus B| = |A| - |B|$

Law of Product

Law of Product: If counting the number of ways of making a sequence of choices, and the choices are **independent** (the number available at each stage does not depend on the choices made previously), then the total number of ways of making the sequence of choices is the **product** of the number of choices at each stage.

This is true because $|A \times B| = |A| \cdot |B|$

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Factorial Function

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Note: The factorial function is extended to $\mathbb{N}$ by setting $0! = 1$

In fact it can be generalized to a function on $[0, \infty)$ using

$$\boldsymbol{\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, \mathrm{d}t.}$$

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Answer $\frac{n!}{k!(n-k)!}$

Combinatorial Coefficients

Definition
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(the number of ways to select k out of n distinguishable objects).

Think of $\binom{\cdot}{\cdot}$ as a partial function $\mathbb{N}^2 \to \mathbb{N}_+$

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By counting the same set in different ways, we can often prove facts about combinatorial formulae, e.g.

Claim $$\mathbf{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}}$$

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Answer

$$d_n = \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \cdots + (-1)^n \frac{n!}{n!}$$

Ceiling and Floor

The floor function "rounds down" real numbers to integers:

$$\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z} \qquad\qquad \lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$$

The ceiling function "rounds up" real numbers to integers:

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<u>Answer</u> 733

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Claim $n! = \binom{n}{0} d_0 + \binom{n}{1} d_1 + \binom{n}{2} d_2 + \cdots + \binom{n}{n} d_n$

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Answer As long as $n_1 + n_2 + \cdots + n_g = n$,

$$\frac{n!}{n_1! n_2! \cdots n_g!}$$

These are called the multinomial coefficients, and the symbol used is

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Example How many different bridge deals are there (allocation of 52 playing cards to four players, 13 each)?

Answer $\binom{52}{13\ 13\ 13\ 13} = \frac{52!}{13!^4} = 53,644,737,765,488,792,839,237,440,000$

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End of Chapter 3