DISCRETE MATHEMATICS

GABRIEL MOISE

MT 2018

PROGRAM SHEET 2

Chapter 2 (Functions) and 3 (Counting)

$\boxed{2.1.}$ (i) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+3$

As $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} 2x+3 = -\infty$, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} 2x+3 = \infty$ and because f is obviously continuous, $\mathrm{Im}(f) = (-\infty, \infty) = \mathbb{R}$

1-1: Let $x_1, x_2 \in \mathbb{R}$ such that

$f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2 \Rightarrow f$ is 1-1.

ONTO: For each $y \in \mathbb{R}$, there is an $x = \frac{y-3}{2} \in \mathbb{R}$ such that $f(x) = y$:

$f\left(\frac{y-3}{2}\right) = 2\frac{y-3}{2} + 3 = y - 3 + 3 = 3 \Rightarrow f$ is onto

Therefore, f has an inverse and $f^{-1}: \mathbb{R} \to \mathbb{R}$, $f^{-1}(y) = \frac{y-3}{2}$.

(ii) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 2x^2 + x$

As $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x^3 - 2x^2 + x = -\infty$, $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x^3 - 2x^2 + x = \infty$ and because f is continuous, $\mathrm{Im}(f) = (-\infty, \infty) = \mathbb{R}$

1-1: We can easily see that $f(0) = 0^3 - 2\cdot 0^2 + 0 = 0$ and

$f(1) = 1^3 - 2\cdot 1^2 + 1 = 1 - 2 + 1 = 0 \Rightarrow f(0) = f(1) \Rightarrow f$ is not 1-1.

ONTO: As f is continuous and $\mathrm{Im}\, f = \mathbb{R}$, then, by applying Darboux's property we get that for every $y \in \mathbb{R}$, $(\exists)\, x \in \mathbb{R}$ such that $f(x) = y$. Therefore, f is onto.

So, f doesn't have an inverse.

(iii) $f: \mathbb{R} \to [-\frac{1}{2}, \frac{1}{2}]$, $f(x) = (\sin x)(\cos x) = \frac{1}{2}\sin(2x)$

As f is continuous, $\mathrm{Im}(f)$ is an interval. Also, as $\sin(2x) \in [-1, 1]$, with all values being obtained, $\mathrm{Im}(f) = [-\frac{1}{2}, \frac{1}{2}]$.

1-1: $f(0) = \frac{1}{2}\sin 0 = \frac{1}{2}$

$f(\pi) = \frac{1}{2}\sin(2\pi) = \frac{1}{2}$ $\Rightarrow f$ is not 1-1

$0 \neq \pi$

ONTO: $\mathrm{Im}\, f = [-\frac{1}{2}, \frac{1}{2}] \Rightarrow f$ is onto

As f is not 1-1, f doesn't have an inverse.

(iv) $f:\mathbb{R}\to(-1,1)$, $f(x)=\dfrac{x}{1+|x|}$

$$f(x)=\begin{cases}\dfrac{x}{1-x}, & x<0\\ \dfrac{x}{1+x}, & x\geq 0\end{cases}$$

As f is continuous ($1+|x|>0$ for all $x\in\mathbb{R}$) and because $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}\dfrac{x}{1-x}=$
$=\lim\limits_{x\to-\infty}\dfrac{1}{\frac{1}{x}-1}=\dfrac{1}{0-1}=-1$ and also $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}\dfrac{x}{1+x}=\lim\limits_{x\to\infty}\dfrac{1}{\frac{1}{x}+1}=\dfrac{1}{0+1}=1$, from Darboux's property we can conclude that $\mathrm{Im}(f)=(-1,1)$

1-1: Let $x_1,x_2\in\mathbb{R}$ such that (we can suppose $x_1<x_2$)

$f(x_1)=f(x_2)$

Case 1: $x_1,x_2<0 \Rightarrow \dfrac{x_1}{1-x_1}=\dfrac{x_2}{1-x_2}\Rightarrow x_1-\cancel{x_1x_2}=x_2-\cancel{x_1x_2}\Rightarrow \underline{x_1=x_2}$

Case 2: $x_1,x_2\geq 0 \Rightarrow \dfrac{x_1}{1+x_1}=\dfrac{x_2}{1+x_2}\Rightarrow x_1+\cancel{x_1x_2}=x_2+\cancel{x_1x_2}\Rightarrow \underline{x_1=x_2}$

Case 3: $x_1<0, x_2\geq 0\Rightarrow \dfrac{x_1}{1-x_1}=\dfrac{x_2}{1+x_2}$. But $\dfrac{x_1}{1-x_1}<0$, $\dfrac{x_2}{1+x_2}>0$, so this equality (and whole case) is impossible.

Therefore, $x_1=x_2\Rightarrow f$ is 1-1.

ONTO: $\mathrm{Im}(f)=(-1,1)\Rightarrow f$ is onto

So, f has an inverse $f^{-1}:(-1,1)\to\mathbb{R}$, $f^{-1}(y)=\begin{cases}\dfrac{y}{y+1}, & y<0\\ -\dfrac{y}{y-1}, & y\geq 0\end{cases}=f^{-1}(y)=\dfrac{y}{1-|y|}$

it can be calculated by cases:

$x<0\Rightarrow \dfrac{x}{1-x}=y\Rightarrow x=y-xy\Rightarrow x+xy=y\Rightarrow x(1+y)=y\overset{(y\neq-1)}{\Rightarrow} x=\dfrac{y}{y+1}$

$x\geq 0\Rightarrow \dfrac{x}{1+x}=y\Rightarrow x=y+xy\Rightarrow x-xy=y\Rightarrow x(1-y)=y\overset{(y\neq1)}{\Rightarrow} x=\dfrac{y}{1-y}$

$\boxed{2.2}$ $a\in\mathbb{R}$, $f:(-a,a)\to\mathbb{R}$, $f(x)=\tan x$

(i) As $\tan x=\dfrac{\sin x}{\cos x}$, f is well-defined when $\cos x\neq 0$. That means $x\notin\left\{\frac{(2k+1)\pi}{2}\,/\,k\in\mathbb{Z}\right\}$.
Therefore, the largest value of a is $\frac{\pi}{2}$.

(ii) So, $f:(-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R}$, which is obviously continuous (as $\sin x$ and $\cos x$ are, and $\cos x\neq0$). We have $\lim\limits_{\substack{x\to-\frac{\pi}{2}\\ x>-\frac{\pi}{2}}}f(x)=\lim\limits_{\substack{x\to-\frac{\pi}{2}\\ x>-\frac{\pi}{2}}}\dfrac{\sin x}{\cos x}=-\infty$ and $\lim\limits_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}f(x)=\lim\limits_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}}\dfrac{\sin x}{\cos x}=\infty$ and from Darboux's property we conclude that $\mathrm{Im}(f)=(-\infty,\infty)=\mathbb{R}\Rightarrow f$ is onto.

Let's take $x_1,x_2\in(-\frac{\pi}{2},\frac{\pi}{2})$ such that

$f(x_1)=f(x_2)\Rightarrow \tan x_1=\tan x_2\Rightarrow \dfrac{\sin x_1}{\cos x_1}=\dfrac{\sin x_2}{\cos x_2}\Rightarrow \sin x_1\cos x_2=\sin x_2\cos x_1\Rightarrow$
$\Rightarrow \sin x_1\cos x_2-\sin x_2\cos x_1=0\Rightarrow \sin(x_1-x_2)=0$

As we have $x_1, x_2 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow x_1 - x_2 \in (-\pi, \pi)$

Therefore $\sin(x_1-x_2)=0$ implies $x_1-x_2=0 \Rightarrow x_1=x_2 \Rightarrow f$ is 1-1.

So, f is also a bijection.

(iii) As f is a bijection, f has an inverse $g=f^{-1}$ and because $f\circ g = g\circ f$ and $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, we can deduce that $g: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

(iv) As $g(x) = \arctan x$ and we want its codomain to be $(-1,1)$, we can simply multiply g with the constant $\frac{2}{\pi}$. This way g is still a bijection (we multiplied the function by a constant $\Rightarrow \frac{2}{\pi} g$ is still <u>1-1</u> and also as the codomain of g was $(-\frac{\pi}{2}, \frac{\pi}{2})$ and g was onto, the $\frac{2}{\pi} g$ has the codomain $(-1,1)$ and it's also <u>onto</u> $\Rightarrow \frac{2}{\pi} g$ is a bijection for $\mathbb{R}$ to $(-1,1)$). So, the function is $h: \mathbb{R} \to (-1,1)$, $h(x) = \frac{2}{\pi} \arctan x$.

$\boxed{2.3}$ $f: \mathbb{N}^2 \to \mathbb{Z}$

$f(x,y) = x^2 - 4y^2$

<u>1-1</u>: We can see that $f(0,0) = 0^2 - 4\cdot 0^2 = 0$ and $f(2,1) = 2^2 - 4\cdot 1^2 = 4-4=0 \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} f(0,0) = f(2,1) \\ (0,0) \neq (2,1) \end{array}\right| \Rightarrow f \text{ is } \underline{\text{not}} \text{ 1-1.}$$

<u>ONTO</u>: As $x \in \mathbb{N}$, x^2 can either be a multiple of 4, or a (multiple of 4) + 1. Let's take cases to see that this is true: $(k \in \mathbb{Z})$

<u>Case 1</u>: $x = 4k \Rightarrow x^2 = 16k^2 = 4\cdot(4k^2) =$ multiple of 4

<u>Case 2</u>: $x = 4k+1 \Rightarrow x^2 = 16k^2 + 8k + 1 = 4\cdot(4k^2+2k)+1 =$ (multiple of 4) + 1

<u>Case 3</u>: $x = 4k+2 \Rightarrow x^2 = 16k^2 + 16k + 4 = 4\cdot(4k^2+4k+1) =$ multiple of 4

<u>Case 4</u>: $x = 4k+3 \Rightarrow x^2 = 16k^2 + 24k + 9 = 4\cdot(4k^2+6k+2)+1 =$ (multiple of 4) + 1

So, x^2 is either a multiple of 4, or a (multiple of 4) + 1.

$$\left.\begin{array}{l} \text{I } x^2 = \text{multiple of } 4 \Rightarrow x^2-4y^2 = \text{multiple of } 4 \Rightarrow x^2-4y^2 \in \{4k \mid k \in \mathbb{Z}\} \\ \text{II } x^2 = (\text{multiple of } 4)+1 \Rightarrow x^2-4y^2 = (\text{multiple of } 4)+1 \Rightarrow x^2-4y^2 \in \{4k+1 \mid k\in\mathbb{Z}\} \end{array}\right| \Rightarrow$$

$$\left.\begin{array}{l} x^2-4y^2 \in \{4k \mid k\in\mathbb{Z}\} \cup \{4k+1 \mid k\in\mathbb{Z}\} \\ \text{Let's suppose that there are some } x \text{ and } y \text{ for which } x^2-4y^2 = 2 \end{array}\right| \Rightarrow$$

$$\Rightarrow 2 \in \{4k \mid k\in\mathbb{Z}\} \cup \{4k+1 \mid k\in\mathbb{Z}\} \text{ (False)} \Rightarrow \left.\begin{array}{l} 2 \notin \operatorname{Im}(f) \\ f: \mathbb{N}^2 \to \mathbb{Z} \end{array}\right| \Rightarrow f \text{ is } \underline{\text{not}} \text{ onto.}$$

$\boxed{2.4.}$ (i) $A = \{1,2,3,4,5\}$, $B = \{1,2\}$

The number of functions from A to B is equal to $|B|^{|A|} = 2^5 = 32$.

Let's suppose there exists an $f: A \to B$, such that f is 1-1. Then $f(1)$, $f(2)$ and $f(3)$ are all distinct. But $f(1), f(2), f(3) \in \{1,2\}$ so this is impossible (Pigeonhole principle). Therefore no function from A to B is 1-1.

For a function f from A to B to be onto, $\operatorname{Im}(f) = \{1,2\}$. The only functions that do not have this property are $f: A \to B, f(x)=1$ and $g: A \to B, g(x)=2$. Therefore, there are $2^5 - 2 = 30$ onto functions from A to B.

(ii) $|A| = m$, $|B| = n$

There are n^m functions from A to B (for each value from A we have n possibilities, so by using the product rule we get n^m)

If $m > n$, there is no 1-1 function (Pigeonhole principle).

If $m \leq n$, we assign to A a set A' with $|A'| = |A| = m$, and $A' \subseteq B$, the set of different values f reaches for each argument from A.

If $f: A \to A'$ is 1-1, with $A = \{x_1, x_2, \dots, x_m\}$ and $A' = \{y_1, y_2, \dots, y_m\}$, the number of functions f is the number of different permutations of m distinguishable objects, which is $m!$.

From B we can choose A' in $\binom{n}{m}$ ways, so the total number of functions f which are 1-1 is $\binom{n}{m} m! = \frac{n!}{\cancel{m!} \cdot (n-m)!} \cdot \cancel{m!} = \frac{n!}{(n-m)!}$

$\boxed{2.5}$ (i) $\overline{abc}$, with $a+b+c=8$

$\overline{ab}$, with $a+b=8 \Rightarrow \{80, 71, 62, 53, 44, 35, 26, 17\} \to \boxed{8}$

$\overline{a}$, with $a=8 \Rightarrow \{8\} \to \boxed{1}$

1) $a=8 \Rightarrow b=c=0 \Rightarrow \{800\} \to \boxed{1}$
2) $a=7 \Rightarrow b+c=1 \Rightarrow \boxed{2}$
3) $a=6 \Rightarrow b+c=2 \Rightarrow \boxed{3}$
4) $a=5 \Rightarrow b+c=3 \Rightarrow \boxed{4}$
5) $a=4 \Rightarrow b+c=4 \Rightarrow \boxed{5}$
6) $a=3 \Rightarrow b+c=5 \Rightarrow \boxed{6}$
7) $a=2 \Rightarrow b+c=6 \Rightarrow \boxed{7}$
8) $a=1 \Rightarrow b+c=7 \Rightarrow \boxed{8}$

$\Rightarrow \frac{8 \cdot 9}{2} = \boxed{36}$

$\Rightarrow \boxed{45}$ numbers

(ii) $\overline{abc}$, with $a>4$, $b \leq 4$, $c \leq 4$ and $a+b+c=8$

1) $a=8 \Rightarrow \{800\} \Rightarrow \boxed{1}$
2) $a=7 \Rightarrow b+c=1 \Rightarrow \boxed{2}$
3) $a=6 \Rightarrow b+c=2 \Rightarrow \boxed{3}$
4) $a=5 \Rightarrow b+c=3 \Rightarrow \boxed{4}$

$\Rightarrow \boxed{10}$ numbers

(iii) $a \Rightarrow a=8 \Rightarrow \boxed{1}$

$\overline{ab} \Rightarrow \{53; 62; 71; 80; 17; 26; 35\} \Rightarrow \boxed{7}$

$\overline{abc}$: $a>4$, $b \leq 4$, $c \leq 4$ and $a+b+c=8 \xRightarrow{(ii)} \boxed{10}$

$b>4$, $a \leq 4$, $c \leq 4$ and $a+b+c=8 \Rightarrow \boxed{6}$:

1) $b=8 \Rightarrow a=0$ (False)
2) $b=7 \Rightarrow a=1 \Rightarrow \{170\} \Rightarrow \boxed{1}$

3) $b=6 \Rightarrow \{161, 260\} \Rightarrow \boxed{2}$

4) $b=5 \Rightarrow \{152, 251, 350\} \Rightarrow \boxed{3}$

Case where $c>4$, $a\leq 4$, $b\leq 4$ is the same as with $b>4$, $a,c\leq 4 \Rightarrow \boxed{6}$

In total, we have $1+7+10+6+6=\boxed{30}$

(iv) $\overline{a}$, $a=14$ NO

$\overline{ab}$, $a+b=14 \Rightarrow \{59, 68, 77, 86, 95\} \Rightarrow \boxed{5}$

$\overline{abc}$, $a+b+c=14$

1) $a=9 \Rightarrow b+c=5 \Rightarrow \boxed{6}$
2) $a=8 \Rightarrow b+c=6 \Rightarrow \boxed{7}$
3) $a=7 \Rightarrow b+c=7 \Rightarrow \boxed{8}$
4) $a=6 \Rightarrow b+c=8 \Rightarrow \boxed{9}$
5) $a=5 \Rightarrow b+c=9 \Rightarrow \boxed{10}$
6) $a=4 \Rightarrow b+c=10 \Rightarrow \boxed{9}$
7) $a=3 \Rightarrow b+c=11 \Rightarrow \boxed{8}$
8) $a=2 \Rightarrow b+c=12 \Rightarrow \boxed{7}$
9) $a=1 \Rightarrow b+c=13 \Rightarrow \boxed{6}$

$\Rightarrow \boxed{70}$

$\Rightarrow \boxed{75}$ numbers

$\boxed{2.6}$ $\overline{abc}$ is divisible by at least one number from $\{5, 6, 8\}$

$A = \{\text{numbers divisible by } 5\} \Rightarrow A = \{5\cdot 20, 5\cdot 21, \dots, 5\cdot 199\} \Rightarrow |A| = 180$

$B = \{\text{numbers divisible by } 6\} \Rightarrow B = \{6\cdot 17, 6\cdot 18, \dots, 6\cdot 166\} \Rightarrow |B| = 150$

$C = \{\text{numbers divisible by } 8\} \Rightarrow C = \{8\cdot 13, 8\cdot 14, \dots, 8\cdot 124\} \Rightarrow |C| = 112$

$D = \{\text{numbers divisible by } 5 \text{ and } 6\} = \{\text{numbers divisible by } 30\} = \{30\cdot 4, 30\cdot 5, \dots, 30\cdot 33\} \Rightarrow$
$\Rightarrow |D| = 27$; $D = A\cap B \Rightarrow |A\cap B| = 30$

$E = \{\text{numbers divisible by } 5 \text{ and } 8\} = \{\text{numbers divisible by } 40\} = \{40\cdot 3, 40\cdot 4, \dots, 40\cdot 24\} \Rightarrow$
$\Rightarrow |E| = 22$; $E = A\cap C \Rightarrow |A\cap C| = 22$

$F = \{\text{numbers divisible by } 6 \text{ and } 8\} = \{\text{numbers divisible by } 24\} = \{24\cdot 5, 24\cdot 6, \dots, 24\cdot 41\} \Rightarrow$
$\Rightarrow |F| = 37$; $F = B\cap C \Rightarrow |B\cap C| = 37$

$G = \{\text{numbers divisible by } 5, 6 \text{ and } 8\} = \{\text{numbers divisible by } 120\} = \{120\cdot 1, 120\cdot 2, \dots, 120\cdot 8\} \Rightarrow$
$\Rightarrow |G| = 8$; $G = A\cap B\cap C \Rightarrow |A\cap B\cap C| = 8$

We want to calculate the number of elements from $A\cup B\cup C = \{\text{numbers divisible by at least one number from } \{5,6,8\}\}$. We will use the inclusion-exclusion principle:

$|A\cup B\cup C| = (|A|+|B|+|C|) - (|A\cap B| + |A\cap C| + |B\cap C|) + |A\cap B\cap C|$

$|A\cup B\cup C| = 361$.

$\boxed{2.7}$ We want to prove that $(m \cdot n)!$ is divisible by $(m!)^n$, for all $m, n \in \mathbb{N}$

By using the formula of a multinomial coefficient, we can create a card game, we'll call it bridge2, where instead of 52 playing cards, we have $(m \cdot n)$ cards, and we have n players, instead of 4, each being dealt a number of m playing cards.

If we want to calculate how many different bridge2 deals there are, we can do that by using the formula: $\dfrac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_g!}$, in our case n is $(m \cdot n)$, $n_1, n_2, \ldots, n_g$ are actually all m (and $g = n$), and we'll obtain

$$\frac{(m \cdot n)!}{\underbrace{m! \cdot m! \cdot \ldots \cdot m!}_{n \text{ times}}} = \frac{(m \cdot n)!}{(m!)^n} \in \mathbb{N}_+,$$

as the number of different bridge2 deals is a whole positive integer.

Therefore, $(m!)^n$ must divide $(m \cdot n)!$.

$\boxed{2.4}$ QUESTION: How can we calculate the number of onto functions from A to B, with $|A| = m$ and $|B| = n$?