INTRODUCTION TO FORMAL PROOF 2018

QUESTION 7

(a) Proof by contradiction (RAA) :

$$\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \text{RA}$$

$\neg\phi \to \bot \vdash \phi$

1.	$\neg\phi \to \bot$	premiss
2.	$\phi \vee \neg\phi$	Law of the Excluded Midde
3.	ϕ	assumption
4.	$\neg\phi$	assumption
5.	$\bot$	$\to$- elim 1
6.	ϕ	$\bot$ - elim 5
7.	ϕ	$\vee$ - elim 2, 3 , 4 - 6

$\neg\neg$- elim can be proven as

$\neg\neg\phi \vdash \phi$

1.	$\neg\neg\phi$	premiss
2.	$\neg\phi$	assumption
3.	$\bot$	$\neg$- elim 2, 1
4.	ϕ	RAA 2 - 3

(b) $\neg(\phi \vee \psi) \vdash \neg\phi \wedge \neg\psi$

1.	$\neg(\phi \vee \psi)$	premiss
2.	ϕ	assumption
3.	$\phi \vee \psi$	$\vee$- intro - L 2
4.	$\bot$	$\neg$- elim 3, 1
5.	$\neg\phi$	$\neg$- intro 2-4
6.	ψ	assumption
7.	$\phi \vee \psi$	$\vee$- intro - R 6
8.	$\bot$	$\neg$- elim 7, 1
9.	$\neg\psi$	$\neg$- intro 6-8
10.	$\neg\phi \wedge \neg\psi$	$\wedge$- intro 5, 9

(c) $(P \to Q) \to P \vdash P$

1.	$(P \to Q) \to P$	premiss
2.	$\neg P$	assumption
3.	$\neg(P \to Q)$	MT - Denying the Consequent 2, 1
4.	$P \wedge \neg Q$	theorem 3.
5.	P	$\wedge$- elim L 4
6.	$\bot$	$\neg$- elim 5, 2
7.	P	RAA 2 - 6

Proving the theorem:

$\neg(P \to Q) \vdash P \wedge \neg Q$

1.	$\neg(P \to Q)$	premiss
2.	$\neg(P \wedge \neg Q)$	assumption
3.	$\neg P \vee \neg\neg Q$	theorem (2) 2.
4.	$\neg P \vee Q$	$\neg\neg$- elim 3
5.	$\neg P$	assumption
6.	$\neg Q$	assumption
7.	$\neg P$	id 5
8.	$\neg Q \to \neg P$	$\to$- intro 6 - 7
9.	$P \to Q$	MT 8
10.	$\bot$	$\neg$- elim 9, 1
11.	Q	assumption
12.	P	assumption
13.	Q	id 11
14.	$P \to Q$	$\to$- intro 12 - 13
15.	$\bot$	$\neg$- elim 14, 1
16.	$\bot$	$\vee$- elim 4, 5 - 10, 11 - 15
17.	$P \wedge \neg Q$	RAA 2 - 16

Theorem 2 :

$\neg(P \wedge \neg Q) \vdash \neg P \vee Q$

1.	$\neg(P \wedge \neg Q)$	premiss
2.	$Q \vee \neg Q$	EM
3.	Q	assumption
4.	$\neg P \vee Q$	$\vee$- intro - R 3
5.	$\neg Q$	assumption
6.	P	assumption
7.	$P \wedge \neg Q$	$\wedge$- intro 6, 5
8.	$\bot$	$\neg$- elim 7, 1
9.	$\neg P$	$\neg$- intro 6 - 8
10.	$\neg P \vee Q$	$\vee$- intro - L 9
11.	$\neg P \vee Q$	$\vee$- elim 2, 3 - 4, 5 - 10

There must be an easier method !

1.	$(P \to Q) \to P$	premiss
2.	$\neg P$	assumption
3.	P	assumption
4.	$\bot$	$\neg$- elim 2, 3
5.	Q	$\bot$ - elim 4
6.	$P \to Q$	$\to$- intro 3 - 5
7.	P	$\to$- elim 6, 1
8.	$\bot$	$\neg$- elim 7, 2
9.	P	RAA 2 - 8

(d) P exclusive or Q is $P \veebar Q$, true when exactly one of P, Q is true

(i) introduction rules:

$$\frac{\varphi \quad \neg\psi}{\varphi \veebar \psi}\ \veebar\text{-intro-L} \qquad\qquad \frac{\neg\varphi \quad \psi}{\varphi \veebar \psi}\ \veebar\text{-intro-R}$$

Elimination rule:

$$\frac{(\varphi \veebar \psi) \quad \boxed{\begin{array}{c}\varphi \\ \neg\psi \\ \vdots \\ \kappa\end{array}} \quad \boxed{\begin{array}{c}\neg\varphi \\ \psi \\ \vdots \\ \kappa\end{array}}}{\kappa}\ \veebar\text{-elim}$$

The introduction rules are designed so that we have exclusive or when exactly one of the two inputs is true, and the elimination is similar to the $\vee$-elim rule:

- $\varphi \veebar \psi \Rightarrow$ one of them is true, one false
- the two boxes ensure that κ happens no matter the configuration

(ii) $\neg P \veebar Q \vdash P \veebar \neg Q$

1.	$\neg P \veebar Q$	premiss
2.	$\neg P$	assumption 1
3.	$\neg Q$	assumption 2
4.	$P \veebar \neg Q$	$\veebar$-intro-R 2,3
5.	$\neg\neg P$	assumption 1
6.	Q	assumption 2
7.	P	$\neg\neg$-elim 5
8.	$\neg\neg Q$	$\neg\neg$-intro 6
9.	$P \veebar \neg Q$	$\veebar$-intro-L 7,8
10.	$P \veebar \neg Q$	$\veebar$-elim 1, 2-4, 5-9

QUESTION 8

- R = unary relation symbol
- P, Q = binary relation simbols

(a) A variable is fresh in a proof if it doesn't appear free in any hypothesis or in the conclusion

(b) (i) $\forall x \cdot R(x) \vdash \neg \exists y \cdot \neg R(y)$

1.	$\forall x \cdot R(x)$	premiss
2.	$\exists y \cdot \neg R(y)$	assumption
	v fresh	
3.	$\neg R(v)$	assumption
4.	$R(v)$	$\forall$ - elim 1 (v free for x in $R(x)$)
5.	$\perp$	$\neg$ - elim 4,3
6.	$\perp$	$\exists$ - elim (2) 3-5
7.	$\neg \exists y \cdot \neg R(y)$	$\neg$ - intro 2-6

(ii) $\neg\exists x \cdot R(x) \vdash \forall y \cdot \neg R(y)$

1.	$\neg \exists x \cdot R(x)$	premiss
	v fresh	
2.	$R(v)$	assumption
3.	$\exists x \cdot R(x)$	(v free of x in $R(x)$)
4.	$\perp$	$\neg$ - elim 3,1
5.	$\neg R(v)$	$\neg$ - intro 2-4
6.	$\forall y \cdot \neg R(y)$	$\forall$ - intro 2-5

(iii) $\neg \exists x \cdot \neg R(x) \vdash \forall y \cdot R(y)$

1.	$\neg \exists x \cdot \neg R(x)$	premiss
2.	$\forall y \cdot \neg\neg R(y)$	theorem (b) (ii)
3.	$\forall y \cdot R(y)$	$\neg\neg$ - elim 2

(c) $\forall x \cdot \forall y \cdot P(x,y) \vdash \forall y \cdot \forall x \cdot P(x,y)$

1.	$\forall x \cdot \forall y \cdot P(x,y)$	premiss
	fresh v	
	fresh w	
2.	$\forall y \cdot P(w,y)$	$\forall$ - elim 1 (w free for x in $P(x,y)$)
3.	$P(w,v)$	$\forall$ - elim 2 (v free for y in $P(w,y)$)
4.	$\forall x \cdot P(x,v)$	$\forall$ - intro 2-3
5.	$\forall y \cdot \forall x \cdot P(x,y)$	$\forall$ - intro 2-4

(d) $\forall x. \forall y. (P(x,y) \to Q(x,y)) \vdash (\exists y. \exists x. \neg Q(x,y)) \to (\exists y. \exists x. \neg P(x,y))$

1.	$\forall x. \forall y. (P(x,y) \to Q(x,y))$	premiss
2.	$\exists y. \exists x. \neg Q(x,y)$	assumption
	v fresh	
3.	$\exists x. \neg Q(x, v)$	assumption
	w fresh	
4.	$\neg Q(w, v)$	assumption
5.	$\forall y. (P(w,y) \to Q(w,y))$	$\forall$-elim 1 (w is free for x)
6.	$P(w, v) \to Q(w, v)$	$\forall$-elim 2 (v is free for y)
7.	$\neg Q(w,v) \to \neg P(w,v)$	MT 6.
8.	$\neg P(w,v)$	$\to$-elim 4,7
9.	$\exists x. \neg P(x, v)$	$\exists$-intro 8 (w is free for x)
10.	$\exists y. \exists x. \neg P(x,y)$	$\exists$-intro 9 (v is free for y)
11.	$\exists y. \exists x. \neg P(x,y)$	$\exists$-elim (3) 4–10
12.	$\exists y. \exists x. \neg P(x,y)$	$\exists$-elim (2) 3–11
13.	$(\exists y. \exists x. \neg Q(x,y)) \to (\exists y. \exists x. \neg P(x,y))$	$\to$-intro 2–12