Linear Algebra — Week 5

1. Calculate the determinants of the following matrices:

   (a)
   $$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

   (b)
   $$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ 4 & 5 & 1 \end{bmatrix}$$

   (c)
   $$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 6 & -1 \\ 4 & 5 & 1 \end{bmatrix}$$

   (d)
   $$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{bmatrix}$$

   (e)
   $$\begin{bmatrix} 2 & 1 & -2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 2 & 1 & -3 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

   (f)
   $$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. A square matrix Q satisfies $Q^\top Q = \mathcal{I}$, where $\mathcal{I}$ is the identity matrix. Prove that $\det(Q) = \pm 1$.

3. Find the determinant of
   $$A = \begin{bmatrix} 4-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}.$$
   For which values of λ is A singular?

4. If every row of A sums to zero, prove that $\det(A) = 0$. If every row of A sums to 1, show that $\det(A - I) = 0$. Show, by example, that this does not imply $\det(A) = 1$.

5. We are given two square matrices of real numbers that satisfy $AB = -BA$. By using properties of determinants determine when at least one of A and B must be singular.

6. A permutation is a matrix that has all entries zero except for: (i) eactly one entry with the value 1 in each row; and (ii) exactly one entry with the value 1 in each column. In lectures you saw that a non-singular matrix A could be factorised as

$$PA = LU,$$

where P is a permutation matrix, L is a lower-triangular matrix, and U is an upper triangular matrix. Explain how this factorisation may be used to calculate the determinant of A.

7. Use determinants to calculate the area of the triangle with nodes at $(1, 2)$, $(2, 3)$ and $(5, 5)$.

8. What is the equation of the plane that passes through the points $(0, 0, 0)$, $(1, 1, 1)$ and $(2, 4, 6)$?

9. Calculate $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$

10. Use LU decomposition to solve the linear system given by

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

11. Use PLU decomposition to solve the linear system given by

$$\begin{pmatrix} 2 & 1 & 4 \\ 4 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 10 \end{pmatrix}.$$