INTRODUCTION TO FORMAL PROOF

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TUTORIAL SHEET 2

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QUESTION 1

(a) $C = \{ \text{Mary} \}$

$P = \{ P(\cdot), S(\cdot), L(\cdot), \cdot \text{ attended } \cdot, \cdot \text{ admires } \cdot \}$

i. Mary admires every professor.

$\forall x \cdot P(x) \rightarrow \text{Mary admires } x$

ii. Some professors admire Mary.

$\exists x \cdot P(x) \wedge x \text{ admires Mary}$

iii. Mary admires herself.

Mary admires Mary

iv. No student attended every lecture.

$\forall x \cdot S(x) \rightarrow (\exists y \cdot L(y) \wedge \neg (x \text{ attended } y))$

v. No lecture was attended by every student.

$\forall y \cdot L(y) \rightarrow (\exists x \cdot S(x) \wedge \neg (x \text{ attended } y))$

vi. No lecture was attended by any student.

$\forall y \cdot \forall x \cdot S(x) \wedge L(y) \rightarrow \neg (x \text{ attended } y)$

(b) $\cdot = \cdot \in P$ means equality

i. Mary admires one professor.

$\exists x \cdot P(x) \wedge \text{Mary admires } x$

ii. Mary doesn't admire more than one professor.

$\neg (\exists x \cdot \exists y \cdot P(x) \wedge P(y) \wedge \neg (x = y) \wedge (\text{Mary admires } x) \wedge (\text{Mary admires } y))$ ← can't admire two or more professors

iii. There is a student who is also a professor.

$\exists x \cdot S(x) \wedge P(x)$

iv. There is just one student who is a professor.

$$\underbrace{(\exists x \cdot S(x) \wedge P(x))}_{\text{there exists one}} \wedge \underbrace{\neg (\exists x \cdot \exists y \cdot (\neg x = y) \wedge (S(x) \wedge P(x)) \wedge (S(y) \wedge P(y)))}_{\text{don't exist two (or more) different ones}}$$

OR

$\exists x \cdot (S(x) \wedge P(x)) \wedge (\forall y \cdot (S(y) \wedge P(y)) \rightarrow (x = y)))$

↑ there exists a student who is also a professor and all other students who are also professors are equal to him

(c) The model is built around all "things". We have the following signature:

$C = \{ box \}$

$\mathcal{F} = \{\}$

$P = \{$ Red(·) – the thing is red,

Box (·) – the thing is in the box,

Blue (·) – the thing is blue,

Shed(·) – the thing is in the shed,

Green(·) – the thing is green,

Mam (·) – the thing is a mammal,

Horse (·) – the thing is a horse,

Prim (·) – the thing is a primate,

· won ·,

Girl(·) – the thing is a girl,

Prize (·) – the thing is a prize,

U(·) – the thing is in the universe,

$\cdot = \cdot \}$

i. All the red things are in the box.

$\forall x \cdot \ Red(x) \rightarrow Box(x)$

ii. Only the red things are in the box.

$\forall x \cdot \ Red(x) \longleftrightarrow Box(x)$

iii. Only red things are in the box.

$\forall x \cdot \ Red(x) \longleftarrow Box(x)$

iv. The box is blue.

$Blue(box)$

v. There's a green thing in the box, and everything else is in the shed.

$\underbrace{(\exists x \cdot \ Green(x) \land Box(x))}_{\text{the exists at least one}} \land \underbrace{\neg(\exists x \cdot \exists y \cdot \neg(x=y) \land (Green(x) \land Box(x)) \land (Green(y) \land Box(y))}_{\text{there don't exist 2 or more}} \land$

$\land \underbrace{(\forall x \cdot \neg(Green(x) \land Box(x)) \longrightarrow Shed(x))}_{\text{everything else is in the shed}}$

vi. No mammal is both a horse and a primate.

$\neg(\exists x \cdot \ Mam(x) \land Horse(x) \land Prim(x))$

vii. The prizes were all won by girls.

$\forall x \cdot \ Prize(x) \rightarrow (\exists y \cdot Girl(y) \land y \text{ won } x)$

viii. A girl won all the prizes

$\underbrace{(\exists x \cdot \ Girl(x) \land (\forall y \cdot Prize(y) \rightarrow x \text{ won } y))}_{\text{there is a girl who won all prizes}} \land \underbrace{\neg(\exists x \cdot \exists y \cdot \exists z \cdot Girl(x) \land Girl(y) \land Prize(z) \land (x \text{ won } z) \land (y \text{ won } z) \land \neg(x=y))}_{\text{no prize can be won by 2 different girls}}$

ix. There is nothing in the universe.

$\forall x \cdot \ \neg Univ(x)$

x. There are at least two things in the universe.

$\exists x \cdot \exists y \cdot \ Univ(x) \land Univ(y) \land \neg(x=y)$

xi. There are exactly two things in the universe.

$\underbrace{(\exists x \cdot \exists y \cdot \neg(x=y) \land Univ(x) \land Univ(y))}_{\text{there are at least two different things in the universe}} \land$

$\underbrace{\neg(\exists x \cdot \exists y \cdot \exists z \cdot \neg(x=y) \land \neg(y=z) \land \neg(z=x) \land \land Univ(x) \land Univ(y) \land Univ(z))}_{\text{there can't be 3 or more different things in the universe}}$

QUESTION 2

(a)

i. $R(j),\ \forall x \cdot (R(x) \to S(x)) \vdash S(j)$

1.	$R(j)$	premiss
2.	$\forall x \cdot (R(x) \to S(x))$	premiss
3.	$R(j) \to S(j)$	$\forall$-e 2 (j free for x)
4.	$S(j)$	$\to$-e 1,3

ii. $\forall x \cdot (R(x) \to S(x)) \vdash (\forall y \cdot R(y)) \to (\forall z \cdot S(z))$

1.	$\forall x \cdot (R(x) \to S(x))$	premiss
2.	$\forall y \cdot R(y)$	assumption
	fresh v	
3.	$R(v)$	$\forall$-e 2
4.	$R(v) \to S(v)$	$\forall$-e 1
5.	$S(v)$	$\to$-e 3,4
6.	$\forall z \cdot S(z)$	$\forall$-i 3-5
7.	$(\forall y \cdot R(y)) \to (\forall z \cdot S(z))$	$\to$-i 2-6

iii. $\forall x \cdot (R(x) \to S(x)),\ \forall y \cdot (S(y) \to T(y)) \vdash \forall z \cdot (R(z) \to T(z))$

1.	$\forall x \cdot (R(x) \to S(x))$	premiss
2.	$\forall y \cdot (S(y) \to T(y))$	premiss
	fresh v	
3.	$R(v) \to S(v)$	$\forall$-e 1
4.	$S(v) \to T(v)$	$\forall$-e 2
5.	$R(v)$	assumption
6.	$S(v)$	$\to$-e 5,3
7.	$T(v)$	$\to$-e 6,4
8.	$R(v) \to T(v)$	$\to$-i 5-7
9.	$\forall z \cdot (R(z) \to T(z))$	$\forall$-i 3-8

iv. $(\forall x \cdot R(x)) \wedge (\forall y \cdot S(y)) \vdash \forall z \cdot (R(z) \wedge S(z))$

1.	$(\forall x \cdot R(x)) \wedge (\forall y \cdot S(y))$	premiss
2.	$\forall x \cdot R(x)$	$\wedge$-e_L 1
3.	$\forall y \cdot S(y)$	$\wedge$-e_R 1
	fresh v	
4.	$R(v)$	$\forall$-e 2
5.	$S(v)$	$\forall$-e 3
6.	$R(v) \wedge S(v)$	$\wedge$-i 4,5

7. $\forall z \cdot (R(z) \wedge S(z))$ $\quad$ $\forall$-i 4-6

V. $\forall x \cdot P(a,x,x)$, $\forall x \cdot \forall y \cdot \forall z \cdot (P(x,y,z) \rightarrow P(f(x), y, f(z))) \vdash P(f(a), a, f(a))$

1. $\forall x \cdot P(a,x,x)$ $\quad$ premiss
2. $\forall x \cdot \forall y \cdot \forall z \cdot (P(x,y,z) \rightarrow P(f(x), y, f(z)))$ $\quad$ premiss
3. $\forall y \cdot \forall z \cdot (P(a,y,z) \rightarrow P(f(a), y, f(z)))$ $\quad$ $\forall$-e 2 (a is free for x)
4. $\forall z \cdot (P(a,a,z) \rightarrow P(f(a), a, f(z)))$ $\quad$ $\forall$-e 3 (a is free for y)
5. $P(a,a,a) \rightarrow P(f(a), a, f(a))$ $\quad$ $\forall$-e 4 (a is free for z)
6. $P(a,a,a)$ $\quad$ $\forall$-e 1 (a is free for x)
7. $P(f(a), a, f(a))$ $\quad$ $\rightarrow$-e 6,5

VI. $\forall x \cdot P(a,x,x)$, $\forall x \cdot \forall y \cdot \forall z \cdot (P(x,y,z) \rightarrow P(f(x), y, f(z))) \vdash \exists z \cdot (P(f(a), z, f(a)))$

1. $\forall x \cdot P(a,x,x)$ $\quad$ premiss
2. $\forall x \cdot \forall y \cdot \forall z \cdot (P(x,y,z) \rightarrow P(f(x), y, f(z)))$ $\quad$ premiss
3. $P(f(a), a, f(a))$ $\quad$ theorem from previous task
4. $\exists z \cdot (P(f(a), z, f(a)))$ $\quad$ $\exists$-i 3 (a is free for z)

(b)

i. $\exists x \cdot R(x) \vdash \neg \forall y \cdot \neg R(y)$

1. $\exists x \cdot R(x)$ $\quad$ premiss

fresh $\surd$

2. $R(\surd)$ $\quad$ assumption
3. $\forall y \cdot R(y)$ $\quad$ $\forall$-i 2
4. $\forall y \cdot \neg R(y)$ $\quad$ assumption
5. $\neg R(T)$ $\quad$ $\forall$-e 4 (T is free for y)
6. $R(T)$ $\quad$ $\forall$-e 3
7. $\perp$ $\quad$ $\neg$-e 6,5
8. $\neg \forall y \cdot \neg R(y)$ $\quad$ $\neg$-i 4-7
9. $\neg \forall y \cdot \neg R(y)$ $\quad$ $\exists$-e (1) 2-8

ii. $\neg \exists x \cdot \neg R(x) \vdash \forall y \cdot R(y)$

1. $\neg \exists x \cdot \neg R(x)$ $\quad$ premiss

fresh $\surd$

2. $\neg R(\surd)$ $\quad$ assumption
3. $\exists x \cdot \neg R(x)$ $\quad$ $\exists$-i 2
4. $\perp$ $\quad$ $\neg$-e 3,1
5. $\neg\neg R(\surd)$ $\quad$ $\neg$-i 2-4
6. $R(\surd)$ $\quad$ $\neg\neg$-e 5
7. $\forall y \cdot R(y)$ $\quad$ $\forall$-i 2-6

iii. $\neg \forall x \cdot R(x) \vdash \exists y \cdot \neg R(y)$

1.	$\neg \forall x \cdot R(x)$	premiss
2.	$\neg \exists y \cdot \neg R(y)$	assumption
3.	$\forall x \cdot R(x)$	theorem from (b). ii.
4.	$\bot$	$\neg$-e 3, 1
5.	$\neg\neg \exists y \cdot \neg R(y)$	$\neg$-i 2–4
6.	$\exists y \cdot \neg R(y)$	$\neg\neg$-e 5

(c) "If there are any tax payers, then all politicians are tax payers."

$(\exists x \cdot T(x)) \rightarrow (\forall y \cdot P(y) \rightarrow T(y))$

$T(x)$ - tax payer
$P(x)$ - politician
$F(x)$ - philanthropist

"If there are any philanthropists, then all taxpayers are philanthropists"

$(\exists x \cdot F(x)) \rightarrow (\forall y \cdot T(y) \rightarrow F(y))$

"So if there are any tax-paying philanthropists, then all politicians are philanthropists."

$(\exists x \cdot T(x) \land F(x)) \rightarrow (\forall y \cdot P(y) \rightarrow F(y))$

So, we want to prove that:

$$(\exists x \cdot T(x)) \rightarrow (\forall y \cdot P(y) \rightarrow T(y)),\ (\exists x \cdot F(x)) \rightarrow (\forall y \cdot T(y) \rightarrow F(y))$$
$$\vdash (\exists x \cdot T(x) \land F(x)) \rightarrow (\forall y \cdot P(y) \rightarrow F(y))$$

1.	$(\exists x \cdot T(x)) \rightarrow (\forall y \cdot P(y) \rightarrow T(y))$	premiss
2.	$(\exists x \cdot F(x)) \rightarrow (\forall y \cdot T(y) \rightarrow F(y))$	premiss
3.	$\exists x \cdot T(x) \land F(x)$	assumption
	fresh $\sqrt{}$	
4.	$T(\sqrt{}) \land F(\sqrt{})$	assumption
5.	$T(\sqrt{})$	$\land$-e_L 4
6.	$\exists x \cdot T(x)$	$\exists$-i 5
7.	$F(\sqrt{})$	$\land$-e_R
8.	$\exists x \cdot F(x)$	$\exists$-i 7
9.	$(\exists x \cdot T(x)) \cdot (\exists x \cdot F(x))$	$\land$-i 6, 8
10.	$(\exists x \cdot T(x)) \land (\exists x \cdot F(x))$	$\exists$-e(3) 4-9
11.	$\exists x \cdot T(x)$	$\land$-e_L 10
12.	$\exists x \cdot F(x)$	$\land$-e_R 10
13.	$\forall y \cdot P(y) \rightarrow T(y)$	$\rightarrow$-e 13,1
14.	$\forall y \cdot T(y) \rightarrow F(y)$	$\rightarrow$-e 14,2
	fresh w	
15.	$P(w) \rightarrow T(w)$	$\forall$-e 13
16.	$T(w) \rightarrow F(w)$	$\forall$-e 14

17.	$P(w)$	assumption
18.	$T(w)$	$\rightarrow$-e 17,15
19.	$F(w)$	$\rightarrow$-e 18,16
20.	$P(w) \rightarrow F(w)$	$\rightarrow$-i 17-19
21.	$\forall y \cdot P(y) \rightarrow F(y)$	$\forall$-i 15-20
22.	$(\exists x \cdot T(x) \wedge F(x)) \rightarrow (\forall y \cdot P(y) \rightarrow F(y))$	$\rightarrow$-i 3-21

QUESTION 3

Let us consider the model as the set of integers which are greater than 1 and negative. This implies that our model has no values in it.

We have:

$C = \{\}$

$\mathcal{F} = \{ succ(\cdot), pred(\cdot) \}$

$P = \{ \cdot = \cdot, Even(\cdot), \cdot > \cdot \}$

We'll first prove that $\forall x \cdot P(x) \vdash \exists x \cdot P(x)$

1. $\forall x \cdot P(x)$ premiss
2. $P(T)$ $\forall$-e 1 (T free for x)
3. $\exists x \cdot P(x)$ $\exists$-i 2

If we apply this to our model, we obtain from the fact that $\forall x \cdot Even(x)$, which is correct, since every number in the model is even (the model is empty, therefore this proposition is true), but that would imply that $\exists x \cdot Even(x)$, which means that there must exist an even number in our model, which can't be true since there is no number in the model.

Therefore, the rules are not sound when applied to empty domains.

Q2 (a)

vi. $\forall x \cdot P(a,x,x), \forall x \cdot \forall y \cdot \forall z \cdot (P(x,y,z) \rightarrow P(f(x),y,f(z))) \vdash \exists z \cdot (P(f(a), z, f(f(a)))$

1.	$\forall x \cdot P(a,x,x)$	premiss
2.	$\forall x \cdot \forall y \cdot \forall z \cdot (P(x,y,z) \rightarrow P(f(x),y,f(z)))$	premiss
3.	$P(a,f(a),f(a))$	$\forall$-e 1 ($f(a)$ is free for x)
4.	$\forall y \cdot \forall z \cdot (P(a,y,z) \rightarrow P(f(a),y,f(z)))$	$\forall$-e 2 (a is free for x)
5.	$\forall z \cdot (P(a, f(a), z) \rightarrow P(f(a),f(a),f(z)))$	$\forall$-e 4 ($f(a)$ is free for y)
6.	$P(a, f(a), f(a)) \rightarrow P(f(a), f(a), f(f(a)))$	$\forall$-e 5 ($f(a)$ is free for z)
7.	$P(f(a), f(a), f(f(a)))$	$\rightarrow$-e 3,6
8.	$\exists z \cdot P(f(a), z, f(f(a)))$	$\exists$-i 7 ($f(a)$ is free for z)