Continuous Maths HT 2019: Problem Sheet 0

Basic differentiation: reminders and exercises

The course assumes that you will be proficient at differentiating powers, exponentials, logarithms, trigonometric functions, compositions, products, and quotients. If you have not seen these before, or are feeling rusty, there are many online resources where you can learn or remind yourself. Or for a very gentle (and cheap) introduction, try the book

Hugh Neill. Calculus: A Complete Introduction. Teach Yourself, 2013.

Chapters 5, 6, 8, & 9 are relevant.

In this sheet, the derivative of a function $y(x)$ is notated by $\frac{\mathrm{d}y}{\mathrm{d}x}$; in some books it is written $y'(x)$. We use n for a positive integer constant, and a and b for arbitrary constants. Note that $\exp(x)$ is another way to write e^x.

Do not forget to look for simplifcations, both before and after you find the derivatives.

0.1 Write down $\frac{\mathrm{d}y}{\mathrm{d}x}$ where

(i) $y = 1$, (ii) $y = 2x^2 + 3x^3$, (iii) $y = \frac{1}{x} - \frac{3}{x^2} + \sin x + e^x$,

(iv) $y = \ln x$, (v) $y = \log_2 x$, (vi) $y = \sum_{i=1}^{n} \frac{x^i}{i}$.

0.2 The *chain rule* could be written

$$\frac{\mathrm{d}(g \circ f)}{\mathrm{d}x} = \left(\frac{\mathrm{d}g}{\mathrm{d}x} \circ f\right) \frac{\mathrm{d}f}{\mathrm{d}x}.$$

For each of the following functions, identify the functions f and g such that $y = g \circ f$, and compute $\frac{\mathrm{d}y}{\mathrm{d}x}$:

(i) $y = e^{\sin x}$, (ii) $y = \ln(1 - x^a)$, (iii) $y = \sin \cos x$.

Now use the chain rule to compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ where

(iv) $y = \sqrt{\cos x^2}$, (v) $y = \exp\big(\exp(x^{-1})\big)$, (vi) $y = \sin^2\left(\frac{e^{-x}}{1+e^{-x}}\right) + \cos^2\left(\frac{1}{1+e^{x}}\right)$.

0.3 The *product rule* could be written

$$\frac{\mathrm{d}(fg)}{\mathrm{d}x} = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}.$$

Use it to compute $\frac{\mathrm{d}x}{\mathrm{d}t}$ where

(i) $x = \sin t \cos t$, (ii) $x = (1 - 2t + 3t^2)e^{4t}$, (iii) $x = t(\sin t)e^t$.

0.4 Applying the product rule to $f(x)$ and $\frac{1}{g(x)}$, and also using the chain rule, derive the *quotient rule*:

$$\frac{\mathrm{d}(f/g)}{\mathrm{d}x} = \frac{g\frac{\mathrm{d}f}{\mathrm{d}x} - f\frac{\mathrm{d}g}{\mathrm{d}x}}{g^2}.$$

Use it to compute $\frac{\mathrm{d}l}{\mathrm{d}z}$ where

(i) $l = \dfrac{z+1}{z-1}$, (ii) $l = \dfrac{\sin \pi z}{\pi z}$, (iii) $l = \dfrac{e^z}{\ln z}$.

0.5 Using the chain rule, product rule, and quotient rule as appropriate, compute $\frac{\mathrm{d}f}{\mathrm{d}x}$ where

(i) $f = \exp(x \ln x)$, (ii) $f = e^{\sin^2 \ln x} e^{\cos^2 \ln x}$, (iii) $f = (x^4 - 1)^3 (x^3 + 1)^4$,

(iv) $f = \log\left(\dfrac{1+x^2}{1-x^2}\right)$, (v) $f = \displaystyle\sum_{i=1}^{n} \ln\bigl(1 + e^{ix}\bigr)$, (vi) $f = -x \log_2 x - (1-x) \log_2(1-x)$.

0.6 Assume that y is a constant. Compute $\frac{\mathrm{d}f}{\mathrm{d}x}$ where

(i) $f = x^2 + y^2 + 2xy$, (ii) $f = 2x \cos y + 3y \sin x$, (iii) $f = \sqrt{4x + 4y}$,

(iv) $f = \dfrac{x-y}{x+y}$, (v) $f = \dfrac{e^{xy}}{xy}$, (vi) $f = \displaystyle\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.

Now differentiate the each of these functions again, but assuming that y is the variable and x is a constant.
These are called *partial derivatives*, and are written $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively.

0.7 Compute $\frac{\partial f}{\partial x_j}$ where

(i) $f = \sum_{i=1}^{n} x_i^2$, (ii) $f = \left(\sum_{i=1}^{n} x_i\right)^2$, (iii) $f = \dfrac{\sum_{i=1}^{n} x_i}{\sqrt{\sum_{i=1}^{n} x_i^2}}$.

According to the last exercise, to do this you should act as if all the $x_1, \ldots, x_n$, except for x_j, are constants. Then differentiate with respect to x_j.

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