Linear Algebra MT18 - Week 2

Chapter 2 (Independence and Orthogonality)

1. Express $\mathbf{u} \in \mathbb{R}^n$ as a linear combination of vectors $\mathbf{v}_i \in \mathbb{R}^n$, $i = 1, \ldots, n$:

   (a) $n = 3$ and

   $$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

   (b) $n = 3$

   $$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

   (c) $n = 4$

   $$\mathbf{u} = \begin{bmatrix} 9 \\ 5 \\ -2 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

2. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m)$ be a set of vectors in vector space V.

   (a) Define $span(S)$.

   (b) Prove that $span(S)$ is a subspace of V.

3. (optional) Suppose $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are in vector space V and $S = \{\mathbf{v} - \mathbf{u}, \mathbf{w} - \mathbf{v}, \mathbf{u} - \mathbf{w}\}$. Investigate wether S is linearly dependent or independent.

4. Let $\mathbf{u}, \mathbf{v}$ are vectors in a vector space V and suppose that $S = \{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Prove that $S' = \{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ is linearly independent.

5. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ is a basis for a vector space V, define $S' = \{c\mathbf{v}_1, c\mathbf{v}_2, \ldots, c\mathbf{v}_n)$, where c is a nonzero scalar. Prove that S' is also a basis for V.

6. (optional) Suppose $\mathbf{u}$ and $\mathbf{s}$ are vectors in $\mathbb{R}^n$. Find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{s}$ if

   (a) $n = 2$, $\mathbf{u} = [1, 2]^T$ and $\mathbf{s} = [2, 1]^T$,

   (b) $n = 3$, $\mathbf{u} = [1, 3, -2]^T$ and $\mathbf{s} = [0, -1, 1]^T$.

7. Assume that $\mathbf{u}$ and $\mathbf{s}$ are vectors in $\mathbb{R}^n$ and that $\mathbf{p_s}(\mathbf{u})$ denotes the orthogonal projection of $\mathbf{u}$ onto $\mathbf{s}$. Prove that $\mathbf{u} - \mathbf{p_s}(\mathbf{u})$ is orthogonal to $\mathbf{s}$.

Applications

of the *classical Gram-Schmidt algorithm*

1. Suppose that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent vectors in $\mathbb{R}^3$. Describe the steps of the *classical Gram-Schmidt algorithm* that takes vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ as input and produces the orthonormal set $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ in $\mathbb{R}^3$.

2. Apply the algorithmic steps described above for input vectors

$$\mathbf{u}_1 = [1, 0, 0]^T$$

$$\mathbf{u}_2 = [1, 1, 1]^T$$

$$\mathbf{u}_3 = [1, 1 - 1]^T$$

   to generate the the orthonormal set $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. Verify the correctness of your calculations by evaluating $\mathbf{q}_1^T\mathbf{q}_2$, $\mathbf{q}_1^T\mathbf{q}_3$ and $\mathbf{q}_2^T\mathbf{q}_3$.

3. (optional) Use Gram-Schmidt to form an orthonormal basis for the subspace $W = span\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

4. (optional) Using Gram-Schmidt, find an orthogonal basis for $\mathbb{R}^3$ that contains the vector $\mathbf{v}_1 = [1, 2, 3]^T$.