LINEAR ALGEBRA

MT 18

WEEK 3

GABRIEL MOISE

Chapter 3: Matrices

1. $A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 2 \end{bmatrix}$

- $A^3 = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ -8 & 25 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix}$
- $A^T A = \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ -5 & 25 \end{bmatrix}$
- $B - C^T = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -5 & -4 \\ -2 & -2 & -3 \end{bmatrix}$
- $DB = \begin{bmatrix} 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 16 & -4 & 10 \end{bmatrix}$
- $B^T A = \begin{bmatrix} 4 & 0 \\ -2 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ -8 & 10 \\ 0 & 15 \end{bmatrix}$
- $BA = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}$. As $B \in \mathbb{R}^{2\times 3}$ and $A \in \mathbb{R}^{2\times 2}$, $B\cdot A$ cannot be defined!
- $CD^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 32 \end{bmatrix}$

2. $A, B \in \mathbb{R}^{n\times n}$ are upper triangular $\Rightarrow$ $A_{ij} = \begin{cases} a_{ij} & \text{for } i \le j \\ 0 & \text{for } i > j \end{cases}$ and $B_{ij} = \begin{cases} b_{ij} & \text{for } i \le j \\ 0 & \text{for } i > j \end{cases}$

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = c_{ij}$$

Let's take two cases:

I $i \le j \Rightarrow c_{ij} = \sum\limits_{k=1}^{n} a_{ik} b_{kj}$ OK

II $i > j \Rightarrow c_{ij} = \sum\limits_{k=1}^{n} a_{ik} b_{kj} = \sum\limits_{k=1}^{i-1} a_{ik} b_{kj} + \sum\limits_{k=i}^{n} a_{ik} b_{kj} = \sum\limits_{k=1}^{i-1} 0 \cdot b_{kj} + \sum\limits_{k=i}^{n} a_{ik} \cdot 0 = 0 \Rightarrow$

$\Rightarrow (AB)_{ij} = \begin{cases} c_{ij} & \text{for } i \le j \\ 0 & \text{for } i > j \end{cases} \Rightarrow AB$ is also upper triangular.

3. (a) $A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

$[A|i_2] = \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right]$

$E_1[A|i_2] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right]$ REF

$E_2(E_1[A|i_2]) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 0 & -2 & -1 & 1 \end{array}\right]$

$E_3(E_2E_1[A|i_2]) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{cc|cc} 2 & 0 & 0 & 1 \\ 0 & -2 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -2 & -1 & 1 \end{array}\right]$

$E_4(E_3E_2E_1[A|i_2]) = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \cdot \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & -2 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$ RREF

So, $A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix} \in \mathbb{R}^{3\times3}$

$[B|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right]$

$E_1 \cdot [B|I_3] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right]$

$E_2 \cdot (E_1 \cdot [B|I_3]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -3 & 7 & -1 & 0 & 1 \end{array}\right]$

$E_3 \cdot (E_2 \cdot E_1 \cdot [B|I_3]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -3 & 7 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right]$ REF

$E_4 \cdot (E_3 \cdot E_2 \cdot E_1 \cdot [B|I_3]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right]$

$E_5 \cdot (E_4 E_3 E_2 E_1 [B|I_3]) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 22 & -9 & -3 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right]$

$E_6 \cdot (E_5 E_4 E_3 E_2 E_1 [B|I_3]) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 22 & -9 & -3 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & -23 & -7 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right]$ RREF

So, $B^{-1} = \begin{bmatrix} 54 & -23 & -7 \\ -16 & 7 & 2 \\ -7 & 3 & 1 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix} \in \mathbb{R}^{3\times3}$

$[C|I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$

$E_1 \cdot [C|I_3] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$

$E_2 \cdot (E_1 [C|I_3]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 5 & -5 & -2 & 0 & 1 \end{array}\right]$

$E_3 \cdot (E_2 \cdot E_1 [C|I_3]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{5}{4} & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 5 & -5 & -2 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -4 & -3 & 1 & 0 \\ 0 & 0 & 0 & \frac{7}{4} & -\frac{5}{4} & 1 \end{array}\right]$

a zero row, so C is not invertible

(d) $D = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix} \in \mathbb{R}^{3\times3}$

$[D|I_3] = \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right]$

$E_1 \cdot [D|I_3] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{a} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 1 & a & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & -\frac{1}{a} & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right]$

$E_2 \cdot (E_1 \cdot [D|I_3]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{a} & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & -\frac{1}{a} & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & -\frac{1}{a} & 1 & 0 \\ 0 & 0 & a & \frac{1}{a^2} & -\frac{1}{a} & 1 \end{array}\right]$

$E_5 \cdot E_4 \cdot E_3 (E_2 \cdot E_1 \cdot [D|I_3]) = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a} \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & -\frac{1}{a} & 1 & 0 \\ 0 & 0 & a & \frac{1}{a^2} & -\frac{1}{a} & 1 \end{array}\right] =$

$$= \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{a} \end{bmatrix} \cdot \left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 0 & -\frac{1}{a} & 1 & 0 \\ 0 & 0 & a & \frac{1}{a^2} & -\frac{1}{a} & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{a^2} & \frac{1}{a} & 0 \\ 0 & 0 & 1 & \frac{1}{a^3} & -\frac{1}{a^2} & \frac{1}{a} \end{array}\right]$$

So, if $a \neq 0$ D is invertible and $D^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{1}{a^2} & \frac{1}{a} & 0 \\ \frac{1}{a^3} & -\frac{1}{a^2} & 0 \end{bmatrix}$ and if $a = 0$, D is not invertible.

4. $A \in \mathbb{R}^{m \times m}$

We want to show $C(A)$ is a subspace of $\mathbb{R}^m$. This happens if $C(A)$ is closed under addition and scalar multiplication.

Let $A = \left[x_1 \,|\, x_2 \,|\, \dots \,|\, x_m \right]$ with $x_1, x_2, \dots, x_m$ the column vectors of A. Also, let $u, v \in C(A)$. Therefore, $\left.\begin{array}{l} u = c_1 x_1 + c_2 x_2 + \dots + c_m x_m \\ v = d_1 x_1 + d_2 x_2 + \dots + d_m x_m \end{array}\right| \overset{(+)}{\Rightarrow} (u+v) = (c_1+d_1) x_1 + (c_2+d_2) x_2 + \dots + (c_n+d_n) x_m \Rightarrow$

$\Rightarrow (u+v) \in C(A)$ because we wrote $(u+v)$ as a linear combination of the column vectors of A.

Also, let $\alpha \in \mathbb{R}$ a scalar, $\alpha u = (\alpha c_1) x_1 + (\alpha c_2) x_2 + \dots + (\alpha c_m) x_m \Rightarrow \alpha u \in C(A)$. Therefore $C(A)$ is a subspace of $\mathbb{R}^m$.

Now, let $A \in \mathbb{R}^{m \times m}$ invertible $\Rightarrow \{x_1, x_2, \dots, x_m\}$ are linearly independent. Let's also take a vector $v \in \mathbb{R}^m$, which we suppose that $v \notin \operatorname{span}\{x_1, x_2, \dots, x_m\} = C(A)$. Then $\{x_1, x_2, \dots, x_m, v\}$ is linearly independent. But this set would span an $(m+1)$-dimensional space, which is false as we are talking about vectors from $\mathbb{R}^m$. Therefore $v \in C(A) \Rightarrow C(A) \supseteq \mathbb{R}^m$. As all the vectors from $C(A)$ are m-dimensional, $C(A) \subseteq \mathbb{R}^m$. In conclusion $C(A) = \mathbb{R}^m$.

5. $\mathcal{N}(A) = \{ v \in \mathbb{R}^n \mid Ax = 0 \}$

Let $u, v \in \mathcal{N}(A) \Rightarrow \left.\begin{array}{l} Au = 0 \\ Av = 0 \end{array}\right| \overset{(+)}{\Rightarrow} A(u+v) = 0 \Rightarrow (u+v) \in \mathcal{N}(A)$

Let $u \in \mathcal{N}(A), \alpha \in \mathbb{R} \Rightarrow Au = 0 \overset{(\cdot\alpha)}{\Rightarrow} \alpha A u = 0 \Rightarrow A(\alpha u) = 0 \Rightarrow (\alpha u) \in \mathcal{N}(A)$ $\Bigg| \Rightarrow \mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

Now, we have A invertible and we want to show that $\mathcal{N}(A) = 0$.

As $A = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_m \end{bmatrix}$ and $\{x_1, x_2, \dots, x_m\}$ linearly independent $\Rightarrow (\forall)\ v = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathcal{N}(A)$ is actually the zero vector as $c_1 x_1 + c_2 x_2 + \dots + c_n x_m = 0$ has only one solution, which is the trivial one. So, $\mathcal{N}(A) = 0$.

Applications

1.

$$P = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} \frac{5}{10} & \frac{4}{10} & \frac{6}{10} \\ \frac{2}{10} & \frac{2}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{4}{10} & \frac{1}{10} \end{bmatrix}$$

a) Each element a_{ij} denotes the probability that an elephant is going from reserve i to reserve j after a month

b) The vector containing the information about what probability there is for the herd of elephants to be in reserve R_1, R_2 or R_3 is $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$. (for the i-th month is $V_i = \begin{bmatrix} V_{1i} \\ V_{2i} \\ V_{3i} \end{bmatrix}$)

At the beginning we know $V_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

After 12 months, we will have $V_{12} = P^{12} V_0 = \frac{1}{10^{12}} \begin{bmatrix} 5 & 4 & 6 \\ 2 & 2 & 3 \\ 3 & 4 & 1 \end{bmatrix}^{12} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2.

a) $P = \begin{array}{c} \text{children} \\ \text{youths} \\ \text{adults} \end{array} \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$ - the Leslie matrix for the frog population

$V = \begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix}$

b) $V_0 = V$ (at the beginning)

$V_1 = PV_0 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 40 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 240 \\ 20 \\ 10 \end{bmatrix}$

$V_2 = PV_1 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 240 \\ 20 \\ 10 \end{bmatrix} = \begin{bmatrix} 110 \\ 120 \\ 5 \end{bmatrix}$

$V_3 = PV_2 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 110 \\ 120 \\ 5 \end{bmatrix} = \begin{bmatrix} 495 \\ 55 \\ 30 \end{bmatrix}$

$V_4 = PV_3 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 495 \\ 55 \\ 30 \end{bmatrix} = \begin{bmatrix} 310 \\ 247 \\ 13 \end{bmatrix}$

$V_5 = PV_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 310 \\ 247 \\ 13 \end{bmatrix} = \begin{bmatrix} 1027 \\ 155 \\ 61 \end{bmatrix}$

It looks like the frogs are going to survive (as the children, youths and adults are more in year 5 than in year 0).

c) $X_0 = \begin{bmatrix} 20 \\ 20 \\ 5 \end{bmatrix}$

$X_1 = PX_0 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 5 \end{bmatrix} = \begin{bmatrix} 95 \\ 10 \\ 5 \end{bmatrix}$

$X_2 = PX_1 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 95 \\ 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 55 \\ 47 \\ 2 \end{bmatrix}$

$X_3 = PX_2 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 55 \\ 47 \\ 2 \end{bmatrix} = \begin{bmatrix} 194 \\ 27 \\ 11 \end{bmatrix}$

$$x_4 = px_3 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} 194 \\ 27 \\ 11 \end{bmatrix} = \begin{bmatrix} 141 \\ 97 \\ 6 \end{bmatrix}$$

$$x_5 = px_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} 141 \\ 97 \\ 6 \end{bmatrix} = \begin{bmatrix} 406 \\ 70 \\ 24 \end{bmatrix}$$

Again, in year 5 it is a big change in population from year 0, so, because there are more children, youths and adults, it looks like the frogs will survive.

d) Now, the new Leslie matrix is:

$$p' = \begin{array}{c} \text{children} \\ \text{youths} \\ \text{adults} \end{array} \begin{bmatrix} 0 & 1 & 0.75 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

Let's take a random population of a children, b youths and c adults

$$v_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$v_1 = p'v_0 = \begin{bmatrix} 0 & 1 & 0.75 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b + \frac{3c}{4} \\ \frac{a}{2} \\ \frac{b}{4} \end{bmatrix}$$

$$v_2 = p'v_1 = \begin{bmatrix} 0 & 1 & 0.75 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} b + \frac{3c}{4} \\ \frac{a}{2} \\ \frac{b}{4} \end{bmatrix} = \begin{bmatrix} \frac{a}{2} + \frac{3b}{16} \\ \frac{b}{2} + \frac{3c}{8} \\ \frac{a}{8} \end{bmatrix}$$

We can see that in all cases, the population reduces, after two years from $(a+b+c)$ to $\frac{5a}{8} + \frac{11b}{16} + \frac{3c}{8}$. This way, in a finite number of years, all frog will eventually die.

3. L_1, L_2, L_3 — OFF (STATE 1), GREEN (STATE 2), RED (STATE 3)

S_1, S_2, S_3 switches $(1 \to 2 \to 3 \to 1)$

$S_1 \to L_1, L_2$

$S_2 \to L_1, L_2, L_3$

$S_3 \to L_2, L_3$

a) We need to work in a $\mathbb{Z}_m^n$ vector space. Because there are only 3 possible states, we'll work in $\mathbb{Z}_3$, but as we work with 3 lights, we need a tridimensional space, so we'll work on $\mathbb{Z}_3^3$, so $n = m = 3$.

b) We'll assign — STATE 1 to 0, STATE 2 to 1, STATE 3 to 2 — as they need to be in $\mathbb{Z}_3$.

The vectors corresponding to the switches are: $s_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $s_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $s_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

c) Initially, we have all 3 lights off, so $y = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and we want to know whether we can get to the y' configuration (after a finite number of switches), where $y' = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Let's suppose it can be done after a switches of type 1, b of type 2 and c of type 3. Therefore, $y' = as_1 + bs_2 + cs_3 + y$:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a+b \\ a+b+c \\ b+c \end{bmatrix} \text{ in } \mathbb{Z}_3^3 \Rightarrow \begin{cases} a+b=0 \\ a+b+c=1 \\ b+c=2 \end{cases} \left| \Rightarrow c = 1 \right. \left| \Rightarrow b = 1 \right. \quad a+b=0 \left| \Rightarrow a = 2 \right.$$

So, if we press switch 1 two times, switch 2 one time and switch 3 one time we get the y' configuration, starting from y.