GABRIEL MOISE

PROBABILITY

PROBLEM SHEET 2

1. $\Omega = \{(i,j) \mid 1 \le i,j \le 6\}$

$A = \{\text{first die shows } 3\}$

$B = \{\text{the second die shows an even number}\}$

$C = \{\text{the sum is even}\}$

(a) $A = \{(3,1),(3,2),\dots,(3,6)\} \Rightarrow \mathbb{P}(A) = \frac{1}{6}$

$B^c = \{\text{the second die shows an odd number}\}$ and $\mathbb{P}(B) = \mathbb{P}(B^c) \Rightarrow \mathbb{P}(B) = \frac{\mathbb{P}(\Omega)}{2} = \frac{1}{2}$

Let $D = \{\text{both dies are even}\} \Rightarrow \mathbb{P}(D) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

Let $E = \{\text{both dies are odd}\} \Rightarrow \mathbb{P}(E) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$\mathbb{P}(C) = \mathbb{P}(D \cup E) = \mathbb{P}(D) + \mathbb{P}(E) = \frac{1}{2}$

A and B are independent if $\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$

$A\cap B = \{\text{first die is 3 and the second is even}\} = \{(3,2),(3,4),(3,6)\} \Rightarrow \mathbb{P}(A\cap B) = \frac{1}{12}$ $\Bigg| \Rightarrow$

$\Rightarrow \frac{1}{12} = \frac{1}{6}\cdot\frac{1}{2}$ True

(b) B and C are independent if $\mathbb{P}(B\cap C) = \mathbb{P}(B)\mathbb{P}(C)$

$B\cap C = \{\text{second die is even and the sum is even}\} = \{\text{both dice are even}\} = D \Rightarrow$

$\Rightarrow \mathbb{P}(B\cap C) = \frac{1}{4} \Rightarrow \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2}$ True

(c) A and C are independent if $\mathbb{P}(A\cap C) = \mathbb{P}(A)\mathbb{P}(C)$

$A\cap C = \{\text{first die is 3 and the sum is even}\} = \{(3,1),(3,3),(3,5)\} = \frac{1}{12} \Rightarrow$

$\Rightarrow \frac{1}{12} = \frac{1}{6}\cdot\frac{1}{2}$ True

(d) A, B and C are independent if $\mathbb{P}(A\cap B\cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$

$A\cap B\cap C = \{\text{first die is 3 and the second is even and the sum is even}\} = \emptyset \Rightarrow$

$\Rightarrow \mathbb{P}(A\cap B\cap C) = 0$

But, $\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{24}$ $\Bigg| \Rightarrow$ A, B and C are not independent.

2. We will create the sets:

$A = \{\text{positive test}\}$ $\quad C = \{\text{healthy person}\}$

$B = \{\text{negative test}\}$ $\quad D = \{\text{unhealthy person}\}$

$\Bigg| \Rightarrow \mathbb{P}(A) + \mathbb{P}(B) = 1$, $\mathbb{P}(C) + \mathbb{P}(D) = 1$, $A\cap B = \emptyset$, $C\cap D = \emptyset$

We need to find the probability that a randomly tested person has the disease, given that their result is positive.

From the information we can gather from the hypothesis of the problem, we have:

$\mathbb{P}(A|D) = 0.95$

$\mathbb{P}(A|C) = 0.01$

$\mathbb{P}(D) = 0.005$

$1 = \mathbb{P}(\Omega) = \mathbb{P}(C \cup D) = \mathbb{P}(C) + \mathbb{P}(D) \Rightarrow \mathbb{P}(C) = 0.995$

$\mathbb{P}(A|D) = \dfrac{\mathbb{P}(A \cap D)}{\mathbb{P}(D)} \Rightarrow \mathbb{P}(A \cap D) = \mathbb{P}(A|D)\mathbb{P}(D) = 0.95 \cdot 0.005 = 0.00475$

$\mathbb{P}(A|C) = \dfrac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \Rightarrow \mathbb{P}(A \cap C) = \mathbb{P}(A|C)\mathbb{P}(C) = 0.01 \cdot 0.995 = 0.00995$

We need to calculate

$$\mathbb{P}(D|A) = \frac{\mathbb{P}(D \cap A)}{\mathbb{P}(A)} \overset{\text{Law of Total Probability } (\{C, D\} \text{ partition})}{=} \frac{\mathbb{P}(A \cap D)}{\mathbb{P}(A|C)\mathbb{P}(C) + \mathbb{P}(A|D)\mathbb{P}(D)} = \frac{0.00475}{0.01 \cdot 0.995 + 0.95 \cdot 0.005}$$

$$\mathbb{P}(D|A) = \frac{0.00475}{0.0147} = \frac{475}{1470} = 0.323$$

3. m red balls

n blue balls

2 balls are extracted from the urn

(a) When we draw the first ball from the $(m+n)$ already existing ones, we have m cases that it's red, so the probability is $\dfrac{m}{m+n}$.

(b) We'll define $A = \{\text{first ball is red}\}$, $B = \{\text{first ball is blue}\}$ $\Rightarrow A \cap B = \emptyset$, $A \cup B = \Omega$

Because we originally have m red balls and n blue balls, we can say that $\mathbb{P}(A) = \dfrac{m}{m+n}$ and $\mathbb{P}(B) = \dfrac{n}{m+n}$.

Now we will create 4 more sets:

$C = \{\text{second ball is blue, after first one was red}\}$

$D = \{\text{second ball is blue, after first one was blue}\}$

$E = \{\text{second ball is red, after first one was red}\}$

$F = \{\text{second ball is red, after first one was blue}\}$

We will explain $\mathbb{P}(C)$ and the rest ones are done in the same manner: if we first drew a ball which was red, then we are left with $(m-1)$ red balls and n blue balls. So, the probability of choosing a blue ball now is $\dfrac{n}{n+m-1}$. But the probability to have the second draw red, after the first one was red is actually $\mathbb{P}(A \cap E) = \mathbb{P}(A) \cdot \mathbb{P}(E)$ (as A and E are independent)

So, $\mathbb{P}(C) = \dfrac{n}{n+m-1}$, $\mathbb{P}(D) = \dfrac{n-1}{n+m-1}$, $\mathbb{P}(E) = \dfrac{m-1}{n+m-1}$; $\mathbb{P}(F) = \dfrac{m}{n+m-1}$

Let $A' = \{\text{second ball is red}\}$ and $B' = \{\text{second ball is blue}\}$

Then, $P(E) = P(A' \mid A) = \dfrac{P(A' \cap A)}{P(A)} = \dfrac{m-1}{m+n-1} \Rightarrow P(A' \cap A) = \dfrac{m(m-1)}{(m+n-1)(m+n)}$

$P(F) = P(A' \mid B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{m}{m+n-1} \Rightarrow P(A' \cap B) = \dfrac{mn}{(m+n-1)(m+n)}$

The probability of the second ball being drawn is the sum of $P(A' \cap A) + P(A' \cap B)$ as $\{A, B\}$ is a partition of Ω (the law of total probability).

$$P(A') = P(A' \mid A) \cdot P(A) + P(A' \mid B) \cdot P(B) = P(A' \cap A) + P(A' \cap B) = \frac{m^2 + mn - m}{(m+n-1)(m+n)} = \frac{m\cancel{(m+n-1)}}{\cancel{(m+n-1)}(m+n)} = \frac{m}{m+n}$$

(c) We want to calculate $P(A \mid A') = \dfrac{P(A \cap A')}{P(A')} = \dfrac{m(m-1)}{\cancel{(m+n-1)(m+n)}} \cdot \dfrac{\cancel{(m+n-1)(m+n)}}{m^2+mn-m}$

$$P(A \mid A') = \frac{m^2 - m}{m^2 + mn - m} = \frac{\cancel{m}(m-1)}{\cancel{m}(m+n-1)} = \frac{m-1}{m+n-1}.$$

4. We'll form the sets:
(part 1)

$A = \{$the person who voted was a Conservative member$\}$
$B = \{$the person who voted was a Labour member$\}$ $\Rightarrow A \cap B = \emptyset$, $A \cup B = \Omega$

$P(A) = p \Rightarrow P(B) = 1 - p$; $A^c = B$

We will also form the sets:

$C = \{$the second vote is the same as the first$\}$
$D = \{$the second vote is different from the first$\}$ $\Rightarrow C \cap D = \emptyset$, $C \cup D = \Omega$; $C^c = D$

As $\{A, B\}$ is a partition of Ω, we have (from the law of total probability) that

$P(C) = P(C \mid A) P(A) + P(C \mid B) P(B)$

We know that $P(C \mid A)$ is the probability that the second vote is the same as the first, given the fact that the person who voted was a Conservative member, so $P(C \mid A) = 1$ as they never change their vote. Also, $P(C \mid B)$ is the probability that the second vote is the same as the first, given the fact that the person who voted was a Labour member, so $P(C \mid B) = 1 - r$ (as the probability to change his/her vote is r).

So, $P(C) = 1 \cdot p + (1-r)(1-p) = \cancel{p} + 1 - r - \cancel{p} + rp = pr - r + 1$.

For D, we proceed the same way to obtain $P(D) = r - pr$. Obviously, $P(C) + P(D) = 1$, as $\{C, D\}$ is also a partition for Ω.

Now, let's define two additional sets:

~~$E = \{$the third vote is the same as the first two$\}$~~
~~$F = \{$the third vote is different from the first two$\}$ $\Rightarrow E \cap F = \emptyset$, $E \cup F = \Omega$; $E^c = F$~~

~~We want to calculate the probability that the third vote will be the same as the first two, given the fact that the second vote is the same as the first, which is $P(E \mid C)$.~~

~~$P(E \mid C) = \dfrac{P(E \cap C)}{P(C)}$. As we know $P(C)$, we will proceed to find $P(E \cap C)$.~~

From the law of total probability (with partition $\{A,B\}$) we have:

$$P(E\cap C) = P((E\cap C)|A)P(A) + P((E\cap C)|B)P(B)$$

In general, $P((X\cap Y)|Z) = P(X|Z)\cdot P(Y|Z)$ as we want both X and Y to happen, given that Z happens. So,

$$P(E\cap C) = P(E|A)P(C|A)P(A) + P(E|B)P(C|B)P(B)$$

$P(E|A)$ = the probability that the third vote is the same as the first two, given the fact that the person who voted was a Conservative member. As a Conservative members votes the same, this probability is 1.

$P(C|A) = 1$, $P(A) = p$ (we already know them)

$P(E|B)$ = the probability that the third vote is the same as the first two, given the fact that the person who voted was a Labour member. First of all, we need the first two votes to be equal, otherwise the third will have probability 0 to be equal to both, so $(1-r)$ and also we have the $(1-r)$ probability that the Labour member doesn't change his/her vote. So,

$$P(E|B) = 0\cdot r + (1-r)(1-r) = r^2-2r+1 = (1-r)^2$$

$P(C|B) = 1-r$, $P(B) = 1-p$ (we already know them)

$P(C) = pr - r + 1$

So, $$P(E|C) = \frac{P(E|A)P(C|A)P(A) + P(E|B)P(C|B)P(B)}{P(C)} = \frac{1\cdot 1\cdot p + (1-r)^2(1-r)(1-p)}{pr-r+1}$$

$$P(E|C) = \frac{p + (1-3r+3r^2-r^3)(1-p)}{pr-r+1} = \frac{\cancel{p}+1-3r+3r^2-r^3-\cancel{p}+3pr-3pr^2+pr^3}{pr-r+1}$$

$$\boxed{P(E|C) = \frac{-r^3+pr^3-3pr^2+3pr+3r^2-3r+1}{pr-r+1}}$$

In a similar way, we get $P(F|C) = \dfrac{-pr^3+3pr^2+r^3-2pr-3r^2+2r}{pr-r+1}$ and $P(E|C)+P(F|C) = 1$, which confirms that these probabilities are complementary (as they should be from the context!)

5. (a) After each toss we can either get a Heads (H) or Tails (T). The number of tosses is $n=26$. The number of favorable cases is the number of configurations of exactly 13 T and 13 H. This number is equal to $\binom{26}{13} = \frac{26!}{13!\cdot 13!}$. The total number of cases is 2^{26}. So, the probability is $P_a = \dfrac{26!}{13!\cdot 13!\cdot 2^{26}}$

(b) We have a pack of 52 cards, consisting of 26 red and 26 black cards. After dealing 26 cards from the pack, we are dealt x red cards and y black cards. We want to know the probability of the event $x=y=13$.

The probability of being dealt 13 red cards and 13 black cards is equal to

$$p_b = \frac{(26\cdot 25\cdot 24\cdot \ldots \cdot 14)\cdot(26\cdot 25\cdot 24\cdot \ldots \cdot 14)}{52\cdot 51\cdot 50\cdot \ldots \cdot 27}\cdot\binom{26}{13}$$

This happens because each draw from the deck alters the state in which the deck will be, but the order doesn't matter, as we can interchange the numberators of the fraction so they look like we drew the cards in a different order.

Let's treat the case where we draw the first 13 cards all red, then all black.

$$\underbrace{R-\left(\text{probability of } \frac{26}{52}\right) \Rightarrow R-\left(\text{probability of } \frac{25}{51}\right) \Rightarrow \ldots \Rightarrow R-\left(\text{probability of } \frac{14}{40}\right)}_{13\ R's} \Rightarrow$$

$$\Rightarrow \underbrace{B-\left(\text{probability of } \frac{26}{39}\right) \Rightarrow B-\left(\text{probability of } \frac{25}{38}\right) \Rightarrow \ldots \Rightarrow B-\left(\text{probability of } \frac{14}{27}\right)}_{13\ B's}$$

So the probability we get $\underbrace{RRR\ldots RR}_{13\ R's}\underbrace{BBB\ldots BB}_{13\ B's}$ is $\frac{(26\cdot\ldots\cdot 14)^2}{(52\cdot\ldots\cdot 27)}$. To obtain the probability of any other configuration of 13 R's and 13 B's, we just need to change the order of the numerators (as the denominators are always the same), as when we get for example a B after a R as the second draw, we will have $\frac{26}{52}\cdot\frac{26}{51}$, instead of $\frac{26}{52}\cdot\frac{25}{51}$ (in the case of RR) as we are left with a different number of red and black cards after each draw.

As we can have $\binom{26}{13}$ different configurations, all with the same probability, we will have

$$p_b = \frac{(26\cdot 25\cdot\ldots\cdot 14)^2}{52\cdot 51\cdot\ldots\cdot 27}\cdot\frac{26!}{13!\cdot 13!} = \frac{26!\cdot 26!\cdot 26!\cdot 26!}{52!\cdot 13!\cdot 13!\cdot 13!} = \frac{(26!)^4}{(13!)^3\cdot 52!}$$

Now, if we want to compare p_a with p_b

$$p_a \boxed{<} p_b$$

$$\frac{\cancel{26!}}{\cancel{13!}\cdot\cancel{13!}\cdot 2^{26}} \boxed{<} \frac{\cancel{26!}\cdot 26!\cdot 26!\cdot 26!}{\cancel{13!}\cdot\cancel{13!}\cdot 13!\cdot 13!\cdot 52!}$$

$$\frac{1}{2^{26}} \boxed{<} \frac{26!\cdot 26!\cdot 26!}{13!\cdot 13!\cdot 52!}$$

Now we'll use Stirling's formula to compare those two $(n! \simeq \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}})$

$$\frac{1}{2^{26}} \boxed{<} \frac{\left(\sqrt{2\pi}\cdot e^{-26}\cdot 26^{\frac{53}{2}}\right)^3}{\left(\sqrt{2\pi}\cdot e^{-13}\cdot 13^{\frac{27}{2}}\right)^2\cdot\left(\sqrt{2\pi}\cdot e^{-52}\cdot 52^{\frac{105}{2}}\right)}$$

$$\frac{1}{2^{26}} \boxed{<} \frac{2\pi\cancel{\sqrt{2\pi}}\cdot\cancel{e^{-78}}\cdot 26^{\frac{159}{2}}}{2\pi\cancel{\sqrt{2\pi}}\,\cancel{e^{-78}}\cdot 13^{27}\cdot 52^{\frac{105}{2}}}$$

$$\frac{1}{2^{26}} \boxed{<} \frac{26^{\frac{159}{2}}}{13^{27} \cdot 52^{\frac{105}{2}}} \quad |()^2$$

$$\frac{1}{2^{52}} \boxed{<} \frac{26^{159}}{13^{54} \cdot 52^{105}}$$

$$13^{54} \cdot 52^{105} \boxed{<} 2^{52} \cdot 26^{159}$$

$$13^{54} \cdot 2^{105} \cdot 26^{105} \boxed{<} 2^{52} \cdot 26^{159} \quad | : (2^{52} \cdot 26^{105})$$

$$13^{54} \cdot 2^{53} \boxed{<} 26^{54}$$

$$13 \cdot (13 \cdot 2)^{53} \boxed{<} 26 \cdot 26^{53}$$

$$13 \cdot \cancel{26^{53}} \boxed{<} 26 \cdot \cancel{26^{53}}$$

$$13 \boxed{<} 26$$

So, $p_a < p_b$. My intuition told me a was less likely to happen as the amount of cases in which we can draw 13 red cards and 13 black cards seemed big enough to surpass p_a (although the probabilities were very close in the end: $p_a = 0.1549$, $p_b = 0.2181$)

4. (part 2)

$E = \{$first three votes are the same$\}$

$F = \{$first three votes are different$\}$

As $\{A, B\}$ and $\{C, D\}$ are both partitions of Ω, we will calculate $\mathbb{P}(E)$ in two ways.

1) $\mathbb{P}(E) = \mathbb{P}(E \cap A) + \mathbb{P}(E \cap B) = \mathbb{P}(E|A)\mathbb{P}(A) + \mathbb{P}(E|B)\mathbb{P}(B) = 1 \cdot p + (1-n)^2(1-p)$

2) $\mathbb{P}(E) = \mathbb{P}(E \cap C) + \mathbb{P}(E \cap D) = \mathbb{P}(E|C)\mathbb{P}(C) + \mathbb{P}(E|D)\mathbb{P}(D) = \mathbb{P}(E|C)(pn - n + 1) + 0 \cdot (n - pn) \Rightarrow$

$$\Rightarrow \mathbb{P}(E|C) = \frac{\mathbb{P}(E)}{(pn - n + 1)} \Rightarrow \boxed{\mathbb{P}(E|C) = \frac{p + (1-n)^2(1-p)}{pn - n + 1}}.$$

6. (Euler's formula for the Riemann zeta function)

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$s > 1$ fixed, $X : \mathbb{N}_+ \to \mathbb{R}$ with $p_X(n) = \mathbb{P}(X = n) = \frac{1}{n^s} \cdot \frac{1}{z(s)}$, $n \geq 1$

(a) $k \geq 2$, $k \in \mathbb{N}$,

$$\mathbb{P}(X \text{ divisible by } k) = \sum_{a=1}^{\infty} \mathbb{P}(X = ak) = \sum_{a=1}^{\infty} \frac{1}{(ak)^s} \cdot \frac{1}{z(s)} = \cancel{\frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^s}}} \cdot \frac{1}{k^s} \cdot \cancel{\sum_{a=1}^{\infty} \frac{1}{a^s}}$$

$$\mathbb{P}(X \text{ divisible by } k) = \frac{1}{k^s}$$

(b) $D_k = \{X$ divisible by $k)$

<u>aim:</u> $\{D_p : p \text{ prime}\}$ are independent

From (a), we have $\mathbb{P}(D_p) = \mathbb{P}(X \text{ divisible by } p) = \frac{1}{p^s}$.

Now, let's imagine a union of m sets: $D_{i_1}, D_{i_2}, \dots, D_{i_m}$, where $i_1, i_2, \dots, i_m$ are prime numbers.

It is known that for a number to be divisible by two or more prime numbers, it needs to be divisible by their product

If we say $\mathcal{P} = \{ i \in \mathbb{N}_+ : i \text{ is prime} \}$ and $J \subset \mathcal{P}$ a finite subset, then $\{D_p : p \text{ prime}\}$ are independent if

countable infinite set ($\mathcal{P} \subset \mathbb{N}_+$)

$$\mathbb{P}\Big(\bigcap_{i \in J} D_i\Big) = \prod_{i \in J} \mathbb{P}(D_i)$$

$$\mathbb{P}(D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_m}) = \mathbb{P}(X \text{ divisible with } i_1, i_2, \dots, i_m) = \mathbb{P}(X \text{ divisible with } (i_1 \cdot i_2 \cdot \ldots \cdot i_m)) \overset{(a)}{=} \frac{1}{(i_1 \cdot i_2 \cdot \ldots \cdot i_m)^s}$$

$$\prod_{\alpha=1}^{m} \mathbb{P}(D_{i_\alpha}) = \prod_{\alpha=1}^{m} \frac{1}{i_\alpha^{\,s}} = \frac{1}{(i_1 \cdot i_2 \cdot \ldots \cdot i_m)^s}$$

$\Rightarrow \{D_p : p \text{ prime}\}$ are independent

(c)

<u>Claim:</u> $\{A_i, i \in I\}$ are independent $\Longrightarrow \{A_i^c, i \in I\}$ are independent

We'll prove this by induction:

$$P(m): \mathbb{P}(A_1^c \cap A_2^c \cap \dots \cap A_m^c) = \prod_{i=1}^{m} \mathbb{P}(A_i^c), \quad m \geq 1$$

We'll start with $P(2)$, as $P(1)$ is obvious: $\mathbb{P}(A_1^c) = \mathbb{P}(A_1^c)$

$$P(2): \mathbb{P}(A_1^c \cap A_2^c) = \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2^c)$$

$$\mathbb{P}(A_1^c \cap A_2^c) = \mathbb{P}((A_1 \cup A_2)^c) = 1 - \mathbb{P}(A_1 \cup A_2) = 1 - \big(\mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)\big) \Rightarrow$$

But, A_1 and A_2 are independent $\Rightarrow \mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$

$$\Rightarrow \mathbb{P}(A_1^c \cap A_2^c) = 1 - \mathbb{P}(A_1) - \mathbb{P}(A_2) + \mathbb{P}(A_1)\mathbb{P}(A_2) = (1 - \mathbb{P}(A_1))(1 - \mathbb{P}(A_2)) = \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2^c),$$

so $P(2)$ is True

Now, let's suppose $P(n)$ is True, and we'll prove $P(n+1)$.

$$P(n): \mathbb{P}(A_1^c \cap A_2^c \cap \dots \cap A_n^c) = \prod_{i=1}^{n} \mathbb{P}(A_i^c) \quad \text{is True}$$

$$P(n+1): \mathbb{P}(A_1^c \cap A_2^c \cap \dots \cap A_n^c \cap A_{n+1}^c) = \prod_{i=1}^{n+1} \mathbb{P}(A_i^c)$$

$$P = \mathbb{P}\big((A_1^c \cap A_2^c \cap \dots \cap A_n^c) \cap A_{n+1}^c\big) = \mathbb{P}\big((A_1 \cup A_2 \cup \dots \cup A_n)^c \cap A_{n+1}^c\big), \quad B = A_1 \cup A_2 \cup \dots \cup A_n$$

We will show that B and A_{n+1} are independent:

$$\mathbb{P}(B \cap A_{n+1}) = \mathbb{P}((A_1 \cup A_2 \cup \dots \cup A_n) \cap A_{n+1}) = \mathbb{P}\big((A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \dots \cup (A_n \cap A_{n+1})\big) \overset{\text{inclusion-exclusion formula}}{=}$$

$$\sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{n+1}) - \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j \cap A_{n+1}) + \dots + (-1)^{n+1} \mathbb{P}\Big(\bigcap_{1 \leq i \leq n} A_i \cap A_{n+1}\Big) \overset{\text{the events are all independent}}{=} \mathbb{P}(A_{n+1}) \cdot \sum_{i=1}^{n} \mathbb{P}(A_i) -$$

$-\mathbb{P}(A_{n+1}) \cdot \sum_{1\le i<j\le n} \mathbb{P}(A_i \cap A_j) + \dots + (-1)^{n+1} \mathbb{P}\left(\bigcap_{1\le i\le n} A_i\right) = \mathbb{P}(A_{n+1}) \cdot \mathbb{P}\left(\bigcup_{1\le i\le n} A_i\right) = \mathbb{P}(B) \cdot \mathbb{P}(A_{n+1}).$

So, B and A_{n+1} are independent. Using the same reasoning as for $P(2)$, we have B^c and A_{n+1}^c independent. $\Rightarrow (A_1 \cup A_2 \cup \dots \cup A_n)^c$ and A_{n+1}^c are independent

So, $\mathbb{P}((A_1 \cup A_2 \cup \dots \cup A_n)^c \cap A_{n+1}^c) = \mathbb{P}((A_1 \cup A_2 \cup \dots \cup A_m)^c) \cdot \mathbb{P}(A_{m+1}^c) = \mathbb{P}(A_1^c \cap A_2^c \cap \dots A_m^c) \cdot$

$\cdot \mathbb{P}(A_{n+1}^c) \overset{P(n)}{=\!=} \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2^c) \cdot \dots \cdot \mathbb{P}(A_m^c) \cdot \mathbb{P}(A_{n+1})^c = \prod_{i=1}^{n+1} \mathbb{P}(A_i^c)$

Therefore, $P(n+1)$ is True.

In conclusion, if the family $\{A_i, i \in I\}$ of events is independent, then so is the family $\{A_i^c, i \in I\}$.

Now we want to prove Euler's formula: for every $s > 1$

$$z(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \text{ prime}} \left(1 - \mathbb{P}(\Delta_p)\right)^{-1} = \frac{1}{\prod_{p \text{ prime}} \mathbb{P}(\Delta_p^c)} \underset{+\text{HINT}}{\overset{(b) - \{\Delta_p^c : p \text{ prime}\} \text{ are independent}}{=\!=}} \frac{1}{\mathbb{P}\left(\bigcap_{p \text{ prime}} \Delta_p^c\right)} =$$

$$= \frac{1}{\mathbb{P}(X \text{ is not divisible by any prime number})} = \frac{1}{\mathbb{P}(X = 1)} = \frac{1}{\frac{1}{1^s} \cdot \frac{1}{z(s)}} = \frac{1}{\frac{1}{z(s)}} = z(s)$$

So, Euler's formula is proven for every $s > 1$.

! My notation for $\zeta(s)$ was $z(s)$ for the entire problem, as it is easier to write.