- Chapter 1: Vectors and vector spaces

  (Week 1, Lectures 1-3)

- Chapter 2: Independence and orthogonality

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- Chapter 3: Matrices

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- Chapter4: Systems of linear equations

  (Week 4, Lectures 10-12)

Chapter 4

Systems of Linear Equations

4.1. Introduction

Suppose we have two unknown real numbers $\boldsymbol{x} \in \mathbb{R}$ and $\boldsymbol{y} \in \mathbb{R}$ satisfying

$$\boldsymbol{eq1: \quad x + 2y = 1}$$

and

$$\boldsymbol{eq2: \quad 2x - y = 7}.$$

We find the x, y by combining the two equation so that one of the unknowns, say y is eliminated. In this case

$$eq1 + (2 \times eq2) \quad => \quad 5x + 0y = 15 \quad => \quad x = 3.$$

We now can find y by inserting the now known value of x into one of the two equations. Giving

$$3 + 2y = 1 \quad => \quad y = -1.$$

This is known as back substitution.

In general we will have n unknowns, say $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^n$, and m equations.

4.2. Geometry of Linear Equations

Recall in two space dimensions with coordinates $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{\boldsymbol{2}}$ the equation of a straight line is given by

$$\boldsymbol{L1:} \quad \boldsymbol{y} = \boldsymbol{a}_1\boldsymbol{x} + \boldsymbol{a}_0, \quad \boldsymbol{a}_0, \boldsymbol{a}_1 \in \mathbb{R}.$$

We are given a second straight line

$$\boldsymbol{L2:} \quad \boldsymbol{y} = \boldsymbol{b}_1\boldsymbol{x} + \boldsymbol{b}_0, \quad \boldsymbol{b}_0, \boldsymbol{b}_1 \in \mathbb{R},$$

and wish to find where they intersect (if they do).

Using the process given in the first section we have

$$\boldsymbol{L1 - L2 \quad => \quad 0 = (a_1 - b_1)x + (a_0 - b_0)}.$$

We now have to consider the following cases:

1. $\boldsymbol{a_1 \neq b_1}$ and $\boldsymbol{a_0 \neq b_0}$

   In this case we have

$$\boldsymbol{x = \frac{b_0 - a_0}{a_1 - b_1}}.$$

   Using back substitution we obtain

$$\boldsymbol{y = \frac{b_0 - a_0}{a_1 - b_1} a_1 + a_0 \quad => \quad y = \frac{b_0 a_1 - a_0 b_1}{a_1 - b_1}}.$$

2. $\boldsymbol{a_1 = b_1}$ and $\boldsymbol{a_0 \neq b_0}$

   In this case we have two distinct parallel lines. NO SOLUTION.

3. $\boldsymbol{a}_1 = \boldsymbol{b}_1$ and $\boldsymbol{a}_0 = \boldsymbol{b}_0$

They are the same line, INFINITE SOLUTIONS.

4.3. Gaussian Elimination

Consider the problem: Find $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^3$ such that

$$\begin{array}{lrcrcrcr} eq1: & 2x & + & y & - & z & = & 3 \\ eq2: & x & & & + & 5z & = & 6 \\ eq3: & -x & + & 3y & - & 2z & = & 3 \end{array}$$

The process of Gaussian elimination is to firstly reduce the above system of equations so that the unknown $\boldsymbol{x}$ is removed from the last two equations, as follows:

$$\begin{array}{rrcrcrcr} eq1: & 2x & + & y & - & z & = & 3 \\ eq2 \;=>\; eq1 - 2 \times eq2: & & & y & - & 11z & = & -9 \\ eq3 \;=>\; eq1 + 2 \times eq3: & & & 7y & - & 5z & = & 9 \end{array}$$

We now remove the unknown $\boldsymbol{y}$ from the last equation, as follows:

$$\begin{aligned} & \boldsymbol{eq1: 2x+y-z} &= \ & \mathbf{3} \\ & \boldsymbol{eq2: \quad\quad y-11z} &= \ & \mathbf{-9} \\ \boldsymbol{eq3 \ => \ 7 \times eq2 - eq3:} & \quad\quad\quad \boldsymbol{-72z} &= \ & \mathbf{-72} \end{aligned}$$

This system is said to be Upper Triangular. We call this particular form Row Echelon.

We can now use back substitution to obtain $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. In this case

$$\boldsymbol{z = 1, \ y - 11 = -9 \ \ => \ \ y = 2, \ 2x + 2 - 1 = 3, \ \ => \ \ x = 1}.$$

When does Gaussian elimination breakdown?

Consider the following system

$$\begin{array}{lcccccccc} eq1: & x & + & y & + & z & = & a \\ eq2: & 2x & + & 2y & + & 5z & = & b \\ eq3: & 4x & + & 6y & + & 8z & = & c \end{array}$$

This firstly reduces to

$$\begin{array}{lcccccccc} eq1: & x & + & y & + & z & = & a \\ eq2: & & & & & 3z & = & b' \\ eq3: & & & 2y & + & 4z & = & c' \end{array}$$

This can easily be switch into row echelon form by switching rows two and three.

Now consider the following system

$$\begin{array}{rrcrcrcl} \boldsymbol{eq1:} & \boldsymbol{x} & + & \boldsymbol{y} & + & \boldsymbol{z} & = & \boldsymbol{a} \\ \boldsymbol{eq2:} & \boldsymbol{2x} & + & \boldsymbol{2y} & + & \boldsymbol{5z} & = & \boldsymbol{b} \\ \boldsymbol{eq3:} & \boldsymbol{4x} & + & \boldsymbol{4y} & + & \boldsymbol{8z} & = & \boldsymbol{c} \end{array}$$

This firstly reduces to

$$\begin{array}{rrcrcrcl} \boldsymbol{eq1:} & \boldsymbol{x} & + & \boldsymbol{y} & + & \boldsymbol{z} & = & \boldsymbol{a} \\ \boldsymbol{eq2:} & & & & & \boldsymbol{3z} & = & \boldsymbol{b'} \\ \boldsymbol{eq3:} & & & & & \boldsymbol{4z} & = & \boldsymbol{c'} \end{array}$$

This system may have no solution, if $\boldsymbol{eq2}$ and $\boldsymbol{eq3}$ are not consistent. Otherwise the system has infinitely many solutions.

4.4. Characterization of a system of linear equations

A system of m linear equations with n variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\cdot \qquad \cdot \qquad \cdot \\
\cdot \qquad \cdot \qquad \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m.
\end{aligned}
$$

This system can be represented by a matrix vector multiplication,

$$\boldsymbol{A}\mathrm{x} = \mathrm{b},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\mathrm{x} \in \mathbb{R}^{n\times 1}$, $\mathrm{b} \in \mathbb{R}^{m\times 1}$, $\boldsymbol{A} = (a_{ij})$, and $\{x_j\}_{j=1}^{n}$, $\{b_i\}_{i=1}^{m}$ are components of x, b.

The system of linear equations is called homogeneous if

$$\boldsymbol{b}_1 = \boldsymbol{b}_2 = \ldots = \boldsymbol{b}_m = \mathbf{0}.$$

In this case the system of linear equations is represented by

$$\boldsymbol{A}\mathbf{x} = \mathbf{0}.$$

The system of linear equations is overdetermined if it has more equations than variables. Therefore in

$$\boldsymbol{A}\mathbf{x} = \mathbf{b},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, we have $\boldsymbol{m} > \boldsymbol{n}$.

Example 4.4.1

$$\begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}.$$

The system of linear equations is underdetermined if it has more variables than equations. Therefore in

$$\boldsymbol{A}\mathrm{x} = \mathrm{b},$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, we have $m < n$.

Example 4.4.2

$$\begin{bmatrix} -3 & 4 & 1 \\ -4 & 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

A system of linear equations is called consistent if it has at least one solution and inconsistent if it has no solution.

4.4.1. Homogeneous systems of linear equations

The matrix form of a homogeneous system of equations is

$$\boldsymbol{A}\mathbf{x} = \mathbf{0}.$$

Recall that we defined the nullspace of $\boldsymbol{A}$, $\boldsymbol{\mathcal{N}}(\boldsymbol{A})$ based on vectors $\mathbf{x}$ that are solutions to the corresponding homogeneous system.

Formally,

$$\boldsymbol{\mathcal{N}}(\boldsymbol{A}) = \{\mathbf{x} \in \mathbb{R}^n \mid \boldsymbol{A}\mathbf{x} = \mathbf{0}\}.$$

$\boldsymbol{\mathcal{N}}(\boldsymbol{A})$ is a subspace of $\mathbb{R}^{\boldsymbol{n}}$.

proof 4.4.3 *To see this note that* $\mathcal{N}(\boldsymbol{A}) \subseteq \mathbb{R}^n$ *since* $\mathbf{x}$ *is an* $\boldsymbol{n}$ *by* $\mathbf{1}$ *vector given that* $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$ *and* $\mathcal{N}(\boldsymbol{A}) \neq \emptyset$ *since the system* $\boldsymbol{A}\mathbf{x} = \mathbf{0}$ *always have the trivial solution,* $\mathbf{x} = \mathbf{0}$*. Furthermore* $\mathcal{N}(\boldsymbol{A})$ *is closed under addition and scalar multiplication, as* $\boldsymbol{A}\mathbf{u} = \mathbf{0}$ *and* $\boldsymbol{A}\mathbf{v} = \mathbf{0}$ *implies that*

$$\boldsymbol{A}(\mathbf{u} + \mathbf{v}) = \boldsymbol{A}\mathbf{u} + \boldsymbol{A}\mathbf{v} = \mathbf{0},$$

$$\boldsymbol{A}(\boldsymbol{c}\mathbf{u}) = \boldsymbol{c}(\boldsymbol{A}\mathbf{u}) = \mathbf{0},$$

where $\boldsymbol{c}$ *is a scalar.* $\square$

Remark 4.4.4 *Therefore, a homogeneous system is always consistent because a homogeneous system of linear equations,* $\boldsymbol{A}\mathbf{x} = \mathbf{0}$ *has at least one solution,* $\mathbf{x} = \mathbf{0}$*. This is called the trivial solution.*

To find the solution space for $\boldsymbol{A}\mathbf{x} = \mathbf{0}$ first construct the augmented matrix $[\boldsymbol{A} \mid \mathbf{0}]$ and apply elementary row operations until the left hand side is in reduced row echelon form. (Note that the right hand side of the augmented matrix will not change.) Next, investigate the system of linear equations corresponding to the reduced row echelon form.

The following examples will demonstrate how to find the solution space for the $\boldsymbol{A}\mathbf{x} = \mathbf{0}$ system and subsequently a basis for $\boldsymbol{\mathcal{N}}(\boldsymbol{A})$.

Example 4.4.5 *Consider the following system of equations:*

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ 3x_1 - x_2 \qquad &= 0 \\ 2x_1 - 4x_2 - 5x_3 &= 0 \end{aligned}$$

Next, we form the augmented matrix,

$$[A \mid 0] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 2 & -4 & -5 & 0 \end{array}\right],$$

and perform the following elementary row operations:

$$\begin{array}{c} r_2 \leftarrow r_2 + 3r_1 \\ r_3 \leftarrow r_3 + 2r_1 \end{array} => \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -2 & -3 & 0 \end{array}\right],$$

$$r_3 \leftarrow r_3 + r_2 => \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$r_1 \leftarrow r_1 - \frac{1}{2} r_2 => \begin{bmatrix} -1 & 0 & -\frac{1}{2} & | & 0 \\ 0 & 2 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix},$$

$$\begin{matrix} r_1 \leftarrow (-1) r_1 \\ r_2 \leftarrow \frac{1}{2} r_2 \end{matrix} => \begin{bmatrix} 1 & 0 & \frac{1}{2} & | & 0 \\ 0 & 1 & \frac{3}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

From the reduced row echelon form, we can write down the system of equations:

$$\begin{aligned} x_1 \qquad + \frac{1}{2} x_3 &= 0, \\ x_2 + \frac{3}{2} x_3 &= 0. \end{aligned}$$

Choosing $x_3 = s$, $s \in \mathbb{R}$ *we have,*

$$
\begin{aligned}
x_1 \qquad + \frac{1}{2}s &= 0 \Rightarrow x_1 = -\frac{1}{2}s, \\
x_2 + \frac{3}{2}s &= 0 \Rightarrow x_2 = -\frac{3}{2}s.
\end{aligned}
$$

Therefore, the solution space is defined by

$$
\mathrm{x} = [-\frac{1}{2}s, -\frac{3}{2}s, s]^T = [-t, -3t, 2t]^T = \begin{bmatrix} -t \\ -3t \\ 2t \end{bmatrix} = t \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix},
$$

where $t = \frac{1}{2}s$*. Thus the basis for the nullspace is*

$$
\left\{ \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \right\}.
$$

Therefore, $dim(\mathcal{N}(A)) = 1$*.*

Example 4.4.6 *For the next homogeneous system of equations, consider:*

$$\begin{aligned} \boldsymbol{x_1 - 2x_2 + 3x_3} &= \mathbf{0}, \\ \boldsymbol{-3x_1 + 6x_2 - 9x_3} &= \mathbf{0}. \end{aligned}$$

As usual we start with forming the corresponding augmented matrix,

$$[\boldsymbol{A} \mid \mathbf{0}] = \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-2} & \mathbf{3} & \mathbf{0} \\ \mathbf{-3} & \mathbf{6} & \mathbf{-9} & \mathbf{0} \end{array}\right]$$

and we only need to perform a single elementary row operation:

$$\boldsymbol{r_2 \leftarrow r_2 + 3r_1} => \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-2} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right],$$

leading to the system of equations:

$$\boldsymbol{x_1 - 2x_2 + 3x_3} = \mathbf{0}.$$

Next, let

$$\boldsymbol{x_3 = s, \quad x_2 = t, \quad s, t} \in \mathbb{R}$$

and we have

$$\boldsymbol{x_1 - 2t + 3s = 0 => x_1 = 2t - 3s}.$$

Therefore, the general solution to the problem can be written in the form

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{2t - 3s} \\ \boldsymbol{t} \\ \boldsymbol{s} \end{bmatrix} = \boldsymbol{t} \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix} + \boldsymbol{s} \begin{bmatrix} \mathbf{-3} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix},$$

and the basis for the nullspace is

$$\left\{ \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{-3} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right\}.$$

Therefore, $\boldsymbol{dim(\mathcal{N}(A)) = 2}$.

Notice that in previous Examples 4.4.5 and 4.4.6 the number of columns of $\boldsymbol{A}$ was equal to the number of nonzero rows in its reduced row echelon form plus the dimension of the nullspace. Note that the number of nonzero rows in the (reduced) echelon form of the matrix indicates the number of linearly independent columns (rows) in the matrix and is equal to $\boldsymbol{dim(\mathcal{C}(A))}$ or $\boldsymbol{rank(A)}$. The next theorem formalizes these observations.

Theorem 4.4.7 *For any $\boldsymbol{m}$ by $\boldsymbol{n}$ matrix $\boldsymbol{A}$, the following statement holds:*

$$\boldsymbol{n = rank(A) + dim(\mathcal{N}(A))}.$$

4.4.2. Nonhomogeneous systems of linear equations

The matrix form of a nonhomogeneous system of equations is

$$\boldsymbol{A}\mathbf{x} = \mathbf{b}, \qquad \mathbf{b} \neq \mathbf{0}.$$

We may define the solution space corresponding to this nonhomogeneous system as

$$\boldsymbol{\mathcal{S}}_{nh}(\boldsymbol{A}) = \{\mathbf{x} \in \mathbb{R}^n \mid \boldsymbol{A}\mathbf{x} = \mathbf{b}, \;\; \mathbf{b} \neq \mathbf{0}\}.$$

Given that $\boldsymbol{A}$ is $\boldsymbol{m}$ by $\boldsymbol{n}$ it is certainly true that $\boldsymbol{\mathcal{S}}_{nh}(\boldsymbol{A}) \subseteq \mathbb{R}^n$ but $\boldsymbol{\mathcal{S}}_{nh}(\boldsymbol{A})$ is not a subspace of $\mathbb{R}^n$ as it clearly does not include the additive identity because $\mathbf{b} \neq \mathbf{0}$.

There is still a relationship between $\boldsymbol{\mathcal{S}}_{nh}(\boldsymbol{A})$ and $\boldsymbol{\mathcal{N}}(\boldsymbol{A})$!

Recall that for a nonhomogeneous system

$$\boldsymbol{A}\mathbf{x} = \mathbf{b},$$

we can write the general solution in the form,

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n,$$

where $\mathbf{x}_p$ is the particular solution and $\mathbf{x}_n \in \mathcal{N}(\boldsymbol{A})$.

This is true since

$$\boldsymbol{A}\mathbf{x} = \boldsymbol{A}(\mathbf{x}_p + \mathbf{x}_n) = \boldsymbol{A}\mathbf{x}_p + \boldsymbol{A}\mathbf{x}_n = \boldsymbol{A}\mathbf{x}_p$$

and $\boldsymbol{A}\mathbf{x}_n = \mathbf{0}$ because $\mathbf{x}_n \in \mathcal{N}(\boldsymbol{A})$.

We proceed with two examples of nonhomogeneous systems.

Example 4.4.8 *Consider the following system of equations:*

$$\begin{aligned} x_1 + 3x_2 + 10x_3 &= 18 \\ -2x_1 + 7x_2 + 32x_3 &= 29 \\ -x_1 + 3x_2 + 14x_3 &= 12 \\ x_1 + x_2 + 2x_3 &= 8 \end{aligned}$$

Next, we form the augmented matrix,

$$[A \mid \mathrm{b}] = \left[\begin{array}{ccc|c} 1 & 3 & 10 & 18 \\ -2 & 7 & 32 & 29 \\ -1 & 3 & 14 & 12 \\ 1 & 1 & 2 & 8 \end{array}\right],$$

and perform the following elementary row operations:

$$\begin{array}{l} r_2 \leftarrow r_2 + 2r_1 \\ r_3 \leftarrow r_3 + \ r_1 \\ r_4 \leftarrow r_4 - \ r_1 \end{array} => \left[\begin{array}{ccc|c} 1 & 3 & 10 & 18 \\ 0 & 13 & 52 & 65 \\ 0 & 6 & 24 & 30 \\ 0 & -2 & -8 & -10 \end{array}\right],$$

$$\boldsymbol{r_2 \leftarrow \frac{1}{13} r_2 => \left[\begin{array}{cccc} 1 & 3 & 10 & | & 18 \\ 0 & 1 & 4 & | & 5 \\ 0 & 6 & 24 & | & 30 \\ 0 & -2 & -8 & | & -10 \end{array}\right]},$$

$$\boldsymbol{\begin{array}{l} r_3 \leftarrow r_3 - 6r_2 \\ r_4 \leftarrow r_4 + 2r_2 \end{array} => \left[\begin{array}{ccccc} 1 & 3 & 10 & | & 18 \\ 0 & 1 & 4 & | & 5 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{array}\right]},$$

$$\boldsymbol{r_1 \leftarrow r_1 - 3r_2 => \left[\begin{array}{ccccc} 1 & 0 & -2 & | & 3 \\ 0 & 1 & 4 & | & 5 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{array}\right]}.$$

Therefore, we have

$$\begin{aligned} \boldsymbol{x_2 + 4x_3} &= \boldsymbol{5} \\ \boldsymbol{x_1 - 2x_3} &= \boldsymbol{3}, \end{aligned}$$

and choose $\boldsymbol{x_3}$ *as a free variable,* $\boldsymbol{x_3 = s}$, $\boldsymbol{s} \in \mathbb{R}$. *This leads to*

$$\begin{aligned} \boldsymbol{x_2 + 4x_3} &= \boldsymbol{5 => x_2 = 5 - 4s} \\ \boldsymbol{x_1 - 2x_3} &= \boldsymbol{3 => x_1 = 3 + 2s}. \end{aligned}$$

Therefore the general solution has the form: $\mathbf{x} = \boldsymbol{[3 + 2s, 5 - 4s, s]^T}$.
Given that the system has (at least one) solution it is <u>*consistent*</u>.
Furthermore notice that

$$\mathbf{x} = \begin{bmatrix} \boldsymbol{3 + 2s} \\ \boldsymbol{5 - 4s} \\ \boldsymbol{s} \end{bmatrix} = \boldsymbol{s} \begin{bmatrix} \boldsymbol{2} \\ \boldsymbol{-4} \\ \boldsymbol{1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{3} \\ \boldsymbol{5} \\ \boldsymbol{0} \end{bmatrix} = \mathbf{x}_{\boldsymbol{n}} + \mathbf{x}_{\boldsymbol{p}},$$

where $\boldsymbol{x_n} \in \boldsymbol{\mathcal{N}(A)}$.

Example 4.4.9 *For the next example, consider the following system of equations:*

$$\begin{aligned} \mathbf{3}x_1 - \mathbf{2}x_2 + \mathbf{16}x_3 - \mathbf{2}x_4 &= -\mathbf{7} \\ -x_1 + \mathbf{5}x_2 - \mathbf{14}x_3 + \mathbf{18}x_4 &= \mathbf{29} \\ \mathbf{3}x_1 - x_2 + \mathbf{14}x_3 + \mathbf{2}x_4 &= \mathbf{1} \end{aligned}$$

First, we form the augmented matrix,

$$[A \mid \mathrm{b}] = \left[\begin{array}{cccc|c} 3 & -2 & 16 & -2 & -7 \\ -1 & 5 & -14 & 18 & 29 \\ 3 & -1 & 14 & 2 & 1 \end{array}\right].$$

Next, we perform the following elementary row operations:

$$\begin{array}{l} r_2 \leftarrow r_2 + \frac{1}{3} r_1 \\ r_3 \leftarrow r_3 - \ \ r_1 \end{array} => \left[\begin{array}{cccc|c} 3 & -2 & 16 & -2 & -7 \\ 0 & \frac{13}{3} & -\frac{26}{3} & \frac{52}{3} & \frac{80}{3} \\ 0 & 1 & -2 & 4 & 8 \end{array}\right],$$

$$r_2 \leftarrow \frac{3}{13} r_2 => \left[\begin{array}{cccc|c} 3 & -2 & 16 & -2 & -7 \\ 0 & 1 & -2 & 4 & \frac{80}{13} \\ 0 & 1 & -2 & 4 & 8 \end{array}\right],$$

$$r_3 \leftarrow r_3 - r_2 => \left[\begin{array}{cccc|c} 3 & -2 & 16 & -2 & -7 \\ 0 & 1 & -2 & 4 & \frac{80}{13} \\ 0 & 0 & 0 & 0 & \frac{24}{13} \end{array}\right],$$

$$r_3 \leftarrow \frac{13}{24} r_3 => \left[\begin{array}{cccc|c} 3 & -2 & 16 & -2 & -7 \\ 0 & 1 & -2 & 4 & \frac{80}{13} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$r_3 \leftarrow r_2 - \frac{80}{13} r_3 => \left[\begin{array}{cccc|c} 3 & -2 & 16 & -2 & -7 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$r_1 \leftarrow r_1 + 7r_3 => \left[\begin{array}{cccc|c} 3 & -2 & 16 & -2 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$r_1 \leftarrow r_1 + 2r_2 => \left[\begin{array}{cccc|c} 3 & 0 & 12 & 6 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$\boldsymbol{r_1 \leftarrow \frac{1}{3} r_1 => \left[\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 0 \\ 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]}.$$

Based on the reduced row echelon form of the augmented matrix, it is clear that the system has no solution. Therefore, the system is <u>inconsistent</u>.

Recall that

$$\boldsymbol{A}\mathbf{x} = \mathbf{b}$$

can be written as a linear combination of vectors,

$$\boldsymbol{x}_1 \begin{bmatrix} \\ \mathbf{a}_1 \\ \\ \end{bmatrix} + \boldsymbol{x}_2 \begin{bmatrix} \\ \mathbf{a}_2 \\ \\ \end{bmatrix} + \cdots + \boldsymbol{x}_n \begin{bmatrix} \\ \mathbf{a}_n \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \mathbf{b} \\ \\ \end{bmatrix}.$$

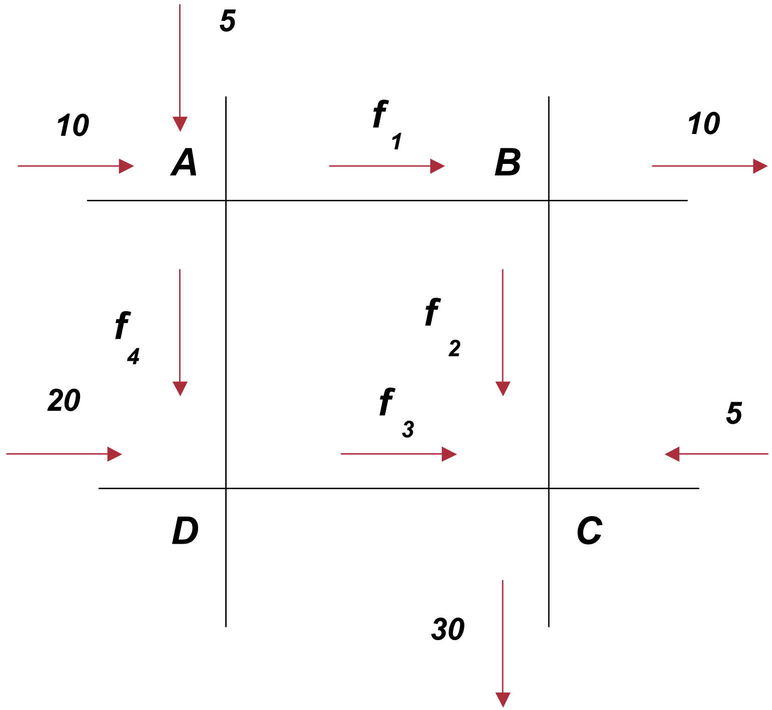
Theorem 4.4.10 *A system of linear equations is consistent if and only if* $\mathbf{b}$ *is in* $\boldsymbol{C}(\boldsymbol{A})$*.*

4.5. Applications

4.5.1. Networks

Four junctions A, B, C and D have two exits and two entrances. The flow in/out of the junctions is given by the diagram below. It is known that the number of cars flowing in at a junction is always equal to the number cars leaving the junction, Conservation Law.

To find the unknown flows f_1, f_2, f_3, f_4 we use the linear system of equations, arising from the conservation law:


5
10
f
1
A
B
10
f
4
f
2
20
f
3
5
D
C
30

$$
\begin{aligned}
\boldsymbol{f_1 + f_4} &= \mathbf{15} \\
\boldsymbol{f_1 - f_2} &= \mathbf{10} \\
\boldsymbol{f_2 + f_3} &= \mathbf{25} \\
\boldsymbol{f_3 - f_4} &= \mathbf{20}
\end{aligned}
$$

Then consider, the augmented matrix,

$$
[\boldsymbol{A} \mid \mathrm{b}] = \left[\begin{array}{cccc|c}
1 & 0 & 0 & 1 & \mathbf{15} \\
1 & -1 & 0 & 0 & \mathbf{10} \\
0 & 1 & 1 & 0 & \mathbf{25} \\
0 & 0 & 1 & -1 & \mathbf{20}
\end{array}\right].
$$

Next, we perform the following elementary row operations:

$$
\boldsymbol{r_2} \leftarrow \boldsymbol{r_2} - \boldsymbol{r_1} \;\; => \left[\begin{array}{cccc|c}
1 & 0 & 0 & 1 & \mathbf{15} \\
0 & -1 & 0 & -1 & \mathbf{-5} \\
0 & 1 & 1 & 0 & \mathbf{25} \\
0 & 0 & 1 & -1 & \mathbf{20}
\end{array}\right],
$$

$$r_2 \leftarrow (-1)r_2 \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 0 & 25 \\ 0 & 0 & 1 & -1 & 20 \end{array}\right],$$

$$r_3 \leftarrow r_3 - r_2 \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 1 & -1 & 20 \end{array}\right],$$

$$r_3 \leftarrow r_4 - r_3 \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, we have a free variable, f_4, we have infinitely many solutions, letting $f_4 = s$ we have

$$f_1 = 15 - s, \quad f_2 = 5 - s, \quad f_3 = 20 + s.$$

4.5.2. Electrical Networks

Ohm's law

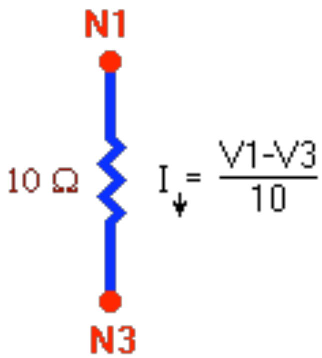
$$V = I \times R,$$

V: Voltage in volts, I: Current in amps, R: Resistance in ohms, e.g.

$$1V = 1A \times 1\Omega.$$

Example 4.5.1 *Using Ohm's law and voltage $V1, V3$ at $N1$ and $N3$ respectively, we obtain the current*

$$I = \frac{V1 - V3}{10}.$$

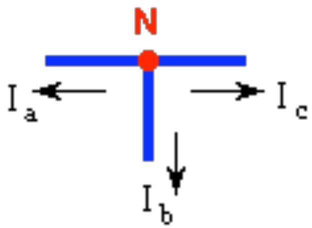


Current Law

The sum of the currents flowing into one node is equal to the sum of the currents flowing out of one node.

Example 4.5.2 *Using the current law we obtain*

$$\boldsymbol{I_a + I_b + I_c = 0}.$$



Voltage Law

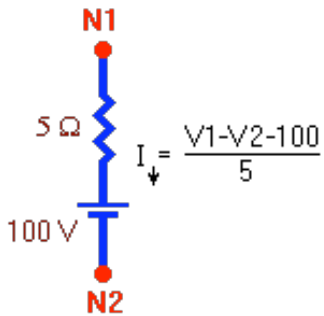
The sum of the voltage drops around any circuit is equal to the total voltage around the circuit.

Example 4.5.3 *Assuming a downward current, applying ohm's law and the voltage law we have*

$$\textit{Total voltage} \;=\; \boldsymbol{V1 - V2 - 100}$$

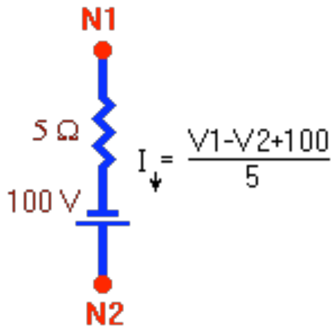
$$\textit{Ohm's Law} \;=>\; \mathbf{5}\boldsymbol{I} \;=\; \textit{Total voltage}$$

$$\frac{\boldsymbol{V1 - V2 - 100}}{\mathbf{5}} = \boldsymbol{I}.$$

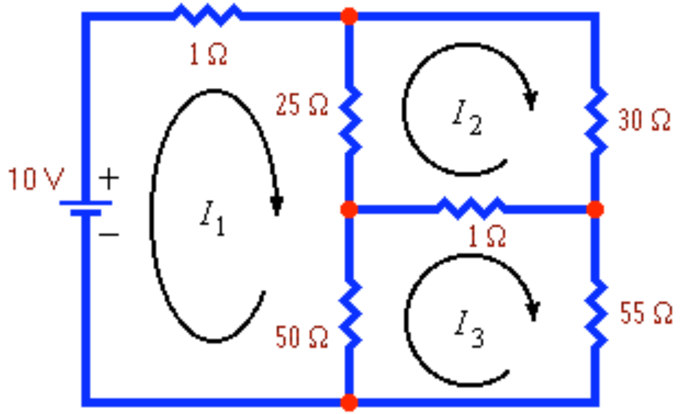


Assuming a downward current, applying ohm's law and the voltage law we have

$$\frac{\boldsymbol{V1 - V2 + 100}}{\mathbf{5}} = \boldsymbol{I}.$$



Example 4.5.4 *This final example shows how we reduce current calculations within a circuit to a system of linear equations*



Applying the previous laws in each loop, we have

$$\mathbf{1}\boldsymbol{I_1} + \mathbf{25}(\boldsymbol{I_1} - \boldsymbol{I_2}) + \mathbf{50}(\boldsymbol{I_1} - \boldsymbol{I_3}) = \mathbf{10},$$

$$\mathbf{25}(\boldsymbol{I_2} - \boldsymbol{I_1}) + \mathbf{30}\boldsymbol{I_2} + \mathbf{1}(\boldsymbol{I_2} - \boldsymbol{I_3}) = \mathbf{0},$$

$$\mathbf{50}(\boldsymbol{I_3} - \boldsymbol{I_1}) + \mathbf{1}(\boldsymbol{I_3} - \boldsymbol{I_2}) + \mathbf{55}\boldsymbol{I_3} = \mathbf{0}.$$

The effective resistance of the circuit is given by applying Ohm's law to current going around the circuit and the total voltage drop. In this case since the we have only one battery this could be calculated based on the current $\boldsymbol{I_1}$ and the total voltage drop, which is $\mathbf{10}\boldsymbol{V}$.

4.5.3. Global Positioning System

Let us assume the earth is a unit sphere (has radius one). Any point on the earth can be expressed as (x, y, z) such that $x^2 + y^2 + z^2 = 1$.

A lost Hiker is at the point (x, y, z) on the earths surface. Four satellites, with known positions (x_i, y_i, z_i), $i = 1, 2, 3, 4$, send signals to your GPS at times t_i seconds,$i = 1, 2, 3, 4$, respectively and the hiker's GPS system receives the signals at the time T seconds. It is known that the signal travels at a speed of 0.5 Earth Radii per 10^{-2} seconds.

We can now calculate the distances, $\boldsymbol{d_i}$ (units are Earth Radii), of each satellite from the hiker.

$$\boldsymbol{d_i = (0.5)\times(T - t_i)(\times 100) = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}.}$$

Squaring

$$\boldsymbol{50^2(T - t_i)^2 = x^2 - 2xx_i + x_i^2 + y^2 - 2yy_i + y_i^2 + z^2 - 2zz_i + z_i^2.}$$

Using the fact that $\boldsymbol{x^2 + y^2 + z^2 = 1}$ we obtain

$$\boldsymbol{1 - 50^2(T - t_i)^2 + (x_i^2 + y_i^2 + z_i^2) = 2x_i x + 2y_i y + 2z_i z.}$$

So we have four unknowns $\boldsymbol{x, y, z}$ and $\boldsymbol{T}$ and four non-linear equations.

Expanding $(T - t_i)^2$ we see that the quadratic term in T is the same for all four equations. Hence, subtracting the first equation, $i = 1$, from each of the remaining three we obtain

$$(2x_2 - 2x_1)x + (2y_2 - 2y_1)y + (2z_2 - 2z_1)z - 5000(t_2 - t_1)T = f_2,$$

$$(2x_3 - 2x_1)x + (2y_3 - 2y_1)y + (2z_3 - 2z_1)z - 5000(t_3 - t_1)T = f_3,$$

$$(2x_4 - 2x_1)x + (2y_4 - 2y_1)y + (2z_4 - 2z_1)z - 5000(t_4 - t_1)T = f_4,$$

where the f_i, $i = \{2, 3, 4\}$ are known.
A LINEAR system of three equations for four unknowns.
By using T as a free variable we may obtain an expression, in terms of T, for (x, y, z).
Finally, we substitute these expressions into the original first equation to obtain a quadratic in T and solve.

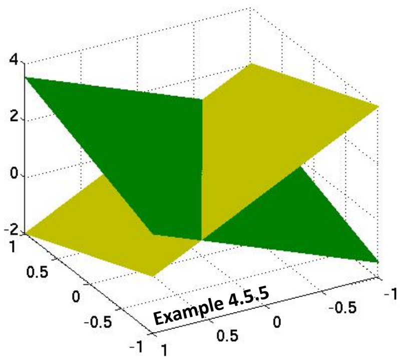
4.5.4. Intersection of planes in $\mathbb{R}^3$

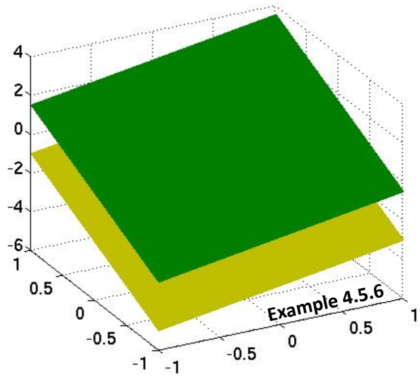
Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ be real nonzero numbers. Then $\boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{cz} = \boldsymbol{d}$ is a linear equation in three variables. The set of solutions of this linear equation is a 2-dimensional plane.

In this section we will use a series of examples to demonstrate some possible scenarios for the solution space for systems of two equations (two planes) and three equations (three planes) and provide geometric interpretation of the outcomes.

Two equations

The system of two equations may be consistent, in which case the two planes intersect in a line (Example 4.5.5 and figure below) or the two planes are identical. If the system is inconsistent then the two planes do not have a common point and they are parallel (Example 4.5.6 and figure below).





Example 4.5.5 *Consider the following system:*

$$\begin{aligned} \boldsymbol{x} + \mathbf{2}\boldsymbol{y} + \boldsymbol{z} &= \mathbf{1} \\ \mathbf{2}\boldsymbol{x} + \mathbf{3}\boldsymbol{y} - \mathbf{2}\boldsymbol{z} &= -\mathbf{2}. \end{aligned}$$

To investigate the solution space, we first form the augmented matrix,

$$[\boldsymbol{A} \mid \mathrm{b}] = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} & | & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & -\mathbf{2} & | & -\mathbf{2} \end{bmatrix},$$

and perform the following elementary row operations:

$$\boldsymbol{r}_2 \leftarrow \boldsymbol{r}_2 - \mathbf{2}\boldsymbol{r}_1 \implies \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} & | & \mathbf{1} \\ \mathbf{0} & -\mathbf{1} & -\mathbf{4} & | & -\mathbf{4} \end{bmatrix},$$

$$\boldsymbol{r}_1 \leftarrow \boldsymbol{r}_1 + \mathbf{2}\boldsymbol{r}_2 \implies \begin{bmatrix} \mathbf{1} & \mathbf{0} & -\mathbf{7} & | & -\mathbf{7} \\ \mathbf{0} & -\mathbf{1} & -\mathbf{4} & | & -\mathbf{4} \end{bmatrix},$$

$$\boldsymbol{r}_2 \leftarrow (-\mathbf{1})\boldsymbol{r}_2 \implies \begin{bmatrix} \mathbf{1} & \mathbf{0} & -\mathbf{7} & | & -\mathbf{7} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} & | & \mathbf{4} \end{bmatrix},$$

Here we choose z *as a free variable,* $z = s$, $s \in \mathbb{R}$.

$$\begin{aligned} z &= s \\ y + 4s &= \quad 4 => y = -4s + 4 \\ x - 7s &= -7 => x = 7s - 7. \end{aligned}$$

Therefore the general solution has the form: $\mathrm{x} = [7s - 7, -4s + 4, s]^T$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7s - 7 \\ -4s + 4 \\ s \end{bmatrix} = s \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ 4 \\ 0 \end{bmatrix}.$$

Given that the system has (at least one) solution it is <u>consistent</u> and the solution is the line in $\mathbb{R}^3$*, in which the two planes intersect.*

Example 4.5.6 *Next consider the system:*

$$\begin{aligned} \boldsymbol{x} + \mathbf{2}\boldsymbol{y} - \boldsymbol{z} &= \mathbf{2} \\ -\mathbf{2}\boldsymbol{x} - \mathbf{4}\boldsymbol{y} + \mathbf{2}\boldsymbol{z} &= \mathbf{1}. \end{aligned}$$

To investigate the solution space, we first form the augmented matrix,

$$[\boldsymbol{A} \mid \mathrm{b}] = \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{2} & -\mathbf{1} & \mathbf{2} \\ -\mathbf{2} & -\mathbf{4} & \mathbf{2} & \mathbf{1} \end{array}\right],$$

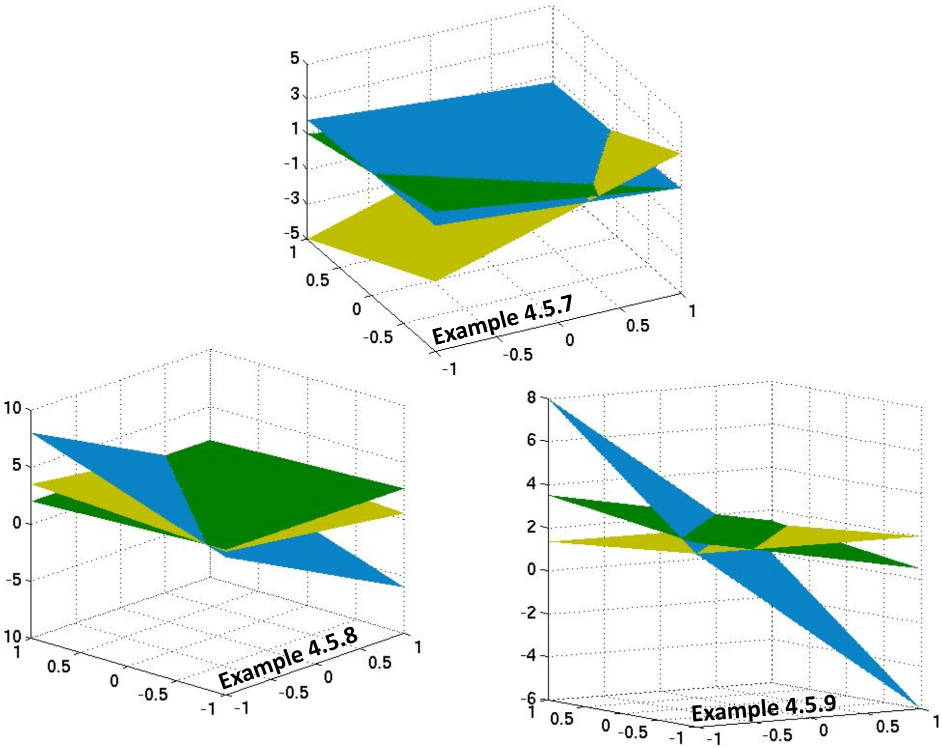
and perform the following elementary row operations:

$$\boldsymbol{r}_2 \leftarrow \boldsymbol{r}_2 + \mathbf{2}\boldsymbol{r}_1 \;\; => \left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{2} & -\mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{5} \end{array}\right].$$

The system has no solution and it is <u>inconsistent</u>. Therefore the two planes have no common point.

Three equations

Each equation has a set of solutions that is a plane. If there is only one point that is a solution of all three equations then the system is consistent and has a unique solution. This case is demonstrated by Example 4.5.7. The next examples include some cases when the intersection of planes are parallel lines. Example 4.5.8 illustrates the case when the parallel lines coincide corresponding to a consistent system of equations. In Example 4.5.9 the parallel lies are distinct and the system of equations is inconsistent. Note that these examples do not cover all possibilities (e.g. three or two planes are parallel).



Planes intersect in one point (top), in a line (bottom left) or the intersection of planes are parallel distinct lines (bottom right).

Example 4.5.7

$$\begin{aligned} x + y + z &= 1 \\ -2x + 2y + z &= -1 \\ 3x + y + 5z &= 7. \end{aligned}$$

As usual we start with forming the augmented matrix,

$$[A \mid \mathrm{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -2 & 2 & 1 & -1 \\ 3 & 1 & 5 & 7 \end{array}\right],$$

and perform the following elementary row operations:

$$\begin{array}{l} r_2 \leftarrow r_2 + 2r_1 \\ r_3 \leftarrow r_3 - 3r_1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & -2 & 2 & 4 \end{array}\right],$$

$$r_3 \leftarrow r_3 + \tfrac{1}{2}r_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 0 & \frac{7}{2} & \frac{9}{2} \end{array}\right],$$

$$r_3 \leftarrow \frac{2}{7}r_3 \;=> \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 4 & 3 & | & 1 \\ 0 & 0 & 1 & | & \frac{9}{7} \end{bmatrix},$$

$$\begin{matrix} r_2 \leftarrow r_2 - 3r_3 \\ r_1 \leftarrow r_3 \end{matrix} \;=> \begin{bmatrix} 1 & 1 & 0 & | & -\frac{2}{7} \\ 0 & 4 & 0 & | & -\frac{20}{7} \\ 0 & 0 & 1 & | & \frac{9}{7} \end{bmatrix},$$

$$r_2 \leftarrow \frac{1}{4}r_2 \;=> \begin{bmatrix} 1 & 1 & 0 & | & -\frac{2}{7} \\ 0 & 1 & 0 & | & -\frac{5}{7} \\ 0 & 0 & 1 & | & \frac{9}{7} \end{bmatrix},$$

$$r_1 \leftarrow r_1 - r_2 \;=> \begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{7} \\ 0 & 1 & 0 & | & -\frac{5}{7} \\ 0 & 0 & 1 & | & \frac{9}{7} \end{bmatrix},$$

Therefore the unique solution is

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{3}}{\mathbf{7}} \\ -\frac{\mathbf{5}}{\mathbf{7}} \\ \frac{\mathbf{9}}{\mathbf{7}} \end{bmatrix}.$$

The three planes intersect in this point. The system of equations is *<u>consistent</u>.*

Example 4.5.8

$$\begin{aligned} 2x - y + 2z &= 4 \\ -y - 3z &= -7 \\ -4x + 3y - z &= -1. \end{aligned}$$

Again, first we form the augmented matrix,

$$[A \mid \mathrm{b}] = \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & -1 & -3 & -7 \\ -4 & 3 & -1 & -1 \end{array}\right],$$

and perform the following elementary row operations:

$$r_3 \leftarrow r_3 + 2r_1 \;\; => \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & -1 & -3 & -7 \\ 0 & 1 & 3 & 7 \end{array}\right],$$

$$r_3 \leftarrow r_3 + r_2 \;\; => \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & -1 & -3 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$r_2 \leftarrow (-1)r_2 \implies \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$r_1 \leftarrow r_1 + r_2 \implies \left[\begin{array}{ccc|c} 2 & 0 & 5 & 11 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$r_1 \leftarrow \tfrac{1}{2}r_1 \implies \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & \frac{11}{2} \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

Here we choose z *as a free variable,* $z = s$*,* $s \in \mathbb{R}$*.*

$$\begin{aligned} z &= s \\ y + 3s &= 7 \implies y = -3s + 7 \\ x + \frac{5}{2}s &= \frac{11}{2} \implies x = -\frac{5}{2}s + \frac{11}{2}. \end{aligned}$$

Therefore the general solution has the form:

$\mathbf{x} = [-\frac{5}{2}s + \frac{11}{2}, -3s + 7, s]^T$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{5}{2}s + \frac{11}{2} \\ -3s + 7 \\ s \end{bmatrix} = s \begin{bmatrix} -\frac{5}{2} \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{11}{2} \\ 7 \\ 0 \end{bmatrix},$$

representing the line in which all three planes intersect. Therefore, the system of equations is <u>*consistent*</u>.

Next consider the previous system, and only change the right hand side.

Example 4.5.9

$$\begin{aligned} 2x - y + 2z &= 4 \\ -y - 3z &= -5 \\ -4x + 3y - z &= -1. \end{aligned}$$

Again, first we form the augmented matrix,

$$[A \mid \mathrm{b}] = \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & -1 & -3 & -5 \\ -4 & 3 & -1 & -1 \end{array}\right],$$

and perform the following elementary row operations:

$$r_3 \leftarrow r_3 + 2r_1 \;\; => \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & -1 & -3 & -5 \\ 0 & 1 & 3 & 7 \end{array}\right],$$

$$r_3 \leftarrow r_3 + r_2 \;\; => \left[\begin{array}{ccc|c} 2 & -1 & 2 & 4 \\ 0 & -1 & -3 & -5 \\ 0 & 0 & 0 & 2 \end{array}\right],$$

The system is <u>inconsistent</u> with no solution.