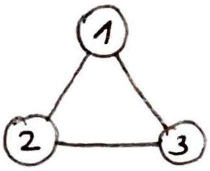
GABRIEL MOISE

PROBABILITY
PROBLEM SHEET 5

1. Let p_m be the probability that the frog is on ① after n jums. In the same manner, let q_m an s_m be the probabilities that the frog is on ② or on ③, respectively, after m jums.

Obviously, $p_m+q_m+s_m=1$, for all $m\in\mathbb{N}$.



We want to prove that $q_m=s_m$ for all $m\in\mathbb{N}$ by induction on m:

The base case $Q(0)$: $q_0=s_0$.

At first, the frog is on ①, therefore $p_0=1$, $q_0=0$ and $s_0=0 \Rightarrow q_0=s_0$.

The inductive step

IH: We know that $Q(K)$: $q_K=s_K$ and we want to prove $Q(K+1)$: $q_{K+1}=s_{K+1}$.

From the fact that the frog jumps with probability $\frac{1}{2}$ from a vertex to either of ith adjacent vertices, we know that $q_{K+1}=\frac{1}{2}(p_K+s_K)$ and $s_{K+1}=\frac{1}{2}(p_K+q_K)$. However, these two are equal because $s_K=q_K$ from IH. So, $q_{K+1}=s_{K+1}$.

Now, we have
$$p_m=\frac{1}{2}(q_{m-1}+s_{m-1})\overset{Q(n-1)}{=}q_{m-1}=\frac{p_{n-2}+s_{m-2}}{2}\overset{Q(n-2)}{=}\frac{p_{n-2}+q_{n-2}}{2}=\frac{p_{n-2}+\frac{1}{2}(q_{n-2}+s_{m-2})}{2}=$$
$$=\frac{p_{n-2}+p_{n-1}}{2}\Rightarrow 2p_m-p_{n-1}-p_{n-2}=0 \text{ for all } m\geq 2$$

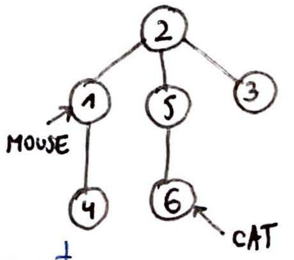
Let $p_m=A\lambda^n \Rightarrow 2A\lambda^n-A\lambda^{n-1}-A\lambda^{n-2}=0 \mid :A\lambda^{n-2}\neq 0$

$(\lambda\neq 0)$
$(A\neq 0)$
$$2\lambda^2-\lambda-1=0 \Rightarrow \lambda_1=1,\ \lambda_2=-\frac{1}{2}\Rightarrow p_m=A_1\cdot 1^m+A_2\cdot\left(-\frac{1}{2}\right)^m$$

From $p_0=1$ and $p_1=0$ we get
$$\begin{cases}A_1+A_2=1\\A_1-\frac{1}{2}A_2=0\end{cases}\Rightarrow \begin{matrix}A_1=\frac{1}{3}\\A_2=\frac{2}{3}\end{matrix}\Rightarrow \boxed{p_m=\frac{1}{3}+\frac{2}{3}\left(-\frac{1}{2}\right)^m} \text{ for all } m\in\mathbb{N}.$$

When $n\to\infty$ we have
$$\lim_{n\to\infty}p_m=\lim_{n\to\infty}\frac{1}{3}+\frac{2}{3}\cdot\left(-\frac{1}{2}\right)^m=\frac{1}{3}+\frac{2}{3}\cdot 0=\frac{1}{3}\Rightarrow\boxed{\lim_{m\to\infty}p_m=\frac{1}{3}}$$

2. The floor plan of the house can be represented by a graph with 6 nodes (each representing a room), where the edges represent the fact that the mouse can get from one room to another directly.



Let X be the number of minutes until the mouse gets in the 6th room. We write e_i for the expectation of X when the mouse starts from the i^{th} room, with $i\in\{1,2,3,4,5,6\}$. We know the mouse walks randomly, so we can say that if he has m choices for the next step, then each of them has a probability of $\frac{1}{m}$ to be followed.

We want to find e_1.

Now,
$$e_1=\frac{1}{2}(e_4+1)+\frac{1}{2}(e_2+1)$$
$$e_2=\frac{1}{3}(e_1+1)+\frac{1}{3}(e_5+1)+\frac{1}{3}(e_3+1)$$
$$e_3=e_2+1$$
$$e_4=e_1+1$$

$e_5 = \frac{1}{2}(e_2+1) + \frac{1}{2}(e_6+1)$

$e_6 = 0$ (the mouse already is in the 6^{th} room)

By solving the linear system we get $e_2 = 16, e_3 = 17, e_4 = 20, e_5 = 9, e_6 = 0$ and

$\boxed{e_1 = 19}$.

3. (Gambler's ruin, symmetric case)

The gambler starts with £m, where $m \in \{1, 2, \dots, M-1\}$, he wins £1 with probability $\frac{1}{2}$ and oses £1 with the same probability. The game ends when he reaches £0 or £M.

(a) Let X denote the amount of money the gambler ends up with. We write e_m for the expectation of X when he starts with £m. Then

$$e_m = \frac{1}{2}\mathbb{E}(X \mid \text{first game win}) + \frac{1}{2}\mathbb{E}(X \mid \text{first game lose})$$

$$e_m = \frac{1}{2}(e_{n+1}+1) + \frac{1}{2}(e_{n-1}-1)$$

$$e_n = \frac{1}{2}e_{n+1} + \frac{1}{2}e_{n-1} \Rightarrow e_{n+1} - 2e_n + e_{n-1} = 0$$

Let $e_m = A\lambda^n$, $A \neq 0$, $\lambda \neq 0$

$$\Rightarrow A\lambda^{n+1} - 2A\lambda^n + A\lambda^{n-1} = 0 \;\Big|\; : A\lambda^{n-1} \neq 0$$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda_{1/2} = 1 \Rightarrow \lambda = 1$$

Then, $e_m = (A + Bm)\lambda^m$

From $e_0 = 0$ and $e_M = M$ we have $(A + B\cdot 0)\lambda^0 = 0 \Rightarrow A = 0$ and $(A + B\cdot M)\lambda^M = M \Rightarrow$

$\Rightarrow B\cdot M\cdot 1^M = M \Rightarrow B = 1$.

Therefore, $\boxed{e_m = m}$

(b) We want to calculate the conditional probability of the fact that he won £1 on his first game, given the fact that he ends the game with £M.

these two are not independent

$$\mathbb{P}(\text{win on } 1^{st} \text{ game} \mid \text{ends with £}M) = \frac{\mathbb{P}(\{\text{win on } 1^{st} \text{ game}\} \cap \{\text{ends with £}M\})}{\mathbb{P}(\{\text{ends with £}M\})} =$$

$\leftarrow$ from lectures, this is $\frac{m}{M}$

$$= \frac{\mathbb{P}(\{\text{win on } 1^{st} \text{ game}\} \cap \{\text{ends the game with £}M\text{, starting from £}(m+1)\})}{\mathbb{P}(\{\text{ends the game with £}M\text{, starting from £}m\})} =$$

$\leftarrow$ these 2 are independent

$$= \frac{\mathbb{P}(\text{win on } 1^{st} \text{ game})\,\mathbb{P}(\text{ends with £}M\text{, starting from £}(n+1))}{\mathbb{P}(\text{ends with £}M\text{, starting from £}m)} = \frac{\frac{1}{2}\cdot\frac{m+1}{\cancel{M}}}{\frac{m}{\cancel{M}}} = \frac{m+1}{2m} \Rightarrow$$

$$\Rightarrow \boxed{\mathbb{P}(\text{win £1 on first game} \mid \text{he ends the game with £}M\text{, starting from £}m) = \frac{m+1}{2m}}$$

(c) Now X denotes the number of steps (length) until the game ends. We now write e_m for the expectation of X when the game starts from £m. Then

$$e_m = \frac{1}{2}\mathbb{E}(X \mid \text{first round is a win}) + \frac{1}{2}\mathbb{E}(X \mid \text{first round is a lose})$$

$$e_m = \frac{1}{2}(e_{m+1}+1) + \frac{1}{2}(e_{m-1}+1)$$

$$e_m = \frac{1}{2}e_{m+1} + \frac{1}{2}e_{m-1} + 1 \Rightarrow \frac{1}{2}e_{m+1} - e_m + \frac{1}{2}e_{n-1} = -1 \quad \Big| \cdot 2$$

$$e_{m+1} - 2e_m + e_{m-1} = -2$$

From (a) we know that $\boxed{W_m = m}$.

Now we'll try to find the particular solution V_m. Trying $V_m = Cm^2 + Dm$ gets us to:

$$C(m+1)^2 + D(m+1) - 2Cm^2 - 2Dm + C(m-1)^2 + D(m-1) = -2$$

$$\cancel{Cm^2} + \cancel{2Cm} + C + \cancel{Dm} + \cancel{D} - \cancel{2Cm^2} - \cancel{2Dm} + \cancel{Cm^2} - \cancel{2Cm} + C + \cancel{Dm} - \cancel{D} = -2$$

$$2C = -2 \Rightarrow \boxed{C = -1}$$

Therefore, we have $\boxed{V_m = -m^2 + Dm} \Rightarrow e_m = -m^2 + Dm + m$.

By using the boundary conditions to find D, we get $e_0 = 0$, which happens for all $D \in \mathbb{R}$, and for $e_M = 0$, we obtain $-M^2 + DM + M = 0 \quad \Big| : M \neq 0$

$$-M + D + 1 = 0 \Rightarrow D = M - 1 \Rightarrow e_m = -m^2 + (M-1)m + m \Rightarrow$$

$\Rightarrow e_m = -m^2 + Mm \Rightarrow \boxed{e_m = m(M-m)}$ for each $m \in \{1, 2, \dots, M-1\}$ (it works for $m=0$ and $m=M$, too).

We want to know for which value of m, e_m is the largest.

For that we'll consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = xM - x^2$, which is a continuous function. We have $f'(x) = M - 2x$, so for $x \in (-\infty, \frac{M}{2})$ $f'(x) > 0 \Rightarrow f(x)$ is strictly increasing on $(-\infty, \frac{M}{2})$

for $x \in (\frac{M}{2}, \infty)$ $f'(x) < 0 \Rightarrow f(x)$ is strictly decreasing on $(\frac{M}{2}, \infty)$

for $x = \frac{M}{2}$ $f'(x) = 0 \Rightarrow x = \frac{M}{2}$ is a critical point

x	$-\infty$		$\frac{M}{2}$		∞
$f'(x)$		$+ + + + + + + + +$	0	$- - - - - -$	
$f(x)$		$\nearrow$	$f(\frac{M}{2})$	$\searrow$	

Therefore, $\max(f(x)) = f(\frac{M}{2})$.

Now, coming back to the problem, we can deduce that if M is $\boxed{\text{even}}$, then $\max(e_m) = e_{\frac{M}{2}} \Rightarrow$

$\Rightarrow \boxed{\max(e_m) = \frac{M}{2} \cdot \frac{M}{2}}$ and if M is even, then we can deduce that $\max(e_n) = e_{\lfloor \frac{M}{2} \rfloor} \underset{\uparrow}{=} e_{\lceil \frac{M}{2} \rceil} \Rightarrow$

easy to prove ⊛

$\Rightarrow \boxed{\max(e_n) = \lfloor \frac{M}{2} \rfloor \cdot \lfloor \frac{M}{2} \rfloor}$

⊛ **Proof**: Let $M = 2p+1 \Rightarrow \lfloor \frac{M}{2} \rfloor = p$, $\lceil \frac{M}{2} \rceil = p+1$. $e_{\lfloor \frac{M}{2} \rfloor} = e_p = p(2p+1-p) \Rightarrow$

$\Rightarrow e_{\lfloor \frac{M}{2} \rfloor} = p(p+1)$ and $e_{\lceil \frac{M}{2} \rceil} = e_{p+1} \Rightarrow$

$\Rightarrow e_{p+1} = (p+1)(2p+1-p-1) = (p+1)p \Rightarrow e_p = e_{p+1}$

$\Rightarrow \boxed{e_{\lfloor \frac{M}{2} \rfloor} = e_{\lceil \frac{M}{2} \rceil}}$

In conclusion, $\boxed{\max(e_m) = \lfloor \frac{M}{2} \rfloor^2}$ for all $M \in \mathbb{N}$ (for M even $\lfloor \frac{M}{2} \rfloor = \frac{M}{2}$) and it happens when $\boxed{m = \lfloor \frac{M}{2} \rfloor}$.

1. (a) Let $X \sim Geom(p) \Rightarrow P_k = p(1-p)^{k-1}$, $k = 1, 2, \dots$

We have $G_X(s) = \sum_{k=1}^{\infty} s^k p(1-p)^{k-1} = \sum_{k=1}^{\infty} (sp)\left(s(1-p)\right)^{k-1} \underset{\substack{\uparrow \\ a=k-1}}{=} (ps) \sum_{a=0}^{\infty} \underset{\substack{\uparrow}}{(s(1-p))^a} = (ps) \cdot \frac{1}{1-s(1-p)}$

Therefore, $G_X(s) = \frac{ps}{1-(1-p)s}$, provided that $|s| < \frac{1}{1-p}$.

this has to be less than 1 (the absolute value), so $|s| < \frac{1}{1-p}$.

(b) $\mathbb{E}(X) = G_X'(1)$

$$G_X'(s) = \frac{p(1-s+ps) - ps(-(1-p))}{(1-s+ps)^2} = \frac{p - \cancel{ps} + \cancel{p^2 s} + \cancel{ps} - \cancel{p^2 s}}{(1-s+ps)^2} = \frac{p}{(1-s+ps)^2} \Bigg| \Rightarrow$$

$$\Rightarrow \mathbb{E}(X) = \frac{p}{(1-1+p)^2} = \frac{p}{p^2} \Rightarrow \boxed{\mathbb{E}(X) = \frac{1}{p}}$$

$$G_X''(s) = \left(\frac{p}{(1-s+ps)^2}\right)' = \frac{-p \cdot 2\,\cancel{(1-s+ps)} \cdot (-1+p)}{(1-s+ps)^4} = \frac{(-2p)(p-1)}{(1-s+ps)^3} = \frac{2p(1-p)}{(1-s+ps)^3}$$

$$Var(X) = G_X''(1) + G_X'(1) - \left(G_X'(1)\right)^2$$

$$Var(X) = \frac{2p(1-p)}{p^3} + \frac{\overset{p^2}{1}}{p} - \frac{\overset{p}{1}}{p^2} = \frac{2p - 2p^2 + p^2 - p}{p^3} = \frac{p-p^2}{p^3} \Rightarrow \boxed{Var(X) = \frac{1-p}{p^2}}$$

5. (a) A fair coin is tossed n times $\Rightarrow p(H) = p(T) = \frac{1}{2}$. Let r_n be the probability that the sequence of tosses never has a head followed by a head.

So, $r_n = \mathbb{P}(n\text{-sequence OK}) = \mathbb{P}(n\text{-sequence ok} \mid T \text{ last}) \cdot \mathbb{P}(T \text{ last}) + \mathbb{P}(n\text{-sequence ok} \mid TH \text{ last } 2) \cdot \mathbb{P}(HT \text{ last } 2) + \mathbb{P}(n\text{-sequence ok} \mid HH \text{ last } 2) \cdot \mathbb{P}(HH \text{ last } 2)$

$$r_n = \mathbb{P}((n-1)\text{-sequence ok}) \cdot \frac{1}{2} + \mathbb{P}((n-2)\text{-sequence ok}) \cdot \frac{1}{4} + 0 \cdot \frac{1}{4}$$

$\boxed{r_n = \frac{1}{2} r_{n-1} + \frac{1}{4} r_{n-2}}$ for $n \geq 2$ and $r_0 = r_1 = 1$.

$r_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$: favorable cases $= 3$ (HT, TH, TT) / total cases $= 4$ (HH, HT, TH, TT).

(b) Let X denote the number of coin tosses needed until we first get two heads in a row. We want to calculate $\mathbb{P}(X=k)$, for $k \in \mathbb{N}, k \geq 2$. So, in order to first get two heads in a row after exactly k coin tosses we need that we didn't get any HH for the first $k-3$ tosses and the last 3 results must be T, H, H. Therefore,

$\boxed{\mathbb{P}(X=k) = r_{k-3} \cdot \frac{1}{8}}$, for $k \geq 3$, and for $k=2$ it is $\mathbb{P}(X=2) = \frac{1}{4}$ (HT)

(a) $r_n - \frac{1}{2} r_{n-1} - \frac{1}{4} r_{n-2} = 0$

Let $r_n = A\lambda^n$, $A \neq 0, \lambda \neq 0$

$$\Rightarrow A\lambda^n - \frac{1}{2} A\lambda^{n-1} - \frac{1}{4} A\lambda^{n-2} = 0 \;\Big|\; : A\lambda^{n-2} \neq 0$$

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{4} = 0 \;\Big|\; \cdot 4$$

$$4\lambda^2 - 2\lambda - 1 = 0$$

$$(\Delta = 4 + 16 = 20) \Bigg| \Rightarrow \lambda_1 = \frac{1+\sqrt{5}}{4}, \; \lambda_2 = \frac{1-\sqrt{5}}{4}.$$

Therefore, $r_n = A\lambda_1^n + B\lambda_2^n$ and by using the boundary conditions we get:

$$\left\{\begin{array}{l} A+B=1 \Rightarrow B=1-A \\ A\cdot\frac{1+\sqrt5}{4}+B\cdot\frac{1-\sqrt5}{4}=1 \end{array}\right| \Rightarrow A\cdot\frac{1+\sqrt5}{4}+\frac{1-\sqrt5}{4}-A\cdot\frac{1-\sqrt5}{4}=1$$

$$A\cdot\left(\frac{1+\sqrt5}{4}-\frac{1-\sqrt5}{4}\right)=1-\frac{1-\sqrt5}{4}$$

$$A\cdot\frac{2\sqrt5}{4}=\frac{3+\sqrt5}{4} \Rightarrow A=\frac{3+\sqrt5}{2\sqrt5} \Rightarrow \boxed{A=\frac{5+3\sqrt5}{10}} \Rightarrow \boxed{B=\frac{5-3\sqrt5}{10}}$$

So, $\boxed{r_n=\frac{5+3\sqrt5}{10}\cdot\left(\frac{1+\sqrt5}{4}\right)^n+\frac{5-3\sqrt5}{10}\cdot\left(\frac{1-\sqrt5}{4}\right)^n}$

$r_2=\frac{5+3\sqrt5}{10}\cdot\frac{6+2\sqrt5}{16}+\frac{5-3\sqrt5}{10}\cdot\frac{6-2\sqrt5}{16}=\frac{30+2\cancel{8}\sqrt5+30}{160}+\frac{30-2\cancel{8}\sqrt5+30}{160}=\frac{120}{160}=\frac34$, so it's correct!

(c) $P(X=k)=\frac{r_{k-3}}{8}$ from (b), for $k\geq3$

For $k=0$ we have $P(X=0)=0$, for $k=1$ $P(X=1)=0$ and for $P(X=2)=\frac14$.

Now,

$$G_X(s)=\sum_{k=0}^{\infty}s^kP(X=k)=0\cdot s^0+0\cdot s^1+\frac14\cdot s^2+\sum_{k=3}^{\infty}s^k\cdot\frac18 r_{k-3}=\frac14 s^2+\frac18\sum_{k=3}^{\infty}s^k\cdot r_{k-3}$$

if we replace $k-3$ with a we get:

$$G_X(s)=\frac14 s^2+\frac18\sum_{a=0}^{\infty}s^{a+3}\cdot r_a=\frac14 s^2+\frac18 s^3\cdot\sum_{a=0}^{\infty}s^a\cdot r_a$$

But $r_a=A\cdot\lambda_1^a+B\cdot\lambda_2^a$, so

$$G_X(s)=\frac14 s^2+\frac18 s^3\cdot\left(A\sum_{a=0}^{\infty}(s\lambda_1)^a+B\sum_{a=0}^{\infty}(s\lambda_2)^a\right)$$

The two sums converge only if $|s\lambda_1|<1$ and $|s\lambda_2|<1$ and that implies $|s|<\frac{4}{1+\sqrt5}$.
We will need the function value (the derivative) for $s=1$, so we can calculate it.

$$G_X(s)=\frac14 s^2+\frac18 s^3\cdot\left(\frac{A}{1-s\lambda_1}+\frac{B}{1-s\lambda_2}\right)=\frac14 s^2+\frac18 s^3\cdot\frac{A(1-s\lambda_2)+B(1-s\lambda_1)}{(1-s\lambda_1)(1-s\lambda_2)}$$

Now we'll make use of these results:

$A+B=1$, $\lambda_1+\lambda_2=\frac12$, $\lambda_1\lambda_2=-\frac14$, $A\lambda_1+B\lambda_1=-\frac12$, so

$$G_X(s)=\frac14 s^2+\frac18 s^3\cdot\frac{(A+B)-s(A\lambda_2+B\lambda_1)}{1-s(\lambda_1+\lambda_2)+(\lambda_1\lambda_2)s^2}=\frac14 s^2+\frac18 s^3\cdot\frac{{}^{4/}1+\frac12 s}{1-\frac12 s-\frac14 s^2}=\frac{s^2}{4}+\frac{s^3}{8}\cdot\frac{4+2s}{4-2s-s^2}$$

$$G_X(s)=\frac{s^2}{4}+\frac{2s^4+4s^3}{-8s^2-16s+32}\backslash 2=\frac{s^2}{4}+\frac{s^4+2s^3}{-4s^2-8s+16}=\frac{s^2(-s^2-2s+4)+s^4+2s^3}{-4s^2-8s+16}$$

$$G_X(s)=\frac{-\cancel{s^4}-\cancel{2s^3}+4s^2+\cancel{s^4}+\cancel{2s^3}}{-4s^2-8s+16}\backslash 4 \Rightarrow \boxed{G_X(s)=\frac{s^2}{-s^2-2s+4}}, \text{provided that } |s|<\frac{4}{1+\sqrt5}$$

If we want to calculate the mean of X, we need
$G'_X(1)$, so we'll calculate $G'_X(s)=\left(\frac{s^2}{-s^2-2s+4}\right)'=\frac{2s(-s^2-2s+4)-s^2(-2s-2)}{(s^2+2s-4)^2}$

$$G'_X(s)=\frac{-\cancel{2s^3}-4s^2+8s+\cancel{2s^3}+2s^2}{(s^2+2s-4)^2}=\frac{-2s^2+8s}{(s^2+2s-4)^2}\Rightarrow\boxed{G'_X(s)=\frac{2s(4-s)}{(s^2+2s-4)^2}}$$

The answer I got at Q6 on Sheet 3 was $\frac{p+1}{p^2}$, where p was the probability to get heads. Here, $p=\frac{1}{2} \Rightarrow \mathbb{E}(X) = \frac{\frac{3}{2}}{\frac{1}{4}} = 6$.

Calculating $G_X'(1) = \frac{2\cdot 1\cdot 3}{(1+2-4)^2} = \frac{6}{(-1)^2} = 6 \Rightarrow \boxed{\mathbb{E}(X)=6}$ (checks with the answer!)

(d) Let Y denote the number of coin tosses needed until we first see a TH.

Proceeding the same way as before, let t_n be the probability that we didn't obtain any TH in a sequence of n tosses. Then, $t_0=t_1=1$ and

$$t_n = \mathbb{P}(n\text{-sequence OK}) = \mathbb{P}(n\text{-sequence ok} \mid \text{last } T)\,\mathbb{P}(\text{last } T) + \mathbb{P}(n\text{-sequence ok} \mid \text{last 2 } TH)\,\mathbb{P}(TH) + \mathbb{P}(n\text{-sequence ok} \mid \text{last 2 } HH)$$

↖ if we have last 2 HH, all tosses must be H

$$t_n = t_{n-1}\cdot\frac{1}{2} + 0\cdot\frac{1}{4} + \frac{1}{2^n} \Rightarrow \boxed{t_n = \frac{1}{2}t_{n-1} + \frac{1}{2^n}}, \text{ with } n \geq 1$$

Solving it, we obtain $\boxed{t_n = \frac{n+1}{2^n}}$, for all $n \in \mathbb{N}$ (can be proven by induction)

$\mathbb{P}(Y>n)$ is the probability to get the first TH in a sequence with more than n tosses ⇒ ⇒ it is the probability to not get a TH in a sequence of n tosses ⇒

$$\Rightarrow \mathbb{P}(Y>n) = t_n \Rightarrow \boxed{\mathbb{P}(Y>n) = \frac{n+1}{2^n}}$$

In the same manner, $\mathbb{P}(X>n) = r_n \Rightarrow \mathbb{P}(X>n) = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{4}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{4}\right)^n$.

We will calculate $\frac{r_n}{r_{n-1}}$, as because of the fact that when $n \to \infty$ we get close to the point where $\frac{r_n}{r_{n-1}} \approx \frac{r_{n-1}}{r_{n-2}} = a$. We have $a := \frac{r_n}{r_{n-1}} = \frac{\frac{1}{2}r_{n-1} + \frac{1}{4}r_{n-2}}{r_{n-1}} = \frac{1}{2} + \frac{1}{4}\frac{r_{n-2}}{r_{n-1}} = \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{a}$

$$a = \frac{1}{2} + \frac{1}{4a} \Rightarrow 4a^2 = 2a+1$$

$$4a^2 - 2a - 1 = 0 \Rightarrow a_{1/2} = \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

But $a>0 \Rightarrow a = \frac{1+\sqrt{5}}{4}$, so r_n decreases with a factor of $\frac{4}{1+\sqrt{5}}$ in general.

Now, $\frac{t_n}{t_{n-1}} = \frac{n+1}{2^n}\cdot\frac{2^{n-1}}{n} = \frac{n+1}{2n} \xrightarrow{n\to\infty} \frac{1}{2}$, so t_n decreases with a factor of 2.

From the fact that $2 > \frac{4}{1+\sqrt{5}}$ ($2+2\sqrt{5} > 4 \Leftrightarrow 2\sqrt{5} > 2 \Leftrightarrow \sqrt{5} > 1$ YES) we deduce that $t_n < r_n \Rightarrow \boxed{\mathbb{P}(Y>n) < \mathbb{P}(X<n)}$, which is not surprising because it is definitely more plausible to get a HH later than a TH in a configuration of tosses.

6. We consider a symmetric random walk on a cycle with N sites, labelled $0, 1, 2, \dots, N-1$. We start at 0 and from i we can go to $(i+1) \bmod N$ or $(i-1) \bmod N$ with $p = \frac{1}{2}$ (for each) independently.

We want to find the expected number of steps until every site has been visited.

(a) We want to find the expected number of steps until we have visited different m sites (the last visit being a new site). Let X_m denote the number of steps until we have visited m different sites have been visited. Therefore, let e_m be the expected number of steps until m different sites have been visited.

Notice that because from one site we can only go to one of its neighbours, the set of visited sites is always of the form $\{N-j, N-j+1, \dots, N-2, N-1, 0, 1, 2, \dots, m-j-1\}$ when we have visited m different sites, with $j \in \{0, 1, 2, \dots, n+1\}$. We can renumber them for $\{1, 2, \dots, n\}$. We now are at 1 or m (after e_m number of steps). We want to find the expected number of steps until we either get to 0 or to $(n+1)$. We can view this problem similarly to the Gambler's Ruin problem. You start from 1 or m, the ground level is 0, the top level is $m+1$.

From 3.c), the expected number of steps is $1(m + \cancel{1} - \cancel{1}) = m(\cancel{m} + 1 - \cancel{m}) = m$, therefore we have $\boxed{e_{m+1} = e_m + m}$, for $m \in \mathbb{N}_+$ and $e_1 = 0$. So, $\boxed{e_m = \frac{(m-1)m}{2}}$, and as we want to see the number of steps needed for us to visit all the N different sites, then the result is $\boxed{e_N = \frac{(N-1)N}{2}}$.

(b) Now we want to calculate the probability that k is the last site to be visited, for each $k = 1, 2, \dots, N-1$.

$$\mathbb{P}(\text{last site was } k) = \mathbb{P}(\text{last site was } k \mid \text{before, we visited } k-1)\,\mathbb{P}(\text{before, we visited } k-1) + \mathbb{P}(\text{last site was } k \mid \text{before, we visited } k+1)\ \mathbb{P}(\text{before, we visited } k+1).$$

① $\mathbb{P}(\text{last site was } k \mid \text{before, we visited } k-1) = ?$

So, now we are in $k-1$ and we want to get to k, but without going through the way with $k+1 \to k$. For an easier reasoning, we'll renumber the sites (we'll add k to each, and then mod m). So, instead of having $0, 1, 2, \dots, N-1$, we'll have $k, k+1, \dots, 0, \dots, k-1$. We'll also use the fact that $\mathbb{P}(\text{last visited was } k) = \mathbb{P}(\text{last visited was } N-k)$, as the directions in which we go through the cycle are left or right and they equal probability.

Now, we basically want to get to the site $\underset{0}{\underset{\shortparallel}{(N-K+K)}}$ from site k, by going through 1 (its right). We can see this as a gambler's ruin game, where the bottom is 0, we are at 1, the top is N (we want to win N). So the probability to win here is $\frac{1}{N}$.

② $\mathbb{P}(\text{before, we visited } k-1) = ?$

In our reordering, we want to first get to $N-K+1+K$, which is 1. So, the bottom is now 1, we are at K, the top is N, so basically we can say that we want to lose a game where the bottom is 0, the position is $k-1$ and the top is $N-1$, where the probability is $\left(1 - \frac{K-1}{N-1}\right)$

③ $\mathbb{P}$(last site was K | before, we visited $K+1$) = ?

So, now we are at $N-1$ and we want to get to 0 from its left (or to N if we replace it with N, as $N \bmod N = 0$ anyways). So, we want to lose the game where the ground is 0, our position is $N-1$ and the top is N (we don't want to get to N). Therefore, the probability is $1 - \frac{N-1}{N} = \frac{1}{N}$.

④ $\mathbb{P}$(before, we visited $K+1$) = ?

In our reordering, we first want to get to $N-1$, starting from K, without going through 0. So, we want to win the game where the ground is 0, our position is K and the top is $N-1$, so the probability is $\frac{K}{N-1}$.

Therefore, our probability is:

$$\mathbb{P}(\text{last site was } K) = \frac{1}{N}\left(1 - \frac{K-1}{N-1}\right) + \frac{1}{N}\cdot\frac{K}{N-1} = \frac{1}{N}\left(1 - \frac{K-1}{N-1} + \frac{K}{N-1}\right) = \frac{1}{\cancel{N}}\cdot\frac{\cancel{N}}{N-1} = \boxed{\frac{1}{N-1}}.$$

As K takes values from 1 to $N-1$, we conclude that there's an equal probability for any site to be visited last (starting from site 0).

The reordering transformed the sequence:

$$K-2, K-1, \overset{\leftarrow \text{FINISH}}{\boxed{K}}, \underset{\substack{\uparrow \\ \text{Case 2}}}{K+1}, K+2, \dots, N-1, \overset{\leftarrow \text{START}}{0}, 1, \dots, \underset{\substack{\uparrow \\ \text{Case 1}}}{K-1}, \overset{\leftarrow \text{FINISH}}{\boxed{K}}, K+1$$

into:

$$N-2, N-1, \overset{\leftarrow \text{FINISH}}{\boxed{0}}, \underset{\substack{\uparrow \\ \text{Case 1}}}{1}, 2, \dots, K-1, \overset{\leftarrow \text{START}}{K}, K+1, \dots, \underset{\substack{\uparrow \\ \text{Case 2}}}{N-1}, \overset{\leftarrow \text{FINISH}}{\boxed{0}}, 1$$