Linear Algebra — Week 7

1. Let λ be an eigenvalue of the matrix A.

   (a) Prove that λ^k is an eigenvalue of A^k for $k = 2, 3, 4, \ldots$.

   (b) If A is non-singular prove that $1/\lambda$ is an eigenvalue of A^{-1}.

   (c) Prove that, for any scalar α, $\lambda + \alpha$ is an eigenvalue of the matrix $A + \alpha\mathcal{I}$.

2. Suppose the matrix B is given by $B = A^\top A$ for some matrix A.

   (a) Explain, using a result from the lectures, why the eigenvalues of B are real.

   (b) Prove that the eigenvalues of B are non-negative. [Hint: you may want to diagonalise B, and then consider $\|A\mathbf{v}\|^2$].

3. Suppose the matrix B is given by $B = A^\top A$ for some non-singular matrix A.

   (a) Prove that the eigenvalues of B are positive.

   (b) Let $\mathbf{v}$ be any vector. Prove that

   $$\lambda_{\min}\|\mathbf{v}\|^2 \leq \mathbf{v}^\top B\mathbf{v} \leq \lambda_{\max}\|\mathbf{v}\|^2,$$

   where $\lambda_{\min}$ is the minimum eigenvalue of B, and $\lambda_{\max}$ is the maximum eigenvalue of B.

   (c) Prove that some $\mathbf{w}$ exists such that

   $$\mathbf{w}^\top B\mathbf{w} = \lambda_{\max}\|\mathbf{w}\|^2,$$

   and another $\mathbf{y}$ exists such that

   $$\mathbf{y}^\top B\mathbf{y} = \lambda_{\min}\|\mathbf{y}\|^2.$$

   We will use these results in a later lecture.

4. For the following matrices, find S such that $S^{-1}AS$ is diagonal, or explain why no such S exists.

   (a) $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

   (b) $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

5. In all subparts of this question, the matrix A is given by

$$A = \begin{pmatrix} -\frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{3}{4} \end{pmatrix}.$$

(a) Find an orthogonal matrix P and a diagonal matrix D such that $D = P^\top AP$.

(b) Write down an expression for A^n, where n is a positive integer.

(c) Write down an expression for A^n as n becomes very large. Hint: you will need to consider the cases n odd and n even separately.

(d) Suppose $\mathbf{u}$ is an eigenvector corresponding to the eigenvalue with smallest modulus. Explain why $A^n\mathbf{u}$ will approach zero as n increases.

(e) Use the matrices P and D to find the general solution of the differential equations given by

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= -\frac{3}{4}y_1 + \frac{1}{4}y_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= \frac{1}{4}y_1 - \frac{3}{4}y_2 \end{aligned}$$