LINEAR ALGEBRA

GABRIEL MOISE

MT 18

WEEK 5

1.

(a) $\det\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = 3+2=5$ (b) $\det\begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \\ 4 & 5 & 1 \end{pmatrix} = 3+20+12-24+15+2=28$

(c) $\det\begin{pmatrix} 1 & -1 & 2 \\ 3 & 6 & -1 \\ 4 & 5 & 1 \end{pmatrix} = 6+30+4-48+5+3=0$ (d) $\det\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{pmatrix} = 3-24-4-4=-29$

(e) $\det\begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 2 & 1 & -3 \\ 0 & 4 & 0 & 1 \end{pmatrix} \overset{2^{nd}\text{ definition of a determinant}}{=} 2\cdot\det\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & -3 \\ 4 & 0 & 1 \end{pmatrix} = 2\cdot(3-24-4-4) = 2\cdot(-29) = -58$

(f) $\det\begin{pmatrix} 2 & 1 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\text{upper triangular matrix}}{=} 24$

2. $Q^TQ = I_n \Rightarrow \det(Q^T\cdot Q) = \det(I_n) \Rightarrow \det(Q^T)\cdot\det(Q) = 1 \Rightarrow (\det(Q))^2 = 1 \Rightarrow$
$\Rightarrow \boxed{\det(Q) = \pm 1}$

3. $A = \begin{bmatrix} 4-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}$

$\det(A) = \det\begin{pmatrix} 4-\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda)+2 = 12-7\lambda+\lambda^2+2 = \lambda^2-7\lambda+14$

A is singular $\Leftrightarrow \det(A)=0 \Leftrightarrow \lambda^2-7\lambda+14=0$ $(\Delta = 49-56=-7)$ $\Leftrightarrow \boxed{\lambda = \frac{7\pm i\sqrt{7}}{14}}$

4. Firstly, we'll prove that if every row of A sums to 0, then $\det(A)=0$. If we add the first column to the last column and then the second column to the last column... and then the (last − 1) column to the last column, then each element of the last column will be 0 (as every row sums to 0) and the determinant will be unchanged (Property 8). From property 5, that would imply that $\det(A)=0$.

Secondly, we'll prove that if every row of A sums to 1, then $\det(A-I)=0$. By substracting I from A, we get to the matrix $B = A-I$, whose rows sum to 0, as we know that I has a 1 value on each row, and each row of A sums to 1. Therefore, as we've already proven, $\det(B)=0$, or $\det(A-I)=0$. Let $A = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$, with the sum on each row equal to 1. However, $\det(A)=5$, therefore $\det(A)$ is not always 1 if each row of A sums to 1.

5. We have $A,B \in \mathbb{R}^{n\times n}$ with $AB=-BA \Rightarrow \det(AB) = \det(-BA) \Rightarrow$

$\Rightarrow \det(A)\det(B) = (-1)^n\det(B)\det(A) \Rightarrow (1-(-1)^n)\det(A)\det(B)=0.$ (*)

If n is even, then $1-(-1)^n = 1-1=0 \Rightarrow$ (*) always takes place so neither of A or B has to be singular.

If $\boxed{n \text{ is odd}}$, then $1-(-1)^n = 1+1=2 \Rightarrow 2\det(A)\det(B)=0 \Rightarrow$ at least one of the determinants is 0, therefore at least one of A and B must be singular.

6. By using the PLU factorisation, you obtain $PA = LU$, therefore $\det(PA) = \det(LU) \Rightarrow$
$\Rightarrow \det(P)\det(A) = \det(L)\det(U)$

Knowing the fact that $P^{-1} = P^T \Rightarrow \det(P^{-1}) = \det(P^T) = \det(P) \Rightarrow \det(P) = \det(P^{-1})$
But, $\det(P)\cdot\det(P^{-1}) = \det(I) = 1$ $\Big\} \Rightarrow$

$\Rightarrow (\det(P))^2 = 1 \Rightarrow \det(P) = \pm 1$, depending if the permutation is even (then $\det(P) = 1$), or odd (then $\det(P) = -1$).
Also, $\det(L) = L_{11}\cdot L_{22}\cdot\ldots\cdot L_{NN}$, as L is lower triangular (Property 6) and $\det(U) = U_{11}\cdot U_{22}\cdot\ldots\cdot U_{NN}$, as U is upper triangular (Property 6). Therefore, $\det(A) = \pm(L_{11}\cdot L_{22}\cdot\ldots\cdot L_{NN})(U_{11}\cdot U_{22}\cdot\ldots\cdot U_{NN})$ and if the PLU factorisation fails, we know that A is singular, so $\det(A) = 0$.

7. The area of the triangle with nodes $(1,2)$, $(2,3)$ and $(5,5)$ is

$$\left|\frac{1}{2}\det\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 5\\ 2 & 3 & 5\end{pmatrix}\right| = \left|\frac{1}{2}(10+3+10-4-15-5)\right| = \left|\frac{1}{2}(-1)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

8. The equation of the plane that passes through the points $(0,0,0)$, $(1,1,1)$ and $(2,4,6)$ is

$$\det\begin{pmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & x\\ 0 & 1 & 4 & y\\ 0 & 1 & 6 & z\end{pmatrix} = \det\begin{pmatrix}1 & 2 & x\\ 1 & 4 & y\\ 1 & 6 & z\end{pmatrix} = 4z + 6x + 2y - 4x - 6y - 2z = 2x - 4y + 2z = 0 \Rightarrow$$

$\Rightarrow \boxed{x - 2y + z = 0}$

9. $a = (1 \;\; 2 \;\; 3)$ $\quad b = (1 \;\; 2 \;\; 4)$

$$a \times b = \begin{pmatrix} i & j & k\\ 1 & 2 & 3\\ 1 & 2 & 4\end{pmatrix} = 8i + \cancel{2k} + 3j - \cancel{2k} - 6i - 4j = 2i - j = 2\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}.$$

10. U and V are vector spaces
We define $T: U \to V$ to be a linear transformation if the following conditions are satisfied:
① if $u \in U$, then $T(u) \in V$
② if $u_1, u_2 \in U$, then $T(u_1 + u_2) = T(u_1) + T(u_2)$
③ if $u \in U$ and c is any scalar, then $T(cu) = cT(u)$

(a) We want to prove that ② and ③ are equivalent to

(*) if $u_1, u_2 \in U$ and c is any scalar, then $T(u_1 + cu_2) = T(u_1) + cT(u_2)$

②,③ $\Rightarrow$ (*): As $u_2 \in U$ and $c \in \mathbb{R} \Rightarrow cu_2 \in U$ (Closure under scalar multiplication)
Therefore $u_1, cu_2 \in U \overset{②}{\Rightarrow} \underline{T(u_1 + cu_2)} = T(u_1) + T(cu_2) \overset{③}{=} \underline{T(u_1) + cT(u_2)}$, which is exactly (*)

②,③ $\Leftarrow$ (*) For $c = 1 \in \mathbb{R}$, we get $\underline{T(u_1 + u_2) = T(u_1) + T(u_2)}$, for all $u_1, u_2 \in U$, which is exactly ②
Now, we know that U is a vector space, so it has a zero vector, denoted 0.
Then, $T(0 + c0) = T(0) + cT(0)$
$\cancel{T(0)} = \cancel{T(0)} + cT(0) \Rightarrow cT(0) = 0$ for all $c \in \mathbb{R} \Rightarrow \underline{T(0) = 0}$

10.

$$\overset{A}{\begin{pmatrix}1&4&7\\2&5&8\\3&6&10\end{pmatrix}}\overset{x}{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}=\overset{b}{\begin{pmatrix}2\\1\\3\end{pmatrix}}$$

$$A=\begin{pmatrix}1&4&7\\2&5&8\\3&6&10\end{pmatrix}$$

$$E_1A=\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&4&7\\2&5&8\\3&6&10\end{pmatrix}=\begin{pmatrix}1&4&7\\0&-3&-6\\3&6&10\end{pmatrix}$$

$$E_2E_1A=\begin{pmatrix}1&0&0\\0&1&0\\-3&0&1\end{pmatrix}\begin{pmatrix}1&4&7\\0&-3&-6\\3&6&10\end{pmatrix}=\begin{pmatrix}1&4&7\\0&-3&-6\\0&-6&-11\end{pmatrix}$$

$$E_3E_2E_1A=\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\begin{pmatrix}1&4&7\\0&-3&-6\\0&-6&-11\end{pmatrix}=\begin{pmatrix}1&4&7\\0&-3&-6\\0&0&1\end{pmatrix}=U$$

$$L=E_1^{-1}E_2^{-1}E_3^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\3&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}=\begin{pmatrix}1&0&0\\2&1&0\\3&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}=\begin{pmatrix}1&0&0\\2&1&0\\3&2&1\end{pmatrix}$$

$$\left.\begin{matrix}Ax=b\\A=LU\end{matrix}\right| \Rightarrow LUx=b$$

$$Ly=b\Rightarrow\begin{cases}y_1=2\\2y_1+y_2=1\\3y_1+2y_2+y_3=3\end{cases}\Big|\Rightarrow y_2=-3\Big|\Rightarrow y_3=3$$

$$Ux=y\Rightarrow\begin{cases}x_1+4x_2+7x_3=2\\-3x_2-6x_3=-3\\x_3=3\end{cases}\Big|\Rightarrow x_2=-5\Big|\Rightarrow x_1=1\Rightarrow x=\begin{pmatrix}1\\-5\\3\end{pmatrix}.$$

11.

$$\overset{A}{\begin{pmatrix}2&1&4\\4&2&1\\4&1&5\end{pmatrix}}\overset{x}{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}=\overset{b}{\begin{pmatrix}7\\7\\10\end{pmatrix}}_c\quad\Big|\cdot P=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

$$\overset{PA}{\begin{pmatrix}2&1&4\\4&1&5\\4&2&1\end{pmatrix}}\overset{x}{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}}=\begin{pmatrix}7\\10\\7\end{pmatrix}$$

$$E_1(PA)=\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2&1&4\\4&1&5\\4&2&1\end{pmatrix}=\begin{pmatrix}2&1&4\\0&-1&-3\\4&2&1\end{pmatrix}$$

$$E_2E_1(PA)=\begin{pmatrix}1&0&0\\0&1&0\\-2&0&1\end{pmatrix}\begin{pmatrix}2&1&4\\0&-1&-3\\4&2&1\end{pmatrix}=\begin{pmatrix}2&1&4\\0&-1&-3\\0&0&-7\end{pmatrix}=U$$

$$L=E_1^{-1}E_2^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\2&1&0\\2&0&1\end{pmatrix}=L$$

$$\left.\begin{matrix}PAx=c\\PA=LU\end{matrix}\right|\Rightarrow LUx=c$$

$$Ly=c\Rightarrow\begin{cases}y_1=7\\2y_1+y_2=10\\2y_1+y_3=7\end{cases}\Big\|\begin{matrix}\Rightarrow y_2=-4\\\Rightarrow y_3=-7\end{matrix}$$

$$Ux=y\Rightarrow\begin{matrix}2x_1+x_2+4x_3=7\\-x_2-3x_3=-4\\-7x_3=-7\Rightarrow x_3=1\end{matrix}\Big|\Rightarrow x_2=1\Big|\Rightarrow x_1$$

So, $x=\begin{pmatrix}1\\1\\1\end{pmatrix}$.