PROBABILITY

GABRIEL MOISE

PROBLEM SHEET 4

1. We have $\mathbb{P}(X=k) = \frac{1}{n}$ for $k = 1, 2, \dots, n$.

The mean of X is

$$\mathbb{E}(X) = \sum_{k=1}^{n} k\,\mathbb{P}(X=k) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{\sum_{k=1}^{n} k}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

The variance of X is

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

$$\mathbb{E}(X^2) = \sum_{k=1}^{n} k^2\,\mathbb{P}(X=k) = \sum_{k=1}^{n} k^2 \cdot \frac{1}{n} = \frac{\sum_{k=1}^{n} k^2}{n} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}.$$

Then, $\mathrm{Var}(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4} = \frac{4n^2+\cancel{6n}+2-3n^2-\cancel{6n}-3}{12} = \frac{n^2-1}{12}.$

2.

Y \ X	−1	0	1
−1	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{2}{27}$
0	$\frac{2}{27}$	$\frac{6}{27}$	$\frac{1}{27}$
1	$\frac{3}{27}$	$\frac{2}{27}$	$\frac{4}{27}$

The marginal distribution of X is

$$P_X(x) = \sum_{y\in\{-1,0,1\}} P_{X,Y}(x,y) \Rightarrow P_X(-1) = P_{X,Y}(-1,-1) + P_{X,Y}(-1,0) + P_{X,Y}(-1,1)$$

$$P_X(-1) = \frac{1}{27} + \frac{2}{27} + \frac{3}{27} = \frac{6}{27}$$

$$\boxed{P_X(-1) = \frac{2}{9}}$$

$$P_X(0) = P_{X,Y}(0,-1) + P_{X,Y}(0,0) + P_{X,Y}(0,1)$$

$$P_X(0) = \frac{6}{27} + \frac{6}{27} + \frac{2}{27}$$

$$\boxed{P_X(0) = \frac{14}{27}}$$

$$P_X(1) = P_{X,Y}(1,-1) + P_{X,Y}(1,0) + P_{X,Y}(1,1)$$

$$P_X(1) = \frac{2}{27} + \frac{1}{27} + \frac{4}{27}$$

$$\boxed{P_X(1) = \frac{7}{27}}$$

The marginal distribution of Y is

$$P_Y(-1) = P_{X,Y}(-1,-1) + P_{X,Y}(0,-1) + P_{X,Y}(1,-1)$$

$$P_Y(-1) = \frac{1}{27} + \frac{6}{27} + \frac{2}{27} = \frac{9}{27}$$

$$\boxed{P_Y(-1) = \frac{1}{3}}$$

$$P_Y(0) = P_{X,Y}(-1,0) + P_{X,Y}(0,0) + P_{X,Y}(1,0)$$

$$P_Y(0) = \frac{2}{27} + \frac{6}{27} + \frac{1}{27} = \frac{9}{27}$$

$$\boxed{P_Y(0) = \frac{1}{3}}$$

$$P_Y(1) = P_{X,Y}(-1,1) + P_{X,Y}(0,1) + P_{X,Y}(1,1)$$

$$P_Y(1) = \frac{3}{27} + \frac{2}{27} + \frac{4}{27} = \frac{9}{27}$$

$$\boxed{P_Y(1) = \frac{1}{3}}$$

The covariance of X and Y is

$$\operatorname{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\mathbb{E}(X) = \sum_{x\in\{-1,0,1\}} x\, p_X(x) = (-1)\cdot\frac{2}{9} + 0\cdot\frac{14}{27} + 1\cdot\frac{7}{27} = \frac{7}{27} - \frac{6}{27} = \frac{1}{27}$$

$$\mathbb{E}(Y) = \sum_{y\in\{-1,0,1\}} y\, p_Y(y) = (-1)\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = 0$$

$$\mathbb{E}(XY) = \sum_{x\in\{-1,0,1\}}\ \sum_{y\in\{-1,0,1\}} xy\, p_{X,Y}(x,y) = (-1)\cdot(-1)\cdot\frac{1}{27} + (-1)\cdot 0\cdot\frac{2}{27} + (-1)\cdot 1\cdot\frac{3}{27} + 0\cdot(-1)\cdot\frac{6}{27} + 0\cdot 0\cdot\frac{6}{27} +$$

$$+ 0\cdot 1\cdot\frac{2}{27} + 1\cdot(-1)\cdot\frac{2}{27} + 1\cdot 0\cdot\frac{1}{27} + 1\cdot 1\cdot\frac{4}{27} = \frac{1}{27} - \frac{3}{27} - \frac{2}{27} + \frac{4}{27} = 0.$$

Therefore, $\operatorname{cov}(X,Y) = 0 - \frac{1}{27}\cdot 0 \Rightarrow \boxed{\operatorname{cov}(X,Y)=0}$

The two discrete random variables X and Y are independent if

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y), \text{ for all } x,y\in\{-1,0,1\}$$

However, for $x=y=0$ we have

$p_{X,Y}(0,0) = \frac{6}{27}$, but $p_X(0)\,p_Y(0) = \frac{14}{27}\cdot\frac{1}{3} = \frac{14}{81}$ and these two are not equal.

Therefore, X and Y are <u>not</u> independent.

3. X and Y are independent random variables and

$X \sim P_o(\lambda)$, $Y \sim P_o(\mu)$

(a) $\mathbb{P}(X=k, Y=m) = p_{X,Y}(k,m) \overset{\text{X and Y are independent}}{=} p_X(k)\, p_Y(m) = \mathbb{P}(X=k)\,\mathbb{P}(Y=m) = \frac{e^{-\lambda}\lambda^k}{k!}\cdot\frac{e^{-\mu}\mu^m}{m!}$

(b) $\mathbb{P}(X+Y=n) \overset{\text{disjoint events}}{=} \sum_{k=0}^{n} \mathbb{P}(X=k, Y=n-k) \overset{\text{independence}}{=} \sum_{k=0}^{n} \mathbb{P}(X=k)\cdot\mathbb{P}(Y=n-k) = \sum_{k=0}^{n} \frac{e^{-\lambda}\cdot\lambda^k}{k!}\cdot\frac{e^{-\mu}\cdot\mu^{n-k}}{(n-k)!} =$

$$= \frac{\sum_{k=0}^{n}\frac{\lambda^k\cdot\mu^{n-k}}{k!\cdot(n-k)!}}{e^{\lambda+\mu}} = \frac{\sum_{k=0}^{n}\binom{n}{k}\cdot\lambda^k\mu^{n-k}}{n!\cdot e^{\lambda+\mu}} = \frac{(\lambda+\mu)^n}{n!\cdot e^{\lambda+\mu}}$$ and this is a Poisson distribution

$(X+Y) \sim P_o(\lambda+\mu)$.

(for $k\in(-\infty,0)\cup(n,\infty)$ we have probability 0)

(c) $\mathbb{P}(X=k \mid X+Y=n) = \frac{\mathbb{P}(\{X=k\}\cap\{X+Y=n\})}{\mathbb{P}(X+Y=n)} \overset{\text{independence}}{=} \frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)} = \frac{\cancel{e^{-\lambda}}\lambda^k}{k!}\cdot\frac{\cancel{e^{-\mu}}\mu^{n-k}}{(n-k)!}\cdot\frac{n!\cdot\cancel{e^{\lambda+\mu}}}{(\lambda+\mu)^n} =$

$$= \frac{n!}{k!(n-k)!}\cdot\frac{\lambda^k\cdot\mu^{n-k}}{(\lambda+\mu)^n} = \binom{n}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\cdot\left(\frac{\mu}{\lambda+\mu}\right)^{n-k} \Bigg| \Rightarrow$$

If we say $p = \frac{\lambda}{\lambda+\mu} \in [0,1] \Rightarrow \frac{\mu}{\lambda+\mu} = 1-p$

$\Rightarrow \mathbb{P}(X=k \mid X+Y=n) = \binom{n}{k}p^k\cdot(1-p)^{n-k} \Rightarrow$ this has a binomial distribution: $(X \mid X+Y=n) \sim Bin\left(n, \frac{\lambda}{\lambda+\mu}\right)$

d) $\mathbb{E}(X|X+Y=n)$

We know that $(X|X+Y=n) \sim Bin(n, \frac{\lambda}{\lambda+\mu})$ and a binomial distribution $Bin(n,p)$ has the mean equal to np (from sheet 3 - exercise 1).

Therefore, $\mathbb{E}(X|X+Y=n) = \frac{n\lambda}{\lambda+\mu}$.

4. Let X and Y be independent random variables with $X \sim Geom(p)$ and $Y \sim Geom(p)$.

a) $\mathbb{P}(X=k | X+Y=n+1)$, for $k \in \{1, 2, \dots, n\}$

$$\mathbb{P}(X=k|X+Y=n+1) = \frac{\mathbb{P}(\{X=k\} \cap \{X+Y=n+1\})}{\mathbb{P}(X+Y=n+1)} = \frac{\mathbb{P}(\{X=k\} \cap \{Y=n-k+1\})}{\mathbb{P}(X+Y=n+1)} \overset{X \text{ and } Y \text{ are independent r.v.}}{=} \frac{\mathbb{P}(X=k)\mathbb{P}(Y=n-k+1)}{\mathbb{P}(X+Y=n+1)}$$

$$\mathbb{P}(X+Y=n+1) \overset{\text{disjoint events}}{=} \sum_{k=1}^{n} \mathbb{P}(X=k)\mathbb{P}(Y=n-k+1) = \sum_{k=1}^{n} p(1-p)^{k-1} \cdot p(1-p)^{n-k} = p^2 \sum_{k=1}^{n} (1-p)^{n-1} = n p^2 (1-p)^{n-1}$$

Now, $\mathbb{P}(X=k|X+Y=n+1) = \frac{\cancel{p}(1-p)^{k-1} \cdot \cancel{p}(1-p)^{n-k}}{n \cancel{p^2}(1-p)^{n-1}} = \frac{(1-p)^{n-1}}{n(1-p)^{n-1}} \Rightarrow$

$$\Rightarrow \boxed{\mathbb{P}(X=k | X+Y=n+1) = \frac{1}{n}}$$

b) From Sheet 3 - exercise 3 a) we have $X \sim Geom(p) \Rightarrow \boxed{\mathbb{P}(X>k) = (1-p)^k}$ (same for Y)

We want to calculate:

$$\mathbb{P}(\min\{X,Y\}=k) = \mathbb{P}\Big((\{X=k\} \cap \{Y>k\}) \cup (\{X>k\} \cap \{Y=k\}) \cup (\{X=k\} \cap \{Y=k\})\Big) \leftarrow \text{disjoint events}$$

$$\mathbb{P}(\min\{X,Y\}=k) = \mathbb{P}(X=k, Y>k) + \mathbb{P}(X>k, Y=k) + \mathbb{P}(X=k, Y=k) \leftarrow X,Y \text{ independent}$$

$$\mathbb{P}(\min\{X,Y\}=k) = \mathbb{P}(X=k)\mathbb{P}(Y>k) + \mathbb{P}(X>k)\mathbb{P}(Y=k) + \mathbb{P}(X=k)\mathbb{P}(Y=k)$$

$$\mathbb{P}(\min\{X,Y\}=k) = p(1-p)^{k-1} \cdot (1-p)^k + (1-p)^k p(1-p)^{k-1} + p(1-p)^{k-1} p(1-p)^{k-1}$$

$$\mathbb{P}(\min\{X,Y\}=k) = 2p(1-p)^{2k-1} + p^2(1-p)^{2k-2}$$

$$\mathbb{P}(\min\{X,Y\}=k) = (1-p)^{2k-2}(2p(1-p)+p^2)$$

$$\mathbb{P}(\min\{X,Y\}=k) = (1-p)^{2k-2}(2p-2p^2+p^2)$$

$$\mathbb{P}(\min\{X,Y\}=k) = (1-p)^{2k-2}(2p-p^2)$$

$$\boxed{\mathbb{P}(\min\{X,Y\}=k) = p(2-p)(1-p)^{2k-2}}$$

6. Let X and Y be discrete random variables. We want to show that (i) and (ii) are equivalent:

(i) X and Y are independent if $\mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\mathbb{P}(Y=y)$ for all $x, y \in \mathbb{R}$

(ii) X and Y are independent if $\mathbb{P}(X\in A, Y\in B) = \mathbb{P}(X\in A)\mathbb{P}(Y\in B)$ for all $A, B \subseteq \mathbb{R}$

(i) $\Rightarrow$ (ii)

$$\mathbb{P}(X\in A, Y\in B) = \mathbb{P}(\{X\in A\}\cap\{Y\in B\}) = \mathbb{P}\left(\left(\bigcup_{x\in A}\{X=x\}\right)\cap\left(\bigcup_{y\in B}\{Y=y\}\right)\right) \overset{\text{distributivity}}{=}$$

$$= \mathbb{P}\left(\bigcup_{x\in A, y\in B}(\{X=x\}\cap\{Y=y\})\right) \overset{\text{disjoint events}}{=} \sum_{x\in A, y\in B}\mathbb{P}(\{X=x\}\cap\{Y=y\}) = \sum_{x\in A, y\in B}\mathbb{P}(X=x, Y=y) \overset{(i)}{=}$$

$$\overset{(i)}{=} \sum_{x\in A, y\in B}\mathbb{P}(X=x)\mathbb{P}(Y=y) = \sum_{x\in A}\sum_{y\in B}\mathbb{P}(X=x)\mathbb{P}(Y=y) = \sum_{x\in A}\mathbb{P}(X=x)\sum_{y\in B}\mathbb{P}(Y=y) =$$

$$\overset{\text{disjoint events}}{=} \mathbb{P}\left(\bigcup_{x\in A}\{X=x\}\right)\mathbb{P}\left(\bigcup_{y\in B}\{Y=y\}\right) = \mathbb{P}(X\in A)\mathbb{P}(Y\in B)$$

(ii) $\Rightarrow$ (i)

For all $x, y \in \mathbb{R}$, we can choose $A=\{x\}$ and $B=\{y\}$, therefore

$$\mathbb{P}(X=x, Y=y) = \mathbb{P}(X\in A, Y\in B) \overset{(ii)}{=} \mathbb{P}(X\in A)\mathbb{P}(Y\in B) = \mathbb{P}(X=x)\mathbb{P}(Y=y).$$

Now, we know that X and Y are independent and we want to prove that for any functions $f, g: \mathbb{R}\to\mathbb{R}$, we have $f(X)$ and $g(Y)$ independent.

We'll start with:

$$\mathbb{P}(f(X)=x) = \mathbb{P}(X=a, f(a)=x) \overset{\text{independence}}{=} \mathbb{P}(X=a)\mathbb{P}(f(a)=x) = \begin{cases} 0, \text{ if } f(a)\neq x \\ \mathbb{P}(X=a), \text{ if } f(a)=x \end{cases} \Rightarrow$$

$$\Rightarrow \boxed{\mathbb{P}(f(X)=x) = \sum_{f(a)=x}\mathbb{P}(X=a)} \quad ①$$

In the same manner we can get to $\boxed{\mathbb{P}(g(Y)=y) = \sum_{g(b)=y}\mathbb{P}(Y=b)}$ ②.

Now,

$$\mathbb{P}(f(X)=x, g(Y)=y) = \mathbb{P}(X=a, f(a)=x, Y=b, g(b)=y) \overset{\text{independence}}{=} \mathbb{P}(X=a)\mathbb{P}(f(a)=x)\mathbb{P}(Y=b)\mathbb{P}(g(b)=y) =$$

$$= \begin{cases} 0, \text{ if } f(a)\neq x \text{ OR } g(b)\neq y \\ \mathbb{P}(X=a)\mathbb{P}(Y=b) \text{ if } f(a)=x \text{ AND } g(b)=y \end{cases} \Rightarrow \underline{\mathbb{P}(f(X)=x, g(Y)=y)} = \sum_{\substack{f(a)=x \\ g(b)=y}}\mathbb{P}(X=a)\mathbb{P}(Y=b) =$$

$$= \sum_{f(a)=x}\mathbb{P}(X=a)\cdot\sum_{g(b)=y}\mathbb{P}(Y=b) \overset{①②}{=} \underline{\mathbb{P}(f(X)=x)\mathbb{P}(g(Y)=y)}.$$

As this happens for all $x, y \in \mathbb{R}$, we proved that $f(X)$ and $g(Y)$ are independent for any functions $f, g: \mathbb{R}\to\mathbb{R}$.

7.

(a) $u_{n+1} = 3u_n + 2 \Rightarrow u_{n+1} - 3u_n = 2$ with $u_0 = 0$

The homogeneous equation is $w_{n+1} = 3w_n$ with the general solution $w_n = 3^n w_0 = 3^n A$

By trying $v_n = c$ as a particular solution to the initial equation, we get

$$v_{n+1} - 3v_n = 2$$

$$c - 3c = 2 \Rightarrow c = -1$$

So, the general solution is $u_n = 3^n A - 1$. Using the boundary value $u_0 = 0$, we get that

$3^0 \cdot A - 1 = 0 \Rightarrow A = 1 \Rightarrow \boxed{u_n = 3^n - 1}$

(b) $u_{n+1} = 2u_n + n \Rightarrow u_{n+1} - 2u_n = n$ with $u_0 = 1$

The homogeneous equation is $w_{n+1} = 2w_n$ with the general solution $w_n = 2^n w_0 = 2^n A$.

By trying $v_n = Cn + D$

$$C(n+1) + D = 2Cn + 2D + n$$

$$Cn + C + D = 2Cn + 2D + n$$

$$Cn + D + n - C = 0$$

$$(C+1)n + (D-C) = 0 \text{ for all } n \Rightarrow \left\{\begin{array}{l} C+1=0 \Rightarrow C=-1 \\ D-C=0 \end{array}\right| \Rightarrow D = -1$$

So, $v_n = -n - 1$

Therefore, the general solution to the initial equation is

$$u_n = -n - 1 + 2^n A$$

Using the boundary value, we get $-0 - 1 + 2^0 \cdot A = 1$

$$A = 2 \Rightarrow u_n = -n - 1 + 2^{n+1}$$

$$\boxed{u_n = 2^{n+1} - n - 1}$$

(c) $u_{n+1} - 5u_n + 6u_{n-1} = 2$, with $u_0 = u_1 = 1$

The homogeneous equation is $w_{n+1} - 5w_n + 6w_{n-1} = 0$. We try $w_n = \lambda^n$, so we get $(\lambda \neq 0)$

$\lambda^{n+1} - 5\lambda^n + 6\lambda^{n-1} = 0 \quad | : (\lambda^{n-1} \neq 0)$

$\lambda^2 - 5\lambda + 6 = 0 \rightarrow$ the auxiliary equation

$(\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$

The general solution to the H.E. is $w_n = A\lambda_1^n + B\lambda_2^n = A2^n + B3^n$

Now, we try to find the particular solution. If we take $v_n = C$ we get

$$C - 5C + 6C = 2 \Rightarrow 2C = 2 \Rightarrow C = 1 \Rightarrow v_n = 1$$

Therefore, the general solution is $u_n = 1 + A \cdot 2^n + B \cdot 3^n$.

Using $u_0 = u_1 = 1$, we get $\left.\begin{array}{l} 1 = 1 + A \cdot 2^0 + B \cdot 3^0 \Rightarrow A + B = 0 \\ 1 = 1 \cdot A \cdot 2^1 + B \cdot 3^1 \Rightarrow 2A + 3B = 0 \end{array}\right| \Rightarrow A = B = 0 \Rightarrow$

$\Rightarrow \boxed{u_n = 1}$

(d) $u_{n+1} - 3u_n + 2u_{n-1} = 1$, with $u_0 = u_1 = 0$

The homogeneous equation is $w_{n+1} - 3w_n + 2w_{n-1} = 0$ and we try $w_n = \lambda^n$, $\lambda \neq 0$, so we get

$\lambda^{n+1} - 3\lambda^n + 2\lambda^{n-1} = 0 \quad | : \lambda^{n-1} \neq 0$

$\lambda^2 - 3\lambda + 2 = 0$ – the auxiliary equation

$(\lambda-1)(\lambda-2) = 0 \Rightarrow \lambda_1 = 1, \ \lambda_2 = 2$

The general solution to the H.E. is $w_n = A + 2^n \cdot B$

Now, we try to find the particular solution. If we take $v_n = Cn$, we get

$C(n+1) - 3Cn + 2C(n-1) = 1$

$\cancel{Cn} + C - 3\cancel{Cn} + 2\cancel{Cn} - 2C = 1$

$-C = 1 \Rightarrow C = -1 \Rightarrow$ the particular solution is $v_n = -n$.

Therefore, the general solution is $u_n = A + 2^n \cdot B - n$

By using the boundary conditions, we get:

$$\left.\begin{array}{lll} A + 2^0 \cdot B - 0 = 0 & \Rightarrow & A + B = 0 \\ A + 2^1 \cdot B - 1 = 0 & \Rightarrow & A + 2B = 1 \end{array}\right| \Rightarrow B = 1 \text{ and } A = -1 \Rightarrow$$

$$\Rightarrow \boxed{u_n = 2^n - n - 1}$$

5. (a) Let X_i denote the number of typos that appear on page $\underbrace{i}_{i \in \{1,2,\dots,n\}}$. We know that $X_i \sim Po(\lambda)$, so $\mathbb{P}(X_i = k) = \frac{e^{-\lambda}\lambda^k}{k!}$. Now, for each X_i let I_i be the indicator of X_i, with

$$I_i = \begin{cases} 0 & \text{if } X_i > 0 \Rightarrow \text{probability of } 1-p \\ 1 & \text{if } X_i = 0 \Rightarrow \text{probability of } p \end{cases}$$

where $p = \mathbb{P}(X_i = 0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$

Let Y denote the number of pages with 0 typos. Obviously, after our notations, $Y = I_1 + I_2 + \dots + I_n$, as each I_i is 1 for a page with no typo, and 0 otherwise

Additionally, $\mathbb{E}(Y) = \mathbb{E}(I_1 + I_2 + \dots + I_n) = \mathbb{E}(I_1) + \mathbb{E}(I_2) + \dots + \mathbb{E}(I_n)$

As $I_i \sim Ber(e^{-\lambda}) \Rightarrow \mathbb{E}(I_i) = e^{-\lambda}$ for all $i \in \{1, 2, \dots, n\}$ $\Bigg| \Rightarrow$

$\Rightarrow \mathbb{E}(Y) = n \cdot e^{-\lambda}$ or $\boxed{\mathbb{E}(Y) = \frac{n}{e^{\lambda}}}$

(b) We detect a typo with probability p. If M denotes the number of typos on a specific page $\Rightarrow M \sim Po(\lambda)$ and D denotes the number of typos we detect on that page, then we can say that $\boxed{\mathbb{P}(D = k \mid M = m) = \binom{m}{k} p^k (1-p)^{m-k}}$ as we have m trials and k successes, so this is a binomial distribution with m trials and probability p.

$$\mathbb{P}(D = k) = \sum_{m=0}^{\infty} \mathbb{P}(D = k \mid M = m)\,\mathbb{P}(M = m) = \sum_{m=k}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} \frac{e^{-\lambda}\lambda^m}{m!} =$$

for $m<k$ the probability is 0

$$= \sum_{m=k} \frac{\cancel{m!}}{k!(m-k)!} \cdot p^k (1-p)^{m-k} \cdot \frac{e^{-\lambda}\lambda^m}{\cancel{m!}} = \frac{p^k}{k! \cdot e^{\lambda}} \sum_{m=\infty}^{\infty} \frac{(1-p)^{m-k} \cdot \lambda^m}{(m-k)!} =$$

$$= \frac{p^k}{k!\, e^{\lambda}} \cdot \sum_{a=0}^{\infty} \frac{(1-p)^a \cdot \lambda^{a+k}}{a!} = \frac{p^k \cdot \lambda^k}{k!\, e^{\lambda}} \sum_{a=0}^{\infty} \frac{[(1-p)\lambda]^a}{a!} = \frac{p^k \lambda^k}{k!\, e^{\lambda}} e^{\lambda - \lambda p} =$$

$\uparrow$

$a = m - k$

$$= \frac{p^k \lambda^k \cancel{e^{\lambda}}}{k! \cdot \cancel{e^{\lambda}} e^{\lambda p}} = \frac{(p\lambda)^k e^{-(p\lambda)}}{k!} \Rightarrow \Delta \sim P_o(p\lambda).$$