DISCRETE MATHEMATICS GABRIEL MOISE

MT 2018

PROBLEM SHEET 1

Chapter 1: Sets

1.1. (i) $\{n \mid n \in \mathbb{Z} \text{ and } n^2 < 0\} = \emptyset$ (all perfect squares are non-negative)

(ii) $\{n \mid n \in \mathbb{N} \text{ and } n^4 - 3n^2 + 2n = 0\} = \{0; 1\}$

$n^4 - 3n^2 + 2n = 0$

$n(n^3 - 3n + 2) = 0$

$n(n-1)^2(n+2) = 0 \Rightarrow n \in \{0; 1; -2\}$. But, $(-2) \in \mathbb{N}$, so we don't count it too.

(iii) $\{n^2 - n \mid n \in \mathbb{Z}_5\} = \{0; 1; 2\}$ $\quad 0^2 - 0 = 0 \quad 2^2 - 2 = 2 \quad 4^2 - 4 = 2$ $\quad 1^2 - 1 = 1 \quad 3^2 - 3 = 1$

(iv) $\{1\} \cup (\bigcup_{i=2}^{\infty} A_i)$, $A_i = \{2i; 3i; \ldots\}$, $\mathcal{U} = \mathbb{N}_+$

Each natural number can be prime or composite. Let's take a prime number p. It can only be obtained from $p \cdot 1$ or $1 \cdot p$. However, there is no A_i, $i \geq 2$, which contains it because each A_i starts from $2i$ and contains all multiples of i. On the other hand, a composite number has a divisor d, which is neither itself, nor the number 1. Let's say x composite number $\Rightarrow \exists d \in \{2; 3; \ldots; x-1\}$ so that $x = d \cdot d'$, where d' is also a divisor of x, $d' = \frac{x}{d}$. So, $x \in A_d$ and $x \in A_{d'}$.

So, $\{1\} \cup (\bigcup_{i=2}^{\infty} A_i) = \{x \in \mathbb{N} \mid x \text{ is composite or } x = 1\} = \{x \in \mathbb{N}_+ \mid x \text{ is not prime}\}$

1.2. $|A| = m$, $|B| = n$

(i) $|A \cup B|$

The maximum value for $|A \cup B|$ is $(m+n)$, which happens when A and B are disjoint. The minimum value for $|A \cup B|$ is when $A \subseteq B$ or $B \subseteq A$, so it is $\max(m, n)$.

(ii) $|A \cap B|$

The maximum value is when $A \subseteq B$ or $B \subseteq A$, so it is $\min(m, n)$, and the minimum value is 0, when A and B are disjoint

(iii) $|A \setminus B|$

The maximum value is m, when A and B are disjoint, and the minimum is 0, when $A \subseteq B$.

(iv) $|A \oplus B|$

The maximum value is $m+n$, when A and B are disjoint, and the minimum value is 0, when $A = B$.

(v) $|A \times B|$

The cartesian product creates pairs between the elements of A and of B, and

$|A \times B| = |A| \cdot |B|$, for all A and B. So the maximum and minimum values of $|A \times B|$ are $m \cdot n$ in both cases. (the pairs are formed of one element of A and one of B; supposing there exist two equal pairs $(x_1, y_1), (x_2, y_2)$ would mean that x_1 and x_2 are equal and both in A, which is impossible, as A is a set)

(vi) $|\mathcal{P}(A)|$

So, $A = \{a_1; a_2; \dots; a_m\}$

We will assign each subset $M \subseteq A$ a sequence of m 0's and 1's this way: if a_i is in M, then the i^{th} element of the sequence is 1, otherwise it's 0. This way, we can easily understand that there are 2^m sequences, thus 2^m subsets (each position in the sequence can be 0 or 1, so 2 possibilities)

$|\mathcal{P}(A)| = 2^m$ if $|A| = m$, so the maximum and minimum values of $|\mathcal{P}(A)|$ are both 2^m.

1.3. (i) $A \oplus A = A$

From the definition of symmetric difference: $A \oplus B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \notin A \text{ and } x \in B)\}$

So, $A \oplus A = \emptyset$, so (i) is False

(ii) $A \oplus B = B \oplus A$, which is True from the definition

(iii) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$, which is True because LHS and RHS both describe the set of all elements x which are in only one set (A, B or C)

(iv) $A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$, which is False because for $A = \{1; 2; 3; 4\}$ $B = \{2; 4; 5; 7\}$; $C = \{2; 4; 5; 8; 9\}$: LHS $= \{1; 2; 3; 4; 7; 8; 9\}$ and RHS $= \{7; 8; 9\}$

(v) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$, which is True because they both describe the set of all elements from A that are also either in B, or in C (but not in both!)

1.4. <u>Claim</u>: $A \subseteq B$ and $A \subseteq C \iff A \subseteq B \cap C$

"$\Rightarrow$": $A \subseteq B$ and $A \subseteq C \Rightarrow$ every element of A is in both B and $C \Rightarrow$ every element of A is in $B \cap C \Rightarrow A \subseteq B \cap C$

"$\Leftarrow$": $A \subseteq B \cap C \Rightarrow$ every element of A is in $B \cap C \Rightarrow$ every element of A is in both B and $C \Rightarrow A \subseteq B$ and $A \subseteq C$

In conclusion, $A \subseteq B$ and $A \subseteq C \iff A \subseteq B \cap C$

Let's take $A = \{1; 2\}$; $B = \{1; 2; 3; 4\}$; $C = \{1; 2; 5; 6\}$

We have that $A \subset B$ and $A \subset C$, but $B \cap C = A$, so $A \subset B \cap C$ would imply $A \subset A$, which is False.

1.5. (i) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

(ii) $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

We will prove (i):

M is an element from $\mathcal{P}(A\cap B) \Leftrightarrow$ M is a subset of $A\cap B \Leftrightarrow M\subseteq A$ and $M\subseteq B \Leftrightarrow M\in \mathcal{P}(A)$ and $M\in \mathcal{P}(B) \Leftrightarrow M\in \mathcal{P}(A)\cap\mathcal{P}(B) \Leftrightarrow$ M is an element from $\mathcal{P}(A)\cap\mathcal{P}(B)$

We obtained that M is an element of $\mathcal{P}(A\cap B) \Leftrightarrow$ M is an element from $\mathcal{P}(A)\cap\mathcal{P}(B)$; so that means $\mathcal{P}(A\cap B)=\mathcal{P}(A)\cap\mathcal{P}(B)$, so (i) is True.

For (ii) we will give a counterexample:

$A=\{1\}$; $B=\{2\} \Rightarrow \mathcal{P}(A)=\{\emptyset,\{1\}\}$, $\mathcal{P}(B)=\{\emptyset,\{2\}\}$, $A\times B=\{(1,2)\}$

$$\left.\begin{array}{l}\mathcal{P}(A\times B)=\{\emptyset,\{(1,2)\}\}\\ \mathcal{P}(A)\times\mathcal{P}(B)=\{(\emptyset,\emptyset),(\emptyset,\{2\}),(\{1\},\emptyset),(\{1\},\{2\})\}\end{array}\right| \Rightarrow \mathcal{P}(A\times B)\neq\mathcal{P}(A)\times\mathcal{P}(B),$$

so (ii) is False.

1.6 $A\setminus((C\cap A)\cup B)=A\setminus(B\cup C)$

$A\setminus((C\cap A)\cup B)=(A\setminus(C\cap A))\cap(A\setminus B)$ (De Morgan's laws)

$(A\setminus(C\cap A))\cap(A\setminus B)=((A\setminus C)\cup(A\setminus A))\cap(A\setminus B)$ (De Morgan's laws)

$((A\setminus C)\cup(A\setminus A))\cap(A\setminus B)=(A\setminus C)\cap(A\setminus B)$ $(A\setminus A=\emptyset)$

$(A\setminus C)\cap(A\setminus B)=A\setminus(C\cup B)$ (De Morgan's laws)

$A\setminus(C\cup B)=A\setminus(B\cup C)$ (Commutativity of "$\cup$")

So, $A\setminus((C\cap A)\cup B)=A\setminus(B\cup C)$

1.7. $A\setminus(B\cap C)\subseteq(A\setminus B)\cap(A\setminus C)$:

$x\in A\setminus(B\cap C) \Rightarrow x\in A$ and $x\notin B\cap C$ Correct!

$\Rightarrow x\in A$ and $x\notin B$ and $x\notin C$ Wrong! (x can be in $B\setminus C$ or in $C\setminus B$)

<u>Counterexample</u>: $A=\{1;4;5;6\}$

$B=\{2;4;6;7\}$

$C=\{3;5;6;7\}$

$A\setminus(B\cap C)=A\setminus\{6;7\}=\{1;4;5\}$

$(A\setminus B)\cap(A\setminus C)=\{1;5\}\cap\{1;4\}=\{1\}$

if $A\setminus(B\cap C)\subseteq(A\setminus B)\cap(A\setminus C)$ was True, that would mean that $\{1;4;5\}\subseteq\{1\}$, which is False!