Linear Algebra MT 2018 Gabriel Moise

Week 1

Chapter 1: Vectors and Vector Spaces

1. $u, v, w \in \mathbb{R}^n$, $c \in \mathbb{R}$

a) $u \cdot v + w$ makes no sense because $u \cdot v$ is a real number, whereas w is a n-dimensional vector, so we can't add these two together

b) $u \cdot (v \cdot w)$ makes no sense because $v \cdot w$ is a real number and we can't calculate a dot product between an n-dimensional vector u and a real number.

c) $c \cdot (u + w)$ makes no sense because $u+w$ is a n-dimensional vector, c is a scalar, and we can't calculate the dot product between them.

2. $u, v \in \mathbb{R}^2/\mathbb{R}^3$

$$\|u+v\| = \|u\| + \|v\|$$

First of all, we will treat the case where $u, v \in \mathbb{R}^2$. So, $u = \begin{bmatrix} a \\ b \end{bmatrix}$, $v = \begin{bmatrix} c \\ d \end{bmatrix}$, with $a, b, c, d \in \mathbb{R}$ We need $\|u+v\| = \|u\| + \|v\|$, which can be rewritten as:

$\sqrt{(a+c)^2+(b+d)^2} = \sqrt{a^2+b^2} + \sqrt{c^2+d^2}$ and because the radicals are positive, we can square the equality

$$(a+c)^2+(b+d)^2 = a^2+b^2+2\sqrt{(a^2+b^2)(c^2+d^2)}+c^2+d^2$$

$$\cancel{a^2}+2ac+\cancel{c^2}+\cancel{b^2}+2bd+\cancel{d^2} = \cancel{a^2}+\cancel{b^2}+2\sqrt{a^2c^2+a^2d^2+b^2c^2+b^2d^2}+\cancel{c^2}+\cancel{d^2} \quad \Big/ :2$$

$$ac+bd = \sqrt{a^2c^2+a^2d^2+b^2c^2+b^2d^2}$$

For this equality to work, we need $\boxed{ac+bd \geq 0}$, as the right term of the equality is ≥ 0.

With this condition we can continue by squaring the equality:

$$\cancel{a^2c^2}+2abcd+\cancel{b^2d^2} = \cancel{a^2c^2}+a^2d^2+b^2c^2+\cancel{b^2d^2}$$

$$a^2d^2-2abcd+b^2c^2 = 0$$

$(ad-bc)^2 = 0 \iff ad-bc=0$, so we also need that $\boxed{ad=bc}$

Now we will treat the case where $u, v \in \mathbb{R}^3$

We have $\|u+v\| = \|u\| + \|v\| \quad \Big/ ()^2$

$$(u+v)(u+v) = u^2+2\|u\|\cdot\|v\|+v^2$$

$$\cancel{u^2}+2u\cdot v+\cancel{v^2} = \cancel{u^2}+2\|u\|\cdot\|v\|+\cancel{v^2} \quad \Big/ :2$$

$$u \cdot v = \|u\|\cdot\|v\|$$

We will replace u with $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and v with $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ($a, b, c, x, y, z \in \mathbb{R}$) and we'll obtain:

$ax+by+cz=\sqrt{a^2+b^2+c^2}\cdot\sqrt{x^2+y^2+z^2}$

So, the first condition is that $\boxed{ax+by+cz\geq 0.}$

By squaring the equation we get:

$$\cancel{a^2x^2}+\cancel{b^2y^2}+\cancel{c^2z^2}+2abxy+2bcyz+2acxz=\cancel{a^2x^2}+a^2y^2+a^2z^2+b^2x^2+\cancel{b^2y^2}+b^2z^2+c^2x^2+c^2y^2+\cancel{c^2z^2}$$

$$(a^2y^2+b^2x^2-2abxy)+(b^2z^2+c^2y^2-2bcyz)+(a^2z^2+c^2x^2-2acxz)=0$$

$$(ay-bx)^2+(bz-cy)^2+(az-cx)^2=0$$

As $(ay-bx)^2\geq 0$, $(bz-cy)^2\geq 0$, $(az-cx)^2\geq 0$ $\Rightarrow \boxed{ay=bx};\ \boxed{bz=cy};\ \boxed{az=cx}$

3. $\lambda\in\mathbb{R}$, $u,v\in\mathbb{R}^m$

$$0\leq\|u-\lambda v\|^2$$

For $\lambda=\frac{\|u\|}{\|v\|}\in\mathbb{R}$ we get $\left\|u-\frac{\|u\|}{\|v\|}v\right\|\geq 0 \quad |()^2$

$$\left\|u-\frac{\|u\|}{\|v\|}v\right\|^2\geq 0$$

$$\left(u-\frac{\|u\|}{\|v\|}v\right)^2\geq 0$$

$$u^2-2\cdot u\cdot\frac{\|u\|}{\|v\|}v+\frac{\|u\|^2}{\|v\|^2}v^2\geq 0 \quad /:\|u\|^2>0$$

$$\underbrace{\frac{u^2}{\|u\|^2}}_{1}-2\cdot\frac{u}{\|u\|}\cdot\frac{v}{\|v\|}+\underbrace{\frac{v^2}{\|v\|^2}}_{1}\geq 0\Rightarrow 2\geq 2\cdot\frac{u\cdot v}{\|u\|\cdot\|v\|} \quad |:2$$

$$\|u\|\cdot\|v\|\geq u\cdot v$$

<u>Case 1:</u> $u\cdot v\geq 0\Rightarrow |u\cdot v|=u\cdot v\Rightarrow\|u\|\cdot\|v\|>|u\cdot v|$

<u>Case 2:</u> $u\cdot v<0\to|u\cdot v|=-u\cdot v$

In this case we will choose $\lambda=-\frac{\|u\|}{\|v\|}$ and we will proceed the same way reaching

$$\frac{u^2}{\|u\|^2}+2\cdot\frac{u}{\|u\|}\cdot\frac{v}{\|v\|}+\frac{v^2}{\|v\|^2}\geq 0\to 2\geq-2\cdot\frac{u\cdot v}{\|u\|\cdot\|v\|} \quad |:2$$

$$\|u\|\cdot\|v\|\geq -u\cdot v=|u\cdot v|$$

So, in both cases we have $\|u\|\cdot\|v\|\geq|u\cdot v|$, which is the Cauchy-Schwartz inequality.

To prove the triangle inequality, we start with

$$\|u+v\|^2=(u+v)\cdot(u+v)=u^2+2u\cdot v+v^2=\|u^2\|+\|v\|^2+2u\cdot v$$

Because $u\cdot v\leq|u\cdot v|$, we can continue with

$$\|u\|^2+\|v\|^2+2\cdot u\cdot v\leq\|u\|^2+\|v\|^2+2|u\cdot v| \quad \Big|\Rightarrow$$

From Cauchy-Schwartz inequality we have $|u\cdot v|\leq\|u\|\cdot\|v\|$

$\Rightarrow \|u\|^2 + \|v\|^2 + 2|u \cdot v| \leq \|u\|^2 + \|v\|^2 + 2\|u\| \cdot \|v\| = (\|u\| + \|v\|)^2$

So, we got that $\|u+v\|^2 \leq (\|u\| + \|v\|)^2$

All lengths are >0, so we can apply $\sqrt{(\ )}$

$\|u+v\| \leq \|u\| + \|v\|$, which is the triangle inequality.

4. Let's say that the first line (L_1) has two points $A(x_1, y_1)$ and $B(x_2, y_2)$. The vector $\vec{AB}$, which is part of the line L_1 is $[x_2 - x_1, y_2 - y_1]^T$. We replace $(x_2 - x_1)$ with a and $(y_2 - y_1)$ with b. So, $\vec{AB} = [a, b]^T$. We do the same thing for line 2, with $\vec{CD} = [c, d]^T$

We have to prove that if the two slopes of the lines m_1 and m_2 have the property $m_1 m_2 = -1$ then $L_1 \perp L_2$ and vice-versa:

"$\Rightarrow$": We know that $m_1 = \frac{b}{a}$ and $m_2 = \frac{d}{c}$, thus $\frac{bd}{ac} = -1 \Rightarrow bd = -ac \Rightarrow ac + bd = 0$

The angle between L_1 and L_2 is the same as the angle between $\vec{AB}$ and $\vec{CD}$. Using the formula of calculating the cosine of that angle, we get

$$\cos\theta = \cos\left(\widehat{[a,b]^T, [c,d]^T}\right) = \frac{ac+bd}{\sqrt{a^2+c^2} \cdot \sqrt{b^2+d^2}} = 0 \Rightarrow \vec{AB} \perp \vec{CD} \Rightarrow L_1 \perp L_2$$

"$\Leftarrow$": We know that $L_1 \perp L_2 \Rightarrow \vec{AB} \perp \vec{CD} \Rightarrow [a,b]^T \perp [c,d]^T \Rightarrow \cos\left(\widehat{[a,b]^T, [c,d]^T}\right) = 0$

$$\Rightarrow \frac{ac+bd}{\sqrt{a^2+c^2} \cdot \sqrt{b^2+d^2}} = 0 \Rightarrow ac + bd = 0 \Rightarrow bd = -ac \Rightarrow \frac{bd}{ac} = -1 \Rightarrow \frac{b}{a} \cdot \frac{d}{c} = -1 \Rightarrow m_1 \cdot m_2 = -1.$$

There is a different discussion when $m_1 = 0$ and m_2 is infinite, as the product $m_1 \cdot m_2$ cannot be defined.

5. We start by defining the angle θ between two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ as the angle between $\vec{n_1}$ and $\vec{n_2}$, where $\vec{n_1}$ and $\vec{n_2}$ are the normal vectors of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively: $\vec{n_1} = [\alpha_1, \beta_1, \gamma_1]^T$, $\vec{n_2} = [\alpha_2, \beta_2, \gamma_2]^T$

$\mathcal{P}_1: a_1 x + b_1 y + c_1 z = d_1$

$\mathcal{P}_2: a_2 x + b_2 y + c_2 z = d_2$

We know that $\vec{n_1}$ is perpendicular to any vector $\vec{PX}$ from $\mathcal{P}_1$, with $P = [x_0, y_0, z_0]^T$ and $X = [x, y, z]^T$. So, $\vec{PX} = [x - x_0, y - y_0, z - z_0]^T$

From the equation of the plane $\mathcal{P}_1$ we know that $\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_1 x_0 + b_1 y_0 + c_1 z_0 = d_1 \end{cases} \Bigg| \overset{(-)}{\Longrightarrow}$

$$\left.\begin{array}{l} \Rightarrow a_1(x - x_0) + b_1(y - y_0) + c_1(z - z_0) = 0 \\ \vec{n_1} \perp \vec{PX} \Leftrightarrow \vec{n_1} \cdot \vec{PX} = 0 \Leftrightarrow \alpha_1(x - x_0) + \beta_1(y - y_0) + \gamma_1(z - z_0) = 0 \end{array}\right| \Rightarrow \begin{array}{l} \alpha_1 = a_1 \\ \beta_1 = b_1 \\ \gamma_1 = c_1 \end{array}$$

So, $\vec{n_1} = [a_1, b_1, c_1]^T$ and $\vec{n_2} = [a_2, b_2, c_2]^T$ (same reasoning)

By calculating the angle θ between $\vec{n_1}$ and $\vec{n_2}$ we get:

$$\cos\theta = \frac{\vec{m_1}\cdot\vec{m_2}}{\|m_1\|\,\|m_2\|} = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}}$$

① If $\cos\theta = 0 \Rightarrow$ the two planes are perpendicular

② If $\cos\theta = \pm1 \Rightarrow$ the two planes are parallel

③ If $\cos\theta \in (-1,0)\cup(0,1)$ the planes are neither perpendicular, nor parallel

We have $\mathcal{P}_1 : 4x - y + 5z = 2 \Rightarrow \vec{m_1} = [4,-1,5]^T$

a) $\mathcal{P}_2 : 2x+3y-z = 1 \Rightarrow \vec{m_2} = [2,3,-1]^T$

$$\cos\theta_a = \frac{\vec{m_1}\cdot\vec{m_2}}{\|m_1\|\|m_2\|} = \frac{8-3-5}{\sqrt{42\cdot14}} = 0 \Rightarrow ①$$

b) $\mathcal{P}_2 : 4x - y + 5z = 0 \Rightarrow \vec{m_2} = [4,-1,5]^T$

$$\cos\theta_b = \frac{\vec{m_1}\cdot\vec{m_2}}{\|m_1\|\cdot\|m_2\|} = \frac{16+1+25}{\sqrt{42\cdot42}} = \frac{42}{42} = 1 \Rightarrow ②$$

c) $\mathcal{P}_2 : x - y - z = 3 \Rightarrow \vec{m_2} = [1,-1,-1]^T$

$$\cos\theta_c = \frac{\vec{m_1}\cdot\vec{m_2}}{\|m_1\|\|m_2\|} = \frac{4+1-5}{\sqrt{42\cdot3}} = 0 \Rightarrow ①$$

d) $\mathcal{P}_2 : 4x + 6y - 2z = 0 \Rightarrow \vec{m_2} = [4,6,-2]^T$

$$\cos\theta_d = \frac{\vec{m_1}\cdot\vec{m_2}}{\|m_1\|\|m_2\|} = \frac{16-6-10}{\sqrt{42\cdot56}} = 0 \Rightarrow ①$$

6. For a set V to be a vector space, it needs to respect the 10 axioms that apply for $(\forall)\ u,v,w \in V$ and $\alpha,\beta \in \mathbb{R}$:

(1) $u+v \in V$
(2) $u+v = v+u$
(3) $(u+v)+w = u+(v+w)$
(4) $(\exists)\ 0 \in V$ such that $u+0 = u$
(5) $(\forall)\ u \in V\ (\exists)\ (-u) \in V$ such that $u+(-u) = 0$
(6) $\alpha u \in V$
(7) $\alpha(u+v) = \alpha u + \alpha v$
(8) $(\alpha+\beta)u = \alpha u + \beta u$
(9) $\alpha(\beta u) = (\alpha\beta)u$
(10) $1u = u$

a) $V = \mathbb{R}^2$ with standard addition and:

$$\alpha\begin{bmatrix}x\\y\end{bmatrix} := \begin{bmatrix}x\\ \alpha y\end{bmatrix}$$

Because we have that standard addition: $(x,y)+(z,t) = (x+z, y+t)$ the following axioms are true: (1), (2), (3), (4), (5)

(6) is true because $\begin{bmatrix}x\\ \alpha y\end{bmatrix} \in \mathbb{R}^2, (\forall)\ x,y,\alpha \in \mathbb{R}$

(7) $\alpha(u+v) = \alpha([u_1,u_2]^T + [v_1,v_2]^T) = \alpha[u_1+v_1, u_2+v_2]^T = \begin{bmatrix}u_1+v_1\\ \alpha(u_2+v_2)\end{bmatrix}$

$\alpha u + \alpha v = \alpha[u_1,u_2]^T + \alpha[v_1,v_2]^T = \begin{bmatrix}u_1\\ \alpha u_2\end{bmatrix} + \begin{bmatrix}v_1\\ \alpha v_2\end{bmatrix} = \begin{bmatrix}u_1+v_1\\ \alpha(u_2+v_2)\end{bmatrix}$ $\Bigg|\Rightarrow$ OK

(8) $(\alpha+\beta)u = \begin{bmatrix}u_1\\ (\alpha+\beta)u_2\end{bmatrix}$

$\alpha u + \beta u = \begin{bmatrix}u_1\\ \alpha u_2\end{bmatrix} + \begin{bmatrix}u_1\\ \beta u_2\end{bmatrix} = \begin{bmatrix}2u_1\\ (\alpha+\beta)u_2\end{bmatrix}$ $\Bigg|\Rightarrow u_1 = 2u_1$ False

So, this set with the standard addition and chosen scalar multiplication is <u>not</u> a vector space.

b) $V = \mathbb{R}^2$ with standard scalar multiplication $\Rightarrow \alpha[x, y]^T = [\alpha x, \alpha y]^T$

and $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}$

The (1) axiom is True, as $\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$

However, for the (2) axiom we have

$$\left.\begin{array}{l} u + v = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \\ v + u = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_1 \end{bmatrix} \end{array}\right| \to$$

these two are not equal for every v and u from $\mathbb{R}^2$, so $V = \mathbb{R}^2$ with the standard scalar multiplication and chosen addition is <u>not</u> a vector space

c) $n \geq 1, n \in \mathbb{Z}$, $\mathcal{P}^n$: the set of all polynomials with maximum degree n or less with real coefficients.

So, all polynomials of degree n or less have the form:

$P(x): a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, with $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$

Because the operations are the standard addition and scalar multiplication, we have axioms (1), (2), (3) obviously True, we have $0 = 0x^n + 0x^{n-1} + \dots + 0x + 0 \in \mathcal{P}^n$ the zero element, we have the opposite of $P(x)$, which is $-p(x) = -a_n x^n - a_{n-1} x^{n-1} - \dots - a_1 x - a_0$, the axioms (6), (7), (8), (9) are True because we always obtain polynomials with a degree between 0 and n, inclusively, and also $1 = 0x^n + 0x^{n-1} + \dots + 0x + 1 \in \mathcal{P}^n$.

So, $\mathcal{P}^n$ with standard addition and scalar multiplication is a vector space

d) The difference from c) is that now it is required that the degree of every polynomial is exactly n, then $a_n \neq 0$. But, there are cases here when axiom (1) is not True, let's take $P_1(x): a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ and $P_2(x): b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$, with $a_n + b_n = 0$, $a_n \neq 0$, $b_n \neq 0$

We need $P_1(x) + P_2(x)$ to be a polynomial of degree exactly n, but we get $P_1(x) + P_2(x) = \underbrace{(a_n + b_n)}_{= 0} x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \dots + (a_1 + b_1) x + (a_0 + b_0)$, which isn't. So, axiom (1) is False, therefore this set with standard addition and scalar multiplication is <u>not</u> a vector space.

7. a) $\mathcal{P}^{n-1} \subseteq \mathcal{P}^n$

$\mathcal{P}^{n-1}$ is a subspace of $\mathcal{P}^n$ $\iff$ for all $P(x), Q(x) \in \mathcal{P}^{n-1}$ $P(x)+Q(x) \in \mathcal{P}^{n-1}$ and for any given scalar α we have $\alpha P(x) \in \mathcal{P}^{n-1}$ for all $P(x) \in \mathcal{P}^{n-1}$.

So, $P(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 x + a_0$

$Q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1 x + b_0$; $a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1} \in \mathbb{R}$

$P(x)+Q(x) = (a_{n-1}+b_{n-1})x^{n-1} + (a_{n-2}+b_{n-2})x^{n-2} + \dots + (a_1+b_1)x + (a_0+b_0) \in \mathcal{P}^{n-1}$ OK.

Also, $\alpha P(x) = \alpha a_{n-1}x^{n-1} + \alpha a_{n-2}x^{n-2} + \dots + \alpha a_1 x + \alpha a_0 \in \mathcal{P}^{n-1}$ OK $\Rightarrow$

$\Rightarrow \mathcal{P}^{n-1}$ is a vector subspace of the vector space $\mathcal{P}^n$

b) $V = \mathbb{R}^2$

$A = \{[x,y]^T \mid x^2 = y\}$

it is obvious that $A \subseteq V$

We need that for all $x, y, z, t \in \mathbb{R}$, $[x,y]^T, [z,t]^T \in A$ to have that $[x,y]^T + [z,t]^T \in A$

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ t \end{bmatrix} = \begin{bmatrix} x \\ x^2 \end{bmatrix} + \begin{bmatrix} z \\ z^2 \end{bmatrix} = \begin{bmatrix} x+z \\ x^2+z^2 \end{bmatrix} \in A \iff (x+z)^2 = x^2+z^2$$

$$\cancel{x^2} + 2xz + \cancel{z^2} = \cancel{x^2} + \cancel{z^2}$$

$2xz = 0$ False (there are cases when this is not true)

So, A is not a vector subspace of the vector space $\mathbb{R}^2$.

c) $V = \mathbb{R}^3$

$B = \{[x,y,z]^T \mid x = 3y,\ z = -2y\}$.

it is obvious that $B \subseteq V$

We need that for all $x, y, z, a, b, c \in \mathbb{R}$, $[x,y,z]^T, [a,b,c]^T \in B$ to have that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in B \iff \begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} + \begin{bmatrix} 3b \\ b \\ -2b \end{bmatrix} \in B \iff \begin{bmatrix} 3(y+b) \\ y+b \\ -2(y+b) \end{bmatrix} \in B$$

if we replace $(y+b)$ with t we obtain a vector $[3t, t, -2t]^T$, which is indeed in B.

We also need that for all $a, b, c, \alpha \in \mathbb{R}$, $[a,b,c]^T \in B$ to have that

$$\alpha \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in B \iff \alpha \begin{bmatrix} 3b \\ b \\ -2b \end{bmatrix} \in B \iff \begin{bmatrix} 3b\alpha \\ b\alpha \\ -2b\alpha \end{bmatrix} \in B$$

if we replace $(b\alpha)$ with k we obtain a vector $[3k, k, -2k]^T$, which is indeed from B.

So, B is a vector subspace of the vector space $\mathbb{R}^3$.

d) $V = \mathbb{R}^3$

$C = \{[x,y,z]^T \mid x = 3y+1,\ z = -2y\}$

It is obvious that $C \subseteq V$.

We need that for all $x,y,z,a,b,c \in \mathbb{R}$, $[x,y,z]^T, [a,b,c]^T \in C$ to have that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in C \iff \begin{bmatrix} 3y+1 \\ y \\ -2y \end{bmatrix} + \begin{bmatrix} 3b+1 \\ b \\ -2b \end{bmatrix} \in C \iff \begin{bmatrix} 3b+3y+2 \\ b+y \\ -2b-2y \end{bmatrix} \in C \iff$$

$$\iff 3b + 3y + 2 = 3(b+y) + 1$$

$$\cancel{3b} + \cancel{3y} + 2 = \cancel{3b} + \cancel{3y} + 1$$

$$2 = 1 \quad \text{False}$$

So, C is <u>not</u> a vector subspace of the vector space $\mathbb{R}^3$.

<u>Applications</u>:

1. $x \in \mathbb{Z}_{10}^{16}$

$$x = 5412 \quad 3456 \quad 7890 \quad 432d$$

$$c = [2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1]^T$$

In this example, x is a valid card number, so it respects

$$c \cdot x + h = 0 \text{ in } \mathbb{Z}_{10}$$

No matter what digit d is, h is still 4 (digits from positions 1, 7, 9, 11)

So, we have $2\cdot 5 + 1\cdot 4 + 2\cdot 1 + 1\cdot 2 + 2\cdot 3 + 1\cdot 4 + 2\cdot 5 + 1\cdot 6 + 2\cdot 7 + 1\cdot 8 + 2\cdot 9 + 1\cdot 0 + 2\cdot 4 + 1\cdot 3 + 2\cdot 2 + 1\cdot d$

$$= \underbrace{10}_{0 \text{ in } \mathbb{Z}_{10}} + 4 + 2 + 2 + \underbrace{(6+4)}_{0 \text{ in } \mathbb{Z}_{10}} + \overset{0 \text{ in } \mathbb{Z}_{10}}{10} + \underbrace{(6+14)}_{0 \text{ in } \mathbb{Z}_{10}} + 8 + 18 + 0 + 8 + 3 + 4 + d = 9 + d \text{ in } \mathbb{Z}_{10}$$

So,
$$9 + d + h = 0$$
$$9 + 4 + d = 0$$
$$\left.\begin{array}{l} 3 + d = 0 \text{ in } \mathbb{Z}_{10} \\ d \in \mathbb{Z}_{10} \end{array}\right| \Rightarrow \boxed{d = 7}$$

We'll start with an $x = [x_1, x_2, x_3, \dots, x_{14}, x_{15}, d]^T \in \mathbb{Z}_{10}^{16}$ which is a valid card number. So, $x \cdot c + h = 0$ in $\mathbb{Z}_{10}$

Now we will interchange two adjacent numbers in x, let's say x_i and x_{i+1}, $i \in \{1;2;\dots;15\}$

We know from the beginning that

$$(*) \quad \sum_{j=1}^{15} c_j x_j + h = 0 \text{ in } \mathbb{Z}_{10}$$

Here, we distinguish two cases:

1. i is odd $\Rightarrow (i+1)$ is even

Then, the sum becomes $\displaystyle\sum_{j=1}^{15} c_j x_j - 2x_i - x_{i+1} + 2x_{i+1} + x_i + \overset{\text{it is different here}}{h'} = 0$ in $\mathbb{Z}_{10}$

If we substract $(*)$ from this we obtain

$$-2x_i - x_{i+1} + x_i + 2x_{i+1} + h' - h = 0$$
$$x_{i+1} - x_i + h' - h = 0$$

There are 4 cases here:

i) $x_i > 4$, $x_{i+1} > 4 \Rightarrow h' = h$

ii) $x_i > 4$, $x_{i+1} \leq 4 \Rightarrow h' = h - 1$

iii) $x_i \leq 4$, $x_{i+1} > 4 \Rightarrow h' = h + 1$

iv) $x_i \leq 4$, $x_{i+1} \leq 4 \Rightarrow h' = h$

In case i) for example, $h' = h \Rightarrow x_{i+1} - x_i = 0$

But we interchanged them, so they're not equal $\Big| \Rightarrow$ the error is detected

In case ii), however, $h' = h - 1 \Rightarrow x_{i+1} - x_i - 1 = 0$
$$x_{i+1} = x_i + 1$$

So, if $x_{i+1} = x_i + 1$, the result is still 0 in $\mathbb{Z}_{10}$, so the error is not detected.

in the example, if $i = 11$, $i+1 = 12$ and we interchange the 9 and 0, the sum remains 0, thus the check is not always effective.

2. $b = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, X]^T$

$c = [10, 9, 8, 7, 6, 5, 4, 3, 2, 1]^T$

$c \cdot b = 0$ in $\mathbb{Z}_{11}$

$b = [0, 5, 3, 4, 4, 2, 2, 0, 0, x]^T$

We need that $c \cdot b = 0$

$$10 \cdot 0 + 9 \cdot 5 + 8 \cdot 3 + 7 \cdot 4 + 6 \cdot 4 + 5 \cdot 2 + 4 \cdot 2 + 3 \cdot 0 + 2 \cdot 0 + 1 \cdot x = 0$$
$$45 + 24 + 28 + 24 + 10 + 8 + x = 0$$
$$139 + x = 0 \text{ in } \mathbb{Z}_{11} \Rightarrow 7 + x = 0 \text{ in } \mathbb{Z}_{11} \Rightarrow \boxed{x = 4}$$

$b = [0, 8, 3, 7, 0, 9, 9, 0, 2, 6]^T$

$c \cdot b = 236 = 5$ in $\mathbb{Z}_{11} \Rightarrow$ an error has occured

We know that this error is a transposition of two adjacent numbers.

The current $c \cdot b$ is 5 in $\mathbb{Z}_{11}$.

if we swap 0 and 8 we get $c \cdot b + 8 - 0 = 3$ in $\mathbb{Z}_{11}$

if we swap 8 and 3 we get $c \cdot b + 3 - 8 = 0$ in $\mathbb{Z}_{11}$, so the two adjacent numbers were 8 and 3 $\Rightarrow$ the original ISBN code was

$$b = [0, 3, 8, 7, 0, 9, 9, 0, 2, 6]^T.$$