GABRIEL MOISE

CONTINUOUS MATHEMATICS

HT 2019

PROBLEM SHEET 1

Derivatives, Taylor's Theorem, 1-dimensional optimization

$\boxed{1.1.}$ If f and g are differentiable on $\mathbb{R}$ and $k \in \mathbb{N}$ then

$$\frac{d}{dx}\left(\frac{f}{g^k}\right) = \frac{g\frac{df}{dx} - kf\frac{dg}{dx}}{g^{k+1}}$$

$$\frac{d}{dx}\left(\frac{f}{g^k}\right) \overset{\substack{\text{quotient rule}\\ \downarrow}}{=} \frac{g^k\frac{df}{dx} - f\frac{dg^k}{dx}}{g^{2k}} \overset{\substack{\text{chain rule}\\ \downarrow}}{=} \frac{g^k\frac{df}{dx} - f\cdot kg^{k-1}\cdot\frac{dg}{dx}}{g^{2k}} = \frac{g\frac{df}{dx} - kf\frac{dg}{dx}}{g^{k+1}}$$

For $k \leq 0$ we need that the codomain of g does not include 0, and the result is the same.
For $k \notin \mathbb{Z}$, we need g to be positive on $\mathbb{R}$, and the result is the same.

$f, g : \mathbb{R}^n \to \mathbb{R}$

$$\frac{d}{d\underline{x}}\left(\frac{f}{g^k}\right) \overset{\substack{\text{quotient rule}\\ \downarrow}}{=} \frac{g^k\frac{df}{d\underline{x}} - f\frac{dg^k}{d\underline{x}}}{g^{2k}} \overset{\substack{\text{chain rule}\\ \downarrow}}{=} \frac{g^k\frac{df}{d\underline{x}} - f\cdot k\cdot g^{k-1}\cdot\frac{dg}{d\underline{x}}}{g^{2k}} = \frac{g\frac{df}{d\underline{x}} - k\cdot f\cdot\frac{dg}{d\underline{x}}}{g^{k+1}}$$

The proof is the same, but be obtain a vector this time.

$\boxed{1.2.}$ $\underline{\underline{A}}$ symmetric $n\times n$ matrix , $\underline{x} \neq \underline{0} \Rightarrow \underline{\underline{A}} = \underline{\underline{A}}^T$

$$f(\underline{x}) = \frac{\underline{x}^T\underline{\underline{A}}\underline{x}}{\underline{x}^T\underline{x}}$$

$$\frac{df}{d\underline{x}} \overset{\substack{\text{quotient rule}\\ \downarrow}}{=} \frac{(\underline{x}^T\underline{x})(\underline{\underline{A}}+\underline{\underline{A}}^T)\underline{x} - (\underline{x}^T\underline{\underline{A}}\underline{x})\cdot 2\underline{x}}{(\underline{x}^T\underline{x})^2} = \frac{2(\underline{x}^T\underline{x})\underline{\underline{A}}\underline{x} - 2(\underline{x}^T\underline{\underline{A}}\underline{x})\underline{x}}{(\underline{x}^T\underline{x})^2}$$

($\Rightarrow$)

$\frac{df}{d\underline{x}} = \underline{0} \Rightarrow x$ is an eigenvector of A

$$\frac{2(\underline{x}^T\underline{x})\underline{\underline{A}}\underline{x} - 2(\underline{x}^T\underline{\underline{A}}\underline{x})\underline{x}}{(\underline{x}^T\underline{x})^2} = \underline{0} \Rightarrow (\underline{x}^T\underline{x})\underline{\underline{A}}\underline{x} = (\underline{x}^T\underline{\underline{A}}\underline{x})\underline{x} \quad \Big| : (\underline{x}^T\underline{x}) \neq 0$$

$$\underline{\underline{A}}\underline{x} = \frac{\underline{x}^T\underline{\underline{A}}\underline{x}}{\underline{x}^T\underline{x}}\underline{x}$$

Let $\lambda = \frac{\underline{x}^T\underline{\underline{A}}\underline{x}}{\underline{x}^T\underline{x}} \in \mathbb{R} \Rightarrow \underline{\underline{A}}\underline{x} = \lambda\underline{x} \Rightarrow x$ is an eigenvector of A

($\Leftarrow$)

x is an eigenvector of $\underline{\underline{A}} \Rightarrow (\exists)\, \lambda \in \mathbb{R}$ such that $\underline{\underline{A}}\underline{x} = \lambda\underline{x}$

$$\frac{2(\underline{x}^T\underline{x})\underline{\underline{A}}\underline{x} - 2(\underline{x}^T\underline{\underline{A}}\underline{x})\underline{x}}{(\underline{x}^T\underline{x})^2} = \frac{2(\underline{x}^T\underline{x})\lambda\underline{x} - 2(\underline{x}^T\lambda\underline{x})\underline{x}}{(\underline{x}^T\underline{x})^2} = \frac{2\lambda(\underline{x}^T\underline{x})\underline{x} - 2\lambda(\underline{x}^T\underline{x})\underline{x}}{(\underline{x}^T\underline{x})^2} = \underline{0}.$$

So, $\frac{df}{d\underline{x}} = \underline{0} \iff \underline{x}$ is an eigenvector of $\underline{\underline{A}}$

$\boxed{1.3.}$ $f(\underline{x}) = \max(x_i)$, $\underline{x} = (x_1\ x_2 \dots x_n)^T$

$$\ell(\underline{x}) = \ln\left(\sum_{i=1}^{n} e^{x_i}\right)$$

(a) $\max x_i \leq \ell(\underline{x}) \leq \max x_i + \ln n$

$$\max x_i \leq \ln\left(\sum_{i=1}^{n} e^{x_i}\right) \leq \max x_i + \ln n$$

$$e^{\max x_i} \leq \sum_{i=1}^{n} e^{x_i}$$

Let $\max x_i = x_k$

$$0 \leq \sum_{i=1}^{k-1} e^{x_i} + \sum_{i=k+1}^{n} e^{x_i} \quad \underline{\text{OK}}$$

$$\ln\left(\frac{\sum_{i=1}^{n} e^{x_i}}{n}\right) \leq \max x_i$$

$$\frac{\sum_{i=1}^{n} e^{x_i}}{n} \leq e^{\max x_i}$$

$$\sum_{i=1}^{n} e^{x_i} \leq n e^{\max x_i}$$

Let $\max x_i = x_k \Rightarrow$

$$\begin{aligned} e^{x_1} &\leq e^{x_k} \\ e^{x_2} &\leq e^{x_k} \\ &\dots \\ e^{x_n} &\leq e^{x_k} \quad (+) \\ \hline \sum_{i=1}^{n} e^{x_i} &\leq n e^{x_k} \quad \underline{\text{OK}} \end{aligned}$$

(b) As $e^{x_i} > 0\ (\forall)\ x_i \in \mathbb{R} \Rightarrow \sum_{i=1}^{n} e^{x_i} > 0 \Rightarrow \ln\left(\sum_{i=1}^{n} e^{x_i}\right)$ is differentiable

$$\frac{\partial \ell}{\partial x_j} \overset{\text{chain rule}}{=} \frac{1}{\sum_{i=1}^{n} e^{x_i}} \cdot e^{x_j} = \frac{e^{x_j}}{\sum_{i=1}^{n} e^{x_i}}$$

$$\frac{\partial^2 \ell}{\partial x_j^2} \overset{\text{quotient rule}}{=} \frac{e^{x_j} \cdot \sum_{i=1}^{n} e^{x_i} - e^{x_j} \cdot e^{x_j}}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2} = \frac{e^{x_j}\left(\sum_{i=1}^{n} e^{x_i} - e^{x_j}\right)}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2}$$

$$\frac{\partial^2 \ell}{\partial x_k \partial x_j} \overset{\text{quotient rule}}{=} \frac{-e^{x_j} \cdot e^{x_k}}{\left(\sum_{i=1}^{n} e^{x_i}\right)^2}$$

$\boxed{1.4.}$ $H : [0,1] \to \mathbb{R}$

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x), \qquad H(0) = H(1) = 0$$

(a) product rule, chain rule

$$\frac{dH}{dx} = -\cancel{x}\cdot\frac{1}{\cancel{x}\ln 2} - \log_2 x - \left((1-\cancel{x})\cdot\frac{1}{(1-\cancel{x})\ln 2}\cdot(-1) + (-1)\log_2(1-x)\right)$$

$$\frac{dH}{dx} = -\frac{\cancel{1}}{\cancel{\ln 2}} - \log_2 x + \frac{\cancel{1}}{\cancel{\ln 2}} + \log_2(1-x) = \log_2\left(\frac{1-x}{x}\right)$$

chain rule, quotient rule

$$\frac{d^2H}{dx^2} = \frac{1}{\frac{1-x}{x}\ln 2}\cdot\frac{(-1)x-(1-x)}{x^2} = \frac{x}{(1-x)\ln 2}\cdot\frac{-1}{x^2} = \frac{-1}{x(1-x)\ln 2}$$

quotient rule, product rule

$$\frac{d^3H}{dx^3} = \left(-\frac{1}{\ln 2}\right)\cdot\frac{-((1-x)+x\cdot(-1))}{(x(1-x))^2} = \frac{-1}{\ln 2}\cdot\frac{-1+x+x}{x^2(1-x)^2} = \frac{1-2x}{x^2(1-x)^2\ln 2}$$

quotient rule, product rule

$$\frac{d^4H}{dx^4} = \frac{-2x^2(1-x)^2-(1-2x)(2x-6x^2+4x^3)}{x^4(1-x)^4\ln 2} = \frac{-2x^2+4x^3-2x^4-2x+6x^2-4x^3+4x^2-12x^3+8x^4}{x^4(1-x)^4\ln 2}$$

$$\frac{d^4H}{dx^4} = \frac{6x^4-12x^3+8x^2-2x}{x^4(1-x)^4\ln 2} = \frac{6x^3-12x^2+8x-2}{x^3(1-x)^4\ln 2} = \frac{-6x^2+6x-2}{x^3(1-x)^3\ln 2} = \frac{6x^2-6x+2}{x^3(x-1)^3\ln 2}$$

(b) Taylor's theorem

$$H(x) = H(x_0) + (x-x_0)\frac{dH}{dx}(x_0) + \frac{(x-x_0)^2}{2!}\frac{d^2H}{dx^2}(x_0) + \frac{(x-x_0)^3}{3!}\frac{d^3H}{dx^3}(x_0) + \frac{(x-x_0)^4}{4!}\frac{d^4H}{dx^4}(\xi)$$

$x_0 = \frac{1}{2}$

$$H\left(\tfrac{1}{2}\right) = -\tfrac{1}{2}\log_2\tfrac{1}{2} - \tfrac{1}{2}\log_2\tfrac{1}{2} = -\tfrac{1}{2}(-1) - \tfrac{1}{2}(-1) = \tfrac{1}{2}+\tfrac{1}{2} = 1$$

$$\frac{dH}{dx}\left(\tfrac{1}{2}\right) = \log_2\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = \log_2 1 = 0$$

$$\frac{d^2H}{dH^2}\left(\tfrac{1}{2}\right) = \frac{-1}{\frac{1}{2}\cdot\frac{1}{2}\ln 2} = \frac{-1}{\frac{1}{4}\ln 2} = -\frac{4}{\ln 2}$$

$$\frac{d^3H}{dH^3}\left(\tfrac{1}{2}\right) = 0$$

$$H(x) = 1 + \left(x-\tfrac{1}{2}\right)\cdot 0 + \frac{\left(x-\frac{1}{2}\right)^2}{2}\cdot\frac{-4}{\ln 2} + \frac{\left(x-\frac{1}{2}\right)^3}{6}\cdot 0 + \frac{\left(x-\frac{1}{2}\right)^4}{24}\cdot\frac{6\xi^2-6\xi+2}{\xi^3(\xi-1)^3\ln 2}$$

$$H(x) = 1 - \frac{(2x-1)^2}{2\ln 2} + \frac{\left(x-\frac{1}{2}\right)^4}{24}\cdot\frac{6\xi^2-6\xi+2}{\xi^3(\xi-1)^3\ln 2}, \text{ where } \xi\in\left(\tfrac{1}{2}, x\right)$$

(c) There is no Taylor expansion for $H(x)$ at $x=0$ because H is not differentiable there.

$\boxed{1.5.}$ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x} + e^{-x}$

(a)
$$\frac{df}{dx} = e^{x} - e^{-x}$$
$$\frac{d^2f}{dx^2} = e^{x} + e^{-x}$$
$$\frac{d^3f}{dx^3} = e^{x} - e^{-x}$$
$$\vdots$$
$$\frac{d^kf}{dx^k} = \begin{cases} e^{x} + e^{-x}, & k = \text{even} \\ e^{x} - e^{-x}, & k = \text{odd} \end{cases}, \ (\forall)\, k \in \mathbb{N}$$

(b)
$$f(x_0) = e^{x_0} + e^{-x_0}$$
$$\frac{d^kf}{dx^k}(x_0) = \begin{cases} e^{x_0} + e^{-x_0}, & k = \text{even} \\ e^{x_0} - e^{-x_0}, & k = \text{odd} \end{cases}$$

$$x_0 = 1 \Rightarrow \frac{d^kf}{dx^k}(1) = \begin{cases} e^{1} + e^{-1} = e + \frac{1}{e} = \frac{e^2+1}{e}, & k = \text{even} \\ e^{1} - e^{-1} = e - \frac{1}{e} = \frac{e^2-1}{e}, & k = \text{odd} \end{cases}$$

$$f(x) \overset{\text{Taylor's theorem}}{=} \underbrace{\frac{e^2+1}{e} + (x-1)\frac{e^2-1}{e} + \frac{(x-1)^2}{2!}\frac{e^2+1}{e} + \frac{(x-1)^3}{3!}\frac{e^2-1}{e} + \ldots + \frac{(x-1)^k}{k!}\left(e + (-1)^k\frac{1}{e}\right)}_{\text{order-}k \text{ Taylor polynomial for } f(x)} + \underbrace{\frac{(x-1)^{k+1}}{(k+1)!}\left(e^{\xi} + \frac{(-1)^{k+1}}{e^{\xi}}\right)}_{\substack{\text{Lagrange remainder term} \\ e_{k+1}(x),\ \xi \in (1,x)}}$$

(c) $x \in (0,1) \Rightarrow x-1 \in (-1,0)$, $\xi \in (0,1)$

$\underline{k = \text{odd}}$ $\Rightarrow (x-1)^{k+1} \in (0,1) \Rightarrow \frac{(x-1)^{k+1}}{(k+1)!} \in \left(0, \frac{1}{(k+1)!}\right)$

$$e^{\xi} + \frac{(-1)^{k+1}}{e^{\xi}} = e^{\xi} + \frac{1}{e^{\xi}} < e + \frac{1}{e}$$

$$\Rightarrow 0 < e_{k+1}(x) < \frac{\left(e + \frac{1}{e}\right)}{(k+1)!}$$

$\underline{k = \text{even}}$ $\Rightarrow (x-1)^{k+1} \in (-1,0)$

$$\frac{(x-1)^{k+1}}{(k+1)!} \in \left(-\frac{1}{(k+1)!}, 0\right)$$
$$e^{\xi} + \frac{(-1)^{k+1}}{e^{\xi}} = e^{\xi} - \frac{1}{e^{\xi}} < e - \frac{1}{e}$$

$$\Rightarrow -\frac{\left(e - \frac{1}{e}\right)}{(k+1)!} < e_{k+1}(x) < 0$$

$$\frac{\left(\frac{1}{e} - e\right)}{(k+1)!} < e_{k+1}(x) < 0.$$

(d)
```
object Problem {
def main (args: Array [String]) =
{ var ok = 0 ; var k = 1
  val boundodd : Double = 3.08616127     // boundodd = e + e^-1
  val boundeven: Double = 2.35040239     // boundeven = e - e^-1
  val precision : Double = 0.000000000000001    // precision = 10^-15
  val error : Double = 0.0
  while (ok == 0)
  { if (k % 2 == 0)
      { error = boundeven
        for (i <- 1 to (k+1)) error = error / i     // error = (e - e^-1)/(k+1)!
        if (error <= precision) ok = 1 }
    else { error = boundodd
           for (i <- 1 to (k+1)) error = error / i     // error = (e + e^-1)/(k+1)!
           if (error <= precision) ok = 1 }
    k = k+1 }
  k = k-1
  println ("The order of the Taylor polynomial is " +k+ " and the error is " +error) }
}
```

The order of the Taylor polynomial is 17 and the error is 4.820339161456492E-16.

$\boxed{1.6.}$ $\underline{\underline{A}}$, $\underline{b}$, $c \in \mathbb{R}$ fixed, $\underline{\underline{A}}$ symmetric

$$f(\underline{x}) = \sin\left(\underline{x}^T \underline{\underline{A}}\,\underline{x} + \underline{x}^T \underline{b} + c\right)$$

(a) $$\frac{df}{d\underline{x}} \overset{\text{chain rule}}{=} \cos\left(\underline{x}^T \underline{\underline{A}}\,\underline{x} + \underline{x}^T \underline{b} + c\right) \cdot \left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right)$$

$$\underline{\underline{H}}(f) = \underline{\underline{J}}\left(\frac{df}{d\underline{x}}\right) = J\left(\cos\left(\underline{x}^T \underline{\underline{A}}\,\underline{x} + \underline{x}^T \underline{b} + c\right) \cdot \left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right)\right)$$ (product of scalar and vector rule)

$$\underline{\underline{H}}(f) = \left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right)\left(-\sin\left(\underline{x}^T \underline{\underline{A}}\,\underline{x} + \underline{x}^T \underline{b} + c\right) \cdot \left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right)\right)^T + \cos\left(\underline{x}^T \underline{\underline{A}}\,\underline{x} + \underline{x}^T \underline{b} + c\right) \cdot \underline{\underline{J}}\left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right),$$

where $\underline{\underline{J}}\left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right) = \underline{\underline{J}}\left(2\underline{\underline{A}}\,\underline{x}\right) + \underline{\underline{J}}\left(\underline{b}\right) = 2\underline{\underline{A}}$

$$\underline{\underline{H}}(f) = \left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right)\left((-1) \cdot \sin\left(\underline{x}^T \underline{\underline{A}}\,\underline{x} + \underline{x}^T \underline{b} + c\right)\left(2\underline{\underline{A}}\,\underline{x} + \underline{b}\right)\right)^T + 2\cos\left(\underline{x}^T \underline{\underline{A}}\,\underline{x} + \underline{x}^T \underline{b} + c\right) \cdot \underline{\underline{A}}$$

(b) $$f_2(\underline{x}) = f(\underline{x_0}) + (\underline{x} - \underline{x_0})^T \frac{df}{d\underline{x}}(\underline{x_0}) + \frac{1}{2}(\underline{x} - \underline{x_0})^T \underline{\underline{H}}(f)(\underline{x_0})(\underline{x} - \underline{x_0}) + e_3$$ (multivariate Taylor's theorem)

We have: $(\underline{x_0} = \underline{0})$

$$f(\underline{0}) = \sin c$$

$$\frac{df}{d\underline{x}}(\underline{0}) = \cos c \cdot \underline{b}$$

$$\underline{\underline{H}}(f)(\underline{0}) = \underline{b} \cdot \left((-\sin c) \cdot \underline{b}^T \right) + 2\cos c \cdot \underline{\underline{A}}$$

Then, we obtain:

$$f_2(\underline{x}) = \sin c + \cos c \cdot \underline{x}^T \underline{b} + \underline{x}^T \left(\cos c \cdot \underline{\underline{A}} - \frac{\sin c}{2} \underline{b}\,\underline{b}^T \right) \underline{x} + e_3,$$

where e_3 is the remainder term q.e.d.

$\boxed{1.7.}$ $a, b \in \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^4 - 4(a+b)x^3 + 6abx^2$

$f'(x) = 12x^3 - 12(a+b)x^2 + 12abx = 12x(x-a)(x-b) \Rightarrow \boxed{f'(0)=0}$, $\boxed{f'(a)=0}$, $\boxed{f'(b)=0}$

$f''(x) = 36x^2 - 24(a+b)x + 12ab \Rightarrow \boxed{f''(0) = 12ab}$

$f'(x) = 0 \Rightarrow f$ has a stationary point at $x=0$

Case 1: $x=0$ is a local maximum

$f''(0) < 0 \Rightarrow 12ab < 0 \Rightarrow \boxed{ab < 0}$

Case 2: $x=0$ is a local minimum

$f''(0) > 0 \Rightarrow 12ab > 0 \Rightarrow \boxed{ab > 0}$

Case 3: $x=0$ is a stationary point of inflexion

$f''(0) = 0 \Rightarrow 12ab = 0 \Rightarrow \boxed{ab = 0}$ (however for $a=b=0$, $x=0$ is Case 2)

As $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (3x^4 - 4(a+b)x^3 + 6abx^2) = \lim\limits_{x\to\infty} 3x^4\left(1 - \underbrace{\frac{4(a+b)}{3x}}_{x\to\infty,\ \to 0} + \underbrace{\frac{2ab}{x^2}}_{x\to\infty,\ \to 0}\right) = +\infty$ and $f(0) = 0$, $x=0$ cannot be the global maximum of f.

Now, knowing the fact that $x=0$ is a local minimum i.e. $ab>0$, we want to determine the values of a and b for which it is also the global minimum for f.

We have:

$f(a) = 3a^4 - 4(a+b)a^3 + 6aba^2 = 3a^4 - 4a^4 - 4a^3b + 6a^3b = 2a^3b - a^4 = a^3(2b-a)$

$f(b) = 3b^4 - 4(a+b)b^3 + 6abb^2 = 3b^4 - 4ab^3 - 4b^4 + 6ab^3 = 2ab^3 - b^4 = b^3(2a-b)$

In order for $x=0$ to be a global minimum, $f(0)$ has to be smaller or equal to $f(a)$ and $f(b)$, which are the only ones that could be points of global minimum for f. (Supposing that $(\exists)$ $c \in \mathbb{R}$, $c \neq 0, c \neq a, c \neq b$, with $f(c)$ global minimum $\Rightarrow f'(c) = 0 \Rightarrow f'$ has 4 roots, but f' is a polynomial of degree 3, so we reach a contradiction).

We'll treat two cases here:

Case 1 : $a,b>0$

We need $f(a)\geq 0 \Rightarrow a^3(2b-a)\geq 0 \Rightarrow 2b-a\geq 0 \Rightarrow \boxed{a\leq 2b}$

and $f(b)\geq 0 \Rightarrow b^3(2a-b)\geq 0 \Rightarrow 2a-b\geq 0 \Rightarrow \boxed{b\leq 2a}$

Case 2 : $a,b<0$

We need that $f(a)\geq 0 \Rightarrow a^3(2b-a)\geq 0 \Rightarrow 2b-a\leq 0 \Rightarrow \boxed{a\geq 2b}$

and $f(b)\geq 0 \Rightarrow b^3(2a-b)\geq 0 \Rightarrow 2a-b\leq 0 \Rightarrow \boxed{b\geq 2a}$

Therefore $x=0$ is a global minimum if $a,b>0$ and $a\leq 2b$, $b\leq 2a$ OR if $a,b<0$ and $a\geq 2b$, $b\geq 2a$.