conjecture

INTRODUCTION TO FORMAL PROOF TUTORIAL SHEETA

TT 2019

QUESTION 1

Vi.
$$(P \rightarrow \overline{Q} \rightarrow R) \rightarrow P \wedge Q \rightarrow R$$
 - this is a

(b) i. The logic lectures have started but the concurrency lectures have not.

L = The logic lectures have started

C = The concurrency lectures have started

ii. If Bernard is at a logic lecture then he is not at a concurrency lecture.

L= Bernard is at a logic lecture

c = Bernard is at a concurrency lecture

iii. Either Bernard teaches a logic lecture on he stays in bed; he can't do both

L = Bernard teaches a logic lecture

B = Bernard stays in bed

$$(L \vee B) \wedge \neg (L \wedge B)$$

iv. If Dick is moving then Jame is stationary and if Jame is moving then Dick is stationary

D = Dick is moving

J = Jame is moving

$$(\Delta \Gamma \leftarrow \Gamma) \wedge (\Gamma \Gamma \leftarrow Q)$$

v. When Dick has drunk he doesn't drive and when he drives he hasn't drunk

P = Dick has drunk

Q = Dick drives

$$(P \rightarrow \neg Q) \land (Q \rightarrow \neg P)$$

vi. Carol is jealous of Yvonne on mot in a good mood.

J = Conol is jealous of Yvonne

M = Canol is in a good mood

J V (¬M)

B = The basometer falls R = It will nain It will smow B -> RVS VIII. If you have been to the lectures and read the first two sections of the notes then you should be able to do the first few tutorial questions; otherwise you will not be able to. L = You have been to the lectures N = You read the first two sections of the notes T = You should be able to do the first few tutorial questions $(L \land N \rightarrow T) \land (\neg (L \land N) \rightarrow \neg T)$ ix. The formula is always true if its subformulae are always true. F = The formula is always true S = its subformulae are always true F - S x. The formula is always true only if its subformulae are always true. $F \longrightarrow S$ xi. The formula is always true if and only if its subformulal are always true. F + S i. Tomorrow it will either rain or snow. RVS It only snows when it is cold. S - C It will be woum tomorrow. 7C Therefore it will rain tomorow. II. Either the bongs repeller on the footler is broken. BVF There is always a bad smell when the bongo repeller is broken. B -> S There isn't a bad smell. 75 Therefore the footler is broken F iii. Either the UFOs are secret enemy weapons on they are spaceships from an alien world. WVS If they are enemy weapons, then enemy technology is (contrary to current thinking) superior to ours. W -> E If they are alien space on ft then they display a technology more sophisticated than anything we can imagine. $S \rightarrow A$ In any case, therefore, their builders are technologically more sophisticated than we are. EVA

VII. When the banometer falls it will rain on snow.

QUESTION 2

- (a) i. $P \vdash Q \rightarrow (Q \land P)$
 - 1. P

premiss

- Q
- assumption
- QAP
- 1-intro 2,1

assumption

-- elim 2,1

1- elim - 1 3

-> - intro 2-4

assumption

-> -elim 6,1 1 - elim-1 7

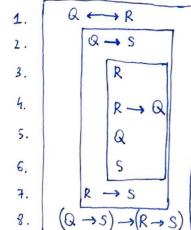
- 4. Q -> (QAP) -- intro 2-3
- iv. $P \rightarrow (Q \land R) \vdash (P \rightarrow Q) \land (P \rightarrow R)$
 - 1. P → (QAR) premiss

3.

- P
- QNR
- Q
- P-Q 5.
- 6. P

- 9 P-R
- QAR
- -- intro 6-8
- 10. (P → Q) A (P → R) A-intro 5,9
- V. $(P \rightarrow Q) \land (P \rightarrow R) \vdash P \rightarrow Q \land R$
 - 1. $(P \rightarrow Q) \land (P \rightarrow R)$ premiss
 - P
 - 3. P -> Q
 - 4. Q
 - 5. P - R
 - 6.
 - R QAR
 - 8. P -> (QAR)

- assumption
- 1-elim-l 1
- -- elim 2,3
- 1- elim-1
- ->-elim 2,5
- 1- intro 4,6
- -> intro 2 7
- (b) i. $F(Q \longleftrightarrow R) \to (Q \to S) \to (R \to S)$



- assumption assumption
- assumption abb- - lim-l 1
- - elim 3,4
- -> elim 5,2
- intro 2-7
- 9. (Q ←> R) → (Q → S) → (R → S) → intro 1-8

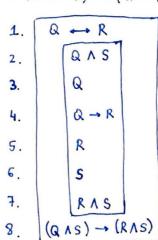
- $P \mapsto Q \rightarrow (P \land Q)$
 - 1. premiss
 - assumption 2. Q
 - 1-intro 1,2 3. PAQ
 - Q -> (PAQ) -- intro 2-3
- iii Q PP -> (PAQ)

4. P -> (PAQ)

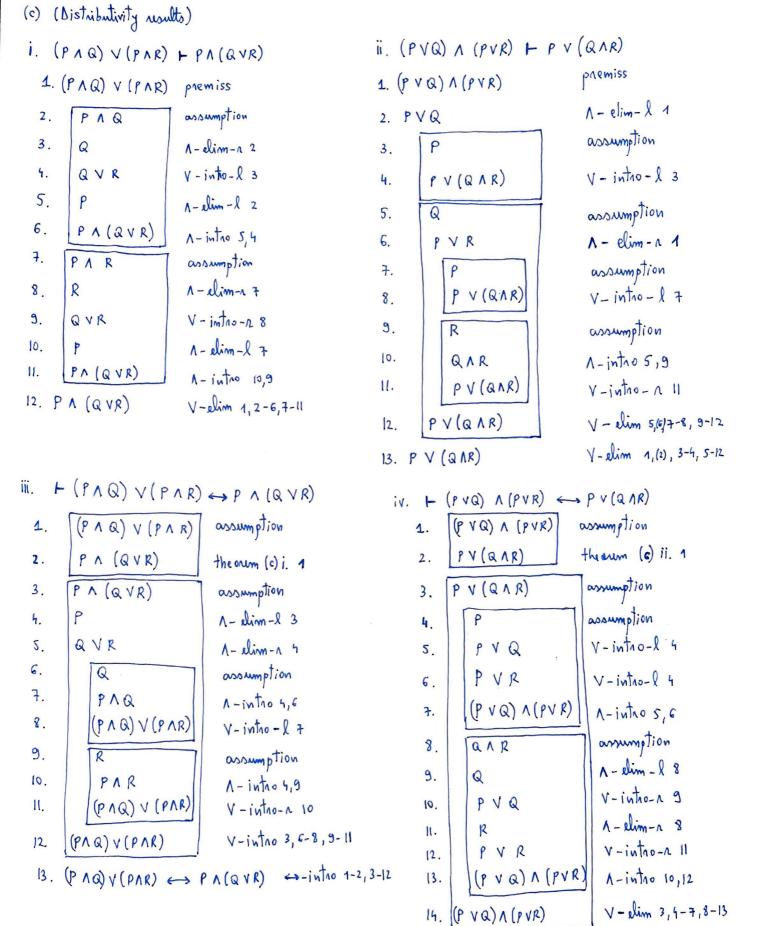
- premiss 1. Q
- assumption 2.
- 1-intro 2,1 PAQ

→-intro 2-3

ii. $\vdash (Q \leftrightarrow R) \rightarrow (Q \land S) \rightarrow (R \land S)$

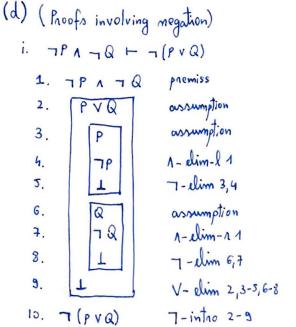


- assumption assumption
- 1- dim-12
- abb- -- lim -1 1
- ->- elim 3,4
- 1 dim -1 2
- 1- intro 5,6
- -> intro 2-7
- 9. (Q -R) (QAS) -(RAS) -- intro 1-8



4

15. (PVQ) A (PVR) - PV (QAR) - intao 1-2,3-1



V- elim 3,4-7,8

9. U

Vii.
$$(P \rightarrow Q) \land (Q \rightarrow R), \neg R \vdash \neg P$$

1. $(P \rightarrow Q) \land (Q \rightarrow R)$ primiss

2. $\neg R$ primiss

3. $P \rightarrow Q$ $\land - \text{dim} - 1 \land 1$

4. $Q \rightarrow R$ $\land - \text{elim} - 1 \land 1$

5. $P \rightarrow Q$ $\land - \text{elim} - 1 \land 1$

6. $Q \rightarrow R$ $\rightarrow - \text{elim} 5, 3$

7. $R \rightarrow - \text{elim} 6, 4$

1.
$$(P \rightarrow Q) \land (Q \rightarrow R)$$
 primiss

2. $\neg R$

3. $P \rightarrow Q$

4. $Q \rightarrow R$

5. $P \rightarrow Q \rightarrow Q$

6. $Q \rightarrow Q \rightarrow Q \rightarrow Q$

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9. $Q \rightarrow Q \rightarrow$

Viii.
$$(P \rightarrow Q) \land (Q \rightarrow \neg P) \vdash \neg P$$

1. $(P \rightarrow Q) \land (Q \rightarrow \neg P) \vdash \neg P$

2. $P \rightarrow Q$

3. $Q \rightarrow \neg P$

4. P

4. P

5. Q

6. $\neg P$

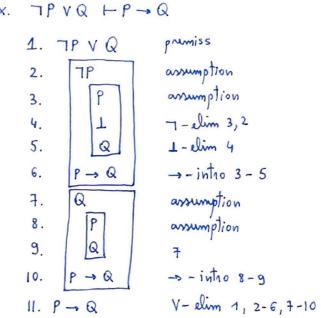
7. P

8. $\neg P$

7. P

7. P

8. P



QUESTION 3

We want to introduce a new binary logical connective, which is "exclusive or", written &, which is true when exactly one of its arguments are true.

From this, we can deduce its table of truth-values:

P	Q	PYQ
F	F	F
F	T	T
T	F	Т
T	T	F
		•

(a) We can extend eval to also incorporate this connective this way:

PY9 -> if eval vp then not (eval vg) else eval vg

(b) Elimination rules for V

Y Ø Y Y Z-dim-r

We will prove that the proof trees are valid in both cases, with the function proves: proves: Proof -> Conjecture -> Bool

p'proves' (ps + c) = conclusion p == c le valid ps p where we have valid defined for these two rules as:

valid :: [Prop] -> Proof -> Bool valid hs proof = case proof of

InferBy "Y-elim-l" [l, Λ] $C \rightarrow$ valid hs ler valid hs Λ re

case conclusion Λ of $(p \vee q) \rightarrow \text{conclusion } l == p$ re $- \rightarrow \text{False}$

Jufu By " y-elim-1" [l,n] c ->

Valid hs l le valid hs n se

conse conclusion n of

(p y g) -> conclusion l == g se c == 7p

-> False

As both thees are valid, we can say that the rules are sound. \$ V Y is equivalent (from (a)) to (\$ 1 - 4) V (- \$ 14), we can also prove the two rules with conjectures: Ψ , $(\emptyset \land \neg \ \ \ \) \lor (\neg \emptyset \land \ \ \) \vdash \neg \emptyset$ ϕ , $(\phi \wedge \neg \forall) \vee (\neg \phi \wedge \forall) \vdash \neg \forall$ 1. 4 premiss prumiss 1. Ø 2. (\$ A 7 +) V (7 \$ A +) 2. (Ø A 7 4) V (¬ Ø A 4) prumiss premiss assumption assumption 3. assumption Ø 17 TY assumption 4 . 5. 7 4 5. N-elim-14 1-dim-1 4 6. 7-elim 1,5 7-dim 3,5 assumption 7014 7. assumption 1-lim-l-7 8. 7 \$ 8. 7 \$ 1-elim-17 7-elim 3,8 9. 9. 7-elim 1,8 V-dim 2, 4-6,7-T V- elim 2, 4-6,7-9 10. 10 1 7-intro 3-10 7-intro 3-10 11. TØ 11. 7 Y We can also do the same thing for the introduction rule $\frac{\phi \vee \psi \qquad \neg (\phi \wedge \psi)}{\phi \vee \psi} \vee -into$ InfuBy " y-intro" [l,n] (pyg) -> valid hs I se valid hs a se conduion l == p v g ll conclusion 1 == 7 (P N9) We'll also prove it using a conjecture

1-intro 3,9

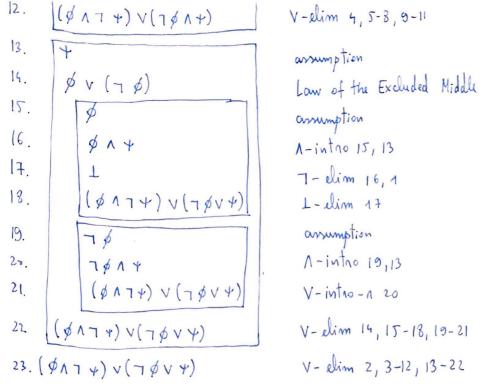
V-intro-2 10

 $\phi \vee +$, $\neg (\phi \wedge \psi) + (\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$ 1. 7 (Ø N +) premiss Ø V Y premiss 2. arrumption Ø 3. Law of the Excluded Middle 4 V (7 Y) 4. arsumption 5. 4 1-intro 3,5 DA+ 6. 7-elim 6,1 7. (\$ 1 7 +) v (7 \$ 1 +) 8. 1-elim 7 assumption 9.

Ø N 74

(\$ A 7 +) V (7 \$ A +)

10.



(c) At (b) we also proved the rules as conjecture (not just soundness implied by being valid proof trees;

Question 4

(a) We want to prove that

First, we'll prove that TPFP-1:

- 1. 7 9 premiss
- 2. P hyp
 3. 1 7-e 2,1

Therefore, we have

We know say that

$$\frac{1}{1} + 4 \rightarrow T \qquad 4 \rightarrow T + C \qquad \text{cot}$$

Since the left branch is our theorem and the right branch is our premiss, we can use the cut rule to get to our desired result.

Now, let's extend this to

$$\frac{T, P \rightarrow L + C}{T \rightarrow P + C} \rightarrow F$$

Will start from the obvious fact that P, TP L (7-e). We can therefore say that T, P, 7P + 1.

Then, we have

$$\frac{T, \, \Psi, \, \neg \, \Psi \vdash \bot}{T, \, \neg \, \Psi \vdash \Psi \rightarrow 1} \rightarrow -i$$

And if we also use our premiss, we get

ii) We want to show that F 4 → T + 1 We'll start by proving that 4 - 1 F 7 4 1. P → 1 premiss PV TP law of the Excluded Middle 6. TP hyp 7. 7 \ V-e 2,3-5,6 Therefore, we now want to show that <u>+ 7 → 1 + 7 ♥</u> cuT And it is correct because of the CUT rule. We can do the same thing as we did at (i) and show that アトヤートファトコ From above we have post to the left-hand side, mothing changes T, P-> 1 1- TP Then, again by wing the cut rule we get We will do this by using a proof tree as follows: $\frac{T, \forall \rightarrow \bot, \forall \vdash \forall }{T, \forall \rightarrow \bot, \forall \vdash \forall \rightarrow \bot} \xrightarrow{hyp} \frac{T, \forall \rightarrow \bot, \forall \vdash \forall \rightarrow \bot}{T, \forall \rightarrow \bot, \forall \vdash \bot} \xrightarrow{P} \frac{T, \forall \rightarrow \bot, \forall \vdash \bot}{T, \forall \rightarrow \bot, \forall \vdash \neg \forall} \xrightarrow{hyp} \frac{T, \forall \rightarrow \bot, \forall \vdash \forall \rightarrow \bot}{T, \forall \rightarrow \bot, \forall \vdash \neg \forall}$ THY CUT

EM* = law of the Excluded Middle

(m)
$$\frac{P \vdash \neg \neg P}{P, P \rightarrow \bot \vdash \bot}$$

$$\frac{P, P \rightarrow \bot \vdash \bot}{P, \neg P \vdash \bot}$$

$$\frac{P, P \rightarrow \bot \vdash \bot}{P, \neg P \vdash \bot}$$

$$\frac{P \vdash \neg \neg P}{P \vdash \neg \neg P}$$

$$\frac{\neg (PVQ) \vdash \neg P \land \neg Q}{\neg (PVQ), P \vdash P} \frac{\neg (PVQ), P \vdash P}{\neg (PVQ), P \vdash PVQ} \frac{\neg (PVQ), P \vdash PVQ}{\neg (PVQ), P \vdash PVQ} \frac{\neg (PVQ), Q \vdash PVQ}{\neg (PVQ), Q$$

$$\frac{(P - Q) \wedge (Q \rightarrow TP) \cdot P + P}{(P - Q) \wedge (Q \rightarrow TP) \cdot P + P} \stackrel{hyp}{(P \rightarrow Q) \wedge (Q \rightarrow TP) \cdot P + Q} \stackrel{hyp}{(P \rightarrow Q) \wedge (Q \rightarrow TP) \cdot P + Q} \stackrel{hyp}{(P \rightarrow Q) \wedge (Q \rightarrow TP) \cdot P + TP} \stackrel{hyp}{(P \rightarrow Q) \wedge (Q \rightarrow TP) \cdot P} \stackrel{hyp}{(P \rightarrow Q) \wedge (Q \rightarrow TP) \cdot P} \vee - e$$

$$(P \rightarrow Q) \Lambda(Q \rightarrow TP) P + P \rightarrow Q$$

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