 GABRIEL MOISE

1. U and V are vector spaces. $T: U \to V$ is a linear transformation if:

① if $u \in U$, then $T(u) \in V$

② if $u_1, u_2 \in U$, then $T(u_1+u_2) = T(u_1) + T(u_2)$ and

③ if $u \in U$ and c is any scalar, then $T(cu) = cT(u)$

(a) We want to prove that ② and ③ are equivalent to

(*) if $u_1, u_2 \in U$ and c is any scalar, then $T(u_1 + cu_2) = T(u_1) + cT(u_2)$

②,③ $\Rightarrow$ (*): As $u_2 \in U$ and $c \in \mathbb{R} \Rightarrow cu_2 \in U$ (Closure under scalar multiplication)

Therefore $u_1, cu_2 \in U \overset{②}{\Longrightarrow} \underline{T(u_1+cu_2)} = T(u_1) + T(cu_2) \overset{③}{=} \underline{T(u_1) + cT(u_2)}$, which is exactly (*)

②,③ $\Leftarrow$ (*) For $c=1$ we get $\underline{T(u_1+u_2) = T(u_1) + T(u_2)}$, for all $u_1, u_2 \in U$, which is exactly ②

Now, we know that U is a vector space, so it has a zero vector, denoted 0.

Then, $T(0+c0) = T(0) + cT(0) \Rightarrow \cancel{T(0)} = \cancel{T(0)} + cT(0) \Rightarrow T(0) = 0$

Now, for $u_1 = 0$, we have $T(0 + cu_2) = T(0) + cT(u_2) \Rightarrow T(cu_2) = cT(u_2)$ for all $u_2 \in U$ and for any scalar c, which is exactly ③.

So, ②,③ $\Leftrightarrow$ (*)

(b) We know that (i) $T: U \to V$ is a linear transformation

(ii) $u_1, u_2, \dots, u_m \in U$ and

(iii) $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars

and we want to prove that $T\left(\sum_{i=1}^{m} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i T(u_i)$

From ② we have $T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_m u_n) = T(\alpha_1 u_1 + \dots + \alpha_{n-1} u_{n-1}) + T(\alpha_n u_n) = \dots = T(\alpha_1 u_1) + \dots + T(\alpha_n u_n)$

From ③ we have $T(\alpha_1 u_1) = \alpha_1 T(u_1)$, $T(\alpha_2 u_2) = \alpha_2 T(u_2)$, ..., $T(\alpha_n u_n) = \alpha_n T(u_n) \overset{\Sigma}{\Longrightarrow}$

$\Rightarrow T(\alpha_1 u_1) + \dots + T(\alpha_m u_n) = \alpha_1 T(u_1) + \dots + \alpha_n T(u_n)$

So, $T(\alpha_1 u_1 + \dots + \alpha_n u_n) = \alpha_1 T(u_1) + \dots + \alpha_n T(u_n)$ or $T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i T(u_i)$.

2. $U = \mathbb{R}^2$, $u \in U$, $u = \begin{pmatrix} x \\ y \end{pmatrix}$, $T: U \to U$

(a) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ y-1 \end{pmatrix}$

$T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2+1 \\ y_1+y_2-1 \end{pmatrix}$

$T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1+1 \\ y_1-1 \end{pmatrix} + \begin{pmatrix} x_2+1 \\ y_2-1 \end{pmatrix} = \begin{pmatrix} x_1+x_2+2 \\ y_1+y_2-2 \end{pmatrix}$

$\Rightarrow T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} \neq T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow$

$\Rightarrow T$ is NOT a linear transformation

(b) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x+y \end{pmatrix}$

$$\left.\begin{array}{l} T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} (x_1+x_2)(y_1+y_2) \\ x_1+x_2+y_1+y_2 \end{pmatrix} = \begin{pmatrix} x_1y_1+x_2y_1+x_1y_2+x_2y_2 \\ x_1+x_2+y_1+y_2 \end{pmatrix} \\ T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1y_1 \\ x_1+y_1 \end{pmatrix} + \begin{pmatrix} x_2y_2 \\ x_2+y_2 \end{pmatrix} = \begin{pmatrix} x_1y_1+x_2y_2 \\ x_1+x_2+y_1+y_2 \end{pmatrix} \end{array}\right| \Rightarrow T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \neq T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} \Rightarrow$$

$\Rightarrow$ T is NOT a linear transformation

(c) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2x+3y \end{pmatrix}$

$$\left.\begin{array}{l} T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2-y_1-y_2 \\ 2x_1+2x_2+3y_1+3y_2 \end{pmatrix} = \begin{pmatrix} x_1-y_1+x_2-y_2 \\ 2x_1+3y_1+2x_2+3y_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ T\begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} cx-cy \\ 2cx+3cy \end{pmatrix} = \begin{pmatrix} c(x-y) \\ c(2x+3y) \end{pmatrix} = c\begin{pmatrix} x-y \\ 2x+3y \end{pmatrix} = cT\begin{pmatrix} x \\ y \end{pmatrix} \end{array}\right| \Rightarrow T \text{ is a linear transformation.}$$

(d) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$

$$\left.\begin{array}{l} T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2+y_1+y_2 \\ x_1+x_2+y_1+y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1+x_2+y_2 \\ x_1+y_1+x_2+y_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ T\begin{pmatrix} cx \\ cy \end{pmatrix} = \begin{pmatrix} cx+cy \\ cx+cy \end{pmatrix} = \begin{pmatrix} c(x+y) \\ c(x+y) \end{pmatrix} = cT\begin{pmatrix} x \\ y \end{pmatrix} \end{array}\right| \Rightarrow T \text{ is a linear transformation.}$$

3. Let $F = \{ f(x) \mid \int_0^1 f(x)\,dx$ takes a finite value$\}$.

(a) We want to prove that F is a vector space:

1. $\int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx$ has a finite value $\Rightarrow f(x)+g(x) \in F$
2. $f(x)+g(x) = g(x)+f(x)$ ok!
3. $f(x)+(g(x)+h(x)) = (f(x)+g(x))+h(x)$ ok!
4. $f(x)=0$ is the zero element
5. If $f(x) \in F \Rightarrow -f(x) \in F$
6. $\alpha f(x) \in F$ as $\int_0^1 \alpha f(x)dx = \alpha \int_0^1 f(x)dx$ has a finite value
7. $\alpha(f(x)+g(x)) = \alpha f(x) + \alpha g(x)$ ok!
8. $(\alpha+\beta)f(x) = \alpha f(x) + \beta f(x)$ ok!
9. $\alpha(\beta f(x)) = (\alpha\beta) f(x)$ ok!
10. $1 \cdot f(x) = f(x)$ ok!

Therefore, F is a vector space.

(b) $T: F \to \mathbb{R}$

$$T(f(x)) = \int_0^1 f(x)\,dx$$

$$T(x^2) = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$T(\sin \pi x) = \int_0^1 \sin \pi x\,dx = \frac{-\cos \pi x}{\pi}\Big|_0^1 = \frac{-\cos\pi + \cos 0}{\pi} = \frac{2}{\pi}$$

(c) We know that $\int_0^1 f(x)\,dx \in \mathbb{R}$ because it takes a finite value.

$$\left.\begin{array}{l} T(f(x)+g(x)) = \int_0^1 (f(x)+g(x))\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = T(f(x)) + T(g(x)) \\ T(cf(x)) = \int_0^1 cf(x)\,dx = c\int_0^1 f(x)\,dx = cT(f(x)) \end{array}\right| \Rightarrow$$

$\Rightarrow$ T is a linear transformation.

4. (a) $T(u)=v$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{rot} \\ y_{rot} \end{pmatrix} \Rightarrow A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

(b) $T(u) = v$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{ref} \\ y_{ref} \end{pmatrix} \Rightarrow \left.\begin{array}{l} x_{ref} = -x \\ y_{ref} = y \end{array}\right| \Rightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(c) The unit normal vector of the line $y=x$ is $n = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

if $v = T(u) \Rightarrow v = u - 2\cdot(u\cdot n)n$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} - 2\cdot\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}\right)n = \begin{pmatrix} x \\ y \end{pmatrix} - \left(x\sqrt{2} - y\sqrt{2}\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x-y \\ y-x \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

So, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(d) • a reflection in the line $y=x \Rightarrow A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

• a rotation of an angle θ about the origin $\Rightarrow R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

• a reflection in the y-axis $\Rightarrow A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A_2RA_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}.$$

5. (a) The matrix representing a translation of the point $\binom{2}{3}$ to the origin is

$$A_1 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The matrix representing a rotation anticlockwise through an angle of $\frac{\pi}{2}$ about the origin is

$$R = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) The matrix representing a translation of the origin to the point $\binom{2}{3}$ is

$$A_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

d) $A_2RA_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$

6. $T: U \to V$ a composite linear transformation, $u \in U$

$$T(u) = CBAu,$$

where $\dim(U) = m$, $A, B, C \in \mathbb{R}^{m \times m}$

a) $Au = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij} u_j$

In total we have m^2 terms to add $\Rightarrow m^2$ additions and m^2 multiplications (each term is a product $A_{ij} u_j$).

b) $v = T(u)$, for $i = 1, 2, \dots, m$:

$$v_i = \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{p=1}^{m} C_{ij} B_{jk} A_{kp} u_p.$$

So, we have

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FOR i=1 TO m
  FOR j=1 TO m
    FOR k=1 TO m
      FOR p=1 TO m   v_i = C_ij B_jk A_kp u_p + v_i
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We have 4 nested FORs, so m^4 operations. Each operation needs 3 multiplications: $C_{ij} B_{jk}$, then $(C_{ij} B_{jk}) A_{kp}$, then $((C_{ij} B_{jk}) A_{kp}) u_p \Rightarrow 3m^4$ multiplications and it needs one addition $\Rightarrow m^4$ additions.

(c) $T(u) = C(B(Au))$

For calculating Au, we need m^2 additions and m^2 multiplications (from (a)). And we obtain a m-dimensional vector u'. The Bu' product needs the same number of operations, as we can replace the A with B and u with u' from (a) to get the same result. $Bu' = u''$ and for Cu'' we have the same result.

Summing up, we need $3m^2$ additions and $3m^2$ multiplications to evaluate $T(u)$ this way.

(d) I would definitely use the method from (c) as it needs less additions (it needs $3m^2$ additions compared to m^4 additions at (b)) and less multiplications (it needs $3m^2$ multiplications compared to $3m^4$ multiplications at (b)). So the method from (c) is better and definitely more efficient. (for $m=1$ it's not a noticeable difference).

7. $T_1 : U \to U$, $T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2x+3y \end{pmatrix}$

$T_2 : U \to U$, $T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$, where $U = \mathbb{R}^2$

a) $T_1\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{array}{l} x-y=0 \Rightarrow x=y \\ 2x+3y=0 \end{array} \Big| \Rightarrow 5x=0 \Rightarrow x=0 \Rightarrow y=0 \Rightarrow \ker(T_1) = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$

$T_2\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x+y=0 \Rightarrow y=-x \Rightarrow \ker(T_2) = \left\{\begin{pmatrix} x \\ -x \end{pmatrix} \Big| x \in \mathbb{R}\right\}$

b) $T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ 2x+3y \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow \operatorname{Im}(T_1) = \left\{x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} -1 \\ 3 \end{pmatrix} \Big| x,y \in \mathbb{R}\right\} \overset{(c)}{=} \mathbb{R}^2$

$T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix} \Rightarrow \operatorname{Im}(T_2) = \left\{\begin{pmatrix} x \\ x \end{pmatrix} \Big| x \in \mathbb{R}\right\}$

c) Let $v \in U$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\begin{cases} x-y=v_1 \\ 2x+3y=v_2 \end{cases} \Rightarrow \begin{cases} x-y=v_1 \\ 5y = v_2-2v_1 \Rightarrow y = \dfrac{v_2-2v_1}{5} \end{cases} \Big| \Rightarrow x = \dfrac{v_2-2v_1}{5} + v_1 = \dfrac{3v_1+v_2}{5}$

So, for any vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in U$, we have

$T_1\begin{pmatrix} \frac{3v_1+v_2}{5} \\ \frac{-2v_1+v_2}{5} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v \Rightarrow \operatorname{Im}(T_1) = U \Rightarrow T_1$ is an ONTO transformation

As $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin \operatorname{Im}(T_2) \Rightarrow \operatorname{Im}(T_2) \neq U \Rightarrow T_2$ is NOT an ONTO transformation

d) Suppose $T_1\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow \begin{array}{l} x_1-y_1 = x_2-y_2 \Rightarrow x_2 = x_1-y_1+y_2 \\ 2x_1+3y_1 = 2x_2+3y_2 \end{array} \Big| \Rightarrow$

$\Rightarrow \cancel{2x_1}+3y_1 = \cancel{2x_1}-2y_1+2y_2+3y_2 \Rightarrow 5y_1 = 5y_2 \Rightarrow y_1 = y_2$

$x_2 = x_1-y_1+y_2 \Big| \Rightarrow x_2 = x_1 \Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Rightarrow$

$\Rightarrow T_1$ is a 1-1 transformation

We have $T_2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $T_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so T_2 is NOT a 1-1 transformation

8. $T : \mathbb{R}^5 \to \mathbb{R}^3$

$T(u) = Au$,

where $u \in \mathbb{R}^5$ and

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{pmatrix}.$$

We have $u = \begin{pmatrix} p \\ q \\ r \\ s \\ t \end{pmatrix}$ and $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, so

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 2 & -1 & 1 & 2 & 2 \\ 3 & 0 & 6 & 0 & -3 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \\ t \end{pmatrix}, \text{ or}$$

$$\begin{pmatrix} a \\ b-2a \\ c-3a \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 0 & -3 & -9 & 0 & -6 \\ 0 & -3 & -9 & -3 & -15 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \\ t \end{pmatrix}, \text{ or}$$

$$\begin{pmatrix} a \\ b-2a \\ c-b-a \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 & 1 & 4 \\ 0 & -3 & -9 & 0 & -6 \\ 0 & 0 & 0 & -3 & -9 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \\ t \end{pmatrix} \quad (*)$$

Suppose $u \in \mathrm{Ker}(T) \Rightarrow$

$$\left.\begin{matrix} p+q+5r+s+4t=0 \\ -3q-9r \quad -6t=0 \\ -3s-9t=0 \Rightarrow s=-3\alpha \\ t=\alpha \end{matrix}\right| \Rightarrow \left.\begin{matrix} p+q+5r=-\alpha \\ -3q-9r=6\alpha \Rightarrow q=-2\alpha-3\beta \\ r=\beta \end{matrix}\right| \Rightarrow p=\alpha-2\beta$$

Therefore, $u = \begin{pmatrix} \alpha-2\beta \\ -2\alpha-3\beta \\ \beta \\ -3\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathrm{Ker}(T) = \left\{ \alpha \begin{pmatrix} 1 \\ -2 \\ 0 \\ -3 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\}$.

So, from that we can deduce the the NULLITY of T is 2.

By using the theorem that says that

nulity of T + rank of $T = \dim(\mathbb{R}^5) = 5 \Rightarrow$ the RANK of T is 3.

As the system formed from $(*)$ is under-determined, then there is an infinite number of solutions $\begin{pmatrix} p \\ q \\ r \\ s \\ t \end{pmatrix}$ for it, so $\mathrm{Im}(T) = \left\{ \begin{pmatrix} a \\ b-2a \\ c-b-a \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\}$, or we can write the image of T as

$\mathrm{Im}(T) = \left\{ a \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{R} \right\}$ and from here we can see and check that the rank of T is indeed 3. (therefore we can also deduce that $\mathrm{Im}(T) = \mathbb{R}^3$)