GABRIEL MOISE

DISCRETE MATHEMATICS

MT 2018

PROBLEM SHEET 3

Chapter 4 (Relations) and 5 (Sequences)

$\boxed{3.1}$ (i) We'll choose the $<$ relation and we'll prove the following:

1. $<$ is <u>antisymmetric</u> $\Leftrightarrow$ for all $a, b \in \mathbb{N}$, if $a<b$ and $b<a$ (which is always False), then $a=b$. As the condition is never met, the whole statement is considered to be True.

2. $<$ is <u>irreflexive</u> $\Leftrightarrow$ for all $a \in \mathbb{N}$, we have $a \not< a$, which is obviously True as a cannot be less than itself

3. $<$ is <u>transitive</u> $\Leftrightarrow$ for all $a, b, c \in \mathbb{N}$, if we have $a<b$ and $b<c$, then we'll prove that $a<c$. From $a<b$, we get that $b = a+k$, with $k \in \mathbb{N}_+$ and from $b<c$, we get that $c = b+p$, with $p \in \mathbb{N}_+$. Then, $c = a+(k+p) = a+t$, where $t = k+p \in \mathbb{N}_+$, therefore $a<c$.

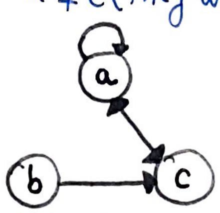
4. $<$ is <u>serial</u> $\Leftrightarrow$ for every $a \in \mathbb{N}$ there is some $b \in \mathbb{N}$ with $a<b$. To show that, we will choose $b = a+1$, and as $a \in \mathbb{N}$, $b = a+1 \in \mathbb{N}$, too and also $a < a+1 = b$, for every $a \in \mathbb{N}$

5. $<$ is <u>not symmetric</u> $\Leftrightarrow$ there is a pair $(a,b) \in \mathbb{N} \times \mathbb{N}$ such that $a<b$ and $b \not< a$. We can choose here $a=0$ and $b=1$ and we have $(0,1) \in \mathbb{N} \times \mathbb{N}$, $0<1$, but $1 \not< 0$.

(ii) $A = \{a, b, c\}$. We will show that the relation $R = \{(a,a), (b,c), (a,c), (c,a)\}$, which is a subset of $A \times A$ is:

1. <u>not symmetric</u>: We have bRc, but $c \not R b$
2. <u>not antisymmetric</u>: We have aRc and cRa, but $a \neq c$ (they are different elements of the A set)
3. <u>not reflexive</u>: We have $b \not R b$
4. <u>not irreflexive</u>: We have aRa

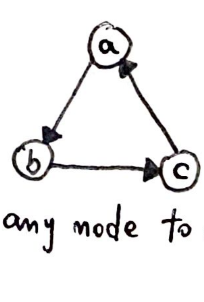
The relation R can be drawn as:



(iii) $A = \{a, b, c\}$. We will show that the relation $R = \{(a,b), (b,c), (c,a)\}$, which is a subset of $A \times A$:

1. is <u>not symmetric</u>: We have bRc, but $c \not R b$.
2. has a <u>symmetric transitive closure</u>: $R^+ = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\} = A \times A$, as we can get from each of a, b, c to each of a, b, c by using the relation R. Additionally, R^+ is obviously symmetric as $R^+ = A \times A$, therefore R has a symmetric transitive closure.

The relation R can be drawn as:



(We can also deduce from the draw that you can get from any node to any node by going counterclockwise in the graph).

$\boxed{3.2}$ On the set $A=\{0,1\}$, we can have $2^{2^2}=2^4=16$ relations which is equal to the number of subsets of $A\times A$.
As $A\times A$ has 4 elements, then the total number of relations is equal to the number of elements from $P(A\times A)$ which is $2^4=16$. $R\subseteq A\times A$ is reflexive if for all $x\in A$ we have xRx, so $(0,0)\in R$ and $(1,1)\in R$.
Therefore, the number of relations is equal to the number of elements from $P(A\times A\setminus\{(0,0),(1,1)\})$, which is $P(\{(0,1),(1,0)\})$, which is $2^2=4$. So, we have 4 reflexive relations on $A=\{0,1\}$.

- Now, we want to know how many relations on A are symmetric.
  First, if $R=\emptyset$, then R is symmetric by the definition of symmetric relations.
  Now, if $(x,y)\in R$, with $x,y\in\{0,1\}$, then also $(y,x)\in R$.

Case 1: $x=y \Rightarrow$ if $(x,x)\in R$, then also $(x,x)\in R$, which is always True, so we can have (x,x) in R or not without affecting R's symmetry.

Case 2: $x\neq y \Rightarrow$ if $(x,y)\in R$, then also $(y,x)\in R$. As $x,y\in\{0,1\} \Rightarrow (0,1)$ and $(1,0)\in R$ (both) or $(0,1)\notin R$ and $(1,0)\notin R$ (neither)

So, we have two cases for Case 2 and 4 cases for Case 1, so by using the product law we get $2\cdot 4=8$ symmetric functions ($\emptyset$ is also included when neither $(0,0)$, nor $(1,1)$ are in R (case 1) and neither $(1,0)$, nor $(0,1)$ are in R (case 2))

- Now, we want to know how many relations on A are antisymmetric.
  Again, we start with $R=\emptyset$, which is by definition antisymmetric.
  Now, if $(x,y)\in R$ and $(y,x)\in R$, then $x=y$ (with $x,y\in A$)

Case 1: $x=y \Rightarrow$ if $(x,x)\in R$ and $(x,x)\in R$, then $x=x$ (so, any (x,x) pair can be in R without affecting its antisymmetry).

Case 2: $x\neq y$: if $(x,y)\in R$ and $(y,x)\in R$, then $x=y$. So, that means we can't have both $(0,1)$ and $(1,0)$ in R. We either have none, or just one of them, so 3 possibilities.

So, by the product law, we have $\underset{\text{Case 1}}{4}\cdot\underset{\text{Case 2}}{3}=12$ antisymmetric functions on A ($\emptyset$ is also included here).

- Now, we want to calculate the number of transitive relations on A.
  $R=\emptyset$ holds.
  If $(x,y)\in R$ and $(y,z)\in R$, then $(x,z)\in R$, for all $x,y,z\in A$

Case 1: $(0,1)\in R$

Subcase 1.1: $(1,0)\in R \Rightarrow (0,0)\in R$ and $(1,1)\in R$ (from the definition of transitivity) $\Rightarrow R=A\times A$

Subcase 1.2: $(1,0)\notin R \Rightarrow (0,0)$ and $(1,1)$ do not affect R's transitivity $\Rightarrow$ 4 possibilities:
$R=\{(0,1)\}$, $R=\{(0,1),(0,0)\}$, $R=\{(0,1),(1,1)\}$, $R=\{(0,1),(0,0),(1,1)\}$

Case 2: $(0,1)\notin R$

Subcase 2.1: $(1,0)\in R \Rightarrow (0,0)$ and $(1,1)$ do not affect R's transitivity $\Rightarrow$ 4 possibilities:
$R=\{(1,0)\}$, $R=\{(1,0),(0,0)\}$, $R=\{(1,0),(1,1)\}$, $R=\{(1,0),(0,0),(1,1)\}$.

Subcase 2.2: $(1,0)\notin R \Rightarrow (0,0)$ and $(1,1)$ do not affect R's transitivity $\Rightarrow$ 4 possibilities
$R=\emptyset$, $R=\{(0,0)\}$, $R=\{(1,1)\}$, $R=\{(0,0),(1,1)\}$

So, in total we have 13 transitive relations on $A=\{0,1\}$.

- Now, we consider A to be a set with $|A| = n$ and we want to calculate the total number of antisymmetric relations on A.

Let R be a relation with $R \subseteq (A \times A)$.

$A = \{x_1, x_2, \dots, x_n\}$ and R is antisymmetric.

Then, if $(x_i, x_j) \in R$ and $(x_j, x_i) \in R$, then x_i must be equal to x_j. In other words, we cannot have both (x_i, x_j) and (x_j, x_i) in R if $x_i \neq x_j$ (or $i \neq j$) (here, $i, j \in \{1, 2, \dots, n\}$).

STATEMENT

For each pair of (x_i, x_j), we can have $(x_i, x_j) \notin R$ and $(x_j, x_i) \notin R$ or
$(x_i, x_j) \in R$ and $(x_j, x_i) \notin R$ or
$(x_i, x_j) \notin R$ and $(x_j, x_i) \in R$.

To avoid counting the same cases more than once, we will suppose that $i < j$ in the statement above.

We have $\binom{n}{2}$ pairs of (i, j) with $i < j$, so $\frac{(n-1)n}{2}$ pairs and for each pair we have 3 possible cases, so, by using the product law, we get $3^{\frac{(n-1)n}{2}}$ relations.

However, we did not say anything about the pairs (x_i, x_i) which do not affect R's antisymmetry. So, for each of these relations we have 2^n possibilities of including the (x_i, x_i) pairs in R.

Therefore, again by the product law, we obtain that the total number of antisymmetric relations on A, where $|A| = n$, is $3^{\frac{(n-1)n}{2}} \cdot 2^n$.

$\boxed{3.3}$ We define the relation $m \sim n$ on $\{1, 2, 3, \dots, 16\}$ by $m \sim n$ if $n = 2^k m$ if $n = 2^k m$ for some $k \in \mathbb{Z}$.

(i) We want to prove that $\sim$ is an equivalence relation.

First of all, we can see that $\sim$ is reflexive: $m \sim m$ for all $m \in A = \{1, 2, \dots, 16\}$, as $m = 2^0 \cdot m$, for all $m, n \in A$;
symmetric: if $m \sim n$, then $n = 2^k m$ for some $k \in \mathbb{Z} \Rightarrow m = 2^{-k} n$, with $(-k) \in \mathbb{Z}$, so also $n \sim m$ and
transitive: if $m \sim n$ and $n \sim p \Rightarrow n = 2^k m$ and $p = 2^a n$, for some $a, k \in \mathbb{Z} \Rightarrow p = 2^a \cdot 2^k \cdot m = 2^{(a+k)} \cdot m$, with $(a+k) \in \mathbb{Z} \Rightarrow m \sim p$, for all $m, n, p \in A$. In conclusion, $\sim$ is an equivalence relation on A.

(ii) We partition A in: $B_1 = \{1, 2, 4, 8, 16\}$; $B_2 = \{3; 6; 12\}$; $B_3 = \{5; 10\}$; $B_4 = \{7, 14\}$; $B_5 = \{9\}$; $B_6 = \{11\}$; $B_7 = \{13\}$ and $B_8 = \{15\}$. We can see that this way, if we have an $x \in B_i$, with $i \in \{1, 2, \dots, 8\}$, and $x \sim y$, with $y \in A \Rightarrow y \in B_i$ too.

We have
$[1] = [2] = [4] = [8] = [16] = B_1$
$[3] = [6] = [12] = B_2$
$[5] = [10] = B_3$
$[7] = [14] = B_4$
$[9] = B_5$
$[11] = B_6$
$[13] = B_7$
$[15] = B_8$

So, $B_1, B_2, \dots, B_8$ are all the equivalence classes of A.

If we define the relation $m \sim n$ on $\{1, 2, \dots, 2N\}$, where N is a fixed positive integer by $m \sim n$ if $n = 2^k m$ for some $k \in \mathbb{Z}$, there will be N equivalence classes and this can be proven by induction: The base case is for $N = 1 \Rightarrow$ the set $\{1, 2\}$ has only one equivalence class which is $\{1, 2\}$. By saying that the set $\{1, 2, \dots, 2N\}$ has N equivalence classes, then $\{1, 2, \dots, 2N+2\}$ has the same equivalence classes plus the class of $2N+1$, which is $\{2N+1\}$ as no number from 1 to $2N$ can have $2N+1$ in its class, and $2N+2$ is in the class of $N+1$, as $(N+1) \sim (2N+2)$. So now we have $N+1$ classes, therefore the inductive step is proven. In conclusion, for a set $\{1, 2, \dots, 2N\}$ there are N equivalence classes.

3.4 $F_0=0,\ F_1=1,\ F_{m+2}=F_{m+1}+F_m$, for all $m\in\mathbb{N}$

(i) We'll prove by induction on m that

$$F_0+F_1+\ldots+F_m=F_{m+1}-1, \text{ for } m\in\mathbb{N}$$

The base case

$S(0)$: $F_0=F_1-1$

$0=1-1$ (True)

The inductive step

iH: We know $S(k)$ is true $\Rightarrow F_0+F_1+\ldots+F_k=F_{k+2}-1$

We'll prove $S(k+1)$: $F_0+F_1+\ldots+F_k+F_{k+1}=F_{k+3}-1$

$(F_0+F_1+\ldots+F_k)+F_{k+1}=F_{k+3}-1$ (Addition is associative)

$(F_{k+2}-1)+F_{k+1}=F_{k+3}-1$ (Using the inductive hypothesis (iH))

$F_{k+1}+F_{k+2}=F_{k+3}$ (Addition is commutative and associative and we added 1 to LHS and RHS)

And this is the recurrence relation for the Fibonacci sequence.

So, we proved $S(k+1)$.

Therefore, the initial stament is true for all $m\in\mathbb{N}$.

(ii) We'll prove by induction on m that

$$0\cdot F_0+1\cdot F_1+\ldots+m\cdot F_m=m\cdot F_{m+2}-F_{m+3}+2 \text{ for } m\in\mathbb{N}$$

The base case

$S(0)$: $0\cdot F_0=0\cdot F_2-F_3+2$

$0=2-F_3=2-(F_1+F_2)=2-(F_1+(F_0+F_1))=2-(1+0+1)=0$ (True)

The inductive step

iH: We know $S(k)$ is true $\Rightarrow 0\cdot F_0+1\cdot F_1+\ldots+k\cdot F_k=k\cdot F_{k+2}-F_{k+3}+2$

We'll prove $S(k+1)$: $0\cdot F_0+1\cdot F_1+\ldots+k\cdot F_k+(k+1)\cdot F_{k+1}=(k+1)\cdot F_{k+3}-F_{k+4}+2$

$(0\cdot F_0+1\cdot F_1+\ldots+k\cdot F_k)+(k+1)\cdot F_{k+1}=(k+1)F_{k+3}-F_{k+4}+2$ (Addition is associative)

$k\cdot F_{k+2}-F_{k+3}+\cancel{2}+(k+1)F_{k+1}=(k+1)F_{k+3}-F_{k+4}+\cancel{2}$ (iH)

$F_{k+4}-(k+2)F_{k+3}+k\cdot F_{k+2}+(k+1)F_{k+1}=0$ (rearrange)

$F_{k+3}+F_{k+2}-(k+2)F_{k+3}+k\cdot F_{k+2}+(k+1)F_{k+1}=0$ (recurrence relation)

$-(k+1)F_{k+3}+(k+1)F_{k+2}+(k+1)F_{k+1}=0$ (divide by $(k+1)\neq 0$)

$F_{k+3}=F_{k+2}+F_{k+1}$ (true) (recurrence relation)

So, we proved $S(k+1)$.

Therefore, the initial statement is true for all $m\in\mathbb{N}$.

$\boxed{3.5}$ (i) We want to prove, by induction on n, that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \text{ for } n \in \mathbb{N}_+$$

Base case

$$S(1): \sum_{i=1}^{1} i = \frac{1 \cdot 2}{2}$$

$$1 = \frac{2}{2} \quad \text{(True)}$$

The inductive step

iH: We know that $S(k)$ is true: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

We'll prove $S(k+1)$: $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\sum_{i=1}^{k} i + (k+1) = \frac{(k+1)(k+2)}{2} \quad \text{(Addition is associative)}$$

$$\frac{k(k+1)}{2} + {}^{2)}(k+1) = \frac{(k+1)(k+2)}{2} \quad \text{(Using iH)}$$

$$\frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)}{2} \quad \text{(True) (Multiplication is commutative)}$$

So, we proved $S(k+1)$.

Therefore, the initial statement is true for all $n \in \mathbb{N}_+$.

(ii) We want to prove, by induction on n, that

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}, \text{ for } n \in \mathbb{N}_+$$

Base case

$$S(1): \sum_{i=1}^{1} i(i+1) = \frac{1 \cdot 2 \cdot 3}{3}$$

$$1 \cdot 2 = 2 \quad \text{(True)}$$

The inductive step

iH: We know that $S(k)$ is true: $\sum_{i=1}^{k} i(i+1) = \frac{k(k+1)(k+2)}{3}$

We'll prove $S(k+1)$: $\sum_{i=1}^{k+1} i(i+1) = \frac{(k+1)(k+2)(k+3)}{3}$

$$\sum_{i=1}^{k} i(i+1) + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3} \quad \text{(Addition is associative)}$$

$$\frac{k(k+1)(k+2)}{3} + {}^{3)}(k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

$$\frac{(k+1)(k+2)(k+3)}{3} = \frac{(k+1)(k+2)(k+3)}{3} \quad \text{(True)}$$

So, we proved $S(k+1)$.

Therefore, the initial statement is true for all $n \in \mathbb{N}_+$.

(iii) We can generalise these results and state that

$$\sum_{i=1}^{n} i(i+1)(i+2)\cdot\ldots\cdot(i+m-1) = \frac{n(n+1)(n+2)\cdot\ldots\cdot(n+m)}{m+1}, \text{ with } m \in \mathbb{N}_+ \text{ and } n \in \mathbb{N}_+.$$

We will prove that by induction on n.

<u>The base case</u>

$$S(1): \quad \sum_{i=1}^{1} i(i+1)(i+2)\cdot\ldots\cdot(i+m-1) = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot(m+1)}{m+1}$$

$$1\cdot 2\cdot 3\cdot\ldots\cdot m = 1\cdot 2\cdot 3\cdot\ldots\cdot m \quad \text{(True)}$$

<u>The inductive step</u>

IH: We know $S(k)$ is true: $\displaystyle\sum_{i=1}^{k} i(i+1)\cdot\ldots\cdot(i+m-1) = \frac{k(k+1)\cdot\ldots\cdot(k+m)}{m+1}$

We'll prove $S(k+1)$: $\displaystyle\sum_{i=1}^{k+1} i(i+1)\cdot\ldots\cdot(i+m-1) = \frac{(k+1)(k+2)\cdot\ldots\cdot(k+m+1)}{m+1}$

$$\sum_{i=1}^{k} i(i+1)\cdot\ldots\cdot(i+m-1) + (k+1)(k+2)\cdot\ldots\cdot(k+m) = \frac{(k+1)(k+2)\cdot\ldots\cdot(k+m+1)}{m+1} \quad \text{(Addition is associative)}$$

$$\frac{k(k+1)\cdot\ldots\cdot(k+m)}{m+1} + {}^{m+1}\!/\,(k+1)(k+2)\cdot\ldots\cdot(k+m) = \frac{(k+1)(k+2)\cdot\ldots\cdot(k+m+1)}{m+1} \quad \text{(Using the IH)}$$

$$\frac{(k+1)(k+2)\cdot\ldots\cdot(k+m)(k+m+1)}{m+1} = \frac{(k+1)(k+2)\cdot\ldots\cdot(k+m+1)}{m+1} \quad \text{(True)}$$

So, we proved $S(k+1)$.

Therefore, the initial statement is true for all $n \in \mathbb{N}_+$ and $m \in \mathbb{N}_+$

$\boxed{3.6}$ We consider the recurrence $a_1 = 1$, $a_n = n\, a_{\lfloor n/2 \rfloor}$, for $n \geq 2$. Therefore, we have

$a_2 = 2\cdot a_1 = 2$ $\quad a_5 = 5\cdot a_2 = 10$

$a_3 = 3\cdot a_1 = 3$ $\quad a_6 = 6\cdot a_3 = 18$ $\quad$ ... and so on.

$a_4 = 4\cdot a_2 = 8$ $\quad a_7 = 7\cdot a_3 = 21$

We want to prove, by using strong induction on n, that

$$a_n \leq n^{\log_2 n} \text{ for all } n \in \mathbb{N}_+.$$

<u>The base case</u>

$$S(1): \quad a_1 \leq 1^{\log_2 1}$$

$$1 \leq 1^0 = 1 \quad \text{(True)}$$

<u>The inductive step</u>

IH: We know that for all $i \in \{1, 2, \ldots, k\}$ $S(i)$ is true: $a_i \leq i^{\log_2 i}$

We'll prove $S(k+1)$: $a_{k+1} \leq (k+1)^{\log_2 (k+1)}$

Case 1: k is even $\Rightarrow$ we can write k as $2p$, where $p \in \mathbb{N}_+ \Rightarrow k+1 = 2p+1$

We want to prove that $a_{2p+1} \leq (2p+1)^{\log_2(2p+1)}$

But, $a_{2p+1} = (2p+1)\, a_{\lfloor (2p+1)/2 \rfloor} = (2p+1)a_p$

Additionally, from the inductive hypothesis, as $p \in \{1,2,\dots,2p\}$, we know that $S(p)$ is true, therefore $a_p \leq p^{\log_2 p}$. Now, we can write:

$$a_{2p+1} = (2p+1)a_p \leq (2p+1)\,p^{\log_2 p}$$

However, $(2p+1)\,p^{\log_2 p} \underset{\substack{\uparrow \\ \log_2 p \,\leq\, \log_2(p+\frac{1}{2}) \\ (p \geq 1)}}{\leq} (2p+1)\,p^{\log_2(p+\frac{1}{2})} = (2p+1)\,p^{\log_2(2p+1)-1} \underset{\substack{\uparrow \\ p \,\leq\, 2p+1}}{\leq} (2p+1)\cdot(2p+1)^{\log_2(2p+1)-1} = (2p+1)^{\log_2(2p+1)}$

Therefore, $(2p+1)p^{\log_2 p} \leq (2p+1)^{\log_2(2p+1)}$ or $\underline{a_{2p+1} \leq (2p+1)^{\log_2(2p+1)}}$.

Case 2: k is odd $\Rightarrow$ we can write k as $2p-1$, where $p \in \mathbb{N}_+ \Rightarrow k+1 = 2p$

We want to prove that $a_{2p} \leq (2p)^{\log_2(2p)}$

But, $a_{2p} = (2p)\cdot a_{\lfloor 2p/2 \rfloor} = (2p)a_p$

Additionally, from the inductive hypothesis, as $p \in \{1,2,\dots,2p\}$, we know that $S(p)$ is true, therefore $a_p \leq p^{\log_2 p}$. Now, we can write:

$$a_{2p} = (2p)\,a_p \leq (2p)\cdot p^{\log_2 p}$$

However, $(2p)\cdot p^{\log_2 p} \underset{\substack{\uparrow \\ p \,\leq\, 2p}}{\leq} (2p)\cdot(2p)^{\log_2 p} = (2p)^{\log_2 p + 1} = (2p)^{\log_2(2p)}$

Therefore, $(2p)\cdot p^{\log_2 p} \leq (2p)^{\log_2(2p)}$ or $\underline{a_{2p} \leq (2p)^{\log_2(2p)}}$.

In both cases we showed that $a_{k+1} \leq (k+1)^{\log_2(k+1)}$, so we proved $S(k+1)$.
Therefore, the initial statement is true for all $n \in \mathbb{N}_+$.

$\boxed{3.7}$ Let b_n denote the number of binary trees with n nodes.

(i) A binary tree with n nodes can be formed from a root and two other binary trees (with less nodes than n, one to the left and one to the the right of the root. The total number of nodes of the left tree plus the total number of nodes of the right tree must be equal to $n-1$ (we counted out the root).

So, we can form a tree with n nodes from a subtree (on the left) with i nodes and a subtree (on the right) with $n-i-1$ nodes.

Let's say that we have a configuration of n nodes, the left subtree has i nodes and the right subtree has $n-i-1$ nodes. How many configurations do we have with this property? The number of configurations of the left subtree is b_i and the number of configurations for the right subtree is b_{n-i-1}, so, by using the product law we get $b_i \cdot b_{n-i-1}$ configurations. But i can take any value from $\{0,1,2,\dots,n-1\}$, so the total number of configurations of binary trees with n nodes is:

$$b_n = \sum_{i=0}^{n-1} b_i \cdot b_{n-i-1} = b_0 b_{n-1} + b_1 b_{n-2} + \dots + b_{n-2} b_1 + b_{n-1} b_0,$$ for all $n>1$, as for $n=0$ and $n=1$ we already know that $b_0 = b_1 = \underline{1}$.

(ii) By using our formula we get:

$b_2 = b_0 b_1 + b_1 b_0 = 1+1 = 2$

$b_3 = b_0 b_2 + b_1 b_1 + b_2 b_0 = 2+1+2 = 5$

$b_4 = b_0 b_3 + b_1 b_2 + b_2 b_1 + b_3 b_0 = 5+2+2+5 = 14$

$b_5 = b_0 b_4 + b_1 b_3 + b_2 b_2 + b_3 b_1 + b_4 b_0 = 14+5+4+5+14 = 42$

So, we found that $\boxed{b_5 = 42}$.