PROBABILITY

GABRIEL MOISE

PROBLEM SHEET 3

1. X has the Binomial distribution with parameters n and p if

$$\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$$

We want to calculate the mean of X, which is

$$\mathbb{E}(X)=\sum_{x\in \operatorname{Im} X} x\,\mathbb{P}(X=x)$$

(i) $\mathbb{E}(X)=\sum_{k=0}^{n} k\cdot\binom{n}{k}p^k(1-p)^{n-k}$

Let $T=\sum_{k=0}^{n} k\cdot\binom{n}{k}p^{n-k}(1-p)^{k}$

$$\overset{(+)}{\Rightarrow} \mathbb{E}(X)+T=n\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}=n(\cancel{p}+1-\cancel{p})^n=n$$

Claim: $\frac{\mathbb{E}(X)}{T}=\frac{p}{1-p}$

Proof: Let $k\in\{0,1,\dots,n\}\Rightarrow k\binom{n}{k}=(n-k+1)\binom{n}{n-k+1}$

$$k\cdot\frac{\cancel{n!}}{k!\cdot(n-k)!}=(n-k+1)\frac{\cancel{n!}}{(k-1)!\,(n-k+1)!}$$

$$\frac{1}{(k-1)!\,(n-k)!}=\frac{1}{(k-1)!\,(n-k)!} \quad \text{OK}$$

So, by multiplying $\mathbb{E}(X)$ with $(1-p)$ and T with p, we obtain two sums of elements which are equal two by two.

Therefore, $T=\frac{1-p}{p}\mathbb{E}(X)\Rightarrow \frac{1-p}{p}\mathbb{E}(X)+\mathbb{E}(X)=n$

$$\frac{1-\cancel{p}+\cancel{p}}{p}\mathbb{E}(X)=n\Rightarrow \boxed{\mathbb{E}(X)=np}$$

(ii) The expectation of having k successes out of n experiments is equal to the sum of expectations of n independent trials (which have Bernoulli distributions).

For a Bernoulli distribution, the mean is always p, so for the Binomial distribution the mean is np.

$$\mathbb{E}(X)=\mathbb{E}(Y_1+Y_2+\dots+Y_n),\ \text{where } Y_i=\mathbb{I}(\{\text{success}\})\Rightarrow\mathbb{E}(Y_i)=p$$

$$\mathbb{E}(X)=\mathbb{E}(Y_1)+\mathbb{E}(Y_2)+\dots+\mathbb{E}(Y_n)=np.$$

4. $\mathbb{E}[f(X)]=\sum_{x\in \operatorname{Im}(X)} f(x)\mathbb{P}(X=x)=\sum_{k=0}^{\infty}f(k)\mathbb{P}(X=k)=\sum_{k=0}^{\infty}e^{\theta k}\cdot\frac{e^{-\lambda}\lambda^k}{k!}=$

$$=\frac{\sum_{k=0}^{\infty}\frac{e^{\theta k}\lambda^k}{k!}}{e^{\lambda}}=\frac{\sum_{k=0}^{\infty}\frac{(e^{\theta}\lambda)^k}{k!}}{e^{\lambda}}=\frac{e^{(e^{\theta}\lambda)}}{e^{\lambda}}=e^{e^{\theta}\lambda-\lambda}=e^{\lambda(e^{\theta}-1)} \quad \square.$$

2. $X : \Omega \to \mathbb{N}$

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$$

LHS: $\mathbb{E}(X) = \sum_{k=0}^{\infty} k\, \mathbb{P}(X = k) = \sum_{k=0}^{\infty} k p_X(k) = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) + \dots$

RHS: $\sum_{k=0}^{\infty} \mathbb{P}(X > k) = \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \mathbb{P}(X = i) = \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} p_X(i)$

Let's say we want to know how many times $p_X(a)$ appears in the RHS sum, for any a: as we can figure it out from $\sum_{i=k+1}^{\infty} p_X(i)$, $p_X(a)$ appears exactly once in the sums for $i = 0, 1, \dots, a-1$, so it appears exactly a times. Therefore, LHS = RHS.

3. $\mathbb{P}(X = k) = p(1-p)^{k-1}$, $k \geq 1$

(a) $\mathbb{P}(X > k)$, $k \geq 0$

$$\mathbb{P}(X > k) = \sum_{a=k+1}^{\infty} \mathbb{P}(X = a) = \sum_{a=k+1}^{\infty} p(1-p)^{a-1} = \sum_{b=k}^{\infty} p(1-p)^{b} =$$

$$= \sum_{b=0}^{\infty} p(1-p)^{b} - \sum_{b=0}^{k-1} p(1-p)^{b} = \frac{\cancel{p}}{\cancel{1-(1-p)}} - \frac{\cancel{p}(1-(1-p)^k)}{\cancel{1-(1-p)}} = \cancel{1} - (\cancel{1} - (1-p)^k) =$$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \qquad \sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}$$

$$= (1-p)^k \Rightarrow \boxed{\mathbb{P}(X > k) = (1-p)^k}$$

(b) $\mathbb{P}(X = k+n \mid X > k) = \mathbb{P}(X = n)$

if $X = k+n \overset{\text{always! } (n>0)}{\Longrightarrow} X > k \Rightarrow \{X = k+n\} \cap \{X > k\} = \{X = k+n\}$

$$\frac{\mathbb{P}(\{X = k+n\} \cap \{X > k\})}{\mathbb{P}(\{X > k\})} = \mathbb{P}(X = n)$$

$$\frac{\mathbb{P}(X = k+n)}{\mathbb{P}(X > k)} = \mathbb{P}(X = n)$$

(a)

$$\frac{\cancel{p}(1-p)^{k+n-1}}{(1-p)^k} = \cancel{p}(1-p)^{n-1}$$

$(1-p)^{n-1} = (1-p)^{n-1}$ YES!

Therefore, $\mathbb{P}(X = k+n \mid X > k) = \mathbb{P}(X = n)$ for all $k \geq 0$, $n \geq 1$

5. $X \sim Bin(m, \frac{\lambda}{n})$, $\lambda > 0 \Rightarrow \mathbb{P}(X=k) = \binom{m}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{m-k}$, $k = 0, 1, 2, \dots, m$

a) For large m we use the approximation $\left(1 - \frac{x}{n}\right)^m \approx e^{-x}$

We want to show that for a fixed $k \geq 0$ and for a large m we have:

$$\mathbb{P}(X=k) \approx \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\left.\begin{aligned} \mathbb{P}(X=k) &= \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\ \frac{e^{-\lambda}\lambda^k}{k!} &\approx \frac{\left(1-\frac{\lambda}{n}\right)^m \lambda^k}{k!}\end{aligned}\right| \Rightarrow \text{We want to prove that:}$$

$$\binom{n}{k}\frac{\cancel{\lambda^k}}{n^k}\cdot\frac{(m-\lambda)^{n-k}}{n^{n-k}} \approx \frac{(m-\lambda)^n \cancel{\lambda^k}}{n^n \cdot k!}$$

$$\frac{n!}{(n-k)!\,\cancel{k!}}\cdot\frac{(n-\lambda)^{n-k}}{\cancel{n^k}} \approx \frac{(n-\lambda)^n}{\cancel{n^k}\cdot\cancel{k!}}$$

$$\frac{n!}{(n-k)!} \approx (m-\lambda)^k$$

k terms

We have: $\lim\limits_{n\to\infty} \underbrace{\frac{n-k+1}{n-\lambda}}_{\to 1} \cdot \underbrace{\frac{n-k+2}{n-\lambda}}_{\to 1} \cdot \dots \cdot \underbrace{\frac{n-1}{n-\lambda}}_{\to 1} \cdot \underbrace{\frac{n}{n-\lambda}}_{\to 1} = 1$

So, for big values of n we have $Bin(n, \frac{\lambda}{n}) \approx P_o(\lambda)$

b) Let X model the number of corrupted characters, out of $n = 1000$.

So, we have $X \sim Bin(1000, 0.001) = Bin(1000, \frac{1}{1000})$

But, from a), we know that, for large n, $Bin(n, \frac{\lambda}{n}) \approx P_o(\lambda)$.

Therefore, we'll calculate $\mathbb{P}(X=0)$, meaning the fact that the file was transferred entirely and without corrupting parts, with $X \sim P_o(1)$ and with $X \sim Bin(1000, \frac{1}{1000})$.

1) $\mathbb{P}(X=0) = \frac{e^{-1}\cdot 1^0}{0!} = e^{-1} = \frac{1}{e}$

2) $\mathbb{P}(X=0) = \binom{1000}{0}\left(\frac{1}{1000}\right)^0\cdot\left(1-\frac{1}{1000}\right)^{1000} = \left(1-\frac{1}{1000}\right)^{1000}$, which, from a), is indeed very close to $\frac{1}{e}$.

6. We have a probability of p to get heads and $(1-p)$ to get tails for each flip.

X denotes the number of flips until we get two heads in a row.

$\{A_1, A_2, A_3\}$ - partition for Ω with $A_1 = \{T \text{ first}\} \Rightarrow \mathbb{P}(A_1) = 1-p$

$A_2 = \{HT \text{ first two}\} \Rightarrow \mathbb{P}(A_2) = p(1-p)$

$A_3 = \{HH \text{ first two}\} \Rightarrow \mathbb{P}(A_3) = p^2$.

From the partition theorem for expectations, we have:

$$\mathbb{E}(X) = \mathbb{E}(X|A_1)\,\mathbb{P}(A_1) + \mathbb{E}(X|A_2)\,\mathbb{P}(A_2) + \mathbb{E}(X|A_3)\,\mathbb{P}(A_3)$$

$\mathbb{E}(X|A_1)$ is the expected value of X (the number of flips until we get two heads in a ow), given A_1 (knowing that the first flip was tails), so $\mathbb{E}(X|A_1) = \mathbb{E}(1+X)$ (as it will be equal to the expected value of $X+1$, which we know it was a fail).

Similarly, $\mathbb{E}(X|A_2) = \mathbb{E}(2+X)$ and $\mathbb{E}(X|A_3) = 2$ (as we only need two flips)

Thus, $\mathbb{E}(X)$ becomes

$$\mathbb{E}(X) = (1-p)\,\mathbb{E}(1+X) + (1-p)p\,\mathbb{E}(2+X) + 2p^2 \quad \Rightarrow$$

Now, we'll use the theorem $\mathbb{E}(aX+b) = a\,\mathbb{E}(X)+b$, for $a, b \in \mathbb{R}$

$$\Rightarrow \mathbb{E}(X) = (1-p)(1+\mathbb{E}(X)) + (1-p)p\,(2+\mathbb{E}(X)) + 2p^2$$

$$\mathbb{E}(X) = 1-p+(1-p)\,\mathbb{E}(X) + 2p-2p^2+(p-p^2)\,\mathbb{E}(X)+2p^2$$

$$\mathbb{E}(X) + (p-1)\,\mathbb{E}(X) + (p^2-p)\,\mathbb{E}(X) = 1-p+2p-\cancel{2p^2}+\cancel{2p^2}$$

$$(\cancel{1}+\cancel{p}-\cancel{1}+p^2-\cancel{p})\,\mathbb{E}(X) = 1+p$$

$$\boxed{\mathbb{E}(X) = \frac{p+1}{p^2}}$$

7. (The coupon collector problem)

(a) As each packet is equally likely to contain any of the toys, the probability that the second packet contains a new toy is $\frac{n-1}{n}$.

Let X denote the number of trials (opened packets) until we find a different toy than the first one $\Rightarrow X \sim \text{Geom}\left(\frac{n-1}{n}\right)$, so

$$\mathbb{P}(X=k) = \left(\frac{1}{n}\right)^{k-1} \cdot \frac{n-1}{n} = \frac{n-1}{n^k}.$$

(b) Now we suppose that we already found $(k-1)$ different types of toy, for some $k \geq 1$. T_k denotes the additional number of packets to open in order to find a new type of toy. Therefore, $T_k \sim \text{Geom}\left(\frac{n-k+1}{n}\right)$, so

$$\mathbb{P}(T_k = a) = \left(\frac{n-k+1}{n}\right) \cdot \left(\frac{k-1}{n}\right)^{a-1}.$$

Let $T = T_1 + T_2 + \dots + T_n$ denote the total number of packets needed to collect the set of different toys.

$$\mathbb{E}(T) = \mathbb{E}(T_1 + T_2 + \dots + T_n) = \mathbb{E}(T_1) + \mathbb{E}(T_2) + \dots + \mathbb{E}(T_n)$$

We'll calculate $\mathbb{E}(T_k)$ for $k = 1, 2, \dots, n$

$$\mathbb{E}(T_k) = \sum_{a=0}^{\infty} a \cdot \mathbb{P}(T_k = a) = \sum_{a=1}^{\infty} a \cdot \frac{n-k+1}{n} \cdot \left(\frac{k-1}{n}\right)^{a-1} = \frac{n-k+1}{n} \cdot \sum_{a=1}^{\infty} a \cdot \left(\frac{k-1}{n}\right)^{a-1} =$$

$$\frac{n-k+1}{n} \cdot \frac{d}{d\left(\frac{k-1}{n}\right)} \left(\sum_{a=0}^{\infty} \left(\frac{k-1}{n}\right)^{a} \right) = \frac{n-k+1}{n} \cdot \frac{d}{d\left(\frac{k-1}{n}\right)} \left(\frac{1}{1 - \left(\frac{k-1}{n}\right)} \right) = \frac{n-k+1}{n} \cdot \frac{1}{\left(1 - \frac{k-1}{n}\right)^2} =$$

$$= \cancel{\frac{n-k+1}{n}} \cdot \frac{1}{\left(\frac{n-k+1}{n}\right)^{\cancel{2}}} = \frac{n}{n-k+1}.$$

Therefore, $\mathbb{E}(T) = \sum_{k=1}^{n} \frac{n}{n-k+1} = n\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} + \frac{1}{n}\right)$

As n becomes large, $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ grows as quickly as $\ln n$, because

$$\lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - \ln n = c = 0.577 \text{ (the Euler-Mascheroni constant)}$$

So, $\mathbb{E}(T)$ grows as quickly as $n \ln n$.