LINEAR ALGEBRA MT18

Gabriel Moise

WEEK 2

Chapter 2: Independence and Orthogonality

1. $u \in \mathbb{R}^n$, $v_i \in \mathbb{R}^n$, $i = 1, \dots, m$

(a) $n=3$ $\quad u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{2}v_1 + \frac{2}{3}v_2 + \frac{3}{4}v_3$$

(b) $n=3$ $\quad u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

$u = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3c_2 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 \\ 4c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 3c_2 + c_3 \\ c_2 + 4c_3 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 1 = c_1 \\ 2 = 3c_2 + c_3 \Rightarrow c_3 = 2 - 3c_2 \\ 3 = c_2 + 4c_3 \end{cases} \Rightarrow 3 = c_2 + 4(2 - 3c_2)$$

$$3 = c_2 + 8 - 12c_2$$

$$11c_2 = 5 \Rightarrow c_2 = \frac{5}{11}$$

$$c_3 = 2 - 3c_2 \Rightarrow c_3 = \frac{7}{11}$$

So, $u = v_1 + \frac{5}{11}v_2 + \frac{7}{11}v_3$

(c) $n=4$ $\quad u = \begin{bmatrix} 9 \\ 5 \\ -2 \\ 2 \end{bmatrix}$, $v_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix}$, $v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$

$u = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$

$$\begin{bmatrix} 9 \\ 5 \\ -2 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 6 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3c_1 \\ 2c_2 + c_3 \\ -4c_2 + 6c_3 \\ 4c_4 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 9 = 3c_1 \Rightarrow c_1 = 3 \\ 5 = 2c_2 + c_3 \\ -2 = -4c_2 + 6c_3 \\ 2 = 4c_4 \Rightarrow c_4 = \frac{1}{2} \end{cases} \longrightarrow \begin{cases} 5 = 2c_2 + c_3 \Rightarrow c_3 = 5 - 2c_2 \\ -2 = -4c_2 + 6c_3 \end{cases} \Rightarrow -2 = -4c_2 + 6(5 - 2c_2)$$

$$-2 = -4c_2 + 30 - 12c_2$$

$$16c_2 = 32 \Rightarrow c_2 = 2$$

$$c_3 = 5 - 2c_2 \Rightarrow c_3 = 1$$

So, $u = 3v_1 + 2v_2 + v_3 + \frac{1}{2}v_4$

2. $S = \{v_1, v_2, \dots, v_m\}$, $v_1, v_2, \dots, v_m \in V$, V vector space

(a) $\text{span}(S) = \{u \in V \mid u = c_1 v_1 + c_2 v_2 + \dots + c_m v_m,\ c_1, c_2, \dots, c_m \in \mathbb{R}\}$ = the set of all linear combinations of $v_1, v_2, \dots, v_m$

(b) First of all, span(S) is not an empty set because it contains v_1, for example.

For span(S) to be a subspace of V, we need to check if span(S) is closed under addition and under scalar multiplication.

For all $u, w \in \text{span}(S)$, we'll prove that $u + w \in \text{span}(S)$

$$\left.\begin{array}{l} u \in \text{span}(S) \Rightarrow u = c_1 v_1 + c_2 v_2 + \dots + c_m v_m,\ \text{with } c_1, c_2, \dots, c_m \in \mathbb{R} \\ w \in \text{span}(S) \Rightarrow w = d_1 v_1 + d_2 v_2 + \dots + d_m v_m,\ \text{with } d_1, d_2, \dots, d_m \in \mathbb{R} \end{array}\right| \Rightarrow$$

$\Rightarrow u + w = (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \dots + (c_m + d_m) v_m$, with $(c_1+d_1), (c_2+d_2), \dots, (c_m+d_m) \in \mathbb{R} \Rightarrow$

$\Rightarrow u + w \in \text{span}(S) \Rightarrow$ span(S) is closed under addition (*)

For all $u \in \text{span}(S)$ and $\alpha \in \mathbb{R}$, we'll prove that $\alpha u \in \text{span}(S)$

$u \in \text{span}(S) \Rightarrow u = c_1 v_1 + c_2 v_2 + \dots + c_m v_m$, with $c_1, c_2, \dots, c_m \in \mathbb{R} \Rightarrow$

$\Rightarrow \alpha u = (\alpha c_1) v_1 + (\alpha c_2) v_2 + \dots + (\alpha c_m) v_m$, with $(\alpha c_1), (\alpha c_2), \dots, (\alpha c_m) \in \mathbb{R} \Rightarrow$

$\Rightarrow \alpha u \in \text{span}(S) \Rightarrow$ span(S) is closed under scalar multiplication (**)

From (*) and (**) we can conclude that span(S) is a subspace of V.

3. $u, v, w \in V$, V vector space

$S = \{v - u, w - v, u - w\}$

From the definition of linear independency, S is linearly independent if

$$\alpha_1 (v - u) + \alpha_2 (w - v) + \alpha_3 (u - w) = 0 \text{ implies that } \alpha_1 = \alpha_2 = \alpha_3 = 0.$$

However, if a set of not all zero $\alpha_1, \alpha_2, \alpha_3$ exists then the set is linearly dependent.

In this case, if $\alpha_1 = \alpha_2 = \alpha_3 = 1$, then

$(v - u) + (w - v) + (u - w) = 0$, which implies that S is linearly dependent.

4. $u, v \in V$, V vector space

$S = \{u, v\}$, S is linearly independent

$S' = \{u + v, u - v\}$

First of all, from the fact that S is linearly independent we get the fact that $u \neq 0$ and $v \neq 0$. If either of them was the zero vector (let's take $u = 0$), then $5u + 0v = 0$, so S is linearly dependent (False).

Now, S' is linearly independent if $\alpha(u + v) + \beta(u - v) = 0$ implies $\alpha = \beta = 0$, where $\alpha, \beta \in \mathbb{R}$.

We'll start with

$$\alpha(u+v)+\beta(u-v)=0$$
$$\alpha u+\alpha v+\beta u-\beta v=0$$
$$(\alpha+\beta)u+(\alpha-\beta)v=0$$

As $S=\{u,v\}$ is linearly independent, the last equality implies that $\alpha+\beta=\alpha-\beta=0$.

$$\left\{\begin{array}{l}\alpha+\beta=0\\ \alpha-\beta=0 \Rightarrow \alpha=\beta\end{array}\right| \Rightarrow 2\alpha=0 \Rightarrow \alpha=0 \qquad \left.\begin{array}{l}\alpha=0\\ \alpha+\beta=0\end{array}\right| \Rightarrow \beta=0.$$

We got to the result $\alpha=\beta=0$, so we can conclude that S' is linearly independent.

5. $S=\{v_1, v_2, \dots, v_m\}$ basis for V, V vector space

$S'=\{cv_1, cv_2, \dots, cv_m\}$, $c\in\mathbb{R}$, $c\neq 0$

S is a basis for V if S is linearly independent and $\mathrm{span}(S)=V$.

First of all, we'll prove that S' is linearly independent.

$\alpha_1(cv_1)+\alpha_2(cv_2)+\dots+\alpha_m(cv_m)=0$, $\alpha_1,\alpha_2,\dots,\alpha_m\in\mathbb{R} \Rightarrow$

$\Rightarrow (\alpha_1 c)v_1+(\alpha_2 c)v_2+\dots+(\alpha_m c)v_m=0$

Since S is basis for V, then S is linearly independent $\Big| \Rightarrow \alpha_1 c=\alpha_2 c=\dots=\alpha_m c=0$, But $c\neq 0$ $\Big| \Rightarrow$

$\Rightarrow \alpha_1=\alpha_2=\dots=\alpha_m \Rightarrow S'$ is linearly independent (*)

Now we'll prove that $\mathrm{span}(S')=V$.

Let u be a vector fro V. As S is a basis for V, u is a linear combination of S. So, $u=\alpha_1 v_1+\alpha_2 v_2+\dots+\alpha_m v_m$, $\alpha_1,\alpha_2,\dots,\alpha_m\in\mathbb{R} \Rightarrow$

$\Rightarrow u=\frac{\alpha_1}{c}\cdot cv_1+\frac{\alpha_2}{c}\cdot cv_2+\dots+\frac{\alpha_m}{c}\cdot cv_m$, as $c\in\mathbb{R}$ and $c\neq 0$

By replacing $\frac{\alpha_i}{c}$ with β_i for $i\in\{1;2;\dots;n\}$, we get

$u=\beta_1(cv_1)+\beta_2(cv_2)+\dots+\beta_m(cv_m)$, $\beta_1,\beta_2,\dots,\beta_m\in\mathbb{R} \Rightarrow$

$\Rightarrow u$ is a linear combination of $S' \Rightarrow \mathrm{span}(S')=V$ (**)

From (*) and (**) we can conclude that S' is also a basis for V.

6. $u, s\in\mathbb{R}^n$

The orthogonal projection of u onto s is $p_s(u)$ and it is equal to $\frac{s^T u}{s^T s}s$

(a) $n=2$, $u=[1,2]^T$, $s=[2,1]^T$

$$p_s(u)=\frac{s^T u}{s^T s}s=\frac{[2,1]^{T}\cdot[1,2]^T}{[2,1]^T\cdot[2,1]^T}[2,1]^T=\frac{4}{5}[2,1]^T=\left[\frac{8}{5},\frac{4}{5}\right]^T=\begin{bmatrix}\frac{8}{5}\\ \frac{4}{5}\end{bmatrix}$$

(b) $m=3$, $u=[1,3,-2]^T$, $s=[0,-1,1]^T$

$$p_s(u)=\frac{s^Tu}{s^Ts}s=\frac{[0,-1,1]^T\cdot[1,3,-2]^T}{[0,-1,1]^T\cdot[0,-1,1]^T}[0,-1,1]^T=\frac{-5}{2}[0,-1,1]^T=[0,\tfrac{5}{2},-\tfrac{5}{2}]^T=\begin{bmatrix}0\\ \frac{5}{2}\\ -\frac{5}{2}\end{bmatrix}$$

7. $u,s\in\mathbb{R}^m$

$p_s(u)=$ the orthogonal projection of u onto s

We need to prove that $u-p_s(u)$ is orthogonal to s. This happens if

$(u-p_s(u))^T\cdot s=0$

Here we apply the distributivity of the dot product to obtain:

$u^T\cdot s-p_s(u)^T\cdot s=0$

$u^T\cdot s=p_s(u)^T\cdot s$

We'll use the fact that $p_s(u)=\frac{s^T u}{s^T\cdot s}s$ $\Big|\Rightarrow u^T\cdot s=\left(\frac{s^T\cdot u}{s^T\cdot s}s\right)^T\cdot s$

But $\frac{s^T\cdot u}{s^T\cdot s}=\alpha$, with $\alpha\in\mathbb{R}\Rightarrow\left(\frac{s^T\cdot u}{s^T\cdot s}s\right)^T=\frac{s^T\cdot u}{s^T\cdot s}s^T$

So, $u^T\cdot s=\frac{s^T\cdot u}{s^T\cdot s}s^T\cdot s$

Since $(\alpha s^T)\cdot s=\alpha(s^T\cdot s)$ from the properties of the dot product $\Big|\Rightarrow$

$\Rightarrow u^T\cdot s=\frac{s^T\cdot u}{s^T\cdot s}(s^T\cdot s)$

as $(s^T\cdot u)$, $(s^T\cdot s)$ are real numbers

But the scalar multiplication is commutative, so $u^T\cdot s=\frac{s^T\cdot s}{s^T\cdot s}(s^T\cdot u)$

Since $\frac{s^T\cdot s}{s^T\cdot s}=1$, we get $u^T\cdot s=s^T\cdot u$ and we apply the commutativity of the dot product to obtain $u^T\cdot s=u^T\cdot s$, which is obviously True.

To conclude, $(u-p_s(u))$ is orthogonal to s.

Applications

of the classical Gram-Schmidt algorithm

1. u_1, u_2, u_3 linearly independent

$u_1, u_2, u_3 \in \mathbb{R}^3$

We get q_1 by normalizing $u_1 \Rightarrow q_1 = \frac{1}{\|u_1\|} u_1$

We extract the q_1 component of u_2 to produce q_2 and then we normalize the result

$$q_2 = \frac{u_2 - (q_1^T u_2) q_1}{\|u_2 - (q_1^T u_2) q_1\|}$$

We calculate q_3 the same way:

$$q_3 = \frac{u_3 - (q_1^T u_3) q_1 - (q_2^T u_3) q_2}{\|u_3 - (q_1^T u_3) q_1 - (q_2^T u_3) q_2\|}$$

By using Gram-Schmidt classical algorithm we started from 3 linearly independent vectors u_1, u_2, u_3 and we obtained the orthonormal set q_1, q_2, q_3. Each vector is normalized, so $q_i \cdot q_i = 0$ $(\forall)\, i \in \{1;2;3\}$ and $q_1 \cdot q_2 = q_2 \cdot q_3 = q_1 \cdot q_3 = 0$ (by replacing q_1, q_2, q_3 in the equations)

2. $u_1 = [1, 0, 0]^T$

$u_2 = [1, 1, 1]^T$

$u_3 = [1, 1, -1]^T$

$$q_1 = \frac{1}{\|u_1\|} u_1 = [1, 0, 0]^T$$

$$q_2 = \frac{u_2 - (q_1^T u_2) q_1}{\|u_2 - (q_1^T u_2) q_1\|} = \frac{[1,1,1]^T - [1,0,0]^T}{\|[1,1,1]^T - [1,0,0]^T\|} = \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$$

$$q_3 = \frac{u_3 - (q_1^T u_3) q_1 - (q_2^T u_3) q_2}{\|u_3 - (q_1^T u_3) q_1 - (q_2^T u_3) q_2\|} = \frac{[1,1,-1]^T - [1,0,0]^T}{\|[1,1,-1]^T - [1,0,0]^T\|} = \left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]^T$$

$q_1 \cdot q_1 = 1$, $q_2 \cdot q_2 = 1$, $q_3 \cdot q_3 = 1$

$$q_1^T q_2 = [1, 0, 0]^T \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 0 \qquad q_2^T q_3 = \left[0 \ \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}\right]^T \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

$$q_1^T q_3 = [1, 0, 0]^T \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

So, $\{q_1, q_2, q_3\}$ orthonormal set.

3. $W = \text{span}\{x_1, x_2, x_3\}$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

By using classical Gram-Schmidt algorithm we get

$$q_1 = \frac{1}{\|x_1\|} x_1 = \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]^T$$

$$q_2 = \frac{x_2 - (q_1^T x_2) q_1}{\|x_2 - (q_1^T x_2) q_1\|} = \frac{[2,1,0,1]^T - \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]^T}{\left\|[2,1,0,1]^T - \left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]^T\right\|} = \frac{\left[\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right]^T}{\left\|\left[\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right]^T\right\|} = \left[\frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}\right]^T$$

$$q_3 = \frac{x_3 - (q_1^T x_3) q_1 - (q_2^T x_3) q_2}{\|x_3 - (q_1^T x_3) q_1 - (q_2^T x_3) q_2\|} = \frac{[2,2,1,2]^T - \left[\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right]^T - \frac{15}{2\sqrt{5}}\left[\frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}\right]^T}{\left\|[2,2,1,2]^T - \left[\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right]^T - \frac{15}{2\sqrt{5}}\left[\frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}\right]^T\right\|} =$$

$$= \frac{\left[\frac{7}{4}, \frac{9}{4}, \frac{5}{4}, \frac{7}{4}\right]^T - \left[\frac{9}{4}, \frac{9}{4}, \frac{3}{4}, \frac{3}{4}\right]^T}{\left\|\left[\frac{7}{4}, \frac{9}{4}, \frac{5}{4}, \frac{7}{4}\right]^T - \left[\frac{9}{4}, \frac{9}{4}, \frac{3}{4}, \frac{3}{4}\right]^T\right\|} = \frac{\left[-\frac{1}{2}, 0, \frac{1}{2}, 1\right]^T}{\left\|\left[-\frac{1}{2}, 0, \frac{1}{2}, 1\right]^T\right\|} = \sqrt{\frac{2}{3}}\left[-\frac{1}{2}, 0, \frac{1}{2}, 1\right]^T =$$

$$= \left[-\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right]^T$$

$\{q_1, q_2, q_3\}$ orthonormal as $q_1^T q_1 = q_2^T q_2 = q_3^T q_3 = 1$ and $q_1^T q_2 = q_2^T q_3 = q_1^T q_3 = 0$.

Also, as we used CGS algorithm to obtain $Q = \{q_1, q_2, q_3\}$ from $X = \{x_1, x_2, x_3\}$ $\Rightarrow$

$\Rightarrow$ $\text{span}(Q) = \text{span}(X) = W$

Q is linearly independent (because it's an orthogonal set) $\Bigg|$ $\Rightarrow$ Q is a basis for W

4. $\mathbb{R}^3$, $v_1 = [1, 2, 3]^T$

So, let's say we start from a set of linearly independent vectors $X = \{x_1, x_2, x_3\}$ and, by using CGS algorithm, we obtain $V = \{v_1, v_2, v_3\}$ orthogonal basis. In order to do that, we'll first find an orthonormal basis $Q = \{q_1, q_2, q_3\}$, which will have $q_1 = \frac{v_1}{\|v_1\|}$, and all we need to do is multiply q_1, q_2, q_3 by $\|v_1\|$ to obtain V, which is orthogonal as $v_i \cdot v_j = 0$ because $(\|v_1\| \cdot q_i)(\|v_1\| q_j) = 0$, which is True. (*) V is not orthonormal, though, because no vector is normalized like in Q.

First of all, $q_1 = \frac{v_1}{\|v_1\|}$, as we want $x_1 = v_1$ for this to work.

Then, $q_1 = \left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]^T$

After that, $q_2 = \frac{x_2 - (q_1^T x_2) q_1}{\|x_2 - (q_1^T x_2) q_1\|}$

Let's say we choose $x_2 = [0, 0, 1]^T$ and $x_3 = [1, 0, 0]^T$ as this way $X = \{x_1, x_2, x_3\}$ is linearly independent.

Then, $q_2 = \dfrac{[0,0,1]^T - \frac{3}{\sqrt{14}}\left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]^T}{\left\| [0,0,1]^T - \frac{3}{\sqrt{14}}\left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]^T \right\|} = \dfrac{\left[-\frac{3}{14}, -\frac{6}{14}, \frac{5}{14}\right]^T}{\left\| \left[\frac{-3}{14}, \frac{-6}{14}, \frac{5}{14}\right]^T \right\|}$

$q_2 = \sqrt{\frac{14}{5}}\left[-\frac{3}{14}, -\frac{6}{14}, \frac{5}{14}\right]^T = \left[-\frac{3}{\sqrt{70}}, -\frac{6}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right]^T$

Finally, $q_3 = \dfrac{x_3 - (q_1^T x_3) q_1 - (q_2^T x_3) q_2}{\| x_3 - (q_1^T x_3) q_1 - (q_2^T x_3) q_2 \|}$

$$q_3 = \dfrac{[1,0,0]^T - \frac{1}{\sqrt{14}}\left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]^T + \frac{3}{\sqrt{70}}\left[-\frac{3}{\sqrt{70}}, -\frac{6}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right]^T}{\left\| [1,0,0]^T - \frac{1}{\sqrt{14}}\left[\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right]^T + \frac{3}{\sqrt{70}}\left[-\frac{3}{\sqrt{70}}, -\frac{6}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right]^T \right\|}$$

$$q_3 = \dfrac{[1,0,0]^T - \left[\frac{1}{14}, \frac{2}{14}, \frac{3}{14}\right] + \left[-\frac{9}{70}, -\frac{18}{70}, \frac{15}{70}\right]^T}{\left\| [1,0,0]^T - \left[\frac{1}{14}, \frac{2}{14}, \frac{3}{14}\right] + \left[-\frac{9}{70}, -\frac{18}{70}, \frac{15}{70}\right]^T \right\|}$$

$$q_3 = \dfrac{\left[\frac{4}{5}, -\frac{2}{5}, 0\right]^T}{\left\| \left[\frac{4}{5}, -\frac{2}{5}, 0\right]^T \right\|} = \frac{\sqrt{5}}{2}\left[\frac{4}{5}, -\frac{2}{5}, 0\right]^T = \left[\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right]^T$$

So, we have the orthonormal basis $Q = \{q_1, q_2, q_3\}$ with $q_1^T q_1 = 1$, $q_2^T q_2 = 1$, $q_3^T q_3 = 1$ and $q_1^T q_2 = 0$, $q_2^T q_3 = 0$, $q_1^T q_3 = 0$.

If we multiply each vector with $\underbrace{\|v_1\|}_{=\sqrt{14}}$ we obtain $V = \{v_1, v_2, v_3\}$, where

$v_1 = \|v_1\| q_1 = [1,2,3]^T$ (as we wanted)

$v_2 = \|v_1\| q_2 = \left[-\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}, \frac{5}{\sqrt{5}}\right]^T$

$v_3 = \|v_1\| q_3 = \left[\frac{2\sqrt{14}}{\sqrt{5}}, -\frac{\sqrt{14}}{\sqrt{5}}, 0\right]^T$

The fact that V is an orthogonal set comes from $(*)$, so V is linearly independent. Also, we formed V from Q by scalar multiplicating every q_i, $i \in \{1;2;3\}$ with $\|v_1\| = \sqrt{14}$, so $\text{span}(V) = \text{span}(Q)$

But, from CGS we know that $\text{span}\, Q = \mathbb{R}^3$ $\Big| \Rightarrow \text{span}(V) = \mathbb{R}^3$

To sum up, we found $V = \left\{ [1,2,3]^T, \left[-\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}, \frac{5}{\sqrt{5}}\right]^T, \left[\frac{2\sqrt{14}}{\sqrt{5}}, -\frac{\sqrt{14}}{\sqrt{5}}, 0\right]^T \right\}$ which contains $v_1 = [1,2,3]^T$ and is a orthogonal basis for $\mathbb{R}^3$.