GABRIEL MOISE

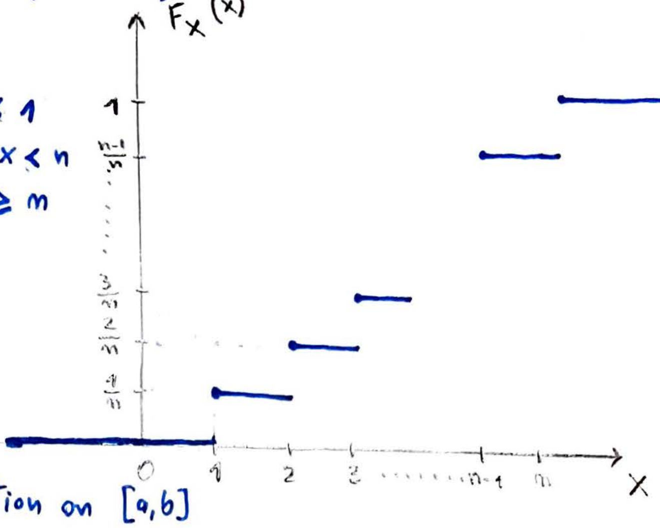
PROBABILITY
PROBLEM SHEET 7

1.

(a) the (discrete) uniform distribution on $\{1,2,\dots,n\}$

$$P_X(u) = \frac{1}{n}, \ (\forall)\, u \in \{1,2,\dots,n\}$$

$$F_X(x) = \sum_{u \le x:\, u \in \operatorname{Im} X} P_X(u) = \begin{cases} 0 & , \text{ if } x < 1 \\ \frac{x}{n} & , \text{ if } 1 \le x < n \\ 1 & , \text{ if } x \ge n \end{cases}$$

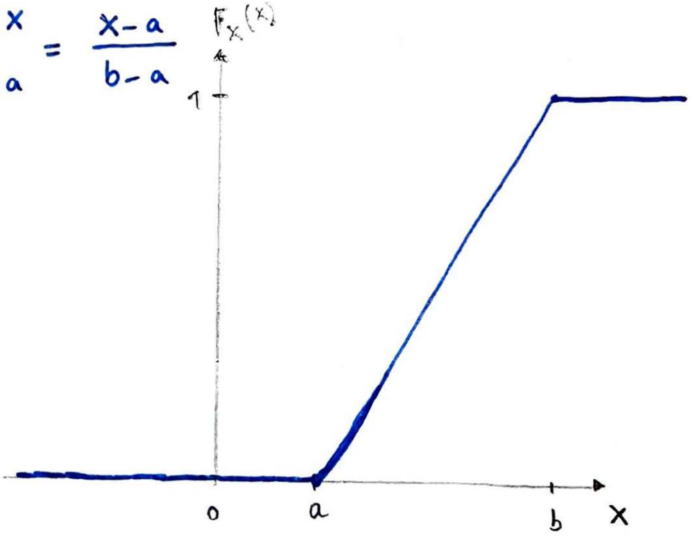


(b) the (continuous) uniform distribution on $[a,b]$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & , \text{ for } a \le x \le b \\ 0 & , \text{ otherwise} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{b-a}\, du = \int_a^x \frac{1}{b-a}\, du = \frac{u}{b-a}\Big|_a^x = \frac{x-a}{b-a}$$

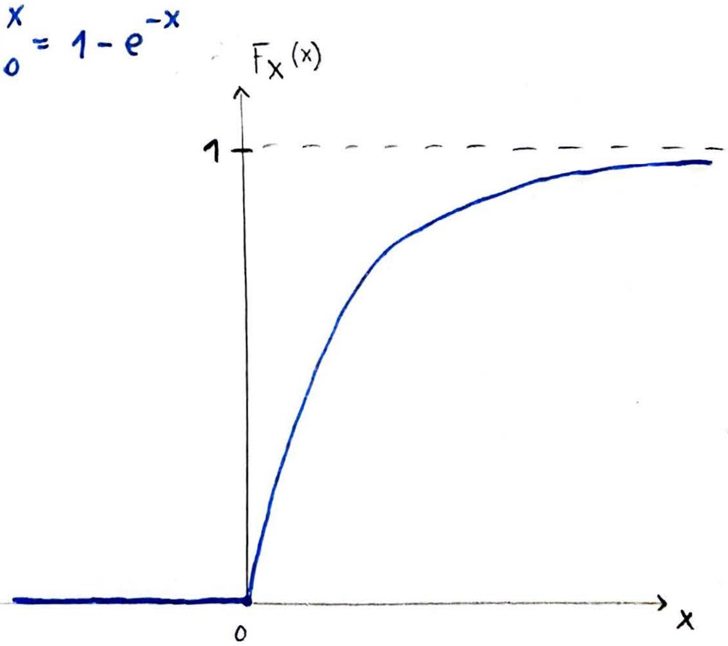
$$F_X(x) = \begin{cases} 0 & , x < a \\ \frac{x-a}{b-a} & , a \le x < b \\ 1 & , b \le x \end{cases}$$



(c) the exponential distribution with parameter 1

$$f_X(x) = e^{-x}, \ x \ge 0$$

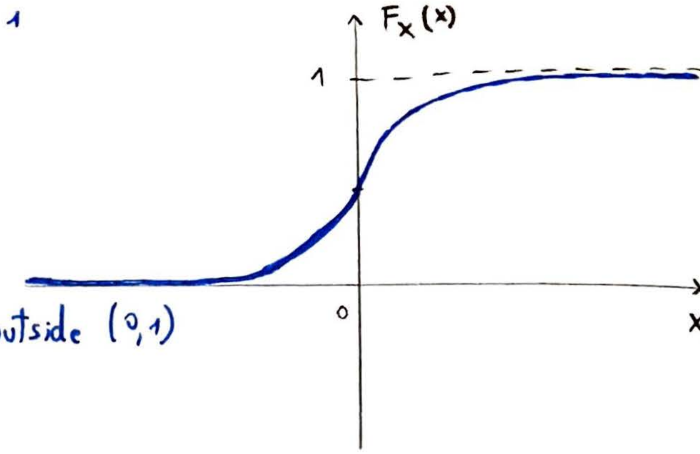
$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = \int_0^x e^{-u}\, du = -e^{-u}\Big|_0^x = 1 - e^{-x}$$



(d) the normal distribution with mean 0 and variance 1

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$



2. c is a constant for each case. Each f is 0 outside (0,1)

(a) $f_1(x) = cx$, for $0 < x < 1$

$f_1(x) \geq 0 \ (\forall)\ x \in (0,1) \Rightarrow c \geq 0$

$\int_{-\infty}^{\infty} f_1(x)\,dx = 1 \Rightarrow \int_0^1 cx\,dx = 1 \Rightarrow c\frac{x^2}{2}\Big|_0^1 = 1 \Rightarrow \frac{c}{2} = 1 \Rightarrow \boxed{c = 2}$

$f_1(x) = 2x$

$F_1(x) = \int_{-\infty}^{x} 2u\,du = \int_0^x 2u\,du = u^2\Big|_0^x = x^2$

$$F_1(x) = \begin{cases} x^2, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $f_2(x) = cx^{-1}$, for $0 < x < 1$

$f_2(x) \geq 0 \ (\forall)\ x \in (0,1) \Rightarrow c \geq 0$

$\int_{-\infty}^{\infty} f_2(x)\,dx = 1 \Rightarrow \int_0^1 \frac{c}{x}\,dx = 1 \Rightarrow \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_{\varepsilon}^{1} \frac{c}{x}\,dx = 1 \Rightarrow \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} c\ln x\Big|_{\varepsilon}^{1} = 1 \Rightarrow$

$\Rightarrow \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} -c\ln\varepsilon = 1 \Rightarrow \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \ln\varepsilon = -\frac{1}{c}$

However, if c constant and $c \neq 0$ we can't obtain. For $c=0$, we get

$f_2(x) = 0 \Rightarrow \int_0^1 0\,dx = t\Big|_0^1 = t - t = 0$, which is not 1. So, no constant c makes f_2 a p.d.f.

(c) $f_3(x) = cx^{-\frac{1}{2}}$, for $0 < x < 1$

$f_3(x) \geq 0 \ (\forall)\ x \in (0,1) \Rightarrow c \geq 0$

$\int_{-\infty}^{\infty} f_3(x)\,dx = \int_0^1 cx^{-\frac{1}{2}}\,dx = c\frac{\sqrt{x}}{\frac{1}{2}}\Big|_0^1 = 2c\sqrt{x}\Big|_0^1 = 2c = 1 \Rightarrow \boxed{c = \frac{1}{2}}$

$f_3(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$F_3(x) = \int_{-\infty}^{x} f_3(u)\,du = \int_0^x \frac{1}{2}u^{-\frac{1}{2}}\,du = \cancel{\frac{1}{2}} \cdot \frac{\sqrt{u}}{\cancel{\frac{1}{2}}}\Big|_0^x = \sqrt{x}$

$$F_3(x) = \begin{cases} \sqrt{x}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(d) $f_4(x) = c(4x^3 - x)$, for $0 < x < 1$

$f_4(x) \geq 0 \ (\forall) x \in (0,1) \Rightarrow c(4x^3 - x) \geq 0$

$4x^3 - x = x(4x^2 - 1) \Rightarrow$ for $x \in (0, \frac{1}{2})$ $\frac{f_4(x)}{c}$ has a sign and for $x \in (\frac{1}{2}, 1)$ $\frac{f_4(x)}{c}$ has the opposite sign. Therefore, $f_4(x)$ cannot be a density function.

3. $U \sim U[0,1]$

(a) $f_U(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

$$\mathbb{E}(U) = \int_{-\infty}^{\infty} x f_U(x) dx = \int_0^1 x\, dx = \frac{x^2}{2}\Big|_0^1 \Rightarrow \boxed{\mathbb{E}(U) = \frac{1}{2}}$$

$$Var(U) = \mathbb{E}(U^2) - \mathbb{E}^2(U)$$

$$\left.\begin{aligned} \mathbb{E}(U^2) &= \int_{-\infty}^{\infty} x^2 f_U(x) dx = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} \\ \mathbb{E}^2(U) &= \frac{1}{4} \end{aligned}\right| \Rightarrow \boxed{Var(U) = \frac{1}{12}}$$

(b) $$\left.\begin{aligned} \mathbb{P}(U < a | U < b) &= \frac{\mathbb{P}(\{U < a\} \cap \{U < b\})}{\mathbb{P}(U < b)} = \frac{\mathbb{P}(U < a)}{\mathbb{P}(U < b)} = \frac{F_U(a)}{F_U(b)} \\ F_U(x) &= \int_{-\infty}^{x} f_U(u) du = \int_0^x 1\, du = u\Big|_0^x = x \end{aligned}\right| \Rightarrow$$

$$\Rightarrow \boxed{\mathbb{P}(U < a | U < b) = \frac{a}{b}}$$

4. Let $X \sim Exp(\lambda)$

(a) $\mathbb{P}(X > x) = 1 - \mathbb{P}(X \leq x) = 1 - F_X(x)$

$$\left.\begin{aligned} f_X(x) &= \lambda e^{-\lambda x}, \quad x \geq 0 \\ F_X(x) &= \int_{-\infty}^{x} f_X(u) du = \int_0^x \lambda e^{-\lambda u} du = -e^{-\lambda u}\Big|_0^x = 1 - e^{-\lambda x} \end{aligned}\right| \Rightarrow \boxed{\mathbb{P}(X > x) = e^{-\lambda x}}$$

(b) $\mathbb{P}(a \leq X \leq b)$ for $0 < a < b$

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(X \leq b) - \mathbb{P}(X < a) = F_X(b) - F_X(a) = 1 - e^{-\lambda b} - 1 + e^{-\lambda a} = e^{-\lambda a} - e^{-\lambda b} \Rightarrow$$

$$\Rightarrow \boxed{\mathbb{P}(a \leq X \leq b) = e^{-\lambda a} - e^{-\lambda b}}$$

(c) $\mathbb{P}(X > a + x | X > a) = \mathbb{P}(X > x)$ for $a, x > 0$

$$\mathbb{P}(X > a + x | X > a) = \frac{\mathbb{P}(\{X > a + x\} \cap \{X > a\})}{\mathbb{P}(X > a)} = \frac{\mathbb{P}(X > a + x)}{\mathbb{P}(X > a)} \stackrel{(a)}{=} \frac{e^{-\lambda(a+x)}}{e^{-\lambda a}} = e^{-\lambda x} \stackrel{(a)}{=} \mathbb{P}(X > x)$$

(d) $\mathbb{P}(\sin X > \frac{1}{2}) = \mathbb{P}\left(\bigcup_{k=0}^{\infty} \{2k\pi + \frac{\pi}{6} \leq X \leq 2k\pi + \frac{5\pi}{6}\} \right) =$

$$= \sum_{k=0}^{\infty} \mathbb{P}\left(2k\pi + \frac{\pi}{6} \leq X \leq 2k\pi + \frac{5\pi}{6}\right) \overset{(b)}{=} \sum_{k=0}^{\infty} e^{-\lambda(2k\pi + \frac{\pi}{6})} - \sum_{k=0}^{\infty} e^{-\lambda(2k\pi + \frac{5\pi}{6})} =$$

$$= e^{-\lambda\frac{\pi}{6}} \sum_{k=0}^{\infty} \left(e^{-2\lambda\pi}\right)^k - e^{-\lambda\frac{5\pi}{6}} \sum_{k=0}^{\infty} \left(e^{-2\lambda\pi}\right)^k = e^{-\lambda\frac{\pi}{6}} \frac{1}{1-e^{-2\lambda\pi}} - e^{-\lambda\frac{5\pi}{6}} \frac{1}{1-e^{-2\lambda\pi}} =$$

$$= \frac{e^{-\frac{\lambda\pi}{6}} - e^{-\frac{5\lambda\pi}{6}}}{1-e^{-2\lambda\pi}}.$$

(e) Let $c>0$.

$$F_{cX}(x) = \mathbb{P}(cX \leq x) = \mathbb{P}\left(X \leq \frac{x}{c}\right) = F_X\left(\frac{x}{c}\right) = 1 - e^{-\frac{\lambda x}{c}}$$

$$f_{cX}(x) = \left(F_{cX}(x)\right)' = -e^{-\frac{\lambda x}{c}} \cdot \left(-\frac{\lambda}{c}\right) = \frac{\lambda}{c} e^{\left(-\frac{\lambda}{c}\right)x} \Rightarrow \boxed{X \sim \mathrm{Exp}\left(\frac{\lambda}{c}\right)}$$

(f) We have $\{\lceil X \rceil = k\} = \{k-1 < X \leq k\}$

$$\mathbb{P}(\lceil X \rceil = k) = \mathbb{P}(k-1 < X \leq k) \overset{(b)}{=} e^{-\lambda(k-1)} - e^{-\lambda k} = \left(e^{-\lambda}\right)^{k-1}\left(1 - e^{-\lambda}\right)$$

if we choose $p = 1 - e^{-\lambda}$ $\Rightarrow$

$$\Rightarrow \mathbb{P}(\lceil X \rceil = k) = (1-p)^{k-1} p \Rightarrow \lceil X \rceil \sim \mathrm{Geom}(p) \Rightarrow \boxed{\lceil X \rceil \sim \mathrm{Geom}\left(1-e^{-\lambda}\right)}$$

5. So, we know that $X \sim N(315, 131^2)$.

Therefore, $f_X(x) = \frac{1}{\sqrt{2\pi \cdot 131^2}} \cdot e^{-\frac{(x-315)^2}{2\cdot 131^2}}$.

(a) $\mathbb{P}(X \leq 300) = \int_{-\infty}^{300} \frac{1}{131\sqrt{2\pi}} e^{-\frac{(x-315)^2}{2\cdot 131^2}} dx \left(= F_X(300)\right)$

if we substitute $\frac{x-315}{131}$ with y we get

$$\mathbb{P}(X \leq 300) = \int_{-\infty}^{\frac{300-315}{131}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \Phi(-0.115) = \boxed{0.454}$$

(b) $\mathbb{P}(X \leq 500) = F_X(500) = \int_{-\infty}^{500} \frac{1}{131\sqrt{2\pi}} e^{-\frac{(x-315)^2}{2\cdot 131^2}} dx = \int_{-\infty}^{\frac{500-315}{131}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy =$

$= \Phi(1.412) = 0.921$

$\mathbb{P}(300 \leq X \leq 500) = F_X(500) - F_X(300) = 0.921 - 0.454 = \boxed{0.467}$.

(c) Now, we test 20 smokers and we want the probability that at most one has a nicotine level higher than 500.

$\mathbb{P}(\text{maximum one} > 500) = \mathbb{P}(\text{none} > 500) + \mathbb{P}(\text{exactly one} > 500) =$

$$= \underbrace{\left(\mathbb{P}(X \leq 500)\right)^{20}}_{\text{all 20 need to be } \leq 500} + \overbrace{20}^{\binom{1}{20}\text{ - number of ways}} \cdot \mathbb{P}(X > 500)\left(\mathbb{P}(X \leq 500)\right)^{19}.$$

if we replace $\mathbb{P}(X \leq 500)$ with p, we get:

$\mathbb{P}(\text{maximum one} > 500) = p^{20} + 20(1-p)p^{19}$

Now, we know from (b) that $p = 0.921$

$\Rightarrow p^{20} + 20(1-p)p^{19} = 0.193 + 20 \cdot 0.079 \cdot 0.209 =$

$= 0.193 + 0.330 = \boxed{0.523}$

6. Let R denote the radius of the circle. We know that $R \sim U[0,b]$. Therefore, we have:

(A) $f_R(x) = \begin{cases} \frac{1}{b}, & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$

(B) $F_R(x) = \begin{cases} \frac{x}{b}, & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$

$\mathbb{E}(R) = \int_{-\infty}^{\infty} x f_R(x)dx = \int_0^b x \cdot \frac{1}{b} dx = \frac{x^2}{2b}\Big|_0^b \Rightarrow$

(C) $\mathbb{E}(R) = \frac{b}{2}$

$\mathbb{E}(R^2) = \int_{-\infty}^{\infty} x^2 f_R(x)dx = \int_0^b \frac{x^2}{b} dx = \frac{x^3}{3b}\Big|_0^b = \frac{b^2}{3}$

$\mathbb{E}^2(R) = \frac{b^2}{2}$

$\Rightarrow \text{Var}(R) = \frac{b^2}{3} - \frac{b^2}{4} \Rightarrow$

(D) $\text{Var}(R) = \frac{b^2}{12}$.

Now, we want the same results for the area A, which we know is equal to πR^2.

- $F_A(x) = \mathbb{P}(A \leq x) = \mathbb{P}(\pi R^2 \leq x) = \mathbb{P}\left(R^2 \leq \frac{x}{\pi}\right) = \mathbb{P}\left(-\sqrt{\frac{x}{\pi}} \leq R \leq \sqrt{\frac{x}{\pi}}\right) = \mathbb{P}\left(R \leq \sqrt{\frac{x}{\pi}}\right) - \mathbb{P}\left(x < -\sqrt{\frac{x}{\pi}}\right) = F_R\left(\sqrt{\frac{x}{\pi}}\right) - F_R\left(-\sqrt{\frac{x}{\pi}}\right)$

We already know that for $x < 0$ we have $F_A(x) = 0$ (area cannot be negative!) and therefore we only worked with $x \geq 0$. We have $F_R\left(-\sqrt{\frac{x}{\pi}}\right) = 0$ from (B). Now,

$F_R\left(\sqrt{\frac{x}{\pi}}\right) = \begin{cases} \sqrt{\frac{x}{\pi}} \cdot \frac{1}{b}, & \text{if } 0 \leq \sqrt{\frac{x}{\pi}} \leq b \\ 0, & \text{otherwise} \end{cases} \Rightarrow F_R\left(\sqrt{\frac{x}{\pi}}\right) = \begin{cases} \frac{\sqrt{x}}{b\sqrt{\pi}}, & \text{if } 0 \leq x \leq \pi b^2 \\ 0, & \text{otherwise} \end{cases} \Rightarrow$

$\Rightarrow F_A(x) = \begin{cases} \frac{\sqrt{x}}{b\sqrt{\pi}}, & \text{if } 0 \leq x \leq \pi b^2 \\ 0, & \text{otherwise.} \end{cases}$

• $f_A(x) = (F_A(x))'$ for $0 < x < \pi b^2$ and 0, otherwise

$$f_A(x) = \left(\frac{\sqrt{x}}{b\sqrt{\pi}}\right)' = \frac{1}{2b\sqrt{\pi x}} \Rightarrow$$

$$f_A(x) = \begin{cases} \frac{1}{2b\sqrt{\pi x}} & , \text{if } 0 < x < \pi b^2 \\ 0 & , \text{otherwise} \end{cases}$$

• $$\mathbb{E}(A) = \int_{-\infty}^{\infty} x f_A(x)dx = \int_0^{\pi b^2} x \cdot \frac{1}{2b\sqrt{\pi x}} dx = \frac{1}{2b\sqrt{\pi}} \int_0^{\pi b^2} \sqrt{x}\, dx = \frac{1}{\cancel{2}b\sqrt{\pi}} \cdot \frac{\cancel{2}x\sqrt{x}}{3} \Big|_0^{\pi b^2} =$$

$$= \frac{1}{b\sqrt{\pi}}\left(\frac{\pi b^2 \cdot b\sqrt{\pi}}{3} - 0\right) = \frac{b^3 \pi \cancel{\sqrt{\pi}}}{3b\cancel{\sqrt{\pi}}} = \frac{\pi b^2}{3}.$$

• $$\mathbb{E}(A^2) = \int_{-\infty}^{\infty} x^2 f_A(x)dx = \int_0^{\pi b^2} x^2 \cdot \frac{1}{2b\sqrt{\pi x}} dx = \frac{1}{2b\sqrt{\pi}} \int_0^{\pi b^2} x^{\frac{3}{2}} dx =$$

$$= \frac{1}{\cancel{2}b\sqrt{\pi}} \cdot \frac{\cancel{2}x^2\sqrt{x}}{5} \Big|_0^{\pi b^2} = \frac{1}{b\sqrt{\pi}}\left(\frac{\pi^2 b^4 \cdot b\sqrt{\pi}}{5} - 0\right) = \frac{\pi^2 \cancel{\sqrt{\pi}}\, b^5}{5b\cancel{\sqrt{\pi}}} = \frac{\pi^2 b^4}{5}$$

$$(\mathbb{E}(A))^2 = \left(\frac{\pi b^2}{3}\right)^2 = \frac{\pi^2 b^4}{9} \quad \Rightarrow$$

$$\Rightarrow \mathrm{Var}(A) = \mathbb{E}(A^2) - \mathbb{E}^2(A) = \frac{\pi^2 b^4}{5} - \frac{\pi^2 b^4}{9} = \frac{4\pi^2 b^4}{45}.$$

7. Let X be a cts. r.v. taking values in $[a,b]$ with c.d.f. F_X which is strictly increasing on $[a,b]$.

(a) We have

$$F_{F_X(X)}(y) = \mathbb{P}(F_X(X) \le y) = \mathbb{P}(X \le F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

Therefore,

$$f_{F_X(X)}(y) = \frac{d}{dy} F_{F_X(X)}(y) = 1 \quad \left(\text{as } F_X : [a,b] \to [0,1] \Rightarrow F_{F_X(X)} : [0,1] \to [0,1], \text{ so } y \in [0,1]\right) \Rightarrow$$

$$\Rightarrow f_{F_X(X)}(y) = \begin{cases} 1 & , \text{if } 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases} \Rightarrow F_X(X) \sim U[0,1].$$

(b) Let $U \sim U[0,1]$

We want to find the distribution of the r.v. $F_X^{-1}(U)$.

First, we know that:

$$f_U(y) = \begin{cases} 1, & \text{for } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_U(y) = \begin{cases} y, & \text{for } 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$F_X : [a,b] \to [0,1]$ is an increasing function $\Rightarrow F_X^{-1}$ is an increasing function

<u>Proof</u>: F_X increasing (strictly) $\Rightarrow F_X' > 0 \Rightarrow$

$$\Rightarrow \left(F_X^{-1}\right)' = \frac{1}{F_X' \circ F_X^{-1}} > 0 \Rightarrow F_X^{-1} \text{ strictly increasing, too.}$$

Now, we have

$$f_{F_X^{-1}(U)}(y) \overset{\text{Theorem 5.24}}{=} f_U\left(\left(F_X^{-1}\right)^{-1}(y)\right) \frac{d}{dy}\left(F_X^{-1}\right)^{-1}(y) = f_U\left(F_X(y)\right) \cdot \frac{d}{dy} F_X(y)$$

$$f_{F_X^{-1}(U)}(y) = f_U\left(F_X(y)\right) \cdot f_X(y)$$

$F_X : [a,b] \to [0,1] \Rightarrow 0 \leq F_X(y) \leq 1$

$f_{F_X^{-1}(U)}(y) = f_X(y) \Rightarrow$ The distribution of $F_X^{-1}(U)$ is the same as the distribution of X. (they have equal p.d.f.s)

(c) $U_1, U_2, \dots, U_m$ are drawn from $U[0,1]$.

We proved at (b) that if X is a cts. r.v. with F_X strictly increasing and $U \sim U[0,1]$, then $F_X^{-1}(U) \sim X$.

Now, we have to simulate a random sample $X_1, X_2, \dots, X_m$ from the distribution with density $f(x) = \mu e^{-\mu x}$, $x \geq 0$, which is $\mathrm{Exp}(\mu)$.

We want $X_i \sim \mathrm{Exp}(\mu)$ and as F_{X_i} is strictly increasing ($F_{X_i}' = f_{X_i} > 0$), we can say that $F_{X_i}^{-1}(U_i) \sim \mathrm{Exp}(\mu)$.

As $F_{X_i}(x) = F_X(x) = 1 - e^{-\mu x}$, $x \geq 0$ for all $i \in \{1, 2, \dots, m\}$ $\Big| \Rightarrow$

$$\Rightarrow X_i = h^{-1}(U_i), \text{ where } h(x) = 1 - e^{-\mu x} \Rightarrow h^{-1}(y) = -\frac{1}{\mu}\log(1-y) \Rightarrow$$

$$\Rightarrow X_i = -\frac{1}{\mu}\log(1 - U_i), \text{ for all } i \in \{1, 2, \dots, m\}.$$