INTRODUCTION TO FORMAL PROOF

TUTORIAL SHEET 1

TT 2019

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QUESTION 1

(a)

i. $\neg\overline{\neg\overline{\neg P}}$

ii. $\overline{\neg\overline{\neg P}} \wedge \overline{\neg Q}$

iii. $\neg P (\neg \vee) Q$ → syntactically incorrect

iv. $P \longleftrightarrow \overline{Q \to Q} \longleftrightarrow P$

v. $\overline{P \to Q} \longleftrightarrow \overline{Q \to P}$

vi. $\overline{(P \to \overline{Q \to R})} \vdash \overline{\overline{P \wedge Q} \to R}$ - this is a conjecture

vii. $\overline{(P \to \overline{Q \to R})} \to \overline{\overline{P \wedge Q} \to R}$

viii. $\overline{\overline{P \wedge Q} \wedge R} \longrightarrow \overline{\overline{P \vee Q} \vee R}$

(b) i. The logic lectures have started but the concurrency lectures have not.

L = The logic lectures have started

C = The concurrency lectures have started

$L \wedge \neg C$

ii. If Bernard is at a logic lecture then he is not at a concurrency lecture.

L = Bernard is at a logic lecture

C = Bernard is at a concurrency lecture

$L \to \neg C$

iii. Either Bernard teaches a logic lecture or he stays in bed; he can't do both

L = Bernard teaches a logic lecture

B = Bernard stays in bed

$(L \vee B) \wedge \neg(L \wedge B)$

iv. If Dick is moving then Jane is stationary and if Jane is moving then Dick is stationary

D = Dick is moving

J = Jane is moving

$(D \to \neg J) \wedge (J \to \neg D)$

v. When Dick has drunk he doesn't drive and when he drives he hasn't drunk

P = Dick has drunk

Q = Dick drives

$(P \to \neg Q) \wedge (Q \to \neg P)$

vi. Carol is jealous of Yvonne or not in a good mood.

J = Carol is jealous of Yvonne

M = Carol is in a good mood

$J \vee (\neg M)$

vii. When the barometer falls it will rain or snow.

B = The barometer falls

R = It will rain

S = It will snow

$B \to R \vee S$

viii. If you have been to the lectures and read the first two sections of the notes then you should be able to do the first few tutorial questions; otherwise you will not be able to.

L = You have been to the lectures

N = You read the first two sections of the notes

T = You should be able to do the first few tutorial questions

$(L \wedge N \to T) \wedge (\neg(L \wedge N) \to \neg T)$

ix. The formula is always true if its subformulae are always true.

F = The formula is always true

S = its subformulae are always true

$F \leftarrow S$

x. The formula is always true only if its subformulae are always true.

$F \longrightarrow S$

xi. The formula is always true if and only if its subformulae are always true.

$F \longleftrightarrow S$

(c) i. Tomorrow it will either rain or snow. $\quad R \vee S$

It only snows when it is cold. $\quad S \to C$

It will be warm tomorrow. $\quad \neg C$

Therefore it will rain tomorrow. $\quad R$

$$\frac{R \vee S \quad S \to C \quad \neg C}{R}$$

ii. Either the bongo repeller or the footler is broken.

There is always a bad smell when the bongo repeller is broken.

There isn't a bad smell.

Therefore the footler is broken

$$\frac{B \vee F \quad B \to S \quad \neg S}{F}$$

iii. Either the UFOs are secret enemy weapons or they are spaceships from an alien world.

If they are enemy weapons, then enemy technology is (contrary to current thinking) superior to ours.

If they are alien spacecraft then they display a technology more sophisticated than anything we can imagine.

In any case, therefore, their builders are technologically more sophisticated than we are.

$$\frac{W \vee S \quad W \to E \quad S \to A}{E \vee A}$$

QUESTION 2

(a) i. $P \vdash Q \rightarrow (Q \wedge P)$

1.	P	premiss
2.	Q	assumption
3.	$Q \wedge P$	∧-intro 2,1
4.	$Q \rightarrow (Q \wedge P)$	→-intro 2-3

ii. $P \vdash Q \rightarrow (P \wedge Q)$

1.	P	premiss
2.	Q	assumption
3.	$P \wedge Q$	∧-intro 1,2
4.	$Q \rightarrow (P \wedge Q)$	→-intro 2-3

iii $Q \vdash P \rightarrow (P \wedge Q)$

1.	Q	premiss
2.	P	assumption
3.	$P \wedge Q$	∧-intro 2,1
4.	$P \rightarrow (P \wedge Q)$	→-intro 2-3

iv. $P \rightarrow (Q \wedge R) \vdash (P \rightarrow Q) \wedge (P \rightarrow R)$

1.	$P \rightarrow (Q \wedge R)$	premiss
2.	P	assumption
3.	$Q \wedge R$	→-elim 2,1
4.	Q	∧-elim-l 3
5.	$P \rightarrow Q$	→-intro 2-4
6.	P	assumption
7.	$Q \wedge R$	→-elim 6,1
8.	R	∧-elim-r 7
9.	$P \rightarrow R$	→-intro 6-8
10.	$(P \rightarrow Q) \wedge (P \rightarrow R)$	∧-intro 5,9

v. $(P \rightarrow Q) \wedge (P \rightarrow R) \vdash P \rightarrow Q \wedge R$

1.	$(P \rightarrow Q) \wedge (P \rightarrow R)$	premiss
2.	P	assumption
3.	$P \rightarrow Q$	∧-elim-l 1
4.	Q	→-elim 2,3
5.	$P \rightarrow R$	∧-elim-r 1
6.	R	→-elim 2,5
7.	$Q \wedge R$	∧-intro 4,6
8.	$P \rightarrow (Q \wedge R)$	→-intro 2-7

(b) i. $\vdash (Q \leftrightarrow R) \rightarrow (Q \rightarrow S) \rightarrow (R \rightarrow S)$

1.	$Q \leftrightarrow R$	assumption
2.	$Q \rightarrow S$	assumption
3.	R	assumption
4.	$R \rightarrow Q$	abb-↔-elim-l 1
5.	Q	→-elim 3,4
6.	S	→-elim 5,2
7.	$R \rightarrow S$	→-intro 3-6
8.	$(Q \rightarrow S) \rightarrow (R \rightarrow S)$	→-intro 2-7
9.	$(Q \leftrightarrow R) \rightarrow (Q \rightarrow S) \rightarrow (R \rightarrow S)$	→-intro 1-8

ii. $\vdash (Q \leftrightarrow R) \rightarrow (Q \wedge S) \rightarrow (R \wedge S)$

1.	$Q \leftrightarrow R$	assumption
2.	$Q \wedge S$	assumption
3.	Q	∧-elim-l 2
4.	$Q \rightarrow R$	abb-↔-elim-r 1
5.	R	→-elim 3,4
6.	S	∧-elim-r 2
7.	$R \wedge S$	∧-intro 5,6
8.	$(Q \wedge S) \rightarrow (R \wedge S)$	→-intro 2-7
9.	$(Q \leftrightarrow R) \rightarrow (Q \wedge S) \rightarrow (R \wedge S)$	→-intro 1-8

(c) (Distributivity results)

i. $(P \land Q) \lor (P \land R) \vdash P \land (Q \lor R)$

1.	$(P \land Q) \lor (P \land R)$	premiss
2.	$P \land Q$	assumption
3.	Q	∧-elim-r 2
4.	$Q \lor R$	∨-intro-l 3
5.	P	∧-elim-l 2
6.	$P \land (Q \lor R)$	∧-intro 5,4
7.	$P \land R$	assumption
8.	R	∧-elim-r 7
9.	$Q \lor R$	∨-intro-r 8
10.	P	∧-elim-l 7
11.	$P \land (Q \lor R)$	∧-intro 10,9
12.	$P \land (Q \lor R)$	∨-elim 1,2-6,7-11

ii. $(P \lor Q) \land (P \lor R) \vdash P \lor (Q \land R)$

1.	$(P \lor Q) \land (P \lor R)$	premiss
2.	$P \lor Q$	∧-elim-l 1
3.	P	assumption
4.	$P \lor (Q \land R)$	∨-intro-l 3
5.	Q	assumption
6.	$P \lor R$	∧-elim-r 1
7.	P	assumption
8.	$P \lor (Q \land R)$	∨-intro-l 7
9.	R	assumption
10.	$Q \land R$	∧-intro 5,9
11.	$P \lor (Q \land R)$	∨-intro-r 11
12.	$P \lor (Q \land R)$	∨-elim 5,(6),7-8, 9-12
13.	$P \lor (Q \land R)$	∨-elim 1,(2), 3-4, 5-12

iii. $\vdash (P \land Q) \lor (P \land R) \leftrightarrow P \land (Q \lor R)$

1.	$(P \land Q) \lor (P \land R)$	assumption
2.	$P \land (Q \lor R)$	theorem (c) i. 1
3.	$P \land (Q \lor R)$	assumption
4.	P	∧-elim-l 3
5.	$Q \lor R$	∧-elim-r 4
6.	Q	assumption
7.	$P \land Q$	∧-intro 4,6
8.	$(P \land Q) \lor (P \land R)$	∨-intro-l 7
9.	R	assumption
10.	$P \land R$	∧-intro 4,9
11.	$(P \land Q) \lor (P \land R)$	∨-intro-r 10
12	$(P \land Q) \lor (P \land R)$	∨-intro 3,6-8,9-11
13.	$(P \land Q) \lor (P \land R) \leftrightarrow P \land (Q \lor R)$	↔-intro 1-2,3-12

iv. $\vdash (P \lor Q) \land (P \lor R) \leftrightarrow P \lor (Q \land R)$

1.	$(P \lor Q) \land (P \lor R)$	assumption
2.	$P \lor (Q \land R)$	theorem (c) ii. 1
3.	$P \lor (Q \land R)$	assumption
4.	P	assumption
5.	$P \lor Q$	∨-intro-l 4
6.	$P \lor R$	∨-intro-l 4
7.	$(P \lor Q) \land (P \lor R)$	∧-intro 5,6
8.	$Q \land R$	assumption
9.	Q	∧-elim-l 8
10.	$P \lor Q$	∨-intro-r 9
11.	R	∧-elim-r 8
12.	$P \lor R$	∨-intro-r 11
13.	$(P \lor Q) \land (P \lor R)$	∧-intro 10,12
14.	$(P \lor Q) \land (P \lor R)$	∨-elim 3,4-7,8-13
15.	$(P \lor Q) \land (P \lor R) \leftrightarrow P \lor (Q \land R)$	↔-intro 1-2, 3-1

(d) (Proofs involving negation)

i. $\neg P \land \neg Q \vdash \neg(P \lor Q)$

1.	$\neg P \land \neg Q$	premiss
2.	$P \lor Q$	assumption
3.	P	assumption
4.	$\neg P$	$\land$-elim-l 1
5.	$\bot$	$\neg$-elim 3,4
6.	Q	assumption
7.	$\neg Q$	$\land$-elim-r 1
8.	$\bot$	$\neg$-elim 6,7
9.	$\bot$	$\lor$-elim 2,3-5,6-8
10.	$\neg(P \lor Q)$	$\neg$-intro 2-9

ii. $\neg P \lor \neg Q \vdash \neg(P \land Q)$

1.	$\neg P \lor \neg Q$	premiss
2.	$P \land Q$	assumption
3.	$\neg P$	assumption
4.	P	$\land$-elim-l 2
5.	$\bot$	$\neg$-elim 4,3
6.	$\neg Q$	assumption
7.	Q	$\land$-elim-r 2
8.	$\bot$	$\neg$-elim 7,6
9.	$\bot$	$\lor$-elim 1, 3-5, 6-8
10.	$\neg(P \land Q)$	$\neg$-intro 2-9

iii. $\neg(P \land Q) \vdash \neg P \lor \neg Q$

1.	$\neg(P \land Q)$	premiss
2.	$\neg(\neg P \lor \neg Q)$	assumption
3.	$\neg\neg P \land \neg\neg Q$	theorem (from task) 2
4.	$P \land \neg\neg Q$	$\neg\neg$-elim 3
5.	$P \land Q$	$\neg\neg$-elim 4
6.	$\bot$	$\neg$-elim 5,1
7.	$\neg\neg(\neg P \lor \neg Q)$	$\neg$-intro 2-6
8.	$\neg P \lor \neg Q$	$\neg\neg$-elim 7

iv. $A, \neg A \vdash B$

1.	A	premiss
2.	$\neg A$	premiss
3.	$\bot$	$\neg$-elim 1,2
4.	B	$\bot$-elim 3

v. $T \lor U, T \to D, \neg D \vdash U$

1.	$T \to D$	premiss
2.	$\neg D$	premiss
3.	$T \lor U$	premiss
4.	T	assumption
5.	D	$\to$-elim 4,1
6.	$\bot$	$\neg$-elim 5,2
7.	U	$\bot$-elim 6
8.	U	assumption
9.	U	$\lor$-elim 3,4-7,8

vi. $\vdash \neg(P \land \neg P)$

1.	$P \land \neg P$	assumption
2.	P	$\land$-elim-l 1
3.	$\neg P$	$\land$-elim-r 1
4.	$\bot$	$\neg$-elim 2,3
5.	$\neg(P \land \neg P)$	$\neg$-intro 1-4

vii. $(P \to Q) \wedge (Q \to R),\ \neg R \vdash \neg P$

1.	$(P \to Q) \wedge (Q \to R)$	premiss
2.	$\neg R$	premiss
3.	$P \to Q$	∧-elim-l 1
4.	$Q \to R$	∧-elim-r 1
5.	P	assumption
6.	Q	→-elim 5, 3
7.	R	→-elim 6, 4
8.	$\bot$	¬-elim 7, 2
9.	$\neg P$	¬-intro 5-8

viii. $(P \to Q) \wedge (Q \to \neg P) \vdash \neg P$

1.	$(P \to Q) \wedge (Q \to \neg P)$	premiss
2.	$P \to Q$	∧-elim-l 1
3.	$Q \to \neg P$	∧-elim-r 1
4.	P	assumption
5.	Q	→-elim 4, 2
6.	$\neg P$	→-elim 5, 3
7.	$\bot$	¬-elim 4, 6
8.	$\neg P$	¬-intro 4-7

ix. $(P \to Q) \vdash \neg P \vee Q$

1.	$P \to Q$	premiss
2.	$P \vee \neg P$	Law of Excluded Middle
3.	P	assumption
4.	Q	→-elim 3, 1
5.	$\neg P \vee Q$	∨-intro-r 4
6.	$\neg P$	assumption
7.	$\neg P \vee Q$	∨-intro-l 6
8.	$\neg P \vee Q$	∨-elim 2, 3-5, 6-7

x. $\neg P \vee Q \vdash P \to Q$

1.	$\neg P \vee Q$	premiss
2.	$\neg P$	assumption
3.	P	assumption
4.	$\bot$	¬-elim 3, 2
5.	Q	⊥-elim 4
6.	$P \to Q$	→-intro 3-5
7.	Q	assumption
8.	P	assumption
9.	Q	7
10.	$P \to Q$	→-intro 8-9
11.	$P \to Q$	∨-elim 1, 2-6, 7-10

QUESTION 3

We want to introduce a new binary logical connective, which is "exclusive or", written $\veebar$, which is true when exactly one of its arguments are true.

From this, we can deduce its table of truth-values:

P	Q	$P \veebar Q$
F	F	F
F	T	T
T	F	T
T	T	F

P	Q	$(P \wedge$	$\neg Q)$	$\vee$	$(\neg P$	$\wedge Q)$
F	F	F	T	F	T	F
F	T	F	F	T	T	T
T	F	T	T	T	F	F
T	T	F	F	F	F	F

Fig. 1.

(a) We can extend eval to also incorporate this connective this way:

```
eval :: (Prop -> Bool) -> Prop -> Bool
eval v prop = case prop of
        ⊥          -> False
        ...
        p ⊻ q      -> if eval v p then not (eval v q) else eval v q
        ...
```

(b) Elimination rules for $\veebar$

$$\frac{\phi \qquad \phi \veebar \psi}{\neg \psi}\ \veebar\text{-elim-}l \qquad\qquad \frac{\psi \qquad \phi \veebar \psi}{\neg \phi}\ \veebar\text{-elim-}r$$

We will prove that the proof trees are valid in both cases, with the function proves:

```
proves :: Proof -> Conjecture -> Bool
p 'proves' (ps ⊢ c) = conclusion p == c && valid ps p
```

where we have valid defined for these two rules as:

```
valid :: [Prop] -> Proof -> Bool
valid hs proof = case proof of
    ...
    InferBy "⊻-elim-l" [l,r] c ->
        valid hs l && valid hs r &&
        case conclusion r of
            (p ⊻ q) -> conclusion l == p && c == ¬q
            _       -> False
    InferBy "⊻-elim-r" [l,r] c ->
        valid hs l && valid hs r &&
        case conclusion r of
            (p ⊻ q) -> conclusion l == q && c == ¬p
            _       -> False
```

As both trees are valid, we can say that the rules are sound.

As $\phi \veebar \psi$ is equivalent (from (a) and Fig. 1) to $(\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$, we can also prove the two rules with conjectures:

$\phi, (\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi) \vdash \neg \psi$

1.	ϕ	premiss
2.	$(\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$	premiss
3.	ψ	assumption
4.	$\phi \wedge \neg \psi$	assumption
5.	$\neg \psi$	$\wedge$-elim-r 4
6.	$\perp$	$\neg$-elim 3,5
7.	$\neg \phi \wedge \psi$	assumption
8.	$\neg \phi$	$\wedge$-elim-l 7
9.	$\perp$	$\neg$-elim 1,8
10.	$\perp$	$\vee$- elim 2,4-6,7-9
11.	$\neg \psi$	$\neg$-intro 3-10

$\psi, (\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi) \vdash \neg \phi$

1.	ψ	premiss
2.	$(\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$	premiss
3.	ϕ	assumption
4.	$\phi \wedge \neg \psi$	assumption
5.	$\neg \psi$	$\wedge$-elim-r 4
6.	$\perp$	$\neg$-elim 1,5
7.	$\neg \phi \wedge \psi$	assumption
8.	$\neg \phi$	$\wedge$-elim-l-7
9.	$\perp$	$\neg$-elim 3,8
10	$\perp$	$\vee$-elim 2,4-6,7-9
11.	$\neg \phi$	$\neg$-intro 3-10

We can also do the same thing for the <u>introduction rule</u>

$$\frac{\phi \vee \psi \qquad \neg(\phi \wedge \psi)}{\phi \veebar \psi}\ \veebar\text{-intro}$$

```
InferBy "⊻-intro" [l,r] (p ⊻ q) ->
    valid hs l && valid hs r &&
    conclusion l == p ∨ q &&
    conclusion r == ¬ (p ∧ q)
```

We'll also prove it using a conjecture

$\phi \vee \psi, \neg(\phi \wedge \psi) \vdash (\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$

1.	$\neg(\phi \wedge \psi)$	premiss
2.	$\phi \vee \psi$	premiss
3.	ϕ	assumption
4.	$\psi \vee (\neg \psi)$	Law of the Excluded Middle
5.	ψ	assumption
6.	$\phi \wedge \psi$	$\wedge$-intro 3, 5
7.	$\perp$	$\neg$-elim 6, 1
8.	$(\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$	$\perp$-elim 7
9.	$\neg \psi$	assumption
10.	$\phi \wedge \neg \psi$	$\wedge$-intro 3, 9
11	$(\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$	$\vee$-intro-l 10

12.	$(\phi \wedge \neg \psi) \vee (\neg \phi \wedge \psi)$	$\vee$-elim 4, 5-8, 9-11
13.	ψ	assumption
14.	$\phi \vee (\neg \phi)$	Law of the Excluded Middle
15.	ϕ	assumption
16.	$\phi \wedge \psi$	$\wedge$-intro 15, 13
17.	$\perp$	$\neg$-elim 16, 1
18.	$(\phi \wedge \neg \psi) \vee (\neg \phi \vee \psi)$	$\perp$-elim 17
19.	$\neg \phi$	assumption
20.	$\neg \phi \wedge \psi$	$\wedge$-intro 19, 13
21.	$(\phi \wedge \neg \psi) \vee (\neg \phi \vee \psi)$	$\vee$-intro-$\wedge$ 20
22.	$(\phi \wedge \neg \psi) \vee (\neg \phi \vee \psi)$	$\vee$-elim 14, 15-18, 19-21
23.	$(\phi \wedge \neg \psi) \vee (\neg \phi \vee \psi)$	$\vee$-elim 2, 3-12, 13-22

(c) At (b) we also proved the rules as conjecture (not just soundness implied by being valid proof trees)

Question 4

(a) We want to prove that

$$\frac{\varphi \to \bot \vdash C}{\neg\varphi \vdash C}\ \neg\vdash$$

First, we'll prove that $\neg\varphi \vdash \varphi \to \bot$:

1.	$\neg\varphi$	premiss
2.	φ	hyp
3.	$\bot$	$\neg$-e 2,1
4.	$\varphi \to \bot$	$\to$-i 2-3

Therefore, we have

$$\frac{}{\neg\varphi \vdash \varphi \to \bot}\ \text{theorem}$$

We know say that

$$\frac{\neg\varphi \vdash \varphi \to \bot \qquad \varphi \to \bot \vdash C}{\neg\varphi \to C}\ \text{CUT}$$

Since the left branch is our theorum and the right branch is our premiss, we can use the CUT rule to get to our desired result.

Now, let's extend this to

$$\frac{\Gamma, \varphi \to \bot \vdash C}{\Gamma, \neg\varphi \vdash C}\ \neg\vdash$$

We'll start from the obvious fact that $\varphi, \neg\varphi \vdash \bot$ ($\neg$-e). We can therefore say that $\Gamma, \varphi, \neg\varphi \vdash \bot$.

Then, we have

$$\frac{\Gamma, \varphi, \neg\varphi \vdash \bot}{\Gamma, \neg\varphi \vdash \varphi \to \bot}\ \to\text{-i}$$

And if we also use our premiss, we get

$$\frac{\Gamma, \neg\varphi \vdash \varphi \to \bot \qquad \Gamma, \varphi \to \bot \vdash C}{\Gamma, \neg\varphi \vdash C}\ \text{CUT}$$

ii) We want to show that

$$\dfrac{\vdash \varphi \to \bot}{\vdash \neg\varphi}\ \vdash\neg$$

We'll start by proving that

$$\varphi \to \bot \vdash \neg\varphi$$

1. $\varphi \to \bot$ premiss
2. $\varphi \lor \neg\varphi$ law of the Excluded Middle
3. $\boxed{\varphi}$ hyp
4. $\bot$ $\to$-e 3, 1
5. $\neg\varphi$ $\bot$-e 4
6. $\boxed{\neg\varphi}$ hyp
7. $\neg\varphi$ $\lor$-e 2, 3-5, 6

Therefore, we now want to show that

$$\dfrac{\vdash \varphi \to \bot \qquad \varphi\to\bot \vdash \neg\varphi}{\vdash \neg\varphi}\ \text{CUT}$$

And it is correct because of the CUT rule.

We can do the same thing as we did at (i) and show that

$$\dfrac{\Gamma \vdash \varphi \to \bot}{\Gamma \vdash \neg\varphi}\ \vdash\neg$$

~~From above we have $\varphi \to \bot \vdash \neg\varphi$, so, by adding Γ to the left-hand side,~~

~~nothing changes $\Gamma, \varphi \to \bot \vdash \neg\varphi$~~

~~Then, again by using the cut rule we get~~

We will do this by using a proof tree as follows:

$$\dfrac{\dfrac{}{\Gamma \vdash \varphi\to\bot}\,\text{premiss} \qquad \dfrac{\dfrac{}{\Gamma,\varphi\to\bot \vdash \varphi\lor\neg\varphi}\,\text{EM}^* \quad \dfrac{\dfrac{\dfrac{}{\Gamma,\varphi\to\bot,\varphi\vdash\varphi}\,\text{hyp} \quad \dfrac{}{\Gamma,\varphi\to\bot,\varphi\vdash\varphi\to\bot}\,\text{hyp}}{\Gamma,\varphi\to\bot,\varphi\vdash\bot}\,\to\text{-e}}{\Gamma,\varphi\to\bot,\varphi\vdash\neg\varphi}\,\bot\text{-e} \quad \dfrac{}{\Gamma,\varphi\to\bot,\neg\varphi\vdash\neg\varphi}\,\text{hyp}}{\Gamma,\varphi\to\bot\vdash\neg\varphi}\,\lor\text{-e}}{\Gamma\vdash\neg\varphi}\,\text{CUT}$$

EM^* = law of the Excluded Middle

(III)

$\boxed{P \vdash \neg\neg P}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P, P\to\bot \vdash P}\,\text{hyp} \qquad \dfrac{}{P, \bot \vdash \bot}\,\text{hyp}}{P, P\to\bot \vdash \bot}\,{\to}\vdash}{P, \neg P \vdash \bot}\,\neg\vdash}{P \vdash \neg\neg P}\,\neg\text{-}i$$

$\boxed{\neg(P\vee Q) \vdash \neg P \wedge \neg Q}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(P\vee Q), P \vdash P}\,\text{hyp}}{\neg(P\vee Q), P \vdash P\vee Q}\,\vee\text{-}i_L \qquad \dfrac{}{\neg(P\vee Q), P \vdash \neg(P\vee Q)}\,\text{hyp}}{\neg(P\vee Q), P \vdash \bot}\,\neg\text{-}e}{\neg(P\vee Q) \vdash \neg P}\,\neg\text{-}i \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg(P\vee Q), Q \vdash Q}\,\text{hyp}}{\neg(P\vee Q), Q \vdash P\vee Q}\,\vee\text{-}i_R \qquad \dfrac{}{\neg(P\vee Q), Q \vdash \neg(P\vee Q)}\,\text{hyp}}{\neg(P\vee Q), Q \vdash \bot}\,\neg\text{-}e}{\neg(P\vee Q) \vdash \neg Q}\,\neg\text{-}i}{\neg(P\vee Q) \vdash \neg P \wedge \neg Q}\,\wedge\text{-}i$$

$\boxed{(P\to Q)\wedge(Q\to\neg P) \vdash \neg P}$

$$\cancel{\dfrac{\dfrac{}{(P\to Q)\wedge(Q\to\neg P) \vdash P\vee\neg P}\,EM^* \qquad \dfrac{\dfrac{\dfrac{}{(P\to Q)\wedge(Q\to\neg P), P \vdash P}\,\text{hyp} \quad \dfrac{}{(P\to Q)\wedge(Q\to\neg P), P \vdash P\to Q}\,\text{hyp}}{(P\to Q)\wedge(Q\to\neg P), P \vdash Q}\,{\to}\text{-}e \quad \dfrac{}{(P\to Q)\wedge(Q\to\neg P), P \vdash Q\to\neg P}\,\text{hyp}}{(P\to Q)\wedge(Q\to\neg P), P \vdash \neg P}\,{\to}\text{-}e \qquad \dfrac{}{(P\to Q)\wedge(Q\to\neg P), \neg P \vdash \neg P}\,\text{hyp}}{(P\to Q)\wedge(Q\to\neg P) \vdash \neg P}\,\vee\text{-}e}$$

$$\dfrac{\dfrac{}{(P\to Q)\wedge(Q\to\neg P) \vdash P\vee\neg P}\,EM^* \qquad \dfrac{\dfrac{\dfrac{}{(P\to Q)\wedge(Q\to\neg P), P \vdash P}\,\text{hyp} \quad \dfrac{}{(P\to Q)\wedge(Q\to\neg P), P \vdash P\to Q}\,\text{hyp}}{(P\to Q)\wedge(Q\to\neg P), P \vdash Q}\,{\to}\text{-}e \quad \dfrac{}{(P\to Q)\wedge(Q\to\neg P), P \vdash Q\to\neg P}\,\text{hyp}}{(P\to Q)\wedge(Q\to\neg P), P \vdash \neg P}\,{\to}\text{-}e \qquad \dfrac{}{(P\to Q)\wedge(Q\to\neg P), \neg P \vdash \neg P}\,\text{hyp}}{(P\to Q)\wedge(Q\to\neg P) \vdash \neg P}\,\vee\text{-}e$$

EM^* = law of Excluded Middle