Continuous Maths Sample Paper 2019

Question 1

(a) To what numerical problems can the method of *interval bisection* be applied? Describe the method and state, without proof, its order of convergence. (5 marks)

(b) What does it mean to say that an iterative method has *quadratic convergence*? Does quadratic convergence always mean rapid convergence? (3 marks)

(c) Simplify the Newton iteration for finding a root of $f(x) = x^2 + ax$, where a is a positive constant. Prove that this iteration converges quadratically to zero, as long as $x_0 > 0$.

Hint: to show that the iteration converges, first prove $0 < x_{n+1} < \frac{x_n}{2}$. (4 marks)

(d) Simplify the Newton iteration for finding a root of $f(x) = x^2 - 2px + p^2$, where p is a positive constant. Prove that this iteration converges linearly to p, for any x_0. Why does this not contradict the standard result about quadratic convergence of Newton's method? (4 marks)

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex decreasing function with a root at x^*. Prove that, no matter what the initial point x_0, applying Newton's method to find x^* produces a sequence $(x_1, \ldots,)$ where $x_i \leq x^*$ for every $i > 0$.

Hint: first use Taylor's theorem to show that, for some ξ,

$$x_{n+1} - x^* = \frac{\frac{1}{2}(x^* - x_n)^2 \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(\xi)}{\frac{\mathrm{d}f}{\mathrm{d}x}(x_n)}.$$

(4 marks)

TURN OVER

Question 2

(a) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. What does it mean to say that f is *convex*? What does it mean to say that f is *concave*? (2 marks)

(b) Show that the following functions are convex. You may use standard results.

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x}$.

(ii) $g : (0, \infty) \to \mathbb{R}$, $g(x) = e^{x - \log x}$.

(iii) $h : \mathbb{R}^d \to \mathbb{R}$, $h(\boldsymbol{x}) = (\boldsymbol{x}^T \boldsymbol{x})^3$.

(4 marks)

Consider methods to approximate the integral

$$I = \int_a^b f(x)\,\mathrm{d}x,$$

where f is a continuous function.

(c) Derive the *Trapezium rule* for this integral, first for a single strip and then (the *composite Trapezium rule*) dividing the interval $[a, b]$ into n equal strips.

Prove that the composite Trapezium rule always equals or over-estimates the value of I when f is convex.

Hint: this can be done without performing a detailed error analysis, and it holds even if f is not differentiable. (5 marks)

(d) Consider the *Midpoint rule* on a single strip $[0, 2l]$. Assuming that f has two continuous derivatives, use Taylor's theorem to prove that the error $\mathrm{err}(M_1)[f, 0, 2l]$ satisfies

$$-\frac{1}{3} l^3 \max_{x \in [0,2l]} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \le \mathrm{err}(M_1)[f, 0, 2l] \le -\frac{1}{3} l^3 \min_{x \in [0,2l]} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}.$$

Use this to prove that the *composite* Midpoint rule, which divides an interval $[a, b]$ into n strips, always equals or under-estimates the value of I when f is convex. (6 marks)

(e) Consider a *Monte Carlo* estimate for I. Quoting appropriate standard results, find the approximate probability that this gives an over-estimate of I, when the number of samples is large. (3 marks)

Question 3

Consider the equality-constrained optimization problem

$$\text{minimize } f(x,y) \text{ subject to } g(x,y) = 0,$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ have continuous derivatives.

(a) What is the *Lagrangian* for this problem? Explain how it is used in an attempt to find the minimum. (2 marks)

(b) Solve the optimization problem in the case when $f(x,y) = 4x^2 + 2xy + 4y^2$ and $g(x,y) = x^2 + xy + y^2 - 3$. (4 marks)

Now consider the function $f : \mathbb{R}^d \to \mathbb{R}$ given by

$$f(\boldsymbol{x}) = -\boldsymbol{x}^T \mathbf{A} \boldsymbol{x} (\boldsymbol{x} - \boldsymbol{v})^T \mathbf{B} (\boldsymbol{x} - \boldsymbol{v})$$

where $\boldsymbol{v}$ is some fixed nonzero vector, and $\mathbf{A}$ and $\mathbf{B}$ are fixed positive definite, symmetric, matrices.

(c) Find $\frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{x}}$ and $\mathbf{H}(f)$, and use them to prove that f has local maxima at $\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{x} = \boldsymbol{v}$.
Hint: for a multivariate scalar function $p : \mathbb{R}^m \to \mathbb{R}$ and a vector function $\boldsymbol{q} : \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{J}(p\boldsymbol{q}) = \boldsymbol{q}\frac{\mathrm{d}p}{\mathrm{d}\boldsymbol{x}}^T + p\mathbf{J}(\boldsymbol{q})$. (6 marks)

(d) Assume that f has a unique local minimum which we wish to approximate using an iterative method. How appropriate are the following methods, in the case when d is large?

(i) Apply Newton's root-finding method to the system of equations $\frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{x}} = \mathbf{0}$.

(ii) Apply Broyden's root-finding method to the system of equations $\frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{x}} = \mathbf{0}$.

(iii) Apply the method of gradient descent to f.

(iv) Apply Newton's minimization method to f.

In each case describe the rate of convergence that you might expect, and how the complexity of the iterative step depends on the dimension d. (6 marks)

(e) Assume that we implement the chosen numerical method in double-precision arithmetic, and that f can be computed within relative error of machine epsilon. Give (without proof) approximately the best accuracy with which you could hope to find:

(i) The *location* of the local minimum of f, and

(ii) The *value* of f at this local minimum?

(2 marks)