Problem Sheet 1

Please send any comments or corrections to martin@stats.ox.ac.uk.

- 1. (a) How many six digit numbers can be formed with the digits 1, 2, ..., 6 without repetition?
 - (b) How many of these begin with 2?
 - (c) In how many of the numbers in (a) do odd and even digits appear alternately?
 - (d) In how many of the numbers in (a) are the digits 1 and 2 separated?
- 2. (a) How many different arrangements are there of the letters of the word EDAMAME?
 - (b) Suppose that you flip a fair coin eight times. What is the probability that exactly five of the flips are heads and three are tails?
 - (c) A fair die is rolled nine times. What is the probability that 1 appears three times, 2 and 3 each appear twice, 4 and 5 once and 6 not at all?
- 3. Let [n+1] be the set defined by $[n+1] = \{1, 2, ..., n+1\}$. Call a subset of [n+1] with r+1 distinct elements an (r+1)-subset. How many (r+1)-subsets of [n+1] have (k+1) as their largest element? Deduce that

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

- 4. Starting from the axioms of probability, $\mathbf{P}_1 \mathbf{P}_3$ from lectures¹, deduce the following results. (Feel free to make use of any set relations that you need.)
 - (a) $\mathbb{P}(\emptyset) = 0$,
 - (b) $\mathbb{P}(A \setminus B) = \mathbb{P}(A) \mathbb{P}(A \cap B)$,
 - (c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) \mathbb{P}(A \cap B)$ (so a generalisation of \mathbf{P}_3 to the case $A \cap B \neq \emptyset$).
- 5. Let A, B and C be events. The event "A and B occur but C does not" may be expressed as $A \cap B \cap C^c$.
 - (a) Find an expression for the event "at least one of B and C occurs but A does not".
 - (b) Show that the probability of the event in (a) is equal to

$$\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C).$$

(c) How many of the numbers $1, 2, \dots, 600$ are divisible by 5 or 7 but not by 4?

¹For convenience:

 $[\]mathbf{P}_1$: For all $A \in \mathcal{F}$, $\mathbb{P}(A) \geq 0$.

 $[\]mathbf{P}_2$: $\mathbb{P}(\Omega) = 1$.

P₃: If $\{A_i, i \in I\}$ is a finite or countably infinite collection of members of \mathcal{F} , and $A_i \cap A_j = \emptyset$ for $i \neq j$, then $\mathbb{P}(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mathbb{P}(A_i)$.

In particular, if $A, B \in \mathcal{F}$ are disjoint, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

- 6. (The birthday problem). There are n people present in a room. Assume that people's birthdays are equally likely to be on any day of the year.
 - (a) What is the probability that at least two of them celebrate their birthday on the same day? How large does n need to be for this probability to be more than $\frac{1}{2}$? (Ignore leap years.)
 - (b) What is the probability that at least one of them celebrates their birthday on the same day as you? How large does n need to be for this probability to be more than $\frac{1}{2}$?
- 7. A confused college porter tries to hang n keys on their n hooks. He does manage to hang one key per hook, but other than this all arrangements of keys on hooks are equally likely. Let A_i be the event that key i is on the correct hook.

We would first like to find the probability that at least one key is on the correct hook, which is $\mathbb{P}(\bigcup_{i=1}^n A_i)$. The generalisation of 4(c) to the case of n events is

$$\mathbb{P}\left(\bigcup_{1 \le i \le n} A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k)$$
$$- \dots + (-1)^{n+1} \mathbb{P}\left(\bigcap_{1 \le i \le n} A_i\right).$$

This is the inclusion-exclusion formula.

- (a) Explain why $\mathbb{P}(A_1) = \frac{(n-1)!}{n!}$ and $\mathbb{P}(A_1 \cap A_2) = \frac{(n-2)!}{n!}$.
- (b) The second sum on the right-hand side above is over all pairs (i, j) which satisfy the condition $1 \le i < j \le n$. Write down the number of such pairs.
- (c) By generalising the ideas in (a) and (b), find the probability that at least one key is on the correct hook.
- (d) Now let $p_n(r)$ denote the probability that exactly r keys are on the correct hook, for $0 \le r \le n$. Find $p_n(0)$. Show that

$$p_n(r) = \frac{1}{r!} \sum_{k=0}^{n-r} \frac{(-1)^k}{k!}.$$

(e) (Optional.) Use induction to prove the inclusion-exclusion formula.