# Linear Algebra MT18 - Week 3

## Chapter 3 (Matrices)

1. Given
$$A = \begin{bmatrix} 3 & 0 \\ -1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 2 & 3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 2 \end{bmatrix},$$
Compute
$$A^3, \quad A^T A, \quad B - C^T, \quad DB, \quad B^T A, \quad AB^T, \quad BA, \quad CD^T$$

2. Given that $A, B \in \mathbb{R}^{n \times n}$ ($n$ by $n$) are upper triangular show that the product $AB$ is also upper triangular.

3. Use the Gauss-Jordan method to find the inverses of the following matrices, if they exist:

   (a)
$$A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$$
   (b)
$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix}$$
   (c)
$$C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$
   (d)
$$D = \begin{bmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 1 & a \end{bmatrix}$$

4. For an $m$ by $n$ matrix $A$ show that $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m$.
   If $A$ is an $n$ by $n$ invertible matrix show that $\mathcal{C}(A) = \mathbb{R}^n$.

5. (optional) Show that $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.
   If $A$ is an invertible matrix show that $\mathcal{N}(A) = \mathbf{0}$.

## Applications

1. We want to follow the movements of a herd of elephants between three reserves in South Africa, denoted $R1, R2, R3$. The monthly transition matrix is given by

$$P = \begin{array}{c} R1 \\ R2 \\ R3 \end{array} \begin{bmatrix} 5/10 & 4/10 & 6/10 \\ 2/10 & 2/10 & 3/10 \\ 3/10 & 4/10 & 1/10 \end{bmatrix}.$$

   (a) Describe the meaning of the entries of $P$.

   (b) Given in January the elephants are in reserve 2, $R2$, give the probability vector for the following January.

2. (optional) The greater oxford pond frog seems to be in danger of extinction. A study showed the frog lives for no more than three years. Defining children to be aged between $0-1$ year, youths to aged between $1-2$ years and adults $2-3$ years. In any given year it was seen that children do not lay eggs; Youths produce on average four children and each adult produces three children.

   From year to year it is expected that fifty percent of the children will die and only twenty five percent of youths will become adults. The population in 2007 had the distribution forty children, forty youths and twenty adults.

   (a) Construct a Leslie matrix for the frog population.

   (b) Give the population over the next five years, always round down between years. Does it look like the frogs will survive?

   (c) Suppose in 2007 only twenty children, twenty youths and five adults existed. Calculate the population over the next five years, always round down between years. Does it look like the frogs will survive?

   (d) A local industry is discharging chemicals into the pond and it is seen to quarter the fertility of the frogs. Create a new Leslie matrix and determine whether, or not the frogs will survive.

3. Consider a row of three lights, $L_1, L_2, L_3$, that can be off (state 1), green (state 2), or red (state 3). Below the lights are three switches

$S_1, S_2, S_3$, these switches change the states of particular lights to the next state, with state 3 going to state 1.

$S_1$ changes the states of $L_1$ and $L_2$, $S_2$ changes the states of all the lights and $S_3$ changes the states of $L_2$ and $L_3$.

(a) Choose a vector space to describe the states of the three lights, $\mathbb{Z}_m^n$?

(b) Give the vectors corresponding to the switches.

(c) All three lights are initially off, is it possible to press the switches so that $L_1$ is off, $L_2$ is green and $L_3$ is red?