# LINEAR ALGEBRA MT 2018

GABRIEL MOISE

## WEEK 8

1. $A$ is a non-singular, upper-triangular matrix. Then, we can write

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\ 0 & A_{22} & A_{23} & \cdots & A_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{NN} \end{pmatrix} \quad \text{with } A_{ii} \neq 0, (\forall)\, i \in \{1,2,\dots,N\}.$$

The linear system $Au=b$ can be written

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\ 0 & A_{22} & A_{23} & \cdots & A_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{NN} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

This may be solved by writing:

$$u_N = \frac{1}{A_{NN}} b_N \quad \longleftarrow 1 \text{ division}$$

$$u_{N-1} = \frac{1}{A_{N-1\,N-1}} \left( b_{N-1} - A_{N-1\,N} u_N \right) \longleftarrow 1 \text{ division, } 1 \text{ subtraction, } 1 \text{ multiplication}$$

$$u_{N-2} = \frac{1}{A_{N-2\,N-2}} \left( b_{N-2} - A_{N-2\,N-1} u_{N-1} - A_{N-2\,N} u_N \right) \longleftarrow 1 \text{ division, } 2 \text{ subtractions, } 2 \text{ multiplications}$$

$$\cdots$$

$$u_1 = \frac{1}{A_{11}} \left( b_1 - A_{12} u_2 - A_{13} u_3 - \dots - A_{1N} u_N \right) \longleftarrow 1 \text{ division, } (N-1) \text{ subtractions, } (N-1) \text{ multiplications}$$

The algorithm gets the desired solutions in a finite number of operations:

- $N$ divisions
- $\frac{(N-1)N}{2}$ multiplications
- $\frac{(N-1)N}{2}$ subtractions

(0 additions)

$\Rightarrow$ In total, we have $N^2$ operations.

2. We have $A = D - L - U$ and $Au = b \Rightarrow (D-L-U)u = b$

$(D-L)u = Uu + b$, where $u$ is the true solution

We have the iterative solutions $u_0$ (initial guess), $u_1, \dots$, therefore we have (from the Gauss-Seidel method) that: $(D-L)u_n = Uu_{n-1} + b$, $n = 1,2,\dots$

By multiplying the equality with $(D-L)^{-1}$, we get

$$u_n = (D-L)^{-1} U u_{n-1} + (D-L)^{-1} b$$

We can rename: $G = (D-L)^{-1} U$ and $c = (D-L)^{-1} b$ to obtain:

$$u_n = G u_{n-1} + c, \quad n = 1,2,\dots$$

3. We have $A = D - L - U$ and $Au = b$.

We may then write the Gauss-Seidel iterative scheme as:

$$u_0 = \text{initial guess}$$

$$D u_m = U u_{n-1} + L u_m + b, \quad m = 1, 2, \dots$$

Suppose we approximate $D u_n$ by

$$D u_n = (1 - w) D u_{n-1} + w (U u_{n-1} + L u_n + b), \quad w \text{ chosen by the user}$$

The iterative scheme becomes:

$$u_0 = \text{initial guess}$$

$$(D - wL) u_n = ((1-w) D + wU) u_{n-1} + wb, \quad m = 1, 2, \dots$$

By multiplying the equality with $(D - wL)^{-1}$, we get

$$u_n = (D - wL)^{-1} ((1-w) D + wU) u_{n-1} + (D - wL)^{-1} w b$$

We can replace: $G = (D - wL)^{-1} ((1-w) D + wU)$ and $c = (D - wL)^{-1} wb$ to get

$$u_n = G u_{n-1} + c, \quad m = 1, 2, \dots$$

4. We are given the linear system:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \; x = \begin{pmatrix} u \\ v \end{pmatrix}, \; b = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

(a) Jacobi's method

We have $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = D - L - U$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$.

The iterative step is $D x_n = (L + U) x_{n-1} + b$, $m = 1, 2, \dots$

$$x_m = G x_{n-1} + c, \text{ where } G = D^{-1}(L+U), \; c = D^{-1} b$$

Now, we study the eigenvalues of $G$.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}, \; U = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow L + U = \begin{pmatrix} 0 & -2 \\ -3 & 0 \end{pmatrix}$$

$$\Rightarrow G = D^{-1}(L+U) = \begin{pmatrix} 0 & -2 \\ -\frac{3}{4} & 0 \end{pmatrix}$$

The eigenvalues of $G$, $\lambda_1$ and $\lambda_2$, have the following property:

$$\det(G - \lambda_1 I) = \det(G - \lambda_2 I) = 0$$

$$\begin{vmatrix} -\lambda & -2 \\ -\frac{3}{4} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{3}{2} = 0 \Rightarrow \lambda^2 = \frac{3}{2} \Rightarrow \lambda_1 = \sqrt{\frac{3}{2}}, \; \lambda_2 = -\sqrt{\frac{3}{2}}.$$

As $|\lambda_1| = |\lambda_2| = \sqrt{\frac{3}{2}} > 1 \Rightarrow$ this method does NOT converge to a result.

(b) the Gauss-Seidel method

The iterative step is $(D - L) u_n = U u_{n-1} + b$, $m = 1, 2, \dots$

Therefore, we have $u_n = G u_{n-1} + c$, with $G = (D - L)^{-1} U$ and $c = (D - L)^{-1} b$.

We'll study the eigenvalues of $G$.

$$D - L = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} \Rightarrow (D-L)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \quad \Bigg| \Rightarrow G = (D-L)^{-1} U = \begin{pmatrix} 0 & -2 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

$$\det(G - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & -2 \\ 0 & \frac{3}{2} - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{3}{2}\lambda = 0 \Rightarrow \lambda\left(\lambda - \frac{3}{2}\right) = 0 \Rightarrow$$

$\Rightarrow \lambda_1 = 0, \ \lambda_2 = \frac{3}{2}$

But $|\lambda_2| = \frac{3}{2} > 1 \Rightarrow$ the Gauss-Seidel method does <u>NOT</u> converge to a result.

(c) <u>the SOR method, with $w = 0.5$</u>

The iterative step is $(D - wL)u_n = ((1-w)D + wU)u_{n-1} + wb, \quad n = 1, 2, \ldots$

Therefore, we have $u_n = G u_{n-1} + c$, where $G = (D - wL)^{-1}((1-w)D + wU)$ and $c = (D - wL)^{-1} wb$.

$$D - wL = D - \frac{1}{2}L = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -\frac{3}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 4 \end{pmatrix} \Rightarrow (D - wL)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

$$((1-w)D + wU) = \frac{1}{2}D + \frac{1}{2}U = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ 0 & 2 \end{pmatrix} \quad \Bigg| \Rightarrow$$

$$\Rightarrow G = (D - wL)^{-1}((1-w)D + wU) = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{3}{16} & \frac{7}{8} \end{pmatrix}$$

$$\det(G - \lambda I) = 0 \Rightarrow \begin{vmatrix} \frac{1}{2} - \lambda & -1 \\ -\frac{3}{16} & \frac{7}{8} - \lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{1}{2} - \lambda\right)\left(\frac{7}{8} - \lambda\right) - \frac{3}{16} = 0$$

$$\frac{7}{16} - \frac{11}{8}\lambda + \lambda^2 - \frac{3}{16} = 0$$

$$\lambda^2 - \frac{11}{8}\lambda + \frac{1}{4} = 0 \Rightarrow \lambda_{1/2} = \frac{11 \pm \sqrt{57}}{16}.$$

As $\lambda_1 > 1$, the SOR method does <u>NOT</u> converge to a result.

5. A non-singular linear system is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

for given constants $a \neq 0, b, c, d \neq 0, p, q$. $\frac{bc}{ad} > 0$.

(a) For the Jacobi's method to converge, we need that the eigenvalues of $G$ are less than 1 (in modulus), where $G = D^{-1}(L+U) = \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix}$

We have $D^{-1} = \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right)^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} \quad \Bigg| \Rightarrow G = \begin{pmatrix} 0 & -\frac{b}{a} \\ -\frac{c}{d} & 0 \end{pmatrix}$

The eigenvalues of $G$ come from

$$\det(G - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & -\frac{b}{a} \\ -\frac{c}{d} & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{bc}{ad} = 0 \Rightarrow \lambda^2 = \frac{bc}{ad} \ (>0) \Rightarrow$$

$$\Rightarrow \lambda_1 = -\sqrt{\frac{bc}{ad}}, \ \lambda_2 = \sqrt{\frac{bc}{ad}}$$

As $|\lambda_1| = |\lambda_2| = \sqrt{\frac{bc}{ad}} < 1 \Rightarrow \boxed{bc < ad}$

(b) For the Gauss-Seidel method to converge, we need that the eigenvalues of G are less than 1 (in modulus), where $G = (D-L)^{-1}U$.

$$\left.\begin{array}{l} (D-L)^{-1} = \left(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\right)^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ \frac{-c}{ad} & \frac{1}{d} \end{pmatrix} \\ U = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} \end{array}\right| \Rightarrow G = \begin{pmatrix} 0 & -\frac{b}{a} \\ 0 & \frac{bc}{ad} \end{pmatrix}$$

The eigenvalues of G come from

$$\det(G - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & -\frac{b}{a} \\ 0 & \frac{bc}{ad} - \lambda \end{vmatrix} = 0 \Rightarrow -\lambda\left(\frac{bc}{ad} - \lambda\right) = 0 \Rightarrow \left.\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = \frac{bc}{ad} \end{array}\right| \Rightarrow$$

$\Rightarrow |\lambda_2| = \frac{bc}{ad} < 1 \Rightarrow \boxed{bc < ad}$

(c) We know that both Jacobi's method and the Gauss-Seidel method converge $\Rightarrow$ we have $bc < ad$.

As $|\lambda_{max}| = \sqrt{\frac{bc}{ad}}$ for Jacobi's method and $|\lambda_{max}| = \frac{bc}{ad}$ for the Gauss-Seidel method, we compare them;

$$\left.\begin{array}{l} \dfrac{|\lambda_{max}|\,(J)}{|\lambda_{max}|\,(GS)} = \dfrac{\sqrt{\frac{bc}{ad}}}{\frac{bc}{ad}} = \dfrac{1}{\sqrt{\frac{bc}{ad}}} \\ \text{As } bc < ad \Rightarrow \frac{bc}{ad} < 1 \Rightarrow \sqrt{\frac{bc}{ad}} < 1 \end{array}\right| \Rightarrow \frac{|\lambda_{max}|\,(J)}{|\lambda_{max}|\,(GS)} > 1 \Rightarrow$$

$\Rightarrow$ the Gauss-Seidel method converges faster than Jacobi's method

6. The iterative step of the SOR method is

$$(D - wL)\,u_m = \left((1-w)D + wU\right)u_{m-1} + wb\,, \quad m = 1, 2, \ldots$$

$$\left.\begin{array}{l} D = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ 0 & A_{22} & 0 & \cdots & 0 \\ 0 & 0 & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{NN} \end{pmatrix} \\ L = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -A_{21} & 0 & 0 & \cdots & 0 \\ -A_{31} & -A_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{N1} & -A_{N2} & -A_{N3} & \cdots & 0 \end{pmatrix} \end{array}\right| \Rightarrow (D - wL) = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ wA_{21} & A_{22} & 0 & \cdots & 0 \\ wA_{31} & wA_{32} & A_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ wA_{N1} & wA_{N2} & wA_{N3} & \cdots & A_{NN} \end{pmatrix}$$

$$U=\begin{pmatrix} 0 & -A_{12} & -A_{13} & \cdots & -A_{1N}\\ 0 & 0 & -A_{23} & \cdots & -A_{2N}\\ 0 & 0 & 0 & \cdots & -A_{3N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix} \Rightarrow (1-w)D+wU=\begin{pmatrix}(1-w)A_{11} & -wA_{12} & -wA_{13} & \cdots & -wA_{1N}\\ 0 & (1-w)A_{22} & -wA_{23} & \cdots & -wA_{2N}\\ 0 & 0 & (1-w)A_{33} & \cdots & -wA_{3N}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & (1-w)A_{NN}\end{pmatrix}$$

$$wb=\begin{pmatrix} wb_1\\ wb_2\\ \vdots\\ wb_m\end{pmatrix}$$

we have $1+2+\ldots+(N-1)=\frac{(N-1)N}{2}$ multiplications, which we'll do only once at the beginning

N multiplications

$$\begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0\\ wA_{21} & A_{22} & 0 & \cdots & 0\\ wA_{31} & wA_{32} & A_{33} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ wA_{N1} & wA_{N2} & wA_{N3} & \cdots & A_{NN}\end{pmatrix}\begin{pmatrix} u_{m1}\\ u_{m2}\\ u_{m3}\\ \vdots\\ u_{mN}\end{pmatrix}=\begin{pmatrix}(1-w)A_{11} & -wA_{12} & -wA_{13} & \cdots & -wA_{1N}\\ 0 & (1-w)A_{22} & -wA_{23} & \cdots & -wA_{2N}\\ 0 & 0 & (1-w)A_{33} & \cdots & -wA_{3N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & (1-w)A_{NN}\end{pmatrix}\begin{pmatrix} u_{m-1\,1}\\ u_{m-1\,2}\\ u_{m-1\,3}\\ \vdots\\ u_{m-1\,N}\end{pmatrix}+\begin{pmatrix} wb_1\\ wb_2\\ wb_3\\ \vdots\\ wb_N\end{pmatrix}$$

$\frac{N(N+1)}{2}$ multiplications and 1 subtraction: $(1-w)$

$$u_{m1}=\frac{1}{A_{11}}\left((1-w)A_{11}u_{m-1\,1}-\sum_{j=2}^{N} wA_{1j}u_{m-1\,j}+wb_1\right)$$

$$u_{m2}=\frac{1}{A_{22}}\left((1-w)A_{22}u_{n-1\,2}-\sum_{j=3}^{N} wA_{2j}u_{n-1\,j}+wb_2-wA_{21}u_{m1}\right)$$

$$u_{m3}=\frac{1}{A_{33}}\left((1-w)A_{33}u_{n-1\,3}-\sum_{j=4}^{N} wA_{3j}u_{n-1\,j}+wb_3-(wA_{31}u_{m1}+wA_{32}u_{m2})\right)$$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - -

$$u_{nN}=\frac{1}{A_{NN}}\left((1-w)A_{NN}u_{n-1\,N}+wb_N-\sum_{j=1}^{N-1} wA_{Nj}u_{mj}\right)$$

We first do $\frac{(N-1)N+2N+N(N+1)}{2}=\frac{N(N-1+N+1+2)}{2}=\frac{N(2N+2)}{2}=N(N+1)$ multiplications and the $1-w$ subtraction to obtain the parameters we work with at each iterative step.

Now each iterative step requires:

- N divisions (each of them at the end)
- $N^2$ multiplications (each element needs N multiplications)
- $2(N-1)$ subtractions ($u_{m1}$ and $u_{mN}$ need 1, others need 2)
- $(N-2)(N-1)+N$ additions ($u_{m1}$ and $u_{mN}$ need $(N-1)$, others need $(N-2)$)
  (or $N^2-2N+2$)

In total, we have $N+N^2+\cancel{2N}-\cancel{2}+N^2-\cancel{2N}+\cancel{2}=2N^2+N$ operations.

7. An iteration of the method of steepest descent is:

- $r_{m-1} = b - Au_{n-1}$
- $u_m = u_{n-1} + \dfrac{r_{n-1}^T r_{n-1}}{r_{n-1}^T A r_{n-1}} r_{n-1}$

First, to calculate $r_{n-1}$ we need a subtraction and for $Au_{n-1}$:

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix} \begin{pmatrix} u_{n-1\,1} \\ u_{n-1\,2} \\ \vdots \\ u_{n-1\,N} \end{pmatrix} = \begin{pmatrix} A_{11} u_{n-1\,1} + A_{12} u_{m-1\,2} + \dots + A_{1N} u_{m-1\,N} \\ A_{21} u_{n-1\,1} + A_{22} u_{n-1\,2} + \dots + A_{2N} u_{n-1\,N} \\ \text{-----------------------------} \\ A_{N1} u_{n-1\,1} + A_{N2} u_{n-1\,2} + \dots + A_{NN} u_{n-1\,N} \end{pmatrix}$$

which requires <u>$N^2$ multiplications</u> and <u>$N(N-1)$ additions</u>.

Secondly, we have:

$$\begin{pmatrix} u_{m1} \\ u_{m2} \\ \vdots \\ u_{mN} \end{pmatrix} = \begin{pmatrix} u_{n-1\,1} \\ u_{m-1\,2} \\ \vdots \\ u_{n-1\,N} \end{pmatrix} + \frac{r_{m-1}^T r_{n-1}}{r_{n-1}^T A r_{n-1}} r_{n-1}$$

$\downarrow$

$r_{n-1}^T r_{n-1}$ requires $(N-1)$ additions and $N$ multiplications

$r_{n-1}^T A r_{n-1}$ requires $N^2+N$ multiplications and $N^2-1$ additions

$\dfrac{r_{n-1}^T r_{n-1}}{r_{n-1}^T A r_{n-1}}$ requires a division

$\dfrac{r_{m-1}^T r_{n-1}}{r_{m-1}^T A r_{n-1}} r_{n-1}$ requires $N$ multiplications

(+)

1 division, $N^2-1+N-1$ additions $(N^2+N-2)$, $N^2+N+N+N$ multiplications $(N^2+3N)$

So, for the second relation we need 1 division, $N^2+2N-2$ additions, $N^2+3N$ multiplications.

In total, we need for an iterative call:

- 1 division
- $2N^2+N-2$ additions
- $2N^2+3N$ multiplications.

8. We want to solve the linear system

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

with the steepest descent method.

For that to work, we need that A is positive definite (otherwise the minimum of F will not be unique, as desired).

We know that A is positive definite <=> it has positive eigenvalues.

Let's calculate the eigenvalues of A:

$$\det(A-\lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 4 = 0 \Rightarrow 1-\lambda_1 = -2 \Rightarrow \lambda_1 = 3$$

$$1-\lambda_2 = 2 \Rightarrow \lambda_2 = -1 < 0 \Rightarrow$$ A is not positive definite, therefore the steepest descent method will not work.

9. The matrix A is given by

$$A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

(a) To calculate $||A||$, we need to calculate the eigenvalues of $A^T A$.

$$A^T A = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} = \begin{pmatrix} 1.25 & 1 \\ 1 & 1.25 \end{pmatrix}$$

$$\det(A^T A - \lambda I) = \begin{pmatrix} 1.25-\lambda & 1 \\ 1 & 1.25-\lambda \end{pmatrix} = 0 \Rightarrow \left(\frac{5}{4}-\lambda\right)^2 = 1 \Rightarrow \left.\begin{array}{l} \frac{5}{4}-\lambda_1 = 1 \Rightarrow \lambda_1 = \frac{1}{4} \\ \frac{5}{4}-\lambda_2 = -1 \Rightarrow \lambda_2 = \frac{9}{4} \end{array}\right| \Rightarrow$$

$$\Rightarrow \lambda_{min} = 0.25, \ \lambda_{max} = 2.25$$

$$||A|| = \sqrt{\lambda_{max}} \Rightarrow \boxed{||A|| = \frac{3}{2} = 1.5}$$

(b) $$||A^{-1}|| = \frac{1}{\sqrt{\lambda_{min}}} = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{1}{2}} \Rightarrow \boxed{||A^{-1}|| = 2}$$

(c) The condition number of A is $K(A) = ||A|| \cdot ||A^{-1}|| = 1.5 \cdot 2 \Rightarrow \boxed{K(A) = 3}$

10. (a) Given a set of linearly independent vectors $x_1, x_2, \dots, x_k \in \mathbb{R}^n$, we begin by constructing the k mutually orthonormal vectors, $q_1, q_2, \dots, q_k \in \mathbb{R}^n$ this way:

$$q_1 = \frac{1}{||x_1||} x_1$$

$$q_2 = \frac{x_2 - (q_1 \cdot x_2) q_1}{||x_2 - (q_1 \cdot x_2) q_1||}$$

. . . . . . . . . . .

$$q_k = \frac{x_k - \sum_{i=1}^{k-1} (q_i \cdot x_k) q_i}{\| x_k - \sum_{i=1}^{k-1} (q_i \cdot x_k) q_i \|}.$$

As we can see, we can easily verify that

$$q_i \cdot q_j = \begin{cases} 1 & , i = j \\ 0 & , i \neq j. \end{cases}$$

Therefore, $q_1, q_2, \dots, q_k \in \mathbb{R}^n$ are mutually orthonormal vectors.

(b) Let $Q = (q_1 \; q_2 \; \dots \; q_m)$, where $q_1, q_2, \dots, q_m$ are mutually orthonormal vectors. Then

$$Q^T \cdot Q = \begin{pmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_m^T \end{pmatrix} (q_1 \, q_2 \dots q_m) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} = I \Rightarrow \text{the inverse of } Q \text{ is } Q^T.$$

(c) $$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \\ 4 & 8 \end{pmatrix}$$

We want to find the QR factorisation of $A$.

$A = (v_1 \; v_2)$, where $v_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$, which are linearly independent.

Using the Gram-Schmidt algorithm, we'll construct the orthonormal vectors $q_1$ and $q_2$.

$$q_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix}$$

$$q_2 = \frac{v_2 - (q_1 \cdot v_2) q_1}{\|v_2 - (q_1 \cdot v_2) q_1\|}$$

$$(q_1 \cdot v_2) = \frac{1}{5}(3 \;\; 0 \;\; 4) \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} = \frac{1}{5}(3 + 32) = 7$$

$$v_2 - (q_1 \cdot v_2) q_1 = \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} - \frac{7}{5} \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 - \frac{21}{5} \\ 3 \\ 8 - \frac{28}{5} \end{pmatrix} = \begin{pmatrix} -\frac{16}{5} \\ 3 \\ \frac{12}{5} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -16 \\ 15 \\ 12 \end{pmatrix}$$

$$\|v_2 - (q_1 \cdot v_2) q_1\| = \sqrt{\frac{1}{25}(256 + 225 + 144)} = \sqrt{\frac{1}{25} \cdot 625} = \sqrt{25} = 5$$

$$q_2 = \frac{1}{25} \begin{pmatrix} -16 \\ 15 \\ 12 \end{pmatrix}$$

Then,

$$A = (v_1 \; v_2) = (q_1 \; q_2) \begin{pmatrix} q_1 \cdot v_1 & q_1 \cdot v_2 \\ 0 & q_2 \cdot v_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{16}{25} \\ 0 & \frac{15}{25} \\ \frac{4}{5} & \frac{12}{25} \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 0 & 5 \end{pmatrix}$$

Therefore, we found $Q = \frac{1}{25} \begin{pmatrix} 15 & -16 \\ 0 & 15 \\ 20 & 12 \end{pmatrix}$ and $R = \begin{pmatrix} 5 & 7 \\ 0 & 5 \end{pmatrix}$.

(d) We want to minimise the least squares function

$$F(x) = \left\| Ax - \begin{pmatrix} 1 \\ 8 \\ 8 \end{pmatrix} \right\|^2 .$$

Therefore, we want to solve the normal equations

$$A^T A x = A^T b, \text{ where } b = \begin{pmatrix} 1 \\ 8 \\ 8 \end{pmatrix}$$

By using the QR factorisation of A, we get:

$$Rx = Q^T b \Rightarrow \begin{pmatrix} 5 & 7 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 15 & 0 & 20 \\ -16 & 15 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 8 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 7 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 175 \\ 200 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{aligned} 5x_1 + 7x_2 &= 7 \\ 5x_2 &= 8 \Rightarrow x_2 = \frac{8}{5} \end{aligned} \quad \Bigg| \Rightarrow 5x_1 = 7 - \frac{56}{5} = -\frac{21}{5} \Rightarrow x_1 = -\frac{21}{25} \Rightarrow$$

$$\Rightarrow x = \begin{pmatrix} -\frac{21}{25} \\ \frac{8}{5} \end{pmatrix}.$$