PROBABILITY
PROBLEM SHEET 1

Gabriel Moise

1. a) For the first digit we have 6 possibilities, for the second we have 5... for the last one we have one possibility, so the answer is $6! = 720$

b) The first digit is always 2, so we have to form a 5-digit number from $\{1;3;4;5;6\}$. The number of ways is $5! = 120$ (same reasoning from a))

c) For the even numbers $\{2,4,6\}$ we know that they can be in positions 1,3,5 or 2,4,6 in the 6-digit number we want to form. There are $3!$ cases to arrange them for both cases and $3!$ ways to arrange the odd numbers. So, the total number of ways is $2 \cdot 3! \cdot 3! = 72$

d) If 1 is on the first position, 2 can be on positions 3,4,5 or 6. This way, for the second position we have 4 possible digits: $\{3;4;5;6\}$, for the third we have still 4 possibilities: $\{2;3;4;5;6\} \setminus$ the digit already chosen and so on.

We basically have:

$$\overline{1abcde}$$
$$\downarrow\downarrow\downarrow\downarrow\downarrow$$
$$4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 96 \text{ ways}$$

If 1 is in the middle, we need 2 to not be neither on its left, nor on its right. The number of cases for each position of 1 in the middle is the same, so we'll only treat case $\overline{ab1cde}$ and multiply the result by 4 (4 middle digits)

For b we have 4 cases $\{3;4;5;6\}$, for c 3 cases $\{3;4;5;6\} \setminus \{b\}$, for a we have 3 cases $\{3;4;5;6;2\} \setminus \{b,c\}$. For d 2 cases and for e 1 case.

So, the result is $4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 72$ ways

We multiply that by 4 and we get $72 \cdot 4 = 288$ ways

The case where 1 is last digit is the same as when 1 is the first digit, so there are 96 ways in this case.

Summing up, we get the result: $96 + 288 + 96 = 480$ ways.

2. a) There are 2 E's, 1 D, 2 A's, 2 M's, so the number of arrangements is

$$\frac{7!}{2! \cdot 2! \cdot 2!} = 630$$

b) The number of favorable cases where we obtain 5 Heads and 3 Tails is equal to the number of arrangements of 5 H's and 3 T's, which is $\frac{8!}{3! \cdot 5!} = \binom{8}{3} = 56$

The total number of cases is $2^8 = 256$, so the probability is $\frac{56}{256} = \frac{7}{32}$.

c) We need to calculate the number of distinct configurations of 9 outcomes of rolling a fair die. Basically, we need to calculate the number of ways we can arrange three 1's, two 2's, two 3's, one 4 and one 5, and this number is equal to $\frac{9!}{3!\cdot 2!\cdot 2!} = 15120$ favorable cases. The total number of cases is 6^9, so the probability is $\frac{9!}{6\cdot 4\cdot 6^9} = \frac{1\cdot 2\cdot 3\cdot \cancel{4}\cdot 5\cdot \cancel{6}\cdot 7\cdot 8\cdot 9}{\cancel{4}\cdot 6^{10\,9}} = \frac{6\cdot 5\cdot 7\cdot 8\cdot 9}{6^9} = \frac{5\cdot 7\cdot 4\cdot 3\cdot 6}{6^8} =$

$= \frac{5\cdot 7\cdot 2\cdot 6}{6^7} = \frac{5\cdot 7\cdot 2}{6^6} = \frac{70}{46656} = \frac{35}{23328}$

3. $[m+1] = \{1;2;3;\dots;m+1\}$

$(n+1)$ is a subset of $[m+1]$ with distinct elements

First of all, we will calculate the total number of $(n+1)$-subsets. Each subset contains $(n+1)$ distinct numbers from $\{1;2;\dots;m+1\}$. So, the result here is the number of ways we choose $(n+1)$ elements from $(m+1)$. The formula for this is $\binom{m+1}{n+1} = \frac{(m+1)!}{(m-n)!\,(n+1)!}$.

Now, each $(n+1)$-subset has an element that is bigger than all the others. Because we have $(n+1)$ distinct elements from $[m+1]$, we know that this element is at least $(n+1)$. So, if we want to calculate the number of $(n+1)$-subsets with the greatest element $(k+1)$, we have the result 0 for all $k < n$ (because there are none). For $k \geq n$, every $(n+1)$-subset needs to have elements from $\{1;2;\dots;k+1\}$. However, they need to have $(k+1)$ in the set so that it is the greatest element from the set. We are left with n elements to be chosen from $\{1;2;\dots;k\}$, so we have $\binom{k}{n}$ in total. For every $(k+1)$, the number of $(n+1)$-subsets is equal to $\binom{k}{n}$ and all $(n+1)$-subsets have an element $(k+1)$ (k from n to m) which is the greatest from the list.

So, we get to the conclusion that

$$\sum_{k=n}^{m} \binom{k}{n} = \binom{m+1}{n+1}$$ (calculating the number of $(n+1)$-subsets in 2 ways)

4. a) $\mathbb{P}(\emptyset) = 0$.

$A_1 = A_1 \cup \emptyset$

A_1 and $\emptyset$ are disjoint, as $A_1 \cap \emptyset = \emptyset$ $\Big| \Rightarrow \mathbb{P}(A_1 \cup \emptyset) = \mathbb{P}(A_1) + \mathbb{P}(\emptyset)$

$\cancel{\mathbb{P}(A_1)} = \cancel{\mathbb{P}(A_1)} + \mathbb{P}(\emptyset)$

$0 = \mathbb{P}(\emptyset)$

b) $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$

We can define $X = A \setminus B$, $Y = B \setminus A$ and $Z = A \cap B$

From there we get that $A = X \cup Z$; $B = Y \cup Z$; $X \cap Y = \emptyset$; $X \cap Z = \emptyset$; $Y \cap Z = \emptyset$

The initial equality becomes

$$\mathbb{P}(X) = \mathbb{P}(X \cup Z) - \mathbb{P}(Z)$$

$$\mathbb{P}(X) = \mathbb{P}(X) + \cancel{\mathbb{P}(Z)} - \cancel{\mathbb{P}(Z)}$$, which is True

c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Again, we use $X = A \setminus B$, $Y = B \setminus A$, $Z = A \cap B$, so $A = X \cup Z$, $B = Y \cup Z$, $X \cap Y = \emptyset$, $Y \cap Z = \emptyset$, $X \cap Z = \emptyset$ and $A \cup B = X \cup Y \cup Z$

The initial equality becomes

$$\mathbb{P}(X \cup Y \cup Z) = \mathbb{P}(X \cup Z) + \mathbb{P}(Y \cup Z) - \mathbb{P}(Z)$$

$$\mathbb{P}(X) + \mathbb{P}(Y) + \mathbb{P}(Z) = \mathbb{P}(X) + \mathbb{P}(Z) + \mathbb{P}(Y) + \cancel{\mathbb{P}(Z)} - \cancel{\mathbb{P}(Z)}$$, which is True

5. a) So, for at least one of B and C to occur we need $B \cup C$. Also, we need that A does not occur, so A^c occurs, so the expression is $(B \cup C) \cap A^c$ or $(B \cup C) \setminus A$.

b) We will form the sets:

M_1 - elements that are only in A, and not in B or C $\Rightarrow M_1 = A \setminus (B \cup C)$

M_2 - elements that are only in B, and not in A or C $\Rightarrow M_2 = B \setminus (A \cup C)$

M_3 - elements that are only in C, and not in A or B $\Rightarrow M_3 = C \setminus (A \cup B)$

M_4 - elements that are only in A and B, but not in C $\Rightarrow M_4 = (A \cap B) \setminus C$

M_5 - elements that are only in A and C, but not in B $\Rightarrow M_5 = (A \cap C) \setminus B$

M_6 - elements that are only in B and C, but not in A $\Rightarrow M_6 = (B \cap C) \setminus A$

M_7 - elements that are in A, B and C $\Rightarrow M_7 = A \cap B \cap C$

So, we get that:

$A = M_1 + M_4 + M_5 + M_7$; $A \cap B = M_4 + M_7$; $A \cup B = M_1 + M_2 + M_4 + M_5 + M_6 + M_7$

$B = M_2 + M_4 + M_6 + M_7$; $B \cap C = M_6 + M_7$; $B \cup C = M_2 + M_3 + M_4 + M_5 + M_6 + M_7$; $A \cap B \cap C = M_7$

$C = M_3 + M_5 + M_6 + M_7$; $A \cap C = M_5 + M_7$; $A \cup C = M_1 + M_3 + M_4 + M_5 + M_6 + M_7$

and $M_1, M_2, M_3, M_4, M_5, M_6, M_7$ are disjoint

The probability from a) is $\mathbb{P}((B \cup C) \cap A^c) = \mathbb{P}((B \cup C) \setminus A)$ and from our notations that equals to $\mathbb{P}(M_2) + \mathbb{P}(M_3) + \mathbb{P}(M_6)$.

The right side of the equality is:

$\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C)$

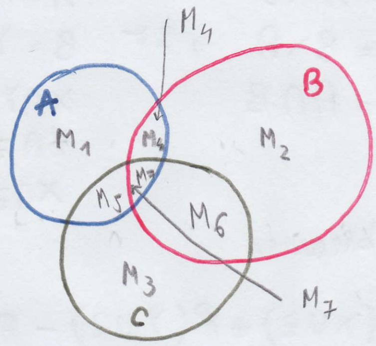
We will replace $\mathbb{P}(M_i) = x_i$, for easier calculus $(i = 1, 2, \dots, 7)$

The right side becomes:

$$(x_2+\cancel{x_4}+\cancel{x_6}+\cancel{x_7})+(x_3+\cancel{x_5}+x_6+\cancel{x_7})-(\cancel{x_6}+\cancel{x_7})-(\cancel{x_5}+\cancel{x_7})-(\cancel{x_4}+\cancel{x_7})+\cancel{x_7}=$$

$= x_2+x_3+x_6 = \mathbb{P}(M_2)+\mathbb{P}(M_3)+\mathbb{P}(M_6)$, which is equal to what we obtained in the left side.

To visualize things easier, we can make a Venn diagram for A, B and C:



c) So, we will have the events:

A – the number is divisible by 4
B – the number is divisible by 5 (for the numbers from $\{1;2;\dots;600\}$)
C – the number is divisible by 7

Therefore, we want to know the probability of $(B\cup C)\setminus A$.

From b), we know that

$\mathbb{P}\big((B\cup C)\setminus A\big) = \mathbb{P}(B)+\mathbb{P}(C)-\mathbb{P}(B\cap C)-\mathbb{P}(A\cap C)-\mathbb{P}(A\cap B)+\mathbb{P}(A\cap B\cap C)$

$\mathbb{P}(B) = \frac{1}{5}$ (as there are 120 numbers divisible by 5 out of 600 in total)

$\mathbb{P}(C) = \frac{85}{600} = \frac{17}{120}$ (as there are 85 numbers divisible by $7 \rightarrow 7\cdot1, 7\cdot2, \dots, 7\cdot85 = 595$ out of 600 in total)

$\mathbb{P}(B\cap C) = \frac{17}{600}$ (17 numbers divisible by 5 and 7 (equivalent to divisible by 35 as $(5,7)=1) \rightarrow 35\cdot1, 35\cdot2, \dots, 35\cdot17 = 595$ out of 600 in total)

$\mathbb{P}(A\cap C) = \frac{21}{600} = \frac{7}{200}$ (21 numbers divisible by 4 and 7 (equivalent to divisible by 28 as $(4,7)=1 \rightarrow 28\cdot1, 28\cdot2, \dots, 28\cdot21 = 588$ out of 600)

$\mathbb{P}(A\cap B) = \frac{30}{600} = \frac{1}{20}$ (30 numbers divisible by 4 and 5 (equivalent to divisible by 20 as $(4,5)=1 \rightarrow 20\cdot1, 20\cdot2, \dots, 20\cdot30 = 600$ out of 600)

$\mathbb{P}(A\cap B\cap C) = \frac{4}{600} = \frac{1}{150}$ (4 numbers divisible by 4, 5 and 7 (equivalent to divisible by 140 as $[4,5,7]=140 \rightarrow 140\cdot1, 140\cdot2, 140\cdot3, 140\cdot4 = 560$ out of 600)

So, the probability we need is:

$$P = \frac{120}{600}+\frac{85}{600}-\frac{17}{600}-\frac{21}{600}-\frac{30}{600}+\frac{4}{600} = \frac{141}{600} = \frac{47}{200}.$$

6. a) The probability of at least two of them to celebrate their birthdays on the same day is actually 1 – the complement of this, which is the probability that all m people have different birthdays.

So, for the first person we choose a random day. The probability that the second person has a different birthday is $\frac{364}{365}$. The third one remains with 363 days out of 365, so the probability is $\frac{363}{365}$ ... the m^{th} person remains with $(366-m)$ days out of 365, so the probability is $\frac{366-m}{365}$.

Let it be $\mathbb{P}(A_i)$ the probability that the i^{th} person doesn't share his/her birthday with anyone before. This way, our probability is

$$\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_m) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 / A_1) \cdot \dots \cdot \mathbb{P}(A_m / A_1 \cap A_2 \cap \dots \cap A_{m-1})$$

Therefore, from what we discussed, $\mathbb{P}(A_i / A_1 \cap A_2 \cap \dots \cap A_{i-1}) = \frac{366-i}{365}$, for i from 1 to m.

Thus, $\mathbb{P}(A_1 \cap A_2 \cap \dots \cap A_m) = \frac{365 \cdot 364 \cdot 363 \cdot \dots \cdot (366-m)}{365^m}$, as $\mathbb{P}(A_1) = \frac{365}{365} = 1$

Our probability will then be

$$P = 1 - \frac{365 \cdot 364 \cdot \dots \cdot (366-m)}{365^m}$$

We want that $P > \frac{1}{2}$ $\Rightarrow \frac{365 \cdot 364 \cdot \dots \cdot (366-m)}{365^m} < \frac{1}{2}$

This happens for $m \geq 23$.

b) In case a) the probability of each event A_i depended on the outcomes of the previous ones, however, in this case, $\mathbb{P}(B_i)$ is the probability of a person i to have a different birthday than you. These events are independent and $\mathbb{P}(B_i) = \frac{364}{365}$, for any i.

So, if there are m people in the room and you are one of them, the probability that nobody shares your birthday is:

$$P = \left(\frac{364}{365}\right)^{m-1}.$$

From $m = 254$ P becomes $< \frac{1}{2}$, so the probability that someone shares your birthday, which is $1-P$, becomes $> \frac{1}{2}$. So, there needs to be at least 254 people, one of which is you for this to happen.

7. $$\mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1\le i<j\le n} \mathbb{P}(A_i\cap A_j) + \sum_{1\le i<j<k\le n} \mathbb{P}(A_i\cap A_j\cap A_k) - \dots + (-1)^{n+1}\,\mathbb{P}\Big(\bigcap_{1\le i\le n} A_i\Big)$$

a) For key 1 to be on hook 1, the number of favorable cases is $(n-1)!$ as key 2 can be on either hooks $2,3,\dots,n$, key 3 the same except the hook for key 2 and so on. Thus, we have $(n-1)\cdot(n-2)\cdot\dots\cdot 1 = (n-1)!$ ways of arranging the keys on hooks such as key 1 is on hook 1. The total number of cases is obviously $n!$, so the probability is $\mathbb{P}(A_1) = \frac{(n-1)!}{n!} = \frac{1}{n}$.

$\mathbb{P}(A_1\cap A_2) = \mathbb{P}(A_1)\cdot \mathbb{P}(A_2 \mid A_1)$

$\mathbb{P}(A_2|A_1)$ is $\frac{1}{n-1}$, each of the remaining hooks: $2,3,\dots,n$ having the same chance to have key 2 on them. So, $\mathbb{P}(A_1\cap A_2) = \frac{1}{n}\cdot\frac{1}{n-1} = \frac{(n-2)!}{n!}$

b) We need to count all pairs (i,j) with $1\le i<j\le n$.

For $i=1$, $j\in\{2;3;\dots;n\}$ – $(n-1)$ pairs

For $i=2$, $j\in\{3;4;\dots;n\}$ – $(n-2)$ pairs

- - - - - - - - - - - -

For $i=n-2$, $j\in\{n-1;n\}$ – 2 pairs

For $i=n-1$, $j=n$ – 1 pair (+)

Total – $\frac{(n-1)n}{2}$ pairs $= \binom{n}{2}$ pairs

c) This probability is actually $\mathbb{P}(\bigcup_{1\le i\le n} A_i)$ and we will calculate it using the inclusion-exclusion formula.

Generalising point a), we find that $\mathbb{P}(\bigcap_{1\le i\le k} A_i) = \frac{(n-k)!}{n!}$ for each k from 1 to n. This happens because if we want to fix the first k keys on the corresponding hooks, the rest of $(n-k)$ keys have $(n-k)!$ different ways to be arranged on the remaining hooks, and this applies for any k keys.

Generalising point b), we obtain that there are $\binom{n}{k}$ k-tuples $(a_1,a_2,\dots,a_k)$ with $a_1<a_2<\dots<a_k$, $k\in\{1;2;\dots;n\}$ and $a_1,a_2,\dots,a_k\in\{A_1;A_2;\dots;A_n\}$ because for each k-tuple we need a combination of k elements from a set of n (and this k-tuples can't have the same elements in different order, as they are ordered increasingly).

So, the sum of probabilities becomes:

$$\mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big) = \binom{n}{1}\cdot\frac{(n-1)!}{n!} - \binom{n}{2}\cdot\frac{(n-2)!}{n!} + \dots + (-1)^n\binom{n}{n-1}\cdot\frac{1!}{n!} + (-1)^{n+1}\binom{n}{n}\cdot\frac{0!}{n!}$$

$$\mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big) = \sum_{k=1}^{n}\binom{n}{k}\cdot\frac{(n-k)!}{n!}\cdot(-1)^{k+1}$$

$$\mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big) = \sum_{k=1}^{n}\frac{\cancel{n!}}{\cancel{(n-k)!}\cdot k!}\cdot\frac{\cancel{(n-k)!}}{\cancel{n!}}\cdot(-1)^{k+1}$$

$$\mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big) = \sum_{k=1}^{n}\frac{1}{k!}\cdot(-1)^{k+1} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^n\cdot\frac{1}{(n-1)!} + (-1)^{n+1}\cdot\frac{1}{n!}$$

d) $p_n(0)$ is the probability that every key is on a wrong hook. Basically, the complement of this is the probability that at least one key is on the correct hook, which we calculated above.

So, $p_n(0) = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^{n-1}\cdot\frac{1}{(n-1)!} + (-1)^n\cdot\frac{1}{n!}$ $\Big(1-\mathbb{P}\big(\bigcup_{1\le i\le n} A_i\big)\Big)$

We'll calculate $p_n(r)$ by approaching the problem this way: Imagine we fix r keys to be on the right hooks. What's the probability that, given the fact that r keys are on the right hooks, the other $(n-r)$ are not on the right hooks, so that <u>exactly</u> r keys are on the right hooks? This probability is exactly $p_{n-r}(0)$ as we are left with a set of $(n-r)$ keys and $(n-r)$ hooks and we want every key to be on a wrong one. However, we fixed the r keys that are correct, so we need to establish what probability it is for r keys to be correct and this is $\sum_{1\le i_1<i_2<\dots<i_r<n}\mathbb{P}(A_{i_1}\cap A_{i_2}\cap\dots\cap A_{i_r})$, which we calculated at c) and it is equal to $\frac{1}{r!}$

We know already that for every $n\ge 1$ $p_n(0) = \sum_{k=2}^{n}\frac{1}{k!}\cdot(-1)^k$, so, by replacing n with $(n-r)$ we get $p_{n-r}(0) = \sum_{k=2}^{n-r}\frac{(-1)^k}{k!}$, which is also equal to $\sum_{k=0}^{n-r}\frac{(-1)^k}{k!}$ because for $k=0$ we get $\frac{1}{1}$ and for $k=1$ we get $\frac{-1}{1}$, which added are 0 and the sum remains the same.

So, $p_n(r) = \frac{1}{r!}\cdot\sum_{k=0}^{n-r}\frac{(-1)^k}{k!}$, which we needed to prove.

e) Case $n=1$

$$\mathbb{P}\Big(\bigcup_{1\le i\le 1} A_i\Big) = \sum_{i=1}^{1} \mathbb{P}(A_i)$$

$$\mathbb{P}(A_1) = \mathbb{P}(A_1) \text{ True}$$

Induction

We assume $p(n)$ holds i.e. $\mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1\le i<j\le n} \mathbb{P}(A_i \cap A_j) + \sum_{1\le i<j<k\le n} \mathbb{P}(A_i\cap A_j\cap A_k) - \dots + (-1)^{n+1}\mathbb{P}\Big(\bigcap_{1\le i\le n} A_i\Big)$ for any n events A_i, $i\in\{1;2;\dots;n\}$

And we need to prove that $p(n+1)$ is True

$$p(n+1): \mathbb{P}\Big(\bigcup_{1\le i\le n+1} A_i\Big) = \mathbb{P}\Big(\bigcup_{1\le i\le n} A_i \cup A_{n+1}\Big) = \mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big) + \mathbb{P}(A_{n+1}) - \mathbb{P}\Big(\Big(\bigcup_{1\le i\le n} A_i\Big)\cap A_{n+1}\Big)$$

(we used $\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cap B)$, which is known to be True from 4. c))

$\mathbb{P}\Big(\Big(\bigcup_{1\le i\le n} A_i\Big)\cap A_{n+1}\Big) = \mathbb{P}\big((A_1\cap A_{n+1})\cup(A_2\cap A_{n+1})\cup\dots\cup(A_n\cap A_{n+1})\big)$ as we applied the distributivity of "$\cap$" over "$\cup$".

Now, the RHS is $p(n)$ for $(A_1\cap A_{n+1}), (A_2\cap A_{n+1}), \dots, (A_n\cap A_{n+1})$, so

$$\mathbb{P}\Big(\Big(\bigcup_{1\le i\le n} A_i\Big)\cap A_{n+1}\Big) = \sum_{1\le i\le n}\mathbb{P}(A_i\cap A_{n+1}) - \sum_{1\le i<j\le n}\mathbb{P}(A_i\cap A_j\cap A_{n+1}) + \sum_{1\le i<j<k\le n}\mathbb{P}(A_i\cap A_j\cap A_k\cap A_{n+1}) - \dots + (-1)^{n+1}\mathbb{P}\Big(\bigcap_{1\le i\le n} A_i\cap A_{n+1}\Big),$$

because if we group k elements from $(A_1\cap A_{n+1})$, $(A_2\cap A_{n+1}), \dots, (A_n\cap A_n)$ and we calculate their intersection, we will have a lot of $A_{n+1}\cap A_{n+1}$, which will be reduced to a single A_{n+1} eventually.

By replacing $\mathbb{P}\Big(\Big(\bigcup_{1\le i\le n} A_i\Big)\cap A_{n+1}\Big)$ into $p(n+1)$ we obtain:

$$\mathbb{P}\Big(\bigcup_{1\le i\le n+1} A_i\Big) = \sum_{1\le i\le n+1}\mathbb{P}(A_i) - \sum_{1\le i<j\le n+1}\mathbb{P}(A_i\cap A_j) + \sum_{1\le i<j<k\le n+1}\mathbb{P}(A_i\cap A_j\cap A_k) - \dots +$$

$$+ (-1)^{n+1}\sum_{1\le i_1<i_2<\dots<i_n\le n+1}\mathbb{P}(A_{i_1}\cap A_{i_2}\cap\dots\cap A_{i_n}) + (-1)^{n+2}\mathbb{P}\Big(\bigcap_{1\le i\le n+1} A_i\Big),$$

which proves that $p(n+1)$ is True (we also used $p(n)$ for replacing $\mathbb{P}\Big(\bigcup_{1\le i\le n} A_i\Big)$)

In conclusion, the "inclusion-exclusion" formula is proven.