Linear Algebra MT18 - Week 4

Chapter 4
(Systems of Linear Equations)

1. Use Gaussian elimination to find all possible solutions to the following systems of equations:

(a)

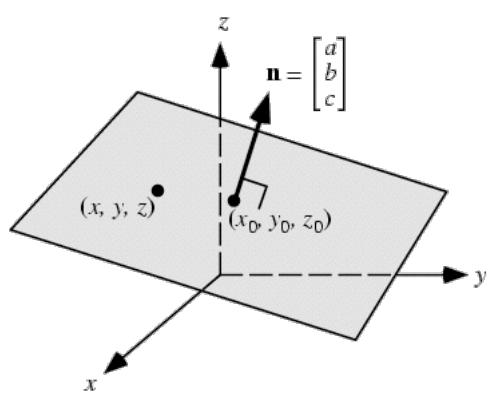
$$\begin{aligned} x + 2y - 3z &= 9 \\ 2x - y + z &= 0 \\ 4x - y + z &= 4 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 + x_2 - x_3 - x_4 + 2x_5 &= 3 \\ x_2 - 2x_3 + x_4 + x_5 &= -1 \\ x_3 + 2x_4 - x_5 &= 2 \end{aligned}$$

(c)

$$\begin{aligned} x_1 - 2x_2 - x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ -3x_1 + x_2 - 2x_3 &= 1 \end{aligned}$$



2. The normal form of the equation of a plane, $\mathcal{P}$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0,$$

where point $\mathbf{x} = [x, y, z]^T$ is lying on $\mathcal{P}$ that is normal to the vector $\mathbf{n} = [a, b, c]^T$ and contains the point $\mathbf{p} = [x_0, y_0, z_0]^T$.

Furthermore, the vector form of the equation of a line, $\mathcal{L}$ is:

$$\mathbf{x} = \mathbf{p} + t\mathbf{d},$$

for some $t \in \mathbb{R}$, and $\mathbf{x} = [x, y]^T$ is any point lying on $\mathcal{L}$ that is parallel to the vector $\mathbf{d} = [a, b]^T$ and contains the point $\mathbf{p} = [x_0, y_0]^T$.

Find the equation of the line where the the two planes

$$3x + 2y + z = -1,$$

and

$$2x - y + 4z = 5$$

intersect.

3. (optional) In four dimensional space, $\mathbb{R}^4$, we have three planes given by

$$u + v + w + z = 6$$

$$u + w + z = 4$$

$$u + w = 2.$$

Describe the intersection of the three planes. If we include the plane

$$u = -1$$

what is the intersection.

Finally, find a fourth plane that leaves us with no solution.

4. Find the nullspace of

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{bmatrix}.$$

by solving $B\mathbf{x} = 0$, where B is the Echelon form of A. What is the dimension of the nullspace? Find the rank of A.

5. (optional) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & -4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}.$$

Given that $\mathbf{v}_i$, $i = 1, 2, 3, 4$ are the transpose of the rows of A consider the following

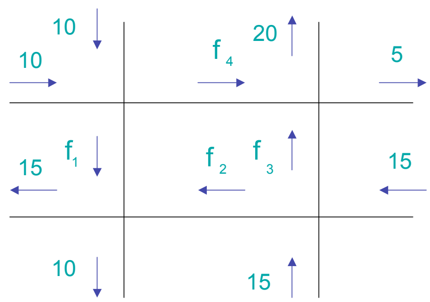
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{b}$$

system of equations, where b is in $\mathbb{R}^5$. Use the Gauss-Jordan method to find the conditions on $\mathbf{b}$ so that the system has a solution.

Next, reduce matrix A to Echelon form and find a basis for the row space of A, $\mathcal{R}(A)$ and a basis for the nullspace of A, $\mathcal{N}(A)$. Verify that vectors in $\mathcal{R}(A)$ satisfy the conditions on $\mathbf{b}$.

Applications

1. Traffic flow along a one way system in Springfield is given by the figure below, where the numbers represent the average number of vehicles per minute entering and leaving a junction.



Traffic Flow on one way streets

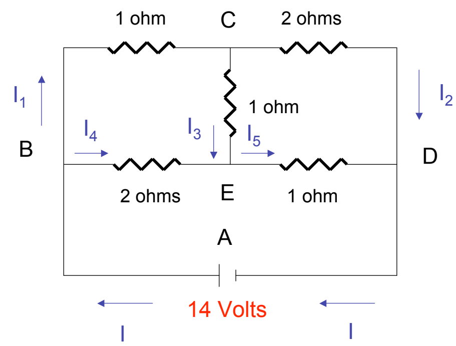
   (a) Set up the system of linear equations for f_1, f_2, f_3 and f_4.

   (b) Solve the system to find possible solutions.

   (c) Traffic flow is regulated so that $f_2 = 10$. Calculate the average flows on the other streets.

   (d) What are the minimum and maximum possible flows on each street.

2. We are given the circuit



(a) Find the currents $I, I_1, I_2, I_3, I_4, I_5$. Hint: Obtain an equation for each basic circuit $ABEDA$, $BCEB$ and $DECD$ using the voltage law, then use the current law at the nodes B, C, D and E.

(b) Find the effective resistance of this circuit.