Seminars 1 & 2 : Complex numbers

1. (a) $x = \pm 3i$ (c) $x = \frac{1}{5}(1 \pm 2i)$
   (b) $x = 1$ (d) $x = 0$, $x = -1$

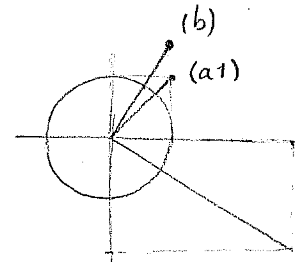
2. (a) $Re\{1+i\} = 1$ $Re\{3-2i\} = 3$
   $Im\{1+i\} = 1$ $Im\{3-2i\} = -2$

   (b) $\underbrace{1+i}_{x}, \underbrace{3-2i}_{y} \to$ $x+y = 4-i$
   $x-y = -2+3i$, $y-x = 2-3i$
   $xy = 5+i$

   (c) $\underbrace{-i}_{x}, \underbrace{4i}_{y} \to$ $x+y = 3i$
   $x-y = -5i$, $y-x = 5i$
   $xy = 4$

   (g) $32+24i$
   (h) $(1+i)' = (1-i)$
   $(i)' = -i$
   $2' = 2$
   $(\sqrt{3}+2i)' = \sqrt{3}-2i$

3. (a) $1+j = \sqrt{2}\, e^{j\pi/4}$, $3-2j = \sqrt{13}\, e^{-j\, 0.588}$
   (b) $1+\sqrt{3}\, j = 2\, e^{j\pi/3}$



4. $|z+7i| = \sqrt{x^2+(y+7)^2}$
   $\left|\frac{z-2}{-z+5-i}\right| = \sqrt{\frac{(x-2)^2+y^2}{(-x+5)^2+(1+y)^2}}$

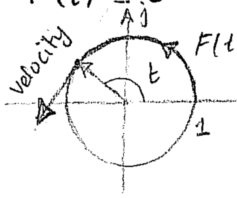
5. (a) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n}$
   $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$

   (b) $\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$
   $= (i\theta)^0 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$ ✱

   (c) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
   $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$ ✱✱

   (d) from ✱ and ✱✱ $\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

6. (a) $F(t) = e^{it}$



(b) $F'(t) = i \underbrace{e^{it}}_{F(t)} = i F(t)$

$|F'(t)| = 1$

(c) $G(t) = \cos t + i \sin t$

$G'(t) = -\sin t + i \cos t = i\,(i \sin t + \cos t) = i G(t)$

(d) $F(0) = e^{i\cdot 0} = 1 \qquad G(0) = \cos 0 + i \sin 0 = 1$

$\Longrightarrow$ same initial condition and same differential equation.

$\Longrightarrow$ same function : $F(t) = G(t)$

7. $$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i \sin\theta \end{aligned}\right\} \Rightarrow \begin{cases} e^{i\theta} + e^{-i\theta} = 2\cos\theta & \Longrightarrow (a) \\ e^{i\theta} - e^{-i\theta} = 2i \sin\theta & \Longrightarrow (b) \end{cases}$$

8. $[\cos\theta + i \sin\theta]^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$

9. $z_1 = r_1 e^{j\theta_1} \quad z_2 = r_2 e^{j\theta_2} \Longrightarrow z_1 z_2 = r_1 r_2 \, e^{j\theta_1} \cdot e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$

$\Longrightarrow |z_1 z_2| = r_1 r_2 = |z_1||z_2|$

$\measuredangle(z_1 z_2) = \theta_1 + \theta_2 = \measuredangle z_1 + \measuredangle z_2$

10. $$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)(\overline{z_1+z_2}) = (z_1+z_2)(\bar z_1 + \bar z_2) \\ &= z_1\bar z_1 + z_2 \bar z_2 + z_1 \bar z_2 + z_2 \bar z_1 \\ &= |z_1|^2 + |z_2|^2 + z_1\bar z_2 + \overline{(z_1 \bar z_2)} \\ &= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}[z_1 \bar z_2] \\ &\le |z_1|^2 + |z_2|^2 + 2|z_1\bar z_2| \qquad (\text{since } \mathrm{Re}\{x\} \le |x|) \\ &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

11. $\mathrm{Re}\{w\bar z\} = \mathrm{Re}\{\underbrace{r_w e^{j\theta_w}}_{w} \cdot \underbrace{r_z e^{-j\theta_z}}_{\bar z}\} = \mathrm{Re}\{r_w \cdot r_z \cdot e^{j(\theta_w - \theta_z)}\}$

$= r_w r_z \cdot \mathrm{Re}\{e^{j(\theta_w-\theta_z)}\} = \underset{|w|}{r_w}\, \underset{|z|}{r_z} \cos \underbrace{(\theta_w - \theta_z)}_{\measuredangle(w,z)}$

$$\Longrightarrow \cos(\theta_w - \theta_z) = \frac{\mathrm{Re}\{w\bar z\}}{|z||w|}$$

12. (d) $e^{j\pi/6} = \frac{\sqrt{3}}{2} + j\frac{1}{2}$

(e) $5e^{j\pi/2} = 5j$

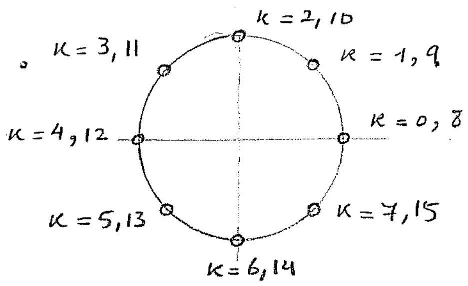
(g) $\tan i = \frac{e^2-1}{e^2+1} j$

13. $\left(e^{j\pi/6}\right)^{-1} = e^{-j\pi/6} = \frac{\sqrt{3}}{2} - j\,\frac{1}{2}$

$\left(5e^{j\pi/2}\right)^{-1} = \frac{1}{5}e^{-j\pi/2} = \frac{-1}{5}j$

$\left(re^{j\theta}\right)^{-1} = \frac{1}{r}e^{-j\theta}$

15.



$$e^{\frac{2\pi(k+8)}{8}i} = e^{\frac{2\pi k}{8}i}$$

In general: $e^{i\theta} = e^{i(\theta+2\pi n)}$

$\Rightarrow e^{i\theta}$ is periodic with period 2π

16. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax-by \\ bx+ay \end{bmatrix} \xrightarrow{\text{equivalent to}} (a+ib)(x+iy)$

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \xrightarrow{\text{equiv. to}} (a+ib)(x+iy) + (c+id)$

18. rotation of angle $\frac{\pi}{2}$: $e^{j\frac{\pi}{2}} = j$

" " $\frac{\pi}{4}$: $e^{j\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}$

" " $60°$: $e^{j\pi/3} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$

" " $30°$: $e^{j\pi/6} = \frac{\sqrt{3}}{2} + j\frac{1}{2}$

19. $2e^{j\pi/4} = \sqrt{2} + j\sqrt{2}$