Review of Complex numbers

1. The purpose of this exercise is to review the historical main motivation for introducing complex numbers: to solve $x^2 = -1$. This led to the number i, which is defined by $i^2 = -1$. With this number on our hands, solve the following quadratic equations.
   (a) $x^2 + 9 = 0$
   (b) $x^2 - 2x + 1 = 0$
   (c) $5x^2 - 2x + 1 = 0$
   (d) $x^3 + 2x^2 + x = 0$

2. The purpose of this exercise is to revise some basic concepts and basic operations on complex numbers. When in this exercise we ask you to compute a complex number, we are asking to express the result in cartesian form that is in the form $x + yi$ where x, y are real numbers.
   (a) Compute the real and imaginary parts of $1 + i$, $3 - 2i$.
   (b) Compute the sum, difference and product of the two complex numbers $1 + i$ and $3 - 2i$.
   (c) Compute the sum, difference and product of the two complex numbers $-i$ and $4i$.
   (d) Compute $\frac{1+i}{3-2i}$. Solve the equation $(3 - 2i)z = 1 + i$.
   (e) Compute $(1 + i)^3$. Compute $(x + iy)^3$ where x, y are real numbers.
   (f) Compute i^n for any natural number n. What is the value of i^{2001} ?
   (g) $-2i(3 + i)(2 + 4i)(1 + i)$
   (h) Compute the complex conjugate of the numbers $1 + i$, i, 2, $\sqrt{3} + 2i$.
   (i) We denote by $\overline{z}$ the complex conjugate of the complex number $z = x + yi$ (x, y are real). Compute $z\overline{z}$.
   (j) Compute $\frac{1}{2}(z + \bar{z})$, $\frac{1}{2i}(z - \bar{z})$.
   (k) Verify that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 z_2} = \overline{z_1}.\overline{z_2}$, $\overline{1/z} = 1/\overline{z}$.
   (l) Compute $\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$.

3. Compute the modulus r and phase θ of the following complex numbers and then write them in polar form $re^{i\theta}$. Represent them graphically in the complex plane.
   (a) $1 + i$, $3 - 2i$.
   (b) $1 + \sqrt{3}i$
   (c) $\cos\phi + i \sin\phi$ where ϕ is a real number.
   (d) $\frac{1+i}{1-i}$.

4. Compute the modulus of $z+7i$ where $z = x+iy$, where x, y are real numbers. Idem with $|\frac{z-2}{-z+5-i}|$ (recall that $|z|$ denotes the modulus of z).

5. Recall Euler's formula $e^{i\theta} = \cos\,\theta + i\sin\theta$ for any real number θ. Let us explain how to derive this formula. Follow the steps:

   (a) Go to your favorite notes on Taylor series and write the Taylor series of $\cos\,\theta$ and $\sin\,\theta$.

   (b) Using them, compute $\cos\,\theta + i\sin\theta$.

   (c) Write the Taylor series for e^x where x is a real number. Accept that the series also holds for e^z where z is a complex number (the series can be used as a definition of e^z). Compute $e^{i\theta}$.

   (d) Conclude $e^{i\theta} = \cos\,\theta + i\sin\theta$.

6. Another view on Euler's formula. Assume that a particle moves in the plane following the trajectory $F(t) = e^{it}$, t being a real number.

   (a) What is the trajectory followed by the particle.

   (b) Compute its velocity vector $F'(t)$ and its speed $|F'(t)|$. For that recall that $\frac{d}{dt}e^{at} = ae^{at}$ where a and t are real numbers. Accept that the result is true when a is a complex number. Show that $F'(t) = iF(t)$.

   (c) Suppose that another particle follows the trajectory $G(t) = \cos\,t + i\sin t$. Show that $G'(t) = iG(t)$.

   (d) Observe that both particles are at the same place at time $t = 0$ and have the same velocity vector. Conclude that $G(t) = F(t)$.

7. Using Euler's formula show that

   (a) $\cos\,\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

   (b) $\sin\,\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

8. Derive De Moivre's formula

$$[\cos\,\theta + i\sin\,\theta]^n = \cos\,(n\theta) + i\sin\,(n\theta),$$

   for any natural number n and any angle θ.

9. Check that if z_1 and z_2 are two complex numbers, then $|z_1z_2| = |z_1||z_2|$. That is, the length of z_1z_2 is the product of the lengths of z_1 and z_2. Check also that the phase of z_1z_2 is the sum of the phase of z_1 plus the phase of z_2.

10. Check that if z_1 and z_2 are two complex numbers, then $|z_1 + z_2| \leq |z_1| + |z_2|$.

11. Let w and z be two complex numbers, both not zero. Show that the cosinus of the angle formed by the two vectors associated to w and z is $\frac{Re(w\bar{z})}{|w||z|}$. Recall that $Re(z)$ denotes the real part of a complex number z.

12. Express in cartesian form, that is, in form $x+iy$, the numbers

    (a) $(1+i)^n + (1-i)^n$ where n is a natural number.

(b) $\left(\frac{-1-i\sqrt{3}}{2}\right)^6$.

(c) $\frac{(1+i\sqrt{3})^3}{(1-i)^2}$.

(d) $e^{i\pi/6}$.

(e) $5e^{i\pi/2}$.

(f) $\sin i$, $\cos i$.

(g) For any complex number z, define $\tan z = \frac{\sin z}{\cos z}$. Express $\tan i$ in cartesian form.

(h) $\tan \frac{1+i}{1-i}$.

13. Compute the inverse of $z = e^{i\pi/6}$, that is, compute $1/z$. Compute the inverses of $5e^{i\pi/2}$, $re^{i\theta}$ (θ is real).

14. Write $(1+i)^{100}$ in polar and in cartesian form.

15. Draw in the plane the points $e^{2k\pi i/8}$, $k = 0, \ldots, 7$. Draw the points when $k = 8, \ldots, 15$. What do you conclude ?

16. Let $a + ib$, $c + id$ be complex numbers, a, b, c, d being real. Represent in matrix form the operation $(x + iy) \to (a + ib)(x + iy)$ (representing complex numbers as column vectors $(x, y)^t$). Represent also using matrix notation the operation $(x + iy) \to (a + ib)(x + iy) + (c + id)$.

17. Consider a complex number in its polar form $re^{i\theta}$. Interpret geometrically the multiplication of another complex number, z, by $re^{i\theta}$. What happens to the plane when $r < 1$? What happens when $r > 1$?

18. Represent with a complex number the rotation of the plane of angle $\frac{\pi}{2}$. Represent the rotations of $\frac{\pi}{4}$, of 60 degrees, and of 30 degrees.

19. Represent with a complex number the transformation of the plane consisting in a rotation of 45 degrees and a change of scale of factor 2.