Assignment 1 - Curvature Estimation via Polygonal Surface Fitting

Geometry Processing - Carleton University

Last update: September 3, 2024

1 Introduction

This is a "warm-up" assignment for learning how to work with triangle meshes through numpy and the geometry processing library GeomProc.

Different methods have been proposed over the years for computing the curvature of triangle meshes. The goal of this assignment is to implement a method based on polygonal surface fitting for computing the Mean and Gaussian curvatures of mesh vertices, and then compare it to the method implemented in the GeomProc library. The method to be implemented consists of two steps:

1. Given a vertex and its neighboring vertices, fit a quadratic surface to all the vertices.
2. Compute the Mean and Gaussian curvature of the vertex based on analytical formulas describing the curvature of the quadratic surface.

2 Quadratic surface fitting

For fitting the quadratic surface, given a vertex v, its normal n, and a set of neighboring vertices, first create a local coordinate system with v as the origin and n as the z axis, and transform all the vertices into this coordinate system. After this transformation, each vertex will have (x, y) coordinates on the plane defined by the normal and a height z to this plane. Note that the given vertex v has coordinates $(0, 0)$ in this local coordinate system.

Then, define the surface in the following form:

$$z = f(x, y) = ax^2 + by^2 + cxy + dx + ey + f. \tag{1}$$

Our goal is to compute the unknown coefficients a, b, c, d, e, f of this quadratic polynomial, based on examples of points on this surface, where the points are given in coordinates (x, y, z) of the local coordinate system. If we have 6 example points and 6 unknowns, we can pose this problem as solving a linear system. If we have more than 6 example points (more equations than unknowns), we can solve the linear system in a least-squares sense.

3 Curvature computation

After finding the quadratic polynomial, we can proceed with computing the curvature at vertex v. Based on the fundamental forms of a surface, the Gaussian (K) and Mean (H) curvatures can be computed as:

$$K = \frac{LN - M^2}{EG - F^2}, \tag{2}$$

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)}, \tag{3}$$

where

$$E = 1 + f_x^2, \tag{4}$$

$$F = f_x f_y, \tag{5}$$

$$G = 1 + f_y^2, \tag{6}$$

$$L = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}, \tag{7}$$

$$M = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}, \tag{8}$$

$$N = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}, \tag{9}$$

and finally

$$f_x = \frac{\partial f}{\partial x}, \tag{10}$$

$$f_y = \frac{\partial f}{\partial y}, \tag{11}$$

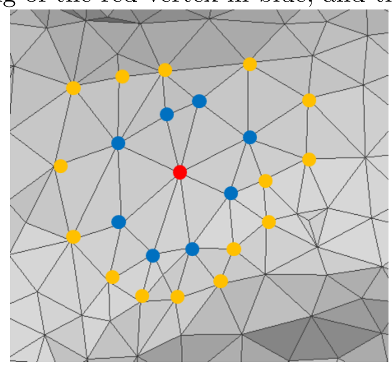
$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \tag{12}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}, \tag{13}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}. \tag{14}$$

4 Implementation details and report

In the assignment, experiment with using the first ring of vertices (neighbors of the given vertex) and first + second ring (neighbors and neighbors of neighbors) as the neighborhood used in the computation. The figure below shows the first ring of the red vertex in blue, and the second ring in yellow.



Compare the values computed with your method and with the method implemented in the GeomProc library to the analytical values computed for known surfaces. Perform the comparison in three manners:

- Compute the difference between the computed values by reporting the mean squared distance for each method, neighborhood size, and curvature type (K and H) to the analytical values.
- Plot a histogram of values for each curvature and method. Use the same bin centers for corresponding histograms to ensure that the histograms are comparable.
- Show a visual comparison of the curvature values based on the two methods.

Experiment with at least three different meshes. You only need to show the visual results and histograms for one mesh. Add the results of these experiments to a report and comment on whether one method seems more accurate than the other.