Exercise 2 (9 points) Let $X_1, X_2, \ldots, X_n$ be independent indicator random variables such that $Pr(X_i = 1) = 1/(4e)$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. By applying the following Chernoff bound, which holds for every $\delta > 0$,

$$Pr(X > (1+\delta)\mu) < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}$$

prove that

$$Pr(X > n/2) < \frac{1}{(\sqrt{2})^n}.$$

Solution: To apply the Chernoff bound we set $n/2$ equal to $(1+\delta)\mu$; since $\mu = E[X] = \sum_{i=1}^{n} E[X_i] = n/(4e)$, we get $\delta = 2e - 1$. Therefore

$$\begin{aligned}
Pr(X > n/2) &= Pr(X > (1 + 2e - 1)\mu) \\
&< \left(\frac{e^{2e-1}}{(2e)^{2e}}\right)^{n/(4e)} \\
&< 1/(2^{2e/4e})^n \\
&= \left(1/\sqrt{2}\right)^n.
\end{aligned}$$

Why this computation? The following explains it exactly:

$Pr(X_i = 1) = \frac{1}{2}$

$X = \sum_{i=1}^{n} X_i = n$

$E[X] = \frac{n}{2}$

↓

$E[X] = \frac{n}{4e}$

$(1+\delta)\frac{\mu}{4e}$

$\frac{n}{2} = (1+\delta)\mu$

$\frac{2\cancel{n}}{\cancel{n}\cancel{2}} = (1+\delta)\frac{n}{4e}\cdot\frac{2}{n}$

$2 = (1+\delta)\cdot\frac{1}{e}$

$2e = (1+\delta)$

$2e - 1 = \delta$