Graph Algorithms

Graphs: the basics

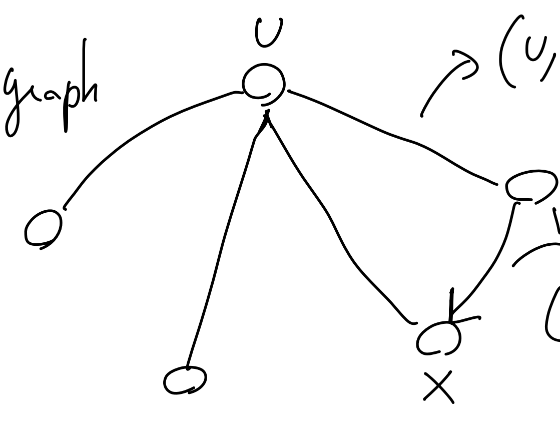
A graph is a representation of the relationships between pairs of objects

$$G = (V, E)$$

V = set of <u>vertices</u> (a.k.a. <u>nodes</u>)

$E \subseteq V \times V$ is a collection of <u>edges</u>, where an edge is a pair of vertices

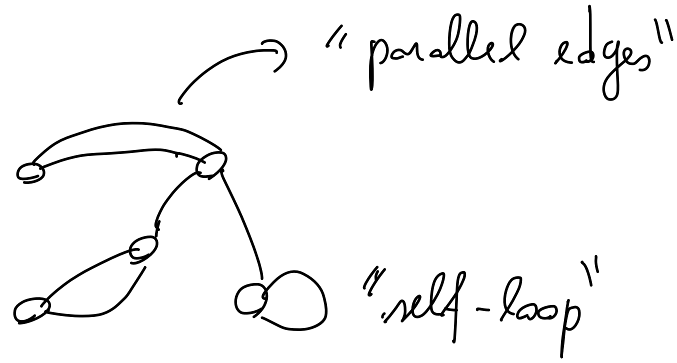
undirected graph



$(u, v) = (v, u)$

if directed: "arc"

$(v, x) \neq (x, v)$



In this course we'll (mostly) use <u>simple</u> graphs:

- no parallel edges
- no self-loops

Terminology:

$e = (u, v)$ → e is <u>incident</u> on u and v

↘ u and v are <u>adjacent</u>

<u>neighbors</u> of a vertex v: all vertices u such that $(u, v) \in E$

<u>degree</u> of a vertex v, denoted $d(v)$ or $\text{degree}(v)$, is the number of edges incident on v

Examples of graphs: in many ways, graphs are the main modality of data we receive from nature:

- road networks → (cities, roads)
- computer networks → (computers, connections)
- WWW → (webpages, hyperlinks)
- social networks → (people, friendship relationships)
- biological networks, e.g. molecules (atoms, chemical bonds)
  brain (neurons, synapses)
- finance → (accounts, transactions)

Concepts:

path: $v_1, v_2, \dots, v_k$ and $(v_i, v_{i+1}) \in E$
$\forall\ 1 \leq i \leq k$

simple path: v_i are all distinct

cycle : simple path s.t. $v_1 = v_k$

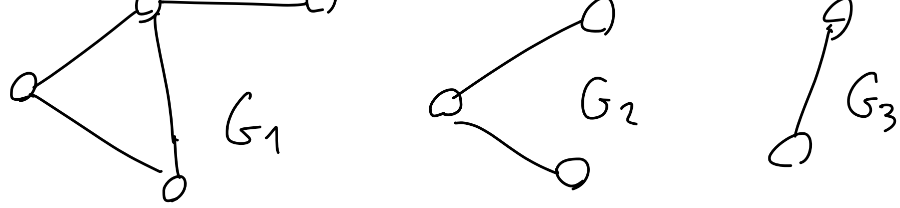
subgraph : $G' = (V', E')$ s.t. $V' \subseteq V$, $E' \subseteq E$ and the edges of E' are incident only on vertices of V'

spanning subgraph : a subgraph with $V' = V$

connected graph : if $\forall u, v \in V$ $\exists$ a path from u to v

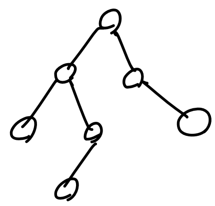
connected components : a partition of G in subgraphs $G_i = (V_i, E_i)$ $\forall$ $1 \leq i \leq k$ s.t.

- G_i is connected $\forall i$
- $V = V_1 \cup V_2 \cup \dots \cup V_k$
- $E = E_1 \cup E_2 \cup \dots \cup E_k$
- $\forall$ $i \neq j$ there is no edge between V_i and V_j



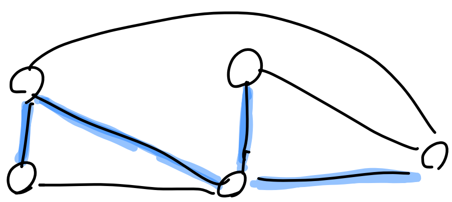
G connected $\Rightarrow$ $k=1$

Tree : connected graph without cycles



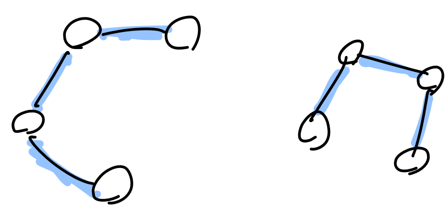
forest : set of trees (disjoint)

spanning tree : a spanning subgraph connected and without cycles



(it exists only if G is connected)

spanning forest : a spanning subgraph without cycles



Basic graph problems:

- traversal
- connectivity
- conn. components
- spanning trees
- minimum-weight spanning trees
- shortest paths

Notation

$n = |V|$

$m = |E|$

size of a graph? $n + m$

Properties of graphs: let $G = (V, E)$ be a simple, undirected graph with n vertices and m edges. Then

1) $\sum_{v \in V} d(v) = 2m$

2) $m \leq \binom{n}{2}$

3) G is a tree $\Rightarrow m = n-1$

4) G is connected $\Rightarrow m \geq n-1$

5) G is acyclic (i.e. is a forest) $\Rightarrow m \leq n-1$

Exercise: prove these properties