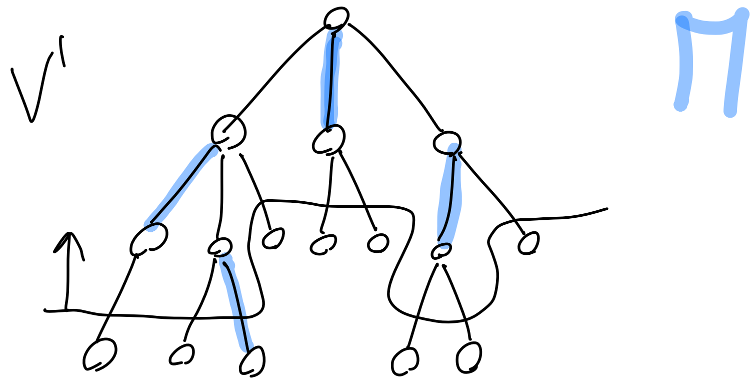
Exercise: an alternative 2-approx for vertex cover

Solution: idea: show a large enough matching in the DFS tree



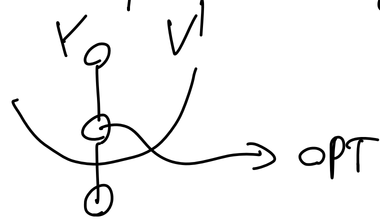

Let $r$ be the root, choose one child $v$ and add $(r,v)$ to the matching $\Pi$. $\forall$ level $i \geq 1$ consider all vertices $v$ not touched by any edge of $\Pi$; choose a child $u$ and add $(v,u)$ to $\Pi$. Repeat up to the leaves.

$$|V'| \leq 2|\Pi|$$

$\forall$ matching $\Pi'$ of $G$, $|V^*| \geq |\Pi'|$

$$|V'| \leq \cancel{2|\Pi|} \leq 2|V^*|$$

Show that the 2 factor is tight :




Exercise: NP-hardness of Metric TSP

1) $w'(u,v) \overset{?}{\leq} w'(u,w) + w'(w,v)$

$w(u,v) + W \overset{?}{\leq} w(u,w) + w(w,v) + 2W$

$w(u,v) \overset{?}{\leq} w(u,w) + w(w,v) + W$

$\underbrace{w(u,w)}_{\geq 0} + \underbrace{w(w,v)}_{\geq 0} + \underbrace{W - w(u,v)}_{\geq 0} \overset{?}{\geq} 0$

2) $\Rightarrow$ : $\exists$ Ham. circuit of cost $k$ in $G$
$\Rightarrow$ the same circuit introduces a $+W$
$\forall$ edge $\Rightarrow$ in $G'$ its cost is $k + nW$

$\Leftarrow$ : just remove the $+W$ $\forall$ edge to obtain a Ham. circuit of cost $k$ in $G$

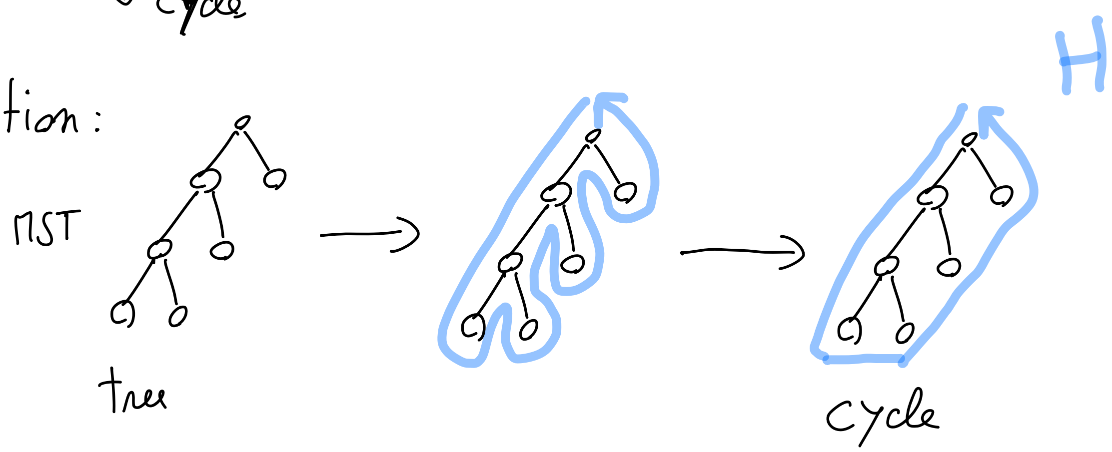
# Metric TSP : a 2-approximation algorithm

V.C. $\rightsquigarrow$ matching

Metric TSP $\rightsquigarrow$ minimum spanning tree

$\downarrow$ cycle

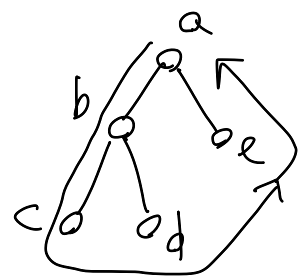
intuition:




tree $\longrightarrow$ cycle?

```
PREORDER(T, v)
    print(v)
    if internal(v) do
        for each u ∈ children(v) do
            PREORDER(u)
    return
```

Example :




$T$

a, b, c, d, e

idea: add to the preorder list the root $\longrightarrow$ Ham. cycle of the original graph.

APPROX_METRIC_TSP $(G)$

$V = \{ v_1, v_2, \dots, v_n \}$

$r \leftarrow v_1$ // root, from which PRIM is run

$T^* \leftarrow \text{PRIM}(G, r)$

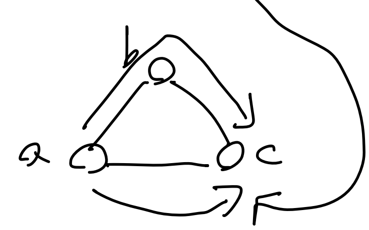
$\langle v_{i_1}, v_{i_2}, \dots, v_{i_n} \rangle = H' \longleftarrow \text{PREORDER}(T^*, r)$

return $\langle H', v_{i_1} \rangle = H$

Analysis of the cost of $H$

intuition: 1) cost of $T^*$ is "low" (actually, the lowest)

2) triangle ineq. $\Longrightarrow$ "shortcuts" do not increase the cost



1) lower bound to the cost of $H^*$ (= optimal tour)

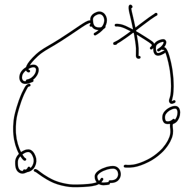
(for v.c.: $|V^*| \geq |A|$)

$w(H^*) \geq$ ?

$\uparrow$

$w(T^*)$

$H^*$



2) upper bound to the cost of $H$

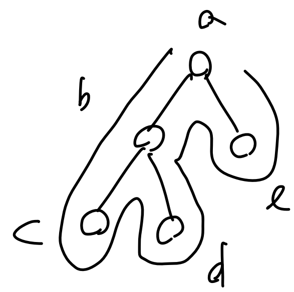
$$w(H) \leq \cancel{\alpha w(T^*)} \leq \alpha w(H^*)$$

$$\alpha = 2$$

$$w(H) \leq 2w(T^*) \; ?$$

Definition: given a tree, a <u>full preorder chain</u> is a list with repetitions of the vertices of the tree which identifies the vertices reached from the recursive calls of PREORDER$(T, v)$.

Example :




$$f.p.c.: \ a, b, c, b, d, b, a, e, a$$

Property: $w(f.p.c.) = 2w(T^*)$

$$\begin{aligned} 2w(T^*) &= w(\langle a, b, c, b, d, b, a, e, a \rangle) \\ &\geq w(\langle a, b, c, d, b, a, e, a \rangle) \\ &\geq w(\langle a, b, c, d, e, a \rangle) \end{aligned}$$

shortcut

triangle ineq.

$$\Rightarrow 2w(T^*) \geq w(H)$$

Putting pieces together :

1) $w(H^*) \geq w(T^*)$

2) $2w(T^*) \geq w(H)$

$$2w(H^*) \geq \cancel{2w(T^*)} \geq w(H)$$

$$\Rightarrow \frac{w(H)}{w(H^*)} \leq 2$$

Exercise: show that the above analysis is tight by giving an example of a graph where APPROX_METRIC_TSP returns a solution of cost $2 \cdot H^*$