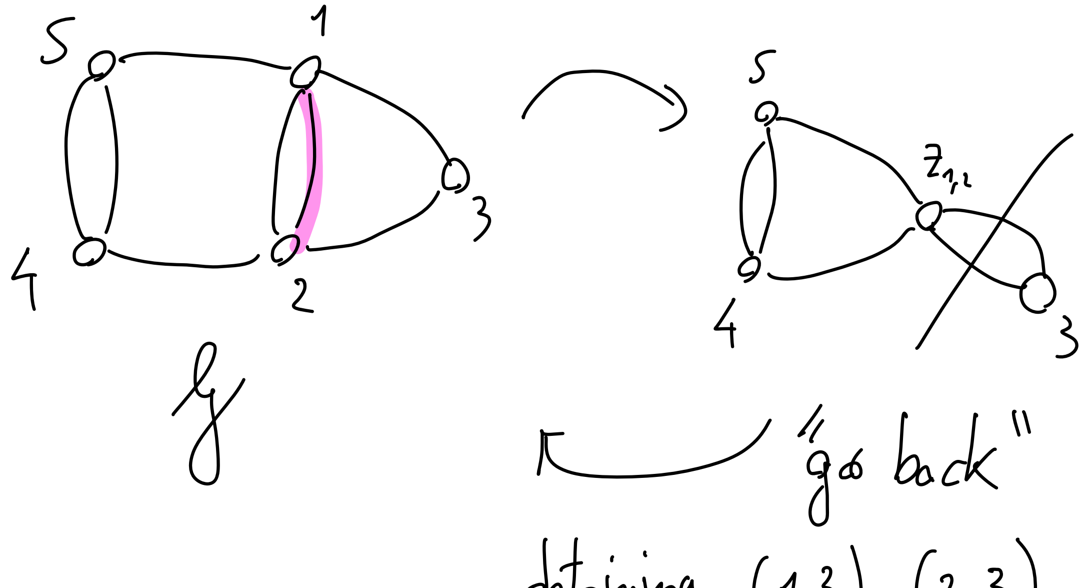
Analysis of Karger's algorithm

Property: $\forall$ cut C' in $G/_e$ $\exists$ a cut C in G of the same cardinality

$\Rightarrow$ $|\text{min-cut in } G/_e| \geq |\text{min-cut in } G|$

Proof: constructive: we'll determine the corresponding cut C in G

C' in $G/_{e=(u,v)}$ $\rightsquigarrow$ C in G by substituting each edge $(z_{u,v}, y)$ in C' with (u,y) or (v,y)

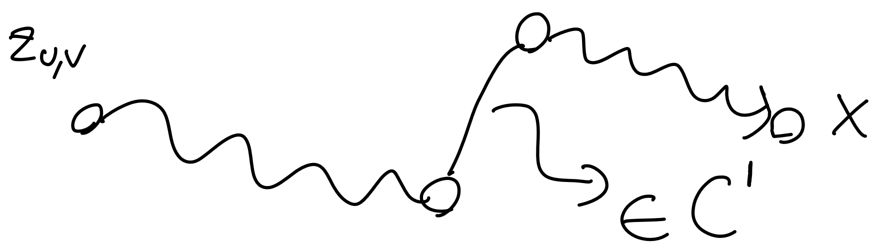


obtaining $(1,3)$, $(2,3)$

$|C'| = |C|$

It remains to show that C is a cut in $\mathcal{G}$.

C' cut in $\mathcal{G}/e = (V', \mathcal{E}') \Rightarrow C'$ separates V' in 2 connected components; let $V_1 \subset V'$ the connected component containing $z_{u,v}$, and let $x \notin V_1$. Then in $\mathcal{G}/e$ every path from $z_{u,v}$ and x must an edge in C':



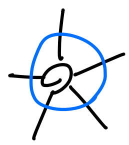
Now we'll show that C in $\mathcal{G}$ disconnects u and v from x: assume by contradiction that C is <u>not</u> a cut in $\mathcal{G} \Rightarrow \exists$ a path between u and x after the removal of C in $\mathcal{G}$. Then the path between $z_{u,v}$ and x "survives" the removal of C' in $\mathcal{G}/e$, i.e. C' is not a cut in $\mathcal{G}/e$: contradiction

What are the cuts that disappear in $\mathcal{G}/e$?

Those hit by the random choice $\Rightarrow$ I want the probability of <u>not</u> hitting edges of a min-cut to be sufficiently high.

Intuition: $|\text{min-cut}|$ is a small fraction of $|\mathcal{E}|$

Property: let $\mathcal{G} = (V, \mathcal{E})$, $|V| = n$. If $\mathcal{G}$ has a min-cut of size t, then $|\mathcal{E}| \geq t\,\frac{n}{2}$

Proof:  $\rightarrow$ $d(v) \geq t \quad \forall v \in V$

$$\sum_{v \in V} \Big\downarrow \sum_{v \in V}$$

$$2|\mathcal{E}| \geq t \cdot n$$

We'll use <u>conditional probabilities</u>

Def.: E_1, E_2 events are <u>independent</u> if

$$\Pr(E_1 \cap E_2) = \Pr(E_1) \cdot \Pr(E_2)$$

Def.: $\Pr(E_1) > 0$ then $\Pr(E_2 | E_1) = \dfrac{\Pr(E_1 \cap E_2)}{\Pr(E_1)}$

extension to k events:

$$\Pr(E_1 \cap E_2 \cap \cdots \cap E_k) = \Pr(E_1)\Pr(E_2 | E_1)\Pr(E_3 | E_1 \cap E_2) \cdots \Pr(E_k | E_1 \cap \cdots \cap E_{k-1})$$

(can be proved by induction on k)

Theorem: The probability that FULL-CONTRACTION returns a minimum cut in $\mathcal{G}$ is at least $2/n(n-1)$

Proof.: Although there may be >1 min-cuts, we will actually prove that, for any min-cut C, the prob. that the alg. returns that particular min-cut C is at least $2/n(n-1)$. So, let C be some specific min-cut

$$t = |C|$$

E_i = in the i-th contraction I did not hit an edge of C

$$\Pr(\bar{E}_1) = \frac{t}{\underbrace{|E|}_{\geq t\frac{n}{2}}} \leq \frac{t}{t\frac{n}{2}} = \frac{2}{n}$$

$$\Pr(E_1) = 1 - \Pr(\bar{E}_1) \geq 1 - \frac{2}{n}$$

$$\Pr(E_2 \mid E_1) \geq 1 - \frac{t}{t\frac{(n-1)}{2}} = 1 - \frac{2}{n-1}$$

$$\vdots$$

$$\Pr(E_i \mid E_1 \cap E_2 \cap \cdots \cap E_{i-1}) \geq 1 - \frac{t}{t\frac{(n-i+1)}{2}} = 1 - \frac{2}{n-i+1}$$

$$\Pr(\text{FULL-CONTR. succeeds}) = \Pr\left(\bigcap_{i=1}^{n-2} E_i\right)$$

i.e., no edge of C is ever contracted

$$\geq \prod_{i=1}^{n-2}\left(1 - \frac{2}{n-i+1}\right)$$

$$= \prod_{i=1}^{n-2} \frac{n-i-1}{n-i+1}$$

$$= \frac{\cancel{n-2}}{n} \frac{\cancel{n-3}}{n-1} \frac{\cancel{n-4}}{\cancel{n-2}} \cdots \frac{\cancel{3}}{\cancel{5}} \frac{2}{\cancel{4}} \frac{1}{\cancel{3}}$$

$$= \frac{2}{n(n-1)}$$

low, but not too low:

KARGER amplifies this probability by repeating FULL-CONTRACTION k times

$$\Pr(k \text{ runs of FULL-CONTR do not return a min-cut})$$

$$\leq \underbrace{\left(1-\frac{2}{n^2}\right)^k}$$

goal: $\frac{1}{n^d}$ for some constant $d>0$

$$\left(1-\frac{2}{n^2}\right)^k \leq \frac{1}{n^d}$$

in this cases it's standard the use of this inequality:

$$\left(1+\frac{x}{y}\right)^y \leq e^x \qquad y \geq 1 \quad y \geq x$$

$$\left(1-\frac{2}{n^2}\right)^{k=n^2} \leq e^{-2} = \frac{1}{e^2} \qquad \text{is \underline{not} in the form } \frac{1}{n^d}$$

recall: $e^{-\ln n^d} = \frac{1}{n^d}$

$$\left(\left(1-\frac{2}{n^2}\right)^{n^2}\right)^{\ln n^d} = \left(1-\frac{2}{n^2}\right)^{n^2 \ln n^d}$$

let's wrap up:

$$\left(1-\frac{2}{n^2}\right)^{\boxed{k=\frac{d n^2 \ln n}{2}}} = \left(\left(1-\frac{2}{n^2}\right)^{n^2}\right)^{\frac{\ln n^d}{2}}$$

$$\leq \left(e^{-2}\right)^{\frac{\ln n^d}{2}} = e^{-\ln n^d} = \frac{1}{n^d}$$

$$\Rightarrow \Pr(\text{KARGER succeeds}) > 1 - \frac{1}{n^d}$$

Complexity:

FULL-CONTR.: $O(n \cdot n) = O(n^2)$

$\Rightarrow$ KARGER: $O(n^4 \log n)$

Speeding up Karger's alg.: (Karger and Stein, 1996) idea : $\Pr(\text{failure of FULL-CONTR.}) = \frac{2}{n} \rightarrow \frac{2}{n-1} \rightarrow \frac{2}{n-2}$

1st contraction 2nd 3rd

do not repeat the first $\sim \frac{n}{\sqrt{2}}$ contractions

$\rightarrow O(n^2 \log^3 n)$ and correct w.h.p.

Current fastest : $O(m \log n)$ (2020)

Programming exercise : implement Karger's alg. and compare it with a deterministic alg.

Exercise : Using the analysis of Karger's alg. show that the n° of distinct min-cuts in a graph is at most $n(n-1)/2$

Also, show that this bound is tight