University of Padova
Master's degree in Computer Science

Advanced Algorithms

Spring 2023

September 13, 2023 – 9:30–11:30

First Part: Theory Questions

Question 1 (4 points) Consider the Union-Find data structure, with the Union operation implemented with union-by-size: show that the complexity of the Find operation is $O(\log n)$, where n is the number of objects in the data structure.

Question 2 (4 points) Consider the following directed and weighted graph, represented by an adjacency matrix where each numerical value represents the weight of the corresponding arc and where the symbol '–' indicates the absence of the arc between the corresponding vertices.

	a	b	c	d	e	f
a	-	3	1	3	-	4
b	-	-	-	-	3	-
c	-	1	-	4	5	-
d	-	-	-	-	-	4
e	2	-	-	-	-	-
f	-	-	-	-	2	-

(a) *Draw the graph.*

(b) *List the lengths of the shortest paths from vertex a to all the other vertices of the graph in the order they are determined by Dijkstra's algorithm.*

Question 3 (4 points) Define what it means for a decision problem A to reduce in polynomial time to a decision problem B.

Second Part: Problem Solving

Exercise 1 (9 points) Given a graph $G = (V, E)$, a *maximal* independent set is an independent set S such that, for each $v \in V \setminus S$, $S \cup \{v\}$ is *not* an independent set.

(a) Give a fast algorithm to return a maximal independent set in G.

(b) Give an example of a graph where there is a maximal independent set of size much smaller than the size of a *maximum* independent set.

Exercise 2 (11 points) Let S be a set of n distinct positive integers, and let WORK(S) be a procedure which, given input S, returns an integer by performing n^2 operations. Now consider the following randomized algorithm:

```
RAND_REC(S)
  if |S| <= 1 then return 1
  x = WORK(S)
  p = RANDOM(S)
  S1 = {s in S such that s < p}
  S2 = {s in S such that s > p}
  if (|S1| >= |S2|) then
     y = RAND_REC(S1)
  else
     y = RAND_REC(S2)
  return x + y
```

Applying the following Chernoff bound show that the complexity of RAND_REC(S) is $O(n^2 \log n)$ with high probability. (Hint: recall the analysis of randomized QuickSort.)

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be independent indicator random variables such that $E[X_i] = p_i, 0 < p_i < 1$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. Then, for $0 < \delta \leq 1$,*

$$\Pr(X < (1-\delta)\mu) < e^{-\mu\delta^2/2}.$$