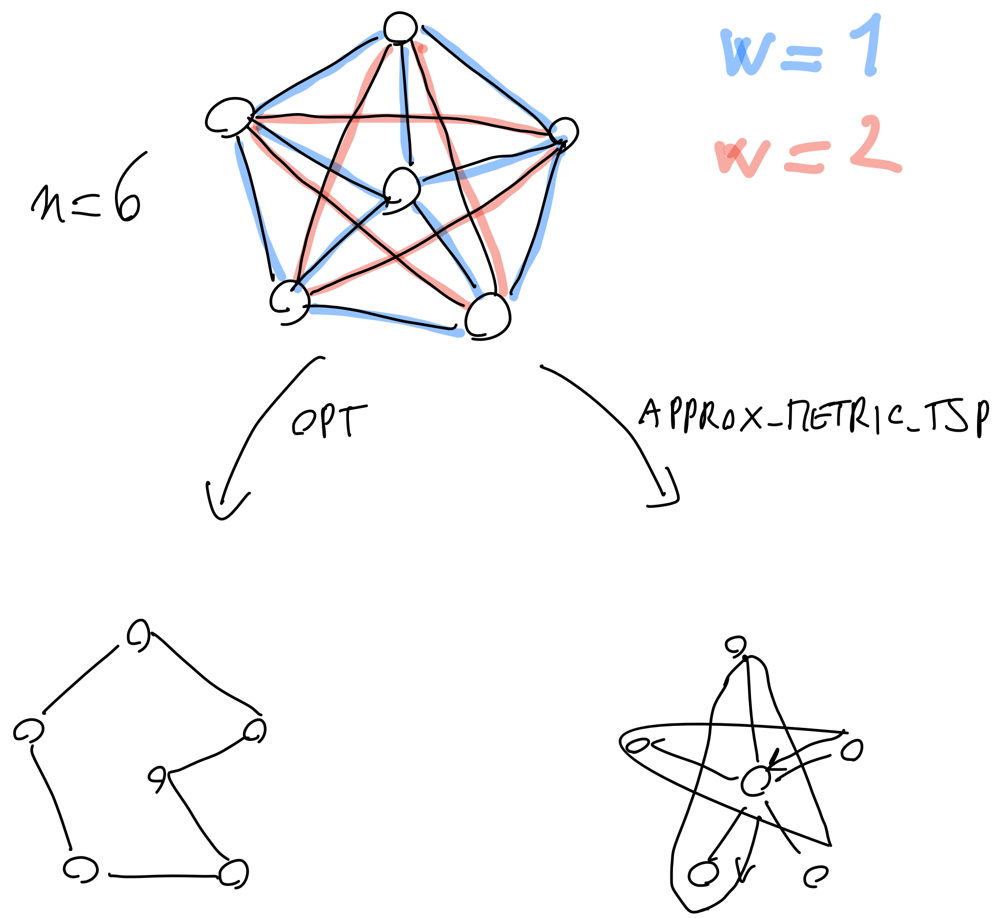
Exercise: show that the above analysis is tight by giving an example of a graph where APPROX_METRIC_TSP returns a solution of cost $2 \cdot H^*$



OPT $\rightarrow$ cost: n

$H \rightarrow$ cost: $2 \cdot n - 2$

as $n \rightarrow +\infty$ $\quad \dfrac{w(H)}{w(OPT)} \longrightarrow 2$

A 3/2-approx for metric TSP

Christofides' algorithm 1976

Reason for 2-approx factor was the fact that the preorder traversal of T^* used every edge of T^* exactly twice. We'll try to improve on this by constructing a tour that traverses MST edges only once.

$\longrightarrow$ Eulerian cycles

Def.: a path (or cycle) is Eulerian if it visits every edge of the graph exactly once.

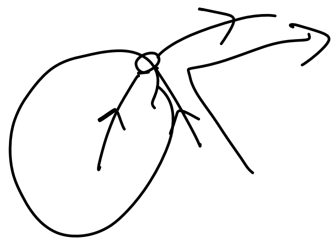
Def.: a connected graph is Eulerian if $\exists$ Eulerian cycle

If the MST was Eulerian (cannot be) then we would have a 1-approx. APPROX_METRIC_TSP is finding a "cheap" Eulerian cycle in the MST, but effectively needs to double its edges.

Question: is there a cheaper Eulerian cycle?

a famous theorem by Euler:

Theorem: a connected graph is Eulerian $\Longleftrightarrow$ every vertex has even degree.



So, let's handle the odd-degree vertices of the MST explicitly.

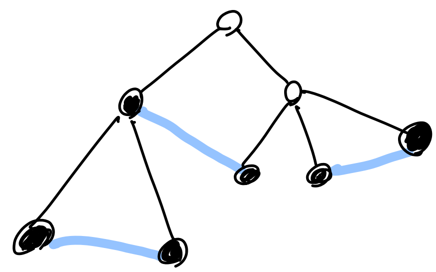
Property: in any (finite) graph the number of vertices of odd degree is even.

proof: $\displaystyle\sum_{v \in V} \deg(v) = 2m$

$$\underbrace{\sum_{\text{even}} \deg(v)}_{\text{even}} + \underbrace{\sum_{\text{odd}} \deg(v)}_{\uparrow} = \underbrace{2m}_{\text{even}}$$

$\Longrightarrow$ must be even

Idea: augment the initial MST T^* with a minimum-weight perfect matching (perfect means that it includes all the vertices) between the vertices that have odd degree in the MST.



$\Longrightarrow$ the resulting graph has only even-degree vertices, i.e. is Eulerian

Christofides(G)

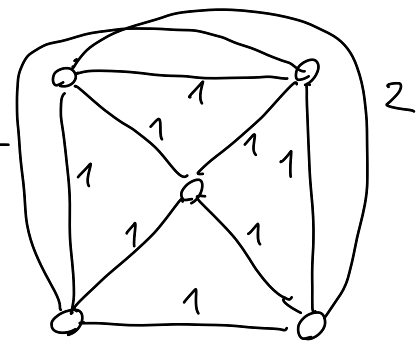
1) $T^* \leftarrow \text{Prim}(G, r)$ // $T^* = (V, E^*)$
2) Let D be the set of vertices of T^* with odd degree. Compute a min.-weight perfect matching M^* on the graph induced by D

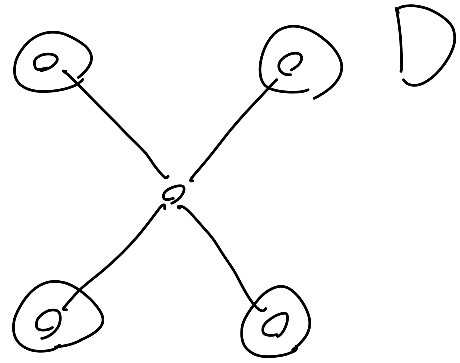
can be done in polynomial time (Edmonds '65)

3) The graph $(V, E^* \cup M^*)$ is Eulerian; compute an Eulerian cycle on this graph

4) Return the cycle that visits all the vertices of G in the order of their first appearence in the Eulerian cycle

Example:

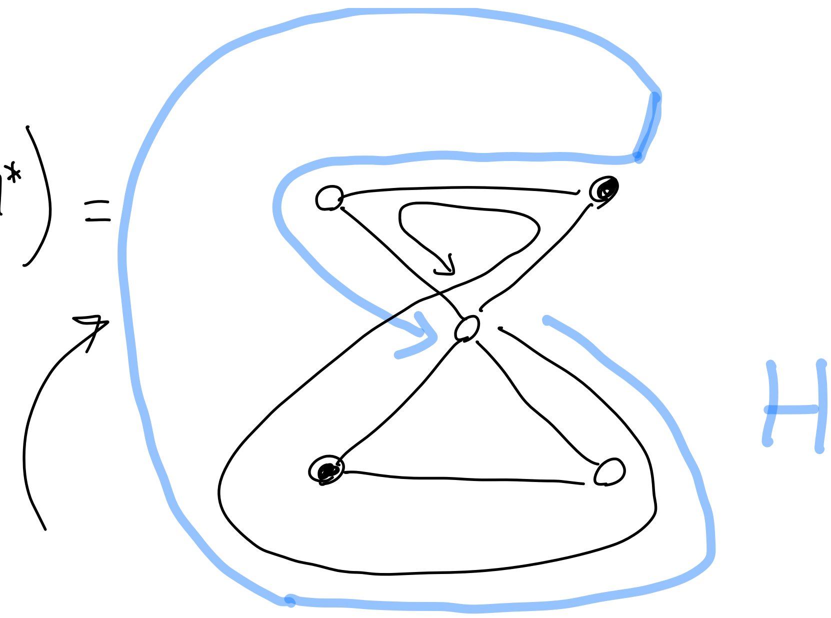


$T^* =$ 

$\cup$

$\Pi^* =$

$(V, T^* \cup M^*) =$



Analysis:

- $w(H) \leq w(T^*) + w(M^*)$
- $w(T^*) \leq w(H^*)$ (last class)

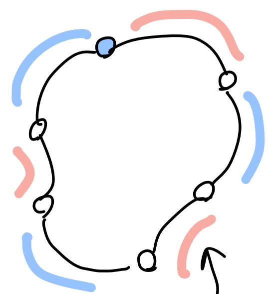
goal: $w(H) \leq \frac{3}{2} w(H^*)$

- $w(M^*) \overset{?}{\leq} \frac{1}{2} w(H^*)$

w(optimal tour of the odd-degree vertices of T^*)

$\leq w(H^*)$

partition this in 2 perfect matchings:



one of these 2 has cost $\leq \frac{w(H^*)}{2}$

Put pieces together:

$$w(H) \leq w(H^*) + \frac{w(H^*)}{2} = \frac{3}{2} w(H^*)$$

- recent advance: $\left(\frac{3}{2} - \varepsilon\right)$ - approx $\quad \varepsilon \sim 10^{-36}$
- approx. ratio $\geq \frac{123}{122}$
- Conjecture: $4/3$