

1. There are  $m = 6n \ln(n)$  jobs assigned at random to  $n$  processors (note: remember  $\ln(x) = \log_e(x)$ ). Consider a processor  $p$  and show, with high probability in parameter  $n$ , processor  $p$  does not receive more than  $12 \ln(n)$  jobs. (Hint: define an appropriate indicator variable for each job and apply the following Chernoff bound).

**Theorem 1.** Siano  $X_1, X_2, \dots, X_n$  variabili indicatore indipendenti con  $E[X_i] = p_i, 0 < p_i < 1$ . Sia  $X = \sum_{i=1}^n X_i$  e  $\mu = E[X]$ . Allora, per ogni  $0 < \delta \leq 1$ ,

$$\Pr(X > (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}.$$

Here, we have  $X_i$  with  $i = 1, 2 \dots 6n \ln(n)$ , an indicator variable with value 1 if the  $i^{th}$  job gets assigned to processor  $p$ . Considering all events  $X_i$  are independent, so we get  $\Pr(X_i = 1) = \frac{1}{n}$ .

Specifically,  $X = \sum_{i=1}^{6n \ln(n)} X_i$ . Given the processor has to receive no more than  $12 \ln(n)$  jobs, this is our bound and in specific we have  $12 \ln(n) = (1 + \delta)\mu$ :

$$\Pr(X > 12 \ln(n))$$

Now, we want to solve for  $\delta$ :

$$12 \ln n = 6 \ln n (1 + \delta)$$

$$\frac{12 \ln n}{6 \ln n} = 1 + \delta$$

$$2 = 1 + \delta$$

$$\delta = 1$$

Then, we substitute everything:

$$\Pr(X > (1 + 1)\mu) = \Pr(X > 12 \ln n) \leq e^{-\frac{1^2 \mu}{3}}$$

$$\Pr(X > 12 \ln n) \leq e^{-\frac{6 \ln n}{3}} = e^{-2 \ln n} = (e^{\ln n})^{-2} = (n)^{-2} = \frac{1}{n^2}$$

Thus we have:

$$\Pr(X > 12 \ln n) \leq \frac{1}{n^2}$$

which means:

$$\Pr(X \leq 12 \ln n) \geq 1 - \frac{1}{n^2}$$

This shows that with high probability, the processor  $p$  does not receive more than  $12 \ln(n)$  jobs

2.

**Exercise 2 (9 points)** For  $n \gg 1$ , let  $X_1, X_2, \dots, X_n$  be independent indicator random variables such that  $\Pr(X_i = 1) = (6 \ln n)/n$  (recall that  $\ln n = \log_e n$ ). Let  $X = \sum_{i=1}^n X_i$  and  $\mu = E[X]$ . By applying the following Chernoff bound

$$\Pr(X > (1 + \delta)\mu) < e^{-\mu \delta^2 / 2} \quad \text{for } 0 < \delta \leq 2e - 1$$

prove that

$$\Pr(X > 10 \ln n) < \frac{1}{n^c}$$

for some positive constant  $c$  to be determined.

Here, given  $\Pr(X_i = 1) = \frac{6 \ln(n)}{n}$ , we have:

$$\mu = E[X] = \sum_{i=1}^n E[X_i] = n \cdot \frac{6 \ln n}{n} = 6 \ln n$$

We then set the Chernoff bounds to be:

$$10 \ln(n) = (1 + \delta)6 \ln(n)$$

Solve for  $\delta$ :

$$\frac{10 \ln n}{6 \ln n} = 1 + \delta$$

$$\frac{10}{6} = 1 + \delta$$

$$\frac{5}{3} = 1 + \delta$$

$$\delta = \frac{5}{3} - 1 = \frac{2}{3}$$

Apply the Chernoff bound:

$$\Pr(X > 10 \ln n) = \Pr(X > (1 + \frac{2}{3})\mu)$$

Using the Chernoff bound formula:

$$\Pr(X > (1 + \delta)\mu) < e^{-\frac{\mu\delta^2}{2}}$$

Substituting  $\delta = \frac{2}{3}$  and  $\mu = 6 \ln n$ :

$$\Pr(X > (1 + \frac{2}{3})6 \ln n) < e^{-\frac{6 \ln n \cdot (\frac{2}{3})^2}{2}}$$

$$= e^{-\frac{6 \ln n \cdot \frac{4}{9}}{2}}$$

$$= e^{-\frac{6 \ln n \cdot \frac{4}{9}}{2}}$$

$$= e^{-\frac{24 \ln n}{18}}$$

$$= e^{-\frac{4 \ln n}{3}}$$

$$e^{-\frac{4 \ln n}{3}} = (e^{\ln n})^{-\frac{4}{3}} = n^{-\frac{4}{3}}$$

Thus, we have:

$$\Pr(X > 10 \ln n) < \frac{1}{n^{4/3}}$$

Therefore, we have correctly proved that:

$$\Pr(X > 10 \ln(n)) < \frac{1}{n^{4/3}}$$

for  $c = \frac{4}{3}$ , bounding correctly for a positive constant  $c$ .

3.

**Exercise 2 (11 points)** Let  $S$  be a set of  $n$  distinct positive integers, and let  $\text{WORK}(S)$  be a procedure which, given input  $S$ , returns an integer by performing  $n^2$  operations. Now consider the following randomized algorithm:

```
RAND_REC(S)
  if |S| <= 1 then return 1
  x = WORK(S)
  p = RANDOM(S)
  S1 = {s in S such that s < p}
  S2 = {s in S such that s > p}
  if (|S1| >= |S2|) then
    y = RAND_REC(S1)
  else
    y = RAND_REC(S2)
  return x + y
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Applying the following Chernoff bound show that the complexity of  $\text{RAND\_REC}(S)$  is  $O(n^2 \log n)$  with high probability. (Hint: recall the analysis of randomized QuickSort.)

**Theorem 1.** Let  $X_1, X_2, \dots, X_n$  be independent indicator random variables such that  $E[X_i] = p_i, 0 < p_i < 1$ . Let  $X = \sum_{i=1}^n X_i$  and  $\mu = E[X]$ . Then, for  $0 < \delta \leq 1$ ,

$$\Pr(X < (1 - \delta)\mu) < e^{-\mu\delta^2/2}.$$

The algorithm basically partitions the original sets into two subsets  $S_1$  and  $S_2$  containing elements from both sets, then calling the procedure on the larger set and returning the sum. We observe the following:

- if the sets are balanced  $S_1$  and  $S_2$ , then  $O(\log(n))$
- we want to ensure that each recursive call reduces the size of the set by a significant fraction
  - o then consider a good pivot ensuring a probability of  $\max(|S_1|, |S_2|) \leq \frac{3n}{4}$

Using Chernoff bounds, we can show that the probability of having a "bad" pivot that doesn't split the set significantly decreases exponentially. For a random pivot  $p$ , the probability that the size of the larger partition the size of the larger partition is at most  $\frac{3n}{4}$  can be derived as follows:

- Let  $X$  be the size of the larger partition.
- For  $0 < \delta \leq 1$ , we have:

$$\Pr\left(X > (1 + \delta)\frac{n}{2}\right) \leq e^{-\frac{\frac{n}{2}\delta^2}{3}}$$

We want to ensure that the size of the larger partition is less than  $\frac{3n}{4}$ :

- Set  $(1 + \delta) \frac{n}{2} = \frac{3n}{4}$

$$1 + \delta = \frac{3}{2} \implies \delta = \frac{1}{2}$$

Using this value in the Chernoff bound:

- $\mu = \frac{n}{2}$

$$\Pr\left(X > \frac{3n}{4}\right) \leq e^{-\frac{\frac{n}{2} \left(\frac{1}{2}\right)^2}{3}} = e^{-\frac{n}{24}}$$

This shows that the probability of a "bad" split is exponentially small.

The total work done at each level of recursion,  $WORK(S)$  performs  $n^2$  operations.

Total work done is:

$$\sum_{i=0}^{O(\log(n))} n^2 = O(n^2 \log(n))$$

4.

**Exercise 2 (9 points)** Suppose you toss  $n \gg 1$  times a coin: applying the following Chernoff bound show that the probability that you obtain more than  $n/2 + \sqrt{6n \ln n}/2$  heads is at most  $1/n$ .<sup>1</sup>

**Theorem 1.** Let  $X_1, X_2, \dots, X_n$  be independent indicator random variables such that  $E[X_i] = p_i, 0 < p_i < 1$ . Let  $X = \sum_{i=1}^n X_i$  and  $\mu = E[X]$ . Then, for  $0 < \delta \leq 1$ ,

$$\Pr(X > (1 + \delta)\mu) \leq e^{-\mu\delta^2/3}.$$

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<sup>1</sup>Recall that  $\ln n = \log_e n$ .

Let  $X_i$  be an indicator random variable for the  $i$ -th coin toss, where  $X_i = 1$  if the outcome is heads and  $X_i = 0$  if the outcome is tails.

Since the coin is fair,  $\Pr(X_i = 1) = \frac{1}{2}$ . Therefore,

$$\mu = \mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X_i] = n \cdot \frac{1}{2} = \frac{n}{2}$$

We need to find  $\delta$  such that  $\frac{n}{2} + \sqrt{6n \ln \left(\frac{n}{2}\right)} = (1 + \delta)\frac{n}{2}$ .

1. Solving for  $\delta$ :

$$\frac{n}{2} + \sqrt{6n \ln \left(\frac{n}{2}\right)} = (1 + \delta)\frac{n}{2}$$

$$1 + \frac{2\sqrt{6n \ln \left(\frac{n}{2}\right)}}{n} = 1 + \delta$$

$$\delta = \frac{2\sqrt{6n \ln \left(\frac{n}{2}\right)}}{n}$$

Simplifying  $\delta$ :

$$\delta = 2\sqrt{\frac{6 \ln \left(\frac{n}{2}\right)}{n}}$$

Since  $\ln \left(\frac{n}{2}\right) = \ln n - \ln 2$ :

$$e^{-4(\ln n - \ln 2)} = e^{-4 \ln n + 4 \ln 2}$$

Simplifying further:

$$= e^{-4 \ln n} \cdot e^{4 \ln 2}$$

$$= (e^{\ln n})^{-4} \cdot 2^4$$

$$= n^{-4} \cdot 16$$

Finally, we get:

$$\Pr \left( X > \frac{n}{2} + \sqrt{6n \ln \left(\frac{n}{2}\right)} \right) \leq \frac{16}{n^4}$$

For  $n \gg 1$ ,  $\frac{16}{n^4} < \frac{1}{n}$ , so we have shown that:

$$\Pr \left( X > \frac{n}{2} + \sqrt{6n \ln \left(\frac{n}{2}\right)} \right) \leq \frac{1}{n}$$

We substitute  $\delta = 2\sqrt{\frac{6 \ln \left(\frac{n}{2}\right)}{n}}$  and  $\mu = \frac{n}{2}$  into the Chernoff bound:

$$\Pr \left( X > \frac{n}{2} + \sqrt{6n \ln \left(\frac{n}{2}\right)} \right) = \Pr (X > (1 + \delta)\mu) \leq e^{-\frac{\mu\delta^2}{3}}$$

Substituting  $\mu$  and  $\delta$ :

$$e^{-\frac{\frac{n}{2} \left( 2\sqrt{\frac{6 \ln \left(\frac{n}{2}\right)}{n}} \right)^2}{3}}$$

$$= e^{-\frac{\frac{n}{2} \cdot \frac{24 \ln \left(\frac{n}{2}\right)}{n}}{3}}$$

$$= e^{-\frac{12 \ln \left(\frac{n}{2}\right)}{3}}$$

$$= e^{-4 \ln \left(\frac{n}{2}\right)}$$

5.

**Exercise 2 (10 points)** Suppose you have a randomized algorithm for a minimization problem  $A$  that returns the correct output with probability at least  $1/n$ , where  $n$  is the input size. Show how to obtain an algorithm for  $A$  that returns the correct output with high probability. (Hint: for the analysis use this inequality:  $(1 + x/y)^y \leq e^x$  for  $y \geq 1, y \geq x$ .)

Characterize the event:

$$\Pr\left(X > \frac{1}{n}\right) > \frac{1}{n^d}$$

We want to find a value for  $k$  such that  $\Pr\left(1 - \frac{1}{n}\right)^k \leq \frac{1}{n^d}$ . In this case, it's standard the use of this inequality:

$$\left(1 + \frac{x}{y}\right)^y \leq e^x, y \geq 1, y \geq x$$

Recall the following:

$$e^{-\ln(n^d)} = \frac{1}{n^d}$$

This inequality is derived from the exponential function and the binomial expansion. It represents an upper bound on the expression  $\left(1 + \frac{x}{y}\right)^y$ , showing that it grows slower than  $e^x$ . The probability of not contracting the minimum cut in each iteration needs to be bounded and manipulated to ensure the overall algorithm's success probability is high.

Recall the following:

$$e^{-\ln(n^d)} = \frac{1}{n^d}$$

By choosing  $k = \frac{dn}{2} \ln(n)$  it follows that:

$$\left(1 - \frac{1}{n}\right)^{k=n} \leq e^{-1} = \frac{1}{e} \rightarrow \text{is not in the form } \frac{1}{n^d}$$

$$\left(\left(1 - \frac{1}{n}\right)^n\right)^{\ln(n^d)} = \left(1 - \frac{1}{n}\right)^{n \ln(n^d)}$$

Let's wrap up:

$$\begin{aligned} \left(1 - \frac{1}{n}\right)^{k=dn \ln(n^d)} &= \left(\left(1 - \frac{1}{n}\right)^n\right)^{\ln(n^d)} \\ &\leq (e^{-1})^{\ln(n^d)} = e^{-\ln(n^d)} = \frac{1}{n^d} \end{aligned}$$

5.

**Problem 2 (10 points)** Suppose you throw  $n$  balls into  $\frac{n}{6 \ln n}$  bins<sup>1</sup> independently and uniformly at random. Applying the following Chernoff bound show that, with high probability, the bin with maximum load (load = number of balls in the bin) contains at most  $12 \ln n$  balls. (Hint: focus first on one arbitrary bin and bound the probability of that bin's load exceeding  $12 \ln n$  ...)

**Theorem 1.** Let  $X_1, X_2, \dots, X_n$  be independent indicator random variables such that  $E[X_i] = p_i, 0 < p_i < 1$ . Let  $X = \sum_{i=1}^n X_i$  and  $\mu = E[X]$ . Then, for  $0 < \delta \leq 1$ ,

$$\Pr(X > (1 + \delta)\mu) \leq e^{-\mu\delta^2/3}.$$

1. Define the Problem Setup:

- Number of balls,  $n$ .
- Number of bins,  $m = \frac{n}{6 \ln n}$ .
- Each ball is independently and uniformly placed in one of the bins.

2. Random Variable Definition:

- Let  $X_i$  be the load (number of balls) in the  $i$ -th bin.
- $X_i$  is a sum of independent indicator random variables, where each indicator variable denotes whether a particular ball lands in the  $i$ -th bin.

3. Expected Load Calculation:

- The probability that a particular ball lands in any specific bin is  $\frac{1}{m}$ .
- Thus, the expected number of balls in any bin (the expected load),  $\mu$ , is:

$$\mu = \frac{n}{m} = \frac{n}{\frac{n}{6 \ln n}} = 6 \ln n$$

4. Applying the Chernoff Bound:

- We need to bound the probability that  $X_i$  exceeds  $12 \ln n$ .
- Let  $X = X_i$ , the load of a particular bin.
- Using the Chernoff bound theorem, set  $\delta$  such that:

$$(1 + \delta)\mu = 12 \ln n$$

- Since  $\mu = 6 \ln n$ :

$$(1 + \delta)6 \ln n = 12 \ln n \implies 1 + \delta = 2 \implies \delta = 1$$

$$\begin{aligned} \Pr(X > 12 \ln n) &= \Pr(X > (1 + 1)6 \ln n) \\ &\leq e^{-\frac{6 \ln n}{3}} \\ &= e^{-2 \ln n} \\ &= e^{\ln n^{-2}} \\ &= 1/n^2. \end{aligned}$$

We now show that, with high probability, the bin with maximum load contains at most  $12 \ln n$  balls. Applying the union bound over all the  $\frac{n}{6 \ln n}$  bins, we have that the probability that at least one bin gets more than  $12 \ln n$  balls is at most

$$\frac{n}{6 \ln n} \cdot \frac{1}{n^2} = \frac{1}{6n \ln n}.$$

In other words, the load of no bin exceeds  $12 \ln n$  with probability at least  $1 - 1/6n \ln n = 1 - o(1/n)$ .

<sup>1</sup>Recall that  $\ln n = \log_e n$ .