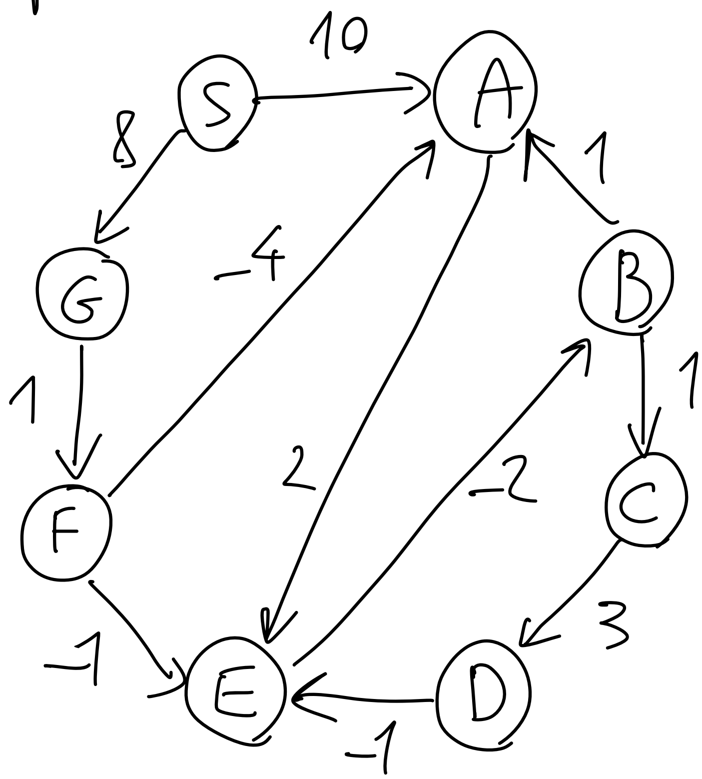
Example :



iterations

vertices	0	1	2	3	4	5	6	7	last
S	0	0	0	0	0	0	0	0	0
A	∞	10	10	5	5	5	5	5	5
B	∞	∞	∞	10	6	5	5	5	5
C	∞	∞	∞	∞	11	7	6	6	6
D	∞	∞	∞	∞	∞	14	10	9	9
E	∞	∞	12	8	7	7	7	7	7
F	∞	∞	9	9	9	9	9	9	9
G	∞	8	8	8	8	8	8	8	8

Correctness of Bellman-Ford:

Let $len(i, v)$ denote the length of a shortest path from s to v that contains $\leq i$ edges.
Since the shortest path from s to v contains $\leq n-1$ edges, it's sufficient to prove that after i iterations $len(v) \leq len(i,v)$

By induction on i

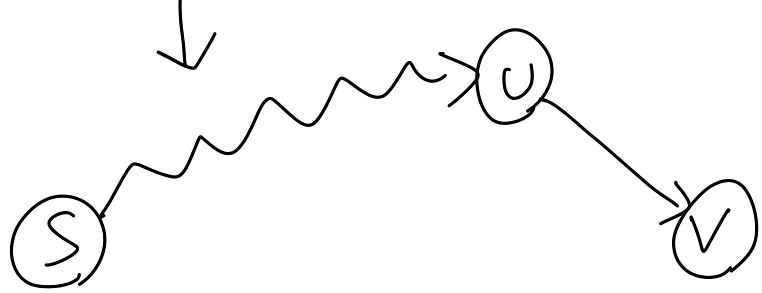
Base case: $i = 0$ $\quad len(s) = 0 \leq len(0, s) = 0$

$$len(v \neq s) = +\infty = len(0, v \neq s)$$

Inductive hypothesis: $len(v) \leq len(k, v) \ \forall 1 \leq k < i$

Take $i \geq 1$ and a shortest path from s to v with $\leq i$ edges. Let (u, v) be the last edge of this path. Then

$$len(i, v) = w(u, v) + \underbrace{len(i-1, u)}$$



why?
by contradiction

By the ind. hyp. $len(u) \leq len(i-1, u)$

In the i-th iteration we update

$$len(v) = \min\{len(v), \underbrace{len(u) + w(u,v)}_{\substack{\leq len(i-1,u) + w(u,v) \\ = len(i,v)}}\}$$

$$\leq len(i,v) \quad \text{as desired}$$

All-Pairs Shortest Paths (APSP)

input : a directed, weighted graph $G=(V,E)$
output : one of the following
a) $dist(u,v)$ $\forall$ ordered vertex pair
b) a declaration that G contains a negative cycle

Obvious solution : invoke B-F once for every vertex

$\longrightarrow O(m \cdot n^2)$ very high

Can we do better?

Yes, using dynamic programming

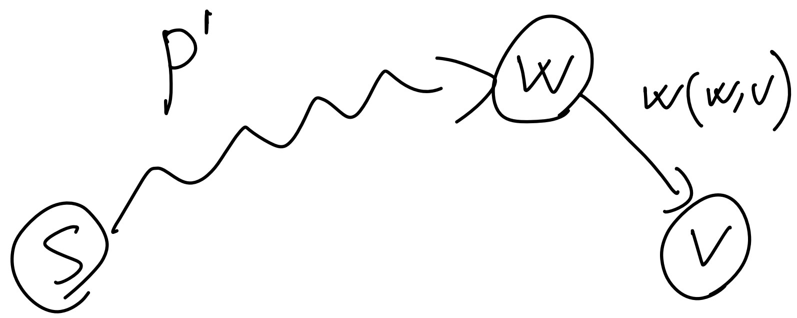
Outline :
1) B-F has a dynamic programming formulation
2) (we won't see) one can adapt that formulation to APSP, obtaining an $O(m \cdot n^2)$ algorithm; an improved formulation can be made to

run in $O(n^3 \log n)$

3) a different dyn. programming strategy gives an $O(n^3)$ algorithm (without full proofs)

Bellman-Ford via dynamic programming

what are the subproblems here?



P = shortest s-v path

observation: P' is a shortest path (to a ≠ destination) with fewer edges than P

why? by contradiction, as before

Then P' can be interpreted as a solution to a smaller subproblem

$\Rightarrow$ idea: introduce a parameter i that restricts the maximum number of edges allowed in a path, with smaller subproblems having smaller edge budgets

serves as a measure of subproblem size

<u>Subproblems</u>: compute len(i, v), the length of a shortest path from s to v that contains at most i edges. (If no such path exists, define len(i,v) as $+\infty$)

↳ $O(n^2)$ subproblems

Obs.: every subproblem works with the full input; the <u>idea</u> is to control the allowable size of the output.

↳ solution to a subproblem

<u>Bellman-Ford recurrence</u>:

$$
len(i,v) = \begin{cases} 0 & i=0 \text{ and } v=s \\ +\infty & i=0 \text{ and } v\neq s \\ \min \begin{cases} len(i-1, v) \\ \min\limits_{(u,v)\in E} \{ len(i-1,u) + w(u,v) \} \end{cases} & \text{otherwise} \end{cases}
$$

(It's easy to transform a dyn. progr. evaluation of this recurrence into our original formulation of B-F.)

This formulation can be adapted to APSP
$\rightarrow \Theta(n^3 \log n)$

The Floyd–Warshall algorithm

Idea: go one step further: instead of restricting the number of edges allowed in a solution, restrict the identities of the vertices that are allowed in a solution. (In other words, now paths can pass through only certain vertices.)

Let's define the subproblems:

Call the vertices $1, 2, \dots, n$

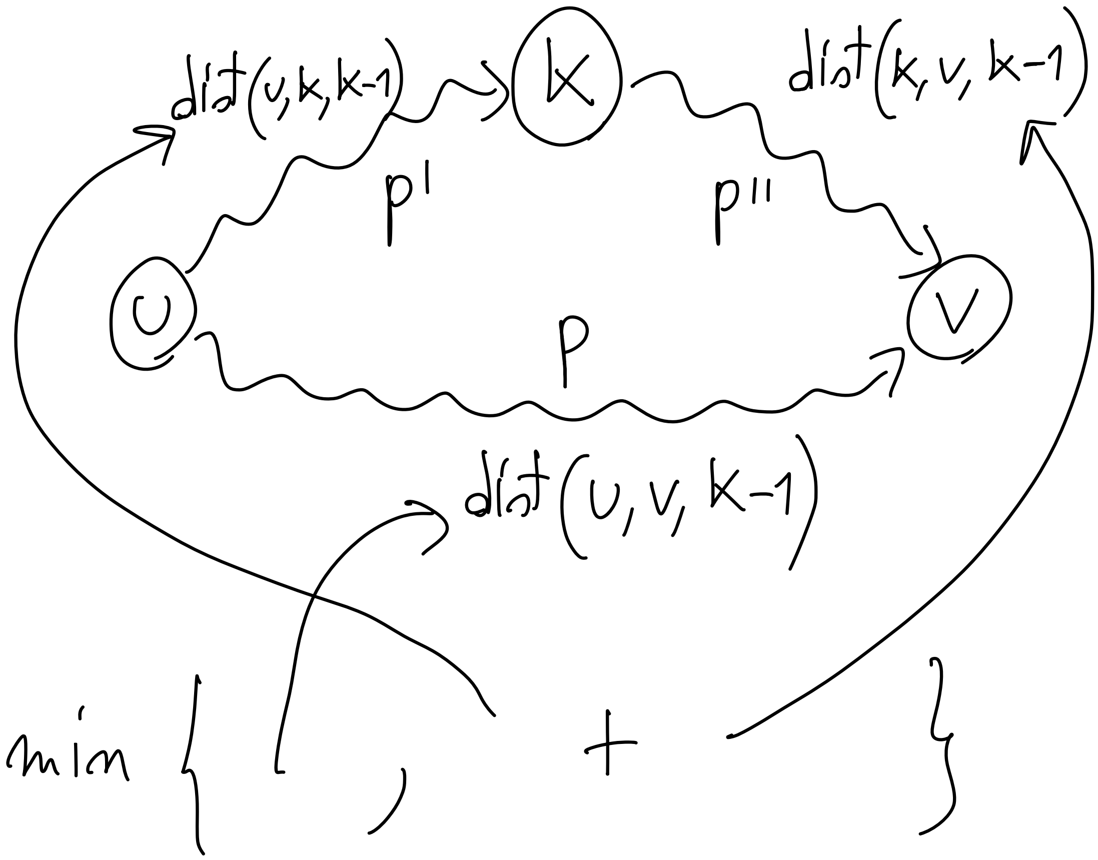
Compute $dist(u, v, k)$ = length of a shortest path from u to v that uses only vertices from $\{1, 2, \dots, k\}$ as internal vertices, and that does not contain a directed cycle.
(If no such path exists, define $dist(u, v, k)$ as $+\infty$)

$k \to$ measures the subproblem size

$\to O(n^3)$ subproblems

Algorithm: expand the set of allowed internal vertices one vertex at a time, until this set is V.

Payoff of defining subproblems in this way: only 2 candidates for the optimal solution to a subproblem, depending on whether it uses vertex k or not :



$\Rightarrow O(1)$ work per subproblem

$\Rightarrow$ Complexity: $O(n^3)$

Floyd-Warshall (G)

label the vertices $V = \{1, 2, \ldots, n\}$ arbitrarily

\\ subproblems (k indexed from 0)

$A = n \times n \times (n+1)$ array

\\ base cases ($k=0$)

for $u = 1$ to n do
    for $v = 1$ to n do
        if $u = v$ then $A[u, v, 0] = 0$
        else if $(u, v) \in E$ then $A[u, v, 0] = w(u, v)$
        else $A[u, v, 0] = +\infty$

\\ solve all subproblems

for $k = 1$ to n do
    for $u = 1$ to n do
        for $v = 1$ to n do

$$A[u, v, k] = \min\{A[u, v, k-1],\ A[u, k, k-1] + A[k, v, k-1]\}$$

```
\\ check for a negative cycle
for v = 1 to n do
    if A[v, v, n] < 0 then
        return "G contains a negative cycle"
```

Is there a truly-subcubic alg. for APSP?

$\underbrace{\text{truly-subcubic}} \downarrow$ $O(n^{3-\varepsilon})$ for some constant $\varepsilon > 0$

Open problem!