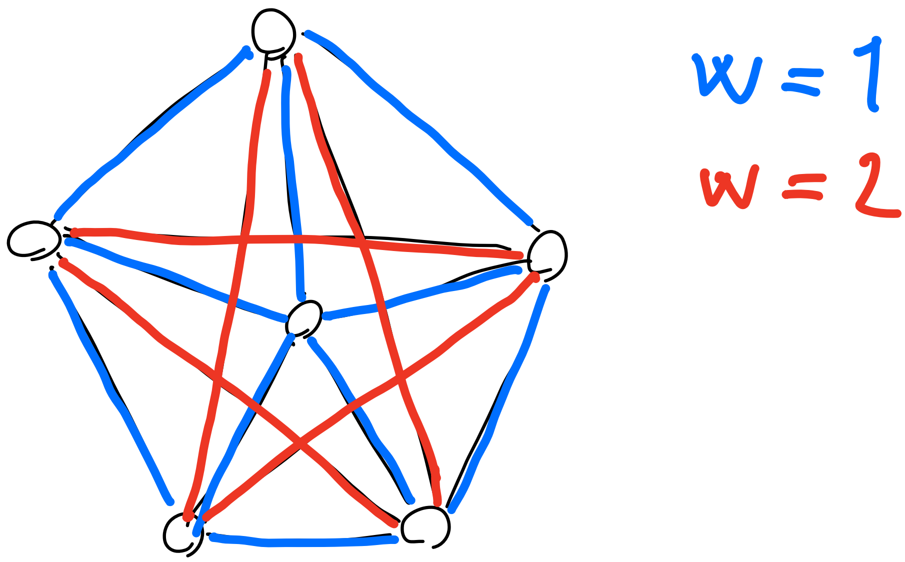
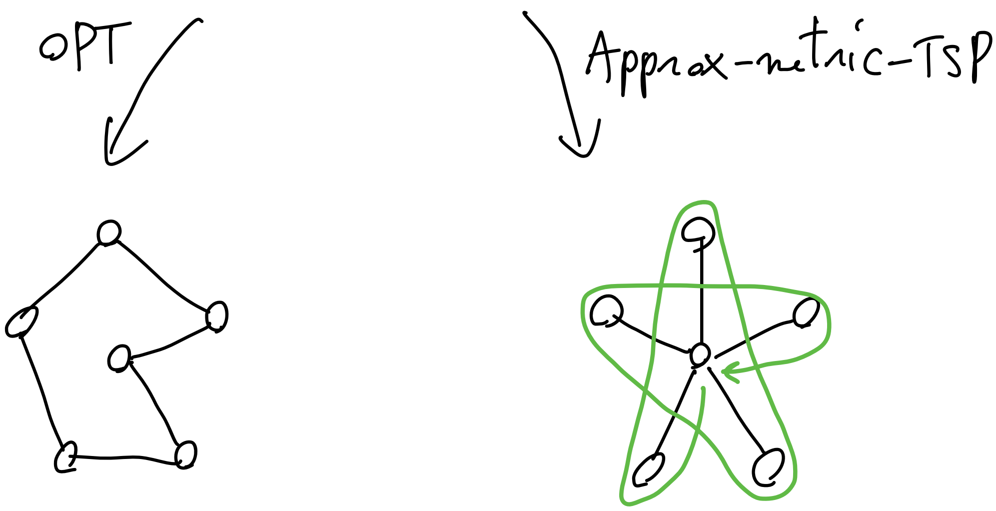
Exercise: show that the above analysis is tight by giving an example of a graph where Approx-metric-TSP returns a solution of cost $2 \cdot w(H^*)$





$w(OPT) = n \qquad w(H) = 2n - 2$

$$\frac{w(H)}{w(OPT)} \xrightarrow[n \to +\infty]{} 2$$

Programming exercise: implement Approx-metric-TSP and run it on TSPLIB → Google it

A 3/2-approx algorithm for metric TSP

Christofides' algorithm 1976 (No CLRS)

Reason for 2-approx factor was that the preorder traversal of T^* crossed every edge of T^* exactly twice. We'll try to improve on this by constructing a tour that traverses MST edges only once.

⟶ Eulerian cycles

Def.: a path (or cycle) is Eulerian if it crosses every edge exactly once.

Def.: a connected graph is Eulerian if ∃ Eulerian cycle

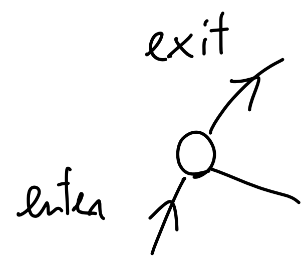
If the MST was Eulerian (cannot be) then we would have a 1-approx. Approx-metric-TSP is finding a "cheap" Eulerian cycle in the MST, but effectively needs to double its edges.

Question: is there a cheaper Eulerian cycle?

A famous theorem by Euler:

Theorem: a connected graph is Eulerian $\iff$ every vertex has even degree.

intuition:



So, let's handle the <u>odd-degree</u> vertices of the MST explicitly.

Property: in any (finite) graph the number of vertices of odd degree is even.

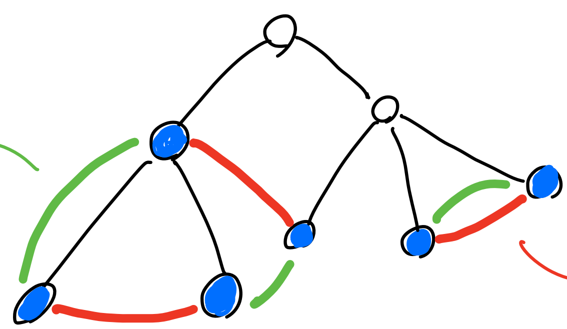
Proof: $\sum_{v \in V} \deg(v) = 2m$

$$\underbrace{\sum_{\text{even}} \deg(v)}_{\text{even}} + \sum_{\text{odd}} \deg(v) = \underbrace{2m}_{\text{even}}$$

$\Longrightarrow$ must be even

Idea: augment the initial MST T^* with a minimum-weight <u>perfect matching</u> (perfect means that it includes <u>all</u> the vertices) between the vertices that have odd degree in the MST:

another perfect matching (using an edge in T^*)



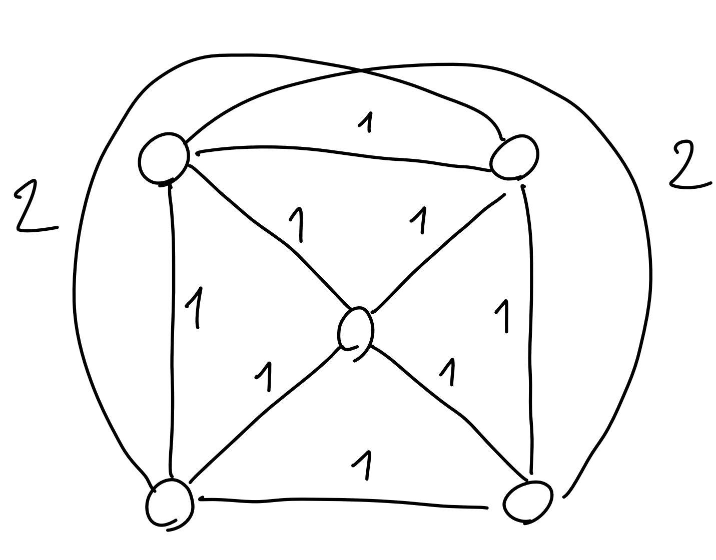
perfect matching

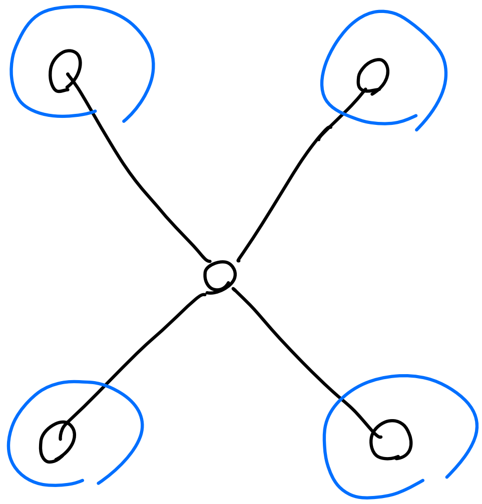
$\Longrightarrow$ the resulting graph has only even-degree vertices i.e., is Eulerian

Christofides (G)

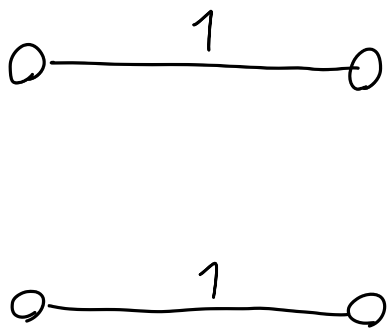
1) $T^* \leftarrow \text{Prim}(G, r)$ // $T^* = (V, E^*)$
2) Let D be the set of vertices of T^* with odd degree. Compute a min-weight perfect matching M^* on the graph induced by D
   // can be done in polynomial time (Edmonds '65)
3) The graph $(V, E^* \cup M^*)$ is Eulerian
   // any edge in both E^* and M^* appears twice in this (multi)graph
   Compute an Eulerian cycle on this graph
4) Return the cycle that visits all the vertices of G in the order of their first appearance in the Eulerian cycle

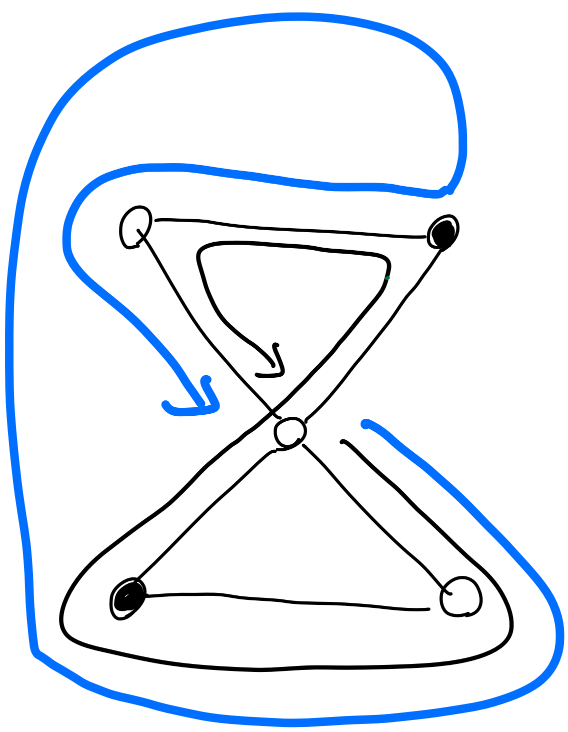
Example:



$T^* =$  D

$\cup$

$\Pi^* =$ 

$(V, E^* \cup \Pi^*) =$  H

Analysis:

1) $w(H) \leq w(T^*) + w(M^*)$ (by triangle ineq.)

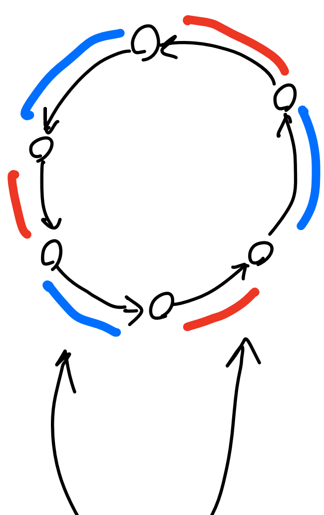
2) $w(T^*) \leq w(H^*)$ (last class)

goal: $w(H) \leq \frac{3}{2} w(H^*)$

3) $w(M^*) \overset{?}{\leq} \frac{1}{2} w(H^*)$

$w($optimal tour of the odd-degree vertices of $T^*)$
$\leq w(H^*)$ (by triangle ineq.)

partition this in 2 perfect matchings:



even n° of vertices

one of these 2 has weight $\leq \frac{w(H^*)}{2}$

both have weight $\geq w(\Pi^*)$, since Π^* is an optimal perfect matching on odd-degree vertices

$$\Longrightarrow \quad w(\Pi^*) \leq \frac{w(H^*)}{2}$$

Put pieces together:

$$w(H) \leq w(H^*) + \frac{w(H^*)}{2} = \frac{3}{2} w(H^*)$$

- recent algorithm: $\left(\frac{3}{2} - \varepsilon\right)$-approx $\quad \varepsilon \sim 10^{-36}$

  ↳ see further reading

- approx ratio $\geq \frac{123}{122}$

- Conjecture: $4/3$