Exercise: Using the analysis of Karger's alg. show that the nº of distinct min-cuts in a graph is at most $n(n-1)/2$
Also, show that this bound is tight

Solution: Let $C_1, C_2, \dots, C_j$ denote the min-cuts of a graph. $j \leq$?

We have shown that FULL-CONTRACTION returns a particular min-cut C_i with probability $\geq 2/n(n-1)$. So, if we denote with A_i the event that C_i is returned by FULL-CONTR.,

$$\Pr(A_i) \geq \frac{2}{n(n-1)}$$

Observe that events $A_1, A_2, \dots, A_j$ are disjoint. Then

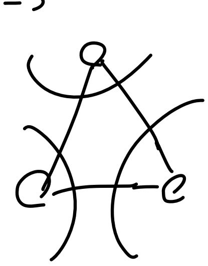
$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_j) = \sum_{i=1}^{j} \Pr(A_i)$$

By definition $\Pr(A_1 \cup A_2 \cup \cdots \cup A_j) \leq 1$, so

$$\sum_{i=1}^{j} \underbrace{\Pr(A_i)}_{\geq \frac{2}{n(n-1)}} \leq 1 \implies j \leq \frac{n(n-1)}{2}$$

This bound is tight: in a cycle on n vertices every pair of edges is a distinct min-cut

$n = 3$



$$\binom{n}{2} = \frac{n(n-1)}{2} \qquad \leq \frac{3 \cdot \cancel{2}}{\cancel{2}} = 3$$

Chernoff Bounds

They're tools from modern probability theory that are frequently used in the analysis of randomized alg.

They're a more powerful variant of Markov's lemma

Phenomenon of "concentration of measure":

Toss a coin

- one time → outcome is unpredictable
- 1000 times → outcome is sharply predictable!

Application: $T(n)$ guaranteed to be concentrated around some value

In many cases the study of $\Pr\left(T(n) > c \cdot f(n)\right)$ can be rephrased as the study of the distribution of a sum of random variables:

Indicator random variable: maps every outcome to either 0 or 1

$$Y = \begin{cases} 1 & \text{if trial is a success} \\ 0 & \text{otherwise} \end{cases}$$

In general

$$X = \sum_{i=1}^{n} X_i \qquad X_i \text{ indicator random var.}$$

we usually have that X_i's are <u>independent</u>

$$Pr(X_i = 1) = p_i$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] =$$

$$= \sum_{i=1}^{n} p_i \overset{\text{notation}}{=} \mu$$

We want to analyze the probability that X deviates from $E[X]$

$$Pr(X > (1+\delta)\mu) \underset{\text{Markov}}{\leq} \frac{E[X]}{(1+\delta)\mu} = \frac{\mu}{(1+\delta)\mu} = \frac{1}{1+\delta}$$

usually <u>not</u> a very good bound

A more powerful probabilistic tool:

Chernoff bound: let $X_1, X_2, \dots, X_n$ be independent indicator random variables where $E[X_i] = p_i$, $0 < p_i < 1$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. Then, $\forall \delta > 0$,

$$\Pr\left(X > (1+\delta)\mu\right) < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

Proof: see book

Example: coin flips

n coin flips $\longrightarrow X_1, X_2, \dots, X_n$

$$\Pr(X_i = \underset{\text{head}}{\underset{\parallel}{1}}) = \frac{1}{2} \quad \forall i$$

$$X = \sum_{i=1}^{n} X_i = \text{n° of heads in } n \text{ coin flips}$$

$$E[X] = \frac{n}{2}$$

Question : what's the probability of getting more than $\frac{3}{4}n$ heads?

Let's apply :

1) Markov

$$\underbrace{\Pr\left(X > \frac{3}{4}n\right) \leq \frac{E[X]}{\frac{3}{4}n} = \frac{\frac{n}{2}}{\frac{3}{4}n} = \frac{2}{3}}_{}$$

<u>constant</u>

2) Chernoff

$$\Pr\left(X > \underbrace{\left(1+\frac{1}{2}\right)\mu}_{=\frac{3}{4}n}\right) < \left(\frac{e^{\frac{1}{2}}}{\left(\frac{3}{2}\right)^{3/2}}\right)^{\frac{n}{2}} < \underbrace{(0.95)^{n}}_{}$$

<u>exponential!</u>

Variants of Chernoff bounds (weaker but easier to state and to use)

1) $\Pr\left(X < (1-\delta)\mu\right) < e^{-\frac{\mu\delta^2}{2}}$ $\quad 0<\delta\leq 1$

2) $\Pr\left(X > (1+\delta)\mu\right) < e^{-\frac{\mu\delta^2}{2}}$ $\quad 0<\delta\leq 2e-1$