Theorem: Maximum Clique is NP-hard

Proof: decision version: input: $\langle G=(V,E), k\rangle$
output: $\exists$ in G a clique of size k?

reduction from max. ind. set

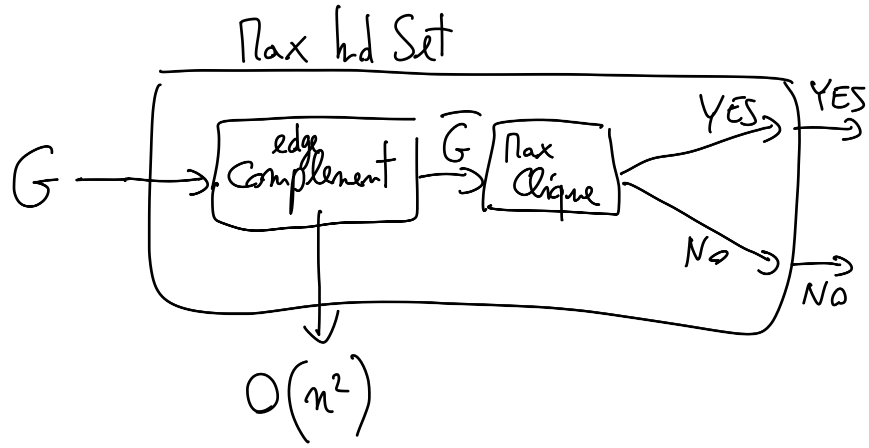
intuition: clique: vertices with all edges between them

max. ind. set: vertices with no edges between them

Def.: given a graph $G=(V,E)$, its edge-complement $\overline{G}=(V,\overline{E})$ has the same vertex set V and an edge set $\overline{E}$ such that $(u,v)\in\overline{E} \iff (u,v)\notin E$

Obs.: a set of vertices S is independent in G $\iff$ S is a clique in G

$\Rightarrow$ the largest ind. set in G has the same size as the largest clique in G



Theorem: Vertex Cover is NP-hard

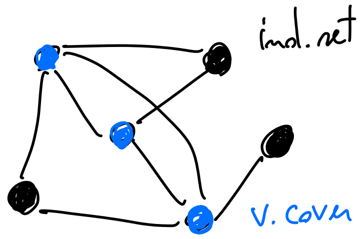
Proof: decision version: input: $\langle G=(V,E), k \rangle$

output: $\exists$ in G a vertex cover of size k?

reduction from max. ind. set

Obs.: a set of vertices S is independent in G

$\iff$ $V \setminus S$ is a vertex cover of G



$\Rightarrow$ the largest ind. set in G has size $n-k$, where k is the size of the smallest vertex cover of G

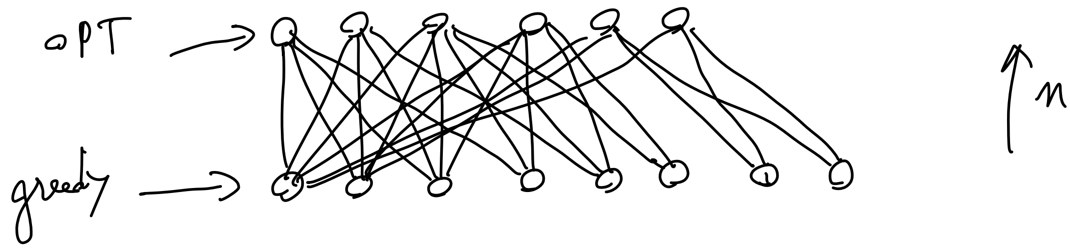
Exercises: show that:

min vertex cover $<_p$ max. ind set

max clique $<_p$ min vertex cover

$\Rightarrow$ these 3 problems are equivalent

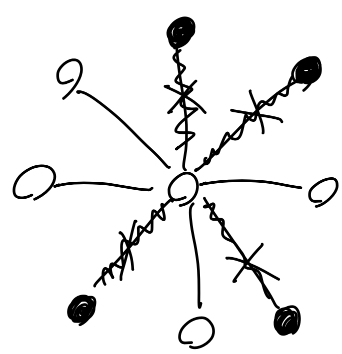
Degree-based greedy approximation for vertex cover



Exercise: show that the approximation factor of Approx_Vertex_Cover is exactly 2:

OPT: just one vertex

Exercise: modify Approx_vertex_cover so as to select only one vertex instead of both of them



"star graph"

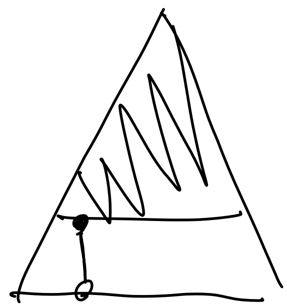
OPT: 1 vertex

our modified alg.: $n-1$ vertices

$\rho \geq n-1$

Exercise: Consider the following approximation algorithm for vertex cover:

1) run DFS from an arbitrary vertex
2) return all the non-leaf vertices of the DFS tree



Show that this is a 2-approximation algorithm for vertex cover.

State of the art:

- $\exists$ $2-\Theta(1/\sqrt{\log n})$ approximation
- vertex cover cannot be approximated better than ~ 1.36
- Conjecture: cannot be approximated better than 2

The Traveling Salesperson Problem (TSP)

Definition: given a complete undirected graph and a function $w: E \to \mathbb{R}^+$, output a tour (i.e. a cycle that visits every vertex exactly once) $T \subseteq E$ minimizing $\sum_{e \in T} w(e)$.

$w: E \to \mathbb{R}^+$ is wlog because every TSP tour has the same number of edges $\Rightarrow$ I can add a large weight to each edge s.t. edges have non-negative weights

Theorem: For any function $\rho(n)$ that can be computed in time polynomial in n, there is no polynomial-time $\rho(n)$-approximation algorithm for TSP, unless $P = NP$.

Proof: reduction from Hamiltonian Circuit

$G \longrightarrow G' = (V, E')$ complete

$$w(e \in E') = \begin{cases} 1 & e \in E \\ \rho \cdot n + 1 & \text{otherwise} \end{cases}$$

idea: weights are far apart

1) G has Hamiltonian circuit $\Rightarrow \exists$ a tour of cost $n \Rightarrow$ TSP algorithm run on G' returns a tour of cost $\boxed{\leq \rho n}$

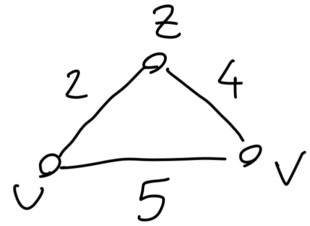
2) G has no Hamiltonian circuit $\Rightarrow$ the TSP algorithm run on G' returns a tour of cost $\geq \rho n + 1$ $\boxed{> \rho n}$

Thus, if we could approximate TSP within a factor of ρ in poly-time, then we would have a poly-time algorithm for Hamiltonian Circuit.

Metric TSP

A special case of TSP where the weight function w satisfies the triangle inequality:

$$\forall\, u, v, z \in V, \text{ it holds that } w(u,v) \leq w(u,z) + w(z,v)$$

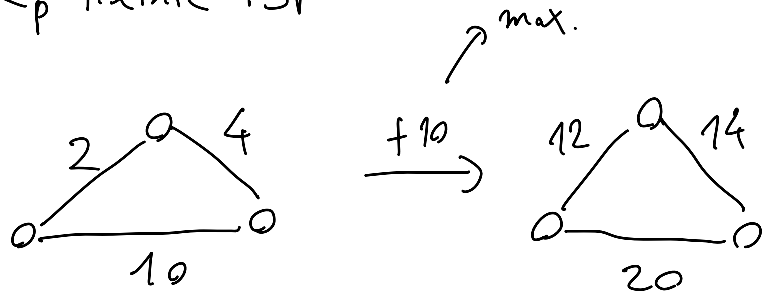


Is metric TSP in P ?

Theorem: Metric TSP is NP-hard

Proof: TSP $<_p$ Metric TSP

idea:



$\langle G = (V, E), w, k \rangle$

$\rightsquigarrow$

$\langle G' = (V, E), w', k' \rangle$

$$w'(u,v) = w(u,v) + W$$

$$W = \max_{u,v \in V} \{ w(u,v) \}$$

$$k' = k + nW$$

to be shown:

1) w' satisfies triangle inequality
2) $\exists$ Ham. circuit of cost k in G $\iff$
   $\exists$ Ham. circuit of cost k' in G'