Def.: given $\Pi \subseteq I \times S$ an algorithm A_Π has complexity $T(n) = O(f(n))$ with high probability (w.h.p.) if $\exists$ constants $c, d > 0$ such that $\forall i \in I$, $|i| = n$,

$$Pr\left(A_\Pi(i) \text{ terminates in } > c \cdot f(n) \text{ steps}\right) \leq \frac{1}{n^d}$$

$$\rightarrow O(f(n)) \text{ w.p. } > 1 - \frac{1}{n^d} \xrightarrow[n \to +\infty]{} 1$$

Def.: given $\Pi \subseteq I \times S$ an algorithm A_Π is correct with high probability (w.h.p.) if $\exists$ constant $d > 0$ such that $\forall i \in I$, $|i| = n$,

$$Pr\left((i, A_\Pi(i)) \notin \Pi\right) \leq \frac{1}{n^d}$$

$$\rightarrow (i, A_\Pi(i)) \in \Pi \text{ w.p. } > 1 - \frac{1}{n^d} \xrightarrow[n \to +\infty]{} 1$$

high probability $\Longrightarrow$ expectation:

Exercise: assume that

1) A_Π LAS VEGAS, with $T_{A_\Pi}(n) = O(f(n))$ w.h.p; in particular
$$Pr\left(T_{A_\Pi}(n) > c \cdot f(n)\right) \leq \frac{1}{n^d}$$

2) A_Π has a worst-case deterministic complexity $O(n^a)$, $a \leq d$, $\forall n$

Show that $E\left[T_{A_\Pi}(n)\right] = O(f(n))$

Apply the following:

<u>Markov's lemma</u>: let T be a non-negative, bounded $\left(= \exists\, b \in \mathbb{N} \text{ s.t. } Pr(T > b) = 0\right)$, and integer random variable. Then $\forall t$ s.t. $0 \leq t \leq b$

$$t \cdot Pr(T \geq t) \leq E[T] \leq t + (b-t) Pr(T \geq t)$$

we obtain

$$E\left[T_{A_\Pi}(n)\right] \leq \underbrace{c \cdot f(n)}_{t} + \underbrace{\left(n^a - c \cdot f(n)\right)}_{b-t} \underbrace{/ n^d}_{\Pr(T \geq t)}$$

$$\leq c \cdot f(n) + \frac{n^a}{n^d} \leq c \cdot f(n) + 1$$

$$= O(f(n))$$

Karger's algorithm for Minimum Cut (1993)

Problem: find a cut of minimum size; in other words, it's the minimum number of edges whose removal disconnects the graph

Applications: network reliability, computer graphics, ...

Remark: unweighted graphs here. Also studied in weighted graphs

We'll actually solve a more general problem: minimum cut on <u>multigraphs</u> (i.e., multiple edges between two vertices are allowed)

Def.: <u>multiset</u> = collection of objects with repetitions allowed

$$S = \{\{ \text{objects} \}\}$$

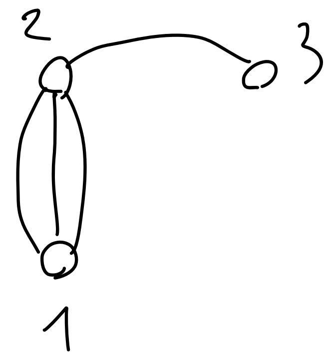
$$\forall \text{ object } \sigma \in S \quad m(\sigma) \in \mathbb{N} \setminus \{0\}$$

$\downarrow$

<u>multiplicity</u>: how many copies of σ are in S

Def.: <u>multigraph</u> = $\mathcal{G} = (V, \mathcal{E})$ such that $V \subseteq \mathbb{N}$ and $\mathcal{E}$ is a multiset of elements (u, v) s.t. $u \neq v$

example:



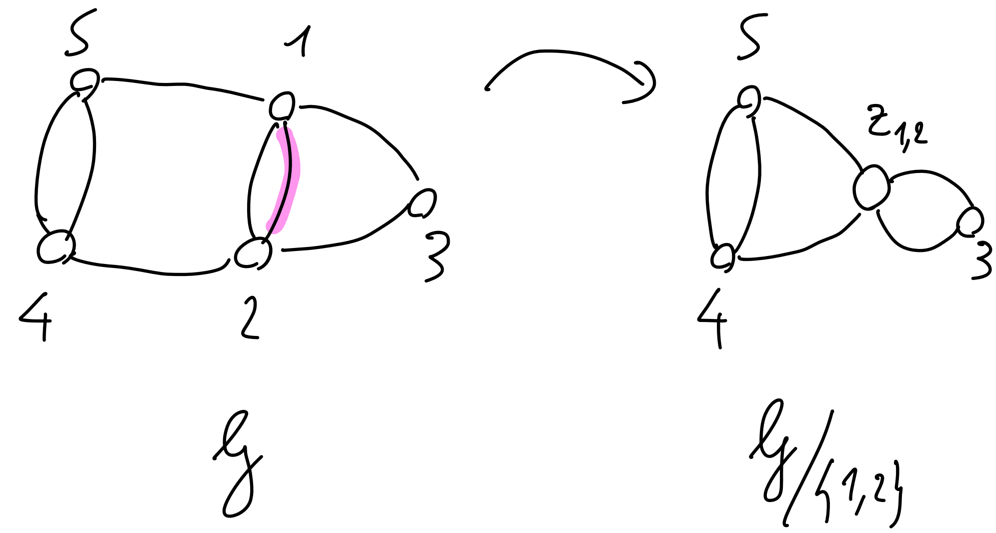
Obs.: a simple graph $G = (V, E)$ is also a multigraph

Def.: given $\mathcal{G} = (V, \mathcal{E})$ connected, a <u>cut</u> $C \subseteq \mathcal{E}$ is a multiset of edges such that $\mathcal{G}' = (V, \mathcal{E} \setminus C)$ is not connected

Karger's idea:

- choose an edge at random
- "contract" the 2 vertices of that edge, removing all the edges incident both vertices

example:



- repeat until only 2 vertices remain: return the edges between them

Def.: given $\mathcal{G}=(V,\mathcal{E})$ and $e=(u,v)\in\mathcal{E}$, the <u>contraction of $\mathcal{G}$ with respect to e</u>, $\mathcal{G}/_e=(V',\mathcal{E}')$, is the multigraph with

$$V' = V\setminus\{u,v\}\cup\{z_{u,v}\} \quad (z_{u,v}\notin V)$$

$$\mathcal{E}' = \mathcal{E}\setminus\{\{(x,y) \text{ s.t. } x=u \text{ or } x=v\}\}$$
$$\cup\,\{\{(z_{u,v},y) \text{ s.t. } (u,y)\in\mathcal{E}$$
$$\text{or } (v,y)\in\mathcal{E},\ y\neq u \text{ and } y\neq v\}\}$$

$|V'| = |V|-1$ $\implies$ n° of iterations needed? $n-2$

$$|\mathcal{E}'| = |\mathcal{E}| - \underset{\geq}{m(e)} \leq |\mathcal{E}|-1$$

FULL-CONTRACTION$(\mathcal{G} = (V, \mathcal{E}))$

for $i = 1$ to $n-2$ do

$e = \text{RANDOM}(\mathcal{E})$

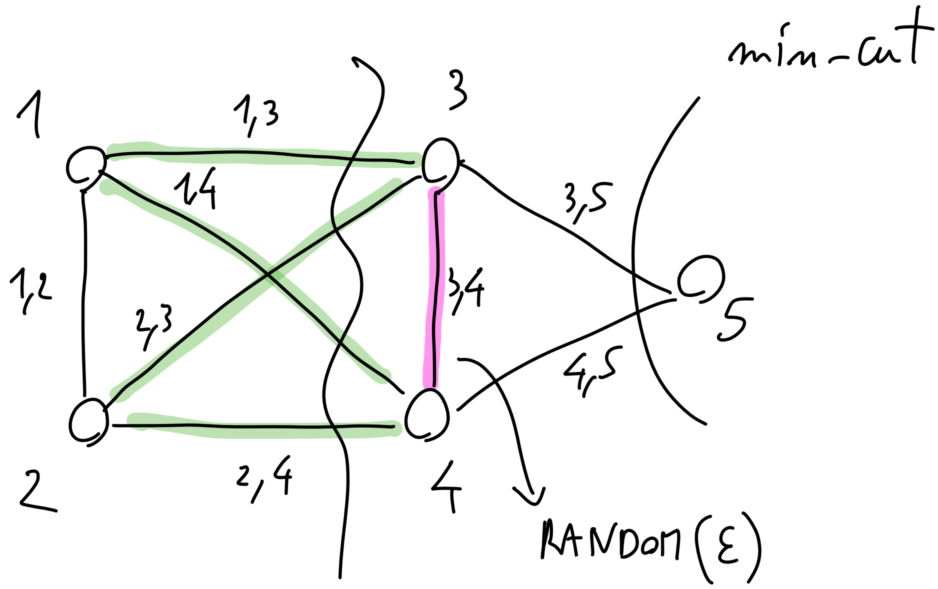
$\mathcal{G}' = (V', \mathcal{E}') \leftarrow \mathcal{G}/e$

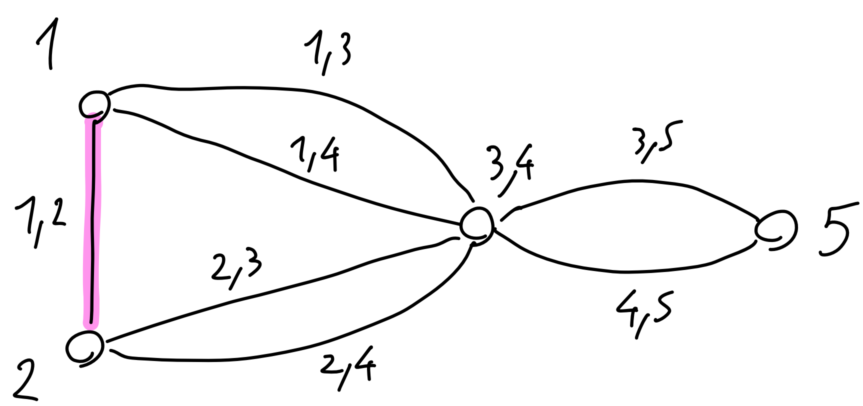
$V = V'$

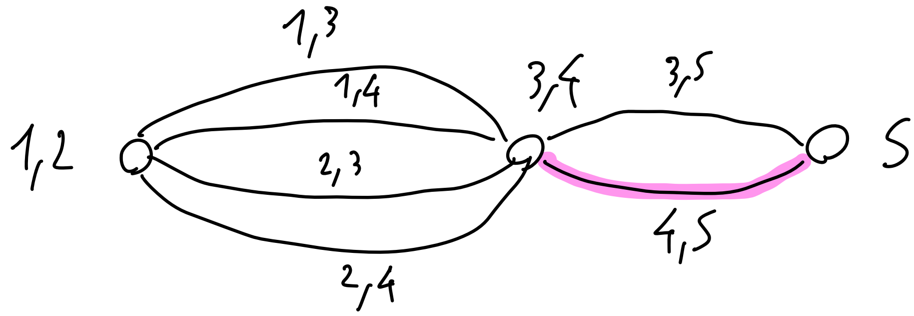
$\mathcal{E} = \mathcal{E}'$

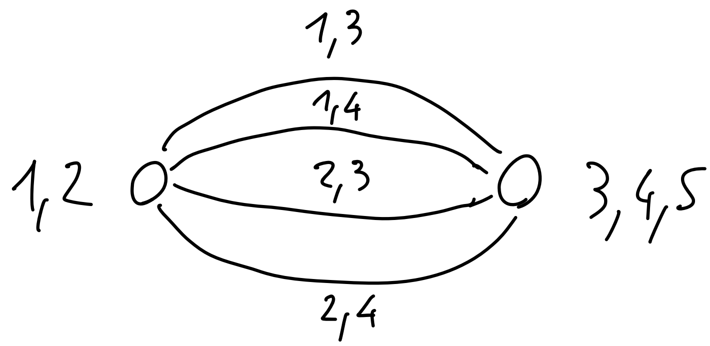
return $\mathcal{E}$

Example:









$\Rightarrow$ <u>unsuccessful</u> run of FULL-CONTRACTION

```
KARGER(G = (V,E), k)
  min-cut = E
  for i = 1 to k do
      t = FULL-CONTRACTION(G)
      if |t| < |min-cut| then
          min-cut = t
  return min-cut
```

$\rightarrow$ repeats FULL-CONTRACTION k times to reduce the probability of error

$\rightarrow$ to be determined by the analysis