Analysis in high probability of randomized QuickSort

RandQuickSort(S) $\qquad\qquad n = |S|$

- if $|S| \leq 1$ then return $\langle S \rangle$
- $p = \text{RANDOM}(S)$ \\\\ pick a "pivot" element uniformly at random from S
- $S_1 = \{x \in S \text{ s.t. } x < p\}$ \\\\ $O(n)$
- $S_2 = \{x \in S \text{ s.t. } x > p\}$ \\\\ $O(n)$
- $Z_1 = \text{RandQuickSort}(S_1)$
- $Z_2 = \text{RandQuickSort}(S_2)$
- return $\langle Z_1, p, Z_2 \rangle$

Is a LAS VEGAS

Suppose p is always the <u>median</u> of S; then

$$T_{RQS}(n) = \begin{cases} 2T_{RQS}\left(\frac{n}{2}\right) + O(n) & n > 1 \\ 0 & n \leq 1 \end{cases}$$

$$T_{RQS}(n) \overset{\text{Master Theorem}}{=} O(n \log n)$$

However, p is the median with probability $\frac{1}{n}$, very low

(Can one find the median in linear time? Yes, deterministically $\Rightarrow$ deterministic QS $O(n \log n)$ exists, but the hidden constant is very high, thus inefficient in practice)

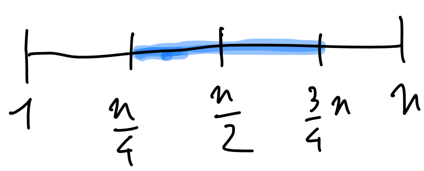
But do we really want <u>exactly</u> the median?

Intuition: if the sizes of S_1 and S_2 are "not too unbalanced", we should be good.

Let's try with a loose request:

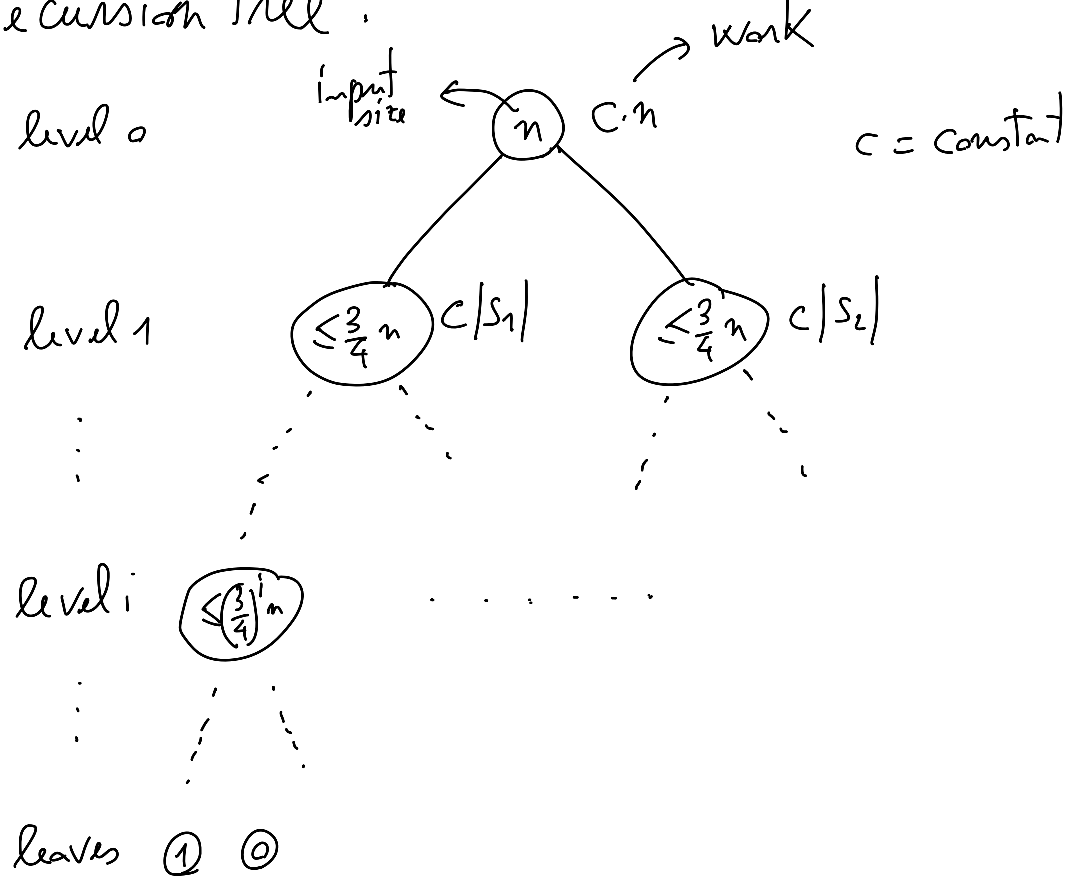
$$\begin{cases} |S_1| \leq \frac{3}{4} n \\ |S_2| \leq \frac{3}{4} n \end{cases}$$

that is, pivot p chosen from



is this good enough for us?

Recursion tree:



- total work at each level is $\leq c \cdot n$
- depth of the recursion tree $= \min \left\{ \text{integers } i \text{ s.t. } \left(\frac{3}{4}\right)^i n \leq 1 \right\} = \lceil \log_{4/3} n \rceil = O(\log n)$

$$\left(\frac{3}{4}\right)^i n \leq 1 \iff \left(\frac{3}{4}\right)^i \leq \frac{1}{n} \iff \left(\frac{4}{3}\right)^i \geq n$$

$$\iff \log_{4/3} \left(4/3\right)^i \geq \log_{4/3} n \iff i \geq \log_{4/3} n$$

$$\Rightarrow T_{RQS}(n) = O(n \log n)$$

that is, it's not necessary that S_1 and S_2 be perfectly balanced. I have $\simeq \frac{n}{2}$ "good" choices for the pivot p.

Analysis

hope: depth of the recursion tree $= O(\log n)$ w.h.p.

that is, <u>all</u> the $\leq n$ distinct root-leaf paths have $O(\log n)$ length w.h.p.

Event "lucky choice of the pivot": pivot chosen between the $\left(\frac{n}{4}+1\right)$-th order statistic and the $\left(\frac{3}{4}n\right)$-th order statistic.

$$\Pr(\text{"lucky choice"}) = \frac{\frac{3}{4}n - \left(\frac{n}{4}+1\right) + 1}{n} = \frac{1}{2}$$

Fix <u>one</u> root-leaf path P: constant

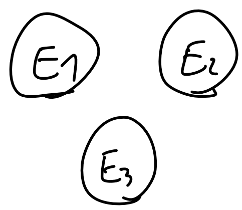
Lemma: $\Pr\left(|P| > a \cdot \log_{4/3} n\right) < \frac{1}{n^3}$

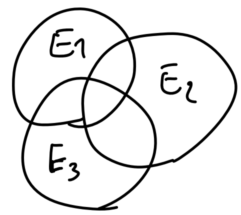
if this is true we're done:

Lemma (union bound): for any random events $E_1, \dots, E_k$:

$$\Pr(E_1 \cup E_2 \cup \dots \cup E_k) \le \Pr(E_1) + \Pr(E_2) + \dots + \Pr(E_k)$$

proof by picture:





$$\Pr(\text{all root-leaf paths have length} \le a \cdot \log_{4/3} n) \ge$$

$$1 - \Pr(\exists \text{ path} > a \log_{4/3} n) \ge \dots$$

E_i = path P_i has length $> a \log_{4/3} n$

union bound

$$\Pr(\exists \text{ path} > a \log_{4/3} n) = \Pr\left(\bigcup_{i=1}^{n} E_i\right) \le$$

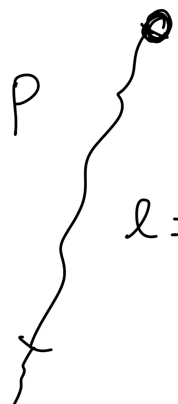
$$\le \sum_{i=1}^{n} \Pr(E_i) \underset{\text{lemma}}{<} n \frac{1}{n^3} = \frac{1}{n^2}$$

$$\dots \ge 1 - \frac{1}{n^2} \implies T_{RQS}(n) = O(n \log n) \text{ w.h.p.}$$

It remains to prove that

$$\Pr\left(|P| > \underbrace{a}_{\text{constant}} \cdot \log_{4/3} n\right) < \frac{1}{n^3}$$

P



$\ell = a \cdot \log_{4/3} n$

E = "in the first $\ell = a \cdot \log_{4/3} n$ nodes of P there have been $< \log_{4/3} n$ lucky choices"

$X_i \qquad 1 \leq i \leq \ell = a \log_{4/3} n$

$X_i = 1$ if at the i-th node of P there is a lucky choice of the pivot

$\Pr(X_i = 1) = \frac{1}{2} \qquad \forall i$

X_i are indipendent

$\Pr\left(\sum_{i=1}^{\ell} X_i < \log_{4/3} n\right)$ to bound

$$X = \sum_{i=1}^{\ell} X_i$$

$$\mu = E[X] = E\left[\sum_{i=1}^{\ell} X_i\right] = \sum_{i=1}^{\ell} E[X_i] = \sum_{i=1}^{\ell} \frac{1}{2} =$$

$$= \frac{a}{2} \log_{4/3} n$$

$$Pr\left(X < (1-\delta)\mu\right) < e^{-\frac{\mu\delta^2}{2}} \qquad 0<\delta\leq 1$$

$$\downarrow$$

$$(1-\delta)\mu = \log_{4/3} n$$

$$(1-\delta)\underbrace{\frac{a}{2}}_{4} \log_{4/3} n = \log_{4/3} n$$

one possible choice is $a = 8$

$\delta = \frac{3}{4}$

$$Pr\left(X < \log_{4/3} n\right) < e^{-\frac{8}{4}\log_{4/3} n \cdot \frac{9}{16}}$$

$$= e^{-\log_{4/3} n \cdot \cancel{\frac{9}{8}}}$$

$$< e^{-\log_{4/3} n}$$

$$= e^{-\frac{\ln n}{\ln 4/3}}$$

$$= \left(e^{-\ln n}\right)^{1/\ln 4/3}$$

$$= \left(\frac{1}{n}\right)^{1/\ln 4/3} \quad \left(1/\ln 4/3 \approx 3{,}47 \right)$$

$$< \frac{1}{n^3}$$