Exercise (high prob. $\Rightarrow$ average): applying Markov's lemma we obtain

$$E[T_{A_\Pi}(n)] \leq \underbrace{c \cdot f(n)}_{t} + \underbrace{(n^a - c \cdot f(n))}_{b-t} / \underbrace{n^d}_{Pr(T \geq t)}$$

$$\leq c \cdot f(n) + \frac{n^a}{n^d} \leq c \cdot f(n) + 1$$

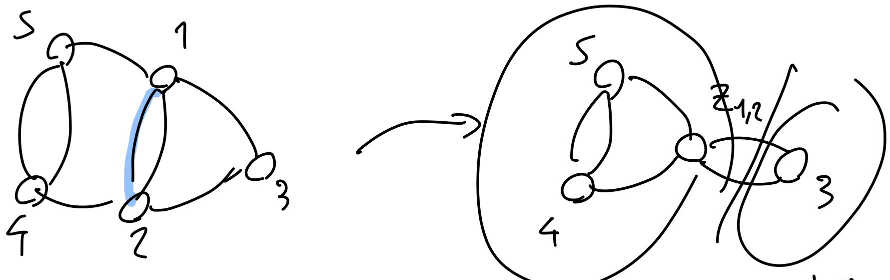
Analysis of Karger's algorithm

Property: $\forall$ cut C' in G/e $\exists$ a cut C in G of the same cardinality

$\Rightarrow$ $|\text{min cut in } G/e| \geq |\text{min cut in } G|$

Proof: constructive: we'll determine the corresponding cut C in G

C' in $G/e=(u,v)$ $\rightsquigarrow$ C in G by substituting each edge $(z_{u,v}, y)$ in C' with (u,y) or (v,y)

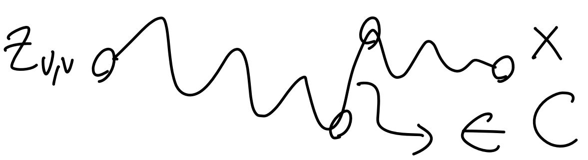


→ "go back" obtaining $(1,3),(2,3)$

$|C'| = |C|$

It remains to be shown that C is a cut in $\mathcal{G}$.

C' cut in $\mathcal{G}/e = (V', \mathcal{E}')$ $\Rightarrow$ C' separates V' in 2 connected components; let $V_1 \subset V'$ the connected component containing $z_{u,v}$, and let $x \notin V_1$. Then in $\mathcal{G}/e$ every path from $z_{u,v}$ and x must use an edge in C'



Now we'll show that C in $\mathcal{G}$ disconnects u and v from x: assume by contradiction that C is <u>not</u> a cut in $\mathcal{G}$ $\Rightarrow$ $\exists$ a path between u and x after the removal of C from $\mathcal{E}$.

Then the path between $z_{u,v}$ and x "survives" the removal of C' in $\mathcal{G}_e$, because "survives" means not using edges in C, that is in

G/e we are not using edges in C' : i.e. C' is not a cut in G/e : contradiction.

What are the cuts that disappear in G/e ? Those hit by the random choice $\Longrightarrow$ I want the probability of <u>not</u> hitting edges of the min. cut to be sufficiently high.

We'll use <u>conditional probabilities</u>

Def.: E_1, E_2 events are <u>independent</u> if

$$\Pr(E_1 \cap E_2) = \Pr(E_1) \cdot \Pr(E_2)$$

Def.: $\Pr(E_1) > 0$ then $\Pr(E_2 | E_1) = \dfrac{\Pr(E_1 \cap E_2)}{\Pr(E_1)}$

extension to k events

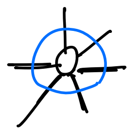
$$\Pr(E_1 \cap E_2 \cap \dots \cap E_k) = \Pr(E_1) \Pr(E_2 | E_1) \Pr(E_3 | E_1 \cap E_2) \cdots \Pr(E_k | E_1 \cap \dots \cap E_{k-1})$$

(can be proved by induction on k)

E_i = in the i-th contraction I did not hit an edge of the min cut

Intuition: |min cut| is a small portion of $|\mathcal{E}|$

Property: let $\mathcal{G} = (V, \mathcal{E})$, $|V| = n$. If $\mathcal{G}$ has a min cut of size t, then $|\mathcal{E}| \geq t \frac{n}{2}$

proof:  $d(v) \geq t \quad \forall v \in V$

$$\sum_{v \in V} \qquad \sum_{v \in V}$$

$$\downarrow$$

$$\underset{\underset{|\mathcal{E}|}{\shortparallel}}{2m} = \sum_{v \in V} d(v) \geq t \cdot n$$

Analysis of FULL-CONTRACTION

let $t = |\text{min cut}|$

E_i = in the i-th contraction I did not hit an edge of the min cut

E_1

$$Pr(\bar{E}_1) = \frac{t}{|\mathcal{E}| \geq t\frac{n}{2}} \leq \frac{t}{t\frac{n}{2}} = \frac{2}{n}$$

$$Pr(E_1) = 1 - Pr(\bar{E}_1) \geq 1 - \frac{2}{n}$$

$$\Pr(E_2 \mid E_1) \geq 1 - \frac{t}{\frac{t(n-1)}{2}} = 1 - \frac{2}{n-1}$$

$\vdots$

$$\Pr(E_i \mid E_1 \cap E_2 \cap \dots \cap E_{i-1}) \geq 1 - \frac{t}{\frac{t(n-i+1)}{2}} = 1 - \frac{2}{n-i+1}$$

$$\Pr(\text{FULL\_CONTR. succeeds}) \geq \Pr\left(\bigcap_{i=1}^{n-2} E_i\right) \geq$$

$$= \prod_{i=1}^{n-2}\left(1 - \frac{2}{n-i+1}\right) = \prod_{i=1}^{n-2} \frac{n-i-1}{n-i+1} =$$

$$= \frac{\cancel{n-2}}{n}\,\frac{\cancel{n-3}}{n-1}\,\frac{\cancel{n-4}}{\cancel{n-2}} \cdots \frac{\cancel{3}}{\cancel{5}}\,\frac{2}{\cancel{4}}\,\frac{1}{\cancel{3}} = \frac{2}{n(n-1)}$$

low, but not too low

KARGER will amplify this probability by repeating FULL_CONTRACTION K times

$$\Pr(\text{the } K \text{ runs of FULL\_CONTR. do not return the min cut})$$

$$\leq \left(1 - \frac{2}{n^2}\right)^K$$

goal $\frac{1}{n^d}$ some constant $d > 0$

$$\left(1-\frac{2}{n^2}\right)^{k} \leq \frac{1}{n^d}$$

in this cases it's standard the use of this ineq.:

$$\left(1+\frac{x}{y}\right)^{y} \leq e^{x} \qquad y \geq 1 \quad y \geq x$$

$$\left(1-\frac{2}{n^2}\right)^{\underline{k=n^2}} \leq e^{-2} = \frac{1}{e^2} \longrightarrow \text{is \underline{not} in the form } \frac{1}{n^d}$$

recall: $e^{-\ln n} = \frac{1}{n}$

$$\left(1-\frac{2}{n^2}\right)^{n^2 \ln n^d} = \left(\left(1-\frac{2}{n^2}\right)^{n^2}\right)^{\ln n^d}$$

let's wrap up:

$$\left(1-\frac{2}{n^2}\right)^{\boxed{k=\frac{d n^2 \ln n}{2}}} = \left(\left(1-\frac{2}{n^2}\right)^{n^2}\right)^{\frac{\ln n^d}{2}}$$

$$\leq \left(e^{-2}\right)^{\frac{\ln n^d}{2}} = e^{-\ln n^d} = \boxed{\frac{1}{n^d}}$$

$\Rightarrow \quad Pr(\text{KARGER succeeds}) > 1 - \frac{1}{n^d}$

Complexity: FULL.CONTR. $O(n^2)$

$\Rightarrow$ KARGER : $O(n^4 \log n)$

this can be improved (Karger-Stein) $\rightarrow O(n^2 \log^3 n)$

idea: $Pr(\underset{\text{FULL.C.}}{\text{failure}}) = \frac{2}{n} \rightarrow \frac{2}{n-1} \rightarrow \frac{2}{n-2}$

1st iteration 2nd 3rd ...

do not repeat the first $\sim \frac{n}{\sqrt{2}}$ iterations

World record : $O(m \log n)$ (2020)