University of Padova
Master's degree in Computer Science

Advanced Algorithms

Spring 2022

June 29, 2022 – 14:30–16:30

First Part: Theory Questions

Question 1 (6 points) Consider the following weighted graph, represented by an adjacency matrix where each numerical value represents the weight of the corresponding edge, and where the symbol '–' indicates the absence of the edge between the corresponding nodes.

	a	b	c	d	e	f
a	-	3	7	-	1	5
b		-	6	8	-	-
c			-	2	9	-
d				-	-	-
e					-	4
f						-

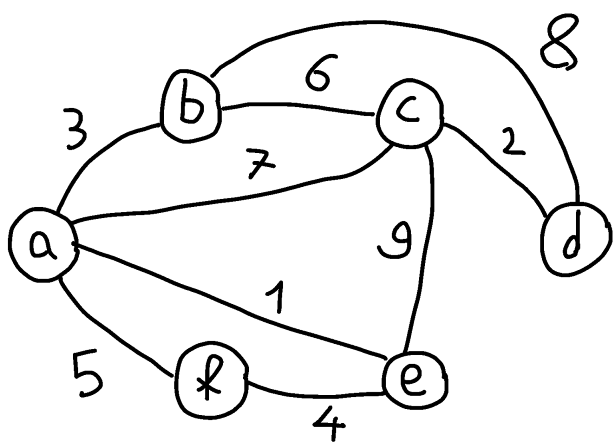
(a) Draw the graph.

(b) List the edges of the minimum spanning tree in the order they are selected by Kruskal's algorithm.

(c) List the edges of the minimum spanning tree in the order they are selected by Prim's algorithm starting at node c.

Solution:

(a)



(b) $(a, e), (c, d), (a, b), (e, f), (b, c)$.

(c) $(c, d), (b, c), (a, b), (a, e), (e, f)$.

Question 2 (7 points) For each of the following problems, say whether it is NP-hard or not and, if not, specify the complexity of the best algorithm seen in class.

(a) Single-source shortest paths

(b) Minimum vertex cover

(c) Connected components

(d) 3SAT

(e) Minimum spanning trees

(f) Metric TSP

(g) Maximum independent set

Solution:

(a) $O((m+n)\log n)$

(b) NP-hard

(c) $O(m+n)$

(d) NP-hard

(e) $O(m \log n)$

(f) NP-hard

(g) NP-hard

Second Part: Problem Solving

Exercise 1 (10 points) A *minimum bottleneck spanning tree* of a connected graph G is a spanning tree of G whose largest edge weight is minimum over all spanning trees of G.

(a) Prove that a minimum bottleneck spanning tree is not necessarily a minimum spanning tree.

(b) Prove that a spanning tree T which is *not* a minimum bottleneck spanning tree cannot be a minimum spanning tree. (Hint: focus on the edge of T of largest weight and try to replace it with an edge from some other suitable spanning tree...)

Solution:

(a) Consider a triangle with two edges of weight 2 and one edge of weight 1. There are three distinct spanning trees, each with largest edge weight 2, hence all three spanning trees are also a minimum bottleneck spanning tree; however, the tree with both edges of weight 2 is not a minimum spanning tree.

(b) Let e be the edge of largest weight in T, and let T_1 and T_2 be the two subtrees of T that remain should e be removed. Now consider a minimum bottleneck spanning tree T', and let e' be an edge of T' that connects T_1 and T_2; since T' is a minimum bottleneck spanning tree and T isn't, the weight of e' is smaller than the weight of e. Then, by replacing e with e' in T we obtain a new spanning tree with total weight smaller than the total weight of T, meaning that T cannot be a minimum spanning tree.

Exercise 2 (9 points) For $n \gg 1$, let $X_1, X_2, \ldots, X_n$ be independent indicator random variables such that $Pr(X_i = 1) = (6 \ln n)/n$ (recall that $\ln n = \log_e n$). Let $X = \sum_{i=1}^{n} X_i$ and $\mu = E[X]$. By applying the following Chernoff bound

$$Pr(X > (1+\delta)\mu) < e^{-\mu\delta^2/2} \quad \text{for } 0 < \delta \leq 2e - 1$$

prove that

$$Pr(X > 10 \ln n) < \frac{1}{n^c}$$

for some positive constant c to be determined.

Solution: To apply the Chernoff bound we set $10 \ln n$ equal to $(1+\delta)\mu$; since $\mu = E[X] = \sum_{i=1}^{n} E[X_i] = n(6 \ln n)/n = 6 \ln n$, we get $\delta = 2/3$. Therefore

$$\begin{aligned} Pr(X > 10 \ln n) &= Pr(X > (1 + 2/3)\mu) \\ &< e^{-\frac{6 \ln n}{2} \frac{4}{9}} \\ &= e^{-\frac{4}{3} \ln n} \\ &= e^{\ln n^{-4/3}} \\ &= n^{-4/3} \\ &= \frac{1}{n^{4/3}}. \end{aligned}$$