Maximum Flows

Originated in the 50s to study rail networks

Definitions:

A flow network is a directed graph $G=(V,E)$ where each edge has a capacity $c(e) \in \mathbb{R}^+$, along with a designated source $s \in V$ and sink $t \in V$.

For convenience, write $c(e)=0$ if $e \notin E$, no edges enter s and no edges leave t.

A flow is a function $f: E \to \mathbb{R}^+$ satisfying the following constraints:

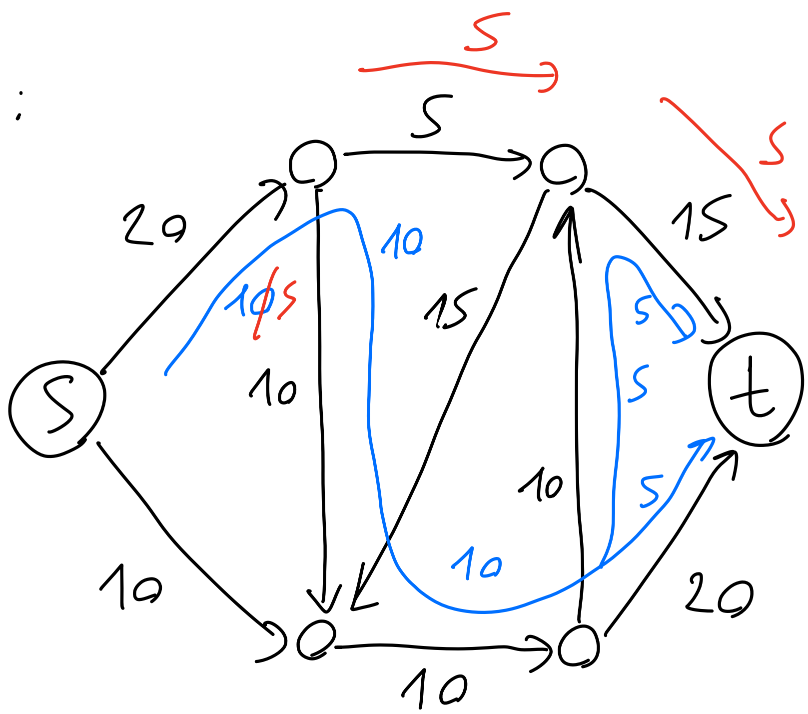
1) (capacity) $\forall e \in E \quad f(e) \leq c(e)$
2) (conservation) $\forall u \in V \setminus \{s,t\}$ we have

$$\sum_{\substack{v \in V \\ \text{s.t. } (v,u) \in E}} f(v,u) = \sum_{\substack{v \in V \\ \text{s.t. } (u,v) \in E}} f(u,v)$$

The <u>value</u> of a flow is $|f| = \sum_{\substack{v \in V \\ s.t. (s,v) \in E}} f(s,v)$

<u>Maximum flow problem</u>: given a flow network find a flow f of maximum value.

Example:



a flow of value 10

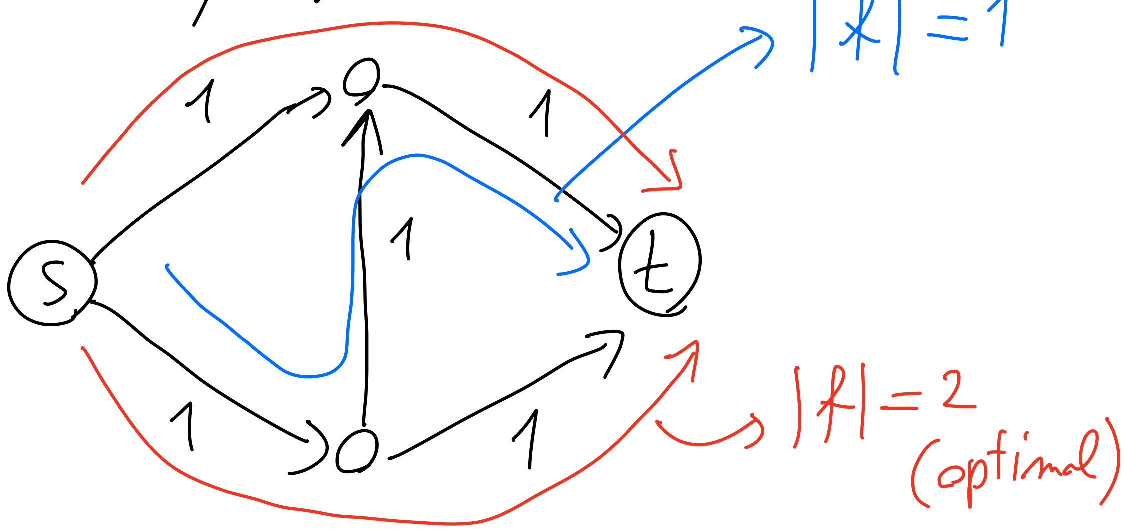
Applications: rail/airline/road networks, electrical networks, communication networks, liquid transportation networks; moreover, it can be applied to solve several other problems in computer science (e.g. bipartite matching)

Max flow reduces to linear programming (like many other problems) but there are more efficient special-purpose algorithms. We'll see one (Ford-Fulkerson) but there are plenty more efficient algorithms → see Further Reading.

A natural idea: be greedy:

- find a path from s to t (in linear time with BFS)
- send as much flow along it as possible
- update capacities
- remove edges that have 0 remaining capacity
- repeat until the graph has no s-t paths

This won't always work:



Idea: revise/undo some of this flow later in the algorithm; how? By "pushing back" some flow through new edges in the reverse direction.

Definition: given a flow network G and a flow f, the residual network of G with respect to flow f, G_f, is a network with vertex set V and with edge set E_f as follows:

for every edge $e = (u, v)$ in G

- if $f(e) < c(e)$, add e to G_f with capacity

$$c_f(e) = c(e) - f(e)$$

- if $f(e) > 0$, add another edge (v, u) to G_f with capacity

$$c_f(e) = f(e)$$

The Ford-Fulkerson algorithm repeatedly finds an s-t path P in G_f (e.g. using BFS) and uses P to increase the current flow. P is called <u>augmenting path</u>.

Ford-Fulkerson (G, s, t) 1956

initialize $f(e) = 0$ for all $e \in G.E$

$G_f = G$

while there exists an augmenting path P in G_f do

  let $\Delta_p = \min_{e \in P} c_f(e)$ // Δ_p is the "bottleneck" capacity in P

  for each edge $e = (u, v) \in P$ do

    if $(u, v) \in G.E$ then

      $f(u, v) = f(u, v) + \Delta_p$

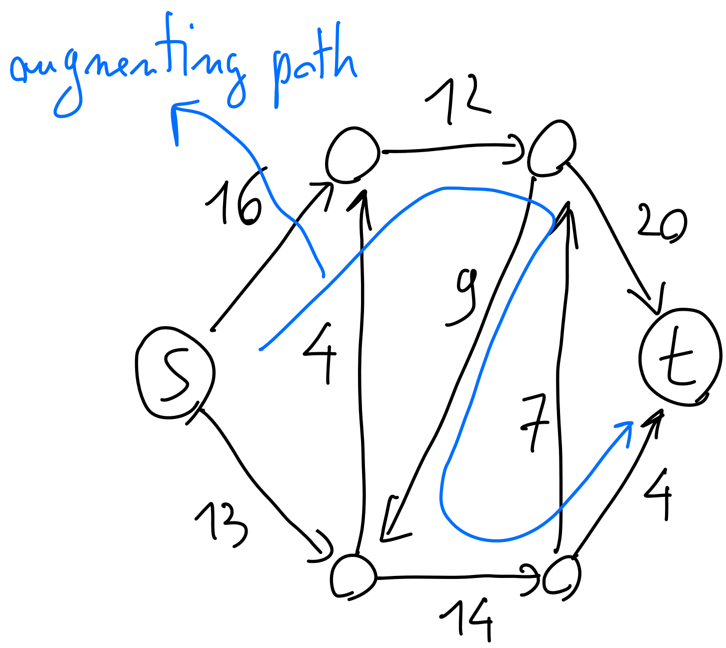
    else $f(v, u) = f(v, u) - \Delta_p$

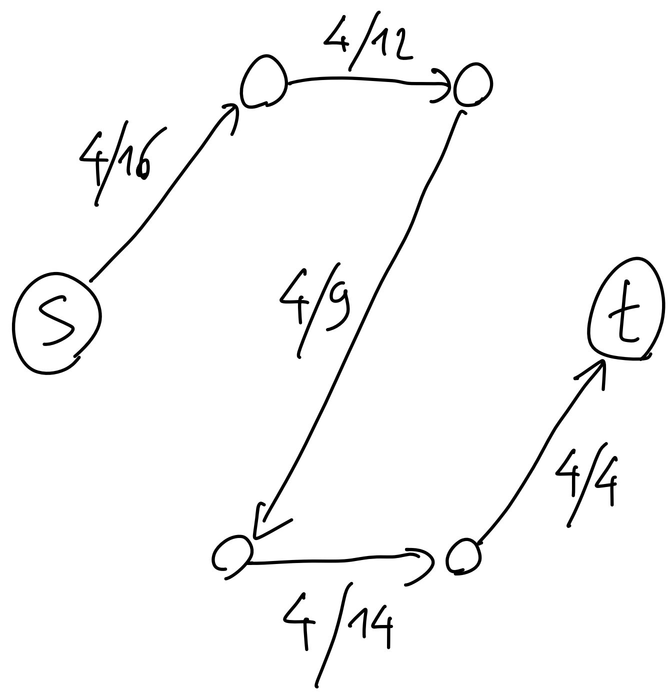
  update the residual graph G_f

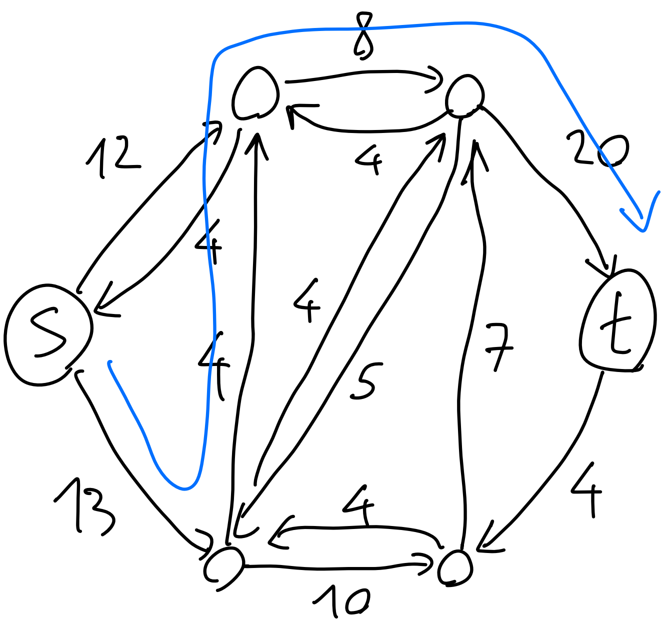
return f

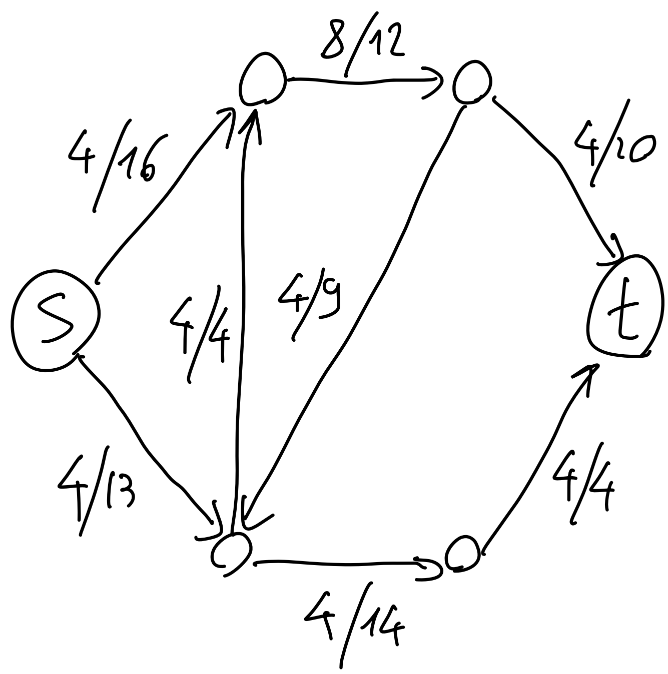
Example:

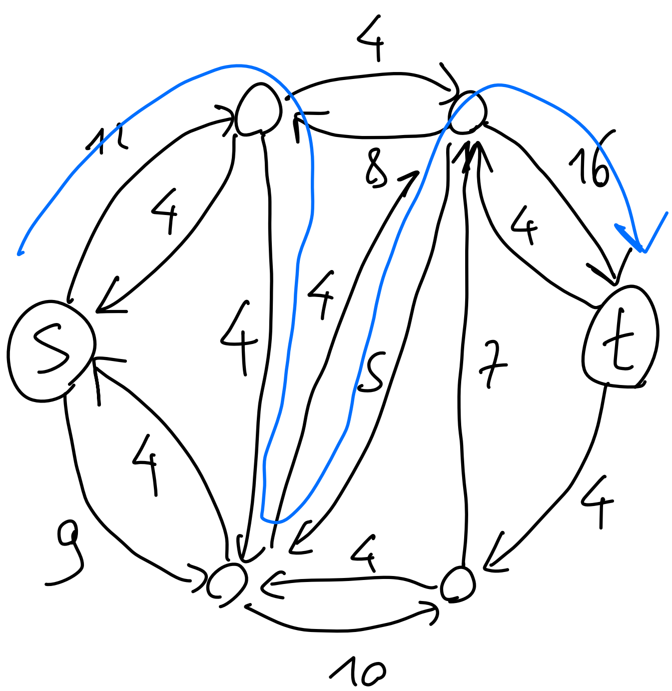
residual graph ↓ new flow ↓







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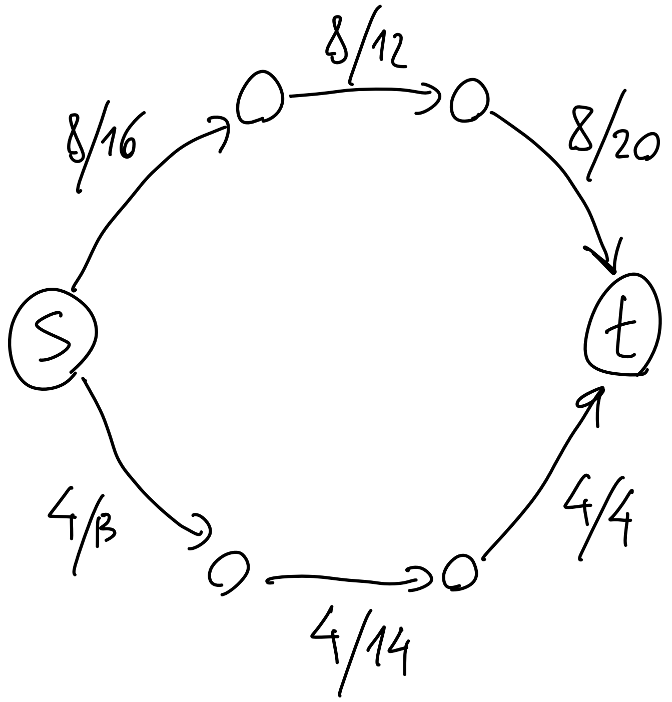
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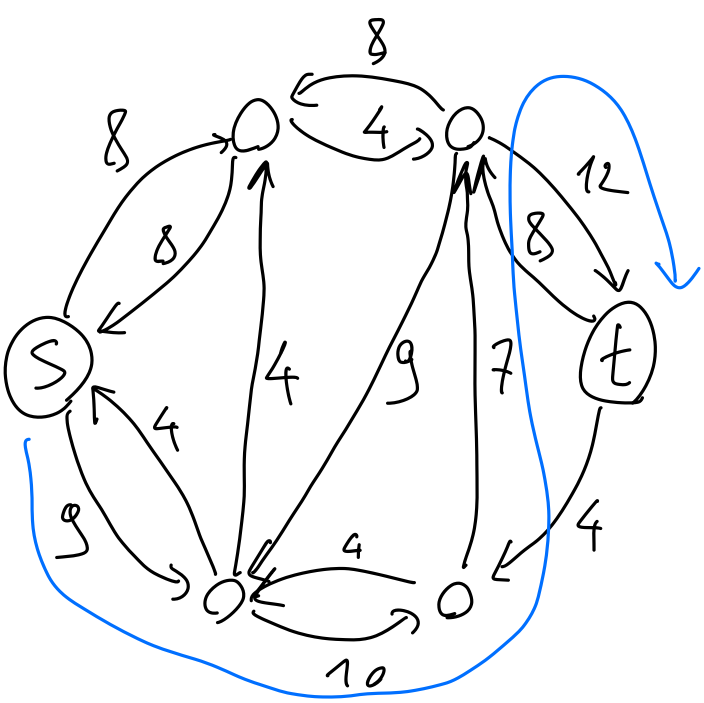
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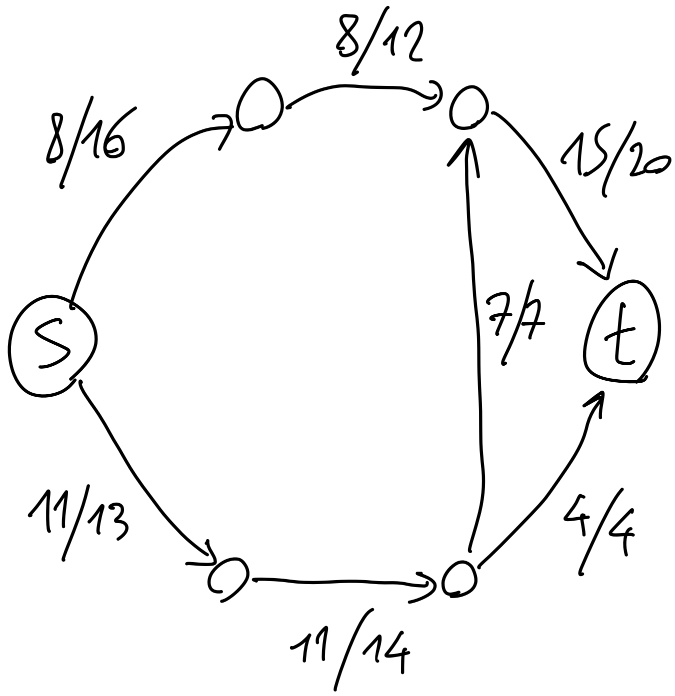
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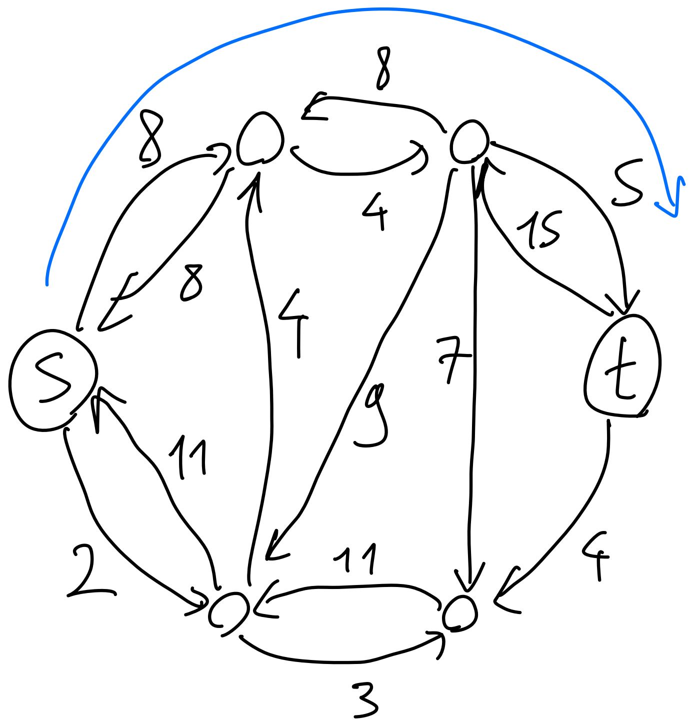
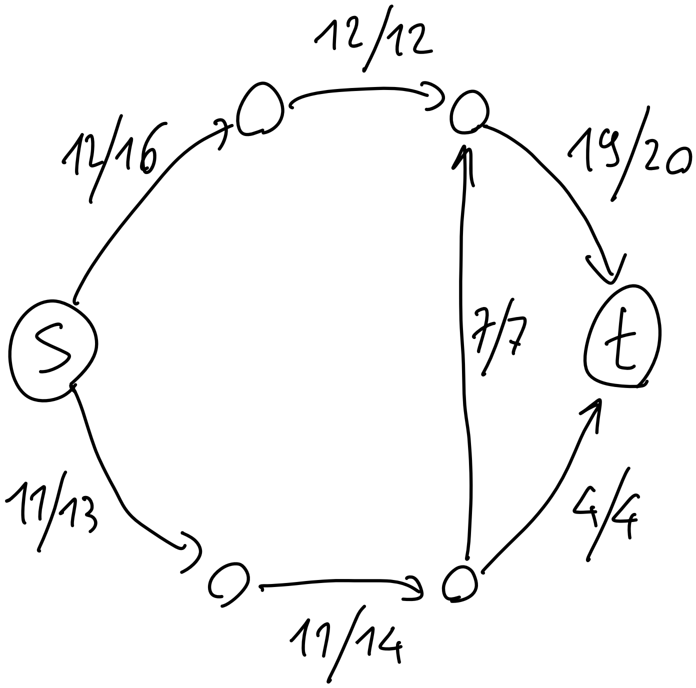
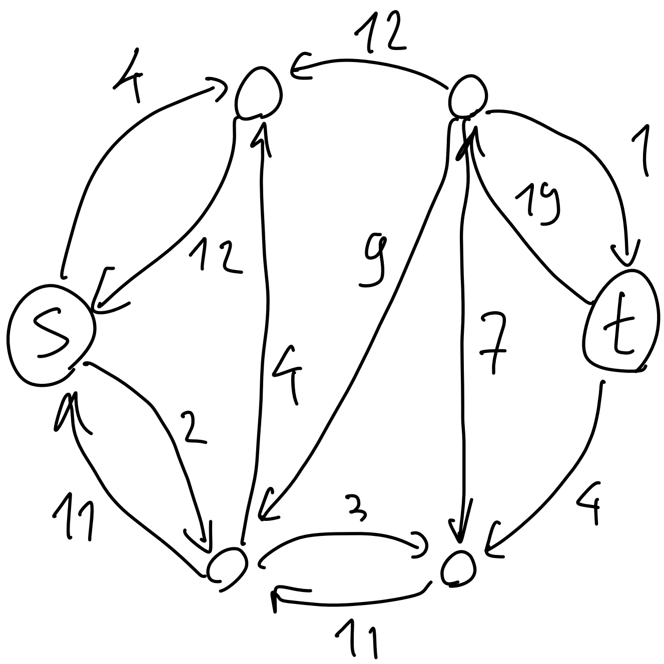






no augmenting paths $\Rightarrow$ is a max flow, of value 23

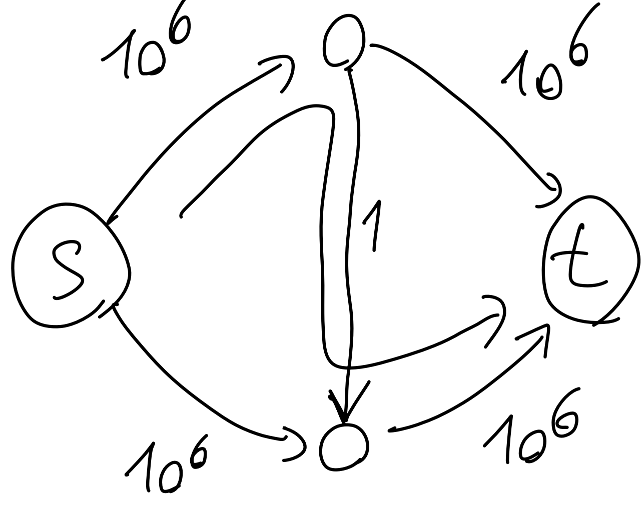
Complexity: assuming capacities are integers; then

- the flow value increases by ≥ 1 in each iteration

- the complexity of each iteration is $O(m)$

$\Rightarrow O(m \cdot |f^*|)$ f^* is a max flow

A flow network for which F-F can take $\Theta(m \cdot |f^*|)$ time:



input size : $O(m \log U)$ U = max capacity

$O(m\,|f^*|) = O(m\,n\,U)$ "<u>pseudo</u>-<u>polynomial</u>"