Theorem: Clique is NP-hard

Proof: decision version:

input: $\langle G = (V, E), k \rangle$
output: $\exists$ in G a clique of size k?

intuition: clique: vertices with <u>all</u> edges between them

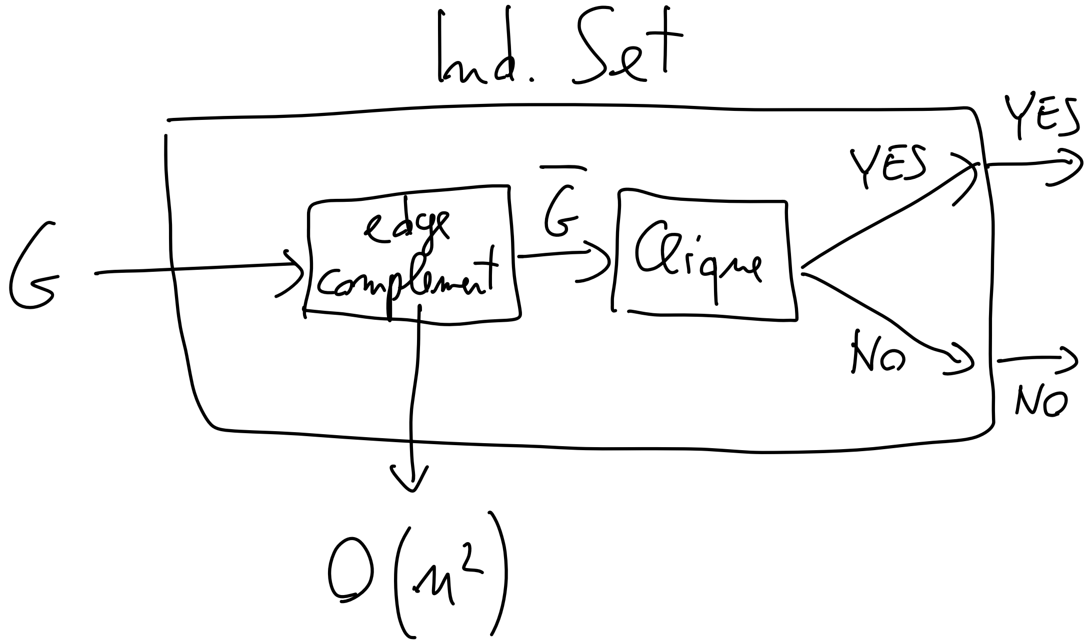
ind. set: vertices with <u>no</u> edges between them

reduction from ind. set

Def.: given a graph $G = (V, E)$, its <u>edge-complement</u> $\overline{G} = (V, E)$ has the same vertex set V and an edge set $\overline{E}$ such that $(u, v) \in \overline{E} \iff (u, v) \notin E$

obs.: a set of vertices is independent in G $\iff$ S is a clique in $\overline{G}$

$\Rightarrow$ the largest ind. set in G has the same site as the largest clique in $\overline{G}$



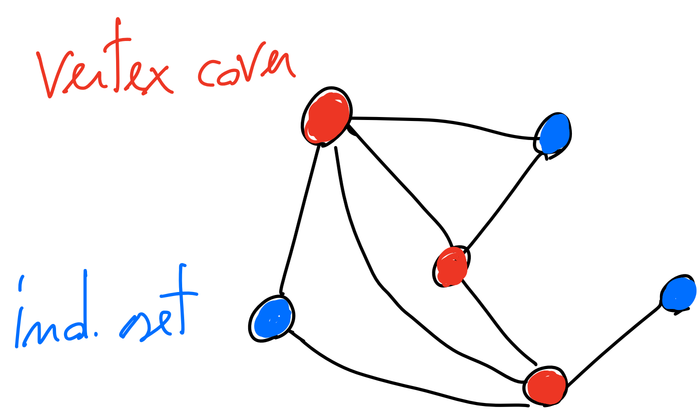
Theorem: Vertex Cover is NP-hard

Proof: decision version: input: $\langle G=(V,E), k \rangle$

output: $\exists$ in G a vertex cover of size k?

reduction from ind. set

obs.: a set of vertices S is independent in G $\iff$ $V \setminus S$ is a vertex cover in G



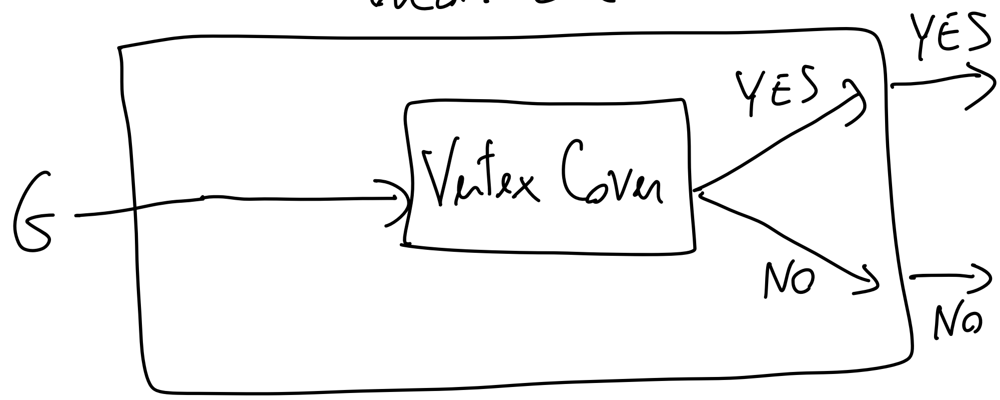
$\Rightarrow$ the largest ind. set in G has size $n-k$, where k is the size of the smallest vertex cover of G

Ind. set:

input: $\langle G=(V,E),\ n-k \rangle$

output: $\exists$ in G an ind. set of size $n-k$?

Ind. Set

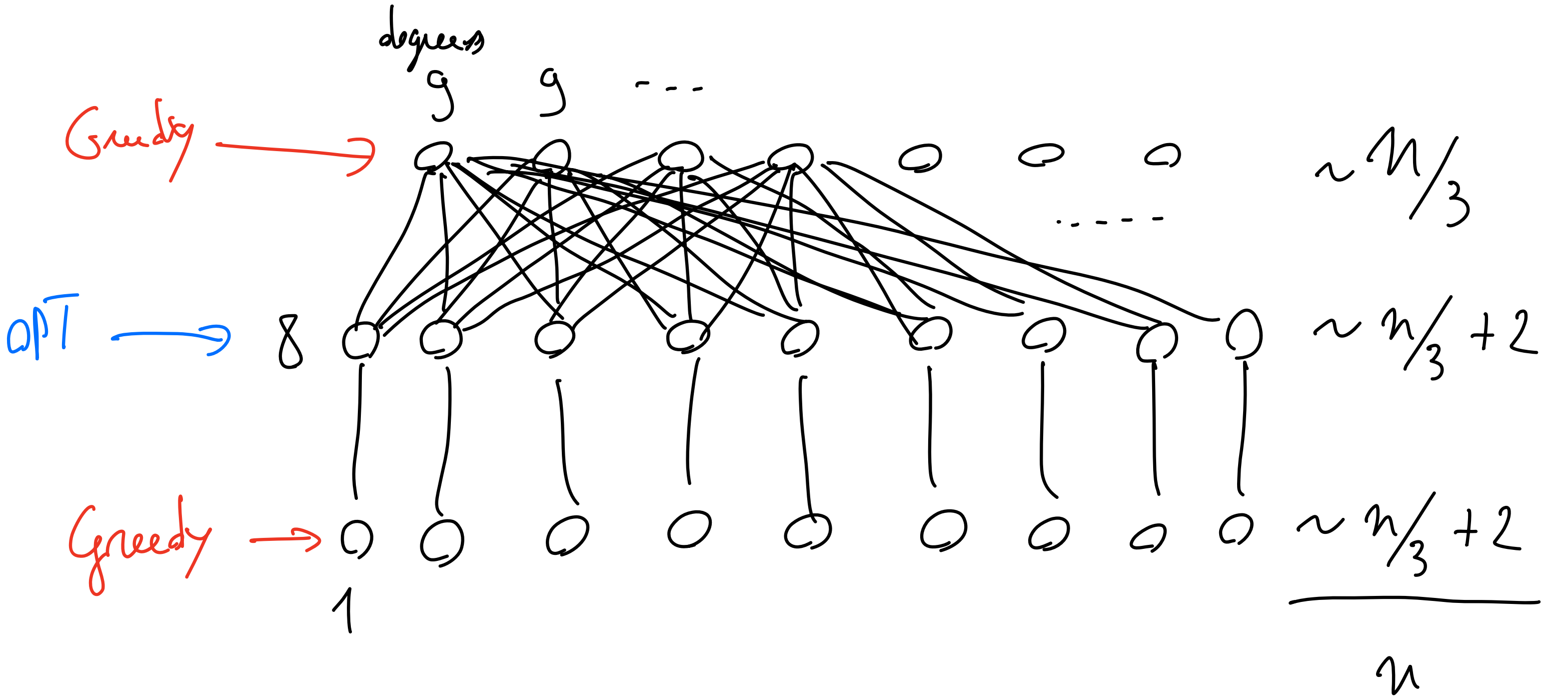


Exercises: show that:

- vertex cover $\le_p$ ind. set
- clique $\le_p$ vertex cover

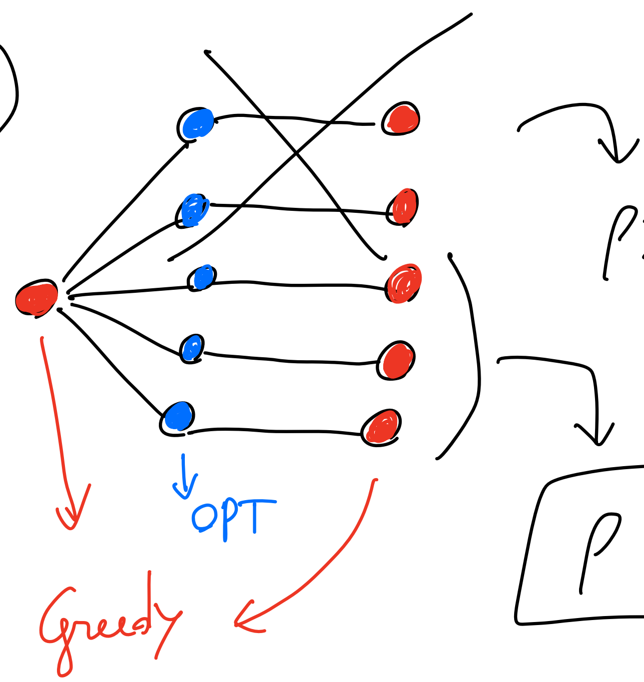
$\Rightarrow$ these 3 problems are equivalent

Degree-based greedy approximation for vertex cover



$$\rho \xrightarrow[n \to +\infty]{} 2$$

a simpler (but lower) lower bound:



$$\rho \geqslant 6/5$$

$$\boxed{\rho \geqslant 4/3}$$

Exercise: show that the approximation factor of Approx-vertex-cover is <u>exactly</u> 2

OPT: just one vertex

Exercise: modify Approx-vertex-cover so as to select only one vertex of the chosen edge instead of both of them. $\rho = ?$

State of the art:

- $2 - \Theta(1/\sqrt{\log n})$ approximation (2009)
- vertex cover cannot be approximated better than $\sqrt{2}$ unless $P = NP$ (2018)
- conjecture: cannot be approximated better than 2

The Traveling Salesperson Problem (TSP)

Definition: given a complete, undirected graph and a function $w: E \to \mathbb{R}^+$, output a tour $T \subseteq E$ (i.e. a cycle that passes through every vertex exactly once) minimizing $\sum_{e \in T} w(e)$

- $w: E \to \mathbb{R}^{+}$ is wlog because every TSP tour has the same number of edges $\Rightarrow$ we can add a large weight to each edge s.t. edges have non-negative weights

Theorem: For any function $\rho(n)$ that be computed in time polynomial in n, there is no polynomial-time $\rho(n)$-approximation algorithm for TSP, unless $P = NP$

Proof: reduction from Hamiltonian Circuit

$$G \longrightarrow G' = (V, E') \text{ complete}$$

$$w(e \in E') = \begin{cases} 1 & e \in E \\ \rho \cdot n + 1 & \text{otherwise} \end{cases}$$

idea: weights are far apart

1) G has a Hamiltonian Circuit $\Rightarrow$
$\exists$ a tour of cost n $\Rightarrow$
TSP algorithm run on G' returns a tour of cost $\boxed{\leq \rho \cdot n}$

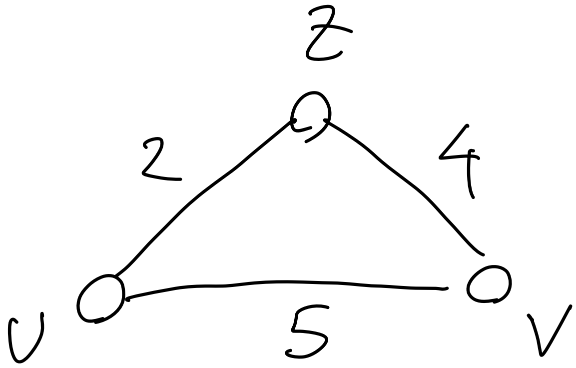
2) G has no Hamiltonian Circuit $\Rightarrow$
the TSP algorithm run on G' returns a tour of cost $\geq \rho \cdot n + 1$ $\boxed{> \rho \cdot n}$

Thus, if we could approximate TSP within a factor of ρ in poly-time, then we would have a poly-time algorithm for Hamiltonian Circuit

Metric TSP

A <u>special case</u> of TSP where the weight function w satisfies the triangle inequality:

$$\forall\, u, v, z \in V: \quad w(u,v) \leq w(u,z) + w(z,v)$$



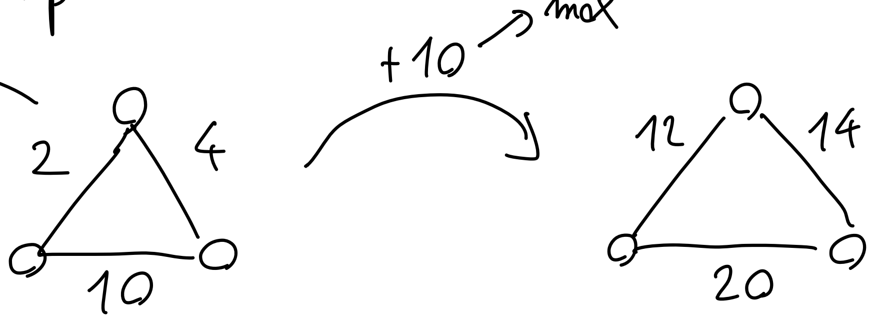
Is metric TSP in P? (often special cases are in P)

Theorem: Metric TSP is NP-hard

Proof: TSP $\leq_p$ Metric TSP

tr. ineq. not necessarily satisfied

idea:



$$\langle G = (V,E), w, k \rangle$$

$$\rightsquigarrow$$

$$\langle G' = (V,E), w', k' \rangle$$

$$w'(u,v) = w(u,v) + W$$

$$W = \max_{u,v \in V} \{ w(u,v) \}$$

$$k' = k + nW$$

to be shown:

1) w' satisfies triangle inequality
2) $\exists$ Ham. circuit of cost k in G $\iff$
   $\exists$ Ham. circuit of cost k' in G'