Graph Algorithms

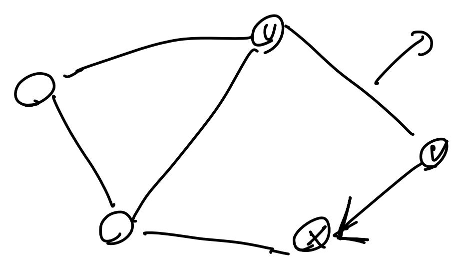
Graphs: the basics

A graph is a representation of the relationships between pairs of objects.

$G = (V, E)$

V = set of vertices (a.k.a. nodes)

E = collection of edges : pair of vertices

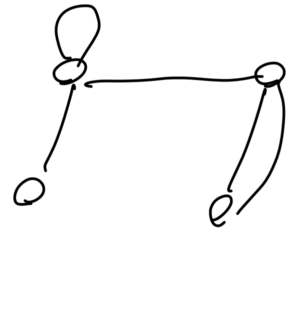


$(u,v) = (v,u)$

$(v,x) \neq (x,v)$

(v,v)

if directed : edge → "arc"



in this course we'll use simple graphs:
- no parallel edges
- no self-loops

Terminology:

$e = (u, v)$ → e is incident on u and v
↘ u and v are adjacent

neighbors of a vertex v: all vertices u s.t. $(v, u) \in E$

degree of a vertex: $d(v)$ / degree(v) is the number of edges incident on v

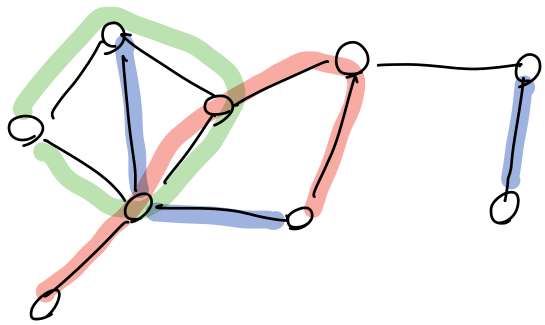
Examples:

road networks → cities/roads
computer networks → computers/connections
e.g. Internet, p2p networks, sensor networks, IoT, ...
WWW → webpages/hyperlinks
social network → people/friendship connection
biological networks
brain

Concepts:

simple path : $v_1, v_2, \dots, v_k$ all distinct and

$(v_i, v_{i+1}) \in E \quad \forall 1 \leq i < k$



cycle : simple path s.t. $v_1 = v_k$

subgraph: $G' = (V', E')$ s.t. $V' \subseteq V$, $E' \subseteq E$

and the edge of E' are incident only on V'

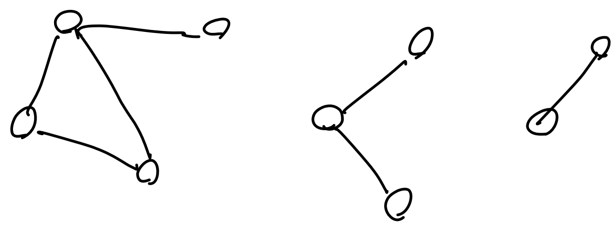
spanning subgraph: a subgraph with $V' = V$

connected graph: if $\forall\, u, v \in V \;\exists$ a path from u to v

Connected components: a partition of G in subgraphs $G_i = (V_i, E_i)$ $\forall 1 \leq i \leq k$ s.t.

- G_i is connected $\forall i$
- $V = V_1 \cup V_2 \cup \dots \cup V_k$

- $E = E_1 \cup E_2 \cup \dots \cup E_k$
- $\forall\, i \neq j$ there is no edge between V_i and V_j

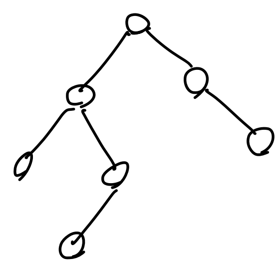


obs:

G_i : is a maximally connected subgraph

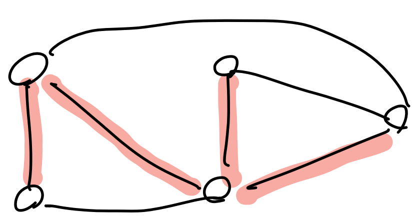
G connected $\Rightarrow k = 1$

tree : connected graph without cycles

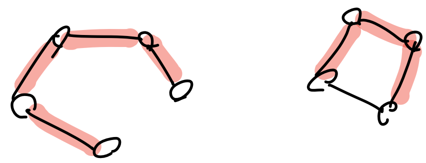


forest : set of trees (disjoint)

spanning tree : is a spanning subgraph connected and without cycles (it exists only if G is connected)



spanning forest: is a spanning subgraph without cycles



Basic graph problems:

- Traversal
- Connectivity
- Conn. components
- Minimum-weight spanning trees
- Shortest paths

Notation

- $n = |V|$
- $m = |E|$

size of a graph? $n + m$

Properties of graphs:

- $\sum_{v \in V} d(v) = 2m$
- $m \leq \binom{n}{2}$
- G is a tree $\implies m = n-1$

- G is connected $\Rightarrow m \geq n-1$
- G is acyclic (i.e., is a forest) $\Rightarrow m \leq n-1$