# Algoritmi e Strutture Dati

* Ricorrenze

  → Sostituzione

  → Master Theorem

* MASTER THEOREM

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$$

$a$: # sottoproblemi $\geq 1$

$b$: dim. dei sottoproblemi $> 1$

$f(n)$: costo della "combina"

$n^{\log_b a}$ vs. $f(n)$

* Master Theorem

Data una ricorrenza

$$T(n) = a\,T\left(\frac{n}{b}\right) + f(n) \qquad \text{con} \quad a \geq 1,\ b > 1$$

vale

1. $T(n) = \Theta(n^{\log_b a})$ se $f(n) = O(n^{\log_b a - \varepsilon})$ per $\varepsilon > 0$
2. $T(n) = \Theta(n^{\log_b a} \log n)$ se $f(n) = \Theta(n^{\log_b a})$
3. $T(n) = \Theta(f(n))$ se $f(n) = \Omega(n^{\log_b a + \varepsilon})$ per $\varepsilon > 0$

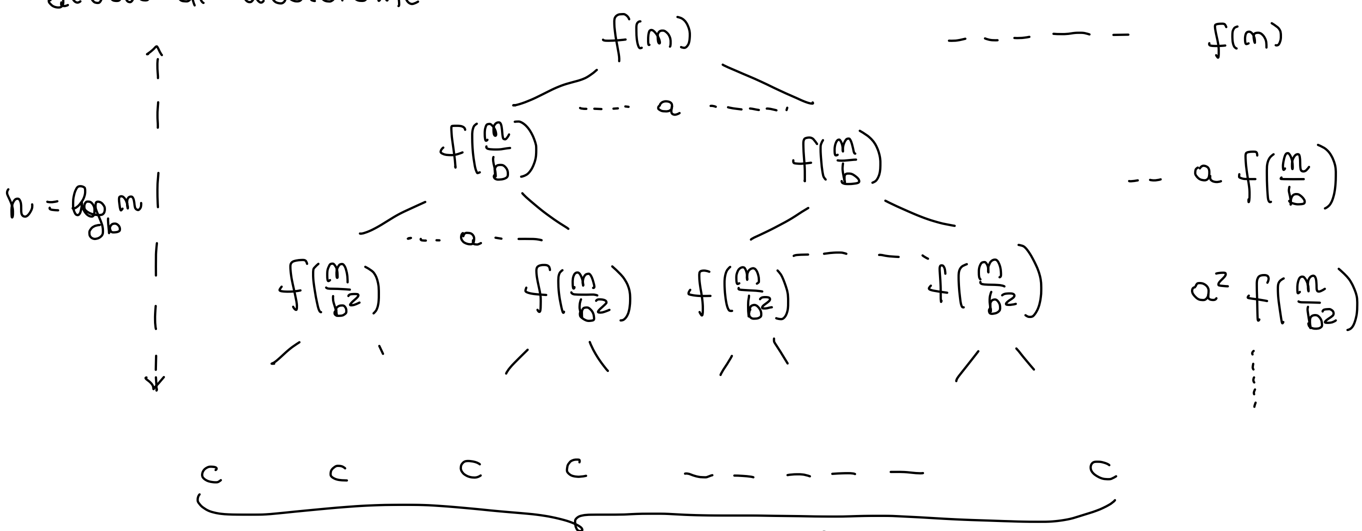
   + condizione di regolarità: esiste $0 < k < 1$, $n_0$ $\forall n \geq n_0$

   $$a\,f\left(\frac{n}{b}\right) \leq k\,f(n)$$

$n^{\log_2 4} = n^2$ $\qquad$ $n^{\log_4 4} = n$

idea : $T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$

albero di ricorsione




$\#$ foglie $\quad a^h = a^{\log_b n}$

$$T(n) = f(n) + a f\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \dots + a^{\log_b n} \, f\Big(\underbrace{\frac{n}{b^{\log_b n}}}_{n}\Big) + c\, a^{\log_b n}$$

$f(1)$

$$a^{\log_b n} = a^{(\log_b a \cdot \log_a n)} = \left(a^{\log_a n}\right)^{\log_b a} = n^{\log_b a}$$

nota: $\log_b n = \log_b a \cdot \log_a n$

$$= \underbrace{f(n) + a f\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \dots + n^{\log_b a}}_{\log_b n} \left(f(1) + c\right)$$

$f(n)$ vs $n^{\log_b a}$

1. "vince" $n^{\log_b a}$ $\rightsquigarrow$ $\Theta\left(n^{\log_b a}\right)$
2. "pareggiano" $\rightsquigarrow$ $\Theta\left(n^{\log_b a} \cdot \log n\right)$
3. vince $f(n)$ $\rightsquigarrow$ $\Theta(f(n))$

$$f(n) + a f\left(\frac{n}{b}\right) + a^2 f\left(\frac{n}{b^2}\right) + \dots$$

$$a f\left(\frac{n}{b}\right) \le k f(n) \qquad a^2 f\left(\frac{n}{b^2}\right) \le k^2 f(n)$$

$$f(n) \underbrace{\left(1 + k + k^2 + k^3 + \dots\right)}_{\text{serie convergente}} \qquad k < 1$$

* data la ricorrenza

$$\boxed{T(n) = a\,T\left(\frac{n}{b}\right) + f(n)} \qquad a \geq 1 \quad b > 1$$

$$T\left(\frac{\cancel{\lceil} n \cancel{\rceil}}{b}\right) \qquad T\left(\frac{n}{\cancel{\lfloor} b \cancel{\rfloor}}\right)$$

→ calcolo $\log_b a$

→ confronto $f(n)$ $\quad n^{\log_b a}$

<u>metodo del limite</u>

tre possibilità:

(a) $\lim_{n\to\infty} \frac{f(n)}{n^{\log_b a}} = K > 0$ $(\neq \infty)$ $\Rightarrow f(n) = \Theta(n^{\log_b a})$

↳ CASO 2 del M.T. $\quad T(n) = \Theta(n^{\log_b a} \log n)$

(b) $\lim_{n\to\infty} \frac{f(n)}{n^{\log_b a}} = 0 \Rightarrow f(n) = O(n^{\log_b a})$ ←
$\neq \Omega(n^{\log_b a})$

⇒ forse siamo nel caso 1 del M.T.

cerca $\varepsilon > 0$ t.c.

$$\lim_{n\to\infty} \frac{f(n)}{n^{\log_b a - \varepsilon}} = 0 \Rightarrow f(n) = O(n^{\log_b a - \varepsilon})$$

⇒ CASO 1 MT e quindi $T(n) = \Theta(n^{\log_b a})$

(c) $\lim_{n\to\infty} \frac{f(n)}{n^{\log_b a}} = \infty \Rightarrow f(n) = \Omega(n^{\log_b a})$
$\neq O(n^{\log_b a})$

⇒ possibile CASO 3 del M.T.

occorre $\varepsilon > 0$ t.c. $\lim_{n\to\infty} \frac{f(n)}{n^{(\log_b a)+\varepsilon}} = \infty \Rightarrow \boxed{f(n) = \Omega(n^{(\log_b a)+\varepsilon})}$

$$\boxed{+ \text{ devo trovare } 0<k<1 \text{ t.c. } \forall n \geq n_0 \quad a f\left(\frac{n}{b}\right) \leq k f(n)}$$

CASO 3 M.T.

$$T(n) = \Theta(f(n))$$

* Esempio : Mergesort

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + an + b$$

"elimina" $\lceil\ \rceil$, $\lfloor\ \rfloor$

$$T(n) = 2\,T\left(\frac{n}{2}\right) + \underbrace{an+b}_{f(n)}$$

$a = 2 \geq 1$ $\quad b = 2 > 1$

$$n^{\log_b a} = n^{\log_2 2} = n \qquad f(n)$$

$$\boxed{f(n)} = an + b = \boxed{\Theta(n^{\log_b a})} = \Theta(n)$$

$\Rightarrow$ $\boxed{\text{CASO 2}}$ di MT

$$T(n) = \Theta\left(n^{\log_b a} \log n\right) = \Theta(n \log n)$$

Esempio :

$$T(n) = 5\,T\left(\frac{n}{2}\right) + \underbrace{2n^2 + n \log n}_{f(n)}$$

$a = 5 \geq 1$ $\quad b = 2 > 1$

$$f(n) = 2n^2 + n \log n \qquad n^{\log_b a} = n^{\log_2 5}$$

$\log_2 5 > 2$

$$\lim_{n\to\infty} \frac{f(n)}{n^{\log_b a}} = \lim_{n\to\infty} \frac{2n^2 + n\log n}{n^{\log_2 5}} = \lim_{n\to\infty} \left( \frac{2n^2}{n^{\log_2 5}} + \frac{n \log n}{n^{\log_2 5}} \right) = 0$$

$$\frac{2n^2}{n^{\log_2 5}} = \frac{2}{n^{(\log_2 5)-2}} \to 0 \qquad \frac{n\log n}{n^{\log_2 5}} = \frac{\log n}{n^{\log_2 5 - 1}} \to 0$$

sospetto CASO 1 del MT

per $\quad 0 < \varepsilon < \log_2 5 - 2$

$$\lim_{n\to\infty} \frac{f(n)}{n^{(\log_b a) - \varepsilon}} = \lim_{n\to\infty} \frac{2n^2 + n\log n}{n^{(\log_2 5) - \varepsilon}} = 0$$

$$\Rightarrow \quad f(n) = O\left(n^{\log_b a - \varepsilon}\right)$$

$\rightarrow$ CASO 1 MT

$$T(n) = \Theta\left(n^{\log_b a}\right) = \Theta\left(n^{\log_2 5}\right)$$

<u>Esempio</u> :

$$T(n) = \underset{a}{5}\, T\left(\frac{n}{\underset{b}{2}}\right) + \underbrace{n^3}_{f(n)}$$

$5 < 8$

$\log_2 8 = 3$ $\Big\}$ $\log_2 5 < \log_2 8 = 3$

$n^{\log_b a} = n^{\log_2 5} \qquad 2 < \log_2 5 < 3$

$$\boxed{f(n) = n^3 = \Omega\left(n^{(\log_b a) + \varepsilon}\right) = \Omega\left(n^{\log_2 5 + \varepsilon}\right)}$$

per $\quad 0 < \varepsilon < 3 - \log_2 5$

inoltre vale la $\boxed{\text{regolarità}}$

$0 < k < 1 \quad n_0 \quad$ t.c. $\forall n \geq n_0$

$$a\, f\left(\frac{n}{b}\right) \leq k\, f(n)$$

$$5\, f\left(\frac{n}{2}\right) \leq k\, f(n)$$

$$\frac{5}{8} n^3 = 5 \left(\frac{n}{2}\right)^3 \leq k\, n^3 \qquad \frac{5}{8} \leq k < 1$$

$n_0$ qualunque

CASO 3 $\qquad T(n) = \Theta(f(n)) = \Theta(n^3)$

ESERCIZIO

$$\boxed{T(n) = \underset{\substack{\uparrow \\ a}}{27}\, T\Big(\frac{n}{\underset{\substack{\uparrow \\ b}}{3}}\Big) + \underbrace{n^3 \log n}_{f(n)}}$$

$$n^{\log_b a} = n^{\log_3 27} = n^3 \quad \text{vs} \quad f(n) = n^3 \log n$$

$$\lim_{n\to\infty} \frac{f(n)}{n^{\log_b a}} = \lim_{n\to\infty} \frac{\cancel{n^3} \log n}{\cancel{n^3}} = \infty$$

sospetto caso 3 del $\boxed{MT}$

dovrei trovare $\varepsilon > 0$ t.c. $\displaystyle\lim_{n\to\infty} \frac{f(n)}{n^{(\log_b a)+\varepsilon}} = \infty$

ma

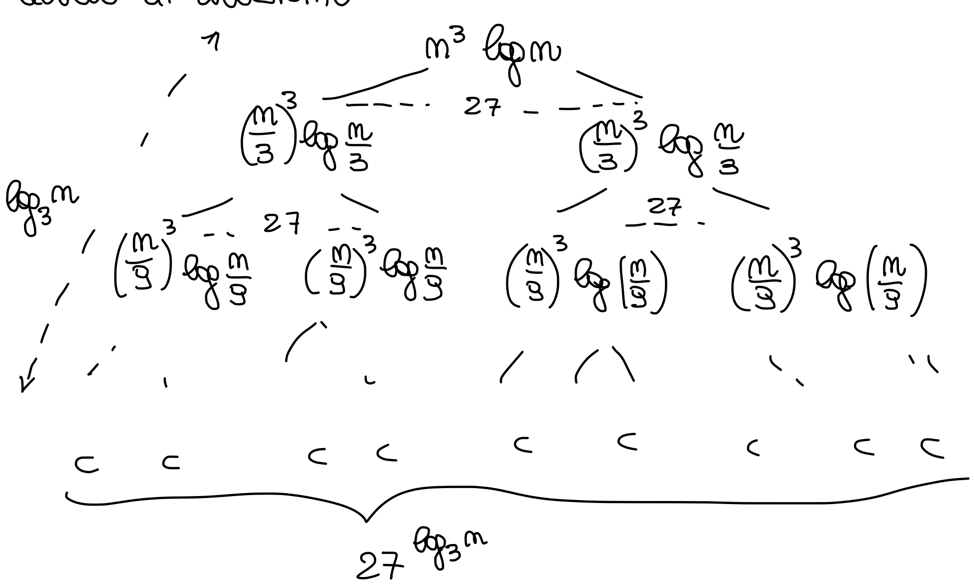
$$\lim_{n\to\infty} \frac{f(n)}{n^{\log_b a+\varepsilon}} = \lim_{n\to\infty} \frac{n^3 \log n}{n^{3+\varepsilon}} = \lim_{n\to\infty} \frac{\log n}{n^{\varepsilon}} = 0 \quad \forall \varepsilon > 0$$

non si applica MT

$*\Rightarrow$ uso sostituzione ...

$$T(n) = 27\, T\Big(\frac{n}{3}\Big) + \underbrace{n^3 \log n}$$

albero di ricorsione




$\cdot\ n^3 \log n$

$\cdot\ 27 \left(\frac{n}{3}\right)^3 \log \frac{n}{3}$

$\cdot\ (27)^2 \left(\frac{n}{9}\right)^3 \log \frac{n}{9}$

$\vdots$

$$\left(27^{\log_3 n}\right) c \sim c\, n^3$$

$$\| $$

$$\left(3^3\right)^{\log_3 n}$$

$$\|$$

$$\left(3^{\log_3 n}\right)^3 = n^3$$

$$T(n) = \left(\sum_{i=0}^{\log_3 n} 27^i \left(\frac{n}{3^i}\right)^3 \log\left(\frac{n}{3^i}\right)\right) + cn^3$$

$$= \left(\sum_{i=0}^{\log_3 n} \cancel{27^i} \frac{n^3}{\cancel{27^i}} (\log n - \log 3^i)\right) + cn^3$$

$$= \left(n^3 \left( \sum_{i=0}^{\log_3 n} (\log n - i \log 3)\right)\right) + cn^3$$

$$= n^3 \left( \underbrace{\sum_{i=0}^{\log_3 n} \log n}_{} - \log 3 \sum_{i=0}^{\log_3 n} i \right) + cn^3$$

$$n^3 \left( (\log n)^2 - \log 3 \underbrace{\frac{((\log_3 n)+1) \log_3 n}{2}}_{\sim (\log n)^2} \right) + cn^3$$

$$\sim n^3 (\log n)^2$$

$$\boxed{T(n) = \Theta(n^3 (\log n)^2)}$$

devo far vedere

1. $T(n) = O(n^3 (\log n)^2)$ $\qquad T(n) \leq c\, n^3 (\log n)^2$
2. $T(n) = \Omega(n^3 (\log n)^2)$ $\qquad T(n) \geq d\, n^3 (\log n)^2$

1. $T(n) = 27\, T\left(\frac{n}{3}\right) + n^3 \log n$ $\qquad$ ip. ind. $T(n) \leq c \left(\frac{n}{3}\right)^3 \left(\log \frac{n}{3}\right)^2$

$$\leq 27 c \left(\frac{n}{3}\right)^3 \left(\log \frac{n}{3}\right)^2 + n^3 \log n$$

$$= \text{— — — —}$$

$$= \text{— — — —}$$

$$= cn^3 (\log n)^2 - n^3 \left( \overbrace{\underbrace{(2c \log 3 - 1)}_{>0} \log n - c (\log 3)^2}^{\geq 0} \right)$$

$$\leq c\, n^3 (\log n)^2$$

$$>0 \Updownarrow c > \frac{1}{2 \log 3}$$