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FORMULA DI TAYLOR

ORDINE DI DERIVAZIONE

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$= \frac{f^{(0)}(x_0)}{(0)!}(x-x_0)^0 + \frac{f^{(1)}(x_0)}{1!}(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2 + \frac{1}{6} f^{(3)}(x_0)(x-x_0)^3 + \underline{o\left((x-x_0)^3\right)}$$

Se $x_0 = 0$ la formula diventa

$$= f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{6} f^{(3)}(0)x^3$$

$$e^x \Rightarrow e^0 = 1$$

$$e^x = 1 + 1\cdot x + \frac{1}{2!}\cdot 1x^2 + \frac{1}{3!}1x^3 + o(x^3)$$

$2!$ $\qquad$ $3!$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$$

$\operatorname{sen} x$: $\quad \underset{f(0)}{\operatorname{sen} 0 = 0} \qquad \underset{f'(0)}{\cos 0 = 1} \qquad \underset{f''(0)}{-\operatorname{sen}(0) = 0}$

$-\cos(0) = -1 \quad \ldots$

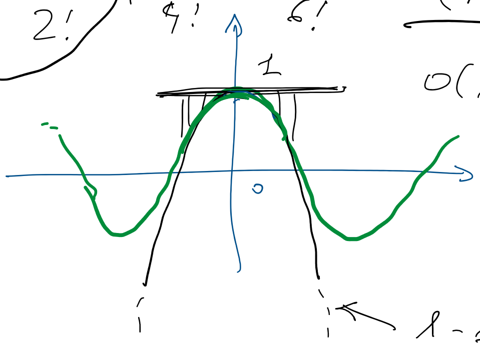
$$\operatorname{sen} x = 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2!} - 1 \frac{x^3}{3!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + o(x^7)$$

$(o(x^8))$

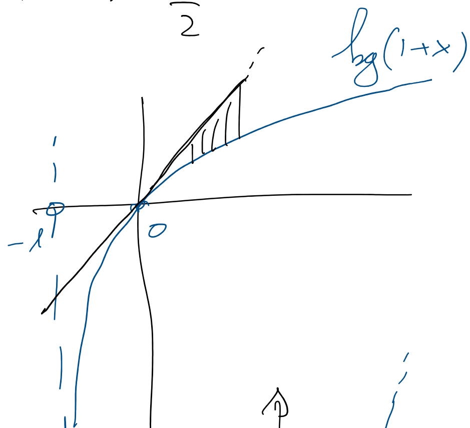
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \underline{o(x^6)}$$

$o(x^7)$

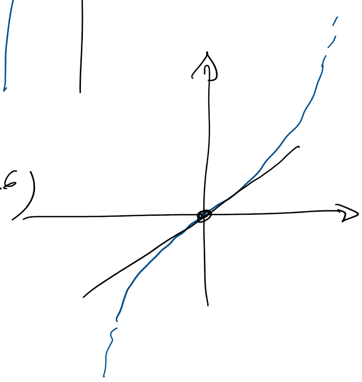


$1 - \frac{x^2}{2}$

$$\log(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$



$$\operatorname{senh} x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)$$



$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + o(x^7)$$

$$f(x) = \frac{1}{1-x} \qquad f(0) = 1$$

$$f'(x) = -\frac{1}{(1-x)^2}(-1)\Big|_{x=0} = 1$$

$$f''(x) = -2\frac{1}{(1-x)^3}(-1)\Big|_{x=0} = 2 = 1\cdot 2$$

$$f^{(3)}(x) = 2(-3)\frac{1}{(1-x)^4}(-1)\Big|_{x=0} = 6 = 1\cdot 2\cdot 3$$

$$\frac{1}{1-x} = 1 + 1\cdot x + \cancel{2}\frac{x^2}{\cancel{2!}} + \cancel{6}\frac{x^3}{\cancel{3!}} + \frac{4!\,x^4}{4!} + \dots$$

$$= 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x} \qquad (|x|<1)$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \dots$$

$$= 1 - x + x^2 - x^3 + \dots$$

$$f(x) = (1+x)^\alpha \qquad f(0) = 1$$

$$f'(x) = \underbrace{\alpha(1+x)^{\alpha-1}}\Big| = \alpha$$

$$f'(x) = \alpha(1+x)^{\alpha-1}\Big|_{x=0} = \alpha$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}\Big|_{x=0} = \alpha(\alpha-1)$$

$$f^{(3)}(x) = \alpha(\alpha-1)(\alpha-2)$$

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} \leftarrow \alpha \quad = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}x^2 + \dots$$

Sviluppi di funzioni composte:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{x-x^2} \quad (x-x^2 \to 0) \quad = 1 + (x-x^2) + \frac{(x-x^2)^2}{2!} + o\left((x-x^2)^2\right)$$

$$= 1 + x - x^2 + \frac{(x^2-2x^3+x^4)}{2} + o(x^2-2x^3+x^4)$$

$$o(x^2)$$

poiché se $f(x) = o(x^2-2x^3+x^4) \iff$

$$\lim_{x\to 0} \frac{f(x)}{x^2-2x^3+x^4} = 0 \quad \text{ma allora}$$

$$\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \left(\frac{f(x)}{x^2-2x^3+x^4}\right)\left(\frac{x^2-2x^3+x^4}{x^2}\right) =$$

$\to 0$ $\qquad \to 1$

$$= 0$$

allora poiché $\lim_{x\to 0} \frac{f(x)}{x^2} = 0 \iff$

$f(x) = o(x^2)$

$$\bigstar = 1 + x \underbrace{- x^2 + \frac{x^2}{2}}_{} + o(x^2)$$

$$= 1 + x - \frac{x^2}{2} + o(x^2) = e^{x - x^2}$$

$$e^{\operatorname{sen} x} = 1 + (\operatorname{sen} x) + \frac{(\operatorname{sen} x)^2}{2!} + o(\operatorname{sen} x)^2$$

ora poniamo $\operatorname{sen} x = x + o(x)$

$$= 1 + (x + o(x)) + \frac{(x + o(x))^2}{2!} + o((x + o(x))^2)$$

$$= 1 + x + o(x) + \frac{1}{2}(x^2 + 2x\, o(x) + o^2(x)) +$$

$$+ \underbrace{o(x^2 + 2 \cdot x \cdot o(x) + o^2(x))}_{o(x^2)}$$

ora: $x \cdot o(x) = o(x^2)$ verificare

$$o(o(x^2)) = o(x^2)$$

$$= 1 + x + o(x) + \frac{1}{2} x^2 + o(x^2)$$

se $\lim_{x\to 0} \frac{f(x)}{x^2} = 0$ allora vale anche se $\lim_{x\to 0} \frac{f(x)}{x^2} = 0$ allora vale anche

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \underbrace{\frac{f(x)}{x^2}}_{0}\cdot x = 0\cdot 0 = 0$$

~~allora~~ $f(x) = o(x)$

$$= \underline{1 + x + o(x)}$$

Sviluppando di più:

$$e^{\operatorname{sen} x} = 1 + \Big(x - \underbrace{\tfrac{x^3}{6}}_{o(x)} + \underbrace{o(x^3)}_{o(x)}\Big) + \underbrace{o\Big(x - \tfrac{x^3}{6} + o(x^3)\Big)}_{o(x)}$$

$$\left(e^t = 1 + t + o(t)\right)$$

$$= 1 + x + o(x)$$

Sviluppando di più entrambe:

$$e^{\operatorname{sen} x} = 1 + \Big(x - \frac{x^3}{6} + o(x^3)\Big) + \frac{1}{2}\Big(x - \frac{x^3}{6} + o(x^3)\Big)^2 +$$

$$+ \underbrace{o\underbrace{\Big(x - \frac{x^3}{6} + o(x^3)\Big)^2}_{x^2}}_{o(x^2)}$$

$$= 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \quad \text{con Taylor:}$$

$$\lim_{x\to 0} \frac{\ldots}{1-\cos x}$$ con taylor:

$$\lim_{x\to 0} \frac{\cancel{1}+\cancel{x}+o(x)+(\cancel{1}-\cancel{x}+o(-x))-\cancel{2}}{\cancel{1}-(\cancel{1}-\frac{x^2}{2!}+o(x^2))}$$

$$\lim_{x\to 0} \frac{\boxed{o(x)+o(x)}}{\frac{x^2}{2}+o(x^2)} =$$

$f=o(-x)$

se $\lim_{x\to 0} \frac{f(x)}{-x}=0$ allora anche

$$\underbrace{\lim_{x\to 0} \frac{f(x)}{x}}_{f=o(x)} = -\lim_{x\to 0}\left(\frac{f(x)}{-x}\right) = 0$$

$$\lim_{x\to 0} \frac{o(x)}{\frac{x^2}{2}+o(x^2)}$$ non si può concludere

allora torniamo indietro e sviluppiamo di più

$$\lim_{x\to 0} \frac{[\cancel{1}+\cancel{x}+\frac{x^2}{2}+o(x^2)]+[\cancel{1}-\cancel{x}+\frac{x^2}{2}+o(x^2)]-\cancel{2}}{1-(1-\frac{x^2}{2!}+o(x^2))}$$

$$\lim_{x\to 0} \frac{x^2+o(x^2)}{\frac{x^2}{2}+o(x^2)} =$$

$$\lim_{x\to 0} \frac{\cancel{x^2}\left(1+\frac{o(x^2)}{x^2}\right)}{\cancel{x^2}\left(\frac{1}{2}+\frac{o(x^2)}{x^2}\right)} = \frac{1}{\frac{1}{2}} = 2$$

$\to 0$

$\downarrow 0$

$$\lim_{x \to 0} \frac{e^x - 1 + \log(1-x)}{x^2 \operatorname{sen} x}$$

$$\lim_{x \to 0} \frac{\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + o(x^3) - \cancel{1} + \left((-\cancel{x}) - \cancel{\frac{(-x)^2}{2}} + \frac{(-x)^3}{3} + o(x^3) \right)}{x^2 \left(x + o(x) \right)}$$

$$\lim_{x \to 0} \frac{\frac{x^3}{3!} + o(x^3) - \frac{x^3}{3} + o(x^3)}{x^3 + o(x^3)}$$

$$\lim_{x \to 0} \frac{x^3 \left(\frac{1}{6} - \frac{1}{3} \right) + o(x^3)}{x^3 + o(x^3)}$$

$$\lim_{x \to 0} \frac{-\frac{1}{6} x^3 + o(x^3)}{x^3 + o(x^3)} = -\frac{1}{6}$$

$$\lim_{x \to 0} \frac{\cos(x^4) - 1}{\sqrt{1+x^8} - \sqrt[3]{1+x^8}}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + o(x^2)$$

$$\lim_{x \to 0} \frac{\cancel{1} - \frac{(x^4)^2}{2!} + o(x^8) - \cancel{1}}{\cancel{1} + \frac{1}{2} x^8 + o(x^8) - \left(\cancel{1} + \frac{1}{3} x^8 + o(x^8) \right)}$$

$$\lim_{x \to 0} \frac{-\frac{1}{2} x^8 + o(x^8)}{\underbrace{x^8 \left(\frac{1}{2} - \frac{1}{3} \right)}_{\frac{1}{6} x^8} + o(x^8)} = \frac{-\frac{1}{2}}{\frac{1}{6}} = -3$$

$$\underbrace{x^8\left(\frac{1}{2}-\frac{1}{3}\right)}_{\frac{1}{6}x^8}+o(x^8) \qquad 6$$

$$\lim_{x\to 0}\frac{e^x\cos x-\operatorname{sen}x-\sqrt[6]{1-x^3}}{x^2-\operatorname{sen}^2 x}$$

$$D(x)=x^2-(\operatorname{sen}x)^2=x^2-(x+o(x))^2=$$

$$=\cancel{x^2}-\cancel{x^2}-\underbrace{2x\,o(x)}_{o(x^2)}+o(x)^2$$

$$=o(x^2)$$

allora

$$D(x)=x^2-\left(x-\frac{x^3}{6}+o(x^3)\right)^2$$

$$=\cancel{x^2}-\left(\cancel{x^2}+2\cdot x\left(-\frac{x^3}{6}\right)+o(x^4)\right)$$

$$=-\frac{1}{3}x^4+o(x^4)$$

$N(x)$:

$e^x\cos x$ sviluppiamo fino a ordine 4:

si otterrà $e^x\cos x=1+x-\frac{1}{3}x^3-\frac{1}{6}x^4+o(x^4)$

In questo modo il numeratore $N(x)$ diventerà

$$N(x)=-\frac{1}{6}x^4+o(x^4)$$

Da cui il limite $\to -\frac{1}{2}$

$$\lim_{x\to 0}\frac{e^{x-x^2}-\operatorname{arctg}x-1}{\sqrt[3]{1+x}-\operatorname{tg}\left(\frac{x}{3}\right)-1}=\frac{9}{2}$$

$x \to 0 \quad \sqrt[3]{1+x} - tg\left(\frac{x}{3}\right) - 1 \qquad (2)$

SALUTI!