15-12-2020

martedì 15 dicembre 2020 10:27

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$$\int f' \cdot g = f \cdot g - \int f \cdot g'$$

$$\int \underbrace{x}_{f'} (\underbrace{\lg x}_{g})^2 \, dx = \underbrace{\frac{x^2}{2}}_{f} \underbrace{\lg^2 x}_{g} - \int \underbrace{\frac{x^{\cancel{2}}}{\cancel{2}}}_{f} \underbrace{\left(\cancel{2} \lg x \, \frac{1}{\cancel{x}}\right)}_{g'} dx$$

$$= \frac{x^2}{2} \lg^2 x - \int \underbrace{x}_{f'} \underbrace{\lg x}_{g} \, dx$$

$$= \frac{x^2}{2} \lg^2 x - \left[\frac{x^2}{2} \lg x - \int \frac{x^{\cancel{2}}}{2} \frac{1}{\cancel{x}} \, dx \right]$$

$$= \frac{x^2}{2} \lg^2 x - \frac{x^2}{2} \lg x + \frac{1}{2} \frac{x^2}{2} + c$$

- $$\int \frac{2x}{\sqrt{1-x^2}} \operatorname{arcsen} x \, dx$$

I° tentativo: $$\int \underbrace{2x}_{g} \underbrace{\operatorname{arcsen} x \cdot \frac{1}{\sqrt{1-x^2}}}_{f'} dx =$$

$$= (\operatorname{arcsen} x)^2 \; 2x \quad \int ((\operatorname{arcsen} x)^2 \ldots$$

$$= \underbrace{\frac{(\text{arcsen}\, x)^2}{2}}_{f} \cdot 2x - \int \frac{(\text{arcsen}\, x)^{\overset{T}{2}}}{2} \cdot 2\, dx$$

$- - - -$

II' tentativo

$$\int \frac{2x}{\sqrt{1-x^2}} \text{arcsen}\, x \, dx =$$

ora invece osservo che $-2x$ è la derivata di $(1-x^2)$

$$= -\int \underbrace{(1-x^2)^{-\frac{1}{2}}(-2x)}_{f'} \cdot \underbrace{\text{arcsen}\, x}_{g} \, dx$$

$$= -\left[\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \cdot \text{arcsen}\, x - \int \frac{\cancel{(1-x^2)^{\frac{1}{2}}}}{\frac{1}{2}} \frac{1}{\cancel{\sqrt{1-x^2}}} dx \right]$$

$$= -\left[2\sqrt{1-x^2}\, \text{arcsen}\, x - \int 2\, dx \right]$$

$$= -2\sqrt{1-x^2}\, \text{arcsen}\, x + 2x + c$$

SOSTITUZIONI

$$\int \frac{e^{2x}}{\sqrt{e^x - 1}} dx \qquad = (e^x)^2$$

$$e^x = t$$

$$x = \log t$$

$\int \sqrt{e^x - 1}$

$$x = \log t \quad\Rightarrow\quad dx = (\log t)' \, dt = \frac{1}{t} dt$$

con l'integrale diventa

$$\int \frac{t^{\cancel{2}}}{\sqrt{t-1}} \frac{1}{\cancel{t}} dt = \int \frac{t}{\sqrt{t-1}} dt =$$

$$= \int \frac{t - 1 + 1}{\sqrt{t-1}} dt = \int \frac{t-1}{\sqrt{t-1}} + \frac{1}{\sqrt{t-1}} dt$$

$$= \int \sqrt{t-1}\, dt + \int \frac{1}{\sqrt{t-1}} dt$$

$$= \int (t-1)^{\frac{1}{2}} dt + \int (t-1)^{-\frac{1}{2}} dt$$

$$= \frac{(t-1)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(t-1)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad (t = e^x)$$

$$= \frac{2}{3}(e^x - 1)^{\frac{3}{2}} + 2(e^x - 1)^{\frac{1}{2}} + c$$

$$\int \frac{1}{\cosh x} dx = \int \frac{1}{\frac{e^x + e^{-x}}{2}} dx =$$

$$= 2\int \frac{1}{e^x + \frac{1}{e^x}}\,dx \qquad e^x = t \quad x = \log t$$

$$dx = \frac{1}{t}\,dt$$

$$= 2\int \frac{1}{t+\frac{1}{t}}\,\frac{1}{t}\,dt$$

$$= 2\int \frac{1}{t^2+1}\,dt = 2\,\mathrm{arctg}\, t + c$$

$$= 2\,\mathrm{arctg}(e^x) + c$$

- $$\int \sqrt{a^2 - x^2}\,dx \qquad (a>0)$$

identità fond. della goniometria

$$x = a\,\mathrm{sen}\,t \longrightarrow dx = (a\,\mathrm{sen}\,t)'\,dt = a\cos t\,dt$$

$$a^2 - (a\,\mathrm{sen}\,t)^2 = a^2(1-\mathrm{sen}^2 t) = a^2\cos^2 t$$

$$\int \sqrt{a^2 - a^2\,\mathrm{sen}^2 t}\; a\cos t\,dt =$$

$$\int a\sqrt{\cos^2 t}\; a\cos t\,dt =$$

$$a^2\int \cos^2 t\,dt = a^2\int \frac{1+\cos 2t}{2}\,dt =$$

$$= a^2\left(\frac{1}{2}t + \frac{1}{2}\,\frac{\mathrm{sen}\,2t}{2}\right) + c$$

$$= a^2\left(\frac{1}{2}(t) + \frac{1}{2}\frac{(\operatorname{sen} 2t)}{2}\right) + c$$

$x = a \operatorname{sen} t$ $\quad \frac{x}{a} = \operatorname{sen} t$ $\quad t = \arcsin \frac{x}{a}$

$$= \frac{a^2}{2} t + \frac{1}{4} 2 \operatorname{sen} t \cos t + c$$

$$= \frac{a^2}{2} t + \frac{1}{2} \operatorname{sen} t \sqrt{1-(\operatorname{sen} t)^2} + c$$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{1}{2} \frac{x}{a} \sqrt{1-\left(\frac{x}{a}\right)^2} + c$$

$$= \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2} \sqrt{a^2-x^2} + c$$

Si può fare anche ponendo $x = a \cos t$

Altro modo: $\quad x = a \operatorname{tgh} t$

$$dx = a \frac{1}{\cosh^2 t} dt \quad - - -$$

• $\int \sqrt{x^2-a^2}\, dx$

$\cosh^2 t - 1 = \operatorname{senh}^2 t$

$\cosh^2 t - \operatorname{senh}^2 t = 1$

usiamo ora l'identità iperbolica

ponendo $x = a \cosh t$ così

$$x^2 - a^2 = a^2 \cosh^2 t - a^2 = a^2(\cosh^2 t - 1)$$

$$= a^2 \operatorname{senh}^2 t$$

Da cui $dx = a \operatorname{senh} t \, dt$, così

$$\int \sqrt{x^2 - a^2}\, dx = \int \sqrt{a^2 \cosh^2 t - a^2}\; a \operatorname{senh} t \, dt$$

$$= \int a \sqrt{\operatorname{senh}^2 t}\; a \operatorname{senh} t \, dt$$

$$= a^2 \int \operatorname{senh}^2 t \, dt \quad \text{(per parti) oppure}$$

$$= a^2 \int \frac{\cosh 2t - 1}{2} \, dt$$

$$= a^2 \frac{1}{4} \operatorname{senh} 2t - \frac{a^2}{2} t$$

$x = a \cosh t$

$t = \operatorname{arcosh} \frac{x}{a}$

$$= a^2 \frac{1}{4}\, 2 \underline{\operatorname{senh} t} \cosh t - \frac{a^2}{2} t$$

$$= \frac{a^2}{2} \sqrt{\cosh^2 t - 1}\, \cosh t - \frac{a^2}{2} t$$

$$= \frac{a^2}{2} \sqrt{\left(\frac{x}{a}\right)^2 - 1}\; \frac{x}{a} - \frac{a^2}{2} \operatorname{arcosh}\left(\frac{x}{a}\right) + c$$

[...]

$$= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right) + c$$

Si poteva fare anche ponendo

$$x = \frac{a}{\cos t} \quad \ldots$$

- $\int \sqrt{x^2+a^2}\,dx$ qui si può porre $x = a\,\mathrm{senh}\,t$

$\cosh^2 t - \mathrm{senh}^2 t = 1$

$\cosh^2 t = \mathrm{senh}^2 t + 1$

$$[\ldots] = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + c$$

Altro modo: $x = a\,\mathrm{tg}\,t$

- $\int \frac{\sqrt{x^2+1}}{x^2}\,dx$

- $\int \frac{1}{\sqrt{x^2+x-1}}\,dx$

- $\int \frac{e^x}{\sqrt{1+e^{2x}}}\,dx$

- $\int \sqrt{\frac{1+x}{1-x}}\,dx$

$\sqrt{\frac{1+x}{1-x}} = t$ isoliamo la x

$\frac{1+x}{1-x} = t^2$ $\quad 1+x = t^2(1-x)$

$$\frac{1+x}{1-x} = t^2 \qquad 1+x = t^2(1-x) = t^2 - t^2 x$$

$$x(1+t^2) = t^2 - 1 \longrightarrow x = \frac{t^2-1}{1+t^2}$$

$$\Rightarrow dx = \left(\frac{t^2-1}{1+t^2}\right)' dt = \frac{2t(1+t^2) - (t^2-1)(2t)}{(1+t^2)^2} dt = \frac{2t + \cancel{2t^3} - \cancel{2t^3} + 2t}{(1+t^2)^2} dt$$

$$= \frac{4t}{(1+t^2)^2} dt$$

Allora

$$\int \sqrt{\frac{1+x}{1-x}}\, dx = \int t \frac{4t}{(1+t^2)^2} dt$$

$$= 4 \int \frac{t^2}{(\underline{1+t^2})^2} dt \qquad (\text{provare con } t = \underline{\text{senh}}\, y)$$

$$= 4 \frac{1}{2} \int t \frac{2t}{(1+t^2)^2} dt$$

$$= 2 \int \underbrace{t}_{g} \underbrace{(1+t^2)^{-2} (2t)}_{f'} dt \qquad \text{per parti:}$$

$$= 2\left[\underbrace{\frac{(1+t^2)^{-2+1}}{-2+1}}_{f}\cdot\underbrace{t}_{g} - \int \underbrace{\frac{(1+t^2)^{-1}}{-1}}_{f}\cdot\underbrace{1}_{g'}\,dt\right]$$

$$= 2\left[-\frac{1}{1+t^2}\,t + \int \frac{1}{1+t^2}\,dt\right]$$

$$= \frac{-2t}{1+t^2} + 2\,\mathrm{arctg}\,t + c \qquad \left(t=\sqrt{\frac{1+x}{1-x}}\right)$$

$$= -2\,\frac{\sqrt{\frac{1+x}{1-x}}}{1+\frac{1+x}{1-x}} + 2\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}} + c$$

...

SOSTITUZIONI PARAMETRICHE

$x = 2\,\mathrm{arctg}\,t$

$t = \mathrm{tg}\,\frac{x}{2}$ $\qquad dx = \frac{2}{1+t^2}\,dt$

$\mathrm{sen}\,x = \frac{2t}{1+t^2}$ $\qquad \cos x = \frac{1-t^2}{1+t^2}$

- $$\int \frac{1}{\mathrm{sen}\,x}\,dx = \int \frac{1}{\frac{\cancel{2}t}{\cancel{1+t^2}}}\,\frac{\cancel{2}}{\cancel{1+t^2}}\,dt$$

$$= \int \frac{1}{t}\,dt = \log|t| = \log\left|\mathrm{tg}\,\frac{x}{2}\right| + c$$

- $$\int \frac{1}{\operatorname{sen}^2 x}\,dx \quad \ldots$$

- $$\int \frac{1+\operatorname{sen} x}{1+\cos x}\,dx = \int \frac{1+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt$$

$$= \int \frac{\frac{1+t^2+2t}{1+t^2}}{\cancel{1}+\cancel{t^2}+\cancel{1}-\cancel{t^2}}\,\cancel{2}\,dt$$

$$= \int \frac{1+t^2+2t}{1+t^2}\,dt = \int 1+\frac{2t}{1+t^2}\,dt$$

$$= t + \log(1+t^2) + c$$

$$= \operatorname{tg}\frac{x}{2} + \log\left(1+\operatorname{tg}^2\frac{x}{2}\right) + c$$

- $$\int \frac{1}{(a^2+b^2)-(a^2-b^2)\cos x}\,dx$$

$$\int \frac{1}{(a^2+b^2)-(a^2-b^2)\,\frac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt$$

$$\int \frac{1}{(a^2+b^2)(1+t^2)-(a^2-b^2)(1-t^2)}\,2dt$$

$$\int \frac{\quad}{(a^2+b^2)(1+t^2)-(a^2-b^2)(1-t^2)}$$

$$2\int \frac{dt}{\cancel{a^2}+a^2t^2+b^2+\cancel{b^2t^2}-\cancel{a^2}+a^2t^2+b^2-\cancel{b^2t^2}}$$

$$2\int \frac{dt}{2a^2t^2+2b^2} = \int \frac{dt}{a^2t^2+b^2} =$$

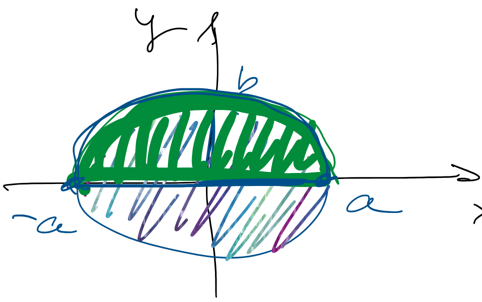
$$= \frac{1}{b^2}\int \frac{dt}{\left(\frac{at}{b}\right)^2+1} = \frac{1}{b^2}\,\frac{b}{a}\int \frac{1}{\left(\frac{a}{b}t\right)^2+1}\,\frac{a}{b}\,dt$$

$$= \frac{1}{ab}\operatorname{arctg}\left(\frac{a}{b}t\right)+c$$

$$= \frac{1}{ab}\operatorname{arctg}\left(\frac{a}{b}\operatorname{tg}\frac{x}{2}\right)+c$$

AREA ELLISSE

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2} \qquad y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y=+\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}$$

$$=b\sqrt{1-\frac{x^2}{a^2}}$$

$$= b\sqrt{1-\frac{x^2}{a^2}}$$

$$\text{AREA} = 2\int_{-a}^{a} b\sqrt{1-\frac{x^2}{a^2}}\,dx =$$

$$= 2b\int_{-a}^{a}\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$= 2b\int_{-a}^{a}\frac{1}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{2b}{a}\int_{-a}^{\boxed{a}}\sqrt{a^2-x^2}\,dx$$

$$x = a\,\text{sen}\,t \qquad dx = a\cos t\,dt$$

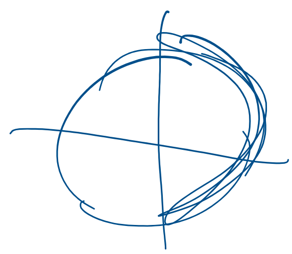
$$x = -a \;\rightarrow\; -a = a\,\text{sen}\,t \qquad -1 = \text{sen}\,t$$

$$t = -\frac{\pi}{2}$$

$$x = \boxed{a} \qquad a = a\,\text{sen}\,t \qquad t = \frac{\pi}{2}$$

$$\Rightarrow \frac{2b}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{a^2-a^2\text{sen}^2 t}\;a\cos t\,dt$$

$$= \frac{2b}{\cancel{a}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cancel{a}\sqrt{\cos^2 t}\;a\cos t =$$



$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt$$

$$\int^{\frac{\pi}{2}} \ldots \cos(2t)\,dt$$

$$= \cancel{2}ab \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2t)}{\cancel{2}}\, dt$$

$$= ab \left[t + \frac{1}{2} \operatorname{sen} 2t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= ab \left[\frac{\pi}{2} + \cancel{\frac{1}{2} \operatorname{sen}\left(2\frac{\pi}{2}\right)} - \left(-\frac{\pi}{2} + \cancel{\frac{1}{2} \operatorname{sen}\left(2\left(-\frac{\pi}{2}\right)\right)} \right) \right]$$

$$= ab \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \pi ab$$

Se $a = b = R$ $\quad \pi R^2$

SALUTI!