21-12-2020

lunedì 21 dicembre 2020 08:26

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$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad n \neq -1$$

$n = -1$

$$\int x^{-1}dx = \int \frac{1}{x}dx = \log|x| + c$$

$a \neq 0,\ b \neq 0$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\int \frac{a\,dx}{ax+b} = \frac{1}{a}\log|ax+b| + c$$

$$(ax+b)' = a$$

$$\int \frac{1}{x^2}dx = \int x^{-2}dx = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x} + c$$

$$\int \frac{1}{x^2+1}dx = \operatorname{arctg} x + c \qquad (\Delta < 0)$$

$$\int \frac{1}{x^2+a^2}dx = \quad (a \neq 0) \quad = \int \frac{dx}{a^2\left(\frac{x^2}{a^2}+1\right)}$$

$$= \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1} \qquad \left(\frac{x}{a}\right)' = \frac{1}{a}$$

$$= \frac{1}{a^{\cancel{2}}}\cancel{a}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\frac{1}{a}dx = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{x^2-1}\,dx = \int \frac{dx}{(x-1)(x+1)} \qquad (\Delta>0)$$

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$= \frac{Ax+A+Bx-B}{x^2-1} = \frac{x(A+B)+A-B}{x^2-1}$$

ora confrontiamo i numeratori:

$$0x+1 = x(A+B)+A-B \qquad \forall x \in \mathbb{R}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \qquad \begin{cases} B=-A \\ 2A=1 \end{cases} \qquad \begin{cases} B=-\frac{1}{2} \\ A=\frac{1}{2} \end{cases}$$

così $\displaystyle \frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}$ e

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1|$$

$$= \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$$

$$\int \frac{dx}{x^2-x} = \int \frac{dx}{x(x-1)} \Rightarrow \qquad (\Delta>0)$$

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \quad \ldots$$

$$\int \frac{dx}{x^2-2x+1} = \int \frac{dx}{(x-1)^2} \qquad \Delta = 0 \qquad (-2)^2-4\cdot 1\cdot 1=0$$

$$= \int (x-1)^{-2}\,dx = \frac{(x-1)^{-2+1}}{-2+1} = -\frac{1}{x-1} + c$$

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + c = -\frac{1}{2}\frac{1}{x^2} + c$$

$$\int \frac{1}{x^3+1}\,dx = \int \frac{dx}{\underbrace{(x+1)}\underbrace{(x^2-x+1)}_{\boxed{\Delta<0}}} \rightarrow$$

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} =$$

$$\frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)} =$$

$$\frac{Ax^2-Ax+A+Bx^2+Bx+Cx+C}{x^3+1} =$$

$$\frac{x^2(A+B)+x(-A+B+C)+A+C}{x^3+1} = \frac{0\cdot x^2+0x+1}{x^3+1}$$

$$\begin{cases} A+B=0 \\ -A+B+C=0 \end{cases} \quad \ldots \quad \begin{cases} A=\frac{1}{3} \\ B=-\frac{1}{3} \end{cases}$$

$$\begin{cases} \dots \\ -A+B+C=0 \\ A+C=1 \end{cases} \quad \cdots \quad \begin{cases} \dots \\ B=-\frac{1}{3} \\ C=\frac{2}{3} \end{cases}$$

Allora

$$\int \frac{1}{x^3+1}\,dx = \int \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{3}\boxed{\int \frac{x-2}{x^2-x+1}\,dx}$$

$\Delta < 0$

ora osserviamo che $(x^2-x+1)' = 2x-1$

$$\int \frac{x-2}{x^2-x+1}\,dx = \frac{1}{2}\int \frac{2x-4}{x^2-x+1}\,dx =$$

$$= \frac{1}{2}\int \frac{(2x-1)+(1-4)}{x^2-x+1}\,dx$$

$$= \frac{1}{2}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{-3}{x^2-x+1}\,dx$$

$$= \frac{1}{2}\log|x^2-x+1| - \frac{3}{2}\boxed{\int \frac{1}{\underbrace{x^2-x+1}_{\Delta<0}}\,dx}$$

$$\int \frac{1}{x^2-x+1}\,dx \overset{*}{=}$$ "completamento del quadrato"

$$(x^2 - x + 1) = \underbrace{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}}_{x^2 - x} + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

doppio prodotto

$$* = \int \frac{1}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx = \int \frac{dx}{\frac{3}{4}\left(\frac{4}{3}\left(x-\frac{1}{2}\right)^2 + 1\right)} =$$

1

$$= \frac{4}{3}\int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}$$

$$\left(\frac{2x-1}{\sqrt{3}}\right)' = \frac{2}{\sqrt{3}}$$

$$= \frac{\overset{2}{\cancel{4}}}{\underset{\sqrt{3}}{\cancel{3}}}\,\frac{\cancel{\sqrt{3}}}{\cancel{2}} \int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1}\cdot\frac{2}{\sqrt{3}}\,dx$$

$$= \boxed{\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right)}$$

In totale (verificare!):

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\log|x+1| - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x-1}{\sqrt{3}}\right) + c$$

$$\int \frac{1}{x^3-1}\,dx = \ldots$$

$$\int \frac{1}{x^4}\,dx = \frac{x^{-4+1}}{-4+1} + c$$

$$\int \frac{1}{x^4-1}\,dx = \int \frac{dx}{\underset{\Delta>0}{(x^2-1)}\,\underset{\Delta<0}{(x^2+1)}} =$$

$$= \int \frac{dx}{(x+1)(x-1)(x^2+1)}$$

decomposizione in frazioni semplici:

$$\frac{1}{x^4-1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1}$$

$= \dots$

alla fine $\displaystyle\int \frac{1}{x^4-1}\,dx = \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\operatorname{arctg} x + c$

$$\int \frac{1}{x^4+1}\,dx = \text{"trucco":}$$

$$\frac{1}{x^4+1} = \frac{1}{x^4+2x^2-2x^2+1} =$$

$$\frac{1}{(x^4+2x^2+1)-2x^2} = \frac{1}{(x^2+1)^2-(\sqrt{2}x)^2} =$$

$$= \frac{1}{\underset{\Delta<0}{(x^2+1-\sqrt{2}x)}\,\underset{\Delta<0}{(x^2+1+\sqrt{2}x)}} =$$

$$(x^2+1-\sqrt{2}x)(x^2+1+\sqrt{2}x)$$
$$\Delta<0 \qquad \Delta<0$$

$$= \frac{Ax+B}{x^2+1-\sqrt{2}x} + \frac{Cx+D}{x^2+1+\sqrt{2}x} \quad \ldots$$

alla fine

$$\int \frac{1}{x^4+1}\,dx = \frac{1}{4\sqrt{2}}\left(\log\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| + 2\arctan(1+\sqrt{2}x) - 2\arctan(1-\sqrt{2}x)\right)$$

$$\int \frac{1}{\underbrace{(x^2+1)}_{\Delta<0}\underbrace{(x+1)^2}_{\Delta=0}}\,dx = -\frac{1}{4}\log(x^2+1) + \frac{1}{2}\log|x+1| - \frac{1}{2}\frac{1}{x+1} + c$$

$$\frac{1}{(x^2+1)(x+1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

...

$$\int \frac{x}{x^2\pm 1}\,dx = \frac{1}{2}\int \frac{2x}{x^2\pm 1}\,dx = \frac{1}{2}\log|x^2\pm 1| + c$$

$$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx =$$

$$= \int 1 - \frac{1}{x^2+1}\,dx = x - \arctan x + c$$

$$\int \frac{x^2+x}{x^2+1}\,dx$$

divisione tra polinomi $\quad$ grado $N \geq$ grado D

$$\begin{array}{rr|l} x^2 + x & & x^2+1 \\ \hline -x^2 & -1 & 1 \\ \hline // \quad x & -1 & \end{array}$$

RESTO

$$\frac{x^2+x}{x^2+1} = 1 + \frac{x-1}{x^2+1}$$

$$\int \frac{x^2+x}{x^2+1}\,dx = \int \left(1 + \frac{x-1}{x^2+1}\right) dx =$$

$$= x + \frac{1}{2}\int \frac{2x}{x^2+1} - \int \frac{1}{x^2+1}\,dx$$

$$= x + \frac{1}{2}\log(x^2+1) - \operatorname{arctg} x + c$$

$$\int \frac{x+1}{x-1}\,dx = \int \frac{x-1+1+1}{x-1}\,dx$$

$$= \int 1 + \frac{2}{x-1}\,dx = x + 2\log|x-1| + c$$

$$\int \frac{ax+b}{cx+d}\,dx \qquad c \neq 0 \quad d \neq 0 \quad a \neq 0$$

$$\begin{array}{l|l} ax + b & cx + d \\ \hline -ax - \frac{a}{c}d & \frac{a}{c} \\ \hline b - \frac{a}{c}d & \end{array}$$

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b-\frac{a}{c}d}{cx+d}$$

$$\int \frac{ax+b}{cx+d}\,dx = \int \frac{a}{c} + \frac{b-\frac{a}{c}d}{cx+d}\,dx =$$

$$\frac{a}{c}x + \left(b-\frac{a}{c}d\right)\frac{1}{c}\int \frac{c}{cx+d}\,dx =$$

$$= \frac{a}{c}x + \frac{bc-ad}{c^2}\log|cx+d| + K$$

$$\int \frac{dx}{ax^2+bx+c} \qquad \Delta = b^2-4ac < 0 \quad (a \neq 0)$$

$$\int \frac{1}{\operatorname{sen}x + \cos x}\,dx$$ con le formule parametriche

$$\operatorname{sen}x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt$$

$$t = \operatorname{tg}\frac{x}{2} \qquad x = 2\operatorname{arctg}t$$

$$\int \frac{1}{\left(\frac{2t}{\cancel{1+t^2}} + \frac{1-t^2}{\cancel{1+t^2}}\right)} \frac{2}{\cancel{(1+t^2)}} dt =$$

$$= \int \frac{2}{2t+1-t^2} dt = -2 \int \frac{dt}{t^2-2t-1}$$

$$\Delta = (-2)^2 - 4(1)(-1) = 4+4 = 8 > 0$$

$$t_{1,2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

decomponiamo:

$$\frac{0t+1}{t^2-2t-1} = \frac{A}{(t-1+\sqrt{2})} + \frac{B}{(t-1-\sqrt{2})} =$$

$$= \frac{A(t-1-\sqrt{2}) + B(t-1+\sqrt{2})}{t^2-2t-1}$$

$$= \frac{t(A+B) + A(-1-\sqrt{2}) + B(-1+\sqrt{2})}{t^2-2t-1}$$

$$\begin{cases} A+B=0 \\ A(-1-\sqrt{2}) + B(-1+\sqrt{2}) = 1 \end{cases} \qquad \begin{cases} B=-A \\ A(-\cancel{1}-\sqrt{2}+\cancel{1}-\sqrt{2}) = 1 \end{cases}$$

$$\begin{cases} B=-A \\ A = -\frac{1}{2\sqrt{2}} \end{cases} \qquad \begin{matrix} B = \frac{1}{2\sqrt{2}} \\ A = -\frac{1}{2\sqrt{2}} \end{matrix}$$

$$-2\int \frac{1}{} dt = -\cancel{2} \int \frac{-\frac{1}{\cancel{2}\sqrt{2}}}{} + \frac{\frac{1}{\cancel{2}\sqrt{2}}}{} dt$$

$$-2\int \frac{1}{t^2-2t-1}\,dt = -\cancel{2}\int \frac{-\frac{1}{\cancel{2}\sqrt{2}}}{t-1+\sqrt{2}} + \frac{\frac{1}{\cancel{2}\sqrt{2}}}{t-1-\sqrt{2}}\,dt$$

$$= \frac{1}{\sqrt{2}}\log|t-1+\sqrt{2}| - \frac{1}{\sqrt{2}}\log|t-1-\sqrt{2}| + c$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right| + c$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{\operatorname{tg}\frac{x}{2}-1+\sqrt{2}}{\operatorname{tg}\frac{x}{2}-1-\sqrt{2}}\right| + c$$

- $\displaystyle\int \frac{dx}{(ax+b)(cx-d)} = \ldots$

- $\displaystyle\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}\,dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+1}\,dx =$

$$x = t^4 \qquad dx = 4t^3\,dt$$

$$= \int \frac{(t^4)^{\frac{1}{2}}}{(t^4)^{\frac{3}{4}}+1}\,4t^3\,dt = \int \frac{t^2}{t^3+1}\,4t^3\,dt$$

$$= 4\int \frac{t^5}{t^3+1}\,dt$$

$$\begin{array}{l|l} t^5 & t^3+1 \\ \hline -t^5-t^2 & t^2 \\ \hline \end{array}$$

$$\begin{array}{r|l} -t^5 - t^2 & t^2 \\ \hline // \quad -t^2 & \end{array}$$

$$= 4 \int \left(t^2 + \frac{-t^2}{t^3+1} \right) dt$$

$$= 4 \frac{t^3}{3} - 4 \int \frac{t^2}{t^3+1} \, dt = \star$$

osserviamo che $(t^3+1)' = 3t^2$

$$\star = \frac{4}{3} t^3 - 4 \cdot \frac{1}{3} \int \frac{3t^2}{t^3+1} \, dt$$

$$= \frac{4}{3} t^3 - \frac{4}{3} \log |t^3+1| + c \qquad \left(\begin{array}{l} x = t^4 \\ t = \sqrt[4]{x} \end{array} \right)$$

$$= \frac{4}{3} x^{\frac{3}{4}} - \frac{4}{3} \log |x^{\frac{3}{4}} + 1| + c$$

SALUTI !