22-12-2020

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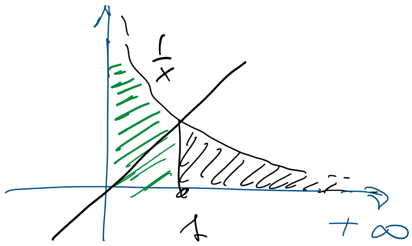
INTEGRALI GENERALIZZATI

$$\int_1^{+\infty} \frac{1}{x}\,dx =$$



$$= \lim_{t\to+\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty} \log|x| \Big]_1^t =$$

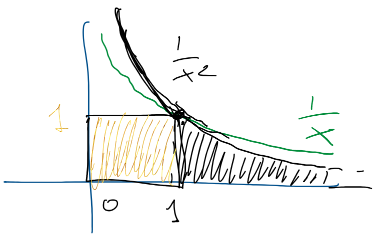
$$= \lim_{t\to+\infty} \left(\log|t| - \log|1|\right) = +\infty$$

$$\int_0^1 \frac{1}{x}\,dx$$ va inteso in senso generalizzato

$$= \lim_{\varepsilon\to0^+}\int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to0^+}\Big[\log|x|\Big]_\varepsilon^1 =$$

$$= \lim_{\varepsilon\to0^+}\Big[\log|1| - \log|\varepsilon|\Big] = -(-\infty) = +\infty$$

$$\int_1^{+\infty} \frac{1}{x^2}\,dx =$$



$$= \lim_{t\to+\infty}\Big[-\frac{1}{x}\Big]_1^t = \lim_{t\to+\infty}\left(\underbrace{\left(-\frac{1}{t}\right)}_{\to 0} - \left(-\frac{1}{1}\right)\right)$$

$$= 0 + 1 = 1$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to0^+}\Big[-\frac{1}{x}\Big]_\varepsilon^1 =$$

$$\lim_{\varepsilon \to 0^+} -1-\left(-\frac{1}{\varepsilon}\right) = +\infty$$

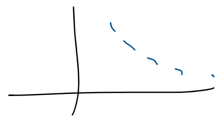
$$\int_1^{+\infty} \frac{\log x}{x}\,dx =$$

Dominio: $x>0$

$$\int (\log x)^1 \cdot \frac{1}{x}\,dx = \frac{(\log x)^2}{2}$$

$$= \lim_{t\to+\infty}\left[\frac{(\log x)^{1+1}}{1+1}\right]_1^t =$$

$$= \lim_{t\to+\infty} \overbrace{\frac{(\log t)^2}{2}}^{\to +\infty} - \frac{(\log 1)^2}{2} = +\infty$$

(anche se $\lim_{x\to+\infty}\frac{\log x}{x} = 0$ )

$$\int_1^{+\infty} \frac{\log x}{x^2}\,dx = \bigstar$$

Dominio: $x>0$

$$\int \frac{\log x}{x^2}\,dx = \int \underbrace{x^{-2}}_{f'}\,\underbrace{\log x}_{g}\,dx =$$

$$= \frac{x^{-2+1}}{-2+1}\log x - \int -\frac{1}{x}\,\frac{1}{x}\,dx$$

$$= -\frac{1}{x}\log x + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\log x - \frac{1}{x}$$

$$\bigstar = \lim_{t\to+\infty}\left[-\frac{1}{x}\log x - \frac{1}{x}\right]_1^t =$$

$$= \lim_{t \to +\infty} \left(\underbrace{-\frac{1}{t}\log t}_{\to 0} - \underbrace{\frac{1}{t}}_{\to 0} - \underbrace{\left(-\frac{1}{1}\log 1 - \frac{1}{1}\right)}_{=0} \right)$$

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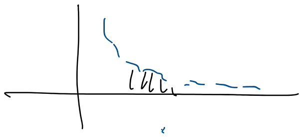
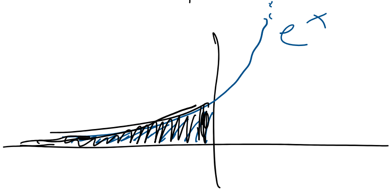
$$= 1$$



$$\int_{-\infty}^{0} e^x \, dx =$$



$$\lim_{t \to -\infty} \left[e^x \right]_t^0 = \lim_{t \to -\infty} \left(e^0 - e^t \right) = 1 - 0 = 1$$

$$\int_0^{+\infty} \frac{1}{1+e^x} \, dx \qquad \text{Dominio} = \mathbb{R}$$

$$\int \frac{1}{1+e^x} \, dx = \int \frac{1+e^x - e^x}{1+e^x} \, dx =$$

$$= \int 1 \, dx - \int \frac{e^x}{1+e^x} \, dx$$

$$= x - \log|1+e^x| \quad \Rightarrow$$

$$\lim_{t \to +\infty} \left[t - \log(1+e^t) - \left(0 - \log(1+e^0)\right) \right]$$

$$= \lim_{t \to +\infty} \left(\underbrace{t}_{+\infty} \underbrace{- \log(1+e^t)}_{-\infty} + \log 2 \right)$$

$$= \lim_{t \to +\infty} t - \log\left(\underbrace{e^t}\cdot\underbrace{(e^{-t}+1)}\right) + \log 2$$

$$= \lim_{t \to +\infty} \left(\cancel{t} - \underbrace{\cancel{\log e^t}}_{t} - \underbrace{\log(e^{-t}+1)}_{\to 0} + \log 2 \right)$$

$$= \log 2$$

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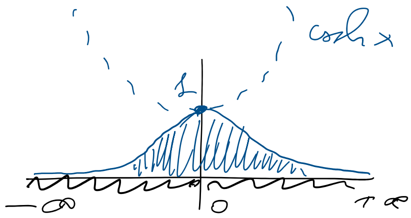
$$\int_{-\infty}^{+\infty} \frac{1}{\cosh x}\,dx = \star$$



$$\int \frac{1}{\cosh x}\,dx = \int \frac{1}{\frac{e^x+e^{-x}}{2}}\,dx =$$

$$= 2\int \frac{1}{e^x + \frac{1}{e^x}}\,dx \qquad e^x = t \qquad x = \log t$$

$$dx = \frac{1}{t}\,dt$$

$$= 2\int \frac{1}{t+\frac{1}{t}}\,\frac{1}{t}\,dt = 2\int \frac{1}{1+t^2}\,dt$$

$$= 2\operatorname{arctg} t = 2\operatorname{arctg}(e^x)$$

$$\star = \lim_{t\to+\infty}\Big[2\operatorname{arctg}(e^x)\Big]_0^t +$$

$$\lim_{y\to-\infty}\Big[2\operatorname{arctg} e^x\Big]_y^0 =$$

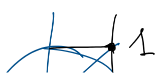
$$= \left(2\cdot\frac{\pi}{2} - 2\operatorname{arctg} e^0\right) + \left(2\operatorname{arctg}(e^0) - 2\operatorname{arctg} 0\right)$$

$$= \pi - \cancel{2\frac{\pi}{4}} + \cancel{2\frac{\pi}{4}} - 2\cdot 0 = \pi$$

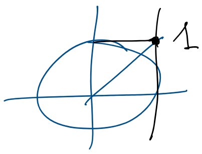
$$\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\operatorname{tg} x}\,\cos^2 x}\,dx = \lim_{\varepsilon\to0^+}\int_\varepsilon^{\frac{\pi}{4}} \frac{dx}{\sqrt{\operatorname{tg} x}\,\cos^2 x} = \star$$

$$\Rightarrow \int (\operatorname{tg} x)^{-\frac{1}{2}}\,\frac{1}{\cos^2 x}\,dx = (\text{immediato})$$

$$= (\operatorname{tg} x)^{-\frac{1}{2}+1} = 2\sqrt{\operatorname{tg} x}$$



$$= \frac{(\operatorname{tg} x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{\operatorname{tg} x}$$



$$☆ = \lim_{\varepsilon \to 0^+} \left(2\sqrt{\operatorname{tg}\frac{\pi}{4}} - 2\sqrt{\operatorname{tg}\varepsilon} \right)$$

$$= 2\cdot\sqrt{1} - 2\cdot\sqrt{0} = 2$$

$$\int_0^{c} \sqrt{\frac{x}{c-x}}\,dx \qquad c > 0$$

Dominio: $\frac{x}{c-x} \geq 0$

$N \geq 0 \quad x \geq 0$

$c - x > 0 \quad x < c$

		0		c	
N	$-$		$+$		$+$
D	$+$		$+$		$-$
	$-$		$+$		$-$

Va inteso in senso generalizzato $x \to c^-$

$$\int \sqrt{\frac{x}{c-x}}\,dx = ☆ \qquad t = \sqrt{\frac{x}{c-x}}$$

$$t^2 = \frac{x}{c-x} \qquad t^2(c-x) = x \qquad x(1+t^2) = t^2 c$$

$$x = \frac{ct^2}{1+t^2} \qquad dx = \left(\frac{ct^2}{1+t^2}\right)' dt$$

$$= \frac{2tc(1+t^2) - ct^2\cdot 2t}{(1+t^2)^2}\,dt$$

$$= \frac{2ct}{(1+t^2)^2}\,dt$$

Cambiamo anche gli estremi: $x \to 0^+ \Rightarrow t \to 0$

$x \to c^- \Rightarrow t \to +\infty$

Con tutto l'integrale diventa:

Con tutto l'integrale diventa :

$$☆ = \int_0^{+\infty} t \cdot \frac{2ct}{(1+t^2)^2}\,dt = 2c \int_0^{+\infty} \frac{t^2}{(1+t^2)^2}\,dt \quad ☆$$

$$\Rightarrow \int \frac{t^2}{(1+t^2)^2}\,dt = \frac{1}{2} \int \underbrace{t}_{g}\, \underbrace{(1+t^2)^{-2}(2t)}_{f'}\,dt$$

derivata $= 2t$

$$= \frac{1}{2}\left[\frac{(1+t^2)^{-2+1}}{-2+1}\, t - \int - \frac{1}{(1+t^2)} \cdot 1\,dt \right]$$

$$= -\frac{1}{2}\frac{t}{1+t^2} + \frac{1}{2}\operatorname{arctg} t$$

$$☆ = 2c \lim_{y \to +\infty} \left[-\frac{1}{2} \underbrace{\frac{y}{1+y^2}}_{\to 0} + \frac{1}{2} \underbrace{\operatorname{arctg} y}_{\to \frac{\pi}{2}} - \frac{1}{2} 0 + \frac{1}{2} 0 \right]$$

$$= 2c\, \frac{1}{2}\, \frac{\pi}{2} = \boxed{c\,\frac{\pi}{2}}$$

$$\int_0^1 \log\left(1+\frac{1}{x}\right)dx$$

$$\int \log\left(1+\frac{1}{x}\right)dx = \int \log\left(\frac{1+x}{x}\right)dx =$$

$$= \int \underbrace{\log(1+x)}_{(1)} - \underbrace{\log x}_{(2)}\,dx = ☆ \text{ per parti:}$$

$$(1) \quad x\log(1+x) - \int x \frac{1}{1+x}\,dx =$$

$$x\log(1+x) - \int \frac{x+1-1}{1+x}\,dx =$$

$$x\log(1+x) - x + \log|1+x|$$

$$x\log(1+x) - x + \log|1+x|$$

② $\int \log x\,dx = x\log x - x$

☆

$$\lim_{t\to 0^+}\Big[\underbrace{x\log(1+x)}_{\to 0} - x + \log(1+x) - \underbrace{x\log x}_{\to 0} + x\Big]_t^1$$

$$1\cdot\log 2 - 1 + \log 2 - 1\cdot\log 1 + 1 -$$

$$-\left(0 - 0 + \log 1 - 0 + 0\right) = 2\log 2$$

$$\int_0^{+\infty}\frac{1}{x^2+4x+9}\,dx \qquad \Delta = 16-4\cdot 9<0$$

Dominio $=\mathbb{R}$

$$\int\frac{1}{x^2+4x+9}\,dx = \int\frac{1}{(x+2)^2-4+9}\,dx$$

$$=\int\frac{dx}{(x+2)^2+5} = \frac{1}{5}\int\frac{dx}{\frac{(x+2)^2}{5}+1}$$

$$=\frac{1}{5}\int\frac{dx}{\left(\frac{x+2}{\sqrt5}\right)^2+1} = \frac{\sqrt5}{5}\int\frac{1}{\left(\frac{x+2}{\sqrt5}\right)^2+1}\,\frac{1}{\sqrt5}\,dx$$

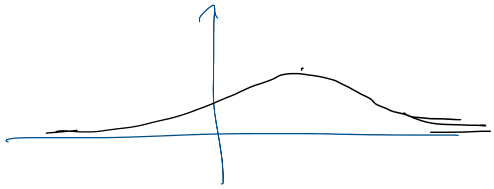
$$=\frac{1}{\sqrt5}\operatorname{arctg}\left(\frac{x+2}{\sqrt5}\right)$$

$$\int_0^{+\infty}\frac{1}{x^2+4x+9}\,dx =$$

$$\lim_{t\to+\infty}\left[\frac{1}{\sqrt5}\operatorname{arctg}\Big(\underbrace{\frac{t+2}{\sqrt5}}_{\to+\infty}\Big) - \frac{1}{\sqrt5}\operatorname{arctg}\left(\frac{2}{\sqrt5}\right)\right] =$$

$$= \frac{1}{\sqrt{5}}\frac{\pi}{2} - \frac{1}{\sqrt{5}}\operatorname{arctg}\left(\frac{2}{\sqrt{5}}\right)$$

$$\int_{-\infty}^{+\infty} \frac{1}{ax^2+bx+c}\,dx \qquad a>0 \quad b\neq 0 \quad c\neq 0 \qquad \Delta<0$$

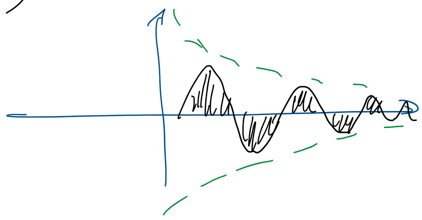


$$\int_{0}^{+\infty} e^{-ax}\operatorname{sen} bx\,dx = \star \qquad a>0 \quad b\in\mathbb{R}$$

Dominio $=\mathbb{R}$

$\Rightarrow \; --- =$ (per parti 2 volte)

$$= \frac{e^{-ax}}{a^2+b^2}\left(-a\operatorname{sen} bx - b\cos bx\right)$$



$$\star = \lim_{t\to+\infty}\left[\frac{\overset{\to 0}{e^{-at}}}{a^2+b^2}\underbrace{\left(-a\operatorname{sen} bt - b\cos bt\right)}_{\to\ \nexists \text{ limite ma è limitata}} - \frac{1}{a^2+b^2}(-b)\right]$$

$\to 0$ poiché prodotto di funzione infinitesima per funzione limitata

$$= \frac{b}{a^2+b^2}$$

$$\int_{0}^{+\infty} \frac{\log x}{(x+1)^{3/2}}\,dx \qquad \text{Dominio } \begin{cases} x>-1 \\ x>0 \end{cases}$$

$\Rightarrow x>0$

$$\int \frac{\log x}{(x+1)^{3/2}}\,dx = \frac{(x+1)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\log x - \int -\frac{2}{\sqrt{x+1}}\,\frac{1}{x}\,dx$$

$$= -\frac{2\log x}{\sqrt{x+1}} + 2\int \frac{1}{x\sqrt{x+1}}\,dx = \$$$

$$\int \frac{1}{x\sqrt{x+1}}\,dx \qquad \sqrt{x+1} = t \qquad x = t^2-1$$

$$dx = 2t\,dt$$

$$\int \frac{1}{(t^2-1)\cancel{t}}\,2\cancel{t}\,dt = 2\int \frac{1}{t^2-1}\,dt = \star$$

$$\frac{1}{t^2-1} = \frac{1}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} =$$

$$= \frac{A(t-1)+B(t+1)}{t^2-1} = \frac{t(A+B)-A+B}{t^2-1}$$

$$\begin{cases} A+B=0 \\ B-A=1 \end{cases} \qquad \begin{cases} B=-A \\ -2A=1 \end{cases} \qquad \begin{array}{l} B=\frac{1}{2} \\ A=-\frac{1}{2} \end{array}$$

$$\star = 2\int \frac{-\frac{1}{2}}{t+1} + \frac{\frac{1}{2}}{t-1} = -\log|t+1| + \log|t-1|$$

$$= -\log\left|\sqrt{x+1}+1\right| + \log\left|\sqrt{x+1}-1\right|$$

$$= \$ \left[-\frac{2\log x}{\sqrt{1+x}} + 2\log\left|\sqrt{x+1}-1\right| - 2\log\left|\sqrt{x+1}+1\right|\right]_0^{+\infty}$$

$$= \left[-\underbrace{\frac{2\log x}{\sqrt{1+x}}}_{\to 0} + 2\log\overbrace{\left|\underbrace{\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}}_{\to 1}\right|}^{\to 0}\right]_0^{+\infty}$$

$$= 0 + (+\infty - \infty)$$

$$= -\lim_{x \to 0^+} \left[\frac{-2 \log x}{\sqrt{1+x}} + 2 \log \left| \sqrt{x+1} - 1 \right| + \right.$$

$$\left. \underbrace{- 2 \log \left| \sqrt{x+1} + 1 \right|}_{\to -2 \log 2} \right]$$

$$\Rightarrow \lim_{x \to 0^+} \left(-2 (1+x)^{-\frac{1}{2}} \log x + 2 \log \left((x+1)^{\frac{1}{2}} - 1 \right) \right)$$

$$= \lim_{x \to 0^+} \left[-2 \left(1 - \frac{1}{2} x + o(x) \right) \log x + \right.$$

$$\left. + 2 \log \left(\cancel{1} + \frac{1}{2} x + o(x) - \cancel{1} \right) \right]$$

$$= \lim_{x \to 0^+} \left[-2 \log x + x \log x - 2\, o(x) \log x + \right.$$

$$\left. + 2 \log \left(\frac{1}{2} x \left(1 + \frac{o(x)}{\frac{x}{2}} \right) \right) \right]$$

$$= \lim_{x \to 0^+} \left[\cancel{-2 \log x} + \underbrace{x \log x}_{\to 0} - 2 \underbrace{o(x) \log x}_{\to 0} + \right.$$

$$\left. 2 \log \frac{1}{2} + \cancel{2 \log x} + 2 \underbrace{\log \left(1 + \frac{o(x)}{x/2} \right)}_{\to 0} \right]$$

Alla fine rimane

$$- \left[2 \log \frac{1}{2} - 2 \log 2 \right] = - \left(-2 \log 2 - 2 \log 2 \right)$$

$$= +4 \log 2$$

$$= \log 16$$

SALUTI!