Basic Concept

The s-m-n theorem (also known as the parametrization theorem) is a fundamental result in computability theory that formalizes the notion of "fixing some arguments" of a computable function.

Formal Statement

Given $m, n \geq 1$, there exists a computable total function $s(m,n) : Nm+1 \to N$ such that for all $e \in N$, $\bar{x} \in Nm$, $\bar{y} \in Nn$:

$\varphi(m+n)e(\bar{x},\bar{y}) = \varphi(n)s(m,n)(e,\bar{x})(\bar{y})$

Simple Explanation

1. Let's say you have a program that takes multiple inputs (m+n inputs)
2. You want to "fix" some of those inputs (m of them) and create a new program that only takes the remaining inputs (n of them)
3. The s-m-n theorem tells us there's a computable way to do this transformation

Example

Suppose you have a program that computes $f(x,y) = x + y$:

- You want to fix $x = 5$ and get a new program that only takes y as input
- The s-m-n theorem guarantees there's a computable function that can generate this new program
- The new program would compute $g(y) = 5 + y$

Key Points

1. The function $s(m,n)$ is:
    - Total (defined for all inputs)
    - Computable (there's an algorithm to compute it)
2. It effectively transforms programs by "hardcoding" some of their parameters
3. This is a powerful tool for proving other results in computability theory

Practical Interpretation

Think of it as a "partial evaluation" of programs - you're taking a program that needs multiple inputs and creating a specialized version that has some inputs pre-filled.

Definition 3.1. An *indexing* of the partial computable functions is an infinite sequence $\varphi_0, \varphi_1, \ldots,$ of partial computable functions that includes all the partial computable functions of one argument (there might be repetitions, this is why we are not using the term enumeration). An indexing is *universal* if it contains the partial computable function φ_{univ} such that

$$\varphi_{univ}(i, x) = \varphi_i(x)$$

for all $i, x \in \mathbb{N}$. An indexing is *acceptable* if it is universal and if there is a total computable function c for composition, such that

$$\varphi_{c(i,j)} = \varphi_i \circ \varphi_j$$

for all $i, j \in \mathbb{N}$.

A very useful property of acceptable indexings is the so-called "*s-m-n Theorem*".

Using the slightly loose notation $\varphi(x_1, \ldots, x_n)$ for $\varphi(\langle x_1, \ldots, x_n \rangle)$, the s-m-n theorem says the following.

Given a function φ considered as having $m + n$ arguments, if we fix the values of the first m arguments and we let the other n arguments vary, we obtain a function ψ of n arguments. Then, the index of ψ depends in a computable fashion upon the index of φ and the first m arguments $x_1, \ldots, x_m$.

We can "pull" the first m arguments of φ into the index of ψ.

Theorem 3.1. *(The "s-m-n Theorem") For any acceptable indexing $\varphi_0, \varphi_1, \ldots$, there is a total computable function s, such that, for all $i, m, n \geq 1$, for all $x_1, \ldots,$ x_m and all $y_1, \ldots, y_n$, we have*

$$\varphi_{s(i,m,x_1,\ldots,x_m)}(y_1, \ldots, y_n) = \varphi_i(x_1, \ldots, x_m, y_1, \ldots, y_n).$$