1. Unbounded Minimalization

Definition

For $f:N^{k+1}\rightarrow N$:

```
h(x̄) = μy.f(x̄,y) = {
    least z such that:
        f(x̄,z) = 0 AND
        f(x̄,z') is defined for all z'<z
    ↑ (undefined) otherwise
}
```

Key Properties

1. Unlike bounded minimalization:
    - Search has no upper bound
    - May never terminate if:
        - No solution exists
        - $f(\bar{x},y)$ is undefined for some y
2. Equivalent to while loops
3. Can produce partial functions even from total functions

Example: Perfect Square Root

```
μy.|x-y²| = {
    √x  if x is a perfect square
    ↑   otherwise
}
```

2. Inverse Functions

Computability of Inverses

For $f:N\rightarrow N$ computable and injective:

```
f⁻¹(x) = {
    y  if f(y)=x
    ↑  if ∄y.f(y)=x
}
```

is computable via:

```
f⁻¹(x) = μy.|f(y)-x|
```

Key Properties

1. If f is total and injective, f^{-1} is computable
2. Domain of f^{-1} is image of f
3. Computability preserved even if f not total

3. Finite Functions

Definition

A function $\theta:N\to N$ is finite if $dom(\theta)$ is finite

Computability

All finite functions are computable. For $\theta=\{(x_1,y_1),\dots,(x_n,y_n)\}$:

```
θ(x) = Σᵢ₌₁ⁿ yᵢ·sg(|x-xᵢ|) + μz.(Πᵢ₌₁ⁿ|x-xᵢ|)
```

4. Partial Recursive Functions (Class R)

Definition

R is the least class containing:

1. Basic Functions:
    - Zero: $z(x) = 0$
    - Successor: $s(x) = x+1$
    - Projections: $U^i_k(x_1,\dots,x_k) = x_i$
2. Closed under:
    - Composition
    - Primitive recursion

- Unbounded minimalization

Fundamental Properties

1. Totalness:
   - Composition of total functions → total
   - Primitive recursion of total functions → total
   - Minimalization may produce partial functions
2. Relationship with URM:
   - R = C (Class of URM-computable functions)
   - Provides alternative characterization of computability

Proof Structure (R = C)

1. R ⊆ C:
   - C is rich (contains basic functions and closed under operations)
   - R is minimal such class
2. C ⊆ R:
   - For f∈C, exists program P computing f
   - Show register contents and instruction counter are in R
   - Use encoding of configurations and program steps

5. Applications

Program Properties

1. Functions computing:
   - Program inputs
   - Program outputs
   - Computation steps

Common Examples

1. Square root function
2. Division with remainder
3. Fibonacci function (using pair encoding)
4. Functions with finite domain

Important Notes

1. Unbounded minimalization is essential for:
   - Expressing while loops
   - Computing partial functions
   - Inverse functions
2. Key Differences from Bounded Operations:
   - May not terminate
   - Can produce partial functions
   - More expressive power
3. Practical Implications:
   - Not all computations guaranteed to halt
   - Need careful handling of undefined cases
   - Balance between expressiveness and totality