12.2. Effective operations on computable functions

The existence of the universal function, together with the *smn* theorem allows us to formalise operations that manipulate programs and derive their effectiveness.

PROPOSITION 12.9 (Effectiveness of product). *There exists a function* $s : \mathbb{N}^2 \to \mathbb{N}$ *total and computable such that for every* $x, y \in \mathbb{N}$

$$\varphi_{s(x,y)} = \varphi_x \cdot \varphi_y$$

PROOF. We define a function $g : \mathbb{N}^3 \to \mathbb{N}$

$$\begin{aligned} g(x, y, z) &= \varphi_x(z) \cdot \varphi_y(z) \\ &= \Psi_U(x, z) \cdot \Psi_U(y, z) \end{aligned}$$

it is computable since it arises as composition of computable functions. By the *smn* theorem there exists $s : \mathbb{N}^2 \to \mathbb{N}$ total computable such that for every x, y, z

$$\varphi_{s(x,y)}(z) = g(x, y, z) = \varphi_x(z) \cdot \varphi_y(z)$$

thus

$$\varphi_{s(x,y)} = \varphi_x \cdot \varphi_y$$

□

PROPOSITION 12.10 (Effectiveness of squaring). *There exists* $k : \mathbb{N} \to \mathbb{N}$ *total and computable such that, for every* $x \in \mathbb{N}$,

$$\varphi_{k(x)} = \varphi_x^2$$

PROOF. $k(x) = s(x, x)$ □

PROPOSITION 12.11 (Effectiveness of primitive recursion). *Recall the notion of primitive recursion*

$$\begin{aligned} h(\vec{x}, 0) &= f(\vec{x}) \\ h(\vec{x}, y + 1) &= g(\vec{x}, y, f(\vec{x}, y)) \end{aligned}$$

We know that if f, g *are computable then* h *is computable. We can derive that there exists* $r : \mathbb{N}^2 \to \mathbb{N}$ *total computable such that, if* $f = \varphi_{e_1}^{(k)}$ *and* $g = \varphi_{e_2}^{(k+2)}$, *then*

$$h = \varphi_{r(e_1,e_2)}^{(k+1)}$$

PROPOSITION 12.12 (Effectiveness of the inverse function). *There exists* $k : \mathbb{N} \to \mathbb{N}$ *total and computable such that*

$$\forall x \in \mathbb{N} \quad \textit{if } \varphi_x \textit{ is injective } \Rightarrow \varphi_{k(x)} = (\varphi_x)^{-1}$$

PROOF. We define a function $g : \mathbb{N}^2 \to \mathbb{N}$

$$\begin{aligned} g(x, y) &= (\varphi_x)^{-1}(y) \\ &= \begin{cases} z & \exists z \text{ s.t. } \varphi_x(z) = y \\ \uparrow & \text{otherwise} \end{cases} \\ &= (\mu\omega \,.\, |\chi_{S(x,(\omega)_1,y,(\omega)_2)} - 1|)_1 \end{aligned}$$

it is computable by minimalisation. Hence, by *smn* theorem, there is a $k : \mathbb{N} \to \mathbb{N}$ total and computable such that for every x, y

$$\varphi_{k(x)}(y) = g(x,y) = (\varphi_x)^{-1}(y)$$

□

Proposition 12.13. *There is a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that, for every x, y*

$$W_{s(x,y)} = W_x \cup W_y$$

Proof. We want $\varphi_{S(x,y)}(z) \downarrow$ iff $\varphi_x(z) \downarrow$ or $\varphi_y(z) \downarrow$. We define a function $g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x,y,z) = \begin{cases} 1 & z \in W_x \vee z \in W_y \\ \uparrow & \text{otherwise} \end{cases}$$

which is computable:

$$g(x,y,z) = \mathbf{1}(\mu\omega\ .\ |\chi_{H(x,z,\omega) \wedge H(y,z,\omega)} - 1|)$$

Hence by *smn* theorem exists $s : \mathbb{N}^2 \to \mathbb{N}$ computable and total such that

$$\varphi_{S(x,y)}(z) = g(x,y,z)$$

□

Proposition 12.14. *There exists a $s : \mathbb{N}^2 \to \mathbb{N}$ computable and total such that*

$$\forall x, y \quad E_{s(x,y)} = E_x \cup E_y$$

Proof. We want the value of $\varphi_{S(x,y)}$ to be the same of the functions $\varphi_x and \varphi_y$. In order to do this, we can simulate φ_x on even numbers and φ_y on odd numbers. We define a function $g : \mathbb{N}^3 \to \mathbb{N}$

$$g(x,y,z) = \begin{cases} \varphi_x(\frac{z}{2}) & \text{if } z \text{ even} \\ \varphi_y(\frac{z-1}{2}) & \text{if } z \text{ odd} \end{cases}$$

computable since

$$\begin{aligned} g(x,y,z) = \quad & (\mu\omega\ . \quad (S(x, z/2, (\omega)_1, (\omega)_2)\ \wedge\ z \text{ even}) \vee \\ & \qquad\qquad (S(y, (z-1)/2, (\omega)_1, (\omega)_2)\ \wedge\ z \text{ odd}))_1 = \\ & (\mu\omega\ . \quad |\max\{\chi_S(x, qt(2,z), (\omega)_1, (\omega)_2)\ \cdot\ \overline{sg}(rm(2,z)), \\ & \qquad\qquad \chi_S(y, qt(2,z), (\omega)_1, (\omega)_2)\ \cdot\ sg(rm(2,z))\} - 1|)_1 \end{aligned}$$

By *smn* theorem there exists $s : \mathbb{N}^2 \to \mathbb{N}$ computable and total such that

$$\varphi_{s(x,y)}(z) = g(x,y,z)$$

for every x, y, z. So

$$\begin{aligned} v \in E_{s(x,y)} &\Leftrightarrow \exists z\ .\ \varphi_{S(x,y)}(z) = g(x,y,z) = v \\ &\Leftrightarrow \exists z\ .\ \begin{cases} z \text{ even and} & \varphi_x(\frac{z}{2}) = v \\ z \text{ odd and} & \varphi_y(\frac{z-1}{2}) = v \end{cases} \\ &\Leftrightarrow \exists z\ .\ \varphi_x(z) = v \wedge \varphi_y(z) = v \Leftrightarrow \omega \in E_x \cup E_y \end{aligned}$$

□

Proposition 12.15. *There is $k : \mathbb{N} \to \mathbb{N}$ computable and total such that $E_{k(x)} = W_x$*

Proof. Define

$$\begin{aligned} g(x,y) &= \begin{cases} y & y \in W_x \\ \uparrow & \text{otherwise} \end{cases} \\ &= \mathbb{1}(\Psi_U(x,y)) \cdot y \end{aligned}$$

it is computable by composition, so by *smn* theorem there exists $k : \mathbb{N} \to \mathbb{N}$ computable and total such that, for every x, y

$$\varphi_{k(x)}(y) = g(x,y)$$

In other words

$$y \in E_{k(x)} \Leftrightarrow \varphi_{k(x)}(y) = y \Leftrightarrow g(x,y) = y \Leftrightarrow y \in W_x$$

□

Proposition 12.16. *Given $f : \mathbb{N} \to \mathbb{N}$ computable, there exists $k : \mathbb{N} \to \mathbb{N}$ computable and total such that, for every x, $W_{k(x)} = f^{-1}(W_x)$*

Proof. Define

$$g(x,y) = \varphi_x(f(y)) = \Psi_U(x, f(y))$$

computable by definition. By the *smn* theorem, there exists $k : \mathbb{N} \to \mathbb{N}$ computable and total such that $\varphi_{k(x)}(y) = g(x,y)$. So

$$\begin{aligned} y \in W_{k(x)} &\Leftrightarrow \varphi_{k(x)}(y) = g(x,y) = \varphi_x(f(y)) \downarrow \\ &\Leftrightarrow f(y) \downarrow \text{ and } f(y) \in W_x \\ &\Leftrightarrow y \in f^{-1}(W_x) \end{aligned}$$

□

Proposition 12.17. *There exists $k : \mathbb{N} \to \mathbb{N}$ computable and total such that if $\varphi_x = \chi_Q$ is the characteristic function of a decidable predicate Q, then $\varphi_{k(x)} = \chi_{\neg Q}$*

Proof. Define

$$g(x,y) = 1 \dot{-} \varphi_x(y) = 1 - \Psi_U(x,y)$$

which is computable by definition. By the *smn* theorem, there exists k computable and total such that

$$g(x,y) = \varphi_{k(x)}$$

In this way, if $\varphi_x = \chi_Q$

$$g(x,y) = 1 - \varphi_x(y) = \varphi_{k(x)}(y) = 1 \Leftrightarrow \varphi_x(y) = 0 \Leftrightarrow \chi_Q(y) = 0$$

therefore

$$\varphi_{k(x)} = \chi_{\neg Q}$$

□