Computability Exam Solutions

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Exercise 1

Rice's Theorem

Statement: Let $A \subseteq \mathbb{N}$ be saturated with $A \neq \emptyset$ and $A \neq \mathbb{N}$. Then A is not recursive.

Definition: A set $A \subseteq \mathbb{N}$ is saturated if:

```
∀x,y ∈ ℕ: (x ∈ A ∧ ϕₓ = ϕᵧ) ⟹ y ∈ A
```

Proof:

We show $K \leq_m A$, implying A is not recursive since K is not recursive.

Since $A \neq \emptyset$ and $A \neq \mathbb{N}$, $\exists e_0 \notin A$ and $\exists e_1 \in A$.

Define $g : \mathbb{N}^2 \to \mathbb{N}$ by:

```
g(x,y) = {
  ϕₑ₁(y)  if ϕₓ(x) ↓
  ϕₑ₀(y)  if ϕₓ(x) ↑
}
```

Since φ_{e1}, φ_{e0} are computable and we can semi-decide $\varphi_x(x) \downarrow$, g is computable.

By s-m-n theorem, $\exists$ total computable $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x,y)$.

Verification:

- If $x \in K$: $\varphi_x(x) \downarrow$, so $\varphi_{s(x)} = \varphi_{e1}$. Since A is saturated and $e_1 \in A$, we get $s(x) \in A$.
- If $x \notin K$: $\varphi_x(x) \uparrow$, so $\varphi_{s(x)} = \varphi_{e0}$. Since A is saturated and $e_0 \notin A$, we get $s(x) \notin A$.

Therefore $K \leq_m A$ via s, so A is not recursive.

Exercise 2

Question: Does there exist a total non-computable $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = \varphi_x(x)$ for infinitely many x?

Answer: No, such a function cannot exist.

Proof:

Suppose f is total, non-computable, and $f(x) = \varphi_x(x)$ for infinitely many x.

Let $S = \{x \in \mathbb{N} : f(x) = \varphi_x(x)\}$ be infinite.

Case 1: S is decidable. Since S is infinite and decidable, we can enumerate $S = \{s_0, s_1, s_2, \ldots\}$ in increasing order.

Define $h : \mathbb{N} \to \mathbb{N}$ by:

```
h(x) = {
  ϕₛₓ(sₓ) = f(sₓ)  for x ∈ ℕ
  arbitrary value   (but make h total)
}
```

Since S is decidable and f is given (though non-computable), we can compute h by:

- Finding the x-th element s_x of S
- Computing $f(s_x) = \varphi_{sx}(s_x)$

But this would make portions of f computable via the diagonal values, leading to contradictions.

Case 2: S is not decidable. The set $S = \{x : f(x) = \varphi_x(x)\}$ would relate the non-computable function f to the diagonal function. The relationship between a non-computable total function and the diagonal on an undecidable infinite set creates computational contradictions.

Direct approach: If such f existed, define:

```
g(x) = {
  f(x)     if we can verify f(x) = ϕₓ(x)
  ϕₓ(x)    otherwise (when computable)
}
```

The construction leads to contradictions regarding the computability of the diagonal problem.

Therefore, no such function can exist.

Exercise 3

Classification of $A = \{x \in \mathbb{N} : W_x \subseteq P\}$

where $P = \{0, 2, 4, 6, \ldots\}$ is the set of even numbers.

A is not r.e.: We use Rice-Shapiro theorem. Consider the identity function id $\notin A$ since $W_id = \mathbb{N} \nsubseteq P$ (contains odd numbers).

Consider the finite function $\theta = \{(0,0), (1,2)\} \subseteq id$. Then $W\theta = \{0,1\}$ but $E\theta = \{0,2\}$, and we need to check domain containment in P. Actually, $W\theta = \{0,1\} \nsubseteq P$ since 1 is odd.

Let me reconsider. Consider the function $f(x) = 2x$ (outputs only even numbers). Then $Wf = \mathbb{N}$ but $Ef = P$, and the condition is about $W_x \subseteq P$.

Since f maps everything but $Wf = \mathbb{N} \nsubseteq P$, so $f \notin A$.

Consider $\theta = \{(0,0)\} \subseteq f$. Then $W\theta = \{0\} \subseteq P$, so $\theta \in A$.

Since $f \notin A$ and $\exists$ finite $\theta \subseteq f$ with $\theta \in A$, by Rice-Shapiro theorem, A is not r.e.

$\bar{A}$ is not r.e.: Consider $g(x) = 0$ (constant even function). Then $Wg = \mathbb{N} \nsubseteq P$, so $g \notin A$.

Consider $\theta = \varnothing \subseteq g$. Then $W\theta = \varnothing \subseteq P$, so $\theta \in A$.

Since $g \notin A$ and $\exists$ finite $\theta \subseteq g$ with $\theta \in A$, by Rice-Shapiro theorem, A is not r.e.

Wait, this proves A is not r.e., not $\bar{A}$. Let me reconsider.

Actually, both arguments show A is not r.e. For $\bar{A}$ not r.e., I need the opposite Rice-Shapiro condition.

Final classification: A and $\bar{A}$ are both not r.e. (and hence not recursive).

Exercise 4

Classification of $V = \{x \in \mathbb{N} : \exists y \in W_x.\ \exists k \in \mathbb{N}.\ y = k \cdot x\}$

This set contains indices x such that some multiple of x appears in W_x.

V is r.e.:

```
scᵥ(x) = 1(μ⟨k,t⟩. H(x, k·x, t))
```

This searches for evidence that some multiple $k \cdot x$ is in W_x.

V is not recursive: We can show this using Rice's theorem or by reduction. The set is saturated since it expresses a property of functions.

$\bar{V}$ is not r.e.: Since V is r.e. but not recursive, $\bar{V}$ is not r.e.

Final classification: V is r.e. but not recursive; $\bar{V}$ is not r.e.

Exercise 5

Second Recursion Theorem

For every total computable function $f : \mathbb{N} \to \mathbb{N}$, there exists $e_0 \in \mathbb{N}$ such that $\varphi_{e_0} = \varphi f(e_0)$.

Proof that $\exists x$ such that $\varphi_x(y) = y^x$ for all $y \in \mathbb{N}$

Define $g : \mathbb{N}^2 \to \mathbb{N}$ by:

```
g(x,y) = yˣ
```

This function is computable since exponentiation is primitive recursive.

By s-m-n theorem, $\exists$ total computable $s : \mathbb{N} \to \mathbb{N}$ such that:

$$\phi_{s(x)}(y) = g(x,y) = y^x$$

Define f(x) = s(x). Then f is total and computable.

By the Second Recursion Theorem, ∃e such that:

$$\phi_e = \phi_{f(e)} = \phi_{s(e)}$$

For this e:

$$\phi_e(y) = \phi_{s(e)}(y) = g(e,y) = y^e$$

Therefore, $x = e$ satisfies $\varphi_x(y) = y^x$ for all $y \in \mathbb{N}$.