Key Truth: Reduction Works on BOTH Saturated and Non-Saturated Sets

You can use reduction on ANY set - the saturation property doesn't determine whether reduction works.

Method Selection Rules:

For NON-SATURATED Sets:

- **Must use direct reduction** ($K \leq_m A$ or $\overline{K} \leq_m A$)
- **Cannot use Rice's theorem** (only applies to saturated sets)
- **Cannot use Rice-Shapiro** (only applies to saturated sets)

For SATURATED Sets:

- **Can use Rice's theorem** (easiest for non-recursiveness)
- **Can use Rice-Shapiro** (for r.e. classification)
- **Can still use direct reduction** (always works, but more work)

Examples from Your Materials:

Non-Saturated Set (must use reduction):

- $A = \{x \mid x^2 \in Ex\}$ ← Your example
- $K = \{x \mid \varphi x(x) \downarrow\}$ ← Classic halting set
- {x | program x has length ≤ 10}

Saturated Set (multiple options):

- $A = \{x \mid \varphi x \text{ total}\}$
  - **Easy way**: Rice's theorem → not recursive
  - **Hard way**: Direct reduction $K \leq_m A$

Strategic Decision Tree:

```
Is the set saturated?
├─ YES → Use Rice/Rice-Shapiro (easier!)
│   └─ Can also use reduction (but why make it hard?)
└─ NO → Must use direct reduction
    └─ Only choice available
```

The Pattern Recognition:

- **Index-dependent conditions** (like x^2, $x \in W_x$, $\varphi_x(x) = x$) → Usually non-saturated → Reduction required
- **Pure function properties** (like φ_x total, W_x infinite) → Usually saturated → Rice theorems available

Bottom line: Reduction is the universal tool that works everywhere, but Rice theorems are shortcuts that only work on saturated sets. When you see non-saturated sets, reduction becomes your only option.

Strategic Method Selection:

Primary Decision Rule:

```
1. Check if set is saturated
   ├─ NOT saturated → MUST use reduction (only option)
   └─ IS saturated → Use Rice/Rice-Shapiro (faster & easier)
```

When to Use Reduction on Saturated Sets:

Use reduction only when Rice theorems don't fit:

1. **Rice gives wrong conclusion**
   - Example: Need to prove A is r.e., but Rice only tells you "not recursive"
2. **Rice-Shapiro is too complex**
   - Sometimes direct reduction $K \leq_m A$ is cleaner than finding finite subfunctions
3. **Need specific r.e. status**
   - Want to show A is r.e. but $\bar{A}$ is not r.e.
   - Rice-Shapiro might require tricky finite subfunction analysis
4. **Emergency backup**
   - Can't remember Rice theorem conditions precisely
   - Reduction construction is clearer to you

Exam Time Management:

DON'T use reduction as double-check because:

- Wastes precious exam time (5-10 minutes)
- Rice theorems are faster and less error-prone
- Double-checking doesn't add points

DO use this priority:

1. **Non-saturated?** → Reduction (forced)
2. **Saturated?** → Rice/Rice-Shapiro (default choice)
3. **Rice doesn't work?** → Fall back to reduction

Example Decision Process:

Set A = {x | φx total}

- Saturated? YES ✓
- Rice applicable? YES → A ≠ ∅, A ≠ $\mathbb{N}$, non-trivial semantic property
- Conclusion: Not recursive (done in 30 seconds)
- **No need for reduction!**

Set B = {x | x² ∈ Ex}

- Saturated? NO ✗ (index-dependent)
- Must use reduction: K ≤m B
- **No choice but reduction**

Bottom Line: Use reduction when forced (non-saturated) or when Rice methods fail, but not as routine double-checking. Save time for other problems!