In computability theory, there are several important functions that frequently appear in proofs and examples, particularly when working with Rice's theorem and other fundamental results. Let's examine some of these notable functions and their significance.

The Identity Function (id)

The identity function is defined as:

```
id(x) = x for all x ∈ N
```

This function plays a crucial role in many proofs, particularly when working with saturated sets. It has several important properties:

1. It is total and computable
2. It has infinitely many indices (programs that compute it)
3. It is often used as a "reference point" when proving properties about sets of computable functions

In the context of Rice's theorem, the identity function frequently appears in proofs involving saturated sets. For example, when proving a set A is not recursively enumerable, we might show that id $\notin$ A but some finite subfunction $\theta \subseteq$ id belongs to A.

The Always Undefined Function (∅ or H)

The always undefined function, often denoted as ∅ or H, is defined as:

```
∅(x)↑ for all x ∈ N
```

This function has several crucial properties:

1. It is partial computable
2. Its domain is empty: dom(∅) = ∅
3. It is a subfunction of every function
4. It has infinitely many indices

The always undefined function is particularly useful in proofs involving Rice-Shapiro's theorem, where we often need to construct finite subfunctions. Since ∅ is a subfunction of every function, it provides a convenient starting point for such constructions.

Role in Rice's Theorem

These functions are instrumental in proving Rice's theorem, which states that any non-trivial saturated set is not recursive. The proof typically proceeds by:

1. Taking a set A that is saturated and proper ($A \neq \emptyset$ and $A \neq N$)
2. Finding $e_0 \in A$ and $e_1 \notin A$
3. Constructing a reduction from the halting problem

In many cases, we can choose:

- e_0 to be an index of the identity function
- e_1 to be an index of the always undefined function

This choice is particularly useful because:

- We know these functions exist and are computable
- They have very different behaviors (total vs. nowhere defined)
- They have clear and well-understood properties

Examples in Practice

Consider proving that the set $T = \{x \mid \varphi_x \text{ is total}\}$ is not recursive using Rice's theorem:

1. T is saturated (depends only on the function, not the specific index)
2. $id \in T$ (the identity function is total)
3. $\emptyset \notin T$ (the always undefined function is not total)
4. Therefore T is non-trivial and by Rice's theorem, not recursive

Similarly, for Rice-Shapiro's theorem, these functions help us construct counterexamples. For instance, when showing a set A is not r.e., we might show that $id \notin A$ but $\emptyset \subseteq id$ and $\emptyset \in A$.