Definition 13.5. Let $A, B \subseteq \mathbb{N}$. We say that the problem $x \in A$ *reduces* to the problem $x \in B$ (or simply that A reduces to B), written $A \leqslant_m B$ if there exists a function $f : \mathbb{N} \to \mathbb{N}$ computable and total such that, for every $x \in \mathbb{N}$

$$x \in A \quad \Leftrightarrow \quad f(x) \in B$$

In this case, we say that f is the *reduction function.*

Observation 13.6. Let $A, B \subseteq \mathbb{N}$ such that $A \leqslant_m B$ then

1 if B is recursive, then A is recursive

2 of A is not recursive, then B is not recursive

Proof. Simply observe that $\chi_A = \chi_B \circ f$. □

We know that $K = \{x \mid x \in W_x\}$ is not recursive. We next observe see how the non-recursiveness of other sets can be proven by reduction to K.

Example 13.7. $K \leqslant_m T = \{x \mid \varphi_x \text{ total}\}$

Proof. We prove that there exists $s : \mathbb{N} \to \mathbb{N}$ computable and total such that $x \in k \Leftrightarrow s(x) \in T$. In other words

$$x \in W_x \Leftrightarrow \varphi_{f(x)} \text{ is total}$$

To do so, we can define

$$g(x, y) = \begin{cases} 1 & x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

which is computable, since

$$g(x, y) = \mathbf{1}(\varphi_x(x)) = \mathbf{1}(\Psi_U(x, x))$$

Then, by the *smn*-theorem we have that there exists $s : \mathbb{N} \to \mathbb{N}$ computable and total such that

$$\varphi_{s(x)}(y) = g(x, y)$$

and

$$x \in K \Rightarrow x \in W_x \Rightarrow \forall y \ \varphi_{s(x)}(y) = g(x, y) = 1 \Rightarrow \varphi_{s(x)} \text{ total } \Rightarrow s(x) \in T$$

$$x \notin K \Rightarrow x \notin W_x \Rightarrow \forall y \ \varphi_{s(x)}(y) = g(x, y) \uparrow \Rightarrow \varphi_{s(x)} \text{ not total } \Rightarrow s(x) \notin T$$

□

Example 13.8 (Input problem). For every $n \in \mathbb{N}$

$$A_n = \{x \mid \varphi_x(n) \downarrow\}$$

is not recursive.

Proof. We will prove that $K \leqslant A_n$. We have to define a function f s.t.

$$x \in K \Leftrightarrow f(x) \in A_n$$

i.e., $x \in W_x \Leftrightarrow \varphi_{f(x)}(n) \downarrow$.

Define

$$g(x,y) = \begin{cases} 1 & x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$
$$= \mathbf{1}(\Psi_U(x,x))$$

Function g is computable, and thus by the *smn*-theorem, there exists $f : \mathbb{N} \to \mathbb{N}$ computable and total such that $g(x,y) = \varphi_{f(x)}(y)$. It is now easy to show that s is the reduction function, i.e.,

$$x \in K \Rightarrow f(x) \in A_n$$
$$x \notin K \Rightarrow f(x) \notin A_n$$

$\square$

Example 13.9 (The output problem). For every $n \in \mathbb{N}$, $B_n = \{x \mid n \in E_x\}$ is not recursive

Proof. We show that $K \leqslant_m B_n$. Define the function

$$g(x,y) = \begin{cases} n & x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$
$$= n \cdot \mathbf{1}(\Psi_U(x,x))$$

Observe that g is computable. Hence by the *smn*-theorem there exists a function $s : \mathbb{N} \to \mathbb{N}$ such that

$$\forall x, y \quad g(x,y) = \varphi_{s(x)}(y)$$

It is now easy to show that s is the reduction function, i.e.,

$$x \in K \Rightarrow s(x) \in B_n$$
$$x \notin K \Rightarrow s(x) \notin B_n$$

$\square$

Observation 13.10. Let $A, B \subseteq \mathbb{N}$ with $A \leqslant_m B$ through an injective reduction function $f : \mathbb{N} \to \mathbb{N}$ (total and computable). One could think that, since f^{-1} is computable, then also $B \leqslant_m A$. This is clearly not the case since f^{-1} is not total and thus it reduces A to a "subproblem" of B (which typically have no clear relation with B).