1. Structure Theorem for Semi-Decidable Predicates

1.1 Statement

A predicate $P(\vec{x})$ is semi-decidable if and only if there exists a decidable predicate $Q(t, \vec{x})$ such that:

```
P(\vec{x}) ⟺ ∃t.Q(t,\vec{x})
```

1.2 Application Template

```
To prove P is semi-decidable:

1. Find decidable Q where P(\vec{x}) = ∃t.Q(t,\vec{x})
2. Show Q is decidable by providing χ_Q
3. Conclude P is semi-decidable via:
   sc_P(\vec{x}) = 1(μt.|χ_Q(t,\vec{x}) - 1|)
```

1.3 Common Examples

```
1. K is semi-decidable:
   x ∈ K ⟺ ∃t.H(x,x,t)
   where H is decidable

2. W_e is semi-decidable:
   x ∈ W_e ⟺ ∃t.H(e,x,t)
   where H is decidable
```

2. Projection Theorem

2.1 Statement

If $P(x, \vec{y})$ is semi-decidable, then $\exists x. P(x, \vec{y})$ is also semi-decidable.

2.2 Template for Application

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To use projection theorem:

1. Start with semi-decidable P(x,\vec{y})
2. By structure theorem:
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   P(x,\vec{y}) = ∃t.Q(t,x,\vec{y})
   for decidable Q
3. Then ∃x.P(x,\vec{y}) = ∃w.Q((w)_1,(w)_2,\vec{y})
4. Conclude ∃x.P(x,\vec{y}) is semi-decidable
```

2.3 Proving Semi-Decidability

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To prove R(\vec{y}) is semi-decidable:

1. Express R as existential quantification:
   R(\vec{y}) = ∃x.P(x,\vec{y})
2. Show P is semi-decidable
3. Apply projection theorem
```

3. Closure Properties

3.1 Semi-Decidable Predicates are Closed Under

1. Conjunction (∧):

```
If P, Q semi-decidable then P ∧ Q semi-decidable:
P(\vec{x}) = ∃t.R(t,\vec{x})
Q(\vec{x}) = ∃s.S(s,\vec{x})
P ∧ Q = ∃w.(R((w)_1,\vec{x}) ∧ S((w)_2,\vec{x}))
```

2. Disjunction (∨):

```
If P, Q semi-decidable then P ∨ Q semi-decidable:
Similar to conjunction
```

3. Existential quantification (∃):

```
By projection theorem
```

3.2 Not Closed Under

1. Negation (¬):

```
Example: K is semi-decidable but K̄ is not
```

2. Universal quantification (∀):

```
Example: Q(x,t) decidable doesn't imply
∀t.Q(x,t) is semi-decidable
```

4. Exercise Strategies

4.1 Proving Semi-Decidability

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Method 1: Structure Theorem
1. Find decidable Q(t,\vec{x})
2. Show P(\vec{x}) = ∃t.Q(t,\vec{x})
3. Conclude P semi-decidable

Method 2: Closure Properties
1. Express P using ∧, ∨, ∃
2. Show components semi-decidable
3. Use closure properties
```

4.2 Proving Not Semi-Decidable

```
1. Assume semi-decidable
2. Use structure theorem
3. Derive contradiction with known results
   (often about K or K̄)
```

5. Common Problem Types

5.1 Combination Problems

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Given: P, Q semi-decidable
Prove: R(x) = P(x) ∧ ∃y.Q(x,y) semi-decidable

Solution:
1. P semi-decidable ⟹ P = ∃t.P'(t,x)
2. Q semi-decidable ⟹ Q = ∃s.Q'(s,x,y)
3. Use closure under ∧ and ∃
```

5.2 Reduction Problems

```
Show: if A semi-decidable then B semi-decidable

1. Express B using A and computable functions
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```
2. Use closure properties
3. Conclude B semi-decidable
```

5.3 Impossibility Problems

```
Show: P not semi-decidable

1. Assume P semi-decidable
2. P = ∃t.Q(t,\vec{x}) for decidable Q
3. Derive contradiction (e.g., with K̄)
```