Core Functions and Notation

Domain and Image

- Wx: domain of φx (all inputs where function is defined)
- Ex: image/codomain of φx (all outputs function can produce)
- cod(f): alternate notation for codomain
- img(f): alternate notation for image

Basic Functions

1. Monus (Truncated Subtraction)

```
x −̇ y = {
    x − y  if x ≥ y
    0      otherwise
}
```

Key point: Always yields non-negative results, defined only for natural numbers

Characteristic Functions

For set A ⊆ N:

```
χA(x) = {
    1  if x ∈ A
    0  if x ∉ A
}
```

Function Cases

```
div(x,y) = {
    1  if x divides y
    0  otherwise
}
```

Function Properties

Totality

- Total function: defined for all possible inputs
- Partial function: defined only for some inputs
- Domain: dom(f) ⊆ N

Special Notation

- φx: primitive recursive k-ary function from x-th step of enumeration
- θ: typically denotes a finite subfunction
- ↓: function converges/is defined (e.g., f(x)↓)
- ↑: function diverges/is undefined (e.g., f(x)↑)

Common Functions Used in Examples

Arithmetic Functions

```
sum(x,y) = x + y
product(x,y) = x * y
div(x,y) = "x divides y"
```

Bounded Functions

For a computable $f:N^{k+1}\to N$:

```
Σz<y f(x̄,z)  // bounded sum
Πz<y f(x̄,z)  // bounded product
```

Sign Functions

```
sg(x) = {     // sign function
    0  if x = 0
    1  if x > 0
}

sḡ(x) = {     // complement sign
    1  if x = 0
    0  if x > 0
}
```

Function Composition Types

Generalized Composition

For $f:N^k \to N$ and $g_1,\ldots,g_k:N^n \to N$:

```
h(x̄) = f(g₁(x̄),...,gₖ(x̄))
```

Defined only if all component functions are defined

Primitive Recursion

For $f:N^k \to N$ and $g:N^{k+2} \to N$:

```
h(x̄,0) = f(x̄)
h(x̄,y+1) = g(x̄,y,h(x̄,y))
```

Bounded Minimalization

```
μz<y.f(x̄,z) = {
    min{z<y : f(x̄,z)=0}  if exists
    y                     otherwise
}
```

Vector Notation

- $\bar{x}$: vector of variables $(x_1,\ldots,x_k)$
- N^k: k-dimensional natural numbers

Important Properties to Remember

1. All arithmetic functions shown are primitive recursive
2. Bounded operations always terminate
3. Characteristic functions must be total
4. When composing functions:
    - All subfunctions must be defined for result to be defined
    - Order of evaluation matters for partial functions