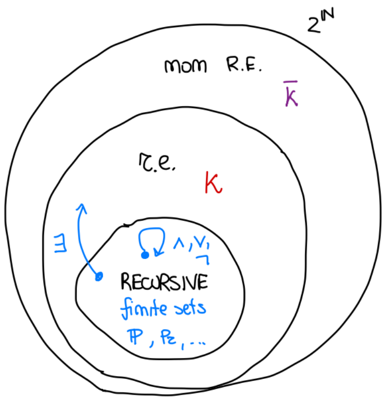
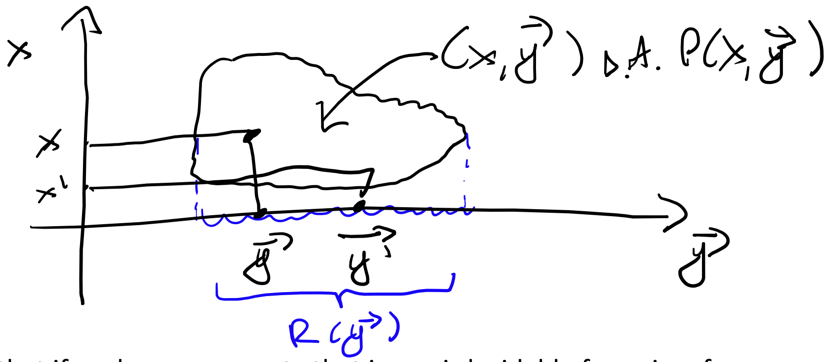
16.4 Projection Theorem

Definition (projection theorem) – closure by existential quantification



Let $P(x, \vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable

Then $R(\vec{y}) \equiv \exists x. P(x, \vec{y})$ is semi-decidable



In essence, the projection theorem tells us that if we have a property that is semi-decidable for pairs of numbers, then we can define another property about the second part of those pairs, and it will also be semi-decidable. It establishes a connection between the semi-decidability of properties involving pairs and the semi-decidability of properties involving only one part of those pairs.

It also shows $\mathcal{RE}$ class is shown with respect to existential quantification.

Proof

Let $P(x, \vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable. Hence, by structure theorem, there is $Q(t, x, \vec{y}) \subseteq \mathbb{N}^{k+2}$ decidable s.t.

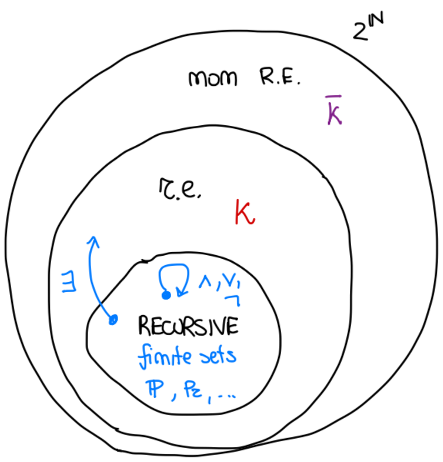
$$P(x, \vec{y}) \equiv \exists t. Q(t, x, \vec{y})$$

Now:

$$\boxed{R(\vec{y}) \equiv} \exists x. P(x, \vec{y}) \equiv \exists x. \exists t. Q(t, x, \vec{y})$$

$$\boxed{\equiv \exists w. \underbrace{Q((w)_1, (w)_2, \vec{y})}_{\text{decidable}}}$$

Hence R is the existential quantification of a decidable predicate $\Rightarrow$ by structure theorem, it is semi-decidable.



<u>Theorem</u> (Closure under conjunction/disjunction – and/or)

Let $P(\vec{x}), Q(\vec{x}) \subseteq \mathbb{N}^k$ semi-decidable predicates. Then:

1) $P(\vec{x}) \wedge Q(\vec{x})$

semi-decidable

2) $P(\vec{x}) \vee Q(\vec{x})$

<u>Proof</u>

Since $P(\vec{x}), Q(\vec{x})$ are semi-decidable, by structure theorem, there are two decidable predicates such that:

$$P(\vec{x}) \equiv \exists t. P'(t, \vec{x})$$

$$Q(\vec{x}) \equiv \exists t. Q'(t, \vec{x})$$

with $P'(t, \vec{x}), Q'(t, \vec{x})$ decidable

1) $P(\vec{x}) \wedge Q(\vec{x}) \equiv \exists t. P'(t, \vec{x}) \wedge \exists t. Q'(t, \vec{x})$

$$\equiv \exists w. \underbrace{\left(P'((w)_1, \vec{x}) \wedge Q'((w)_2, \vec{x})\right)}_{\text{decidable}}$$

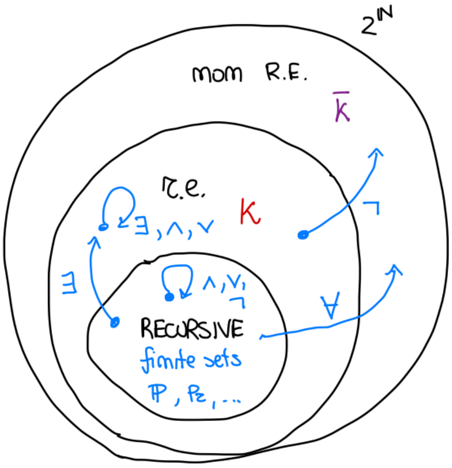
(here, the projection theorem was used, thanks to structure theorem and minimalisation over decidable predicates)

Hence, by the structure theorem, $P(\vec{x}) \wedge Q(\vec{x})$ is semi-decidable.

2) $P(\vec{x}) \vee Q(\vec{x}) \equiv \exists t. P'(t, \vec{x}) \vee \exists t. Q'(t, \vec{x})$

$$\equiv \exists t. \underbrace{\left(P'(t, \vec{x}) \wedge Q'(t, \vec{x})\right)}_{\text{decidable}}$$

Hence, by the structure theorem, $P(\vec{x}) \vee Q(\vec{x})$ is semi-decidable.



* <u>Negation</u> ?

$$Q(x) \equiv "x \in K" \equiv "\phi_x(x) \downarrow"$$

semi-decidable

$$\neg Q(x) \equiv "x \notin K" \equiv "\phi_x(x) \uparrow"$$

not semi-decidable

<u>Theorem</u> (Universal quantification)

$R(t, x) \equiv \neg H(x, x, t)$ decidable

"$x \in \overline{K}$" $\equiv \forall t. R(t, x) \equiv \forall t. \neg H(x, x, t)$ non semi-decidable

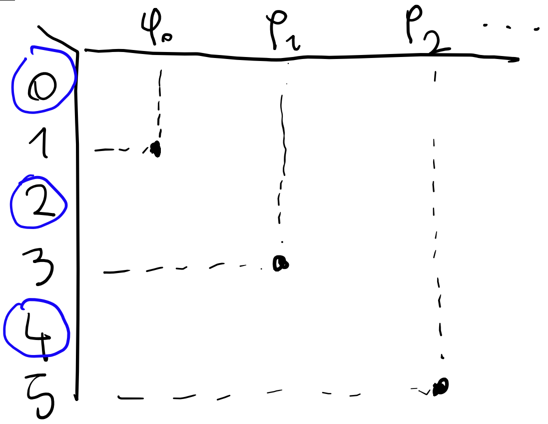
This means that the set of semi-decidable predicates is closed under $\vee, \wedge, \exists$ but not under $\forall$ and $\neg$

- Universal quantification is mangy to deal with because even if a decidable predicate it is universally quantified can become non-semi-decidable. Intuitively this is true because it is indefinite to go and test a predicate on infinite values.
- This is essentially saying that there exists a property R involving universal quantification over terms t and a variable x s.t. R is decidable but you universally quantity over t in the context of $\neg H(x, x, t)$, the resulting property is non-semi-decidable, indicating that determining membership in the complement of set K is not always computationally possible.

16.5 Other exercises from lessons

<u>Exercise</u>: Define a function total and non-computable $f: \mathbb{N} \to \mathbb{N}\ s.t. f(x) = x$ on infinitely many $x \in \mathbb{N}$

<u>1st idea</u>



$$f(x) = \begin{cases} x, & if\ x\ is\ even \\ \phi_{\frac{x-1}{2}}(x) + 1, & if\ x\ is\ odd\ and\ \phi_{\frac{x-1}{2}}(x) \downarrow \\ 0, & if\ x\ is\ odd\ and\ \phi_{\frac{x-1}{2}}(x) \uparrow \end{cases}$$

- f total
- $f(x) = x \quad \forall x$ even (infinite set)
- f not computable (total and $\neq$ from all total computable functions) ($\forall x\ if\ \phi_x$ is total, $f(2x + 1) = \phi_x(2x + 1) + 1 \neq \phi_x(2x + 1)$)

<u>2nd idea</u>

$$f(x) = \begin{cases} \phi_x(x) + 1, & \phi_x(x) \downarrow \\ x, & \phi_x(x) \uparrow \end{cases}$$

- total
- not computable ($\forall x$ if ϕ_x is total, $f(x) = \phi_x(x) + 1 \neq \phi_x(x)$), hence f is different from all total computable functions
- $f(x) = x, \forall x \in \overline{K}$ ($\overline{K}$ is infinite, otherwise it would be recursive and so it will be computable)