22.4 Decidability and Semidecidability

Note: this section requires knowing or remembering at least structure/projection theorem and the definition of semidecidable/decidable/knowing the implications of quantification.

Exercise 4.1. Prove the "structure theorem" of semidecidable predicates, i.e., show that a predicate $P(\vec{x})$ is semidecidable if and only if there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y.\ Q(\vec{x}, y)$.

($\Rightarrow$) Let $P(\vec{x}) \subseteq \mathbb{N}^k$ be semi-decidable

$$sc_P(\vec{x}) = \begin{cases} 1, & if\ P(\vec{x}) \\ \uparrow, & otherwise \end{cases} \quad is\ computable$$

i.e. there is $e \in \mathbb{N}\ s.t.\ sc_P = \phi_e^{(k)}$

$$\text{Observe } P(\vec{x})\ iff\ sc_P(\vec{x}) = 1$$

$$iff\ sc_P(\vec{x}) \downarrow$$

$$iff\ P_e(\vec{x}) \downarrow$$

$$iff\ \exists t. H^{(k)}(e, \vec{x}, t)$$

If we let $Q(t, \vec{x}) = H^{(k)}(e, \vec{x}, t)$ decidable and $P(\vec{x}) \equiv \exists t. Q(t, \vec{x})$

($\Leftarrow$) We assume $P(\vec{x}) \equiv \exists t. Q(t, \vec{x})$ with $Q(t, \vec{x})$ decidable

$$sc_P(\vec{x}) = \begin{cases} 1, & if P(\vec{x}) \Leftrightarrow \exists t. Q(t, \vec{x}) \Leftrightarrow \exists t. \chi_Q(t, \vec{x}) = 1 \\ \uparrow, & otherwise \end{cases}$$

$$= \mathbf{1}\big(\underbrace{\mu t. |\chi_Q(t, \vec{x}) - 1|}_{\substack{t \text{ s.t. } Q(t,\vec{x}) \text{ if it exists} \\ \uparrow \text{ otherwise}}}\big)$$

Exercise 4.2. Prove the "projection theorem", i.e., show that if the predicate $P(x, \vec{y})$ is semidecidable then also $\exists x. P(x, \vec{y})$ is semi-decidable. Does the converse implication hold? Is it the case that if $P(x, \vec{y})$ is decidable then also $\exists x. P(x, \vec{y})$ is decidable? Give a proof or a counterexample.

Proof (Exercise present inside 2017-01-24 exam)

Let $P(x, \vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable. Hence, by structure theorem, there is $Q(t, x, \vec{y}) \subseteq \mathbb{N}^{k+2}$ decidable s.t.

$$P(x, \vec{y}) \equiv \exists t. Q(t, x, \vec{y})$$

Now

$$\boxed{R(\vec{y}) \equiv} \exists x. P(x, \vec{y}) \equiv \exists x. \exists t. Q(t, x, \vec{y})$$

$$\boxed{\equiv \exists w. \underbrace{Q((w)_1, (w)_2, \vec{y})}_{\text{decidable}}}$$

Hence R is the existential quantification of a decidable predicate ⇒ by structure theorem, it is semi-decidable.

Solution: No, the converse is false. Consider, for instance, $P(x, y) = (y = 2x) \wedge (y \notin W_x)$ (or, simply, $P(x, y) = x \notin W_x$), which is not semi-decidable. The existentially quantified version is constant, hence decidable.

Also the second claim is false. Take for instance $P(x, y) = H(y, y, x)$ which is decidable, while $\exists x. P(x, y) \equiv y \in K$ is only semi-decidable, but not decidable. □

Exercise (2015-07-16-solved)

Show that a predicate $P(x, \overrightarrow{y})$ is semidecidable, then $\exists x. P(x, \vec{y})$ is semidecidable. Does the converse hold? Show it or write a counterexample.

Solution

The first one refers to the projection theorem, defined also here. Observe instead that the converse implication is false. Consider, for example, the predicate $P(x, y) = x \in W_x$, which is not semi-decidable.

The predicate obtained through existential quantification $Q(y) = \exists x. P(x, y)$ is consistently true or false (although not relevant to the proof, note that since $\overline{K}$ is nonempty, the predicate $Q(y)$ is consistently true), thus decidable.

As a less "degenerate" example, one may consider $P(x, y) = (y > x) \wedge (y \notin W_x)$ and the quantification $Q(y) = \exists x. (y > x) \wedge (y \notin W_x)$. In this case, note that with $e_0 \in N$, an index for the always indefinite function, we have $Q(y)$ is true for every $y > e_0$, thanks to which $Q(y)$ is decidable.

Exercise (2022-06-17)

c. Show that if predicate $Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ is semi-decidable then also $P(\vec{x}) = \exists y. Q(\vec{x}, y)$ is semi-decidable (do not assume structure and projection theorems). Does the converse hold, i.e., is it the case that if $P(\vec{x}) = \exists y. Q(\vec{x}, y)$ is semi-decidable then $Q(\vec{x}, y)$ is semi-decidable? Provide a proof or a counterexample.

Solution

3. Let $Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ be semi-decidable. Then the semi-characteristic function $sc_Q : \mathbb{N}^{k+1} \to \mathbb{N}$ is computable. Let $e \in \mathbb{N}$ be such that $sc_Q = \varphi_e^{(k+1)}$.

   Then $Q(\vec{x}, y)$ holds iff $\varphi_e^{(k+1)}(\vec{x}, y) = 1$ iff $\varphi_e^{(k+1)}(\vec{x}, y) \downarrow$ iff $\exists t. H^{(k+1)}(e, (\vec{x}, y), t)$.

   Therefore $Q(\vec{x}, y) \equiv \exists t. H^{(k+1)}(e, (\vec{x}, y), t)$ and thus

$$P(\vec{x}) \equiv \exists y. Q(\vec{x}, y) \equiv \exists y. \exists t. H^{(k+1)}(e, (\vec{x}, y), t) \equiv \exists w. H^{(k+1)}(e, (\vec{x}, (w)_1), (w)_2)$$

Therefore $sc_P(\vec{x}) = \mathbf{1}(\mu w. |\chi_{H^{(k+1)}}(e, (\vec{x}, (w)_1)), (w)_2) - 1|)$ is computable, and thus $P(\vec{x})$ is semi-decidable.

The converse implication does not hold. For instance, consider the predicate $Q(x, y) \equiv$ "$\phi_y(x) \uparrow$ ". Then $P(x) \equiv \exists y. Q(x, y) \equiv \exists y. \phi_y(x) \uparrow$ is always true, hence decidable. In fact, if e_0 is an index for the always undefined function, for $y = e_0$ clearly $Q(x, y)$ for every $x \in \mathbb{N}$. Instead $Q(x, y) = \phi_y(x) \uparrow$ is not semi-decidable (it is negation of the halting predicate, which is semi-decidable but not decidable).

<u>Exercise</u> (30-06-2020)

Given two functions $f, g: \mathbb{N} \rightarrow \mathbb{N}$ with f total, define predicate $Q_{f_g}(x) =$ "$f(x) = g(x)$". Show that if f and g are computable, then Q_{f_g} is semidecidable. Does the converse hold, so if $Q_{f\,g}$ is semidecidable, can we deduce f and g are computable?

<u>Solution</u>

Let f, g be computable functions. Let $e_1, e_2 \in \mathbb{N}\ s.t. f = \phi_{e_1}\ and\ g = \phi_{e_2}$.

Then $sc_{f_g} = \mathbf{1}(\mu w. |f(x) - g(x)|$ is computable, hence Q_{f_g} is semidecidable.

If Q_{f_g} is semidecidable and let e be an index of semicharacteristic function of Q, namely $\phi_e = sc_{Q_{f_g}}$

We have $f(x) = (\mu w. H(e, x, (w)_1, (w)_2) \vee H(e, y, (w)_1, (w)_3)$ which shows f and g are computable.

<u>Exercise</u> (2023-02-01)

c. Show that if predicate $Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ is semi-decidable then also $P(\vec{x}) = \exists y. Q(\vec{x}, y)$ is semi-decidable (do not assume structure and projection theorems). Does the converse hold, i.e., is it the case that if $P(\vec{x}) = \exists y. Q(\vec{x}, y)$ is semi-decidable then $Q(\vec{x}, y)$ is semi-decidable? Provide a proof or a counterexample.

<u>Solution</u>

Let $Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ be semi-decidable. Then the semi-characteristic function sc_Q : $\mathbb{N}^{k+1} \rightarrow \mathbb{N}$ is computable. Let $e \in \mathbb{N}$ be such that $sc_Q = \varphi_e^{(k+1)}$.

Then $Q(\vec{x}, y)$ holds iff $\varphi_e^{(k+1)}(\vec{x}, y) = 1$ iff $\varphi_e^{(k+1)}(\vec{x}, y) \downarrow$ iff $\exists t. H^{(k+1)}(e, (\vec{x}, y), t)$.

Therefore $Q(\vec{x}, y) \equiv \exists t.\, H^{(k+1)}(e, (\vec{x}, y), t)$ and thus

$$P(\vec{x}) \equiv \exists y.\, Q(\vec{x}, y) \equiv \exists y. \exists t.\, H^{(k+1)}(e, (\vec{x}, y), t) \equiv \exists w.\, H^{(k+1)}(e, (\vec{x}, (w)_1), (w)_2)$$

Therefore $sc_P(\vec{x}) = \mathbf{1}(\mu w.\ |\chi_{H^{(k+1)}}(e, (\vec{x}, (w)_1)), (w)_2) - 1|)$ is computable, and thus $P(\vec{x})$ is semi-decidable.

The converse implication does not hold. For instance, consider the predicate $Q(x, y) \equiv$ "$\phi_y(x) \uparrow$". Then $P(x) \equiv \exists y. Q(x, y) \equiv \exists y. \phi_y(x) \uparrow$ is always true, hence decidable. In fact, if e_0 is an index for the always undefined function, for $y = e_0$ clearly $Q(x, y)$ for every $x \in \mathbb{N}$. Instead $Q(x, y) = \phi_y(x) \uparrow$ is not semi-decidable (it is negation of the halting predicate, which is semi-decidable but not decidable).