Computability Exam Solutions

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Exercise 1

a. Definition of many-one reducibility

Given sets $A, B \subseteq \mathbb{N}$, we say that $A \leq_m B$ (A is many-one reducible to B) if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$:

```
x ∈ A ⟺ f(x) ∈ B
```

b. Proof: If A is not r.e. and $A \leq_m B$ then B is not r.e.

Assume $A \leq_m B$ via reduction function $f : \mathbb{N} \to \mathbb{N}$ (total and computable). Suppose, by contradiction, that B is r.e. Then the semi-characteristic function scB is computable.

We can define the semi-characteristic function of A as:

```
scA(x) = scB(f(x))
```

Since scB is computable and f is computable, their composition $scA = scB \circ f$ is computable. Therefore A would be r.e., contradicting our assumption that A is not r.e.

Hence, B is not r.e.

c. Question: For all sets $A, B \subseteq \mathbb{N}$, does $A \leq_m A \cup B$ hold?

Answer: No. Counterexample:

Let $A = \emptyset$ and $B = \{0\}$. Then $A \cup B = \{0\}$.

For $A \leq_m A \cup B$ to hold, there must exist a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that:

```
x ∈ A ⟺ f(x) ∈ A ∪ B
```

Since $A = \emptyset$, this becomes:

```
False ⟺ f(x) ∈ {0}
```

This means $f(x) \notin \{0\}$ for all $x \in \mathbb{N}$, i.e., $f(x) \neq 0$ for all x. However, since $A \cup B = \{0\}$ has only one element, any total function $f : \mathbb{N} \to \mathbb{N}$ mapping to $\mathbb{N} \setminus \{0\}$ cannot serve as a reduction function to $\{0\}$.

Therefore, $A \leq_m A \cup B$ does not hold in general.

Exercise 2

Definition of primitive recursive functions

The class PR of primitive recursive functions is the smallest class of functions $PR \subseteq \bigcup_k(\mathbb{N}^k \to \mathbb{N})$ that:

1. **Contains the basic functions:**
   - Zero function: zero(x) = 0
   - Successor function: succ(x) = x + 1
   - Projection functions: $\pi_i^k(x_1,\ldots,x_k) = x_i$ for $1 \leq i \leq k$
2. **Is closed under composition:** If $g_1,\ldots,g_m \in PR$ and $h \in PR$, then $f \in PR$ where $f(\vec{x}) = h(g_1(\vec{x}),\ldots,g_m(\vec{x}))$
3. **Is closed under primitive recursion:** If g, h ∈ PR, then f ∈ PR where:

```
f(x⃗, 0) = g(x⃗)
f(x⃗, y+1) = h(x⃗, y, f(x⃗, y))
```

Proof that f(y) = 2y + 1 is primitive recursive

We show this by constructing f from basic functions using allowed operations:

1. First, observe that the constant function $c_2(x) = 2$ is primitive recursive:

```
c₂(x) = succ(succ(zero(x)))
```

2. The multiplication function mult(x,y) = x·y is primitive recursive (standard result):

```
mult(x, 0) = 0
mult(x, y+1) = mult(x, y) + x
```

3. The addition function add(x,y) = x + y is primitive recursive:

```
add(x, 0) = x
add(x, y+1) = succ(add(x, y))
```

4. Finally, f(y) = 2y + 1 can be constructed as:

```
f(y) = add(mult(c₂(y), y), succ(zero(y)))
     = add(mult(2, y), 1)
     = 2y + 1
```

Since f is obtained through composition of primitive recursive functions, f ∈ PR.

Exercise 3

Classification of $A = \{x \mid W_x \cap E_x \neq \emptyset\}$

The set A is saturated since it can be expressed as $A = \{x \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid dom(f) \cap cod(f) \neq \emptyset\}$.

A is r.e.: The semi-characteristic function of A is computable:

```
scA(x) = 1(μ⟨y,z,t⟩.H(x,y,t) ∧ S(x,z,y,t))
```

where the existential quantification searches for y,z,t such that $\varphi_x(y) = z$ in exactly t steps, meaning $y \in W_x$ and $z \in E_x$, so $y \in W_x \cap E_x$.

A is not recursive: By Rice's theorem, since A is saturated and $A \neq \emptyset$, $A \neq \mathbb{N}$:

- $A \neq \emptyset$: The identity function has domain and codomain $\mathbb{N}$, so their intersection is non-empty
- $A \neq \mathbb{N}$: The everywhere undefined function has empty domain and codomain, so their intersection is empty

Therefore A is not recursive.

Ā is not r.e.: Since A is r.e. but not recursive, by the characterization theorem (A recursive $\Longleftrightarrow$ A, Ā both r.e.), Ā cannot be r.e.

Final classification: A is r.e. but not recursive; Ā is not r.e. (and hence not recursive).

Exercise 4

Classification of $B = \{x \in \mathbb{N} \mid \exists y.\ \varphi_x(y) = x + 1\}$

The set B is saturated since it can be expressed as $B = \{x \mid \varphi_x \in \mathcal{B}\}$ where $\mathcal{B} = \{f \mid \exists y.\ f(y) = x + 1\}$ for some fixed x.

B is r.e.: The semi-characteristic function is computable:

```
scB(x) = 1(μ(y,t).S(x, y, x+1, t))
```

This searches for y,t such that $\varphi_x(y) = x + 1$ in exactly t steps.

B is not recursive: We show $K \leq_m B$. Define:

```
g(x,y) = {
  x + 1  if x ∈ K
  ↑      if x ∉ K
} = (x + 1) · sc_K(x)
```

Since g is computable, by the s-m-n theorem there exists total computable $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x,y)$.

The reduction works as follows:

- If $x \in K$: $\varphi_{s(x)}(y) = x + 1$ for all y, so $s(x) \in B$
- If $x \notin K$: $\varphi_{s(x)}(y) \uparrow$ for all y, so $s(x) \notin B$

Therefore $K \leq_m B$, and since K is not recursive, B is not recursive.

Ā is not r.e.: Since B is r.e. but not recursive, Ā is not r.e.

B is not saturated: Actually, let me reconsider the saturation property. The set B depends on the specific index x in the condition $\varphi_x(y) = x + 1$. This makes B non-saturated because equivalent functions with different indices would have different requirements.

For saturation, we need: if $x \in B$ and $\varphi_x = \varphi_x'$, then $x' \in B$. However, $x \in B$ means $\exists y.\ \varphi_x(y) = x + 1$, while $x' \in B$ would require $\exists y.\ \varphi_x'(y) = x' + 1$. Even if $\varphi_x = \varphi_x'$, the condition changes from $x + 1$ to $x' + 1$.

Final classification: B is r.e. but not recursive; $\bar{A}$ is not r.e.; B is not saturated.