1. Preliminaries

1.1 Key Definitions

- A set $A \subseteq \mathbb{N}$ is **recursive** if its characteristic function χ_A is computable:

  ```
  χ_A(x) = {
    1 if x ∈ A
    0 if x ∉ A
  }
  ```

- A set $A \subseteq \mathbb{N}$ is **recursively enumerable (r.e.)** if its semi-characteristic function sc_A is computable:

  ```
  sc_A(x) = {
    1 if x ∈ A
    ↑ if x ∉ A
  }
  ```

- A set $A \subseteq \mathbb{N}$ is **saturated** if for all $x, y \in \mathbb{N}$:

  ```
  x ∈ A ∧ ϕ_x = ϕ_y ⟹ y ∈ A
  ```

1.2 Key Theorems

1. **Rice's Theorem**: If A is saturated, $A \neq \emptyset$, and $A \neq \mathbb{N}$, then A is not recursive.
2. **Rice-Shapiro Theorem**: If $A \subseteq C$ is a set of computable functions and $A = \{x | \varphi_x \in A\}$ is r.e., then:
   $\forall f (f \in A \iff \exists \theta$ finite function, $\theta \subseteq f \land \theta \in A)$

2. Analysis Framework

2.1 Is a Set Recursive?

To determine if set A is recursive, follow these steps:

1. Try to prove it's recursive by:
   - Writing a computable characteristic function
   - Showing both A and $\bar{A}$ are r.e.
2. If you suspect it's not recursive:

- Try to reduce from a known non-recursive set (often K or $\bar{K}$)
- Show $K \leq_m A$ or $\bar{K} \leq_m A$
- If the set is saturated, apply Rice's theorem

Example:

```
To prove A = {x | ϕ_x is total} is not recursive:
1. Show it's saturated
2. Show A ≠ ∅ (e.g., constant functions are total)
3. Show A ≠ ℕ (e.g., the always undefined function)
4. Apply Rice's theorem
```

2.2 Is a Set R.E.?

To determine if set A is r.e., follow these steps:

1. Try to prove it's r.e. by:
   - Writing a computable semi-characteristic function
   - Using closure properties (union, intersection of r.e. sets is r.e.)
2. If you suspect it's not r.e.:
   - Try to reduce from $\bar{K}$ if possible ($\bar{K} \leq_m A$)
   - Apply Rice-Shapiro theorem for saturated sets
   - Show it contains a finite function but not its total extension (or vice versa)

Example:

```
To prove A = {x | ϕ_x(x)↓} is r.e.:
sc_A(x) = 1(ϕ_x(x)) = 1(Ψ_U(x,x))
which is computable by definition
```

2.3 Is a Set Saturated?

To determine if set A is saturated, follow these steps:

1. Check if the set describes a property of the computed function rather than the program:
   - If it depends on the function's I/O behavior, likely saturated
   - If it depends on program specifics (steps, resources), likely not saturated
2. To prove it's not saturated:
   - Find x, y where $\varphi_x = \varphi_y$ but $x \in A$ and $y \notin A$
   - Often use the Second Recursion Theorem to construct such programs

Example:

```
To prove K is not saturated:
1. Use SRT to find e such that:
   ϕ_e(y) = {
     0 if y = e
     ↑ otherwise
   }
2. Find e' ≠ e with ϕ_e = ϕ_e'
3. Show e ∈ K but e' ∉ K
```

3. Common Reduction Techniques

When reducing $A \leq_m B$:

1. Define a computable function f
2. Use the s-m-n theorem to get a computable total function s
3. Prove that $x \in A \iff s(x) \in B$

Example template:

```
1. Define g(x,y) = {
     [something] if x ∈ K
     ↑ otherwise
   }

2. By s-m-n theorem, ∃s: ℕ→ℕ total computable where
   ϕ_s(x)(y) = g(x,y)

3. Prove x ∈ K ⇔ s(x) ∈ B
```

4. Common Pitfalls

1. Don't assume a set is recursive just because its complement is r.e.
2. Remember that Rice's theorem only applies to saturated sets
3. For Rice-Shapiro, check both conditions:
   - Functions in A with no finite subfunctions in A
   - Finite functions in A with no extension in A