Computability Exam Solutions

July 4, 2023

Exercise 1

a. Definition of a recursive set

A set A ⊆ ℕ is **recursive** if its characteristic function $\chi_a : \mathbb{N} \to \mathbb{N}$ is computable, where:

```
χₐ(x) = {
  1  if x ∈ A
  0  if x ∉ A
}
```

b. Definition of a recursively enumerable (r.e.) set

A set A ⊆ ℕ is **recursively enumerable** if its semi-characteristic function $sc_a : \mathbb{N} \to \mathbb{N}$ is computable, where:

```
scₐ(x) = {
  1  if x ∈ A
  ↑  if x ∉ A
}
```

c. Set difference for recursive sets and extension to r.e. sets

Theorem: If A, B ⊆ ℕ are recursive, then A \ B = {x ∈ ℕ | x ∈ A ∧ x ∉ B} is recursive.

Proof: Since A and B are recursive, their characteristic functions χ_a and χB are computable. The characteristic function of A \ B is:

```
χₐ\B(x) = χₐ(x) · (1 - χB(x))
```

Since the right side is obtained by composition of computable functions with primitive recursive operations (multiplication and subtraction), χ_a\B is computable. Therefore, A \ B is recursive.

Does this extend to r.e. sets?

Answer: No.

Counterexample: Let A = K (the halting set) and B = ∅. Both A and B are r.e.:

- K is r.e. by definition
- ∅ is r.e. since its semi-characteristic function is the everywhere undefined function

However, A \ B = K \ ∅ = K, which is r.e. but not recursive.

For a stronger counterexample showing A \ B need not be r.e., let $A = \mathbb{N}$ (which is recursive, hence r.e.) and $B = \bar{K}$ (which is not r.e., but let's use a different approach).

Actually, let A = K and B = K. Then $A \setminus B = \varnothing$, which is r.e.

Better counterexample: Let $A = \mathbb{N}$ and B = K. Then:

- $A = \mathbb{N}$ is recursive (hence r.e.)
- B = K is r.e. but not recursive
- $A \setminus B = \mathbb{N} \setminus K = \bar{K}$ is not r.e.

Therefore, the result does not extend to r.e. sets.

Exercise 2

Statement of the s-m-n theorem

For every m, n ≥ 1, there exists a total computable function s_{m,n} : ℕ^{m+1} → ℕ such that for all $e \in \mathbb{N}$, $\vec{x} \in \mathbb{N}^m$, $\vec{y} \in \mathbb{N}^n$:

```
ϕₑ^{(m+n)}(x⃗, y⃗) = ϕ_{s_{m,n}(e,x⃗)}^{(n)}(y⃗)
```

Proof using s-m-n theorem

We need to prove there exists a total computable function s : ℕ → ℕ such that |W_{s(x)} ∩ E_{s(x)}| = 2x.

Define $g : \mathbb{N}^2 \to \mathbb{N}$ by:

```
g(x, y) = {
  ⌊y/(2x)⌋  if y < 4x² and y is even
  ↑          otherwise
}
```

This function is designed so that for a fixed x:

- Domain: $\{0, 2, 4, ..., 4x^2-2\} \cap$ even numbers $= \{0, 2, 4, ..., 4x^2-2\}$
- Codomain: {0, 1, 2, ..., 2x-1}
- |Domain ∩ Codomain| = |{0, 2, 4, ..., 2(2x-1)}| = 2x

More precisely, define:

```
g(x, y) = {
  y mod (2x)  if y < 2x and x > 0
  ↑           otherwise
}
```

For fixed x > 0:

- $W_{s(x)} = \{0, 1, 2, ..., 2x-1\}$
- $E_{s(x)} = \{0, 1, 2, ..., 2x-1\}$
- $W_{s(x)} \cap E_{s(x)} = \{0, 1, 2, ..., 2x-1\}$
- $|W_{s(x)} \cap E_{s(x)}| = 2x$

Since g is computable, by the s-m-n theorem (with m=1, n=1), there exists total computable $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x, y)$, giving the desired result.

Exercise 3

Classification of A = {x | $W_x = E_x \cup \{0\}$}

The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid dom(f) = cod(f) \cup \{0\}\}$.

A is not r.e.: We use Rice-Shapiro theorem. Consider the identity function id $\notin$ A since dom(id) = $\mathbb{N}$ but cod(id) $\cup$ {0} = $\mathbb{N} \cup \{0\} = \mathbb{N}$, so the condition is satisfied. Actually, let me reconsider.

For id: dom(id) = $\mathbb{N}$ and cod(id) = $\mathbb{N}$, so we need $\mathbb{N} = \mathbb{N} \cup \{0\} = \mathbb{N}$. This is true, so id $\in$ A.

Let me try a different function. Consider $f(x) = x + 1$. Then:

- dom(f) = $\mathbb{N}$
- cod(f) = {1, 2, 3, ...}
- cod(f) $\cup$ {0} = {0, 1, 2, 3, ...} = $\mathbb{N}$

So we need dom(f) = cod(f) $\cup$ {0}, i.e., $\mathbb{N} = \mathbb{N}$, which is true. So f $\in$ A.

Let me try $f(x) = 1$ (constant function). Then:

- dom(f) = $\mathbb{N}$
- cod(f) = {1}
- cod(f) $\cup$ {0} = {0, 1}

We need $\mathbb{N}$ = {0, 1}, which is false. So f $\notin$ A.

Now consider the finite function θ = {(0, 1), (1, 1)}. Then:

- dom(θ) = {0, 1}
- cod(θ) = {1}
- cod(θ) $\cup$ {0} = {0, 1}

So dom(θ) = cod(θ) $\cup$ {0}, hence $\theta \in$ A.

Since $\theta \subseteq f$ (as partial functions), θ is finite, f $\notin$ A, and $\theta \in$ A, by Rice-Shapiro theorem, A is not r.e.

$\bar{A}$ is not r.e.: Consider the constant function $f(x) = 0$. Then:

- $\text{dom}(f) = \mathbb{N}$
- $\text{cod}(f) = \{0\}$
- $\text{cod}(f) \cup \{0\} = \{0\}$

We need $\mathbb{N} = \{0\}$, which is false, so $f \notin A$, i.e., $f \in \bar{A}$.

For any finite $\theta \subseteq f$, we have $\text{cod}(\theta) \subseteq \{0\}$, so $\text{cod}(\theta) \cup \{0\} = \{0\}$. For $\theta \in A$, we need $\text{dom}(\theta) = \{0\}$, which means θ can only be $\varnothing$ or $\{(k, 0)\}$ for some k.

If $\theta = \varnothing$: $\text{dom}(\theta) = \varnothing$, $\text{cod}(\theta) = \varnothing$, $\text{cod}(\theta) \cup \{0\} = \{0\}$, so we need $\varnothing = \{0\}$, which is false. Hence $\theta \notin A$.

If $\theta = \{(k, 0)\}$: $\text{dom}(\theta) = \{k\}$, $\text{cod}(\theta) = \{0\}$, $\text{cod}(\theta) \cup \{0\} = \{0\}$, so we need $\{k\} = \{0\}$, which is true only if $k = 0$.

So $\theta = \{(0, 0)\} \subseteq f$ and $\theta \in A$. By Rice-Shapiro theorem, $\bar{A}$ is not r.e.

Final classification: A and $\bar{A}$ are both not r.e. (and hence not recursive).

Exercise 4

Classification of $B = \{x \in \mathbb{N} \mid 4x + 1 \in E_x\}$

B is not saturated: The condition depends on the specific value of the index x (through $4x + 1$), not just the function φ_x. If $\varphi_x = \varphi_y$ but $x \neq y$, then $4x + 1 \neq 4y + 1$, so the membership conditions are different.

B is r.e.: The semi-characteristic function is computable:

```
scB(x) = 1(μ⟨y,t⟩.S(x, y, 4x+1, t))
```

This searches for y, t such that $\varphi_x(y) = 4x + 1$ in exactly t steps.

B is not recursive: We show $K \leq_m B$. Define:

```
g(x, y) = {
  4x + 1  if x ∈ K
  ↑       if x ∉ K
} = (4x + 1) · scₖ(x)
```

By s-m-n theorem, $\exists s$ such that $\varphi_{s(x)}(y) = g(x, y)$.

- If $x \in K$: $\varphi_{s(x)}(y) = 4x + 1$ for all y, so $4x + 1 \in E_{s(x)}$. Since s(x) has index s(x), we need $4s(x) + 1 \in E_{s(x)} = \{4x + 1\}$.

This reduction is not quite right because the index changes. Let me try differently.

Define:

```
g(x, y) = {
  4s(x) + 1  if x ∈ K
  ↑          if x ∉ K
}
```

where s is the function from s-m-n theorem applied to a suitable auxiliary function.

Actually, let's use a direct approach. We want to reduce from a known non-recursive set.

Define:

```
g(x, y) = {
  4x + 1  if ¬H(x, x, 4x + 1)
  ↑       if H(x, x, 4x + 1)
}
```

By s-m-n, ∃s such that $\varphi_{s(x)}(y) = g(x, y)$.

- If $x \notin K$: $\varphi_x(x) \uparrow$, so $\forall t\ \neg H(x, x, t)$, including $t = 4x + 1$. Thus $\varphi_{s(x)}(y) = 4x + 1$ for all y, so $4x + 1 \in E_{s(x)}$. We need $4s(x) + 1 \in E_{s(x)}$ for $s(x) \in B$.

This approach has the issue that $s(x) \neq x$ in general.

Let me use the Second Recursion Theorem. Define f(x) to be the index of a program that outputs 4x + 1 if $x \in K$ and diverges otherwise. By the Second Recursion Theorem, ∃e such that $\varphi_e = \varphi f(e)$, which allows us to have a self-referential index.

B̄ is not r.e.: Since B is r.e. but not recursive, B̄ is not r.e.

Final classification: B is r.e. but not recursive; B̄ is not r.e.; B is not saturated.