Complete Set Theory Implications in Recursion Theory

Fundamental Definitions

Set Classes

- **Recursive**: $\chi_a(x)$ is computable
- **Recursively Enumerable (r.e.)**: $sc_a(x)$ is computable
- **Saturated**: $x \in A \land \varphi_x = \varphi_y \Longrightarrow y \in A$

Key Sets

- **K** = $\{x \mid x \in W_x\}$ (halting set)
- **$\bar{K}$** = $\{x \mid x \notin W_x\}$ (complement of halting set)
- **W_x** = domain of φ_x
- **E_x** = codomain of φ_x

Complete Implication Hierarchy

Level 1: Basic Class Relationships

```
Recursive ⊂ r.e. ⊂ All Sets
    ↓          ↓        ↓
  Decidable  Semi-   Undecidable
           decidable
```

Fundamental Inclusion:

- Every recursive set is r.e.
- Not every r.e. set is recursive
- Recursive sets are exactly the intersection of r.e. and co-r.e.

Level 2: Complement Relationships

Theorem: A recursive $\Longleftrightarrow$ A and $\bar{A}$ are r.e.

Implications:

- A recursive $\Longrightarrow$ $\bar{A}$ recursive
- A r.e. $\land$ $\bar{A}$ r.e. $\Longrightarrow$ A recursive
- A r.e. $\land$ A not recursive $\Longrightarrow$ $\bar{A}$ not r.e.
- A not r.e. $\Longrightarrow$ A not recursive
- $\bar{A}$ not r.e. $\Longrightarrow$ A not recursive

Contrapositive Chain:

```
A not r.e. ⟹ A not recursive ⟹ ¬(A and Ā both r.e.) ⟹ Ā not r.e. OR A not r.e.
```

Level 3: Reduction Implications

Many-one Reduction Properties:

- $K \leq_m A \implies A$ not recursive
- $\bar{K} \leq_m A \implies A$ not r.e.
- $A \leq_m B \land A$ not recursive $\implies B$ not recursive
- $A \leq_m B \land A$ not r.e. $\implies B$ not r.e.
- $A \leq_m B \land B$ recursive $\implies A$ recursive
- $A \leq_m B \land B$ r.e. $\implies A$ r.e.

Reduction Transitivity:

- $A \leq_m B \land B \leq_m C \implies A \leq_m C$

Level 4: Rice's Theorem Implications

Rice's Theorem: A saturated $\land$ $A \neq \emptyset \land A \neq \mathbb{N} \implies$ A not recursive

Immediate Consequences:

- Saturated $\land$ non-trivial $\implies$ not recursive
- Saturated $\land$ not recursive $\implies$ either A or $\bar{A}$ (or both) not r.e.
- Any non-trivial property of computable functions is undecidable

Saturation Implications:

- $A = \{x \mid \varphi_x \in \mathcal{A}\}$ for some $\mathcal{A} \subseteq \mathcal{C} \implies A$ saturated
- A saturated $\land$ decidable $\implies A = \emptyset$ or $A = \mathbb{N}$

Level 5: Rice-Shapiro Implications

Rice-Shapiro Theorem: $A = \{x \mid \varphi_x \in \mathcal{A}\}$ r.e. $\implies \forall f(f \in \mathcal{A} \iff \exists \theta \subseteq f$ finite, $\theta \in \mathcal{A})$

Contrapositive Applications:

- $\exists f \in \mathcal{A}, \forall \theta \subseteq f$ finite, $\theta \notin \mathcal{A} \implies A$ not r.e.
- $\exists f \notin \mathcal{A}, \exists \theta \subseteq f$ finite, $\theta \in \mathcal{A} \implies A$ not r.e.

Standard Patterns:

- $id \in \mathcal{A} \land \emptyset \notin \mathcal{A} \implies A$ not r.e.
- Total function $\in \mathcal{A} \land$ no finite function $\in \mathcal{A} \implies A$ not r.e.

Level 6: Specific Set Classifications

Always Recursive:

- $\emptyset$, $\mathbb{N}$ (trivial cases)
- All finite sets
- $\{x \mid x \text{ is even}\}$, $\{x \mid x \text{ is prime}\}$
- Any set with computable characteristic function

r.e. but not Recursive:

- $K = \{x \mid x \in W_x\}$
- $\{x \mid W_x \neq \emptyset\}$
- $\{x \mid \exists y(\varphi_x(y) \downarrow)\}$
- $\{x \mid |W_x| \geq k\}$ for $k \geq 1$

Not r.e.:

- $\bar{K} = \{x \mid x \notin W_x\}$
- $\{x \mid \varphi_x \text{ total}\}$
- $\{x \mid W_x = \mathbb{N}\}$
- $\{x \mid \varphi_x = \text{constant function}\}$

Decision Algorithm for Set Classification

Step 1: Identify Set Structure

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Does definition involve Wₓ or Eₓ only?
├─ YES → Likely saturated, consider Rice-Shapiro
└─ NO → Consider reduction techniques
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Step 2: Apply Appropriate Theorem

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Is the set saturated?
├─ YES → Rice's Theorem (not recursive if non-trivial)
│   ├─ Can construct scₐ? → r.e.
│   └─ Apply Rice-Shapiro → potentially not r.e.
└─ NO → Construct reduction from K or K̄
```

Step 3: Analyze Complement

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If A is r.e. but not recursive → Ā not r.e.
If A is not r.e. → A not recursive
If both A and Ā not r.e. → neither recursive
```

Advanced Implications

Enumeration Equivalences:

- A r.e. $\iff$ A = $\emptyset$ or A is range of total computable function
- A r.e. $\iff$ A = W_e for some e
- A recursive $\iff$ A r.e. and $\bar{A}$ r.e.

Relativization:

- A^B recursive $\implies$ A $\leq_m$ B
- If B r.e., then A^B r.e. $\implies$ A r.e.

Degree Implications:

- deg(A) = deg(B) $\implies$ A $\equiv_m$ B
- A $<_m$ B $\implies$ deg(A) <deg(B)

Practical Application Rules

For Proving Non-Recursiveness:

1. Show saturation + non-triviality (Rice)
2. Construct reduction from K
3. Show A r.e. but $\bar{A}$ not r.e.

For Proving Non-r.e.:

1. Apply Rice-Shapiro with counterexample
2. Construct reduction from $\bar{K}$
3. Show neither A nor $\bar{A}$ is r.e.

For Proving Recursiveness:

1. Construct explicit χ_a
2. Show both A and $\bar{A}$ are r.e.
3. Use closure properties of recursive sets

This complete framework provides the theoretical foundation for systematically approaching any set classification problem in recursion theory, ensuring comprehensive coverage of all logical relationships and implications.