Reducibility and r.e. sets

Given two sets $A, B \subseteq \mathbb{N}$ and $A \leq_m B$:

1) if B is r.e. then A is r.e.

2) if A is not r.e. then B is not r.e.

Etymology theorem

* Recursively enumerable

enumerable / countable $\quad |A| \leq |\mathbb{N}|$

i.e. there is $f: \mathbb{N} \to A$ surjective

$$\underbrace{f(0) \quad f(1) \quad f(2) \quad \ldots}_{\text{enumeration of } A}$$

recursively enumerable $\stackrel{?}{\equiv}$ enumerable via a computable f

Proposition: Let $A \subseteq \mathbb{N}$ be a set

A r.e. iff $A = \emptyset$ or ($A = img(f)$ $\quad f: \mathbb{N} \to \mathbb{N}$ total computable)

$\Rightarrow$

The etymology theorem, which states that a set A is recursively enumerable if and only if A = cod(f) = {f(x) | x ∈ N} for some computable function f : N → N, is important here because it provides a key insight into why the set A in the proposition is not recursively enumerable when A ≠ ∅ and A ≠ N.

The proof proceeds by assuming A is recursively enumerable and then deriving a contradiction. If A is recursively enumerable, then by the etymology theorem, there exists a computable function f : N → N such that A = img(f) = {f(x) | x ∈ N}.

Now, we distinguish two cases:

1. If A = ∅, then f is a total computable function with img(f) = ∅. However, this is not possible, since img(f) must contain at least the elements {f(x) | x ∈ N}. So A cannot be empty.
2. If A ≠ ∅, then fix some a0 ∈ A. Since A = img(f), f must be total, otherwise img(f) ⊂ A which contradicts A = img(f). But then the function F defined by F(x) = a0 if x ∈ A and

$F(x) = a0$ otherwise, is total computable, and $img(F) = \{a0\}$. However, $img(F) \neq A$ since we assumed $A \neq N$. This contradicts the assumption that $A = img(f)$.

Therefore, in both cases we arrive at a contradiction by assuming A is recursively enumerable and $A \neq \emptyset$ and $A \neq N$. The etymology theorem was key in allowing us to characterize A as the image of a total computable function f and leading to these contradictions.

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