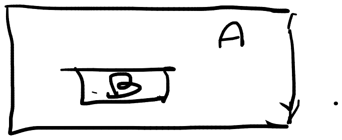
15/01/2025

① DEFINITION OF REDUCIBILITY

$A, B \subseteq \mathbb{N}$ & $A \leq_m B$



$x \in A \mid f(x) \in B$

② SHOW THAT IF A IS R.E. (1)

$A \leq_m B$ (2)

$\Rightarrow$ B IS NOT R.E.

$$\exists\ SC_A = \begin{cases} 1 & x \in A \\ \uparrow & \text{OTHERWISE} \end{cases}$$

$SC_A(x) = SC_B(f(x))$

$x \in A \Rightarrow f(x) \in K$

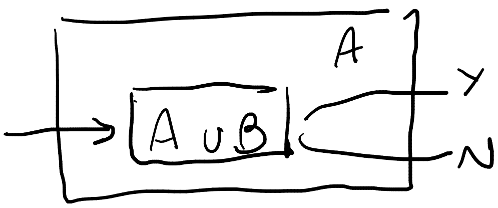
$$SC_K = \begin{cases} 1 & x \in K \\ \uparrow & \text{OTHERWISE} \end{cases}$$

R.E BUT NOT COMPUTABLE (BY DEF.)

③ IS IT TRUE $\forall A, B \subseteq \mathbb{N}$ HOLDS

$A \leq_m A \cup B$? $\Leftrightarrow [\mathbb{N} \leq_m \mathbb{N} \cup \emptyset]$

PROVE IT / PROVIDE A COUNTEREXAMPLE



$A = K$

$B = \overline{K}$

$K \leq_m K \cup \overline{K}$

IT WORKS AS A COUNTEREXAMPLE... $\Rightarrow [x \in K \leq x \in K \cup x \in \overline{K}]$

K IS NOT RECURSIVE

OTHERWISE, BOTH $K / \overline{K}$ WOULD BE RECURSIVE

REDUCTION $\rightarrow$ TRANSITIVE

REFLEXIVITY ...

[SHOW THAT $P(\vec{x})$ SEMI-DECIDABLE

IFF

$\exists\, Q(\vec{x}, y)$ s.t. $P(\vec{x}) \equiv \exists y.\, Q(\vec{x}, y)$]

$P(\vec{x}) \equiv SC_{P(\vec{x})}$ | $\exists e \in \mathbb{N}$ (PROGRAM)

$SC_P(\vec{x}) \equiv \varphi_e^{(k)}$

($\Rightarrow$)

$$= \begin{cases} 1 & x \in P \\ \uparrow & \text{OTHERWISE} \end{cases}$$

$SC_P(\vec{x}) \downarrow$ IFF $\exists y.\, H^{(k)}(e, \vec{x}, y)$

$Q(\vec{x}, y) \equiv \exists y.\, Q(\vec{x}, y)$

$\equiv \exists y.\, H^{(k)}(e, \vec{x}, y)$

($\Leftarrow$)

[$\exists\, Q(\vec{x}, y)$ s.t. $P(\vec{x}) \equiv \exists y.\, Q(\vec{x}, y)$]

ASSUMED...

$SC_A = \mu y. \left[\,|\chi_Q(\vec{x}, y) - 1|\,\right]$

[$P(\vec{x})$ IS SEMI-DECIDABLE] $\equiv$ [$SC_A(x)$]

$A / \bar{A} \to \chi_A(x) = \mu w.\, (S(e, x, y, t)$

$\exists \mu w. (S(e, x, (w)_1, (w)_2)$

<u>6.32</u> $A \to$ RECURSIVE SET

$f_1, f_2 \to$ COMPUTABLE FUNCTIONS

PROVE $\quad f(x) = \begin{cases} f_1(x) & x \in A \\ f_2(x) & x \notin A \end{cases} \iff A$ R.E ?

$e_0 \in A,\ e_1 \notin A \qquad (w)_1 = y,\ (w)_2 = A$

$f(x) = \mu w. (S(e_0, x, (w)_1, (w)_2 \wedge \mathcal{X}_A(x)$
$\Updownarrow$
$|\mathcal{X}_A(x) - 1)$

$\vee (S(e_1, x, (w)_1, (w)_2) \wedge \cancel{sg}(\overline{sg}\mathcal{X}_A(x)))$

GIVEN A R.E ....

[RICE-THEOREM $\to e_0 \in A / e_1 \notin A$
$e_0 = \mathrm{ID} / e_1 = \emptyset$
$e_0 \in A, e_1 \notin A$]

<u>8.5</u>

$A = \{x \in \mathbb{N} : \exists y, z \in \mathbb{N},\ z > 1 \wedge x = y^z\}$

$A / \bar{A}$ REC./R.E

$K \leq_m A \to$ NOT RECURSIVE
$\bar{K} \leq_m A \to$ NOT R.E $\quad sc_{\bar{K}} = \begin{cases} 1 & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$

$g(x,y) = \begin{cases} 1 & x \in K \\ 0 & \text{OTHERWISE} \end{cases} = \mathcal{X}_A(x)$

$[\exists x \in K \mid f(x) \in A]$

$g(x,y) = \varphi_{s(x)}(y) \quad s: \mathbb{N} \to \mathbb{N}$

SMN-THEOREM

$K \leq_m A$

$x \in K \mid \varphi_{s(x)}(y) = 1 \quad \forall y \in \mathbb{N}$

$W_{s(x)} / E_{s(x)} = 1 \subset \mathbb{N}$

$x \notin K \mid \varphi_{s(x)}(y) = \emptyset$

$W_{s(x)} / E_{s(x)} = \emptyset \quad \forall y \in \mathbb{N}$

$\downarrow$

A NOT RECURSIVE

$$A = \{ x \in \mathbb{N} : \exists y, z \in \mathbb{N}, z > 1 \wedge x = y^z \}$$

$\downarrow$

A IS R.E.

$SC_A \approx$ OUTPUT $\equiv E_x \equiv S$ (STOP)

DOMAIN $\equiv W_x \equiv H$

$$SC_A(x) = \mu(y, z, t) . | S(x, y, z, t) \wedge z > 1 \wedge x = y^z |$$

$$= \mu(y, z, t) . | \chi_S(x, y, z, t) - 1 | \wedge (z > 1) \wedge (x = y^z) |$$

$$= \mu w . | \chi_S(x, (w)_1, (w)_2, (w)_3 - 1) \wedge ((w)_2 > 1) \wedge (x = (w)_1^{(w)_2}) |$$

$$A \to [\text{NOT REC.} / \text{R.E.}]$$

$$A = \{ x \in \mathbb{N} : \exists y, z \in \mathbb{N}, z > 1 \wedge x = y^z \}$$

$\downarrow$

$\bar{A} = \{ z < 1 \vee x \neq y^z \} \to$ NOT R.E.

NOT RECURSIVE

6.33

$\exists$ TOTAL / NON-COMPUTABLE FUNCTION S.T.

$IMG(f) = \{ \underline{f(x) \mid x \in \mathbb{N}} \} \Leftrightarrow Pr$ (PRIME NUMBERS)

$E_x \Leftrightarrow Pr$

$\Downarrow$

$$f(x) = \begin{cases} p & \text{if } [x \in W_x] \quad \simeq \chi_K(1)^{-W_K} \\ 2 & \text{OTHERWISE} \end{cases}$$

$p = \min\{p' \in Pr \mid p' > \varphi_x(x)\}$

2 → FIRST NON-PRIME NUMBER

TOTAL (?) → YES

COMPUTABLE (?) $\simeq \chi_K(1)$ → TOTAL, NOT COMPUTABLE

$IMG(F) \subseteq Pr \to \varphi_x(x) \to 1/3/5\ldots$

$2 \notin f$

$\{p > p-1\} = p \in img(f)$

|9.21| → STATE 2 R.T.

→ USE IT TO PROVE

$C = \{x \in \mathbb{N} \mid \varphi_x(x) = x^2\}$

IS NOT SATURATED

$[\exists h : \mathbb{N} \to \mathbb{N},\ \exists e \in \mathbb{N} \text{ s.t. } \varphi_{h(e)} = \varphi_{(e)}]$

2 R.T.

$g(x,y) = x^2$ if $\varphi_x(x)\downarrow$ $(x \in W_x)$

$\varphi_{s(e)}(y) = g(e,y) = e^2$ if $\varphi_e(e)\downarrow$ $(e \in W_e)$

BY SMN-THEOREM $\varphi_{h(e)}(y) = g(e,y)$

h (TOTAL/COMPUTABLE)

$\exists e_0 \in \mathbb{N} \text{ s.t. } \varphi_{h(e_0)}(y) = \varphi_{e_0}$

$e' \neq e$

$\varphi_e \neq \varphi_{e'}$

$\varphi_e = \{(e_0, e^2)\} \simeq e \in C / e_0 \notin C$

SHOW $\forall K \geq 2$, $SUM_K : \mathbb{N}^K \to \mathbb{N}$

$$SUM_K(X_1, \dots X_K) = \sum_{i=1}^{K} X_i$$

IS PRIMITIVE RECURSIVE

PR — [0, S, U_i^K - ZERO / SUCCESSOR / PROJECTION]

PR — [CLOSED UNDER COMPOSITION, PRIMITIVE RECURSION]

$$SUM_K(X_1, \dots X_K) = \sum_{i=1}^{K} X_i \;\Big|\; \forall K \geq 2 \quad SUM_K \in PR$$

$\boxed{K=2}$
$$\begin{cases} SUM_2(X_1, 0) = X_1 \\ SUM_2(X_1, X_2 + 1) = \underset{\uparrow\, +1 \dots}{S}(SUM_2(X_1, X_2)) \end{cases}$$

$\boxed{K>2}$ $SUM_K(X_1 \dots X_K) = SUM_2(SUM_{K-1}(X_1 \dots X_{K-1}), X_K)$

$F = \{\theta \mid \theta : \mathbb{N} \to \mathbb{N} \wedge dom(\theta) \text{ finite}\}$

F IS COUNTABLE → FINITE N. OF ELEMENTS (UNLESS OTHERWISE)

[BOUNDED SUM / BOUNDED PRODUCT]

$\hookrightarrow \bar{\theta} = \prod_{i=1}^{n} p_{x_i}^{y_i}$ p = PRIME NUMBER

$p_n \in \mathbb{N}$

(x_i, y_i) = INPUT / OUTPUT PAIRS

$x \in dom(\theta)$ iff $\bar{\theta} \neq 0$