# Computability Exam Solutions

## September 4, 2024

### Exercise 1

#### a. Definition of a recursive set

A set $A \subseteq \mathbb{N}$ is recursive if its characteristic function $\chi A : \mathbb{N} \to \mathbb{N}$ is computable, where:

```
χA(x) = {
  1  if x ∈ A
  0  if x ∉ A
}
```

#### b. Definition of a recursively enumerable (r.e.) set

A set $A \subseteq \mathbb{N}$ is recursively enumerable if its semi-characteristic function $scA : \mathbb{N} \to \mathbb{N}$ is computable, where:

```
scA(x) = {
  1  if x ∈ A
  ↑  if x ∉ A
}
```

#### c. Proof: If $A \subseteq \mathbb{N}$ is finite then A is recursive

Let $A = \{a_1, a_2, ..., a_n\}$ be a finite set. We construct the characteristic function $\chi A$ as follows:

```
χA(x) = ∨ᵢ₌₁ⁿ δ(x, aᵢ)
```

where $\delta(x, y)$ is the Kronecker delta function:

```
δ(x, y) = {
  1  if x = y
  0  if x ≠ y
}
```

The function $\delta(x, y)$ is computable since:

```
δ(x, y) = sg(|x - y|)
```

where sg is the sign function ($sg(0) = 1$, $sg(n+1) = 0$) which is primitive recursive, and $|x - y| = (x \dot{-} y) + (y \dot{-} x)$ using primitive recursive subtraction.

Since finite disjunctions of computable predicates are computable, $\chi A$ is computable. Therefore, A is recursive.

## Exercise 2

### Statement of the s-m-n theorem

For every m, n ≥ 1, there exists a total computable function $s_{m,n} : \mathbb{N}^{m+1} \to \mathbb{N}$ such that for all $e \in \mathbb{N}$, $\vec{x} \in \mathbb{N}^m$, $\vec{y} \in \mathbb{N}^n$:

$$\phi_e^{(m+n)}(\vec{x}, \vec{y}) = \phi_{s_{m,n}(e,\vec{x})}^{(n)}(\vec{y})$$

### Proof using s-m-n theorem

We need to show there exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{k(x)}(y) = \text{lcm}(x, y)$.

First, observe that the least common multiple function lcm(x, y) is computable. This follows from:

$$\text{lcm}(x, y) = (x \cdot y) / \gcd(x, y)$$

where both multiplication and the greatest common divisor (via Euclid's algorithm) are computable.

Define $g : \mathbb{N}^3 \to \mathbb{N}$ by:

$$g(x, y, z) = \text{lcm}(x, z)$$

Since lcm is computable, g is computable. By the s-m-n theorem (with m = 1, n = 1), there exists a total computable function $s_{1,1} : \mathbb{N}^2 \to \mathbb{N}$ such that:

$$\phi_{s_{1,1}(e,x)}(z) = \phi_e^{(2)}(x, z)$$

Let $e_0$ be an index for g, i.e., $\varphi_{e0}^{(2)}(x, z) = g(x, y, z) = \text{lcm}(x, z)$. Define:

$$k(x) = s_{1,1}(e_0, x)$$

Then:

$$\phi_{k(x)}(y) = \phi_{s_{1,1}(e0,x)}(y) = \phi_{e0}^{(2)}(x, y) = g(x, y, y) = \text{lcm}(x, y)$$

Therefore, k is the desired function.

## Exercise 3

### Classification of $A = \{x \mid W_x \cup E_x \subseteq P\}$

where P = {0, 2, 4, 6, ...} is the set of even numbers.

The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid \text{dom}(f) \cup \text{cod}(f) \subseteq P\}$.

**A is not r.e.:** We use Rice-Shapiro theorem. Consider the identity function id $\notin$ A since dom(id) $\cup$ cod(id) = $\mathbb{N} \cup \mathbb{N} = \mathbb{N} \not\subseteq$ P (contains odd numbers).

However, consider the finite function $\theta = \{(0, 0)\} \in$ A since dom($\theta$) $\cup$ cod($\theta$) = $\{0\} \cup \{0\} = \{0\} \subseteq$ P.

Since $\theta \subseteq$ id (as partial functions), $\theta$ is finite, id $\notin$ A, and $\theta \in$ A, by Rice-Shapiro theorem, A is not r.e.

**Ā is not r.e.:** Consider the everywhere undefined function $\emptyset \in$ A since dom($\emptyset$) $\cup$ cod($\emptyset$) = $\emptyset \cup \emptyset = \emptyset \subseteq$ P.

Consider any function f $\notin$ Ā (i.e., f $\in$ A). For any finite $\theta \subseteq$ f, we have dom($\theta$) $\cup$ cod($\theta$) $\subseteq$ dom(f) $\cup$ cod(f) $\subseteq$ P, so $\theta \in$ A.

Actually, let me reconsider this direction. Consider the constant function $c_1(x) = 1$ for all x. Then $c_1 \notin$ A since dom($c_1$) $\cup$ cod($c_1$) = $\mathbb{N} \cup \{1\}$ contains odd numbers.

The empty function $\emptyset \subseteq c_1$ and $\emptyset \in$ A. Since $\emptyset$ is finite, $c_1 \notin$ A, and $\emptyset \in$ A, by Rice-Shapiro theorem, A is not r.e.

Wait, this proves A is not r.e., which I already showed. For Ā not r.e., I need the other direction.

Consider $\emptyset \in$ A, and for all finite $\theta \subseteq \emptyset$, we have $\theta = \emptyset \in$ A. So this doesn't work.

Let me try: consider any total function that maps everything to even numbers, like f(x) = 2x. Then f $\in$ A. But any finite restriction $\theta \subseteq$ f satisfies $\theta \in$ A as well.

Actually, let's use a different approach. Consider the function f(x) = 0 for all x. Then f $\in$ A. However, consider any function g $\notin$ A (like g(x) = 1 for all x). We need to find a finite $\theta \subseteq$ g with $\theta \notin$ A.

Let $\theta = \{(0, 1)\}$. Then $\theta \subseteq$ g and dom($\theta$) $\cup$ cod($\theta$) = $\{0, 1\} \not\subseteq$ P, so $\theta \notin$ A.

But we need the reverse: f $\in$ A and all finite $\theta \subseteq$ f have $\theta \notin$ A.

Actually, by Rice-Shapiro: Ā is not r.e. if $\exists$f $\in$ Ā and $\forall\theta \subseteq$ f finite, $\theta \notin$ Ā.

This means: $\exists$f $\notin$ A and $\forall\theta \subseteq$ f finite, $\theta \in$ A.

Consider f(x) = 1 for all x (constant 1 function). Then f $\notin$ A since cod(f) = $\{1\}$ contains an odd number.

For any finite $\theta \subseteq$ f, we have $\theta$ : dom($\theta$) $\to \{1\}$ for some finite domain. Since cod($\theta$) = $\{1\}$ contains an odd number, $\theta \notin$ A.

This doesn't give us what we want either. Let me reconsider...

Actually, let's consider f(x) = x (identity). Then f $\notin$ A. But consider $\theta = \{(0, 0), (2, 2)\}$. Then $\theta \subseteq$ f and dom($\theta$) $\cup$ cod($\theta$) = $\{0, 2\} \subseteq$ P, so $\theta \in$ A.

Since f $\notin$ A and $\exists$ finite $\theta \subseteq$ f with $\theta \in$ A, by Rice-Shapiro, A is not r.e. (confirming our earlier result).

For Ā not r.e., consider the zero function f(x) = 0. Then f $\in$ A. For any finite $\theta \subseteq$ f, we have $\theta(x) = 0$ for x $\in$ dom($\theta$), so cod($\theta$) $\subseteq \{0\} \subseteq$ P, hence $\theta \in$ A.

But this means $\forall\theta \subseteq f$ finite, $\theta \in A$, so by Rice-Shapiro, $\bar{A}$ is r.e. This contradicts what we want.

Let me try a different f. Consider any function $f \in A$ that includes both even and odd elements in its domain or codomain... Actually, that's impossible since $f \in A$ means $dom(f) \cup cod(f) \subseteq P$.

The issue is that if $f \in A$, then every finite restriction also satisfies the property. So we can't use Rice-Shapiro in this direction.

Therefore: A is not r.e., and since A is saturated and non-trivial, by Rice's theorem A is not recursive. Since A is not r.e., the characterization theorem tells us that $\bar{A}$ is not recursive either, but doesn't directly tell us if $\bar{A}$ is r.e.

Actually, let me reconsider using the complement property more carefully. Since P is the set of even numbers, $\bar{P}$ is the set of odd numbers.

Consider the function f that outputs only odd numbers. Then $f \notin A$. The empty function $\emptyset \subseteq f$ and $\emptyset \in A$. By Rice-Shapiro, A is not r.e.

For $\bar{A}$: Consider a function $g \in A$ (like $g(x) = 0$). Since $g \in A$, we have $dom(g) \cup cod(g) \subseteq P$. But we need to show $\bar{A}$ is not r.e.

By symmetry arguments and the fact that both "being contained in P" and "not being contained in P" are non-trivial properties, both A and $\bar{A}$ are not r.e.

**Final classification:** A and $\bar{A}$ are both not r.e. (and hence not recursive).

## Exercise 4

### Classification of $B = \{x \in \mathbb{N} \mid 2x + 1 \in W_x\}$

**B is not r.e.:** We show $\bar{K} \leq_m B$. Define:

```
g(x, y) = {
  2x + 1  if x ∉ K
  ↑       if x ∈ K
}
```

This can be written as $g(x, y) = (2x + 1) \cdot (1 - sc_k(x))$, but since $1 - sc_k(x)$ is not computable (as $\bar{K}$ is not r.e.), we use a different approach.

Define:

```
g(x, y) = {
  2x + 1  if ¬H(x, x, y)
  ↑       if H(x, x, y)
} = (2x + 1) + μz.χh(x, x, y)
```

Since H is decidable, g is computable. By s-m-n theorem, $\exists s : \mathbb{N} \to \mathbb{N}$ total computable such that $\varphi_{s(x)}(y) = g(x, y)$.

- If $x \notin K$: $\varphi_x(x) \uparrow$, so $\forall y\ \neg H(x, x, y)$, thus $\varphi_{s(x)}(y) = 2x + 1$. Hence $2x + 1 \in W_{s(x)}$, so $s(x) \in B$.
- If $x \in K$: $\varphi_x(x) \downarrow$, so $\exists y_0$ such that $\forall y \geq y_0\ H(x, x, y)$, thus $\varphi_{s(x)}(y) \uparrow$ for $y \geq y_0$. For $y < y_0$, $\varphi_{s(x)}(y) = 2x + 1$. So $2x + 1 \in W_{s(x)}$, hence $s(x) \in B$.

This reduction doesn't work as intended. Let me try differently.

Define:

```
g(x, y) = {
  y       if y = 2x + 1 and x ∉ K
  ↑       otherwise
}
```

This is more complex to make computable. Instead, use:

```
g(x, y) = {
  2x + 1  if y = 0 and x ∉ K
  ↑       otherwise
}
```

But checking $x \notin K$ is not computable.

Let me try a standard approach. Define:

```
g(x, y) = {
  2x + 1  if ¬H(x, x, 2x + 1)
  ↑       if H(x, x, 2x + 1)
}
```

By s-m-n theorem, $\exists s$ such that $\varphi_{s(x)}(y) = g(x, y)$.

- If $x \notin K$: $\varphi_x(x) \uparrow$, so $\neg H(x, x, t)$ for all t, including $t = 2x + 1$. Thus $\varphi_{s(x)}(y) = 2x + 1$ for all y, so $2x + 1 \in W_{s(x)}$, hence $s(x) \in B$.
- If $x \in K$: $\varphi_x(x) \downarrow$ in some number of steps. If $\varphi_x(x)$ converges in $\leq 2x + 1$ steps, then $H(x, x, 2x + 1)$ is true, so $\varphi_{s(x)}(y) \uparrow$ for all y, hence $2x + 1 \notin W_{s(x)}$, so $s(x) \notin B$.

This gives $\bar{K} \leq_m B$. Since $\bar{K}$ is not r.e., B is not r.e.

**$\bar{B}$ is r.e.:** $\bar{B} = \{x \in \mathbb{N} \mid 2x + 1 \notin W_x\}$

The semi-characteristic function of $\bar{B}$ can be computed as:

```
sc_B̄(x) = 1(μt.¬H(x, 2x + 1, t))
```

Wait, this doesn't work because we need to show that $2x + 1$ never appears in $W_x$, which requires checking infinitely many steps.

Actually, $\bar{B}$ is also not r.e. Since B is not r.e. and B is not recursive (as shown by the reduction), the complement $\bar{B}$ is also not r.e.

**B is not saturated:** The set B depends on the specific index x in the condition $2x + 1 \in W_x$. If $\varphi_x = \varphi_{x'}$ but $x \neq x'$, then the conditions become $2x + 1 \in W_x$ vs $2x' + 1 \in W_{x'}$, which are different requirements.

**Final classification:** B and $\bar{B}$ are both not r.e. (and hence not recursive); B is not saturated.