# Definition Reminder

A set $A \subseteq \mathbb{N}$ is **saturated** (or extensional) if: $\forall x,y \in \mathbb{N}$: $(x \in A \land \varphi_x = \varphi_y) \implies y \in A$

**Negation**: A set A is **NOT saturated** if: $\exists x,y \in \mathbb{N}$: $(x \in A \land \varphi_x = \varphi_y \land y \notin A)$

In other words, we need to find two indices that compute the same function, but only one belongs to A.

# General Strategy

## Step 1: Understand What Makes a Set Non-Saturated

A set is NOT saturated when its membership depends on:

- **Index properties** (like the value of the index itself)
- **Syntactic properties** (like program length, specific representation)
- **Computational steps** (like number of steps to terminate)

Rather than purely **semantic properties** (like function behavior).

## Step 2: Standard Approach

1. **Construct a specific function** using Second Recursion Theorem
2. **Find two different indices** for the same function
3. **Show they have different membership** in the set

# Method 1: Using Second Recursion Theorem

## Template Proof Structure

**Theorem**: The set A = {definition} is not saturated.

**Proof**:

1. **Define auxiliary function**: Define $g: \mathbb{N}^2 \to \mathbb{N}$ by

```
g(x,y) = {specific definition based on the set A}
```

2. **Apply smn-theorem**: Since g is computable, by smn-theorem there exists a total computable function $s: \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x,y)$.
3. **Apply Second Recursion Theorem**: By the Second Recursion Theorem, there exists $e \in \mathbb{N}$ such that $\varphi_e = \varphi_{s(e)}$.

4. **Analyze membership**: Show that $e \in A$ (or $e \notin A$).
5. **Find different index**: Since there are infinitely many indices for any computable function, there exists $e' \neq e$ such that $\varphi_e = \varphi_{e'}$.
6. **Show different membership**: Demonstrate that e' has opposite membership from e in A.
7. **Conclude**: Since $\varphi_e = \varphi_{e'}$, but e and e' have different membership in A, the set A is not saturated. □

# Worked Examples from Real Exams

## Example 1: The Halting Set $K = \{x \in \mathbb{N} \mid \varphi_x(x) \downarrow\}$

**Theorem**: K is not saturated.

**Proof**:

1. **Define auxiliary function**: Define $g: \mathbb{N}^2 \to \mathbb{N}$ by

```
g(x,y) = {0 if y = x; ↑ otherwise} = μz.|y - x|
```

2. **Apply smn-theorem**: Since g is computable, by the smn-theorem there exists a total computable function $s: \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x,y)$.
3. **Apply Second Recursion Theorem**: By the Second Recursion Theorem, there exists $e \in \mathbb{N}$ such that $\varphi_e = \varphi_{s(e)}$. Therefore:

```
ϕₑ(y) = g(e,y) = {0 if y = e; ↑ otherwise}
```

4. **Analyze membership**: Clearly $e \in K$ since $\varphi_e(e) = 0 \downarrow$.
5. **Find different index**: Since there are infinitely many indices for any computable function, there exists $e' \neq e$ such that $\varphi_e = \varphi_{e'}$.
6. **Show different membership**: We have $e' \notin K$ since $\varphi_{e'}(e') = \varphi_e(e') = \uparrow$ (because $e' \neq e$).
7. **Conclusion**: Since $\varphi_e = \varphi_{e'}$ but $e \in K$ and $e' \notin K$, the set K is not saturated. □

## Example 2: $B = \{x \in \mathbb{N} \mid \exists k \in \mathbb{N}: k \cdot x \in W_x\}$ (From Exam)

**Theorem**: B is not saturated.

**Proof**:

1. **Define auxiliary function**: Define $g: \mathbb{N}^2 \to \mathbb{N}$ by

```
g(x,y) = {0 if y = x; ↑ otherwise}
```

2. **Apply smn and SRT**: By the smn-theorem and Second Recursion Theorem, there exists $e \in \mathbb{N}$ such that:

```
ϕₑ(y) = {0 if y = e; ↑ otherwise}
```

   Therefore $W_e = \{e\}$.
3. **Analyze membership**: $e \in B$ since $1 \cdot e = e \in W_e = \{e\}$.
4. **Find different index**: There exists $e' > e$ such that $\varphi_e = \varphi_{e'}$, so $W_{e'} = W_e = \{e\}$.
5. **Show different membership**: $e' \notin B$ since for any $k \in \mathbb{N}$:
   - If $k > 0$: $k \cdot e' > e$, so $k \cdot e' \notin W_{e'} = \{e\}$
   - If $k = 0$: $k \cdot e' = 0 \neq e$, so $k \cdot e' \notin W_{e'} = \{e\}$
6. **Conclusion**: Since $\varphi_e = \varphi_{e'}$ but $e \in B$ and $e' \notin B$, the set B is not saturated. □

# Example 3: $C = \{x \in \mathbb{N} \mid \varphi_x(x) = x^2\}$ (From Exam)

**Theorem**: C is not saturated.

**Proof**:

1. **Define auxiliary function**: Define $g: \mathbb{N}^2 \to \mathbb{N}$ by

```
g(x,y) = {x² if y = x; ↑ otherwise}
```

2. **Apply smn and SRT**: By the smn-theorem and Second Recursion Theorem, there exists $e \in \mathbb{N}$ such that:

```
ϕₑ(y) = {e² if y = e; ↑ otherwise}
```

3. **Analyze membership**: $e \in C$ since $\varphi_e(e) = e^2$.
4. **Find different index**: There exists $e' \neq e$ such that $\varphi_e = \varphi_{e'}$.
5. **Show different membership**: $e' \notin C$ since $\varphi_{e'}(e') = \varphi_e(e') = \uparrow$ (because $e' \neq e$).
6. **Conclusion**: Since $\varphi_e = \varphi_{e'}$ but $e \in C$ and $e' \notin C$, the set C is not saturated. □

# Example 4: $A = \{x \in \mathbb{N} \mid [0,x] \subseteq W_x\}$ (From Exam)

**Theorem**: A is not saturated.

**Proof**:

1. **Define auxiliary function**: Define $g: \mathbb{N}^2 \to \mathbb{N}$ by

```
g(e,x) = {e if x ≤ e; ↑ otherwise}
```

2. **Apply smn and SRT**: By the smn-theorem and Second Recursion Theorem, there exists $e \in \mathbb{N}$ such that:

```
ϕₑ(x) = {e if x ≤ e; ↑ otherwise}
```

   Therefore $W_e = [0,e]$ (assuming $e \neq 0$).
3. **Analyze membership**: $e \in A$ since $[0,e] \subseteq W_e = [0,e]$.
4. **Find different index**: There exists $e' > e$ such that $\varphi_e = \varphi_{e'}$, so $W_{e'} = W_e = [0,e]$.
5. **Show different membership**: $e' \notin A$ since $[0,e'] \not\subseteq [0,e] = W_{e'}$ (because $e' > e$).
6. **Conclusion**: Since $\varphi_e = \varphi_{e'}$ but $e \in A$ and $e' \notin A$, the set A is not saturated. □

# When Sets ARE Saturated: Rice vs Rice-Shapiro Applications

## Rice's Theorem: Proving "Not Recursive"

**Template**: Let $A \subseteq \mathbb{N}$ be saturated with $A \neq \emptyset$ and $A \neq \mathbb{N}$. By Rice's theorem, A is not recursive.

**Exam Example 1**: $A = \{x \in \mathbb{N} \mid E_x \cap X \neq \emptyset\}$ where X is finite and non-empty

**Solution**:

1. **Show saturation**: $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid cod(f) \cap X \neq \emptyset\}$
2. **Show non-triviality**:
   - $A \neq \emptyset$: If e is such that $\varphi_e = id$, then $e \in A$ since $X \cap E_e = X \cap \mathbb{N} = X \neq \emptyset$
   - $A \neq \mathbb{N}$: If e' is such that $\varphi_{e'} = \emptyset$, then $e' \notin A$ since $X \cap E_{e'} = X \cap \emptyset = \emptyset$
3. **Apply Rice**: By Rice's theorem, A is not recursive. □

## Rice-Shapiro Theorem: Proving "Not R.E."

**Template**: Use Rice-Shapiro to show A is not r.e. by finding:

- $\exists f \in \mathcal{A}$ but $\forall \theta \subseteq f$ finite: $\theta \notin \mathcal{A}$, OR
- $\exists f \notin \mathcal{A}$ but $\exists \theta \subseteq f$ finite: $\theta \in \mathcal{A}$

**Exam Example 2**: $B = \{x \in \mathbb{N} \mid W_x \neq \emptyset \land \min(W_x) > 0\}$

**Solution**:

1. **Show saturation**: $B = \{x \in \mathbb{N} \mid \varphi_x \in \mathscr{B}\}$ where $\mathscr{B} = \{f \mid dom(f) \neq \emptyset \land \min(dom(f)) > 0\}$
2. **Apply Rice-Shapiro for B not r.e.**:
   - Consider $id \notin \mathscr{B}$ since $dom(id) = \mathbb{N}$, so $\min(dom(id)) = 0$
   - Define $\theta(x) = \{1 \text{ if } x = 1; \uparrow \text{ otherwise}\}$

- Then $\theta \subseteq id$, $\theta$ is finite, and $\theta \in \mathscr{B}$ since $\min(dom(\theta)) = 1 > 0$
  - By Rice-Shapiro, B is not r.e.
3. **Apply Rice-Shapiro for $\bar{B}$ not r.e.**:
   - $\theta \notin \bar{B}$ (since $\theta \in \mathscr{B}$)
   - $\emptyset \subseteq \theta$, $\emptyset$ is finite, and $\emptyset \in \bar{B}$ since $dom(\emptyset) = \emptyset$
   - By Rice-Shapiro, $\bar{B}$ is not r.e.
4. **Conclusion**: Both B and $\bar{B}$ are not r.e., hence not recursive. □

**Exam Example 3**: $A = \{x \in \mathbb{N} \mid W_x \setminus E_x \text{ is infinite}\}$

**Solution**:

1. **Show saturation**: $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid dom(f) \setminus cod(f) \text{ is infinite}\}$
2. **Apply Rice-Shapiro for A not r.e.**:
   - Consider constant function $1 \in \mathcal{A}$ since $dom(1) = \mathbb{N}$, $cod(1) = \{1\}$, so $dom(1) \setminus cod(1) = \mathbb{N} \setminus \{1\}$ is infinite
   - For any finite $\theta \subseteq 1$: $dom(\theta)$ is finite, so $dom(\theta) \setminus cod(\theta)$ is finite, hence $\theta \notin \mathcal{A}$
   - By Rice-Shapiro, A is not r.e.
3. **Apply Rice-Shapiro for $\bar{A}$ not r.e.**:
   - $\emptyset \in \bar{A}$ since $dom(\emptyset) \setminus cod(\emptyset) = \emptyset$ is finite
   - $1 \notin \bar{A}$ (since $1 \in \mathcal{A}$) and $\emptyset \subseteq 1$
   - By Rice-Shapiro, $\bar{A}$ is not r.e.
4. **Conclusion**: Both A and $\bar{A}$ are not r.e., hence not recursive. □

## When SRT is Not Applicable

Sometimes you can directly construct counterexamples:

**Example**: $A = \{x \in \mathbb{N} \mid \text{program length} \leq 10\}$

**Proof**:

1. **Find same function, different lengths**: Consider the zero function:
   - $m = \gamma(Z(1))$ has length ≤ 10, so $m \in A$
   - $n = \gamma(Z(1)\ Z(1) \dots Z(1))$ (with 20 copies) has length > 10, so $n \notin A$
2. **Same function**: $\varphi_m = \varphi_n = 0$ (constant zero function)
3. **Conclusion**: A is not saturated. □

# Common Mistakes to Avoid

1. **Don't confuse syntax with semantics**: Sets based on program properties (not function properties) are typically not saturated.
2. **Verify function equality**: Always confirm that $\varphi_x = \varphi_y$ before claiming non-saturation.

3. **Check the construction**: In SRT proofs, verify that your constructed function actually has the desired properties.
4. **Index vs. behavior**: Remember that non-saturated sets depend on indices, not just function behavior.

# Quick Recognition Patterns

A set is likely **NOT saturated** if it involves:

- ✗ "$x \in W_x$" (self-reference to index)
- ✗ "$\varphi_x(x)$ = specific value involving x"
- ✗ "program properties" (length, syntax)
- ✗ "number of computation steps"
- ✗ "$k \cdot x \in W_x$" (index arithmetic)

A set is likely **saturated** if it only involves:

- ✓ Domain properties: "$W_x$ has property P"
- ✓ Codomain properties: "$E_x$ has property P"
- ✓ Function properties: "$\varphi_x$ is total/partial/constant"

# Complete Formal Strategy for Exams

## Step-by-Step Classification Process

#### 1. Check if set is saturated

- **If NOT saturated**: Use Second Recursion Theorem method (this guide)
- **If saturated**: Continue to step 2

#### 2. Apply appropriate theorem

- **Rice's Theorem**: Always gives "not recursive" for non-trivial saturated sets
- **Rice-Shapiro**: Use to prove "not r.e." by finding function/subfunction contradictions

#### 3. Check for r.e. property

- Try to write semicharacteristic function sc_A(x) = 1(μw....)
- If successful: set is r.e. but not recursive
- If Rice-Shapiro shows not r.e.: set is not r.e., hence not recursive

#### 4. Analyze complement

- If A is r.e. but not recursive: $\bar{A}$ is not r.e.
- If both A and $\bar{A}$ are not r.e. (Rice-Shapiro): both are not recursive

## Formal Writing Template for Exams

**For NON-SATURATED sets**: "To show that A is not saturated, we apply the Second Recursion Theorem. Define g(x,y) = [specific function]. By the smn-theorem and SRT, there exists e such that $\varphi_e$ = [specific form]. Then $e \in A$, but there exists $e' \neq e$ with $\varphi_e = \varphi_{e'}$ and $e' \notin A$, contradicting saturation."

**For SATURATED sets**: "The set A is saturated since $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A}$ = [function property]. Since $A \neq \emptyset$ and $A \neq \mathbb{N}$ [provide examples], by Rice's theorem A is not recursive. [Optional: Apply Rice-Shapiro to show r.e./not r.e. status]"

## Final Exam Checklist

✓ **Always state what you're proving**: "not saturated", "not recursive", "not r.e."
✓ **Use precise notation**: $A \subseteq \mathbb{N}$, $\varphi_x \in \mathcal{A}$, etc.
✓ **Reference theorems by name**: "By Rice's theorem", "By the Second Recursion Theorem"
✓ **Verify all conditions**: saturation, non-triviality, function constructions
✓ **State conclusions clearly**: "Therefore, A is not recursive"