Computability Exam Solutions

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Exercise 1

Definition of the class PR of primitive recursive functions

The class PR of primitive recursive functions is the smallest class of functions $PR \subseteq \bigcup_k(\mathbb{N}^k \to \mathbb{N})$ that:

1. **Contains the basic functions:**
   - Zero function: zero(x) = 0
   - Successor function: succ(x) = x + 1
   - Projection functions: $\pi_i^k(x_1,...,x_k) = x_i$ for $1 \le i \le k$
2. **Is closed under composition:** If $g_1,...,g_m \in PR$ and $h \in PR$, then $f \in PR$ where $f(\vec{x}) = h(g_1(\vec{x}),...,g_m(\vec{x}))$
3. **Is closed under primitive recursion:** If g, h ∈ PR, then f ∈ PR where:

   ```
   f(x⃗, 0) = g(x⃗)
   f(x⃗, y+1) = h(x⃗, y, f(x⃗, y))
   ```

Proof that χ_a is primitive recursive for $A = \{2^n - 1 : n \in \mathbb{N}\}$

We need to show that the characteristic function $\chi_a(x) = 1$ if $x \in A$, 0 otherwise, is primitive recursive.

Note that $x \in A \iff x = 2^n - 1$ for some $n \iff x + 1 = 2^n$ for some $n \iff x + 1$ is a power of 2.

First, we define auxiliary functions:

1. **Power function pow(x,y) = x^y** (primitive recursive by assumption or standard construction)
2. **Function to check if x + 1 is a power of 2:**

   ```
   isPowerOf2(x) = sg(Σᵢ₌₀ˣ sg(|pow(2,i) - (x+1)|))
   ```

   This sums 1 for each i where $2^i \neq x + 1$, then applies sg to get 0 if any $2^i = x + 1$, and 1 otherwise.
3. **Characteristic function:**

   ```
   χₐ(x) = sḡ(isPowerOf2(x))
   ```

Since isPowerOf2 uses bounded sum, power function, sg, and absolute difference (all primitive recursive), and χ_a is obtained by composition with $\overline{sg}$, we have $\chi_a \in PR$.

Exercise 2

Analysis of $f(x) = x/2$ if $\varphi_x(x)\downarrow$, x+1 otherwise

Answer: The function f is not computable.

Proof: Suppose f were computable. Then we could decide the halting problem as follows:

Given input x, compute f(x):

- If $f(x) = \lfloor x/2 \rfloor$, then $\varphi_x(x) \downarrow$
- If $f(x) = x + 1$, then $\varphi_x(x) \uparrow$

This would give us a decision procedure for $K = \{x : \varphi_x(x) \downarrow\}$, contradicting the fact that K is not recursive.

Therefore, f is not computable.

Exercise 3

Classification of $A = \{x \in \mathbb{N} : W_x = E_x\}$

The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid dom(f) = cod(f)\}$.

A is not r.e.: We use Rice-Shapiro theorem. Consider the identity function $id \in \mathcal{A}$ since $dom(id) = cod(id) = \mathbb{N}$.

However, for any finite function $\theta \subseteq id$, we have $|dom(\theta)| = |cod(\theta)| < \infty$, but unless $dom(\theta) = cod(\theta)$ exactly, $\theta \notin \mathcal{A}$.

Consider $\theta = \{(0,1)\} \subseteq id$. Then $dom(\theta) = \{0\}$ and $cod(\theta) = \{1\}$, so $dom(\theta) \neq cod(\theta)$, hence $\theta \notin \mathcal{A}$.

Since $id \in \mathcal{A}$ and $\exists$ finite $\theta \subseteq id$ with $\theta \notin \mathcal{A}$, by Rice-Shapiro theorem, A is not r.e.

$\bar{A}$ is not r.e.: Consider the function $f(x) = x + 1$. Then $dom(f) = \mathbb{N}$ and $cod(f) = \{1,2,3,...\}$, so $dom(f) \neq cod(f)$, hence $f \notin \mathcal{A}$.

For any finite $\theta \subseteq f$, we have $\theta : dom(\theta) \to cod(\theta) \subseteq \{1,2,3,...\}$. For $\theta \in \mathcal{A}$, we need $dom(\theta) = cod(\theta)$. But if $dom(\theta) \subseteq \mathbb{N}$ and $cod(\theta) \subseteq \{1,2,3,...\}$, then $0 \notin cod(\theta)$, so if $0 \in dom(\theta)$, then $dom(\theta) \neq cod(\theta)$.

The empty function $\emptyset \subseteq f$ has $dom(\emptyset) = cod(\emptyset) = \emptyset$, so $\emptyset \in \mathcal{A}$.

Since $f \notin \mathcal{A}$ and $\exists$ finite $\emptyset \subseteq f$ with $\emptyset \in \mathcal{A}$, by Rice-Shapiro theorem, $\bar{A}$ is not r.e.

Final classification: A and $\bar{A}$ are both not r.e. (and hence not recursive).

Exercise 4

Classification of $B = \{\pi(x,y) : P_x$ terminates on input x in more than y steps$\}$

where $\pi : \mathbb{N}^2 \to \mathbb{N}$ is the pair encoding function.

B is r.e.:

```
scB(z) = 1(μt. let (x,y) = π⁻¹(z) in [¬H(x,x,y) ∧ H(x,x,t) ∧ t > y])
```

This searches for evidence that $P_x(x)$ terminates in more than y steps.

B is not recursive: We show $K \leq_m B$. Define the reduction $f(x) = \pi(x,0)$.

- If $x \in K$: $\varphi_x(x) \downarrow$ in some number of steps $t \geq 1 > 0$, so $\pi(x,0) \in B$
- If $x \notin K$: $\varphi_x(x) \uparrow$, so it doesn't terminate in more than 0 steps, hence $\pi(x,0) \notin B$

Therefore $K \leq_m B$, and since K is not recursive, B is not recursive.

$\bar{B}$ is not r.e.: Since B is r.e. but not recursive, $\bar{B}$ is not r.e.

Final classification: B is r.e. but not recursive; $\bar{B}$ is not r.e.

Exercise 5

Second Recursion Theorem

For every total computable function $f : \mathbb{N} \to \mathbb{N}$, there exists $e_0 \in \mathbb{N}$ such that:

```
ϕₑ₀ = ϕf(e₀)
```

Proof that $C = \{x : 2x \in W_x \cap E_x\}$ is not saturated

Define $f : \mathbb{N} \to \mathbb{N}$ by $f(x) = x + 1$ (total and computable).

By the Second Recursion Theorem, $\exists e$ such that $\varphi_e = \varphi_{e+1}$.

Consider the behavior:

- Since $\varphi_e = \varphi_{e+1}$, we have $W_e = W_{e+1}$ and $E_e = E_{e+1}$
- Therefore: $2e \in W_e \cap E_e \iff 2e \in W_{e+1} \cap E_{e+1}$

However, the conditions for membership in C are:

- $e \in C \iff 2e \in W_e \cap E_e$
- $(e+1) \in C \iff 2(e+1) \in W_{e+1} \cap E_{e+1}$

Since $2e \neq 2(e+1) = 2e + 2$, these are different conditions. Even though $\varphi_e = \varphi_{e+1}$, the membership of e and e+1 in C depends on different values (2e vs 2e+2) being in the same sets.

If we choose e such that exactly one of {2e, 2e+2} is in $W_e = W_{e+1}$, then exactly one of e, e+1 will be in C, violating saturation.

Therefore, C is not saturated.