18.12

OVERVIEW:

- RECURSIVE FUNCTIONALS
  - MYHILL-SHEPHERDSON
  - 1° / 2° REC. THEOREMS
  - EXAM EXERCISES → (28/01)

FUNCTIONALS (OPERATORS) → TRANSFORM STUFF INTO OTHER STUFF... (SMN)

$\Phi : F(\mathbb{N}^k) \to F(\mathbb{N}^h)$

- (RECURSIVE) FUNCTIONALS

$\Phi\, f(\vec{x}) = y$ (SUBFUNCTIONS)

↳ TYPE 2

ACK = (1) + (2)

[MYHILL - SHEPHERDSON]

e = INDEX OF A PROGRAM

$$f = \varphi_e \quad \to \quad \Phi(f) = \varphi_{e'}$$

$\to$ $(e \neq e')$ $\to$ DIFFERENT PROGRAMS

$f : \mathbb{N} \to \mathbb{N}$ (TOTAL)

$$\left[\forall e, e' \to \varphi_{f(e)} = \varphi_{f(e')}\right]$$

MYHILL $\underbrace{\text{ISOMORPHISM}}$ = PROGRAMS HAVE THE SAME SHAPE

1° REC.

Φ REC. FUNCTIONAL

$$\underbrace{\exists e_0 \to \varphi_{e_0} = \varphi_{f(e_0)}}_{\text{TRANSFORMATION ON YOUR INPUT}}$$

2° REC.

$\exists e, \ e \neq e'$

$$\varphi_{f(e)} = \varphi_{f(e')}$$

EXAMPLES $\to$ SOLVED EXAM (2° ND OF LAST YEAR)

$$\left[B = \{x \in \mathbb{N} \mid \exists z.\ \varphi_x(z) > x\}\right]$$

① $B / \bar{B}$ REC/R.E. (?) ② B is saturated

B is saturated $\rightarrow$ $\left[\begin{array}{l} 2^\circ \text{ REC. THEOREM} \\ e, e' \quad e \neq e' \\ \varphi_{(e)} \neq \varphi(e') \end{array}\right]$

$x \mid \exists z.\ \varphi_x(z) > x$ $\qquad (x, z) \rightarrow y$

$B \rightarrow$ R.E $\qquad (w)_1 = y,\ (w)_2 = z,\ (w)_3 = t$

$$SC_B = 1(\mu.(y, z, t).\ S(x, y, z, t) \wedge z > x))$$

$$= 1(\mu\ w.\ S(x, (w)_1, x+1+(w)_2, (w)_3)$$

QUESTION $\rightarrow$ CAN WE USE H?

$$H(\_, \_ . \_) \rightarrow 1(\mu(y, z).\ H(x, y, z) \wedge z > x)$$

$\downarrow$

NOT TOTALLY — WE MISS t = # OF STEPS; WE MISS OUTPUT

$$1(\mu.\underbrace{(y, z, t)}_{w}.\ H(x, z, t) \wedge S(x, y, z, t)\ \wedge (z > x))$$

<u>TOTALLY FORMAL</u> (SOMETIMES BUT NOT STRICT)

$$1(\mu\ w.\ \chi_H(x, (w)_2, (w)_3 \wedge \ldots \ldots)$$

B IS R.E $\rightarrow$ B IS SATURATED (?)

$B = \{x \in \mathbb{N} \mid \exists z . [\varphi_x(z) > x]\}$

e, e' $\in$ INDEXES ? Y / N (2° R.T.)

$$g(x,y) = \begin{cases} x+1 & x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

$\rightarrow e$

SMN $\rightarrow \varphi_{s(x)}(y) = g(x,y) = x+1$

2 R.T $\rightarrow \varphi_{s(e)}(y) = g(e,y) = e+1 \in W_{s(e)}$
$\in \mathbb{N}$

K $\begin{matrix} e' \\ \neq \\ e \end{matrix}$ > OUTPUT (...) $\rightarrow$ NOT SATURATED

2. R.T $\rightarrow$?

$\forall y \in \mathbb{N} \mid \varphi_e(y) = \varphi_{e'}(y) = e+1 \leq e'$

$[e \in \mathbb{N}, e' \in \mathbb{N}] \rightarrow e' \neq e$

$(e' > e)$ $[e \in B, e' \notin B]$

2. R. T.

(PROOF) RICE-THEOREM

RICE-THEOREM $\quad A \neq \emptyset,\ A \neq \mathbb{N}$

(USAGE

2.R.T.)

$[e_0, e_1 \in A] \rightarrow e_0 = 1D$

$e_1 \neq 1D$

$e_0 \in A, e_1 \notin A$

$\boxed{9.6}$

STATE THE 2.R.T. AND USE IT TO PROVE

$\exists x \in \mathbb{N}$ s.t. $\underbrace{\varphi_x(y) = x - y}_{\text{SMN!}}$

1

$g(x, y) = \begin{cases} x - y & \underbrace{x \in N_x}_{H(x,x,y) = \varphi_x(y)} \\ \uparrow & \text{otherwise} \end{cases}$

e $\quad \neq e \quad e \quad e, e \quad e$

$S : \mathbb{N} \rightarrow \mathbb{N}$ TOTAL COMPUTABLE

s.t. $g(x, y) = \varphi_{s(x)}(y) = x - y$

2.R.T $\rightarrow \exists e \in \mathbb{N} \quad \varphi_e = \varphi_{h(e)}$

$h : \mathbb{N} \rightarrow \mathbb{N}$

$g(e, y) = e - y,$

$\forall y \in \mathbb{N}$

$\boxed{9.12}$ → 2.R.T (STATEMENT)

$\exists h : \mathbb{N} \to \mathbb{N}$ (TOTAL / COMPUTABLE)

$\exists e \in \mathbb{N} \to \varphi_{h(e)} = \varphi_e$

$\mathcal{C} = \{ x \in \mathbb{N} \mid \underbrace{x \in E_x} \}$ → PROVE THIS IS NOT SATURATED

① $g(x,y) = \begin{cases} x & x \in W_x \\ \uparrow & \end{cases}$ $[\varphi_x(y)] \to x$ (e)

$s : \mathbb{N} \to \mathbb{N} \to \varphi_{s(x)} = g(x,y)$

$\exists e \to \varphi_{s(e)} = \varphi_e \to \varphi_e(y) = e$

$[\forall e' \neq e,\ E_{e'} = E_e,\ e \in \mathcal{C},\ e' \notin \mathcal{C}]$

↓ $\mathcal{C}$ IS NOT SATURATED

RECURSIVENESS

$A = \{ x \mid W_x \wedge E_x \neq \emptyset \}$ — A, $\bar{A}$ (REC / R.E)

① A IS SATURATED — y / $\boxed{N}$ → 2.R.T.

$A = \{ x \mid \varphi_x \in \mathcal{A} \}$ $\mathcal{A} = \{ f \mid dom(f) \wedge cod(f) \neq \emptyset \}$

② $\begin{bmatrix} W_x \to H \\ E_x \to S \end{bmatrix}$

GENERAL FORM

$SC_A = \mathbb{1}\,(\mu.(y,z,t).\,(H(x,y,t) \wedge S(x,y,z,t)))$

$= \mathbb{1}\,(\mu\, w.\,(\mathcal{X}_H(x,(w)_1,(w)_2) \wedge \mathcal{X}_S(x,(w)_1,$
$(w)_2,(w)_3)$

$\left[\; \varphi_{(x)}(y) > x \to (\underbrace{x+1}) \ldots \quad (\mathcal{X}_S(x,y,z,t)-1) \right]$

LOGIC

$A \to R.E$

REC?

Y — (N)

$[W_x \cap E_x \neq \emptyset]$

NOT RECURSIVE

$\{e, e'\} > OUT$

NOT REC.

RICE $\to A \neq \underbrace{\emptyset}_{e_0},\ A \neq \underbrace{\mathbb{N}}_{e_1}$

$- \emptyset$

$-$ id

$K \leq_m A$

$\underbrace{W_x \cap E_x \neq \emptyset}$ IN

$[e_0 \in A,\ e_1 \notin A]$

$e_0 = id \to e_0 \in A$

$e_1 = \emptyset \quad\; e_1 \notin A$

$\mathbb{1}\,(A \neq \emptyset)$

2 $(A \neq \mathbb{N})$ $[\ell_0 = \emptyset]$ $\underbrace{W_x \wedge E_x \neq \emptyset}_{A}$

$\ell_1 = \mathbb{N})$

$W_{e_0} \wedge E_{e_0} = \emptyset \wedge \emptyset$

$\ell_0 \notin A,\ \ell_1 \in A = \emptyset$

$\rightarrow A$ NOT REC. / A NOT R.E.

$\bar{A}$ NOT. REC. / NOT R.E. $[\bar{A} \rightarrow W_x \cup E_x = \emptyset]$

(PRIM. RECURSIVE)

- DEFINE $\mathbb{P}\mathbb{R}$ (CLASS)

ZERO / SUCC. / PROJ.

COMP. / PRIM. REC.

- USING ONLY DEF.

$a : \mathbb{N}^2 \rightarrow \mathbb{N}$

$$a(x,y) = \begin{cases} [1 & \text{if } x > 0 \wedge y > 0] \\ 0 & \text{otherwise} \end{cases}$$

PR. FUNCTION WHICH GOES 1 $\rightarrow sg(x)$

- $a(x,y) \in \mathbb{P}\mathbb{R}$

$$\begin{cases} a(x,0) = 0 \\ a(x,y+1) = sg(x) \end{cases}$$

- $sg : \mathbb{N} \rightarrow \mathbb{N}$

$$\begin{cases} sg(0) = 0 \\ sg(y+1) = 1 \end{cases}$$

PRIM. REC

$f: \mathbb{N} \to \mathbb{N} \quad \to \quad f(x,y) = \begin{cases} 1 & \text{if } x \le y \\ 0 & \text{otherwise} \end{cases}$

\- DEFINE IPR

$(x \le y) \to sg\,|y \dot- (x+1)|$

$[f(x,y)] = \begin{cases} f(x,0) = sg(x) \\ f(x,y) = sg\,|\underbrace{y - (x+1)}| \end{cases}$

SMN

$\overline{sg}(y \dot- x)$

$(y+1 > x)$

$\to \left[y \dot- 1 \quad x \dot- y \quad \overline{sg} \right]$

$\overline{sg}: \mathbb{N} \to \mathbb{N}$

$\begin{cases} \overline{sg}(0) = 1 = \overset{succ}{succ(0)} \\ \overline{sg}(y+1) = 0 \end{cases}$

$y \dot- 1 \quad \to \begin{cases} 0 \dot- 1 = 0 \\ (y+1) \dot- 1 = y \end{cases}$

$x \dot- y \quad \begin{cases} x \dot- 0 = x - 0 = x \\ x \dot- (y+1) = (x \dot- y) \dot- 1 \end{cases}$

$[y+1 > x]$

$f(x,y) = \overline{sg}(x \dot- y)$

$\overline{sg}$ (PR)... C_1^0

THEORETICAL DEF.

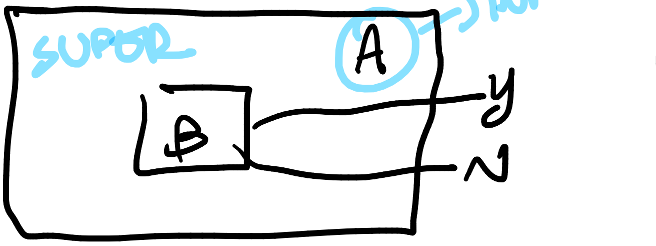
① PROVIDE REDUCTION DEFINITION, $A, B \subseteq \mathbb{N}$, $A \leq_m B$

② if B is R.E, $A \leq_m B$, A is r.e

③ IS IT POSSIBLE $A, B \subseteq \mathbb{N}$, A INFINITE, B FINITE s.t. $A \leq_m B$?

$A \leq_m B \rightarrow$ SUB

③



(KARP REDUCTION)

$A = \{\mathbb{R}\} \rightarrow$ NOT COUNT.

$B = \{\mathbb{N}\} \rightarrow$ COUNT.

$x \in A$ iff $f(x) \in B$

② if B is R.E, $A \leq_m B \Rightarrow A$ is RE

B is R.E $\rightarrow SC_B$

$A \leq_m B \rightarrow f$ computable / total if

$x \in A \Rightarrow f(x) \in B$

$\uparrow f(x)$ $\quad f(x) \in SC_B$

$H = \begin{cases} 1 & x \in W_x \\ 0 & \text{oth...} \end{cases}$

$\uparrow$

$\bar{K}$

$SC_B(x) = \begin{cases} 1 & x \in B \\ \uparrow & \end{cases}$

$SC_{K/\bar{K}} = \begin{cases} 1 & \dots \\ \uparrow & \end{cases}$

$x \in B \mid \underbrace{x \in A}_{①} \rightarrow SC_A$

A r.e / B r.e $\leftarrow$

$SC_A(x) = SC_B(f(x))$

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$A = \{\mathbb{N}\}$ $B = \{\emptyset\}$

$\emptyset = \emptyset \in \mathbb{N}$

$A \leq_m B$ works!