Recursive Sets

A set $A \subseteq \mathbb{N}$ is called recursive (or decidable) if its characteristic function is computable. The characteristic function $\chi_A : \mathbb{N} \to \mathbb{N}$ is defined as:

$\chi_A(x) = 1$ if $x \in A$
0 if $x \notin A$

In other words, a set A is recursive if there exists an algorithm that, given any $x \in \mathbb{N}$, can determine in a finite number of steps whether x belongs to A or not.

Properties of Recursive Sets

1. The class of recursive sets is closed under complement, union, and intersection.
2. If A is recursive and B is finite, then $A \cup B$ and $A \cap B$ are also recursive.
3. Every finite set is recursive.
4. The set $\mathbb{N}$ of natural numbers is recursive.
5. The set of prime numbers is recursive.

Reductions

Given two sets $A, B \subseteq \mathbb{N}$, we say that A is many-one reducible to B, written $A \leq_m B$, if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$,

$x \in A \Leftrightarrow f(x) \in B$

The function f is called a reduction from A to B.

Properties of Reductions

1. If $A \leq_m B$ and B is recursive, then A is also recursive.
2. If $A \leq_m B$ and A is not recursive, then B is not recursive.
3. The relation $\leq_m$ is transitive: if $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$.
4. If $A \leq_m B$ and $B \leq_m A$, then A and B are said to be many-one equivalent, written $A \equiv_m B$.

Exercises

Exercise 1

Prove that a set $A \subseteq \mathbb{N}$ is recursive if and only if $A \leq_m \{0\}$, where $\{0\}$ is the singleton set containing 0.

Solution

(⇒) Assume A is recursive. Define the function $f : \mathbb{N} \to \mathbb{N}$ as $f(x) = 1 - \chi_A(x)$. Then f is computable and $x \in A \Leftrightarrow f(x) = 0 \Leftrightarrow f(x) \in \{0\}$. Thus, $A \leq_m \{0\}$.

(⇐) Assume $A \leq_m \{0\}$ via a computable function f. Then $x \in A \Leftrightarrow f(x) = 0$. Define $\chi_A(x) = 1 - f(x)$. Then χ_A is computable, so A is recursive.

Exercise 2

Let A and B be recursive sets. Prove that $A \times B = \{(a, b) \mid a \in A, b \in B\}$ is also recursive.

Solution

Define the characteristic function of $A \times B$ as follows:
$\chi_{A \times B}(x) = \chi_A(\pi_1(x)) \cdot \chi_B(\pi_2(x))$
where π_1 and π_2 are computable functions that extract the first and second components of a pair, respectively.

Since χ_A, *χ_B, π_1, and π_2 are all computable,* $\chi_{A \times B}$ is also computable (as the product of computable functions is computable). Therefore, $A \times B$ is recursive.

Exercise 3

Prove that if A is recursive and B is recursively enumerable, then $A \cap B$ is recursively enumerable.

Solution

Since A is recursive, its characteristic function χ_A is computable. Since B is recursively enumerable, there exists a computable function f such that $B = \{x \mid \exists y\, f(x, y) = 1\}$.

Define the function $g(x, y) = \chi_A(x) \cdot f(x, y)$. Then g is computable, and
$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
$\Leftrightarrow \chi_A(x) = 1$ and $\exists y\, f(x, y) = 1$
$\Leftrightarrow \exists y\, g(x, y) = 1$

Therefore, $A \cap B$ is recursively enumerable.