Definition 5.1.1 A binary relation, $\leq$, on a set, X, is a *partial order* (or *partial ordering*) iff it is *reflexive*, *transitive* and *antisymmetric*, that is:

(1) (*Reflexivity*): $a \leq a$, for all $a \in X$;

(2) (*Transitivity*): If $a \leq b$ and $b \leq c$, then $a \leq c$, for all $a, b, c \in X$.

(3) (*Antisymmetry*): If $a \leq b$ and $b \leq a$, then $a = b$, for all $a, b \in X$.

A partial order is a *total order (ordering)* (or *linear order (ordering)*) iff for all $a, b \in X$, either $a \leq b$ or $b \leq a$.

When neither $a \leq b$ nor $b \leq a$, we say that *a and b are incomparable*.

A subset, $C \subseteq X$, is a *chain* iff $\leq$ induces a total order on C (so, for all $a, b \in C$, either $a \leq b$ or $b \leq a$).

The *strict order (ordering), $<$, associated with $\leq$* is the relation defined by: $a < b$ iff $a \leq b$ and $a \neq b$.

If $\leq$ is a partial order on X, we say that the pair $\langle X, \leq \rangle$ is a *partially ordered set* or for short, a *poset*.

Remark: Observe that if $<$ is the strict order associated with a partial order, $\leq$, then $<$ is transitive and *anti-reflexive*, which means that

(4) $a \not< a$, for all $a \in X$.

Conversely, let $<$ be a relation on X and assume that $<$ is transitive and anti-reflexive.

Then, we can define the relation $\leq$ so that $a \leq b$ iff $a = b$ or $a < b$.

It is easy to check that $\leq$ is a partial order and that the strict order associated with $\leq$ is our original relation, $<$.

Given a poset, $\langle X, \leq \rangle$, by abuse of notation, we often refer to $\langle X, \leq \rangle$ as the *poset X*, the partial order $\leq$ being implicit.

If confusion may arise, for example when we are dealing with several posets, we denote the partial order on X by $\leq_X$.

Here are a few examples of partial orders.

1. **The subset ordering**. We leave it to the reader to check that the subset relation, $\subseteq$, on a set, X, is indeed a partial order.

   For example, if $A \subseteq B$ and $B \subseteq A$, where $A, B \subseteq X$, then $A = B$, since these assumptions are exactly those needed by the extensionality axiom.

2. **The natural order on $\mathbb{N}$**. Although we all know what is the ordering of the natural numbers, we should realize that if we stick to our axiomatic presentation where we defined the natural numbers as sets that belong to every inductive set (see Definition 1.10.3), then we haven't yet defined this ordering.

However, this is easy to do since the natural numbers are sets. For any $m, n \in \mathbb{N}$, define $m \leq n$ as $m = n$ or $m \in n$.

Then, it is not hard check that this relation is a total order (Actually, some of the details are a bit tedious and require induction, see Enderton [4], Chapter 4).

3. **Orderings on strings**. Let $\Sigma = \{a_1, \ldots, a_n\}$ be an alphabet. The prefix, suffix and substring relations defined in Section 2.11 are easily seen to be partial orders.

   However, these orderings are not total. It is sometimes desirable to have a total order on strings and, fortunately, the lexicographic order (also called dictionnary order) achieves this goal.

In order to define the *lexicographic order* we assume that the symbols in Σ are totally ordered, $a_1 < a_2 < \cdots < a_n$. Then, given any two strings, $u, v \in \Sigma^*$, we set

$$u \preceq v \quad \begin{cases} \text{if } v = uy, \text{ for some } y \in \Sigma^*, \text{ or} \\ \text{if } u = xa_iy, \; v = xa_jz, \\ \text{and } a_i < a_j, \text{ for some } x, y, z \in \Sigma^*. \end{cases}$$

In other words, either u is a prefix of v or else u and v share a common prefix, x, and then there is a differring symbol, a_i in u and a_j in v, with $a_i < a_j$.

It is fairly tedious to prove that the lexicographic order is a partial order. Moreover, the lexicographic order is a total order.