27/11

OVERVIEW OF TODAY

→ RECURSIVE SETS (PARALLELS WITH PREDICATES - MORE LATER)

→ REDUCTION] → 7.X

→ MANY EXERCISES

[RECURSIVE SETS]

$\underbrace{A \subseteq \mathbb{N}}$ → SET = HAS PROPERTIES ↗ TM

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

[HALTS / DOES NOT HALT]

$[\chi_A] \to 1 \; (H(\ldots) \land S(\ldots))$

$A \to A \cup B,\; A \cap B$

[K → HALTING SET]

$$\to \chi_K(x) = \begin{cases} 1 & x \in W_K \\ 0 & x \notin W_K \end{cases}$$

NOT RECURSIVE (BY DEFINITION) → $[A \leq_m K]$

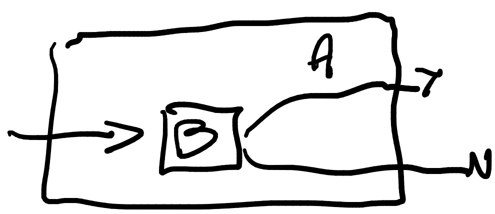
$\{x \mid \varphi_x \text{ TOTAL}\}$

[REDUCTION]

A, B sets

$[A] \leq_m B$ ↙ SUBROUTINE

↑ "BIGGER" PROBLEM



$\rightarrow \left[x \in A \iff f(x) \in B \right]$

$\downarrow$

$A \leq_m B$ $\qquad \left[K \leq_m T \right]$

T = SET OF TOTAL FUNCTIONS $\rightarrow$ [T is not recursive]

① SMN THEOREM $\rightarrow g(x,y) = \begin{cases} 1 & x \in W_x \\ \uparrow & \text{otherwise } (x \notin W_x) \end{cases}$ (x)

$= W_x \rightarrow \varphi_x(x) \qquad g(x,y)$

$= 1(\varphi_x(x)) = 1(\varphi_u(x,x))$ $\rightarrow$ COMPUTABLE FORM

$x \in K$ [iff $\iff$] $f(x) \in T$

T = set of total functions

$\boxed{1}$ $x \in K \Rightarrow x \in W_K$

$\boxed{2}$ $x \notin K \Rightarrow x \notin W_K$

SMN THEOREM $\nearrow$ $\left[\begin{array}{l} s : \mathbb{N} \rightarrow \mathbb{N} \\ \varphi_{s(x)}(y) = g(x,y) \end{array} \right]$

$\boxed{1}$ $\forall y, \varphi_{s(x)}(y) = g(x,y) = 1 \rightarrow \varphi_{s(x)}$ total $1 \in \mathbb{N}$

[2] $\forall y,\ \varphi_{s(x)}(y) = g(x,y) = \uparrow$

$\varphi_{s(x)} \notin T$

$\varphi_{s(x)}\, s(x) = \uparrow$ $[\varnothing]$

EMPTY SET

[7.12] A ⊆ ℕ is recursive iff $A \leq_m \{0\}$

($\Rightarrow$)

If A is recursive (computable), $\chi_A(x)$ exists $\rightarrow \chi_A = \begin{cases} 1 & x \in \mathbb{N} \\ 0 & \end{cases}$

$\chi_A(x) = |\varphi_x(x)|$

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ (reduction function)

$[x \in A \iff f(x) \in B]$ ⇐ POINT OF LOGICS

$x \in A \rightarrow f(x) = \overline{sg}|1 \dot{-} \chi_A(x)|$

$f(x) = 0 \rightarrow 0$ (CORRECT)

($\Leftarrow$) $A \leq_m \{0\} \rightarrow$ A is recursive

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ $f(x) \in \{0\} \rightarrow f(x) = 0$

0 is the output

$\chi_A(x) =$ "$f(x) = 0$"

$= \overline{sg}(f(x)) \rightarrow$ computable

A is recursive

[7.5] - A is a recursive set
- $f : \mathbb{N} \to \mathbb{N}$ total computable function

Is it true → $f(A)$ is r.e?

REC → $\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ ← IT ALWAYS STOPS

R.E. → $SC_A = \begin{cases} 1 & x \in A \\ \uparrow & x \notin A \end{cases}$ ← ∃ (SOMETIMES) AN ALGORITHM (DEPENDING ON THE INPUT) STOPPING

$$\left[f(A) = \{ y \in \mathbb{N} \mid \exists x \in A \; y = f(x) \to 5 \to f(5) = 5 \right]$$

$\{A$ is recursive / f total - computable$\}$

① Is $f(A)$ r.e?

⇒ A is recursive | ∃ χ_A computable

$\chi_A = \begin{cases} 1 & x \in W_x \\ 0 & \text{otherwise} \end{cases}$ → "$f(A)$ is r.e"

$SC_A(x) = [\;\dots\;]$

$H(x, y, t)$ → FUNCTION (PREDICATE) STOPS
x = INPUT, y = OUTPUT, t = # OF STEPS

$$\left[\chi_H = \begin{cases} 1 & \text{if } x \in W_H \\ 0 & \text{otherwise} \end{cases} \right]$$

$$SC_A(x) = \left[y \cdot \chi_H^{(1)}(x, y, t) \right] + \left[y \cdot \overline{sg}\left(\chi_H(x, y, t)\right) \right]^{1}$$

STOPS (↓) AND GETS Y

DOES NOT STOP ON THE SAME CONDITIONS

INTEGER TUPLE NOTATIONS $\rightarrow \pi((w)_1,(w)_2)$

$(w)_1 = y,\ (w)_2 = A$

$= \mu \| [(w)_1 \cdot H(x,(w)_1,(w)_2)] + [(w)_1 \cdot \bar{s}\, H(\dots)]$

ONLY DOING μ (MINIMIZE) ONCE $\rightarrow [\quad]\ 1$

A [RECURSIVE]
A [R.E] $\rightarrow$ PREDICATES ["HI, "AT" ....]

$\downarrow$ PROJECTION

$\downarrow$ STRUCTURE

$[Q] =$ QUANTIFIER $\Rightarrow \exists x.\, Q(\vec{x}, y)$ STOPS

$\Rightarrow \exists A.\, \mathcal{X}_Q(\vec{x}, y, A)$

RECURSIVE $\rightarrow$ DECIDABLE $\quad P(\vec{x},y)\downarrow = \begin{smallmatrix}1\\0\end{smallmatrix}$

R.E $\rightarrow$ SEMIDECIDABLE $\quad P(\vec{x},y)\downarrow = \begin{smallmatrix}1\\\uparrow\end{smallmatrix}$

[3.2] STATE THE SMN-THEOREM

$$\left[\ \varphi_{(e)}^{(m+n)}(\vec{x},\vec{y}) = \varphi^{(n)}_{s_{m,n}(e,\vec{x})}(\vec{y})\ \right]$$

$\exists$ total / computable function $s : \mathbb{N} \rightarrow \mathbb{N}$

$|W_{s(x)}| = 2x \quad , \quad |E_{s(x)}| = x$

$$g(x,y) = \begin{cases} qt(x,y) & \text{if } y < 2x \\ \uparrow & \text{otherwise} \end{cases}$$

$$g(x,y) = sg(y) \cdot qt(x,y) + \mu z. (y+1 \dot{-} 2x)$$

$$\left[g : \mathbb{N} \to \mathbb{N} \to \varphi_{s(x)}(y) = g(x,y) \quad x,y \in \mathbb{N} \right]$$

SMN-THEOREM

① $W_{s(x)} = \{ y \mid g(x,y)\downarrow \}$ ..... $= 2x$

② $E_{s(x)} = \{ g(x,y) \mid x \in W_{s(x)} \}$ .... $= x$

① $W_{s(x)} = \{ y \mid y < 2x \} \to \mathbb{N}$ $\quad y \; \exists \to$ x GOOD

(TOTAL DOMAIN)

② $E_{s(x)} = \{ qt(x,y) \mid y < 2x \}$

$= \{ (y+1) \dot{-} 2x \mid y+1 < 2x \} = [0, 2x)$

EXPLOIT ORDERING OF NATURALS

$\exists y < 2x \mid qt(\cdot)$

[6.28] $\left[\begin{array}{l} \chi_K \to \text{NOT REC.} \\ \text{DIAG.} \to \text{NOT COMP.} \end{array} \right]$

$\exists$ total / not computable s.t. $f(x) = \frac{x}{2}$ (?)

$\forall x \in \mathbb{N}$ (even)

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ even and } x \in W_x \\ \varphi_{\frac{(x-1)}{2}} + 1 & \text{if } x \text{ odd and } x \in W_x \\ 0 & \text{otherwise} \end{cases}$$

$(x \text{ even})$

$W_x = $ DOMAIN

$W_{\frac{(x-1)}{2}}$

$[x-1 < x]\downarrow$

$$\varphi_{\frac{x-1}{2}} = \begin{array}{c} 0 \\ 2 \\ 4 \end{array} \begin{bmatrix} \bullet & & \\ \dots & \bullet & \\ & \dots & \bullet \end{bmatrix}$$

x even

$x \in W_x$

$\leftarrow \dots$

$\frac{x}{2}$

$\rightarrow$ if $f(x)\downarrow$ $(\dots)$ $x \in W_x$ $\varphi_{\frac{(x-1)}{2}}+1$

$\rightarrow$ if $f(x)\uparrow$ $\dots$ $\dots$

9.4 $\in$ IPR

$\left[\begin{array}{l} \text{EVEN} \\ \text{ODD} \end{array}\right\}$ COMPUTABLE $\in \mathbb{N}]$

TOTAL $\rightarrow$ DIVISION OCCURS ONLY BETWEEN ODD AND EVEN

$f \rightarrow$ TOTAL NOT COMPUTABLE

$x \in W_x$

$x \notin W_x$

$[\varphi_x \neq \varphi_{x+1}]$

$$\varphi_{\frac{(x-1)}{2}}(x) \neq \varphi_{\frac{(x-1)}{2}}(x+1)$$

BY DIAGONALIZATION

[6.31] $f: \mathbb{N} \rightarrow \mathbb{N}$ s.t. $dom(f) = K$ $\rightarrow$ WE ARE TRYING TO STOP

$\exists f$ computable?

$[cod(f) = \mathbb{N}]$

$\uparrow$ STOP IS DEFINED FOR SURE

$$g(x,y) = \begin{cases} y & [x \in K] \\ \uparrow & \text{otherwise} \end{cases}$$

$\exists g : \mathbb{N} \to \mathbb{N}$

$x \in K \quad g(x,y)\downarrow = y \quad (x \in W_K)$

$x \notin K, \; g(x,y)\uparrow = \uparrow \quad (x \notin W_K)$

REDUCTION (IN PRACTICE)

[$dom(f) = K$] [$cod(f) = \mathbb{N}$] $\varphi_x(x) = f(x) = \begin{cases} 1 & \text{if } H(x,x,y)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$

$\equiv$

$f(x) = \begin{cases} 1 & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$

$f(x) = (\mu t \,.\, H(x,x,t) - 1)$

(t = STOP ON T ~~#~~ OF STEPS)

① $f(x)\downarrow \Rightarrow H(x,x,t) \quad \exists x \in K \mid \varphi_x(x) =$

$\begin{bmatrix} 1 & \to x = 5 \\ \uparrow & \text{otherwise} \end{bmatrix}$

NOT

② $f(x)\uparrow \Rightarrow \neg H(x,x,t)$

$\bar{K} \quad \chi_K = \begin{cases} 1 & \neg H(x,x,y) \\ 0 & \text{otherwise} \end{cases}$

$\nexists x \in K \mid \varphi_x(x)\uparrow \mid \leftarrow$ BY CONSTRUCTION

$[g \,.\, \neg x \,..]$