06/11/2024 → Computability (MEETING 3)

(OVERVIEW)

→ ENUMERATING URM PROGRAMS
→ ACKERMANN FUNCTION
 - DIAGONALIZATION
 - MINIMIZATION } EXAMPLES → HOW TO WRITE THEM
→ EXERCISES ON PR / URM

PR
- ZERO
- SUCCESSOR
- PROJECTION
- COMPOSITION
- MINIMALIZATION

ACKERMANN → PR → PARTIAL RECURSIVE (Defined on some inputs)

ACKERMANN → TOTAL NOT PRIMITIVE RECURSIVE

Ψ (PSI) →

$$\begin{cases} \psi(0, y) = y+1 \\ \psi(x+1, 0) = \psi(x, 1) \\ \psi(x+1, y+1) = \psi(x, \psi(x+1, y)) \end{cases}$$

MINIMIZATION (μ) = UNBOUNDED
↳ EXPRESS TOTAL THINGS
(EVEN WHEN THEY ARE NOT)

$\underbrace{x} \cdot \underbrace{y}$

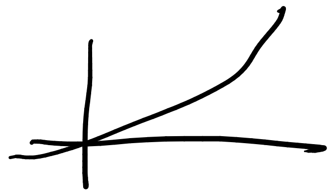
PR → OPERATIONS — O, S, Π

IR → INPUTS

ACKERMANN → START : IR (x, y)

OUTPUT :

n times FOR LOOP



→ ACKERMANN → WHY WE USE

μ - OPERATOR   PR = TRUE

μ = OPERATOR $\mu(z).\, |x = y|$

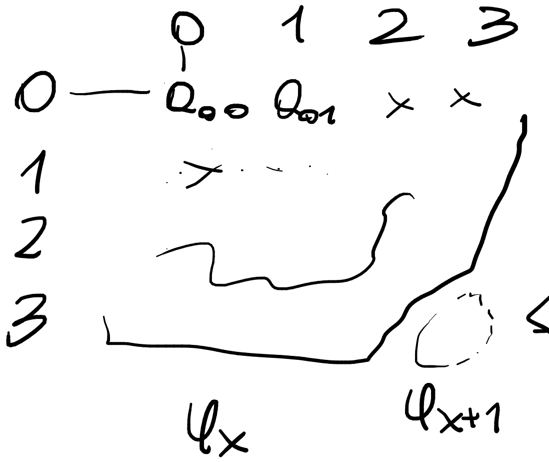
FUNCTION (ψ) — TOTAL — COMPUTABLE — $\psi \notin PR$

IR → PARTIAL → SOME INPUTS

W_n = DOMAIN E_n = CODOMAIN

$[\underbrace{\varphi_x}]$ → VARPHI (PHI)

^ BY CONSTRUCTION ⇒ DIFFERENT FROM EVERY INPUT



$$f(x) = \begin{cases} x & x \in W_x \\ \uparrow & \text{OTHERWISE} \end{cases}$$

$f(x) = x \to$ TOTAL $= \varphi_x$

$f(x) \neq x$

$[\varphi_x] \neq [\varphi_{x+1}]$

$\boxed{5.2}$ → Exercises

$f: \mathbb{N} \to \mathbb{N}$ TOTAL-INCREASING

$\forall x, y \in \mathbb{N} \mid f(x) < f(y)$

(2) $x \circ y$

- PROVE THE SET OF TOTAL INCREASING FUNCTIONS IS NOT COUNTABLE

$[\underbrace{\Pi / \Sigma}_{\text{BOUNDED}} / \mu$ - UNBOUNDED $]$

(2) $[x, y]$

z = SMALLEST VALUE FOR WHICH P_y IS TRUE

$$\mu z . \underbrace{|f(y) \dot{-} f(x)|}_{P_x}$$

$x, y \in \underbrace{\mathbb{N}}_{\text{ORDERED}}$ — WELL-FOUNDED

$(x = y = 1) \to 0$

TO COMPUTE $\Rightarrow$ Π BOUNDED PRODUCT

$(x, y = \dots)$

$$\to \mu z . |\prod_{i=0}^{x} \prod_{i=0}^{y} f(y) \dot{-} f(x)|$$

$[\varphi_x]$ → RECURSION DEFINED BY TERMS OF ITSELF (x)

[SHOW THAT "..." IS NOT COMPUTABLE]

DIAGONALIZATION

$\underline{K}$ = HALTING SET

↑ NOT RECURSIVE - COMPUTABLE

$x, y \in \mathbb{N} \quad f(x) < f(y)$

$g \to x \circ y \qquad g(x) < g(x+1)$

↑ COMPOSITION

W_x = DOMAIN

$$g(x) = \begin{cases} \varphi_x(x) \underline{+1} & x \in W_x] \\ 0 & \text{OTHERWISE (NOT IN DOMAIN)} \end{cases}$$

[- $x \in W_x \to \varphi_x(x)+1$ (x) ← DEFINED IN TERMS OF x

 - $x \notin W_x \to \varphi_x(x)+1 \neq \varphi_x(x+1)+1$]

↑ BUILT TO BE DIFFERENT FROM ALL INPUTS

] SMN - THEOREM

- $\exists$ NON - COMPUTABLE TOTAL FUNCTION

$[f : \mathbb{N} \to \mathbb{N}$ s.t. $img(f) = \mathbb{N} \setminus \underline{\{0\}}$?]

$[img(f) = \{f(x) \mid x \in \mathbb{N}\}]$

IMAGE = CODOMAIN = OUTPUT

$\setminus$ = SET MINUS =

↓

→ $\mu w. \mid f(x) \mid$ → DOES NOT WORK

↑

(w) = "FAKE" VARIABLE FOR MINIMIZATION

SEMI-CHARATERISTIC $\rightarrow$ R.E.

$\left[\begin{matrix} 1 \\ \uparrow \end{matrix} \rightarrow \text{SEMI-CHAR.} \right] \left[\begin{matrix} 1 \\ 0 \end{matrix} \rightarrow \text{RECURSIVE (CHARATERISTIC)} \right]$

$$f(x) = \begin{cases} x+1 & x \in \mathbb{N}_x \\ 0 & \text{OTHERWISE} \end{cases}$$

$= \mu[x].|f(x)+1 \dot{-} x|$

$= \mu w. | 6 - 5| <$

if $x = 5$

$f(x) = x \rightarrow$ IDENTITY FUNCTION

$f(5) = 5] \rightarrow$ EXAMPLE

function
- TOTAL $\rightarrow$ DEFINED FOR ALL INPUTS
- NOT COMPUTABLE $\rightarrow f(x) = x+1$

$(\mathbb{N} \setminus \{0\})$ $x \in \mathbb{N}_x$

$f(x) \geq 0$

$[x \quad x+1]$ 0

$\rightarrow f(x) \neq \mathbb{N}x$

$f(x) \geq 0$ 5 6 7. 0

$\exists w \in \mathbb{N} \mid \mu w. [\ldots\ldots]$

$\mu w. |f(x)|$ NO $\uparrow [x = 0]$

$\mu w. |f(x) = x|$

TRYING TO ACHIEVE

PARTIAL $\rightarrow x = 0$

$$f(x) = \begin{cases} [\varphi_x(x+1) & \text{if } x \in W_x] \\ 0 & \text{otherwise} \end{cases}$$

$\boxed{2.3}$ - Define IPR $] \leftarrow$ 0, S, π_j^i → GENERALIZED COMPOSITION

- Prove half: $\mathbb{N} \to \mathbb{N}$

defined by half $= \left\lfloor \frac{x}{2} \right\rfloor$ is IPR

$$\begin{cases} half(0) = \frac{0}{2} = 0 \\ half(y+1) = \underbrace{half(y)}_{} \end{cases}$$

$[Q(\vec{x}, y)]$ → = PROJECTING "x" OVER "y"

↑ DIVISION (!!) → 0/... (?)

MM → OUTPUT

$Mm_2(x)$ → IPR

$$Mm_2(x) = \begin{cases} Mm_2(0) = 0 & [\text{EVEN}] \\ Mm_2(x+1) = \overline{sg}(Mm_2(x)) & [\text{ODD}] \end{cases}$$

$$sg(x) = \begin{cases} sg(\cancel{0}) = 0 \\ sg(x+1) = 1 \end{cases}$$

$[half(y+1) = half(y) + Mm_2(y)]$ → Mm_2, sg → PR

half → IPR

$$\overline{sg}(x) = \begin{cases} sg(0) = 1 \\ sg(x+1) = 0 \end{cases}$$

COMPOSITION / PROJECTION

MINIMIZATION

- $a = b \rightarrow sg(|a-b|)$
- $a > b \rightarrow \overline{sg}(a-b)$
- $a \geq b \rightarrow \overline{sg}(a+1-b)$
- OR $(+) \rightarrow$ MULTIPLICATIONS
- AND $(\cdot) \rightarrow$ ADDITIONS
- NOT $(!) \rightarrow$ SIGNS / [NEGATED SIGNS] $\nearrow 1$, $\searrow 0$

$\mu w. |\ \ldots\ldots\ldots|$ $\begin{cases} 1 \rightarrow (\ldots)\ (*) \\ 0 \rightarrow \text{UNDEFINED} \end{cases}$

$(*)$

EXAMPLES (TOY EXAMPLES OF READY FUNCTIONS)

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \text{ is even and } x > 0 \\ 2 & \text{if } x \text{ is odd} \end{cases}$$

$sg \rightarrow$ SIGN

BASE CASE $= 0$

$$= \mu w. |[sg(x) + sg(rm(x,2)) + 1]|$$

$(1 \dot{-} sg(x))$

$[N^2 \rightarrow N^2]$

$$f(x,y) = \begin{cases} x & \text{if } x > y \\ y & \text{if } y > x \\ 100 & \text{if } x \notin W_x \end{cases}$$

$(*) = \uparrow$ $\downarrow$ $x < y$ OR $y < x$

$(100 \cdot \overline{sg}|x-y|)$

$$= \mu w. |x \cdot \overline{sg}(x \dot{-} y) + y \cdot \overline{sg}(y \dot{-} x)| \cdot (*)$$

$\downarrow$