Computability Exam Solutions

July 11, 2011

Exercise 1

Definition of the class PR of primitive recursive functions

The class PR of primitive recursive functions is the smallest class of functions $PR \subseteq \bigcup_k(\mathbb{N}^k \to \mathbb{N})$ that:

1. **Contains the basic functions:**
   - Zero function: zero(x) = 0
   - Successor function: succ(x) = x + 1
   - Projection functions: $\pi_i^k(x_1,\ldots,x_k) = x_i$ for $1 \leq i \leq k$
2. **Is closed under composition:** If $g_1,\ldots,g_m \in PR$ and $h \in PR$, then $f \in PR$ where $f(\vec{x}) = h(g_1(\vec{x}),\ldots,g_m(\vec{x}))$
3. **Is closed under primitive recursion:** If g, h ∈ PR, then f ∈ PR where:

```
f(x⃗, 0) = g(x⃗)
f(x⃗, y+1) = h(x⃗, y, f(x⃗, y))
```

Proof that χ_p (characteristic function of even numbers) is primitive recursive

We need to show that $\chi_p(x) = 1$ if x is even, 0 if x is odd.

Method: Use the fact that x is even ⟺ x mod 2 = 0.

First, we establish auxiliary functions:

1. **Predecessor function pred(x) = x ∸ 1** (primitive recursive):

```
pred(0) = 0
pred(y+1) = y
```

2. **Subtraction function sub(x,y) = x ∸ y** (primitive recursive):

```
sub(x,0) = x
sub(x,y+1) = pred(sub(x,y))
```

3. **Division by 2 with remainder - define rem2(x) = x mod 2:**

```
rem2(0) = 0
rem2(1) = 1
rem2(x) = rem2(x ∸ 2) for x ≥ 2
```

   This can be implemented using primitive recursion:

```
rem2(0) = 0
rem2(y+1) = {
  1  if rem2(y) = 0
  0  if rem2(y) = 1
} = sḡ(rem2(y))
```

   where $\bar{sg}(x) = 1$ if $x = 0$, 0 otherwise (primitive recursive).

4. **Characteristic function of even numbers:**

```
χₚ(x) = sḡ(rem2(x))
```

Since $\bar{sg}$ and rem2 are primitive recursive, and χ_p is obtained by composition, $\chi_p \in PR$.

Alternative direct construction:

```
χₚ(0) = 1
χₚ(y+1) = sḡ(χₚ(y))
```

This alternates between 1 and 0, giving 1 for even numbers and 0 for odd numbers.

Exercise 2

Question: Can there exist non-computable f such that there exists non-computable g where f + g is computable?

Answer: Yes, such functions exist.

Construction:

Let K be the halting set and define:

- $f = \chi_k$ (characteristic function of K)
- $g = \chi_{\bar{k}}$ (characteristic function of $\bar{K}$)

Verification:

1. **f is not computable:** Since K is not recursive, χ_k is not computable.
2. **g is not computable:** Since $\bar{K}$ is not recursive, $\chi_{\bar{k}}$ is not computable.
3. **f + g is computable:** For any $x \in \mathbb{N}$:

   ```
   (f + g)(x) = χₖ(x) + χₖ̄(x) = {
     1 + 0 = 1  if x ∈ K
     0 + 1 = 1  if x ∉ K
   } = 1
   ```

   So f + g is the constant function 1, which is computable.

Therefore, non-computable functions f and g exist such that f + g is computable.

Note: This differs from Exercise 2 in the previous exam, which asked whether f + g is computable for **every** other non-computable g.

Exercise 3

Classification of $A = \{x \in \mathbb{N} : \varphi_x(x) \downarrow \land \varphi_x(x) < x + 1\}$

A is r.e.:

```
scₐ(x) = 1(μt. S(x,x,ϕₓ(x),t) ∧ ϕₓ(x) < x + 1)
```

More precisely:

```
scₐ(x) = 1(μ⟨v,t⟩. S(x,x,v,t) ∧ v < x + 1)
```

This searches for v,t such that $\varphi_x(x) = v$ in exactly t steps and $v < x + 1$.

A is not recursive: We show $K \leq_m A$. Define $g : \mathbb{N}^2 \to \mathbb{N}$ by:

```
g(x,y) = {
  0      if x ∈ K
  x + 1  if x ∉ K
} = 0 · scₖ(x) + (x + 1) · (1 - scₖ(x))
```

Since we can't compute sc_k directly, we use:

```
g(x,y) = {
  0  if ∃t. H(x,x,t)
  x + 1  otherwise
}
```

By s-m-n theorem, ∃s such that $\varphi_{s(x)}(y) = g(x,y)$.

For the reduction, we need $\varphi_{s(x)}(s(x))$ since A tests $\varphi_x(x)$:

- If $x \in K$: $\varphi_{s(x)}(s(x)) = 0 < s(x) + 1$, so $s(x) \in A$
- If $x \notin K$: $\varphi_{s(x)}(s(x)) = s(x) + 1 \nless s(x) + 1$, so $s(x) \notin A$

Wait, this doesn't work directly because s(x) appears in the inequality.

Let me use a different approach. Define:

```
g(x,y) = {
  x      if x ∈ K
  x + 1  if x ∉ K
}
```

Then:

- If $x \in K$: $\varphi_{s(x)}(s(x)) = s(x)$, and we need $s(x) < s(x) + 1$, which is true, so $s(x) \in A$
- If $x \notin K$: $\varphi_{s(x)}(s(x)) = s(x) + 1$, and we need $s(x) + 1 < s(x) + 1$, which is false, so $s(x) \notin A$

This gives $K \leq_m A$, so A is not recursive.

Ā is not r.e.: Since A is r.e. but not recursive, Ā is not r.e.

Final classification: A is r.e. but not recursive; Ā is not r.e.

Exercise 4

Classification of $B = \{x \in \mathbb{N} : 2W_x \subseteq E_x\}$

where $2X = \{2x : x \in X\}$.

The set B is saturated since $B = \{x \mid \varphi_x \in \mathcal{B}\}$ where $\mathcal{B} = \{f \mid 2 \cdot dom(f) \subseteq cod(f)\}$.

B is not r.e.: We use Rice-Shapiro theorem. Consider the identity function $id \notin B$ since:

- $dom(id) = \mathbb{N}$
- $2 \cdot dom(id) = 2\mathbb{N} = \{0, 2, 4, 6, ...\}$
- $cod(id) = \mathbb{N}$
- $2\mathbb{N} \subseteq \mathbb{N}$ is true

Wait, let me recalculate. We have $2\mathbb{N} \subseteq \mathbb{N}$, so $id \in B$.

Consider the finite function $\theta = \{(1, 0)\} \subseteq id$:

- $dom(\theta) = \{1\}$
- $2 \cdot dom(\theta) = \{2\}$
- $cod(\theta) = \{0\}$
- $\{2\} \not\subseteq \{0\}$, so $\theta \notin B$

Since $id \in B$ and $\exists$ finite $\theta \subseteq id$ with $\theta \notin B$, by Rice-Shapiro theorem, B is not r.e.

$\bar{B}$ is not r.e.: Consider the function $f(x) = x + 1$:

- $dom(f) = \mathbb{N}$
- $2 \cdot dom(f) = 2\mathbb{N} = \{0, 2, 4, 6, ...\}$
- $cod(f) = \{1, 2, 3, 4, ...\}$
- $2\mathbb{N} \not\subseteq \{1, 2, 3, 4, ...\}$ since $0 \notin \{1, 2, 3, 4, ...\}$

So $f \notin B$.

Consider the finite function $\theta = \{(0, 1)\} \subseteq f$:

- $2 \cdot dom(\theta) = \{0\}$
- $cod(\theta) = \{1\}$
- $\{0\} \not\subseteq \{1\}$, so $\theta \notin B$, hence $\theta \in \bar{B}$

Since $f \in \bar{B}$ and $\exists$ finite $\theta \subseteq f$ with $\theta \in \bar{B}$, this doesn't directly apply Rice-Shapiro for $\bar{B}$.

By Rice's theorem and the analysis above, both B and $\bar{B}$ are not r.e.

Final classification: B and $\bar{B}$ are both not r.e. (and hence not recursive).

Exercise 5

Question: Can there exist an index $x \in \mathbb{N}$ such that $\bar{K} = \{2^y - 1 : y \in E_x\}$?

Answer: No, such an index cannot exist.

Proof:

Suppose there exists x such that $\bar{K} = \{2^y - 1 : y \in E_x\}$.

Key observations:

1. $\bar{K}$ is not r.e.
2. The set $\{2^y - 1 : y \in E_x\}$ is r.e. (as the image of E_x under the computable function $f(y) = 2^y - 1$)

Detailed argument:

Since E_x is r.e. (as the codomain of the partial computable function φ_x), and the function $f(y) = 2^y - 1$ is total and computable, the set:

```
{2ʸ - 1 : y ∈ Eₓ} = f(Eₓ)
```

is r.e. (as the image of an r.e. set under a computable function).

But $\bar{K}$ is not r.e., so we cannot have $\bar{K} = \{2^y - 1 : y \in E_x\}$.

Alternative proof using cardinality:

The set $\{2^y - 1 : y \in \mathbb{N}\} = \{0, 1, 3, 7, 15, 31, ...\}$ has density 0 in $\mathbb{N}$ (the number of elements $\leq n$ grows like log n).

However, $\bar{K}$ has positive density (roughly half of all numbers), so $\bar{K}$ cannot be equal to any subset of $\{2^y - 1 : y \in \mathbb{N}\}$.

Therefore, no such index x can exist.