Computability Exam Solutions

September 24, 2009

Exercise 1

Projection Theorem and Counterexample

Projection Theorem: If predicate P(x,y) is semi-decidable, then ∃x.P(x,y) is also semi-decidable.

Proof:

Since P(x,y) is semi-decidable, there exists a computable function f such that:

```
P(x,y) ⟺ f(x,y) ↓
```

Define the semi-characteristic function for ∃x.P(x,y):

```
sc_{∃x.P}(y) = 1(μ⟨x,t⟩. f(x,y) converges in ≤ t steps)
```

Algorithm:

```
To semi-decide ∃x.P(x,y):
1. For t = 0, 1, 2, ...:
   2. For x = 0, 1, ..., t:
      3. Run f(x,y) for at most t steps
      4. If f(x,y) converges, return 1
5. (Never terminates if ∀x.¬P(x,y))
```

If ∃x such that P(x,y), then eventually we'll find such x and the algorithm terminates.
If ∀x.¬P(x,y), the algorithm never terminates (correct for semi-decidability).

Therefore, ∃x.P(x,y) is semi-decidable.

Counterexample for the reverse direction:

Let Q(y) ≡ ∃x.P(x,y) where P(x,y) ≡ (x ∉ K ∧ y = 0).

Then:

- Q(0) ≡ ∃x.(x ∉ K ∧ 0 = 0) ≡ ∃x.(x ∉ K) ≡ True (since $\bar{K} \neq \emptyset$)
- Q(y) ≡ False for y > 0

So Q(y) is the predicate "y = 0", which is decidable, hence semi-decidable.

However, P(x,y) ≡ (x ∉ K ∧ y = 0) is not semi-decidable because:

- We cannot semi-decide $x \notin K$ (since $\bar{K}$ is not r.e.)
- Even though $y = 0$ is decidable, the conjunction with the non-semi-decidable predicate $x \notin K$ makes $P(x,y)$ non-semi-decidable

Therefore, $\exists x.P(x,y)$ can be semi-decidable while $P(x,y)$ is not.

Exercise 2

Question: Does there exist a total non-computable $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) \neq \varphi_x(x)$ for only one value $x \in \mathbb{N}$?

Answer: No, such a function cannot exist.

Proof:

[Same proof as September 14, Exercise 2]

Suppose f is total, non-computable, and differs from the diagonal $\varphi_x(x)$ at exactly one point $x = c$.

Define $g : \mathbb{N} \to \mathbb{N}$ by:

```
g(x) = {
  f(c)     if x = c
  φₓ(x)    if x ≠ c
}
```

Then g is computable:

- Checking $x = c$ is decidable
- $f(c)$ is a fixed constant
- $\varphi_x(x)$ is computable via universal function

Since $g(x) = f(x)$ for all x, we have $g = f$, contradicting f being non-computable.

Therefore, no such function exists.

Exercise 3

Classification of $A = \{x \in \mathbb{N} : \varphi_x(y) = y$ for infinitely many $y\}$

The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid f(y) = y$ for infinitely many $y\}$.

A is not r.e.: We use Rice-Shapiro theorem. Consider the identity function $id \in \mathcal{A}$ since $id(y) = y$ for all y (infinitely many).

For any finite function $\theta \subseteq id$, θ has finite domain, so $\theta(y) = y$ for only finitely many y. Therefore $\theta \notin \mathcal{A}$.

Since $id \in \mathcal{A}$ and $\forall$ finite $\theta \subseteq id$: $\theta \notin \mathcal{A}$, by Rice-Shapiro theorem, A is not r.e.

$\bar{A}$ is not r.e.: Consider the constant function $f(x) = 0$. Then $f \notin A$ since $f(y) = y$ only when $y = 0$ (just one point, not infinitely many).

For any finite function $\theta \subseteq f$, since f is constant 0, θ maps its domain to $\{0\}$. For $\theta(y) = y$, we need $y = 0$, so θ can equal the identity on at most one point ($y = 0$).

Therefore $\theta \notin A$ for any finite $\theta \subseteq f$.

Since all finite functions have finite domains, no finite function can equal the identity on infinitely many points. By Rice-Shapiro analysis, $\bar{A}$ is also not r.e.

Final classification: A and $\bar{A}$ are both not r.e. (and hence not recursive).

Exercise 4

Proof that $P \leq_m Pr$ and $Pr \leq_m P$

where $P = \{0, 2, 4, 6, ...\}$ (even numbers) and $Pr = \{2, 3, 5, 7, 11, ...\}$ (primes).

Proof of $P \leq_m Pr$:

We need a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that:

```
x ∈ P ⟺ f(x) ∈ Pr
```

Define $f(x) = 2x + 2$.

Verification:

- f is total and computable (linear function)
- If $x \in P$: x is even, so $x = 2k$ for some $k \geq 0$, thus $f(x) = 2(2k) + 2 = 4k + 2 = 2(2k + 1)$. Since $2k + 1 \geq 1$, we have $f(x) \geq 2$. For $k \geq 1$, $f(x) = 2(2k + 1)$ is even and > 2, so $f(x)$ is composite (not prime). For $k = 0$, $f(0) = 2 \in Pr$.

Wait, this doesn't work. Let me try a different approach.

Corrected approach for $P \leq_m Pr$:

Since both P and Pr are infinite decidable sets, we can establish the reduction by using enumeration:

Let $p_0, p_1, p_2, ...$ be the enumeration of primes: 2, 3, 5, 7, 11, ...
Let $e_0, e_1, e_2, ...$ be the enumeration of even numbers: 0, 2, 4, 6, 8, ...

Define $f : \mathbb{N} \to \mathbb{N}$ by:

```
f(x) = {
  p_{x/2}     if x is even
  p₀ + 1 = 3  if x is odd (where 3 is chosen to be composite)
}
```

Wait, 3 is prime. Let me use 4:

```
f(x) = {
  p_{x/2}  if x is even
  4        if x is odd
}
```

Then $x \in P$ (x even) $\iff f(x) \in Pr$.

Proof of $Pr \leq_m P$:

Similarly, define $g : \mathbb{N} \to \mathbb{N}$ using enumerations:

```
g(x) = {
  e_k      if x = pₖ (x is the k-th prime)
  1        if x is not prime (1 is odd, so 1 ∉ P)
}
```

Then $x \in Pr \iff g(x) \in P$.

Both reductions are computable since:

- Prime testing is decidable
- Even/odd testing is decidable
- Enumeration of primes and evens can be computed

Therefore $P \leq_m Pr$ and $Pr \leq_m P$.

Exercise 5

Question: Can there exist an index $x \in \mathbb{N}$ such that $\bar{K} = \{y^2 - 1 : y \in E_x\}$?

Answer: No, such an index cannot exist.

Proof:

Suppose there exists x such that $\bar{K} = \{y^2 - 1 : y \in E_x\}$.

Key observations:

1. $\bar{K}$ is not r.e.
2. The set $\{y^2 - 1 : y \in E_x\}$ is r.e. (as the image of the r.e. set E_x under the computable function $y \mapsto y^2 - 1$)

Contradiction: Since E_x is r.e. (as the codomain of φ_x) and the function $h(y) = y^2 - 1$ is total and computable, the set:

```
{y² - 1 : y ∈ Eₓ} = h(Eₓ)
```

is r.e. (as the image of an r.e. set under a computable function).

But $\bar{K}$ is not r.e., so we cannot have $\bar{K} = \{y^2 - 1 : y \in E_x\}$.

Alternative argument using density: The set $\{y^2 - 1 : y \in \mathbb{N}\} = \{-1, 0, 3, 8, 15, 24, 35, ...\}$ has density 0 (grows like $\sqrt{n}$).

However, $\bar{K}$ has positive density (roughly half of all natural numbers), so $\bar{K}$ cannot equal any subset of $\{y^2 - 1 : y \in \mathbb{N}\}$.

Therefore, no such index x can exist.