20/11/2024

OVERVIEW

- (13/11) UNIVERSAL FUNCTIONS AND MEANING
- (13/11) EFFECTIVE OPERATIONS ON COMPUT. FUNCTIONS
- EXERCISES

[UNIVERSAL FUNCTION]

$$\varphi^{(k)}_{(e)}(\vec{x}) = \Psi_U(e, \vec{x})$$

SMN-THEOREM

e = PROGRAM

K = N. OF INPUTS

(. . . . .)₁ → 1 USED AS SUBSCRIPT

$$g(x,y,z) = \varphi_e(z) * \varphi_y(z)$$

$$= \Psi_U(e,z) \cdot \Psi_U(y,z)$$

e = PROGRAM
OBOX

$H(e,\vec{x},t)$ = HALTING ORACLE =

e PROGRAM

$\vec{x}$ = INPUT

t N. OF STEPS

$S(e,\vec{x},y,t)$ = OUTPUT FUNCTION

WHERE WE STOP ON t steps on y

KNF → KLEENE NORMAL FORM

$$\chi_S(e,(w)_1,(w)_2)$$

$(w)_1 = y$

$(w)_2 = t$

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

A is a set

$f : \mathbb{N} \to \mathbb{N}$ computable and injective]

$f^{-1} : \mathbb{N} \to \mathbb{N}$ is computable]

$[f = \varphi_e]$ → " $\mu.(x, t).\, S(e, x, y, t)$ "

e = INDEX OF PROGRAM

PREDICATES → NO POWER ENOUGH TO USE FUNCTIONS

(π) → ENCODING

$\mu w.[\chi_S(e, x, (w)_1, (w)_2)]_1$ → COMPLETE FORM (KNF)

$f^{-1}(y) = (\mu w.\, |\chi_S(e, x, (w)_1, (w)_2) - 1|)_1$

⇕ EQUIVALENT

$(\mu w.\, |S(e, x, (w)_1, (w)_2)|)_1$

<u>EXERCISE</u> Let $Q(x)$ be a decidable predicate.

$f_1, f_2 : \mathbb{N} \to \mathbb{N}$ computable

define $f(x) = \begin{cases} f_1(x) & \text{if } Q(x) \\ f_2(x) & \text{otherwise} \end{cases}$] $Q(x)$ is TRUE (1)

$(w)_1 = e_1 / e_2$ $\quad (w)_2 = y$

$$f(x) = \mu w.\{(S(e_1, x, y, t) \wedge Q(x)) \vee S(e_2, x, y, t) \wedge \neg Q(x))\}_1$$

e_1, e_2 = PROGRAMS ON $f_1(x), f_2(x)$

$\mathbf{1}(\mu w. | \_ \_ \_ \_ |)_1$ $\downarrow$

$\mu x. | \varphi_x(x) | \rightarrow \mu x. | \Psi_U(x, x) |$

S $\rightarrow$ OUTPUT

H $\longrightarrow$ $f(x) = \begin{cases} 1 & \text{if } \varphi_x(x)\downarrow \\ 0 & \text{otherwise} \end{cases}$

HALTING PROBLEM (PREDICATE) $\rightarrow$ $\mathbf{1}(\mu w. H(x, x, t)) \equiv \varphi_x(x)\downarrow$

$H(x, x, t) \wedge S(x, x, y, t)$

NOTABLE FUNCTIONS — $id(x)$, $\emptyset(x)$

$f(x) = \begin{cases} x \\ \uparrow \end{cases}$ $\quad f(x) = \begin{cases} \uparrow \\ 0 \end{cases}$

(1) $\rightarrow$ BEFORE THE KNF

$\mathbf{1}$ = CONSTANT ONE FUNCTION $\quad \mathbf{1}(x) = \begin{cases} 1 & x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$

$\mathbf{0}(x) = \begin{cases} 0 & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$

$\boxed{6.22}$

$f: \mathbb{N} \rightarrow \mathbb{N}$ defined by: $f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_y(y)\downarrow \ \forall y \leq x \\ 0 & \text{otherwise} \end{cases}$

Is it computable?

$\left[x = e \rightarrow \begin{matrix} \varphi_e(e)+1 \\ 0 \end{matrix} \right] \leftarrow Q(x)$

$= 1 \mu w. \mid (\mathcal{X}_H(\ldots) - 1 \mid)$

$\uparrow$

$H(e, e, \ )$ $\equiv$ $H(x, y, t)$

OR

$H(x, x, y)$

$S(e, e, y, t)$

$$f(x) \triangleq \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_y(y)\downarrow \ \forall y \leq x \\ 0 & \text{otherwise} \end{cases} \ \text{if } x \in W_x$$

ALTERNATIVELY

$\Downarrow$

$[k = \min [x, y)]$

$\varphi_k(k) \neq \varphi_{k+1}(x)$

- f is total $\rightarrow$ defined by cases
- f is not computable

$\rightarrow$ - if $x \in W_x$ $\varphi_x(x) + 1 \leq \varphi_x(x+1) + 1$

- if $x \notin W_x$ $\rightarrow$ $[\varphi_x \neq \varphi_{x+1}]$

$f(x) = 0$

$\varphi_x(x) + 1 \neq \varphi_0(0)$

$\uparrow$

$[\ddots]_0$

(2015-04-20)

STATE THE (1) SMN-THEOREM AND (2) USE IT TO SHOW THERE IS A TOTAL COMPUTABLE FUNCTION $s : \mathbb{N} \to \mathbb{N}$

$[\text{s.t. } \forall x \in \mathbb{N} \mid W_{s(x)} = \{(k+2)^2 \mid k \in \mathbb{N}\}]$

$$g(x, y) = \begin{cases} k & \text{if } (\exists k \text{ s.t. } y = (k+2)^2) \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \mu k \cdot |(x+k)^2 - y|$$ → MINIMALIZATION PART

THIS DOES NOT EXIST YET

$$\left[g(x,y) = \varphi_{s(x)}(y) \quad \forall x,y \in \mathbb{N} \right]$$

$$W_{s(x)} = \{ x \mid g(x,y)\downarrow \} = \{ \exists k \in \mathbb{N} \mid y = (x+k)^2 \}$$
$$= \{ x \mid (x+k)^2 \in \mathbb{N} \}$$ REQUIRED

$$E_{s(x)} = \{ g(x,y)\downarrow \mid x \in W_{s(x)} \}$$
$$= \{ k \mid y = (k+2)^2 \} =$$
$$= \{ \exists k \in \mathbb{N} \mid (x+k)^2 - y \}$$

OPTIONAL (FOR THIS EX. DEF.)

S/H → 2.5 ALSO Q(x) PREDICATES

$1(\ldots)$

PR → DEFINE PR → O, S, P (f) → COMP. PRIM. REC

PROVE $P_{MAX} : \mathbb{N}^2 \to \mathbb{N}$ defined by [MADE FROM SCRATCH]

$$P_{max}(x,y) = max(2^x, 3^y) \in PR$$

$$\to \begin{cases} P_{max}(x,0) = \\ P_{max}(x,y+1) = \end{cases}$$ WHAT WE WANT!

NO

EXP. FUNCTIONS → $EXP_2(x) = 2^x$
$EXP_3(x) = 3^y$] 2 FUNCTIONS

$$\begin{cases} exp_2(0) = 1 \\ exp_3(x+1) = 2 \cdot exp_2(x) \end{cases} \quad \begin{cases} exp_3(0) = 1 \\ exp_3(x+1) = 3 \cdot exp_3(x) \end{cases}$$

MAX → DEFINED BY PR → S(SUCC(x))

$$\begin{cases} max(x,0) = x \\ max(x, y+1) = max(\underline{SUCC(x)}, y) \end{cases}$$

[SUCC(y)]

BASIC OPERATIONS ∈ PR

$\boxed{6.5}$ $f: \mathbb{N} \to \mathbb{N}$ is "decreasing" IS IT COMPUTABLE?
(total)

[$\forall x, y \in \mathbb{N}$
if $x \leq y$ then $f(x) \geq f(y)$]

$x \leq y$] → CONSTANTS
→ NEW FUNCTIONS

$$g(x,y) = \begin{cases} K & if(\ldots) \\ 0 & otherwise \end{cases}$$

$= \mu K. \,|\, (f(x) \geq f(y)) \,|$

$\mu K. \,|\, "(x \leq y) \wedge (f(x) \geq f(y))" \,|$

$(w)_1 = y$ NOT USED DIRECTLY

$1 (\mu K. (x \leq (w)_1) \wedge (\varphi_x(x) \geq \varphi_y(y)) \wedge$] $(w)_2 = t$ if $x \in W_x$; $y \in W_y$

$1 (\mu K. (x \leq (w)_1) \wedge S(e, x, y, t)$

$[x \leq y]$ if $x \in W_x$ $y \in W_y$ 4

$$\left[\neg(\mu k.(x \leq (w)_1) \wedge S(e, x, (w)_1, (w)_2)\right] \vee (x > (w)_1 \wedge S \ldots)$$

A WAY OF THINKING THIS EXERCISE