Exercise 1

Let $A, B \subseteq \mathbb{N}$. Define the notion of reducibility $A \leq_m B$. Prove whether it is true that if A is recursive and B is finite, non-empty then $A \leq_m B$. Analyze the case without the finiteness hypothesis for B.

Solution:

First, let us formally define many-one reducibility: A set A reduces to a set B (written $A \leq_m B$) if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$:
$x \in A \iff f(x) \in B$

For the first case, we prove that with B finite and non-empty, the property does not hold by constructing a counterexample:

Let $A = \mathbb{N} \setminus 0$ (recursive) and $B = 1$ (finite, non-empty). Assume by contradiction that $A \leq_m B$ through some reduction function f. Then:

- For all $x \in A$, $f(x) = 1$ (since $B = 1$)
- For all $x \notin A$, $f(x) \neq 1$ This contradicts the fact that f must be total, as $\overline{B}$ is infinite.

For the general case, we can restore the property by requiring B to be recursive. Then for any recursive A, we can construct a reduction:

$$f(x) = \begin{cases} \mu y.\,(y \in B) & \text{if } x \in A \\ \mu y.\,(y \notin B) & \text{if } x \notin A \end{cases}$$

This f is computable since:

1. A is recursive so χ_A is computable
2. B being recursive means both B and $\overline{B}$ are r.e., allowing us to compute the minima

Exercise 2

State the second recursion theorem and use it to prove that for every $k \geq 0$ there exist two indices $x, y \in \mathbb{N}$ such that $x - y = k$ and $\varphi_x = \varphi_y$.

Solution:

The Second Recursion Theorem states that for any total computable function $h : \mathbb{N} \to \mathbb{N}$, there exists $e \in \mathbb{N}$ such that: $\phi_e = \phi_{h(e)}$

To prove the claim, let $k \geq 0$ and define: $h(x) = x - k$

This function is total and computable. By the Second Recursion Theorem, there exists $e \in \mathbb{N}$ such that: $\phi_e = \phi_{h(e)} = \phi_{e-k}$

Let $x = e$ and $y = e - k$. Then:

1. $x - y = e - (e - k) = k$
2. $\phi_x = \phi_e = \phi_{e-k} = \phi_y$

Therefore, we have found indices satisfying both required conditions.

Exercise 3

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, with f total, define the predicate $Q_{f,g}(x) \equiv$ "$f(x) = g(x)$". Show that if f and g are computable then $Q_{f,g}$ is semidecidable. Does the converse hold, i.e., if $Q_{f,g}$ is semidecidable can we deduce that f and g are computable?

Solution:

Let f and g be computable functions. Then:

1. First part: Let's prove that $Q_{f,g}$ is semidecidable.
   - Since f and g are computable, there exist indices e_1, e_2 such that $f = \phi_{e_1}$ and $g = \phi_{e_2}$
   - The semi-characteristic function of $Q_{f,g}$ can be written as: $sc_{Q_{f,g}}(x) = \begin{cases} 1 & \text{if } f(x) = g(x) \\ \uparrow & \text{otherwise} \end{cases}$
   - This is equivalent to $sc_{Q_{f,g}}(x) = 1(\mu z. |f(x) - g(x)|)$
   - Since f and g are computable, this is computable by composition
   - Therefore $Q_{f,g}$ is semidecidable
2. Second part: The converse does not hold. Let's provide a counterexample:
   - Let $f(x) = x$ (which is computable)
   - Let $g(x) = \begin{cases} x & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$
   - Then $Q_{f,g}$ is semidecidable as it coincides with the halting problem K
   - However, g is not computable as it would solve the halting problem

Exercise 4

Let $\mathbb{P} = \{2k \mid k \in \mathbb{N}\}$ be the set of even numbers. Study the recursiveness of the set $A = \{x \in \mathbb{N} : |W_x \cap \mathbb{P}| \geq 2\}$, i.e., determine if A and $\bar{A}$ are recursive/recursively enumerable.

Solution:

Let $\mathbb{P} = \{2k \mid k \in \mathbb{N}\}$ be the set of even numbers. We want to study the recursiveness of the set $A = \{x \in \mathbb{N} : |W_x \cap \mathbb{P}| \geq 2\}$, i.e., determine if A and $\bar{A}$ are recursive/recursively enumerable.

First, we observe that the set A is saturated, since it can be expressed as $A = \{x \mid \phi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \in \mathcal{C} \mid |dom(f) \cap \mathbb{P}| \geq 2\}$.

We will use the Rice-Shapiro theorem to prove that both A and $\bar{A}$ are not r.e.:

1. A is not r.e. Consider the identity function $id \notin A$, since $dom(id) \cap \mathbb{P} = \mathbb{P}$ is infinite. However, we can find a finite subfunction $\theta \subseteq id$, defined as $\theta(x) = \begin{cases} x & \text{if } x \in 0, 2 \\ \uparrow & \text{otherwise} \end{cases}$ which satisfies $|dom(\theta) \cap \mathbb{P}| = |0, 2| = 2$, and thus $\theta \in A$. Therefore, by the Rice-Shapiro theorem, we conclude that A is not r.e.
2. $\bar{A}$ is not r.e.
   Note that if θ is the function defined above, then $\theta \notin \bar{A}$. However, the always undefined function $\emptyset \subseteq \theta$ and $|dom(\emptyset) \cap \mathbb{P}| = |\emptyset| = 0 < 2$, so $\emptyset \in \bar{A}$. Therefore, by the Rice-Shapiro theorem, we conclude that $\bar{A}$ is not r.e.

In conclusion, neither A nor $\bar{A}$ are recursive, as they are not both r.e.

Exercise 5

State the second recursion theorem and use it to prove that the set $B = \{x \in \mathbb{N} : |W_x| = x + 1\}$ is not saturated.

Solution:

1. Second Recursion Theorem: For any total computable function $f : \mathbb{N} \to \mathbb{N}$, there exists $e_0 \in \mathbb{N}$ such that $\phi_{e_0} = \phi_{f(e_0)}$
2. To prove B is not saturated:
   - Let's define a function $g(x, y) = \{0 \quad \text{if } y \leq x \uparrow \quad \text{otherwise}$
   - By s-m-n theorem, there exists a computable function s such that $\phi_{s(x)}(y) = g(x, y)$
   - By the second recursion theorem, there exists e such that $\phi_e = \phi_{s(e)}$
   - Therefore $|W_e| = e + 1$, so $e \in B$
   - However, there exists $e' \neq e$ such that $\phi_e = \phi_{e'}$ (since every computable function has infinitely many indices)
   - But $|W_{e'}| = e + 1 \neq e' + 1$, so $e' \notin B$
   - Therefore B is not saturated

Exercise 6

Let $A, B \subseteq \mathbb{N}$. Define the notion of reducibility $A \leq_m B$. Consider the set $S_4 = \{4 * n \mid n \in \mathbb{N}\}$, i.e., the set of multiples of 4. Prove that A is recursive if $A \leq_m S_4$.

Solution:

1. First, recall that $A \leq_m B$ means there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall x \in \mathbb{N} : x \in A \iff f(x) \in B$
2. Let's prove that if $A \leq_m S_4$ then A is recursive:
   - S_4 is recursive since its characteristic function is: $\chi_{S_4}(x) = \{1 \quad \text{if } 4|x \; 0 \quad \text{otherwise}$
   - Let f be the reduction function $A \leq_m S_4$

- Then $\chi_A(x) = \chi_{S_4}(f(x))$
- Since both f and χ_{S_4} are computable, χ_A is computable
- Therefore A is recursive

Exercise 7

Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists y \in W_x. \exists z \in E_x. x = y + z\}$, i.e., determine if A and $\bar{A}$ are recursive/recursively enumerable.

Solution:

- First, let's prove that $K \leq_m A$, showing that A is not recursive:
  - Define $g(x, y) = \{1 \quad \text{if } x \in K \uparrow \quad \text{otherwise}$
  - By s-m-n theorem, there exists total computable s such that $\phi_{s(x)}(y) = g(x, y)$
  - Then:
    - $x \in K \implies W_{s(x)} = \mathbb{N} \wedge E_{s(x)} = 1 \implies s(x) \in A$
    - $x \notin K \implies W_{s(x)} = \emptyset \implies s(x) \notin A$
  - Therefore $K \leq_m A$, so A is not recursive
- A is r.e. since:
  - Its semi-characteristic function is: $sc_A(x) = 1(\mu w. (H(x, (w)_1, (w)_3) \wedge S(x, (w)_2, (w)_1 + (w)_2, (w)_3)))$
- Since A is r.e. but not recursive, $\bar{A}$ cannot be r.e.

Exercise 8

Study the recursiveness of the set $B = \{x \in \mathbb{N} : W_x \cup E_x = \mathbb{N}\}$, i.e., determine if B and $\bar{B}$ are recursive/recursively enumerable.

Solution:

- B is saturated since $B = \{x | \phi_x \in \mathcal{B}\}$ where $\mathcal{B} = f \in \mathcal{C} | dom(f) \cup cod(f) = \mathbb{N}$
- By Rice-Shapiro theorem:
  - B is not r.e. because:
    - The identity function $id \in B$ since $dom(id) = cod(id) = \mathbb{N}$
    - No finite subfunction $\theta \subseteq id$ can be in B since $dom(\theta) \cup cod(\theta)$ is finite
  - $\bar{B}$ is not r.e. because:
    - $\emptyset \in \bar{B}$
    - $\emptyset \subseteq id$ but $id \in B$
- Therefore, neither B nor $\bar{B}$ are recursive.

Exercise 9

Let $A, B \subseteq \mathbb{N}$ such that $\bar{A}$ is finite and $B \neq \emptyset, \mathbb{N}$. Prove that $A \leq_m B$.

Solution:

Let $\bar{A} = a_1, \dots, a_n$ be finite and let $b \in B, c \notin B$. Define:

$$f(x) = \begin{cases} c & \text{if } x \in a_1, \dots, a_n \\ b & \text{otherwise} \end{cases}$$

1. f is computable since $\bar{A}$ is finite
2. f is total by definition
3. For all $x \in \mathbb{N}$:
    - $x \in A \iff x \notin \bar{A} \iff f(x) = b \in B$
    - $x \notin A \iff x \in \bar{A} \iff f(x) = c \notin B$

Therefore $A \leq_m B$.

Exercise 10

Consider the set $A = \{x \mid W_x = E_x \cup 0\}$. We need to establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution:

Let's observe that A is saturated since $A = \{x \mid \phi_x \in A\}$ where $A = \{f \mid dom(f) = cod(f) \cup 0\}$.

By Rice-Shapiro's theorem, we can prove that both A and $\bar{A}$ are not r.e., and thus not recursive:

1. A is not r.e.:
    - Consider the identity function $id \notin A$, since $dom(id) = \mathbb{N} \neq \mathbb{N} \cup 0 = cod(id) \cup 0$
    - However, the empty function $\emptyset \subseteq id$ belongs to A, since $dom(\emptyset) = \emptyset = \emptyset \cup 0 = cod(\emptyset) \cup 0$
    - Therefore, by Rice-Shapiro's theorem, A is not r.e.
2. $\bar{A}$ is not r.e.:
    - If we take $f = (0, 0)$, we have $f \notin \bar{A}$ since $dom(f) = 0 = 0 \cup 0 = cod(f) \cup 0$
    - However, consider $g = (0, 1) \subseteq f$. Then $g \in \bar{A}$ since $dom(g) = 0 \neq 1 \cup 0 = cod(g) \cup 0$
    - Therefore, by Rice-Shapiro's theorem, $\bar{A}$ is not r.e.

Exercise 11

Consider the set $B = \{x \in \mathbb{N} \mid 4x + 1 \in E_x\}$. We need to establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution:

We will prove that B is not recursive by showing $K \leq_m B$.

Let us define:

$$g(x, y) = \begin{cases} 4x + 1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

g is computable since $g(x, y) = (4x + 1) \cdot 1(\Psi_U(x, x))$

By the s-m-n theorem, there exists $s : \mathbb{N} \to \mathbb{N}$ computable and total such that:

$$\phi_{s(x)}(y) = g(x, y) \; \forall x, y$$

Then s is a reduction function for $K \leq_m B$ since:

- If $x \in K$: $\phi_{s(x)}(y) = 4x + 1$ thus $4x + 1 \in E_{s(x)}$ and therefore $s(x) \in B$
- If $x \notin K$: $\phi_{s(x)}(y) \uparrow$ for all y, thus $E_{s(x)} = \emptyset$ and therefore $s(x) \notin B$

B is r.e. since its semi-characteristic function is computable:

$$sc_B(x) = 1(\mu(y, t).\, S(x, y, 4x + 1, t))$$

Since B is r.e. but not recursive, by the complementation theorem we can conclude that $\bar{B}$ is not r.e.

Exercise 12

Show that for any $k \geq 2$, the function $sum_k : \mathbb{N}^k \to \mathbb{N}$ defined by $sum_k(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i$ is primitive recursive.

Solution:

First recall that the class of primitive recursive functions $\mathcal{PR}$ is the smallest class containing:

1. Zero function: $z : \mathbb{N}^k \to \mathbb{N}, z(x_1, \ldots, x_k) = 0$
2. Successor function: $s : \mathbb{N} \to \mathbb{N}, s(x) = x + 1$
3. Projections: $U_i^k : \mathbb{N}^k \to \mathbb{N}, U_i^k(x_1, \ldots, x_k) = x_i$

And closed under:

- Composition
- Primitive recursion

For $k = 2$, we can define sum_2 by primitive recursion:

$$sum_2(x_1, 0) = x_1$$
$$sum_2(x_1, x_2 + 1) = s(sum_2(x_1, x_2))$$

For $k > 2$, we can define inductively:

$$sum_k(x_1, \ldots, x_k) = sum_2(sum_{k-1}(x_1, \ldots, x_{k-1}), x_k)$$

Since composition preserves primitive recursiveness and $sum_2 \in \mathcal{PR}$, by induction we have $sum_k \in \mathcal{PR}$ for all $k \geq 2$.

Exercise 13

Given a function $f : \mathbb{N} \to \mathbb{N}$, define: $Z(f) = g : \mathbb{N} \to \mathbb{N} | \forall x \in \mathbb{N}. g(x) = f(x) \vee g(x) = 0$

Show that $Z(id)$ is not enumerable, where id is the identity function. Is it true for all functions f that $Z(f)$ is not enumerable?

Solution:

To prove $Z(id)$ is not enumerable, we proceed by contradiction. Assume $Z(id)$ is enumerable. Then there would exist a surjective function $f : \mathbb{N} \to Z(id)$.

We can construct a bijective correspondence between $\mathcal{P}(\mathbb{N})$ and $Z(id)$ by defining $h : Z(id) \to \mathcal{P}(\mathbb{N})$ as:

$$h(g) = \{x \in \mathbb{N} \mid g(x) = 0\}$$

For any $g \in Z(id)$, h uniquely constructs the set $D \subseteq \mathbb{N}$ where:

$$g(x) = \{0 \quad \text{if } x \in D \; x \quad \text{if } x \notin D$$

At this point, we can define a surjective function $\bar{g} : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ as:

$$\bar{g} = h \circ f$$

This would imply $\mathcal{P}(\mathbb{N})$ is enumerable, which contradicts Cantor's theorem. Therefore, $Z(id)$ cannot be enumerable.

This property does not hold for all functions. For $f = 0$ (constant zero function):

$$Z(0) = 0$$

which is a finite set and thus enumerable. In general, $Z(f)$ is enumerable if and only if $f = 0$.

Exercise 14

Consider $A = \{x \mid W_x \subseteq x\}$. We need to establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution:

Let's show that $K \leq_m A$. Define:

$$g(x, y) = \begin{cases} x & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

By s-m-n theorem, $\exists s : \mathbb{N} \to \mathbb{N}$ computable and total such that $\phi_{s(x)}(y) = g(x, y)$. Then:

- If $x \in K$: $W_{s(x)} = x$, thus $s(x) \in A$
- If $x \notin K$: $W_{s(x)} = \emptyset \subseteq s(x)$, thus $s(x) \notin A$

Therefore A is not recursive. However, A is r.e. since:

$$sc_A(x) = 1(\mu w.\, H(x, w) \wedge (w = x))$$

Thus $\bar{A}$ is not r.e.

Exercise 15

Consider the set $B = \{x \in \mathbb{N} : |W_x| > 1\}$. We need to establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution:

First, observe that B is saturated since $B = \{x \mid \phi_x \in B\}$ where $B = f \in C : |dom(f)| > 1$. Using Rice-Shapiro's theorem, we can prove that both B and $\bar{B}$ are not r.e.

For B not r.e.: Consider the constant function $\mathbf{1}(x) = 1$. Then $\mathbf{1} \notin B$ since $|dom(\mathbf{1})| = 1$. However, if we consider the finite function:

$$\theta(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ \uparrow & \text{otherwise} \end{cases}$$

We have that $\theta \subseteq c_1$ and $\theta \in B$ since $|dom(\theta)| = 2$. Therefore, by Rice-Shapiro's theorem, B is not r.e.

For $\bar{B}$ not r.e.: Consider θ as defined above. Then $\theta \notin \bar{B}$. However, the empty function $\emptyset \subseteq \theta$ and $\emptyset \in \bar{B}$ since $|dom(\emptyset)| = 0 \leq 1$. Therefore, by Rice-Shapiro's theorem, $\bar{B}$ is not r.e.

Exercise 16

State the s-m-n theorem and use it to prove that there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $|W_{s(x)}| = 2x$ and $|E_{s(x)}| = x$.

Solution:

First, let us define:

$$g(x, y) = \begin{cases} y/2 & \text{if } y < 2x \\ \uparrow & \text{otherwise} \end{cases}$$

This function is computable since:

$$g(x, y) = \frac{y}{2} + \mu z.\, \mathrm{sg}(2x - y)$$

By the s-m-n theorem, there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that:

$$\phi_{s(x)}(y) = g(x, y)$$

Therefore:

1. $W_{s(x)} = y | y < 2x$, thus $|W_{s(x)}| = 2x$
2. $E_{s(x)} = \{y/2 \mid y < 2x\} = \{z \mid z < x\}$, thus $|E_{s(x)}| = x$

Exercise 17

State the Second Recursion Theorem and use it to show there exists some $x \in \mathbb{N}$ s.t. $\varphi_x(y) = y^x, \forall y \in \mathbb{N}$

Solution:

By the Second Recursion Theorem, for any total computable function $h : \mathbb{N} \to \mathbb{N}$, there exists $e_0 \in \mathbb{N}$ such that $\phi_{e_0} = \phi_{h(e_0)}$.

Let us define:

$$g(n, y) = y^n$$

By the s-m-n theorem, there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that:

$$\phi_{s(n)}(y) = g(n, y) = y^n$$

Applying the Second Recursion Theorem to s, there exists $x \in \mathbb{N}$ such that:

$$\phi_x = \phi_{s(x)}$$

Therefore:

$$\phi_x(y) = \phi_{s(x)}(y) = y^x \text{ for all } y \in \mathbb{N}$$

Exercise 18

Prove that $F = \{\theta \mid \theta : \mathbb{N} \to \mathbb{N} \wedge \text{dom}(\theta) \text{ finite}\}$ (unary functions with finite domain) set is countable.

Solution:
For any finite function θ ∈ F, we can encode it uniquely as a natural number using the following encoding:

$\tilde{\theta} = \prod_{i=1}^{n} p_{x_i+1}^{y_i+1}$

where:

- $(x_1, y_1), \ldots, (x_n, y_n)$ represents the input-output pairs of θ
- p_i represents the i-th prime number

Given an encoding z = $\tilde{\theta}$:

- $x \in dom(\theta)$ iff $(z)_{x+1} \neq 0$

- $\theta(x) = (z)_{x+1} - 1$ when $x \in dom(\theta)$

This encoding is:

1. Injective (each finite function has a unique encoding)
2. Computable (we can effectively compute the encoding and decoding)

Therefore, we have established a one-to-one correspondence between F and a subset of $\mathbb{N}$, proving that F is countable.

Exercise 19

Study the recursiveness of $B = \{x \mid \phi_x(x) \downarrow \wedge \phi_x(x) \text{ odd}\}$

Solution:

We show that $K \leq_m B$ to prove B is not recursive.

Define $g(x, y)$ as: $g(x,y) = \begin{cases} 1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$

By the s-m-n theorem, there exists $s : \mathbb{N} \to \mathbb{N}$ total computable such that: $\varphi_s(x)(y) = g(x, y)$

Then s is a reduction function $K \leq_m B$ since:

- $x \in K \Rightarrow \varphi_s(x)(s(x)) \downarrow \ = 1(odd) \Rightarrow s(x) \in B$
- $x \notin K \Rightarrow \varphi_s(x)(s(x)) \uparrow \ \Rightarrow s(x) \notin B$

Therefore:

1. B is not recursive
2. B is r.e. since $sc_B(x) = 1(\varphi_x(x)) \cdot sg(mod(\varphi_x(x), 2))$
3. $\bar{B}$ is not r.e. (otherwise B would be recursive)

Exercise 20

State the Second Recursion Theorem and use it to show there exists some $x \in \mathbb{N}$ s.t. $|W_x| = x$

Solution:

By the Second Recursion Theorem, for any total computable $f : \mathbb{N} \to \mathbb{N}$, there exists $e \in \mathbb{N}$ such that $\phi_e = \phi_{f(e)}$.

Define: $h(x, y) = \begin{cases} x & \text{if } y \leq |W_x| \\ \uparrow & \text{otherwise} \end{cases}$

By s-m-n theorem, $\exists s$ total computable such that $\phi_{s(x)}(y) = h(x, y)$.

By Second Recursion Theorem, $\exists e$ such that: $\phi_e = \phi_{s(e)}$

Therefore $|W_e| = e$, proving the existence of such e.

Exercise 21

Define the class of primitive recursive functions. Using only the definition, show that the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(y) = 2y + 1$ is primitive recursive.

Solution:

First, let us recall that the class PR of primitive recursive functions is the smallest class containing:

Base functions:

1. Zero function: $z(x) = 0$
2. Successor function: $s(x) = x + 1$
3. Projection functions: $U_i^k(x_1, \ldots, x_k) = x_i$

And closed under:

1. Composition
2. Primitive recursion

To show $f(y) = 2y + 1$ is primitive recursive, we can construct it using only these operations:

1. Define $double(y)$ by primitive recursion:
   $\begin{cases} double(0) = 0 \\ double(y+1) = double(y) + 2 = s(s(double(y))) \end{cases}$
2. Then $f(y) = s(double(y))$

Therefore, f is primitive recursive as it is constructed using only composition and primitive recursion from the base functions.

Exercise 22

Classify the following set from the point of view of recursiveness $A = \{x \mid W_x \cap E_x \supseteq 0\}$, i.e., establish if A and $\overline{A}$ are recursive/recursively enumerable.

Solution:

A is saturated since $A = \{x \mid \phi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid 0 \in dom(f) \cap cod(f)\}$.

Let's prove $K \leq_m A$. Define: $g(x, y) = \begin{cases} 0 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$

By s-m-n theorem, $\exists s : \mathbb{N} \to \mathbb{N}$ total computable such that $\phi_{s(x)}(y) = g(x, y)$.

Then:

- $x \in K \implies \phi_{s(x)}(0) = 0 \implies 0 \in W_{s(x)} \cap E_{s(x)} \implies s(x) \in A$
- $x \notin K \implies \phi_{s(x)}(0) \uparrow \implies 0 \notin W_{s(x)} \cap E_{s(x)} \implies s(x) \notin A$

Therefore:

1. A is not recursive
2. A is r.e. since $sc_A(x) = 1(\mu y. (H(x, 0, y) \land S(x, 0, 0, y)))$
3. $\overline{A}$ is not r.e.

Exercise 23

Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} \mid \exists y. \phi_x(y) = x + 1\}$, i.e., establish if B and $\overline{B}$ are recursive/recursively enumerable. Also establish if B is saturated.

Solution:

First, let's prove B is r.e. Its semi-characteristic function is: $sc_B(x) = 1(\mu w. (S(x, (\omega)_1, x + 1, (\omega)_2)))$

Now let's show $K \leq_m B$. Define: $g(x, y) = \begin{cases} s(x) & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$

By s-m-n theorem, $\exists s$ total computable such that $\phi_{s(x)}(y) = g(x, y)$.

Then:

- $x \in K \implies \phi_{s(x)}(0) = x + 1 \implies s(x) \in B$
- $x \notin K \implies \phi_{s(x)}(y) \uparrow \text{ for all } y \implies s(x) \notin B$

Therefore:

1. B is not recursive
2. B is r.e.
3. $\overline{B}$ is not r.e.

For saturation: Define by Second Recursion Theorem an index e such that: $\phi_e(y) = \begin{cases} e + 1 & \text{if } y = e \\ \uparrow & \text{otherwise} \end{cases}$

Then $e \in B$. Let $e' \neq e$ such that $\phi_{e'} = \phi_e$. Then $\phi_{e'}(e') = \phi_e(e') \uparrow$, hence $e' \notin B$. Therefore B is not saturated.

Exercise 24

State the s-m-n theorem. Use it to prove that there exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$ it holds that $\phi_{k(x)}(y) = lcm(x, y)$, where lcm is the least common multiple of x and y.

Solution:
First, let's state the s-m-n theorem: For any $m, n \geq 1$, there exists a total computable function $s_n^m : \mathbb{N}^{m+1} \to \mathbb{N}$ such that for all $e \in \mathbb{N}, \vec{x} \in \mathbb{N}^m, \vec{y} \in \mathbb{N}^n$: $\phi_e^{(m+n)}(\vec{x}, \vec{y}) = \phi_{s_n^m(e,\vec{x})}^{(n)}(\vec{y})$

To prove the existence of k, define: $f(x, y) = \mu z \leq x \cdot y.\ (x|z \wedge y|z)$

Then f is computable since: $f(x, y) = \mu z \leq x \cdot y.\ (\overline{sg}(div(x, z)) \cdot \overline{sg}(div(y, z)))$

By the s-m-n theorem, there exists $k : \mathbb{N} \to \mathbb{N}$ total computable such that:
$\phi_{k(x)}(y) = f(x, y) = lcm(x, y)$

Exercise 25

Classify the following set from the point of view of recursiveness $A = x \mid W_x \cup E_x \subseteq \mathbb{P}$, where $\mathbb{P}$ is the set of even numbers, i.e., establish if A and $\overline{A}$ are recursive/recursively enumerable.

Solution: A is saturated since $A = \{x \mid \phi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid dom(f) \cup cod(f) \subseteq \mathbb{P}\}$.

Let's prove $\overline{K} \leq_m A$. Define: $g(x, y) = \begin{cases} 0 & \text{if } x \in \overline{K} \\ 1 & \text{otherwise} \end{cases}$

By s-m-n theorem, $\exists s$ total computable such that $\phi_{s(x)}(y) = g(x, y)$.

Then:

- $x \in \overline{K} \implies \phi_{s(x)}(y) = 0 \implies W_{s(x)} \cup E_{s(x)} \subseteq \mathbb{P} \implies s(x) \in A$
- $x \notin \overline{K} \implies \phi_{s(x)}(y) = 1 \implies W_{s(x)} \cup E_{s(x)} \nsubseteq \mathbb{P} \implies s(x) \notin A$

Therefore:

1. A is not r.e.
2. $\overline{A}$ is r.e. since
   $sc_{\overline{A}}(x) = 1(\mu w.\ (H(x, (\omega)_1, (\omega)_2) \wedge odd((\omega)_1) \vee S(x, (\omega)_1, (\omega)_2, (\omega)_3) \wedge odd((\omega)_2)))$

Exercise 26

Classify $B = \{x \in \mathbb{N} \mid 2x + 1 \in W_x\}$ from the point of view of recursiveness, i.e., establish if B and $\overline{B}$ are recursive/recursively enumerable. Also establish if B is saturated.

Solution:

First, B is r.e. since: $sc_B(x) = 1(\mu w.\ H(x, 2x + 1, w))$

Let's prove $K \leq_m B$. Define: $g(x,y) = \{1 \quad \text{if } y = 2s(x)+1 \land x \in K \uparrow \quad \text{otherwise}$

By s-m-n theorem, $\exists s$ total computable such that $\phi_{s(x)}(y) = g(x,y)$.

Then:

- $x \in K \implies 2s(x)+1 \in W_{s(x)} \implies s(x) \in B$
- $x \notin K \implies W_{s(x)} = \emptyset \implies 2s(x)+1 \notin W_{s(x)} \implies s(x) \notin B$

Therefore:

1. B is not recursive
2. B is r.e.
3. $\overline{B}$ is not r.e.

For saturation: Define by Second Recursion Theorem an index e such that:
$\phi_e(y) = \begin{cases} 1 & \text{if } y = 2e+1 \\ \uparrow & \text{otherwise} \end{cases}$

Then $e \in B$. Let $e' \neq e$ such that $\phi_{e'} = \phi_e$. Then $2e'+1 \notin W_{e'} = 2e+1$, hence $e' \notin B$. Therefore B is not saturated.

Exercise 27

Let us study the recursiveness of the set

$B = \{x \mid k \cdot (x+1) \in W_x \cap E_x \text{ for all } k \in \mathbb{N}\}$

In other words, determine if B and $\bar{B}$ are recursive/recursively enumerable.

Solution:

Let us prove that set B is not recursive by showing that $K \leq_m B$. We will then prove that B is recursively enumerable, which will imply that $\bar{B}$ is not recursively enumerable.

We show that $K \leq_m B$ by constructing a computable reduction function. Let us define:

$$g(x,y) = \begin{cases} 1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

This function is computable since $g(x,y) = sc_K(x)$. By the smn theorem, there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\phi_{s(x)}(y) = g(x,y)$ for all $x, y \in \mathbb{N}$.

We shall prove that s is a reduction function for $K \leq_m B$:

1. If $x \in K$, then:
    - $\phi_{s(x)}(y) = 1$ for all $y \in \mathbb{N}$
    - Therefore $W_{s(x)} = E_{s(x)} = \mathbb{N}$
    - Thus for all k, $k(s(x)+1) \in W_{s(x)} \cap E_{s(x)}$

- Hence $s(x) \in B$

2. If $x \notin K$, then:
   - $\phi_{s(x)}(y) \uparrow$ for all $y \in \mathbb{N}$
   - Therefore $W_{s(x)} = E_{s(x)} = \emptyset$
   - Thus no $k(s(x)+1)$ can be in $W_{s(x)} \cap E_{s(x)}$
   - Hence $s(x) \notin B$

B is recursively enumerable since its semi-characteristic function is computable:

$$sc_B(x) = 1(\mu w.\,(H(x, k \cdot (x+1), (\omega)_2) \wedge S(x, (\omega)_1, k \cdot (x+1), (\omega)_2)))$$

where H and S are the standard halting and computation predicates respectively.

Therefore:

- B is recursively enumerable but not recursive
- Since B is not recursive but is r.e., $\bar{B}$ cannot be r.e. (otherwise B would be recursive)
- Thus $\bar{B}$ is not recursively enumerable

Exercise 28

Does there exist a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that its image $cod(f) = \{y \mid \exists x \in \mathbb{N}.\, f(x) = y\}$ is finite? Provide an example or prove that such a function does not exist.

Solution: Yes, such a function exists. Consider the function:

$$f(x) = \begin{cases} \overline{sg}(\phi_x(x)) & \text{if } x \in W_x \\ 0 & \text{if } x \notin W_x \end{cases}$$

This function has the following properties:

1. It is total, as it provides a value for every input $x \in \mathbb{N}$.
2. It is not computable because for every $x \in \mathbb{N}$, $f(x) \neq \phi_x(x)$. Specifically:
   - If $\phi_x(x) \downarrow$, then $f(x) = \overline{sg}(\phi_x(x)) \neq \phi_x(x)$
   - If $\phi_x(x) \uparrow$, then $f(x) = 0 \neq \phi_x(x)$
3. By construction, $cod(f) \subseteq 0, 1$, which is clearly finite.

Exercise 29

State the s-m-n theorem and use it to prove that there exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that $W_{k(n)} = \{x \in \mathbb{N} \mid x \geq n\}$ and $E_{k(n)} = \{y \in \mathbb{N} \mid y \text{ is even}\}$ for all $n \in \mathbb{N}$.

Solution: Let us first define a computable function of two arguments $f(n, x)$ that satisfies the conditions when viewed as a function of x, with n as a parameter:

$$f(n,x) = \begin{cases} 2(x-n) & \text{if } x \geq n \\ \uparrow & \text{otherwise} \end{cases} = 2(x-n) + \mu z.\,(n-x)$$

By the s-m-n theorem, there exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that $\phi_{k(n)}(x) = f(n,x)$ for all $n, x \in \mathbb{N}$. Therefore:

1. $W_{k(n)} = \{x \mid f(n,x) \downarrow\} = \{x \mid x \geq n\}$
2. $E_{k(n)} = f(n,x) \mid x \in \mathbb{N} = 2z \mid z \in \mathbb{N}$

Exercise 30

State the s-m-n theorem and use it to prove that there exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, $W_{k(n)} = \{z^n \mid z \in \mathbb{N}\}$ and $E_{k(n)}$ is the set of odd numbers.

Solution: Let us define a computable function of two arguments $f(n,x)$ that meets the required conditions:

$$f(n,x) = \begin{cases} 2z+1 & \text{if } x = z^n \text{ for some } z \\ \uparrow & \text{otherwise} \end{cases} = 2 \cdot \mu z.\,|x - z^n| + 1$$

By the s-m-n theorem, there exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that $\phi_{k(n)}(x) = f(n,x)$ for all $n, x \in \mathbb{N}$. Therefore:

1. $W_{k(n)} = \{x \mid f(n,x) \downarrow\} = \{x \mid \exists z \in \mathbb{N}.\, x = z^n\} = \{z^n \mid z \in \mathbb{N}\}$
2. $E_{k(n)} = \{f(n,x) \mid x \in W_{k(n)}\} = \{2z+1 \mid z \in \mathbb{N}\}$

Exercise 31

Do there exist an index $e \in \mathbb{N}$ and a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that, denoting by $dom(f)$ and $cod(f)$ the domain and codomain of f respectively (where $dom(f) = x \mid f(x) \downarrow$ and $cod(f) = y \mid \exists x.\, f(x) = y$), it holds that $dom(f) = W_e$ and $cod(f) = E_e$? Provide an example or prove non-existence.

Additionally, can such a function f with $dom(f) = W_e$ and $cod(f) = E_e$ be found for every $e \in \mathbb{N}$?

Solution: For the first part, consider an index $e \in \mathbb{N}$ of the identity function, where $W_e = E_e = \mathbb{N}$. Define $f : \mathbb{N} \to \mathbb{N}$ as:

$$f(x) = \begin{cases} \phi_x(x) + 1 & \text{if } x \in W_x \\ 0 & \text{otherwise} \end{cases}$$

The function f is total, therefore $dom(f) = \mathbb{N} = W_e$. Furthermore, $dom(f) = \mathbb{N} = E_e$. Indeed, for any $n \in \mathbb{N}$:

- If $n = 0$, consider an index x of the always undefined function, then $f(x) = 0$
- If $n > 0$, consider any index x of the constant function $n - 1$, then $f(x) = (n-1) + 1 = n$

For the second question, the answer is no. For example, if we consider $e \in \mathbb{N}$ such that ϕ_e is the always undefined function, any f such that $dom(f) = W_e = \emptyset$ must coincide with ϕ_e and thus would be computable.

Exercise 32

Provide the definition of the set $\mathcal{PR}$ of primitive recursive functions and prove that the function $cpr : \mathbb{N}^2 \to \mathbb{N}$ defined as

$$cpr(x, y) = |\{p \mid x \leq p < y \wedge p \text{ prime}\}|$$

is primitive recursive, where $cpr(x, y)$ counts the number of primes in the interval $[x, y]$.

Solution: Let us first define $cpr' : \mathbb{N}^2 \to \mathbb{N}$ such that $cpr'(x, k) = |p \mid x \leq p < x + k \wedge p \text{ prime}|$ by primitive recursion:

$$cpr'(x, 0) = 0$$
$$cpr'(x, k + 1) = cpr'(x, k) + \chi_{Pr}(x + k)$$

Then $cpr(x, y) = cpr'(x, y - x)$, which as composition of primitive recursive functions is itself primitive recursive.

Exercise 33

State the s-m-n theorem and use it to prove that there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$, $W_{s(x)} = \{(k + x)^2 \mid k \in \mathbb{N}\}$.

Solution: Define a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that when viewed as a function of y it has the desired properties:

$$g(x, y) = \begin{cases} k & \text{if } \exists k.\,, y = (x + k)^2 \\ \uparrow & \text{otherwise} \end{cases}$$

This can be written as $g(x, y) = \mu k. \, |(x + k)^2 - y|$. This function is computable, therefore by the s-m-n theorem there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\phi_{s(x)}(y) = g(x, y)$ for all $x, y \in \mathbb{N}$.