04/12

OVERVIEW:

- RICE'S THEOREM / SATURATED SETS → RICE-SHAPIRO RECURSIVENESS (8.X)
- R.E. SETS
- [- UNIVERSAL QUANTIFICATION + PROJECTION THEOREM]

→ [SATURATED] (EXTENSIONAL)

$\forall m, n \in \mathbb{N}$

$A \subseteq \mathbb{N}$ (SUBSET) $\Rightarrow m \in A$, $\varphi_m = \varphi_n$
$n \in A$

A (SETS / SUBSETS) ⇒ PROPERTIES

LOOKING AT A SET, A PROPERTY IS FUNCTIONALLY COMPUTABLE

$\mathcal{A}$ = SET OF COMPUTABLE FUNCTIONS

DOMAIN — CODOMAIN ⇒ DEFINED IN TERMS

$\{A \mid n \in \mathcal{A}\}$ ⇒ SATURATED (POINT...)

(EX. OF A SATURATED SET)

$ONE = \{ n \mid \varphi_n \text{ computes } 1\}$

↑ SATURATED → A PROPERTY IS ALWAYS DEFINED

$n, m, \ldots \in 1$

K → NON COMPUTABLE
NOT RECURSIVE

$n, m \in \mathbb{N}$

$[\varphi_n(m) \neq \varphi_m(n)]$

↑
NOT SATURATED

[RICE'S THEOREM] → NON-TRIVIAL PROPERTIES → NOT DECIDABLE
→ ARGUMENTATIONS

A def → $[A \neq \emptyset, A \neq \mathbb{N}, A \subseteq \mathbb{N}]$ (comput.)

if A is saturated, A is not recursive

W_x $\quad$ E_x

DIRECTLY → RICE'S THEOREM → NOT RECURSIVE
REDUCTION FROM K

$\boxed{8.9}$ $A = \{ x \in \mathbb{N} \mid \varphi_x(y) = x \cdot y$ for some "y" $\}$

$A / \bar{A}$ (RECURSIVENESS)

RECURSIVE → $\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

R.E → $sc_A = \begin{cases} 1 & x \in A \\ \uparrow & x \notin A \end{cases}$

↑
PARTIAL = "ON SOME INPUTS" THEY STOP

$A / \bar{A} \to$ RECURSIVE / R.E. (QUESTION?)

$\boxed{8.9}$ $A = \{ x \in \mathbb{N} \mid \varphi_x(y) = x \cdot y$ for some "y" $\}$

$[x, y] \to$ PARTIAL (DEPENDS ON BOTH)

$A \in \mathcal{A} = \{ x \in \mathbb{N} \mid \varphi_U(x, y) = x \cdot y \}$

A is saturated $\Rightarrow$ A not recursive

$\left. \begin{array}{l} A \neq \emptyset,\ A \neq \mathbb{N},\ A \subseteq \mathbb{N} \\ A \text{ saturated} \end{array} \right] \to$ RICE THEOREM A not recursive

$\boxed{A}$ K

RICE

$[K \leq_m A] \to$ REDUCTION

NOT RECURSIVE

$SC_K \to \begin{bmatrix} 1 & x \in W_K \\ \uparrow & x \notin W_K \end{bmatrix}$

SMN-THEOREM

$\uparrow$

$g(x, y) = \begin{cases} 0 & x \in K \\ \uparrow & \text{otherwise} \end{cases}$

$\frac{1}{2}g \mid x \mid$

WRITE IT IN A COMPUT. WAY $= \underbrace{\text{"} x \in K \text{"}}_{Q} = \begin{cases} \\ \mathbf{0}(SC_K(x)) \end{cases}$

CONSTANT ZERO FUNCTION $\to \mathbf{0}(x) = \begin{cases} 0 & x \in \mathbb{N} \\ \uparrow & \text{otherwise} \end{cases}$

$id(x) \begin{cases} x \\ \uparrow \end{cases}$

$\mathbb{O}(\mathcal{C}_K(x))$ $\rightarrow$ SMN-THEOREM

$\forall x,y \in \mathbb{N}\ g(x,y) = \varphi_{s(x)}(y)$

$- x \in K \Rightarrow s(x) \in A$

$\varphi_{s(x)}(y) = g(x,y) = 0,\ \forall y \in \mathbb{N} \rightarrow \varphi_{s(x)}(0) = 0$
$= s(x) \cdot 0$
$= 0 \in A$

$- x \notin K \Rightarrow s(x) \notin A$

$\varphi_{s(x)}(y) = g(x,y) = \uparrow,\ \forall y \in \mathbb{N}$

$\varphi_{s(x)}(y) = x \cdot y = s(x) \cdot y = \uparrow \cdot y = \uparrow$

$W_{s(x)} = E_{s(x)} = \emptyset \notin \mathbb{N}$

$A \rightarrow$ [NOT RECURSIVE / R.E]
$\hookrightarrow$ R.E? $(\mathcal{C}A) = \mathbb{O}(\mathcal{C}_K(x))$

$A = \{x \in \mathbb{N} \mid \varphi_x(y) = x \cdot y\}$

$\bar{A} = \{x \in \mathbb{N} \mid \varphi_x(y) \neq x \cdot y\}$

$\hookrightarrow$ [$\bar{A}$ is not r.e $\Rightarrow$ A WOULD BE RECURSIVE]

$A / \bar{A}$ recursive $\rightarrow$ K recursive

[$\forall m,n \in \mathbb{N} \mid \varphi_m = \varphi_n$ $\nearrow$!]

RECURSIVE → DECIDABLE

NOT RECURSIVE → NOT DECIDABLE

$$SC_Q = \begin{cases} 1 & \text{if } Q(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases} \qquad SC_Q : \mathbb{N}^k \to \mathbb{N}$$

Q = PREDICATE

$$\chi_Q = \begin{cases} 1 & \text{if } Q(\vec{x}) \text{ (is true)} \\ 0 & \text{otherwise} \end{cases}$$

SEMIDECIDABLE → R.E (SEMI CHAR. FUNCTION)

UNIVERSAL QUANTIFICATION → $Q(t, \vec{x}) \subseteq \mathbb{N}^{k+1}$

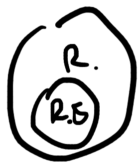
$$P(\vec{x}) \equiv \exists t.\ Q(t, \vec{x})$$

t = # OF STEPS

IF A PREDICATE IS DECIDABLE

↓

CAN BECOME SEMI-DECIDABLE



PROJECTION THEOREM → $P(x, \vec{y}) \subseteq \mathbb{N}^{k+1}$ SEMI-DECIDABLE

↓

$$R(\vec{y}) \equiv \exists x.\ P(x, \vec{y})$$

IS SEMI-DECIDABLE

$[(W)_1, (W)_2] \to [x, y]$

(WHATEVER PROPERTY CAN BE PROVEN TO STOP ON x, y FOR "SOME CONDITIONS")

"SOME CONDITIONS" $\Rightarrow$ x, y STOPS $[Q(\vec{x}, y)\downarrow_y]$

[2022-16-17] - EXAM

$Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ (PREDICATE)

IS SEMIDECIDABLE

$\Downarrow$

$P(\vec{x}, y) = \exists y . Q(\vec{x}, y)$ IS SEMI-DECIDABLE

1

$\Rightarrow Q(\vec{x}, y)$ SEMI-DECIDABLE

$SC_Q(\vec{x})$ IS COMPUTABLE

1 if $Q(\vec{x}, y)$

↑ otherwise

e = INDEX OF A PROGRAM

$H(x, y, t)$

$S(x, y, z, t)$

$[SC_Q, e \in \mathbb{N}, SC_Q = \varphi_e^{(k+1)}]$

ALTERNATIVE

$Q(\vec{x}, y)$ IS SEMI-DECIDABLE

$\Downarrow$

$P(\vec{x}, y) = \exists y . Q(\vec{x}, y)$ IS SEMI-DECIDABLE

TO PROVE

$SC_Q = \varphi_e^{(k+1)} = \exists t . H^{(k+1)}(e, (\vec{x}, y), t)$

STOPPING $Q(\vec{x}, y)$ IN t STEPS

$Q(\vec{x}, y) \equiv \exists t . H^{(k+1)}(e, (\vec{x}, y), t)$

$\searrow$

1 iff $\varphi_e^{(k+1)}(\vec{x}, y)\downarrow \quad = SC_Q$

SUMMARIZE:

$\hookrightarrow P(\vec{x}) = \exists y . Q(\vec{x}, y)$

$(w)_1 = y$

$(w)_2 = t$

$= \exists y . \exists t . H^{(k+1)}(e, (\vec{x}, y), t)$

$$\underbrace{= \exists w . H^{(k+1)}(e, (\vec{x}, (w)_1), (w)_2)}_{\text{1ST PART OF EX.}}$$

② DOES THE CONVERSE HOLD?

$Q(\vec{x}, y)$ IS SEMIDECIDABLE

$\Downarrow$

$P(\vec{x}) = \exists y . Q(\vec{x}, y)$ IS NOT SEMIDECIDABLE

$Q(\vec{x}, y) = SC_Q(\vec{x}, y)$ COMPUTABLE

SOMETHING NOT SEMI-DECIDABLE ?

$SC_K(\vec{x}, y)$

$$\emptyset(x) = \begin{cases} x & x \notin W_x \\ \uparrow & \text{otherwise} \end{cases}$$

$ID(5) = 5$

$$id(x) = \begin{cases} x \\ \uparrow \end{cases}$$

ALWAYS UNDEFINED FUNCTION $\rightarrow \emptyset(x)$

$P(\vec{x}) = \exists y . Q(\vec{x}, y) = \exists y . \varphi_y(x)\uparrow$

$\uparrow$

$SC_K = \begin{cases} 1 & \text{if } H(x, x, y)\uparrow \equiv Q(\vec{x}, y) \\ \uparrow & \text{otherwise} \end{cases}$

e

$\downarrow$

$P(\vec{x}) \rightarrow \exists w . H(e, (\vec{x}, x, y)\uparrow \rightarrow \forall x, y \in \mathbb{N}$

$\exists w . H(e, (\vec{x}, x, (w)_1)\uparrow$ NOT SEMIDECIDABLE

$\forall e \notin \mathbb{N} \mid P(\vec{x})\uparrow$

NOT COMPUTABLE
- $\rightarrow$ NOT RECURSIVE $\rightarrow K$
- DIAGONALIZATION $\rightarrow$ $\varphi_x \neq \varphi_{x+1}$

$\boxed{6.9}$

$\exists f : \mathbb{N} \rightarrow \mathbb{N}$ not-computable s.t. $dom(f) \cap img(f)$ is empty?

$dom(f) \wedge img(f) \rightarrow \emptyset$

$f(x) = \begin{cases} [2 \cdot qt(2, x) + \varphi_x(x)] & dom(f) = \text{EVEN} = rm(x, 2) = 0 \\ \varphi_{x/2}(x) \quad x \text{ odd} & cod(f) = \text{ODD} = rm(x, 2) \neq 0 \end{cases}$

TOTAL $\rightarrow$ DEFINED BY CASES

NOT COMPUTABLE $\rightarrow$ $\exists x \mid f(x)\downarrow$

$2 \cdot qt(2, x) + \varphi_x(x) \neq \dots + \varphi_{x+1}(x)$

$$f(x) = \begin{cases} 2 \cdot \varphi_K^{\lfloor x/2 \rfloor} & x \text{ ODD} \\ \uparrow & \text{OTHERWISE} \end{cases}$$

ODD/EVEN

$dom(f) \to$ SET OF ODD NUMBERS $(x \text{ ODD})$

$$\underline{dom(f) \cap cod(f)} = \emptyset$$

$\wedge$ OTHERWISE, $\frac{x}{2} \Rightarrow num(0)$

f NOT COMPUTABLE $\Rightarrow$ K WOULD BE RECURSIVE OTHERWISE

$\left[\varphi_K \text{ IS DEFINED}\right] = K$ STOPS

$\boxed{8.29}$ $\left[A = \{x \in \mathbb{N} : W_x \cap E_x \neq \emptyset\} \right]$

$W_x =$ DOMAIN $E_x =$ CODOMAIN/IMAGE

$\left[f : \mathbb{N} \to \mathbb{N} \right]$ $A / \bar{A}$ REC./R.E.

"FUNCTION PRACTICAL" MEANING

SATURATED

$A = \{x : \varphi_x \in \mathcal{A}\}$ where

$\mathcal{A} = \{f : \mathcal{C} : W_x \cap E_x \neq \emptyset\} = \{f : \mathcal{C} : dom(f) \cap cod(f) \neq \emptyset\}$

SATURATED $\varphi_{m} \neq \varphi_{n}$

$A \to$ REC. / R.E

RICE THEOREM $\to$ $A \neq \emptyset$, $A \neq \mathbb{N}$, $A \subseteq \mathbb{N}$
[A NOT RECURSIVE]

$K \leq_m A$ (TAKES A LOT OF TIME)

$\boxed{SC_A = \mathbb{1}\,(\mu.(x, y, t).\ H(x, y, t) \wedge S(x, z, y, t)}$

$A = \{W_x \cap E_x \neq \emptyset\}$ $\to$ W_x = DOES IT STOP?
E_x = DOES IT STOP IN A DEFINED N. OF STEPS?

$W_x = H$

$E_x = S$

$(w)_1 = y$, $(w)_2 = z$, $(w_3) = y$

$SC_A = \mathbb{1}\,(\mu w.\ H(x, (w)_1 (w_2)$ ......

x, y [A r.e $\to$ $\bar{A} = \{x \in \mathbb{N} \mid W_x \cap E_x = \emptyset\}$

[A not r.e $A = \{x \in \mathbb{N} \mid W_x \cap E_x \neq \emptyset\}$

A r.e $\to$ STOPS ON x, y ALWAYS (W_x, E_x)

$\bar{A}$ not r.e $\to$ IT DOES NOT STOP (BY DEFINITION)

A r.e / not rec. – $\bar{A}$ not r.e / not recursive

$\bar{A}$ not recursive otherwise A would be recursive!

(both recursive = both stopping always)