22-01

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-> PROVIDE SATURATED SET DEFINITION

$$A \subseteq \mathbb{N}$$

RECURSION THEOREMS (KLEENE) => RECURSIVENESS (R.E / REC)

$\exists m, n \in \mathbb{N} \Rightarrow \varphi_m = \varphi_n$

NOT SATURATED -> 2 REC. THEOREM

(2) STATE THE 2. R. T

TOTAL COMPUTABLE $f : \mathbb{N} \to \mathbb{N}$

$$\exists e \in \mathbb{N} \mid \varphi_e = \varphi_{f(e)}$$

(3) USE 2.R.T TO PROVE

K IS NOT SATURATED

SMN-THEOREM $\varphi_{s(x)}(y) = g(x,y) = \ldots$

$$g(x,y) = \begin{cases} 1 & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases} = \chi_K(x)$$

$$= \mu z. |y - x|$$

SMN - THEOREM $\varphi_{s(x)}(y) = g(x,y)$

2.R.T $\Rightarrow \exists e \in \mathbb{N} \mid \varphi_{R(e)} = \varphi_e$

$$h(e,y) = \begin{cases} 1 & \text{if } e \in W_e \\ \uparrow & \text{otherwise} \end{cases} = \mu z. |y - e|$$

NOT SATURATED $\Rightarrow \varphi_m \neq \varphi_n$

$e' \in \mathbb{N}, e' \neq e, \varphi_{e'} \neq \varphi_e$ $\quad e' \notin K$

$e' \in K$

$\boxed{\text{K IS NOT SATURATED}}$ ← TEMPLATE 2.R.T.

SMN - THEOREM → STATE IT

$\exists s : \mathbb{N} \to \mathbb{N}$ (TOTAL / COMPUTABLE)

$\forall x \in \mathbb{N}, x > 0 \mid W_{s(x)} = \mathbb{P}, E_{s(x)} = 2x$

$\mathbb{P}$: SET OF EVEN NUMBERS

$P_M \neq \mathbb{P}$

- STATE IT

$[S_m^n]$ $\exists\, m, n > 1$

$S_m^n : \mathbb{N}^{m+1} \to \mathbb{N}$ s.t. $\forall e \in \mathbb{N}, \vec{x} \in \mathbb{N}, \vec{y} \in \mathbb{N}^n$

$$\varphi_e^{(m+n)}(\vec{x}, \vec{y}) = \varphi^{(n)}_{s_{m,n}(e, \vec{x})}(\vec{y})$$

$[W_{s(x)} = \mathbb{P},\ |E_{s(x)}| = 2x]$ $\quad g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x,y) = \begin{cases} qt(2,y) \bmod 2x & y \in \mathbb{P},\ x > 0 \\ \uparrow & \text{otherwise} \end{cases}$$

$$= \min(2x, qt(2,y))$$

$$+ \mu z.\ (\mathrm{rm}(2,y) + \overline{sg}(x))$$

$$\overline{sg} = \begin{cases} 0 \\ 1 \end{cases}$$

$|W_{s(x)}| = \{x \in \mathbb{N} \mid g(x,y)\downarrow\}$

$= \{x \in \mathbb{P} \mid y \in \mathbb{P}\} = \{\mathbb{P}\}$

$|E_{(s(x))}| = |\{g(x,y)\downarrow \mid y < 2x\}| = 2x$

SET = PROPERTY $\{A \subseteq \mathbb{N}\}$

[RECURSIVE] $\Longrightarrow$ χ_A

[NOT RECURSIVE] — $K \leq_m A$ / RICE-THEOREM

[NOT R.E] — $\bar{K} \leq_m A$ / RICE - SHAPIRO

[NOT COMPUTABLE $\subseteq$ NOT RECURSIVE]

χ_K — DIAGONALIZATION $\varphi_e \neq \varphi_{e+1}$

[SATURATED $\Rightarrow$ RECURSIVENESS

NOT SATURATED $\Rightarrow$ 2.R.E]

$A = \{x \in \mathbb{N} \mid W_x \neq \emptyset \land W_x \subseteq E_x\}$

REC. / R.E (?)

SATURATEDNESS $\rightarrow$ $\exists$ INPUTS FOR WHICH THE SET CONDITIONS

HOLD IN
A FINITE
N. OF STEPS

$A = \{x \in \mathbb{N} \mid W_x \neq \emptyset \wedge W_x \subseteq E_x\}$

A IS SATURATED $\Rightarrow$ $\mathcal{A}$ = SET OF COMPUTABLE FUNCTIONS

$A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$

$\mathcal{A} = \{f \mid dom(f) \neq \emptyset \wedge dom(f) \subseteq cod(f)\}$

8.7X/6X

$\varphi_x(y) \Rightarrow f_x(y)$

$\varphi_x(x) \Rightarrow f_x(x)$

$A = \{x \in \mathbb{N} \mid W_x \neq \emptyset \wedge W_x \subseteq E_x\}$

A IS NOT R.E. $\rightarrow$ RICE-SHAPIRO (?)

$\rightarrow$ ① $\exists f \in \mathcal{A}, \forall \theta \notin \mathcal{A}$
② $\exists f \notin \mathcal{A}, \exists \theta \subseteq f \in \mathcal{A}$

$ID = \mathbb{N}$ $\quad ID \in A \mid W_x = \mathbb{N} \neq \emptyset \wedge \mathbb{N} \subseteq \mathbb{N}$

[NO FINITE SUBFUNCTION DOES NOT RESPECT $\mathcal{A}$]

$\Downarrow$

ALTERNATIVELY $\rightarrow$ USE PROD $(x) = \begin{cases} x \dot{-} 1 \\ \uparrow \end{cases}$

$\bar{A} = \{x \in \mathbb{N} \mid W_x = \emptyset \vee W_x \not\subseteq E_x\}$

$\downarrow$ RECURSIVE (NOT)?

$\left[ID \notin \bar{A} \mid W_x = \mathbb{N} \neq \emptyset \vee \mathbb{N} \not\subseteq \mathbb{N} \right]$

$\exists f \notin A, \exists \theta \subseteq f \in A \Leftarrow$ (2) RICE-SHAPIRO ...

$\left[\theta = \emptyset \in \bar{A} \mid W_x = \emptyset \vee \emptyset \not\subseteq \emptyset \right]$

$\underbrace{x}_{\text{PROD}} \not\subseteq x-1 \quad \underbrace{\left[\begin{array}{l} x = \{0,1,2\} \\ x = \{0,1\} \end{array} \right]}_{\text{RANGE-IDEA}}$

dom (pred) $\subseteq$ cod (pred)

$A / \bar{A}$ NOT R.E $\rightarrow$ NOT RECURSIVE

✓

PR → STATE WHAT PR MEANS

SHOW

① isqrt : $\mathbb{N} \to \mathbb{N}$ s.t. isqrt(x) = $\lfloor \sqrt{x} \rfloor$

② lp : $\mathbb{N} \to \mathbb{N}$ (LARGEST PRIME DIVISOR OF x)

ASSUME PR FUNCTIONS SEEN IN THE COURSE

PR
- "NORMAL" DEFINITION (INDUCTION)
- MINIMALIZATION (μ - OPERATOR)

isqrt : $\mathbb{N} \to \mathbb{N}$ s.t. isqrt(x) = $\lfloor \sqrt{x} \rfloor$

$$\sqrt{x} = y^2$$

LARGEST y s.t. $y^2 \leq x$

SMALLEST y s.t. $y^2 > x$

$y^2 \leq x \Rightarrow y+1$

$$\text{isqrt}(x) = \mu y < x+1 . ((y+1)^2 > x)$$

$= \mu y < x+1 \, . \, \overline{sg}((y+1)^2 \dot{-} x))$

② $lp : \mathbb{N} \to \mathbb{N}$ (LARGEST PRIME DIVISOR OF x)

BOUNDED SUM / BOUNDED PRODUCT $\in \mathcal{PR}$

$p_i \in Pr$

COUNT $: \mathbb{N} \to \mathbb{N}$ s.t.

$$count(x) = \sum_{i=1}^{x} div(p_i, x)$$

$$lp(x) = count(x) + \overline{sg}(x \dot{-} 1)$$

[EXAMPLES FROM LESSON $\to$ lcd / gcd]

$A = \{x \in \mathbb{N} \mid x+1 \in E_x\}$ $\quad A / \bar{A}$ REC. / R.E.

$K \leq_m A$ $\quad [x \in K \mid f(x) \in A]$

SMN-THEOREM

$$g(x,y) = \begin{cases} y & x \in K \\ \uparrow & \text{otherwise} \end{cases} = y \cdot SC_K(x)$$

$$SC_K = \begin{cases} 1 & x \in K \\ \uparrow & \text{OTHERWISE} \end{cases}$$

$\Downarrow$

$\varphi_{s(x)}(y) = g(x,y)$

- $x \in K$, $\varphi_{s(x)}(y) = g(x,y) = y \quad \forall y \in \mathbb{N}$

$y \equiv s(x) \quad | \quad s(x)+1 \in E_{s(x)} = \mathbb{N}$

so $s(x) \in A$

- $x \notin K$, $\varphi_{s(x)}(y) = g(x,y) = \uparrow$, $\forall y \in \mathbb{N}$

$W_{s(x)} = E_{s(x)} = \emptyset \neq \mathbb{N}$

A NOT RECURSIVE

$e_1, e_2 = \mathrm{Id} / \emptyset$

$\hookrightarrow$ ALTERNATIVELY (RICE'S THEOREM) $\Rightarrow A \neq \emptyset, A \neq \mathbb{N}$

SATURATED

$=$

A NOT RECURSIVE

$A = \{x \in \mathbb{N} \mid x+1 \in \underbrace{E_x}\}$

$\downarrow$ NOT REC. / R.E(?)

$\downarrow$ $S(x,y,z,t)$

$W_x \equiv H(x,y,t)$

$K = H(x,x,y)$

$sc_A(x) =$

$\mathbb{1}(\mu(x,y,t) \cdot S(x,y,x+1,t)) = \varphi_x$

$= \mathbb{1}(\mu(x,y,t) \cdot |\chi_S(x,y,x+1,t) - 1|)$

$(w)_1 = y,\ (w)_2 = t$

$= \mathbf{1}(\mu w. |\chi_S(x, (w)_1, x+1, (w)_2 - 1|)$

$\Downarrow$

$\mathbf{1}(\mu w. S(x, (w)_1, x+1, (w)_2))$

$A \Rightarrow$ NOT REC. / NOT R.E.

$\bar{A} = \{x \in \mathbb{N} \mid x+1 \notin E_x\} \rightarrow$ NOT R.E.

$\bar{A}$ (NOT RECURSIVE $\Rightarrow$ OTHERWISE BOTH WOULD BE)