- **sum** $x + y$
  $x + 0 = x$
  $x + (y+1) = (x+y) + 1$

  $h(x,0) = x$
  $h(x,y+1) = h(x,y) + 1$

  $f(x) = x$
  $g(x,y,z) = z + 1$

- **product** $x \cdot y$
  $x \cdot 0 = 0$
  $x \cdot (y+1) = (x \cdot y) + x$

  $h(x,0) = 0$
  $h(x,y+1) = h(x,y) + x$

  $f(x) = 0$
  $g(x,y,z) = z + y$

- **factorial** $y!$
  $0! = 1$
  $(y+1)! = y! \cdot (y+1)$

  $h(0) = 1$
  $h(y+1) = h(y) \cdot (y+1)$

  $f(0) = 1$
  $g(y,z) = z \cdot (y+1)$

exponential x^y

$x^0 = 1$ $\quad h(x,0) = 1$ $\quad f(x) = 1$
$x^{y+1} = x^y \cdot x$ $\quad h(x,y+1) = h(x,y) \cdot x$ $\quad g(x,y,z) = z \cdot x$

predecessor $x \dot{-} 1$

$0 \dot{-} 1 = 0$ $\quad h(0) = 0$ $\quad f \equiv \underline{0}$
$(x+1) \dot{-} 1 = x$ $\quad h(x+1) = x$ $\quad g(y,z) = y$

subtraction $x \dot{-} y = \begin{cases} x - y & x \geqslant y \\ 0 & \text{otherwise} \end{cases}$

$x \dot{-} 0 = x$ $\quad f(x) = x$
$x \dot{-} (y+1) = (x \dot{-} y) \dot{-} 1$ $\quad g(x,y,z) = z \dot{-} 1$

sign $sg(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$

$sg(0) = 0$ $\quad f \equiv \underline{0}$
$sg(x+1) = 1$ $\quad g(y,z) = 1$

(7) **complement sign** $\bar{sg}(x) = \begin{cases} 1 & x = 0 \\ 0 & x > 0 \end{cases}$
$\bar{sg}(x) = 1 \dot{-} sg(x)$, composition and (6);

(8) $|x - y| = \begin{cases} x - y & x \geqslant y \\ y - x & x < y \end{cases}$
$|x-y| = (x \dot{-} y) + (y \dot{-} x)$ from (1), (6) and composition;

(9) **factorial** $y!$
$0! = 1$ $\quad f \equiv (y+1)! = y! \cdot (y+1)$ $\quad g(y,z) = (y+1) \cdot z$

(10) **minimum** $min(x,y) = x \dot{-} (x \dot{-} y)$;

(11) **maximum** $max(x,y) = (x \dot{-} y) + y$;

(12) **remainder** $rm(x,y) = \begin{cases} y \mod x & x \neq 0 \\ y & x = 0 \end{cases}$
remainder of the integer division of y by x

$$rm(x,0) = 0$$
$$rm(x,y+1) = \begin{cases} rm(x,y) + 1 & rm(x,y) + 1 \neq x \\ 0 & \text{otherwise} \end{cases}$$
$$= (rm(x,y) + 1) \cdot sg((x \dot{-} 1) \dot{-} rm(x,y))$$
$$f(x) = 0 \quad g(x,y,z) = z * sg(x \dot{-} 1 \dot{-} z)$$

(13) **quotient** $qt(x, y) = y$ div x (convention $qt(0, y) = y$), we define:

$$
\begin{aligned}
qt(x, 0) &= 0 \\
qt(x, y+1) &= \begin{cases} qt(x, y) + 1 & rm(x, y) + 1 = x \\ qt(x, y) & \text{otherwise} \end{cases} \\
&= qt(x, y) + sg((x \dot{-} 1) \dot{-} rm(x, y))
\end{aligned}
$$

(14)

$$
\begin{aligned}
div(x, y) &= \begin{cases} 1 & rm(x, y) = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \bar{sg}(rm(x, y))
\end{aligned}
$$