Rice-Shapiro's theorem is a powerful tool for proving that certain sets of computable functions are not recursively enumerable (r.e.). It provides a necessary and sufficient condition for a set of indices $A = \{x \mid \varphi_x \in \mathcal{A}\}$ to be r.e. in terms of the functions in the corresponding set $\mathcal{A} \subseteq C$. The theorem states:

$\forall f \, (f \in \mathcal{A} \Leftrightarrow \exists \theta \text{ finite}, \theta \subseteq f, \theta \in \mathcal{A})$

In other words, a set A is r.e. if and only if for every function f, f belongs to A precisely when there exists a finite subfunction θ of f such that θ is also in A.

This amounts to

① $\exists f \quad f \notin \mathcal{A}$ and $\exists \theta \subseteq f$, θ finite $\theta \in \mathcal{A} \implies A$ not r.e.

② $\exists f \quad f \in \mathcal{A}$ and $\forall \theta \subseteq f$, θ finite $\theta \notin \mathcal{A} \implies A$ not r.e.

Key Steps in Applying Rice-Shapiro's Theorem

To use Rice-Shapiro's theorem in exercises, follow these steps:

1. Identify the set of computable functions A that you want to prove is not r.e.
2. Determine the corresponding set of indices $A = \{x \mid \varphi_x \in A\}$.
3. Find a function f that violates the condition of the theorem, i.e.:
   - Either $f \in A$ but no finite subfunction $\theta \subseteq f$ is in A,
   - Or $f \notin A$ but there exists a finite subfunction $\theta \subseteq f$ such that $\theta \in A$.
4. Conclude that the set A is not r.e.

Examples

Consider the following sets of computable functions:

1. $A = \{f \mid f \text{ is total}\}$
   - The identity function $id \in A$, but no finite subfunction $\theta \subseteq id$ is total, so $\theta \notin A$.
   - Therefore, A is not r.e.
2. $A = \{f \mid f \text{ is not total}\}$
   - The always diverging function $\emptyset(x)\uparrow$ is not in A, but its finite subfunction θ (the empty function) is in A.
   - Thus, A is not r.e.

Differences from Rice's Theorem

While both Rice-Shapiro's theorem and Rice's theorem are used to prove that certain sets are not recursive or r.e., there are some key differences:

1. Rice's theorem applies to non-trivial properties of partial computable functions, while Rice-Shapiro's theorem deals with sets of total computable functions.
2. Rice's theorem uses the concept of extensionality (if two functions compute the same partial function, they are either both in the set or both not in the set), while Rice-Shapiro's theorem uses the notion of subfunctions.
3. Rice's theorem is typically used to prove that a set is not recursive, while Rice-Shapiro's theorem is used to prove that a set is not r.e.

→ Rice's Theorem : only trivial (true / false) extensional properties are decidable

→ Rice-Shapiro's Theorem : an extensional property of programs $\boxed{\text{can be}}$ semidecidable only when it is finitary

(behaviour of the program on a finite amount of inputs