Diagonal Method and Non-Computability

The diagonal method is used to prove the existence of non-computable functions. For any enumeration of computable functions φ_0, φ_1, φ_2, ..., we can construct a function f that differs from each φ_i at position i:

```
f(n) = {
    ϕₙ(n) + 1   if ϕₙ(n)↓
    0            if ϕₙ(n)↑
}
```

This function f is total but not computable, as it differs from every computable function φ_i at position i.

The SMN Theorem (Parametrization Theorem)

For any m,n ≥ 1, there exists a total computable function s(m,n): $N^{m+1} \to N$ such that for all e ∈ N, $\bar{x} \in N^m$, $\bar{y} \in N^n$:

$\varphi_e^{m+n}(\bar{x},\bar{y}) = \varphi_{s(m,n)(e,\bar{x})}^n(\bar{y})$

Key Applications

1. **Parameter Fixing**: Given a computable function g(x,y), there exists a total computable function s such that:

   $\varphi_{s(x)}(y) = g(x,y)$
2. **Reduction Functions**: Often used to create reduction functions by:
   - Finding appropriate g(x,y)
   - Using SMN theorem to obtain s
   - Proving s is the required reduction function

Example Application

To show there exists s: $N^2 \to N$ such that $W_{s(x,y)} = \{z : x * z = y\}$:

1. Define helper function:

```
f(x,y,z) = {
    0   if x * z = y
```

```
    ↑   otherwise
}
```

2. By SMN theorem, get $s: N^2 \to N$ where:
    - $\varphi_{s(x,y)}(z) = f(x,y,z)$
    - Therefore, $z \in W_{s(x,y)} \iff x * z = y$

Universal Function

The universal function $\Psi_u: N^2 \to N$ is defined as:

$\Psi_u(e,x) = \varphi_e(x)$

This represents an interpreter that can simulate any computable function given its index. A key application is replacing $\varphi_x(y)$ with $\Psi_u(x,y)$ when writing semicharacteristic functions:

- Instead of: $sc_a(x) = (...- \varphi_x(x))$
- Write: $sc_a(x) = (...- \Psi_u(x,x))$