Computability Exam Solutions

February 8, 2019

Exercise 1

Definition of the class PR of primitive recursive functions

The class PR of primitive recursive functions is the smallest class of functions $PR \subseteq \bigcup_k(\mathbb{N}^k \to \mathbb{N})$ that:

1. **Contains the basic functions:**
   - Zero function: zero(x) = 0
   - Successor function: succ(x) = x + 1
   - Projection functions: $\pi_i^k(x_1,\dots,x_k) = x_i$ for $1 \le i \le k$
2. **Is closed under composition:** If $g_1,\dots,g_m \in PR$ and $h \in PR$, then $f \in PR$ where $f(\vec{x}) = h(g_1(\vec{x}),\dots,g_m(\vec{x}))$
3. **Is closed under primitive recursion:** If g, h ∈ PR, then f ∈ PR where:

   ```
   f(x⃗, 0) = g(x⃗)
   f(x⃗, y+1) = h(x⃗, y, f(x⃗, y))
   ```

Proof that $sum_k(x_1,\dots,x_k) = \Sigma_{i=1}^k x_i$ is primitive recursive for $k \ge 2$

We proceed by induction on k.

Base case k = 2: The binary sum function $sum_2(x, y) = x + y$ is primitive recursive:

```
sum₂(x, 0) = x = π₁¹(x)
sum₂(x, y+1) = sum₂(x, y) + 1 = succ(π₃³(x, y, sum₂(x, y)))
```

Inductive step: Assume $sum_{k-1} \in PR$. We show $sum_k \in PR$:

```
sumₖ(x₁,...,xₖ) = sumₖ₋₁(x₁,...,xₖ₋₁) + xₖ
                = sum₂(sumₖ₋₁(π₁ᵏ(x₁,...,xₖ),...,πₖ₋₁ᵏ(x₁,...,xₖ)), πₖᵏ(x₁,...,xₖ))
```

Since sum_k is obtained by composition of primitive recursive functions (sum_{k-1}, sum_2, and projections), it is primitive recursive.

Exercise 2

Definition and Analysis of Z(f)

Given $f : \mathbb{N} \to \mathbb{N}$, define:

```
Z(f) = {g : ℕ → ℕ | ∀x ∈ ℕ. g(x) = f(x) ∨ g(x) = 0}
```

Proof that Z(id) is not enumerable

Let id(x) = x be the identity function. Then:

```
Z(id) = {g : ℕ → ℕ | ∀x ∈ ℕ. g(x) = x ∨ g(x) = 0}
```

Each function $g \in Z(id)$ is determined by the set $S = \{x \in \mathbb{N} \mid g(x) = x\}$, since:

```
g(x) = {
  x  if x ∈ S
  0  if x ∉ S
}
```

This establishes a bijection between Z(id) and P(ℕ) (the powerset of ℕ). Since P(ℕ) is uncountable, Z(id) is uncountable and hence not enumerable.

Is Z(f) non-enumerable for every function f?

Answer: No.

Counterexample: Let f(x) = 0 for all x (constant zero function). Then:

```
Z(f) = {g : ℕ → ℕ | ∀x ∈ ℕ. g(x) = 0 ∨ g(x) = 0} = {g : ℕ → ℕ | ∀x ∈ ℕ. g(x) = 0}
```

This set contains only the constant zero function, so $|Z(f)| = 1$, which is clearly enumerable (in fact, finite).

Exercise 3

Classification of $A = \{x \mid W_x \subseteq \{x\}\}$

A is r.e.:

```
scₐ(x) = 1(μ⟨y,t⟩.H(x, y, t) ∧ y ≠ x)
```

This searches for evidence that some $y \neq x$ is in W_x. If found, the computation diverges (making $x \notin A$). If no such y exists, then $W_x \subseteq \{x\}$, so $x \in A$.

Actually, let me reconsider this. We want to check if $W_x \subseteq \{x\}$. This means:

```
x ∈ A ⟺ ∀y ∈ Wₓ. y = x
```

This is equivalent to:

```
x ∈ A ⟺ ¬∃y ≠ x. y ∈ Wₓ
```

The semi-characteristic function can be defined as:

```
scₐ(x) = 1(μ⟨y,t⟩.(y ≠ x ∧ H(x, y, t)))
```

If there exists $y \neq x$ such that $y \in W_x$, this will eventually find it and diverge. Otherwise, it will never converge, which means $x \in A$.

Wait, this is backwards. Let me be more careful:

```
scₐ(x) = {
  1  if Wₓ ⊆ {x}
  ↑  if Wₓ ⊈ {x}
}
```

We can't directly compute this as stated. Instead, consider that A is **not r.e.**

A is not r.e.: We show $\bar{K} \leq_m A$. Define:

```
g(x, y) = {
  x       if x ∉ K
  ↑       if x ∈ K
}
```

By s-m-n theorem, $\exists s$ such that $\varphi_{s(x)}(y) = g(x, y)$.

- If $x \notin K$: $\varphi_x(x) \uparrow$, so $W_{s(x)} = \{x\}$, hence $s(x) \in A$
- If $x \in K$: $\varphi_x(x) \downarrow$, so $W_{s(x)} = \varnothing \subseteq \{s(x)\}$, hence $s(x) \in A$

This doesn't work. Let me try differently:

```
g(x, y) = {
  y       if y = x and x ∉ K
  ↑       otherwise
}
```

- If $x \notin K$: $W_{s(x)} = \{x\}$, so $s(x) \in A$
- If $x \in K$: $W_{s(x)} = \varnothing \subseteq \{s(x)\}$, so $s(x) \in A$

Still doesn't work. Let me reconsider the problem structure.

Actually, let's use a different approach:

```
g(x, y) = {
  0       if x ∉ K
  y       if x ∈ K
}
```

- If $x \notin K$: $W_{s(x)} = \{0\}$, and since s(x) likely ≠ 0, we have $W_{s(x)} \nsubseteq \{s(x)\}$, so $s(x) \notin A$
- If $x \in K$: $W_{s(x)} = \mathbb{N}$, so $W_{s(x)} \nsubseteq \{s(x)\}$, hence $s(x) \notin A$

This gives $\bar{K} \leq_m \bar{A}$, so $\bar{A}$ is not r.e., hence A is not recursive.

Let me try to show A is r.e. more carefully. A is r.e. because:

```
scₐ(x) = lim_{t→∞} [∀y≤t (H(x,y,t) → y=x)]
```

If $W_x \subseteq \{x\}$, then eventually we will have checked all elements of W_x and confirmed they equal x.

Final classification: A is r.e. but not recursive; $\bar{A}$ is not r.e.

Exercise 4

Classification of B = {x ∈ ℕ : |W_x| > 1}

B is r.e.:

```
scB(x) = 1(μ(y₁,y₂,t).(y₁ ≠ y₂ ∧ H(x,y₁,t) ∧ H(x,y₂,t)))
```

This searches for two distinct elements in W_x.

B is not recursive: We show Tot $\leq_m \bar{B}$ where Tot = {x | φ_x total}. Define:

```
g(x, y) = {
  0      if y = 0
  1      if y = 1 and ϕₓ(y) ↓ ∀y
  ↑      otherwise
}
```

- If φ_x is total: $W_{s(x)} = \{0, 1\}$, so $|W_{s(x)}| = 2 > 1$, hence $s(x) \in B$
- If φ_x is not total: $W_{s(x)} = \{0\}$, so $|W_{s(x)}| = 1$, hence $s(x) \notin B$

This gives Tot $\leq_m$ B. Since Tot is not recursive, B is not recursive.

$\bar{B}$ is not r.e.: Since B is r.e. but not recursive, $\bar{B}$ is not r.e.

Final classification: B is r.e. but not recursive; $\bar{B}$ is not r.e.

Exercise 5

Second Recursion Theorem

For every total computable function $f : \mathbb{N} \to \mathbb{N}$, there exists $e_0 \in \mathbb{N}$ such that:

```
ϕₑ₀ = ϕf(e₀)
```

Proof that $A = \{x \mid W_x \subseteq \{x\}\}$ is not saturated

Define $f : \mathbb{N} \to \mathbb{N}$ by:

```
f(x) = x + 1
```

By the Second Recursion Theorem, $\exists e$ such that $\varphi_e = \varphi f(e) = \varphi_{e+1}$.

Now consider the function computed by program e:

- If $e \in A$, then $W_e \subseteq \{e\}$
- Since $\varphi_e = \varphi_{e+1}$, we have $W_e = W_{e+1}$
- If $e \in A$, then $W_{e+1} = W_e \subseteq \{e\} \neq \{e+1\}$ (assuming $e \neq e+1$)
- So $W_{e+1} \nsubseteq \{e+1\}$, which means $e+1 \notin A$

This shows that $e \in A$, $\varphi_e = \varphi_{e+1}$, but $e+1 \notin A$.

Therefore, A is not saturated since it doesn't respect functional equivalence.