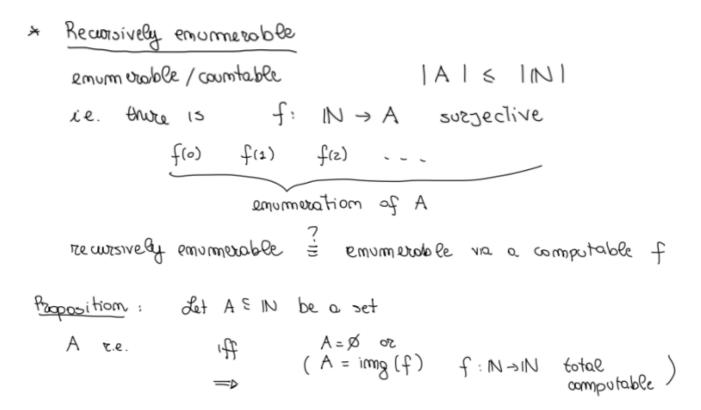
## Reducibility and r.e. sets

Given two sets  $A, B \subseteq \mathbb{N}$  and  $A \leq_m B$ :

- 1) if B is r.e. then A is r.e.
- 2) if A is not r.e. then B is not r.e.

## **Etymology theorem**



The etymology theorem, which states that a set A is recursively enumerable if and only if  $A = cod(f) = \{f(x) \mid x \in N\}$  for some computable function  $f : N \to N$ , is important here because it provides a key insight into why the set A in the proposition is not recursively enumerable when  $A \neq \emptyset$  and  $A \neq N$ .

The proof proceeds by assuming A is recursively enumerable and then deriving a contradiction. If A is recursively enumerable, then by the etymology theorem, there exists a computable function  $f: N \to N$  such that  $A = img(f) = \{f(x) \mid x \in N\}$ .

Now, we distinguish two cases:

- 1. If  $A = \emptyset$ , then f is a total computable function with  $img(f) = \emptyset$ . However, this is not possible, since img(f) must contain at least the elements  $\{f(x) \mid x \in N\}$ . So A cannot be empty.
- 2. If  $A \neq \emptyset$ , then fix some  $a0 \in A$ . Since A = img(f), f must be total, otherwise  $img(f) \subset A$  which contradicts A = img(f). But then the function F defined by F(x) = a0 if  $x \in A$  and

F(x) = a0 otherwise, is total computable, and  $img(F) = \{a0\}$ . However,  $img(F) \neq A$  since we assumed  $A \neq N$ . This contradicts the assumption that A = img(f).

Therefore, in both cases we arrive at a contradiction by assuming A is recursively enumerable and  $A \neq \emptyset$  and  $A \neq N$ . The etymology theorem was key in allowing us to characterize A as the image of a total computable function f and leading to these contradictions.

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