11 - 12

OVERVIEW

- REDUCIBILITY AND R.E SETS
- ALTERNATIVE CHARACTERIZATION OF R.E SETS
- RICE-SHAPIRO'S THEOREM → RECURSIVENESS

PREDICATES
- $\boxed{\text{STRUCTURE}}$ THEOREM
- PROJECTION THEOREM

→ A = N. OF STEPS ON WHICH THEY HALT

$Q(A, \vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable $\chi_Q = \begin{cases} 1 \\ 0 \end{cases}$

⇓

semidecidable

$P(\vec{x}) = \exists A. Q(A, \vec{x})$

THIS IS A LITERAL NOTATION (TO BE USED)

PREDICATES ⟷ FUNCTIONS

A = SET = Holds a property

COMPUTABLE → RECURSIVE $\begin{cases} 1 \\ 0 \end{cases}$

NOT COMPUTABLE
- → NOT RECURSIVE (K)
- DIAGONALIZATION (φ_x)

- RECURSIVE (Q)
||
COMPUTABLE
χ_Q

- R.E (Q)
||
"SEMI" - COMPUTABLE
sc_Q

NOT RECURSIVE { RICE'S THEOREM
K (K $\leq_m$ A)

NOT RECURSIVE

NOT RECURSIVE $\not\Rightarrow$ NOT R.E

IT'S BETTER $\rightarrow$ R.E. SETS

$\exists$ AN INPUT ON WHICH WE STOP

$W_x \rightarrow H$
$E_x \rightarrow S$

SATURATED { m, n $\Rightarrow$ $\varphi_m = \varphi_n$

[RICE - SHAPIRO] $\rightarrow$ NOT R.E

A = set of comput. functions

$\exists f \notin A,\ \exists \theta \in A$
$\exists f \in A,\ \forall \theta \notin A$

θ = SUBFUNCTION (FINITE)

$id(x) = \begin{cases} x \\ \uparrow \end{cases}$ $\quad 1 = \begin{cases} 1 \\ \uparrow \end{cases}$

$\boxed{8.48}$ $A = \{x \in \mathbb{N} \mid \mathbb{P} \subseteq W_x\}$

$[A / \bar{A}$ REC. / R.E$]$? $\mathbb{P}$ = SET OF PRIME NUMBERS

= $\exists$ AN INPUT WHICH IS PRIME

$A \to$ recursive?

$A \neq \emptyset$, $A \neq \mathbb{N}$, A saturated — $\exists\, m, n$ PRIME NUMBERS IN ITS ENTIRE DOMAIN?

UNIV. FUNCTION

$A = \{x \mid \varphi_x \in \mathcal{A}\}$ dom(f)

$= \{x \in \mathbb{N} \mid \mathbb{P} \subseteq W_x\}$

$W_x \to$ dom(f) $\quad E_x \to$ cod(f)

$\varphi_x(x) \to f(x)$

$[\mathcal{A} = \{f \mid \mathbb{P} \subseteq \text{dom}(f)\}]$

① A IS SATURATED

② RICE-SHAPIRO $\to$ NOT R.E

$\underline{A \text{ is not r.e}}$

$$\left[\begin{array}{l}\exists f \notin \mathcal{A},\ \exists \theta \in \mathcal{A} \\ \exists f \in \mathcal{A},\ \forall \theta \notin \mathcal{A}\end{array}\right] \text{R.S.}$$

$$id(x) = \begin{cases} x & x \in W_x \\ \uparrow & \end{cases}$$

$[\mathbb{P} \subseteq \mathbb{N}]$

$id \in \mathcal{A}$, $\forall \theta \notin \mathcal{A}$

$A = \{x \in \mathbb{N} \mid \mathbb{P} \subseteq W_x\}$ → NOT R.E ($\underbrace{id \in A, \forall \theta \notin A}$)

$\bar{A} = \{x \in \mathbb{N} \mid \mathbb{P} \not\subseteq W_x\}$

$[id \notin \bar{A}, \emptyset \in \bar{A}]$ → $\begin{bmatrix} \exists f \notin A, \exists \theta \in A \\ \exists f \in A, \forall \theta \notin A \end{bmatrix}$

$A / \bar{A}$ NOT R.E → REC

$\underline{\text{NOT REC.}}$

(THEOREM...)

OTHERWISE A ($\bar{A}$) WOULD BE RECURSIVE

$\boxed{8.65}$

SET → SATURATED (?) $\forall \theta \notin A$

W_x IN AN $\underline{\text{INFINITE}}$ NUMBER OF STEPS)

SET DIFFERENCE = BACKSLASH

$A = \{x \mid W_x \setminus E_x \text{ INFINITE}\}$

$A / \bar{A}$ REC./RE?

$K \to \varphi_m \neq \varphi_n$ (NOT SATURATED)

SATURATED $\begin{cases} Y \\ N \end{cases}$

$[W_x \setminus E_x]$ INFINITE

$\|$

$[[dom(f) \setminus cod(f)]$ INFINITE$]$

$A = \{x \mid \varphi_x \in \mathcal{A}\}$

$\Downarrow$

$\mathcal{A} = \{f \mid dom(f) \setminus cod(f)$ INFINITE$\}$ (A IS SATUR.)

$[A = \{W_x \setminus E_x$ INFINITE$\}] \to SC_A$

$\bar{A} = \{W_x \setminus E_x$ FINITE$\}$ $[W_x = E_x]$ id

$SC_A = \mathbb{1}(\cancel{\mu w. \mid \chi_H(x, y, A)} \setminus \chi_S(x, z, y, A) \mid)\}$

NOT R.E. $\longrightarrow$ RICE - SHAPIRO

PROF.

$\to \begin{bmatrix} \exists f \notin \mathcal{A}, \exists \theta \in \mathcal{A} \\ \exists f \in \mathcal{A}, \forall \theta \notin \mathcal{A} \end{bmatrix}$ 1 2

$[\mathbb{1} \in \mathcal{A}, \forall \theta \notin \mathcal{A}]$

$A = \{\underline{W_x \setminus E_x \text{ INFINITE}}\}$

$\mathcal{A} = \ldots$ $id \notin \mathcal{A}$, $0 / \mathbb{1}$ ["USUAL" SUBFUNCTIONS]

$\underline{\mathbb{1} \in \mathcal{A}}$ $\begin{cases} DOM(\mathbb{1}) = \mathbb{1} \ (\mathbb{N}) \\ COD(\mathbb{1}) = 1 \end{cases}$

$\theta \in \mathcal{A} \iff \underline{1 \setminus 1}$ INFINITE?

$\bar{A} = \{ W_x \setminus E_x \text{ FINITE } \} \to$ R.E

NOT R.E

$[id \notin A, \; \emptyset = \theta \in A] \to A$

$id \in \bar{A}, \; \exists \underbrace{\theta \notin \bar{A}}$

$\boxed{1 \notin \bar{A}}$

[8.33] $A = \{ x \in \mathbb{N} : \underline{x \in W_x \wedge \varphi_x(x) > x} \}$ $x+1$

$A / \bar{A}$ REC. / R.E (?) COMPUTABLE

$x \in W_x$

$SC_A = 1(\mu w. \boxed{H(x, x, y)} \wedge (\underbrace{f_x(x)}))$

$x+1 > \Phi_U(x,x)$

$= 1(\mu w. (\underbrace{(x+1}_{A \text{ R.E } / A \text{ REC.}} = \Phi_U(x,x)))$

$\mu w. (x+1 = \boxed{\Phi_U(x,x)})$

$\nexists (x+1)$ FOR WHICH $H(x,x,y) \to$ NOT REC.

$K / \bar{K}$ $\chi_K = \begin{cases} 1 \\ \uparrow \end{cases}$ $\chi_H = \begin{cases} 1 & \text{if } H(x,x,y) \\ \uparrow \end{cases}$

$\neg H(x,x,y)$

$A = \{x \in \mathbb{N} : x \in W_x \wedge \varphi_x(x) > x\}$

$[\forall x, \ x+1 > x]$

$\downarrow$

RICE'S THEOREM $\nearrow$ $\searrow$ N

$A \neq \emptyset$, $A \neq \mathbb{N}$, A SATURATED $\to$ NOT REC.

NOT SATURATED

$[x \in W_x \wedge \underbrace{\varphi_x(x)}_{x+1} > x] \to K \leq_m A$

$$g(x,y) = \begin{cases} x+1 & x \in K \\ \uparrow & \end{cases}$$

$\varphi_{s(x)}(y) = g(x,y)$

if $x \in K$, $[y+1]$ $\forall y \in \mathbb{N}$

$s(x) \in W_{s(x)} = \mathbb{N}$

$s(x) \in A$

if $x \notin K$, $g(x,y)\uparrow$

$s(x) \notin W_{s(x)} = \emptyset$

A NOT REC $\to$ R.E. (STOPS IN A PARTICULAR CASE)

$\left[\bar{A} \to \{\underbrace{x \notin W_x}_{\text{NOT DECIDABLE}} \vee \underbrace{\varphi_x(x)}_{\text{DECIDABLE}} \leq x\}\right]$ $\uparrow$ NOT REC. / NOT R.E.

$A \to \{x \in W_x \wedge \varphi_x(x) > x\}$

OTHERWISE BOTH RECURSIVE

$$g(x,y) = \begin{cases} y & \text{if } H(x,x,y) = W_x = \mathcal{X}_H \\ \uparrow & \neg H(x,x,y) \to \emptyset = \mathcal{X}_{\bar{K}} = \bar{K} \end{cases}$$

X - INPUT

y - OUTPUT

$$[f(x) = x] \to \varphi_x(x)$$

$f(x) = x^2$

SECOND REC. (UP NEXT)

$x \to e$ (PROGRAM)

$$\varphi_{s(x)}(y) = g(x,y)$$

$e \neq e'$ $[\varphi_e = \varphi_{e'}]$

SATURATED

NOT SATURATED