<u>Theoretical definitions</u>

Exercise 1

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if A is not r.e. and $A \leqslant_m B$ then B is not r.e.

c. Is it true that for all sets $A, B \subseteq \mathbb{N}$ it holds that $A \leqslant_m A \cup B$? Prove it or provide a counterexample.

a. Given sets $A, B \subseteq N$, we say that A is reducible to B, denoted $A \leq_m B$, if there exists a total computable function $f : N \rightarrow N$ such that for all $x \in N$, $x \in A$ if and only if $f(x) \in B$. The function f is called the reduction function.

b. Suppose A is not r.e. and $A \leq_m B$. Let f be the computable reduction function such that $x \in A$ iff $f(x) \in B$ for all $x \in N$.

If B were r.e., its semi-characteristic function sc_B would be computable. But then we could define $sc_A(x) = sc_B(f(x))$, which would be computable by composition, implying A is r.e. This contradicts the assumption that A is not r.e.

Therefore, if A is not r.e. and $A \leq_m B$, then B cannot be r.e.

c. The statement does not hold in general. As a counterexample, consider $A = N\setminus\{0\}$ and $B = \{0\}$.

We have $A \leq_m B$ via the constant reduction function $f(x) = 0$ for all $x \in N$. However, the converse reduction does not exist because there is no computable function $g : N \rightarrow N$ such that $0 \in B$ iff $g(0) \in A$, since $g(0) \in N\setminus\{0\}$ for any total computable g.

The reducibility relation is transitive (if $A \leq_m B$ and $B \leq_m C$ then $A \leq_m C$) but not symmetric in general. A sufficient condition for symmetry is requiring the reduction function to be bijective. If f reduces A to B and f is a computable bijection, then $f^{(-1)}$ reduces B to A. However, bijective reductions are atypical.

In summary, the reducibility relation captures a notion of one set being "no harder" than another, but it does not imply equivalence unless the reduction is bijective. Sets need not be reducible to each other in both directions.

<u>Exercise</u> (2019-01-24)

Given two sets $A, B \subseteq \mathbb{N}$, define the reduction $A \leq_m B$ and show $A \leq_m B$ and A is not recursive, then B is not recursive. Can a set $A \subseteq \mathbb{N}$ be such that $A \leq_m \overline{A}$? Show an example or show the non-existence of such set.

Given sets A, B ⊆ N, let us define the reduction A ≤_m B as follows: A ≤_m B holds if and only if there exists a total computable function f : N → N such that for all x ∈ N, x ∈ A iff f(x) ∈ B.

Now, suppose A ≤_m B and A is not recursive. We want to show that B is also not recursive.

Assume, for the sake of contradiction, that B is recursive. Since A ≤_m B, there exists a total computable function f : N → N such that x ∈ A iff f(x) ∈ B for all x ∈ N.

Let χ_B be the characteristic function of B, which is computable since B is assumed to be recursive. Then we can define χ_A : N → N as follows:

χ_A(x) = χ_B(f(x))

Observe that χ_A is computable, since it is the composition of the computable functions χ_B and f. Moreover, for any x ∈ N:

- If x ∈ A, then f(x) ∈ B, so χ_B(f(x)) = 1, and thus χ_A(x) = 1.

- If x ∉ A, then f(x) ∉ B, so χ_B(f(x)) = 0, and thus χ_A(x) = 0.

Therefore, χ_A is the characteristic function of A, implying that A is recursive. However, this contradicts our assumption that A is not recursive.

Hence, our initial assumption that B is recursive must be false. We conclude that if A ≤_m B and A is not recursive, then B is also not recursive.

The converse statement does not hold in general. That is, given A ≤_m B, if B is not recursive, A may still be recursive. Here's a counterexample:

Let A = ∅ (the empty set) and B = $\overline{K}$ (the complement of the halting set). We know that $\overline{K}$ is not recursive (since K is not recursive).

Define the function f : N → N as f(x) = 0 for all x ∈ N.

Clearly, f is a total computable function. Moreover, for any x ∈ N:

x ∈ A iff x ∈ ∅ iff false iff 0 ∈ $\overline{K}$ iff f(x) ∈ B

Thus, A ≤_m B. However, A = ∅ is recursive (trivially), while B = $\overline{K}$ is not recursive.

This counterexample demonstrates that the converse of the original statement does not hold. The reduction A ≤_m B, even coupled with the non-recursiveness of B, does not imply the non-recursiveness of A.

c. Show that $P(\vec{x})$ is semi-decidable if and only if there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$.

3. Assume that $P(\vec{x})$ is semi-decidable. Then $\chi_P(\vec{x})$ is computable. Let $e \in \mathbb{N}$ be an index for such function, i.e., $\chi_P = \varphi_e^{(k)}$. Then we have that $P(\vec{x})$ holds iff $sc_P(\vec{x}) = 1$ iff $sc_P(\vec{x}) \downarrow$ iff $\exists y.\ H^{(k)}(e, \vec{x}, y)$. Hence, if we let $Q(\vec{x}, y) = H^{(k)}(e, \vec{x}, y)$, we have

$$P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$$

and $Q(\vec{x}, y)$ is decidable, since $H^{(k)}(e, \vec{x}, y)$ is so.

Assume now that $P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$ with $Q(\vec{x}, y)$ decidable. Let $\chi_Q : \mathbb{N}^{k+1} \to \mathbb{N}$ be its computable characteristic function.

Then $sc_A(\vec{x}) = \mu y. |\chi_Q(\vec{x}, y) - 1|$ is computable, i.e., $P(\vec{x})$ is semi-decidable.

$$1 \dot{-} sc_{\bar{A}}(x) = \begin{cases} 0 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

is computable. Let $e_1, e_0 \in \mathbb{N}$ be such that $\varphi_{e_1} = sc_A$ and $\varphi_{e_0} = 1 \dot{-} sc_{\bar{A}}$. Then $\chi_A(x) = (\mu w.S(e_0, x, (w)_1, (w)_2) \vee S(e_1, x, (w)_1, (w)_2))_1$. This is computable and thus A is recursive.

Non-computable functions

Exercise 6.32. Let A be a recursive set and let $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ be computable functions. Prove that the function $f : \mathbb{N} \to \mathbb{N}$ defined below is computable:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \notin A \end{cases}$$

Does the result hold if we weaken the hypotheses and assume A only r.e.? Explain how the proof can be adapted, if the answer is positive, or provide a counterexample, otherwise.

Solution: Let $e_1, e_2 \in \mathbb{N}$ be indexes for f_1, f_2, respectively, namely $\varphi_{e_1} = f_1$ and $\varphi_{e_2} = f_2$. Observe that we can define f as

$$f(x) = (\mu w.((S(e_1, x, (w)_1, (w)_2) \ \wedge \ \chi_A(x) = 1) \vee (S(e_2, x, (w)_1, (w)_2) \ \wedge \ \chi_A(x) = 0)))_1$$

showing that f is computable. Relaxing the hypotheses to recursive enumerability of A, the result is no longer true. Consider for instance $f_1(x) = 1$, $f_2(x) = 0$ and $A = K$, which is r.e. Then f defined as above would be the characteristic function of K which is not computable. □

Exercise 6.33(p). Is there a total, non-computable function such that $img(f) = \{f(x) \mid x \in \mathbb{N}\}$ is the set Pr of Prime numbers? Justify your answer.

Solution: Yes, it exists. For example, consider:

$$f(x) = \begin{cases} p & \text{if } x \in W_x ep = \min\{p' \in Pr \mid p' > \varphi_x(x)\} \\ 2 & \text{otherwise} \end{cases}$$

Then the function f

- is total;
- it is not computable, since for each $x \in \mathbb{N}$ one has that $f(x) \neq \varphi_x(x)$; in fact, if $\varphi_x(x) \downarrow$ we have that $f(x)$ is a prime larger than $\varphi_x(x)$, and if $\varphi_x(x) \uparrow$ then $f(x) = 2$;
- clearly $img(f) \subseteq Pr$. For the reverse inclusion, consider any prime number $p \in Pr$ and the constant function $g(x) = p - 1$ for each $x \in \mathbb{N}$. The function g is computable, thus $g = \varphi_n$ for a suitable index n. We conclude by noting that $f(n) = \min\{p' \in Pr \mid p' > \varphi_n(n)\} = \min\{p' \in Pr \mid p > p - 1\} = \min\{p' \in Pr \mid p' \geqslant p\} = p$ and thus $p \in img(f)$.

Exercise 6.17. Say if there are total computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(x) \neq \varphi_x(x)$ for each $x \in K$ and $g(x) \neq \varphi_x(x)$ for each $x \notin K$. Justify your answer by providing a example or by proving non-existence.

Solution: The function f does not exist. In fact, for every $x \in K$ we have $f(x) \neq \varphi_x(x)$. Moreover, for every x, if φ_x is total then $x \in K$. It follows that f is different from all total computable function. So, if it is total it is not computable.

The function g exists since we can just take $g(x) = 1$ for all $x \in \mathbb{N}$. In fact, if $x \in \bar{K}$, we have that $g(x) = 1 \neq \varphi_x(x) = \uparrow$. □

Recursiveness

Exercise 8.1. Study the recursiveness of the set $A = \{x \in \mathbb{N} : |W_x| \geqslant 2\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.2. Study the recursiveness of the set $A = \{x \in \mathbb{N} : x \in W_x \cap E_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.1 Solution:

The set $A = \{x \in N : |W_x| \geq 2\}$ is not recursive. To prove this, first observe that A is saturated since $A = \{x \mid \phi_x \in A\}$ where $A = \{f \mid |dom(f)| \geq 2\}$.

Using Rice-Shapiro's theorem, we can deduce that both A and $\bar{A}$ are not r.e.:

1. A is not r.e. No finite function can belong to A, since finite functions have finite domains (and thus domains of size < 2). However, A is non-empty, e.g. the identity function is in A. Therefore, by Rice-Shapiro's theorem, A is not r.e.

2. $\bar{A}$ is also not r.e. The always undefined function ∅ is in $\bar{A}$, but it admits the identity function as an extension which is not in $\bar{A}$. Thus, by Rice-Shapiro's theorem, $\bar{A}$ is not r.e.

Since both A and its complement are not r.e., we conclude that A is not recursive.

Exercise 8.2 Solution:

The set $A = \{x \in N : x \in W_x \cap E_x\}$ is not recursive, but it is r.e.

To show A is not recursive, we prove that $K \leq_m A$. Consider the computable function:

$g(x,y) = 1$ if $x \in K$, and undefined otherwise

By the s-m-n theorem, there exists a total computable function s such that $\phi_{s(x)}(y) = g(x,y)$ for all $x,y \in N$.

The function s is a reduction of K to A:

- If $x \in K$, then $\phi_{s(x)}(y) = g(x,y) = 1$ for all y. So $s(x) \in W_{s(x)} = E_{s(x)} = N$, and thus $s(x) \in A$.

- If $x \notin K$, then $\phi_{s(x)}(y) = g(x,y)$ is undefined for all y. So $W_{s(x)} = E_{s(x)} = \emptyset$, hence $s(x) \notin A$.

Therefore $K \leq_m A$, implying A is not recursive.

However, A is r.e. since its semi-characteristic function

$sc_A(x) = \mu w.H(x,x,(w)_1) \vee S(x,(w)_2,x,(w)_1)$

is computable. The μ-minimalization succeeds if either the machine x halts on input x, or if machine x produces output x for some input, both of which capture the definition of A.

In conclusion, the set A is r.e. but not recursive, so its complement $\bar{A}$ is not r.e.

Exercise 8.5. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists y, z \in \mathbb{N}.\ z > 1 \ \wedge \ x = y^z\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.6. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \phi_x(y) = y \text{for infinities} y\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

<u>Exercise 8.5 Solution:</u>

Let $A = \{x \in N : \exists y, z \in N.\ z > 1 \wedge x = y^z\}$. We will show that A is not recursive by proving that $K \leq_m A$.

Consider the computable function $g(x,y) = 1$ if $x \in K$, undefined otherwise. By the s-m-n theorem, there exists a total computable function $s : N \to N$ such that $\phi_{s(x)}(y) = g(x,y)$ for all $x,y \in N$.

We argue that s is a reduction function for $K \leq_m A$:

- If $x \in K$, then $\phi_{s(x)}(y) = g(x,y) = 1$ for all $y \in N$. So $s(x) = s(x)^1$, with $s(x), 1 \in N$ and $1 > 1$. Thus $s(x) \in A$.

- If $x \notin K$, then $\phi_{s(x)}(y) = g(x,y)$ is undefined for all $y \in N$. So there cannot exist $y, z \in N$ with $z > 1$ such that $s(x) = y^z$. Thus $s(x) \notin A$.

Therefore, $K \leq_m A$, implying A is not recursive.

The set A is r.e., since its semi-characteristic function

$sc_A(x) = \mu w.(S(x,(w)_1,(w)_3) \wedge (w)_2 > 1 \wedge x = (w)_1^{(w)_3})$

is computable. Since A is r.e. but not recursive, its complement $\bar{A}$ is not r.e. (otherwise both would be recursive). Thus $\bar{A}$ is also not recursive.

<u>Exercise 8.6 Solution:</u>

Let $A = \{x \in N : \phi_x(y) = y \text{ for infinitely many } y\}$. The set A is saturated, since $A = \{x : \phi_x \in A\}$, where $A = \{f : f(y) = y \text{ for infinitely many } y\}$.

Applying Rice-Shapiro's theorem, we deduce that both A and $\bar{A}$ are not r.e.:

1. A is not r.e. The identity function $id \in A$, but no finite subfunction $\theta \subseteq id$ can belong to A, since such θ would only satisfy $\theta(y) = y$ for finitely many y. Thus by Rice-Shapiro's theorem, A is not r.e.

2. $\bar{A}$ is not r.e. The always undefined function $\emptyset \in \bar{A}$, but $\emptyset$ admits the identity function id as an extension, and $id \notin \bar{A}$. Thus by Rice-Shapiro's theorem, $\bar{A}$ is not r.e.

Since both A and $\bar{A}$ are not r.e., we conclude that A is not recursive. In summary, A is neither recursive nor recursively enumerable. The same holds for its complement $\bar{A}$.

Exercise 8.12. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *almost total* if it is undefined on a finite set of points. Study the recursiveness of the set $A = \{x \mid \varphi_x \text{ almost total}\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

We start from understanding if the set is saturated or not and use Rice-Shapiro to guess the set is not r.e. We also need to see, for Rice's theorem, it is not empty or not the naturals. $A = \{f \in C \mid f \text{ almost total}\}$ using Rice-Shapiro and guess it is not r.e.

We show there is, using the identity function, which is total and r.e.

As solved by an old tutor mentioned in beginning of this chapter:

$A = \{ x \in \mathbb{N} \mid \varphi_x \in \mathcal{A} \}$

$\mathcal{A} = \{ f \in \mathcal{C} \mid f \text{ almost total} \}$

$\exists f \quad f \in \mathcal{B}$ and $\forall \theta \subseteq f \quad \theta$ FINITE $\theta \notin \mathcal{B} \;] \Rightarrow \mathcal{B}$ is NOT R.E.

$id \in \mathcal{A}$

$\forall \theta \subseteq id \quad \theta$ FINITE $\Rightarrow \theta \notin \mathcal{A}$

$\exists f \quad f \notin \mathcal{B}$ and $\exists \theta \subseteq f \quad \theta$ FINITE $\theta \in \mathcal{B} \Rightarrow \mathcal{B}$

$id \in \mathcal{A} \Rightarrow id \notin \bar{\mathcal{A}}$

$\emptyset \in \bar{\mathcal{A}} \qquad \emptyset(x)\uparrow \; \forall x \in \mathbb{N}$

$\bar{A}$ is NOT R.E.

Smn-theorem

Esercizio 1

Enunciare il teorema s-m-m. Utilizzarlo per dimostrare che esiste $k : \mathbb{N} \to \mathbb{N}$ calcolabile totale tale che, per ogni $n \in \mathbb{N}$, si ha che $W_{k(n)} = \{z^n \mid z \in \mathbb{N}\}$ e $E_{k(n)}$ è l'insieme dei numeri dispari.

Soluzione: Si inizia definendo una funzione calcolabile di due argomenti $f(n,x)$ che rispetti le condizioni quando vista come funzione di x, con n considerato come parametro, ad es.

$$f(n,x) = \begin{cases} 2z+1 & \text{se } x = z^n \text{ per qualche } z \\ \uparrow & \text{altrimenti} \end{cases} = 2 * (\mu z.|x \dot{-} z^n|) + 1$$

Per il teorema smn esiste una funzione totale calcolabile $k : \mathbb{N} \to \mathbb{N}$ tale che $\varphi_{k(n)}(x) = f(n,x)$ per ogni $n, x \in \mathbb{N}$. Pertanto, come desiderato

- $W_{k(n)} = \{x \mid f(n,x) \downarrow\} = \{x \mid \exists z \in \mathbb{N}.\, x = z^n\} = \{z^n \mid z \in \mathbb{N}\}$;
- $E_{k(n)} = \{f(n,x) \mid x \in W_{k(n)}\} = \{2\sqrt[n]{z^n} + 1 \mid z \in \mathbb{N}\} = \{2z+1 \mid z \in \mathbb{N}\}$.

$\square$

Exercise (2020-11-23)

State the smn-theorem. Use it for proving there exists $k: \mathbb{N} \to \mathbb{N}$ total and computable s.t. $\forall n \in \mathbb{N}$ we have $|W_x| = 2^x$ and $|E_x| = x + 1$.

Solution

The smn-theorem states that, given $m, n \geq 1$ there is a computable total function $s_{m,n}: \mathbb{N}^{m+1} \to \mathbb{N}$ $s.t. \forall e \in \mathbb{N}, \vec{x} \in \mathbb{N}^m, \vec{y} \in \mathbb{N}^n$

$$\phi_e^{m+n}(\vec{x}, \vec{y}) = \phi_{s_{m,n}(e,\vec{x})}^{(n)}(\vec{y})$$

Define:

$$g(x,y) = \begin{cases} \lfloor \log_2(y+1) \rfloor & \text{se } y < 2^x \\ \uparrow & \text{altrimenti} \end{cases}$$

which is computable.

Infact, $g(x,y)$ when defined, is the greatest z s.t. $2^x \leq y+1$ and the minimum s.t. $2^{z+1} > y+1$, so:

$$g(x,y) = \mu z.\, \overline{sg}(2^{z+1} \dot{-} (y+1)) + \mu w.(y + 1 \dot{-} 2^x)$$

Exercise 3.3. State the smn theorem and use it to prove that there exists a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that $W_{s(x,y)} = \{z : x * z = y\}$

Given $m, n \geq 1$ there is a total computable function $s_{m,n}: \mathbb{N}^{m+1} \to \mathbb{N}$ such that $\forall \vec{x} \in \mathbb{N}^m, \forall \vec{y} \in \mathbb{N}^n, \forall e \in \mathbb{N}$

$$\phi_e^{(m+n)}(\vec{x}, \vec{y}) = \phi_{s_{m,n}(e,\vec{x})}^{(n)}(\overrightarrow{y})$$

We start by defining a computable function of two arguments $f(n,x)$ which meets the conditions when viewed as a function of x, with n taken as a parameter, e.g.

$$f(n,x) = \begin{cases} \frac{x}{z}, & if y \; multiple \; of \; x \\ \uparrow, & otherwise \end{cases} = qt(x,z) + \mu z.rm(x,z)$$

By the smn-theorem, there is a computable total function $k: \mathbb{N} \to \mathbb{N}$ such that $\phi_{K(n)}(x) = f(n,x)\ \forall n, x \in \mathbb{N}$. Therefore:

- $W_{s(k,y)} = \{x | f(n,x) \downarrow\} = \left\{y : \frac{x}{z}\right\} = \{z : x * z = y\}$

as desired.

Second recursion theorem

Esercizio 5
Enunciare il Secondo Teorema di Ricorsione ed utilizzarlo per dimostrare che esiste $x \in \mathbb{N}$ tale che $|W_x| = x$.

The Second Recursion Theorem states that given any total computable function $h : N \to N$, there exists an $e \in N$ such that $\phi\_e = \phi\_(h(e))$.

To prove the desired result, consider the function $g : N^2 \to N$ defined as follows:

$g(x,y) = x$

This function is clearly computable. Therefore, by the s-m-n theorem, there exists a total computable function $s : N \to N$ such that for all $x,y \in N$:

$\phi\_(s(x))(y) = g(x,y) = x$

Now, by the Second Recursion Theorem applied to the function s, there exists an index $e \in N$ such that:

$\phi\_e = \phi\_(s(e))$

This means that for all $y \in N$:

$\phi\_e(y) = \phi\_(s(e))(y) = g(e,y) = e$

Hence, $W\_e = \{y \mid \phi\_e(y) \downarrow\} = \{y \mid \phi\_(s(e))(y) \downarrow\} = \{y \mid e \downarrow\} = N$

Therefore, $|W\_e| = |N| = e$, as desired.

In summary, by leveraging the power of the Second Recursion Theorem in conjunction with the s-m-n theorem, we have demonstrated the existence of a program index e such that the domain of the e-th computable function ϕ_e has cardinality exactly equal to e itself. This construction relies on the ability to define a suitable computable function that, when combined with the recursion theorem, yields the desired property.

Esercizio 5
Enunciare il Secondo Teorema di Ricorsione ed utilizzarlo per dimostrare che esiste $x \in \mathbb{N}$ tale che $\varphi_x(y) = y^x$, per ogni $y \in \mathbb{N}$.

The Second Recursion Theorem states that for any total computable function $h : N \to N$, there exists an $e \in N$ such that $\phi\_e = \phi\_(h(e))$, where ϕ_e denotes the e-th computable function.

We will leverage this theorem to prove the existence of an $x \in N$ satisfying $\phi\_x(y) = x$^y for all $y \in N$.

Define the function g : N^2 → N as follows:

g(x,y) = x^y

Clearly, g is a computable function. By the s-m-n theorem, there exists a total computable function s : N → N such that for all x,y ∈ N:

ϕ_(s(x))(y) = g(x,y) = x^y

Now, apply the Second Recursion Theorem to the function s. This guarantees the existence of an index e ∈ N satisfying:

ϕ_e = ϕ_(s(e))

Consequently, for all y ∈ N:

ϕ_e(y) = ϕ_(s(e))(y) = g(e,y) = e^y

Thus, by setting x = e, we have successfully constructed an x ∈ N with the property that ϕ_x(y) = x^y for all y ∈ N, as desired.

Exercise 9.21. State the second recursion theorem. Use it for proving that the set $C = \{x \in \mathbb{N} : \varphi_x(x) = x^2\}$ is not saturated.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$, there exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

Concerning the question, as in the case of the proof for K we can find an index e such that $\varphi_e = \{(e, e^2)\}$. Then we have $e \in C$, but any other index for the same function is not in C. $\square$

Exercise 9.22. State the second recursion theorem and use it for proving that there is an index k such that $W_k = \{k * i \mid i \in \mathbb{N}\}$.

Solution: Consider the following function

$$g(x, y) = \begin{cases} 0 & \text{if there exists } i \text{ such that } y = x * i \\ \uparrow & \text{otherwise} \end{cases} \qquad = \mu i.|x \cdot i - y|$$

Primitive recursion

Exercise 2.2(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove that the the characteristic function χ_A of the set $A = \{2^n - 1 : n \in \mathbb{N}\}$ is primitive recursive. You can assume, without proving it, that sum, product, sg and $\overline{sg}$ are in $\mathcal{PR}$.

Solution: Observe that $A = \{a(n) : n \in \mathbb{N}\}$ where $a : \mathbb{N} \to \mathbb{N} \in \mathcal{PR}$ is the function defined by

$$\begin{cases} a(0) & = & 0 \\ a(n+1) & = & 2 \cdot a(n) + 1 \end{cases}$$

Now define $chk : \mathbb{N}^2 \to \mathbb{N}$, in a way that $chk(x, m) = 1$ if there exists $n \leqslant m$ such that $x = a(n)$ and 0 otherwise. It can be defined by primitive recursion as follows:

$$\begin{cases} chk(x, 0) & = & \overline{sg}(x) \\ chk(x, m+1) & = & chk(x, m) + eq(x, a(m+1)) \end{cases}$$

Hence we can deduce that $chk \in \mathcal{PR}$ by the fact that $y \dot{-} 1$ and $x \dot{-} y$ are in $\mathcal{PR}$, and observing that $eq(x, y) = \overline{sg}(x \dot{-} y + y \dot{-} x)$, hence also such function is in $\mathcal{PR}$. We conclude by noting that $\chi_A(x) = chk(x, x)$. $\square$