Computability

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This is a collection of exam exercises, roughly organised by thematic areas. The exercises often come along with a solution, which is sometimes fully detailed and in some other cases only sketched.

The exercises that can be used for the preparation of the intermediate test are marked by a "(p)".

Please report any mistake you might find.

1 URM machine

Exercise 1.1(p). Consider a variant, denoted URM$^-$, of the URM machine obtained replacing the successor instruction $S(n)$ with a predecessor instruction $P(n)$. Executing $P(n)$ replaces the content r_n of register n with $r_n \dot{-} 1$. Determine the relation between the set $\mathcal{C}^-$ of the functions computable by a URM$^-$ machine and the set $\mathcal{C}$ of functions computable by a standard URM machine. Is one contained in the other? Is the inclusion strict? Justify your answer.

Solution: It holds that $\mathcal{C}^- \subseteq \mathcal{C}$ because predecessor is URM-computable. Inclusion is strict because it is possible to prove, inductively on the number of steps, that the maximum of the values contained in the registers at any time is bounded by the maximum value in the initial configuration. As a consequence the successor function is not URM$^-$ computable. □

Exercise 1.2(p). Consider a variant of the URM machine where the jump and successor instructions are replaced by the instruction $JI(m, n, t)$ which compare the content r_m and r_n of of registers R_m and R_n and then:

- if $r_m = r_n$, increment register R_m and jump to the address t (it is intended that if t is outside the program, the execution of the program halts).
- otherwise, continue with the next instruction.

Describe the relation between the set $\mathcal{C}'$ of the functions computable by the new machine and the set $\mathcal{C}$ of the functions that can be computed by a standard URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Solution: Observe that the instructions of each machine can be encoded in the other. Then show, by induction on the length of a program that contains both sets of instructions, that it can be transformed into an equivalent program that contains instructions only of one of the two machines.

In particular, a URM instruction $I_j : S(n)$ can be replaced by

$$I_j : JI(n, n, j + 1)$$

Moreover, if k is any register not used by the program, the instruction $J(m, n, t)$ can be replaced by

$T(m, k)$
$JI(k, n, t)$

Conversely, the new instruction $JI(m, n, t)$ can encoded as a jump $J(m, n, t_1)$ to a subroutine at t_1

$$\begin{array}{ll} t_1 & : S(m) \\ t_1 + 1 & : J(m, m, t) \end{array}$$

□

Exercise 1.3(p). Consider a variant URM^s of URM machine obtained by removing the successor $S(n)$ and jump $J(m, n, t)$ instructions, and inserting the instruction $JS(m, n, t)$, which compares the contents of register m and n, and if they coincide, it jumps to instruction t, otherwise it increments the m-th register and executes the next instruction. Determine the relation between the set $\mathcal{C}^s$ of functions computable by a URM^s machine and the set $\mathcal{C}$ of functions computable by a standard URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Solution: Clearly the instruction $JS(m, n, t)$ can be simulated in the URM machine as

$J(m, n, t)$
$S(m)$

Conversely, the instruction $S(n)$ cannot be simulated. In fact, starting from the configuration in which all registers have value 0, there is no a way of modifying the content of any register: this would require the presence of two registers with different content and there are none. □

Exercise 1.4(p). Consider the subclass of URM programs where, if the i-th instruction is a jump instruction $J(m, n, t)$, then $t > i$. Prove that the functions computable by programs in such subclass are all total.

Solution: Given a program P prove, by induction on t, that the instruction to execute at the $t + 1$-th step has an index greater than t. This implies that the program will end in at most $l(P)$ steps. □

Exercise 1.5. Consider a variant of the URM machine, which includes the jump and transfer instructions and two new instructions

- *A(m, n)* which adds to register m the content of register n, i.e., $r_m \leftarrow r_m + r_n$;
- *C(n)* which replaces the value in register n by its sign, i.e., $r_n \leftarrow sg(r_n)$.

Determine the relation between the set $\mathcal{C}'$ of the functions computable with the new machine and the set $\mathcal{C}$ of the functions that can be computed with the URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Solution: Let us denote by URM* the modified machine. We observe that the URM* machine instructions can be encoded as programs of standard URM machine.

The instruction $I_j : A(m, n)$ can be replaced with a jump to the following routine (where we denote by q the index of the first register not used by the program, hence such register initially contains 0)

$$\begin{array}{lll} SUB & : & J(n, q, j + 1) \\ & & S(m) \\ & & S(q) \\ & & J(1, 1, SUB) \end{array}$$

Similarly, by indicating again with q the index of an unused register, an instruction $I_j : C(m)$ can be replaced by a jump to the subroutine

$$
\begin{array}{lll}
SUB & : & J(n, q, ZERO) \\
 & & Z(n) \\
 & & S(n) \\
ZERO & & J(1, 1, j+1)
\end{array}
$$

More formally, we can prove that $\mathcal{C}^* \subseteq \mathcal{C}$ showing that, for each number of arguments k and for each program P using both sets of instructions we can obtain a URM program P' which computes the same function, i.e., such that $f_{P'}^{(k)} = f_P^{(k)}$.

The proof proceeds by induction on the number h of A and C instructions in the program. The base case $h = 0$ is trivial, since a program P with 0 instructions A and C is already a URM program. Suppose that the result holds for h, let us prove it for $h + 1$. The program P certainly contains at least one A or C instruction. Assume it is a C instruction and call j its index.

$$
\begin{array}{lll}
1 & : & I_1 \\
 & & \dots \\
j & & A(m, n) \\
 & & \dots \\
\ell(P) & : & I_{\ell(P)}
\end{array}
$$

We build a program P'', using a register not referenced in P, say $q = \max\{\rho(P), k\} + 1$

$$
\begin{array}{lll}
1 & : & I_1 \\
 & & \dots \\
j & & J(1, 1, SUB) \\
 & & \dots \\
\ell(P) & : & I_{\ell(P)} \\
 & & J(1, 1, END) \\
SUB & : & J(n, q, ZERO) \\
 & & Z(n) \\
 & : & S(n) \\
ZERO & & J(1, 1, j+1) \\
END & : &
\end{array}
$$

The program P'' is such that $f_{P''}^{(k)} = f_P^{(k)}$ and it contains h instructions of type A or C. By inductive hypothesis, there exists a URM program P' such that $f_{P'}^{(k)} = f_{P''}^{(k)}$, which is the desired program.

If the instruction I_j is of type A, we proceed in a completely analogous way, replacing the instruction with its encoding and using the inductive hypothesis.

The inclusion is strict, i.e., $\mathcal{C} \nsubseteq \mathcal{C}^*$. For example, one can easily see that the successor function is not URM* computable. In fact, it can be shown that, starting from a configuration with all registers at 0, any program URM*, after any number of steps, will produce a configuration with all registers still at 0. A fully formal proof proves the above by induction on the number of steps. □

Exercise 1.6(p). Consider a variant URM^m of the URM machine obtained by removing the successor instruction $S(n)$ and adding the instruction $M(n)$, which stores in the nth register the value $1 + \min\{r_i \mid i \leqslant n\}$, i.e., the successor of the least value contained in registers with index less than or equal to n. Determine the relation between the set $\mathcal{C}^m$ of functions computable by the URM^m machine and the set $\mathcal{C}$ of the functions computable by the ordinary URM machine. Is one included in the other? Is the inclusion strict? Justify your answers.

Solution: Observe that the instruction $M(n)$ can be simulated in the URM machine as follows: store in an "unused" register k, an increasing number, which starts from zero. Such a number is compared with all registers $R_1, \ldots, R_n$ until it coincides with one of them. Then the value in register k will be the minimum of registers $R_1, \ldots, R_m$. Its successor is the value to be stored in R_n

$$
\begin{array}{ll}
 & Z(k) \\
LOOP: & J(1,k,END) \\
 & J(2,k,END) \\
 & \ldots \\
 & J(n,k,END) \\
 & S(k) \\
 & J(1,1,LOOP) \\
END: & S(k) \\
 & T(k,n)
\end{array}
$$

Conversely, the instruction $S(n)$ can be simulated in the URM^m machine as follows. Assume again that k is the number of a register not used by the program. Then the encoding can be the following:

$$
\begin{array}{l}
T(1,k) \\
T(n,1) \\
M(1) \\
T(1,n) \\
T(k,1)
\end{array}
$$

□

Exercise 1.7(p). Define the operation of primitive recursion and prove that the set $\mathcal{C}$ of URM-computable functions is closed with respect to this operation.

2 Primitive Recursive Functions

Exercise 2.1(p). Give the definition of the set $\mathcal{PR}$ of recursive primitive functions and, using only the definition, prove that the function $pow2 : \mathbb{N} \to \mathbb{N}$, defined by $pow2(y) = 2^y$, is primitive recursive.

Solution: We define $pow2 : \mathbb{N} \to \mathbb{N}$:

$$
\begin{cases}
pow2(0) = 1 \\
pow2(y+1) = double(pow2(y))
\end{cases}
$$

where $double(x)$ can be defined by primitive recursion as

$$
\begin{cases}
double(0) = 0 \\
double(y+1) = double(y) + 2 = (double(y)+1)+1
\end{cases}
$$

□

Exercise 2.2(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove that the the characteristic function χ_A of the set $A = \{2^n - 1 : n \in \mathbb{N}\}$ is primitive recursive. You can assume, without proving it, that sum, product, sg and $\overline{sg}$ are in $\mathcal{PR}$.

Solution: Observe that $A = \{a(n) : n \in \mathbb{N}\}$ where $a : \mathbb{N} \to \mathbb{N} \in \mathcal{PR}$ is the function defined by

$$
\begin{cases}
a(0) & = & 0 \\
a(n+1) & = & 2 \cdot a(n) + 1
\end{cases}
$$

Now define $chk : \mathbb{N}^2 \to \mathbb{N}$, in a way that $chk(x, m) = 1$ if there exists $n \leqslant m$ such that $x = a(n)$ and 0 otherwise. It can be defined by primitive recursion as follows:

$$\begin{cases} chk(x, 0) & = & \overline{sg}(x) \\ chk(x, m+1) & = & chk(x, m) + eq(x, a(m+1)) \end{cases}$$

Hence we can deduce that $chk \in \mathcal{PR}$ by the fact that $y \dot{-} 1$ and $x \dot{-} y$ are in $\mathcal{PR}$, and observing that $eq(x, y) = \overline{sg}(x \dot{-} y + y \dot{-} x)$, hence also such function is in $\mathcal{PR}$. We conclude by noting that $\chi_A(x) = chk(x, x)$. □

Exercise 2.3(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove that the $\chi_{\mathbb{P}}$, the characteristic function of the set of even numbers $\mathbb{P}$ is primitive recursive.

Solution: The function $\chi_{\mathbb{P}}$ can be defined as follows:

$$\begin{array}{l} \chi_{\mathbb{P}}(0) = 1 \\ \chi_{\mathbb{P}}(y+1) = \overline{sg}(\chi_{\mathbb{P}}(y)) \end{array}$$

where $\overline{sg}$ can also be defined by primitive recursion:

$$\begin{array}{l} \overline{sg}(0) = 1 \\ \overline{sg}(y+1) = 0 \end{array}$$

□

Exercise 2.4(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove the function $half : \mathbb{N} \to \mathbb{N}$, defined by $half(x) = x/2$, is primitive recursive.

Solution: The set $\mathcal{PR}$ of primitive recursive functions is the smallest set of functions that contains the basic functions:

1. $\mathbf{0} : \mathbb{N} \to \mathbb{N}$defined by $\mathbf{0}(x) = 0$ for each $x \in \mathbb{N}$;
2. $\mathbf{s} : \mathbb{N} \to \mathbb{N}$defined by $\mathbf{s}(x) = x + 1$ for each $x \in \mathbb{N}$;
3. $\mathbf{U}_j^k : \mathbb{N}^k \to \mathbb{N}$defined by $\mathbf{U}_j^k(x_1, \ldots, x_k) = x_j$ for each $(x_1, \ldots, x_k) \in \mathbb{N}^k$.

and which is closed with respect to generalized composition and primitive recursion, defined as follows. Given the functions $f_1, \ldots, f_n : \mathbb{N}^k \to \mathbb{N}$and $g : \mathbb{N}^n \to \mathbb{N}$ their generalized composition is the function $h : \mathbb{N}^k \to \mathbb{N}$ defined by:

$$h(\vec{x}) = g(f_1(\vec{x}), \ldots, f_n(\vec{x})).$$

Given the functions $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^{k+2} \to \mathbb{N}$ the function defined by primitive recursion is $h : \mathbb{N}^{k+1} \to \mathbb{N}$:

$$\begin{cases} h(\vec{x}, 0) = f(\vec{x}) \\ h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y)) \end{cases}$$

We need to prove that the function $half$ can be obtained from the basic functions (1), (2) and (3), using primitive recursion and generalized composition. One can proceed as follows.

First we define the function $\overline{sg} : \mathbb{N} \to \mathbb{N}$ such that $\overline{sg}(x) = 1$ if $x = 0$ and $\overline{sg}(x) = 0$ otherwise:

$$\begin{cases} \overline{sg}(0) & = & 1 \\ \overline{sg}(x+1) & = & 0 \end{cases}$$

Then the function $rm_2 : \mathbb{N} \to \mathbb{N}$ which returns the remainder of the division of x by 2:

$$\begin{cases} rm_2(0) & = & 0 \\ rm_2(x+1) & = & \overline{sg}(rm_2(x)) \end{cases}$$

Finally the function $half : \mathbb{N} \to \mathbb{N}$ can be defined as:

$$\begin{cases} half(0) & = & 0 \\ half(x+1) & = & half(x) + rm_2(x) \end{cases}$$

□

Exercise 2.5(p). Give the definition of the set $\mathcal{PR}$ of primitive recursive functions and, using only the definition, prove that $p_2 : \mathbb{N} \to \mathbb{N}$ defined by $p_2(y) = |y - 2|$ is primitive recursive.

Solution: For the definition of $\mathcal{PR}$ see the book. For the second part, we observe that if we define $p_1(y) = |y - 1|$ then

$$\begin{cases} p_1(0) = 1 \\ p_1(y+1) = |y+1-1| = |y| = y \end{cases}$$

and therefore

$$\begin{cases} p_2(0) = 2 \\ p_2(y+1) = |y+1-2| = |y-1| = p_1(y) \end{cases}$$

Hence p_2 can be defined by primitive recursion starting from basic functions and thus it is in $\mathcal{PR}$.
□

3 SMN Theorem

Exercise 3.1(p). State the smn theorem and prove it (it is sufficient to provide the informal argument using encode/decode functions).

Exercise 3.2(p). State the theorem s-m-n and use it to prove that it exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $|W_{s(x)}| = 2x$ and $|E_{s(x)}| = x$.

Solution: We can define, for instance,

$$f(x, y) = \begin{cases} qt(2, y) & \text{if } y < 2x \\ \uparrow & \text{otherwise} \end{cases}$$

Observe that $f(x, y) = qt(2, y) + \mu z. (y + 1 \dot{-} 2x)$ is computable and finally use the smn theorem to get function $s(x)$. □

Exercise 3.3. State the smn theorem and use it to prove that there exists a total computable function $s : \mathbb{N}^2 \to \mathbb{N}$ such that $W_{s(x,y)} = \{z : x * z = y\}$

Exercise 3.4(p). Prove that there is a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$ it holds that $W_{k(n)} = \mathbb{P} = \{x \in \mathbb{N} \mid x \text{ even}\}$ and $E_{k(n)} = \{x \in \mathbb{N} \mid x \geqslant n\}$.

Solution: We start by defining a computable function of two arguments $f(n, x)$which meets the conditions when viewed as a function of x, with n taken as a parameter, e.g.

$$f(n,x) = \begin{cases} x/2 + n & \text{if } x \text{ even} \\ \uparrow & \text{otherwise} \end{cases} = qt(2,x) + n + \mu z.rm(2,x)$$

By the smn theorem, there is a computable total function $k : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{k(n)}(x) = f(n,x)$ for each $n, x \in \mathbb{N}$. Therefore:

- $W_{k(n)} = \{x \mid f(n,x) \downarrow\} = \{x \mid x \text{ even}\}$
- $E_{k(n)} = \{f(n,x) \mid x \in \mathbb{N}\} = \{n + x/2 \mid x \text{ even}\} = \{n + x \mid x \geqslant 0\} = \{y \mid y \geqslant \mathbb{N}\}$

as desired. □

Exercise 3.5. State the smn theorem. Use it to prove it exists a total computable function $k : \mathbb{N} \to \mathbb{N}$ such that $W_{k(n)} = \{x \in \mathbb{N} \mid x \geqslant n\}$ e $E_{k(n)} = \{y \in \mathbb{N} \mid y \text{ even}\}$ for all $n \in \mathbb{N}$.

Solution: We start by defining a computable function of two arguments $f(n,x)$ which enjoys the property when viewed as a function of x, with n seen as a parameter, e.g.

$$f(n,x) = \begin{cases} 2 * (x \dot{-} n) & \text{if } x \geqslant n \\ \uparrow & \text{otherwise} \end{cases} = 2 * (x \dot{-} n) + \mu z.(n \dot{-} x)$$

By the smn theorem, there is a computable total function $k : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{k(n)}(x) = f(n,x)$ for each $n, x \in \mathbb{N}$. Therefore, as desired

- $W_{k(n)} = \{x \mid f(n,x) \downarrow\} = \{x \mid x \geqslant n\}$;
- $E_{k(n)} = \{f(n,x) \mid x \in \mathbb{N}\} = \{2(x \dot{-} n) \mid x \geqslant n\} = \{2(n + z \dot{-} n) \mid z \geqslant 0\} = \{2z \mid z \in \mathbb{N}\}$.

□

4 Decidability and Semidecidability

Exercise 4.1. Prove the "structure theorem" of semidecidable predicates, i.e., show that a predicate $P(\vec{x})$ is semidecidable if and only if there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y.\ Q(\vec{x}, y)$.

Exercise 4.2. Prove the "projection theorem", i.e., show that if the predicate $P(x, \vec{y})$ is semidecidable then also $\exists x.\, P(x, \vec{y})$ is semi-decidable. Does the converse implication hold? Is it the case that if $P(x, \vec{y})$ is decidable then also $\exists x.\, P(x, \vec{y})$ is decidable? Give a proof or a counterexample.

Solution: No, the converse is false. Consider, for instance, $P(x,y) = (y = 2x) \wedge (y \notin W_x)$ (or, simply, $P(x,y) = x \notin W_x$), which is not semi-decidable. The existentially quantified version is constant, hence decidable.

Also the second claim is false. Take for instance $P(x,y) = H(y,y,x)$ which is decidable, while $\exists x.\, P(x,y) \equiv y \in K$ is only semi-decidable, but not decidable. □

5 Numerability and diagonalization

Exercise 5.1(p). Consider the set F_0 of functions $f : \mathbb{N} \to \mathbb{N}$, possibly partial, such that $cod(f) \subseteq \{0\}$. Is the set F_0 countable? Justify your answer.

Solution: No, such functions are completely determined by the their domain, which is a generic subset of $\mathbb{N}$, and the set of subsets of $\mathbb{N}$ is uncountable. $\square$

Exercise 5.2(p). A function $f : \mathbb{N} \to \mathbb{N}$ is called *total increasing* when it is total and for each $x, y \in \mathbb{N}$, if $x < y$ then $f(x) < f(y)$. Prove that the set of total increasing functions is not countable.

Solution: Given any enumeration of the total increasing functions $\{f_n\}_{n\in\mathbb{N}}$ you can define a function $f : \mathbb{N} \to \mathbb{N}$ as follows

$$f(x) = 1 + \Sigma_{n=0}^{x} f_n(n),$$

Such function is an total increasing and different from all f_n. In fact

- f is clearly total by definition.
- f is increasing, since $f(x+1) = f(x) + f_{x+1}(x+1) > f(x)$. The last inequality is motivated by the fact that f_{x+1} is increasing, and thus $f_{x+1}(x+1) > f_{x+1}(x) \geqslant 0$.
- f is differs from all f_x since for each $x \in \mathbb{N}$,

$$f(x) = 1 + \Sigma_{n=0}^{x} f_n(n) \geqslant 1 + f_x(x) > f_x(x).$$

It follows that no enumeration can contain all total increasing functions.

The same argument would work if we defined $f(x) = 1 + \max\{f_n(n) \mid 0 \leqslant n \leqslant x\}$. $\square$

Exercise 5.3(p). A function $f : \mathbb{N} \to \mathbb{N}$ is called *total increasing* when it is total and for each $x, y \in \mathbb{N}$, if $x \leqslant y$ then $f(x) \leqslant f(y)$. It is called binary if $cod(f) \subseteq \{0, 1\}$. Is the set of binary total increasing functions countable? Justify your answer.

Solution: Let f be an total increasing binary function, different from the constant 0, and define $s(f) = \min\{x \mid f(x) = 1\} \in \mathbb{N}$. It is easy see that $f_1 = f_2$ iff $s(f_1) = s(f_2)$. Hence, indicated by

$$f_i(x) = \begin{cases} 0 & x < i \\ 1 & \text{otherwise} \end{cases}$$

we have that $(f_i)_{i\in\mathbb{N}}$ is an enumeration of the total binary increasing functions, different from the constant 0, which therefore they are countable. When adding the constant 0 the set clearly stays countable. $\square$

6 Functions and Computability

Exercise 6.1(p). Define a function $f : \mathbb{N} \to \mathbb{N}$ total and not computable such that $f(x) = x$ for infinite arguments $x \in \mathbb{N}$ or prove that such a function cannot exist.

Solution: We can define

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in W_x \\ x & \text{if } x \notin W_x \end{cases}$$

Clearly, for all $x \in \mathbb{N}$ we have $\varphi_x(x) \neq f(x)$, hence f is not computable. Moreover $x \notin W_x$ holds true infinitely many times since the empty function has infinitely many indices. Therefore also the last condition is satisfied. $\square$

Exercise 6.2(p). Say that a f *function* $: \mathbb{N} \to \mathbb{N}$ is *increasing* if it is total and for each $x, y \in \mathbb{N}$, if $x \leqslant y$ then $f(x) \leqslant f(y)$. Is there an increasing function which is not computable? Justify your answer.

Solution: Define

$$g(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in W_x \\ 0 & \text{otherwise} \end{cases}$$

and then $f(x) = \Sigma_{y \leqslant x} g(y)$. □

Exercise 6.3(p). Are there two functions $f, g : \mathbb{N} \to \mathbb{N}$ with g not computable such that the composition $f \circ g$ (defined by $(f \circ g)(x) = f(g(x))$) is computable? And requiring that f is also not computable, can the composition $f \circ g$ be computable? Justify your answer, giving examples or proving non-existence.

Solution: Yes, in both cases. In fact, let $g = \chi_k$, not computable, and f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 1 \\ \chi_k(x) & \text{otherwise} \end{cases}$$

not computable too, otherwise χ_K would be computable. It is easy to see that $f \circ g$ is the constant 0, which is computable. □

Exercise 6.4(p). Is there a function $f : \mathbb{N} \to \mathbb{N}$ with finite range, total and increasing (i.e. $f(x) \leqslant f(y)$ for $x \leqslant y$) and not computable? Justify your answer with an example or a proof of non-existence. What if we relax the requirement of totality?

Solution: With the totality requirement, function f cannot exist. Indeed, we can prove that each function $f : \mathbb{N} \to \mathbb{N}$ with all the required properties is computable. The proof proceeds for induction on $M = \max\{f(x) \mid x \in \mathbb{N}\}$.

($M = 0$) Observe that in this case $f(x) = 0$ for all $x \in \mathbb{N}$, i.e. f is the constant 0 and therefore it is computable.

($M > 0$) In this case, let $x_0 = \min\{x \mid f(x) = M\}$. If $x_0 = 0$, the function f is the constant M, and therefore it is computable.

If, on the other hand, $x_0 > 0$, let $M' = f(x_0 - 1)$, i.e., the value assumed by f before M. We can then write $f(x)$ as the sum of two functions

$$f(x) = f'(x) + g(x)$$

where $f' : \mathbb{N} \to \mathbb{N}$ is:

$$f'(x) = \begin{cases} f(x) & \text{if } x < x_0 \\ M' & \text{otherwise} \end{cases}$$

and $g : \mathbb{N} \to \mathbb{N}$ is:

$$g(x) = \begin{cases} 0 & \text{if } x < x_0 \\ M - M' & \text{otherwise} \end{cases} = (M - M')\ sg(x + 1 \dot{-} x_0)$$

The function f' is total, with range included in that of f, whence finite; moreover it is increasing and $\max\{f'(x) \mid x \in \mathbb{N}\} = M' < M$. Hence it is computable by inductive hypothesis. Also g is computable as it can be expressed as a composition of computable functions. Thus f is also computable.

If instead we relax the requirement of totality we can define a function

$$f(x) = \begin{cases} 1 & \text{if } x \notin W_x \\ \uparrow & \text{otherwise} \end{cases}$$

that is increasing, with finite range and not computable since, by diagonalization, it is different from all computable function.

□

Exercise 6.5(p). Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *decreasing* if it is total and for each $x, y \in \mathbb{N}$, if $x \leqslant y$ then $f(x) \geqslant f(y)$. Is there a decreasing function which is not computable? Justify your answer.

Solution: Let $k = \min\{f(x) \mid x \in \mathbb{N}\}$ and let $x_0 \in \mathbb{N}$ be such that $f(x_0) = k$. Therefore, since f is decreasing, $f(x) = k$ for all $x \geqslant x_0$. If we define

$$\theta(x) = \begin{cases} f(x) & \text{if } x < x_0 \\ \uparrow & \text{otherwise} \end{cases}$$

we can write f as

$$f(x) = \begin{cases} \theta(x) & \text{if } x < x_0 \\ k & \text{otherwise} \end{cases}$$

Since θ is finite, it is computable. Let $\theta = \varphi_e$. Therefore

$$f(x) = (\mu w. ((x < x_0 \ \wedge \ S(e, x, (w)_1, (w)_2) \vee (x \geqslant x_0 \ \wedge \ (w)_1 = k)))_1$$

hence it is computable.

An alternative simpler solution, shows that actually all decreasing functions are primitive recursive. One can reuse the previous exercise and observe that $g(x) = f(0) \dot{-} f(x)$ is total, increasing and with finite domain. A direct proof can proceed by (complete) induction on $f(0)$. □

Exercise 6.6(p). Say if there can be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that for any other non-computable function $g : \mathbb{N} \to \mathbb{N}$ the function $f + g$ defined by $(f + g)(x) = f(x) + g(x)$ is computable. Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Solution: It cannot exist otherwise, since the quantification over g is universal, the property should also hold for $g = f$. Thus $f + f = 2f$ would be computable, which implies f computable. □

Exercise 6.7. Say if there can be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that there exists a non-computable function $g : \mathbb{N} \to \mathbb{N}$ for which the function $f + g$ (defined by $(f + g)(x) = f(x) + g(x)$) is computable. Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Solution: Yes, $\chi_K + \chi_{\bar{K}}$ is the constant 1. □

Exercise 6.8(p). Say if there can be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f) \cap img(f)$ is finite. Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Solution: Yes, define

$$f(x) = \begin{cases} \uparrow & \text{if } x \leqslant 1 \\ \chi_K(x) & \text{otherwise, if } x > 1 \end{cases}$$

□

Exercise 6.9. Is there non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f) \cap img(f)$ is empty? Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Solution: Consider the function

$$f(x) = \begin{cases} 2 * \chi_K(\lfloor x/2 \rfloor) & \text{if } x \text{ odd} \\ \uparrow & \text{otherwise} \end{cases}$$

We have that $dom(f)$ is the set of odd numbers, $cod(f) = \{0, 2\}$, then $dom(f) \cap cod(f) = \emptyset$. Also, f is not computable. If it were then also $\chi_K(z) = f(2z+1)/2$ would be computable, while we know that K is not recursive, i.e., χ_K is not is computable. □

Exercise 6.10. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$, such that its image $cod(f) = \{y \mid \exists x \in \mathbb{N}.\ f(x) = y\}$ is finite? Provide an example or show that such a function does not exists.

Solution: Yes, it exists. For example, just consider:

$$f(x) = \begin{cases} \overline{sg}(\varphi_x(x)) & \text{if } x \in W_x \\ 0 & \text{if } x \notin W_x \end{cases}$$

Then the function f

- it is total;
- it is not computable since for each $x \in \mathbb{N}$, one has that $f(x) \neq \varphi_x(x)$; in fact, if $\varphi_x(x) \downarrow$ then $f(x) = \overline{sg}(\varphi_x(x)) \neq \varphi_x(x)$, and if $\varphi_x(x) \uparrow$ then $f(x) = 0 \neq \varphi_x(x)$;
- clearly $cod(f) \subseteq \{0, 1\}$.

□

Exercise 6.11(p). Prove that the function $f : \mathbb{N} \to \mathbb{N}$, defined as

$$f(x) = \begin{cases} \varphi_x(x) & \text{if } x \in W_x \\ x & \text{otherwise} \end{cases}$$

is not computable.

Solution: Observe that

$$g(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in W_x \\ x + 1 & \text{otherwise} \end{cases}$$

is not computable, and, for concluding, use the fact that $g(x) = f(x) + 1$. Hence if f were computable, also g would have been so. □

Exercise 6.12(p). Say if there is a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that, for infinite $x \in \mathbb{N}$ it holds

$$f(x) = \varphi_x(x)$$

If the answer is negative, provide a proof, if the answer is positive, provide an example of such a function.

Solution: We can define

$$f(x) = \begin{cases} \varphi_x(x) & \text{if } x \in W_x \\ 0 & \text{if } x \notin W_x \end{cases}$$

If this were computable, also function: $h : \mathbb{N} \to \mathbb{N}$ defined below, would be computable (by composition)

$$h(x) = f(x) + 1 = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in W_x \\ 1 & \text{if } x \notin W_x \end{cases}$$

Instead, we know that it is not computable. In fact, it is easy to prove that for each $x \in \mathbb{N}$ we have $h \neq \varphi_x$. □

Exercise 6.13. Say if there is a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(x) \neq \varphi_x(x)$$

only on a single argument $x \in \mathbb{N}$. If the answer is negative provide a proof, if the answer is positive give an example of such a function.

Exercise 6.14. Is there non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(x) \neq \varphi_x(x)$$

only on a single $x \in \mathbb{N}$? If the answer is negative provide a proof of non-existence, otherwise give an example of such a function.

Exercise 6.15. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that $cod(f)$ is the set $\mathbb{P}$ of even numbers? Justify your answer response (providing an example of such f, if it exists, or proving that it does not exist).

Solution: Yes, such a function exists. For example, just consider:

$$f(x) = \begin{cases} 2\varphi_x(x) + 2 & \text{if } x \in W_x \\ 2k & \text{if } x \notin W_x \text{e} k = |\{y < x \mid y \notin W_y\}| \end{cases}$$

The domain of f is the set of even numbers, since there are infinitely many functions undefined on their index (e.g. there are infinitely many indices for the function which is always undefined). Furthermore, it is not computable since, by construction, it is different from all computable functions.

Alternatively we could consider

$$f(x) = \begin{cases} 2\varphi_{\frac{x}{2}}(x) + 2 & \text{if } x \in W_x \\ 0 & \text{otherwise} \end{cases}$$

In fact, all even numbers greater than zero will be "covered" by the first case (e.g., all constant functions are computable!). □

Exercise 6.16. Say if there is a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that the set $D = \{x \in \mathbb{N} \mid f(x) \neq \phi_x(x)\}$ is finite. Justify your answer.

Exercise 6.17. Say if there are total computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(x) \neq \varphi_x(x)$ for each $x \in K$ and $g(x) \neq \varphi_x(x)$ for each $x \notin K$. Justify your answer by providing a example or by proving non-existence.

Solution: The function f does not exist. In fact, for every $x \in K$ we have $f(x) \neq \varphi_x(x)$. Moreover, for every x, if φ_x is total then $x \in K$. It follows that f is different from all total computable function. So, if it is total it is not computable.

The function g exists since we can just take $g(x) = 1$ for all $x \in \mathbb{N}$. In fact, if $x \in \bar{K}$, we have that $g(x) = 1 \neq \varphi_x(x) = \uparrow$. $\square$

Exercise 6.18. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} 2x + 1 & \text{if } \varphi_x(x) \downarrow \\ 2x \dot{-} 1 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Solution: The function is not computable, since we can write

$$\chi_K(x) = sg(f(x) \dot{-} 2x).$$

If f were computable, we would deduce that also χ_K is computable, while we know that K is not is recursive and thus χ_K is not computable. $\square$

Exercise 6.19(p). Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} x & \text{sg } \forall y \leqslant x.\ \ \varphi_y \text{total} \\ 0 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Solution: Let $y_0 = \min\{y \mid \varphi_y \text{ is not total}\}$. Note that y_0 is well-defined since the set of non-total computable function is non-empty and natural numbers are well-ordered. Then note that

$$f(x) = \begin{cases} x & \text{if } x < y_0 \\ 0 & \text{otherwise} \end{cases} = x \cdot sg(y_0 - x)$$

is computable. $\square$

Exercise 6.20. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} x + 2 & \text{if } \varphi_x(x) \downarrow \\ x \dot{-} 1 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.21. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} \varphi_x(x+1) + 1 & \text{if } \varphi_x(x+1) \downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.22. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } \varphi_y(y) \downarrow \text{ for each } y \leqslant x \\ 0 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Exercise 6.23. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } \varphi_x(x) \downarrow \\ x+1 & \text{otherwise} \end{cases}$$

Is it computable? Justify your answer.

Solution: The function f is not computable. In fact, since $x^2 \neq x+1$ for each $x \in \mathbb{N}$, if we consider the function $g(x) = \overline{sg}(|f(x) - x^2|)$ we have that $g(x) = \chi_K(x)$. □

Exercise 6.24. A function $f : \mathbb{N} \to \mathbb{N}$ is called *almost total* if it is undefined on a finite set of points. Is there an almost total and computable function $f : \mathbb{N} \to \mathbb{N}$ such that $f \subseteq \chi_K$? Justify your answer by giving an example of such a function in case it exists or a proof of non-existence, otherwise.

Solution: Let f be almost total and assume that $f \subseteq \chi_K$. Note that, if we let $D = dom(f)$, one has that $\overline{D}$ is finite and therefore recursive. Thus also D is recursive. Define $\theta = \chi_K|_{\overline{D}}$, which is a finite function, therefore computable.

Now, we observe that

$$\chi_K(x) = \begin{cases} f(x) & x \in D \\ \theta(x) & \text{otherwise} \end{cases}$$

and conclude that f cannot be computable, otherwise also χ_K would be computable. □

Exercise 6.25. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *almost constant* if there is a value $k \in \mathbb{N}$ such that the set $\{x \mid f(x) \neq k\}$ is finite. Is there an almost constant function which is not computable? Adequately motivate your answer.

Solution: Let $I = \{x \mid f(x) \neq k\}$ and define

$$\theta(x) = \begin{cases} f(x) & \text{if } x \in I \\ \uparrow & \text{otherwise} \end{cases}$$

We can write f as

$$f(x) = \begin{cases} \theta(x) & \text{if } x \in I \\ k & \text{otherwise} \end{cases}$$

Since θ is finite, it is computable. Let $\theta = \varphi_e$. Therefore $f(x) == (\mu w. ((x \in I \wedge S(e, x, (w)_1, (w)_2) \vee (x \notin I \ \wedge (w)_1 = k))_1$ is computable. □

Exercise 6.26. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ with the property that $f(x) = x^2$ for all $x \in \mathbb{N}$ such that $\varphi_x(x) \downarrow$? Justify your answer by providing an example of such function, if it exists, or by proving that it does not exist, otherwise.

Solution: Yes, the function exists and it can be defined as:

$$f(x) = \begin{cases} x^2 & \text{if } \varphi_x(x) \downarrow \\ x^2 + 1 & \text{otherwise} \end{cases}$$

It is not computable since $\chi_K(x) = \overline{sg}(f(x) - x^2)$. □

Exercise 6.27(p). Is there a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that for any non-computable function $g : \mathbb{N} \to \mathbb{N}$ the function $f * g$ (defined as $(f * g)(x) = f(x) \cdot g(x)$) is computable? Justify your answer (providing an example of such f, if it exists, or proving that it does not exist).

Solution: No, the function cannot exist. In fact, assume, by contradiction, that such non-computable function f exists. Then, in particular, we can choose $g = f$ and deduce that $f * f$ is computable. Now

$$(f * f)(x) = f(x) \cdot f(x) = f(x)^2$$

But then also $f(x) = \mu y.|(f * f)(y) - y \cdot y|$ is computable, leading to a contradiction. □

Exercise 6.28(p). Define a function $f : \mathbb{N} \to \mathbb{N}$ total and not computable such that $f(x) = x/2$ for each even $x \in \mathbb{N}$ or prove that such a function does not exist.

Solution: We define

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ even} \\ \varphi_{\frac{x-1}{2}}(x) + 1 & \text{if } x \text{ odd and} x \in W_{\frac{x-1}{2}} \\ 0 & \text{otherwise} \end{cases}$$

Then observe that for each $x \in \mathbb{N}$ it holds that $\varphi_x \neq f$ since $\varphi_x(2x + 1) \neq f(2x + 1)$. □

Exercise 6.29. Is there a total non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that the function $g : \mathbb{N} \to \mathbb{N}$ defined, for each $x \in \mathbb{N}$, by $g(x) = f(x) \dot{-} x$ is computable? Provide an example or prove that such a function does not exist.

Solution: Consider $f(x) = \chi_K(x)$. Then $f(x) \dot{-} x$ is the constant 0 for each $x \geqslant 1$, therefore computable. □

Exercise 6.30(p). Is there may be a non-computable function $f : \mathbb{N} \to \mathbb{N}$ such that for each non-computable function $g : \mathbb{N} \to \mathbb{N}$ the function $f + g$ (defined by $(f + g)(x) = f(x) + g(x)$) is computable? Justify your answer (providing an example of such f, if it exists, or proving that cannot exist).

Solution: No, otherwise, we should have $f + f = 2f$ computable, and thus f computable. □

Exercise 6.31. Is there a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f) = K$ and $cod(f) = \mathbb{N}$? Justify your answer.

Solution: Yes, it exists. For example, consider $f(x) = \varphi_x(x)$. Clearly $dom(f) = K$. Furthermore, for each $k \in N$, if we consider an index of the constant function k we have that $f(e) = \varphi_e(e) = k$. Thus $cod(f) = \mathbb{N}$.

Alternatively one can define

$f(x) = (\mu t.H(x, x, t)) - 1$

Clearly$dom(f) = K$ since $f(x) \downarrow$ if there exists some t such that $H(x, x, t)$, i.e., if $x \in K$. Furthermore, for each $x \in \mathbb{N}$ just take the program Z_k which consists of $Z(1)$ repeated x times. For the corresponding index $y = \gamma(Z_k)$ we will have $f(y) = k - 1$, which shows that $cod(f) = \mathbb{N}$. □

Exercise 6.32. Let A be a recursive set and let $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ be computable functions. Prove that the function $f : \mathbb{N} \to \mathbb{N}$ defined below is computable:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A \\ f_2(x) & \text{if } x \notin A \end{cases}$$

Does the result hold if we weaken the hypotheses and assume A only r.e.? Explain how the proof can be adapted, if the answer is positive, or provide a counterexample, otherwise.

Solution: Let $e_1, e_2 \in \mathbb{N}$ be indexes for f_1, f_2, respectively, namely $\varphi_{e_1} = f_1$ and $\varphi_{e_2} = f_2$. Observe that we can define f as

$$f(x) = (\mu w.((S(e_1, x, (w)_1, (w)_2) \ \wedge \ \chi_A(x) = 1) \vee (S(e_2, x, (w)_1, (w)_2) \ \wedge \ \chi_A(x) = 0)))_1$$

showing that f is computable. Relaxing the hypotheses to recursive enumerability of A, the result is no longer true. Consider for instance $f_1(x) = 1$, $f_2(x) = 0$ and $A = K$, which is r.e. Then f defined as above would be the characteristic function of K which is not computable. □

Exercise 6.33(p). Is there a total, non-computable function such that $img(f) = \{f(x) \mid x \in \mathbb{N}\}$ is the set Pr of Prime numbers? Justify your answer.

Solution: Yes, it exists. For example, consider:

$$f(x) = \begin{cases} p & \text{if } x \in W_x\text{ep} = \min\{p' \in Pr \mid p' > \varphi_x(x)\} \\ 2 & \text{otherwise} \end{cases}$$

Then the function f

- is total;
- it is not computable, since for each $x \in \mathbb{N}$ one has that $f(x) \neq \varphi_x(x)$; in fact, if $\varphi_x(x) \downarrow$ we have that $f(x)$ is a prime larger than $\varphi_x(x)$, and if $\varphi_x(x) \uparrow$ then $f(x) = 2$;
- clearly $img(f) \subseteq Pr$. For the reverse inclusion, consider any prime number $p \in Pr$ and the constant function $g(x) = p - 1$ for each $x \in \mathbb{N}$. The function g is computable, thus $g = \varphi_n$ for a suitable index n. We conclude by noting that $f(n) = \min\{p' \in Pr \mid p' > \varphi_n(n)\} = \min\{p' \in Pr \mid p > p - 1\} = \min\{p' \in Pr \mid p' \geqslant p\} = p$ and thus $p \in img(f)$.

□

7 Reduction, Recursiveness and Recursive Enumerability

Exercise 7.1. Prove that a set A is recursive if and only if there is a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that $x \in A$ if and only if $f(x) > x$.

Solution: Let A be recursive. Then χ_A is computable. Therefore the required function can be $f(x) = x + \chi_A(x)$.

Vice versa, let $f : \mathbb{N} \to \mathbb{N}$ be a computable total function such that $x \in A$ if and only if $f(x) > x$. Then $\chi_A(x) = sg(f(x) \dot{-} x)$ is computable and therefore A is recursive. $\square$

Exercise 7.2. Prove that a set A is recursive if and only if there are two total computable functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for each $x \in \mathbb{N}$

$$x \in A \text{ if and only if } f(x) > g(x).$$

Solution: Let A be recursive. Then χ_A is computable. Therefore the required functions can be $f(x) = \chi_A(x)$ and $g(x) = 0$.

Vice versa, let $f, g : \mathbb{N} \to \mathbb{N}$ be total computable functions such that $x \in A$ if and only if $f(x) > g(x)$. Then $\chi_A(x) = sg(f(x) \dot{-} g(x))$ is computable and therefore A is recursive. $\square$

Exercise 7.3. Prove that a set A is recursive if and only if $A \leqslant_m \{0\}$.

Solution: Let A be recursive. Then χ_A is computable. The reduction function witnessing $A \leqslant_m \{0\}$ can then be $1 - \chi_A(x)$. Conversely, if $A \leqslant_m \{0\}$ and f is the reduction function, then $\chi_A(x) = \bar{sg}(f(x)$. $\square$

Exercise 7.4. Let $A \subseteq \mathbb{N}$ be a set and let $f : \mathbb{N} \to \mathbb{N}$ be a computable function. Prove that if A is r.e. then $f(A) = \{y \in \mathbb{N} \mid \exists x \in A.\ y = f(x)\}$ is r.e. Is the converse also true? That is, from $f(A)$ r.e. can we deduce that A is r.e.?

Solution: Let e, e' be such that $f = \varphi_e$ and $sc_A = \varphi_{e'}$. Then

$$sc_{f(A)}(y) = \mathbf{1}(\mu w.\ H(e', (w)_1, (w)_2) \ \wedge \ S(e, (w)_1, y, (w)_3))$$

hence $f(A)$ is r.e. The converse is not true. For example $\mathbf{1}(\bar{K}) = \{1\}$ is r.e., but $\bar{K}$ is not r.e. $\square$

Exercise 7.5. Let A be a recursive set and $f : \mathbb{N} \to \mathbb{N}$ be a total computable function. Is it true, in general, that $f(A)$ is r.e.? Is it true that $f(A)$ is recursive? Justify your answers with a proof or counterexample.

Solution: We have that $f(A)$ is r.e. since

$$sc_{f(A)}(x) = \mathbf{1}(\mu z.|f(z) - y|)$$

However,$f(A)$ is not recursive. For example, consider the function defined as follows. Take $a \in K$ and define:

$$\begin{aligned} f(x) &= \begin{cases} (x)_1 & \text{if } H((x)_1, (x)_1, (x)_2) \\ a & \text{otherwise} \end{cases} \\ &= (x)_1 \cdot \chi_H((x)_1, (x)_1, (x)_2) + a \cdot \bar{sg}(\chi_H((x)_1, (x)_1, (x)_2)) \end{aligned}$$

computable and total. Moreover $f(\mathbb{N}) = K$. $\square$

Exercise 7.6. Let $A \subseteq \mathbb{N}$ be a set and let $f : \mathbb{N} \to \mathbb{N}$ be a computable function. Prove that if A is recursive then $f^{-1}(A) = \{x \in \mathbb{N} \mid f(x) \in A\}$ is r.e. Is the set $f^{-1}(A)$ also recursive? For the latter give a proof or provide a counterexample.

Solution: The set $f^{-1}(A)$ is r.e. since

$$sc_{f^{-1}(A)}(x) = \chi_A(f(x))$$

It is not recursive in general, since $sc_K^{-1}(\mathbb{N}) = K$. $\square$

Exercise 7.7. Prove that a set A is r.e. if and only if $A \leqslant_m K$.

Solution: Consider If A is r.e., then consider $g(x, y) = sc_A(x)$, and, using the smn theorem, obtain the reduction function. Conversely, if $A \leqslant_m K$, then if f is the reduction function, we have $sc_A(x) = sc_k(f(x))$, which is computable. $\square$

Exercise 7.8. Prove that a set A is r.e. if and only if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $A = img(f)$ (remember that $img(f) = \{y : \exists z.\ y = f(z)\}$).

Solution: If A is r.e., then just take $f(x) = x \cdot sc_A(x)$. Conversely, if $A = img(f)$ for $f : \mathbb{N} \to \mathbb{N}$ computable, say$f = \varphi_e$ for a suitable$e \in \mathbb{N}$ then $sc_A(x) = \mathbf{1}(\mu w.S(e, (w)_1, x, (w)_2))$. $\square$

Exercise 7.9. Given a function $f : \mathbb{N} \to \mathbb{N}$, define the predicate $P_f(x, y) \equiv$ "$f(x) = y$", i.e., $P_f(x, y)$ is true if $x \in dom(f)$ and $f(x) = y$. Prove that f is computable if and only if the predicate $P_f(x, y)$ is semi-decidable.

Solution: Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function. Let $e \in \mathbb{N}$ such that $f = \varphi_e$. Then $sc_P(x, y) = \mathbf{1}(\mu w.|f(x) - y|$ is computable, hence P is semidecidable.

Vice versa, let $P(x, y)$ be semidecidable and let e be an index for the semi-characteristics function of P, namely $\varphi_e^{(2)} = sc_P$. Then we have $f(x) = (\mu w.H^{(2)}(e, (x, (w)_1), (w)_2))_1$. $\square$

Exercise 7.10. Let $A \subseteq \mathbb{N}$. Prove that A is recursive and infinite if and only if it is the image of a function $f : \mathbb{N} \to \mathbb{N}$ computable, total and strictly increasing (i.e., such that for each $x, y \in \mathbb{N}$, if $x < y$ then $f(x) < f(y)$).

Solution: Let A be recursive and infinite. Define the function $g : \mathbb{N} \to \mathbb{N}$ as

$$g(x) = \Sigma_{y<x}\chi_A(y)$$

that is, $g(x)$ counts the number of elements of A below x, or, in other words, it assigns a increasing number to each element of A. The function is computable since χ_A is so. Furthermore, it is easy to see that g is monotone, that is, for each $x \in \mathbb{N}$, $g(x) \leqslant g(x + 1)$. Moreover, $x \in A$ if and only if $g(x + 1) = g(x) + 1$. Since A is infinite, this implies that $img(g) = \mathbb{N}$.

Now we can define $f : \mathbb{N} \to \mathbb{N}$ as

$$\begin{aligned} f(n) &= \mu x.\ (g(x + 1) = n + 1) \\ &= \mu x.\ |n + 1 - g(x + 1)| \end{aligned}$$

The function f is

- computable, since it arises as the minimization of a computable function;
- total, since $img(g) = \mathbb{N}$ and therefore, for all n, the condition $g(x + 1) = n + 1$ is certainly satisfied for some x;
- increasing, since if $n < m$ then $g(f(n) + 1) = n + 1 < m + 1 = g(f(m) + 1)$. Recalling that g is increasing, this implies $f(n) + 1 < g(m) + 1$ and therefore $f(n) < f(m)$.

In addition, $img(f) = A$. In fact, if $x \in img(f)$, then there exists $n \in \mathbb{N}$ such that $f(n) = x$, hence $g(x) = n$ and $g(x+1) = n+1$. Therefore, as observed above, $\chi_A(x) = 1$, i.e., $x \in A$. Conversely, if $x \in A$, then we have $g(x) = n$ and $g(x+1) = n+1$. Therefore $f(n) = x$, and thus $x \in img(A)$.

For the converse implication, let $A = img(f)$ with f total computable and strictly increasing. Clearly A is infinite. In addition, since f is increasing, it is easy to see that for each $x \in \mathbb{N}$ we have $f(x) \geqslant x$ and thus, if there is $z \in \mathbb{N}$ such that $f(z) = x$ then $z \leqslant x$. Therefore the characteristic function of A can be expressed as $\chi_A(x) = \overline{sg}(\Pi_{z=0}^{x} |f(z) - x|)$. □

Exercise 7.11. Let $\pi : \mathbb{N}^2 \to \mathbb{N}$ be the function encoding pairs of natural numbers into the natural numbers. Prove that a function $f : \mathbb{N} \to \mathbb{N}$ is computable if and only if the set $A_f = \{\pi(x, f(x))\ x \in \mathbb{N}\}$ is recursively enumerable.

Exercise 7.12. Prove that a set $A \subseteq \mathbb{N}$ is recursive if and only if $A \leqslant_m \{0\}$.

Exercise 7.13. Let $A \subseteq \mathbb{N}$ be a non-empty set. Prove that A is recursively enumerable if and only if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $dom(f)$ is the set of prime numbers and $img(f) = A$.

Exercise 7.14. Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions such that, denoted by $\mathbf{0}$ and $\mathbf{1}$ the constant functions 0 and1, respectively, we have $\mathbf{0} \notin \mathcal{A}$ and $\mathbf{1} \in \mathcal{A}$. Define $A = \{x : \varphi_x \in \mathcal{A}\}$ and show that either A is not or $\bar{A}$ is not r.e.

Solution: Since neither A nor its complement are empty, by Rice's theorem they are not recursive. Therefore, they cannot be both r.e. □

Exercise 7.15. Establish whether an index $x \in \mathbb{N}$ can exist such that $\bar{K} = \{2^y - 1 : y \in E_x\}$. Justify your answer.

Solution: No, it cannot exist. Let f denote the function $f(y) = 2^y - 1$. Then, we have that $\{2^y - 1 : y \in E_x\} = img(f \circ \varphi_x)$, which implies that such set is r.e., unlike $\bar{K}$. Hence they cannot coincide. □

Exercise 7.16. Given two sets $A, B \subseteq \mathbb{N}$ what $A \leqslant_m B$ means. Prove that given $A, B, C \subseteq \mathbb{N}$ the following hold:

a. if $A \leqslant_m B$ and $B \leqslant_m C$ then $A \leqslant_m C$;

b. if $A \neq \mathbb{N}$ then $\varnothing \leqslant_m A$.

Solution:

a. Observe that if f reduces A to$B$, and g reduces B to C then $g \circ f$ reduces A to C.

b. Consider $a_0 \notin A$ (which exists since $A \neq \mathbb{N}$). Then the reduction function can be $f(x) = a_0$ for each $x \in \mathbb{N}$.

□

Exercise 7.17. Given two sets $A, B \subseteq \mathbb{N}$ define what $A \leqslant_m B$ means. Is it the case that $A \leqslant_m A \cup \{0\}$ for all sets A? If the answer is positive, provide a proof, otherwise, a counterexample. In the second case, identify a condition (specifying whether it is only sufficient or also necessary) that make $A \leqslant_m A \cup \{0\}$ true.

Solution:

In general $A \leqslant_m A \cup \{0\}$ does not hold. For instance $\mathbb{N}\backslash\{0\} \not\leqslant_m \mathbb{N}$, since a total function $f : \mathbb{N} \to \mathbb{N}$ cannot exist such that $x \in \mathbb{N}\backslash\{0\}$ iff $f(x) \in \mathbb{N}$: for an choice of f the consequent is always true, while the antecedent is not!

This is the only counterexample, i.e., for each $A \neq \mathbb{N}\backslash\{0\}$ we have $A \leqslant_m A \cup \{0\}$. In fact, we distinguish two cases:

- if $0 \in A$, then $A \cup \{0\}$ (the reduction function can be the identity).
- if $0 \notin A$, then we can certainly find $x_0 \notin A$, $x_0 \neq 0$ (in fact we know that $A \neq \mathbb{N}$ and $A \neq \mathbb{N}\backslash\{0\}$). Then, the reduction function can be

$$f(x) = \begin{cases} x_0 & \text{if } x = 0 \\ x & \text{otherwise} \end{cases}$$

□

Exercise 7.18. Given two sets $A, B \subseteq \mathbb{N}$ define what $A \leqslant_m B$ means. Prove that, given any $A \subseteq \mathbb{N}$, we have A r.e. iff $A \leqslant_m K$.

Solution: If $A \leqslant_m K$ then A r.e., by reduction, since K is r.e.

Conversely, let A r.e. Define

$$g(x, y) = sc_A(x) = \varphi_{s(x)}(y)$$

with $s : \mathbb{N} \to \mathbb{N}$ computable total, given by the smn theorem. Then s is a reduction function for $A \leqslant_m K$. □

Exercise 7.19. Prove that a set $A \subseteq \mathbb{N}$ is recursive if and only if A and $\bar{A}$ are r.e.

Exercise 7.20. State and prove Rice's theorem(without using the second recursion theorem).

Exercise 7.21. Define what it means for a set $A \subseteq \mathbb{N}$ to be saturated and prove that K is not is saturated.

Exercise 7.22. Let$\mathcal{A} \subseteq \mathcal{C}$be a set of functions computable and let$f \in \mathcal{A}$ such that for any function over$\theta \subseteq f$ is worth $\theta \notin \mathcal{A}$. Prove that $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$ is not r.e.

8 Characterization of sets

Exercise 8.1. Study the recursiveness of the set $A = \{x \in \mathbb{N} : |W_x| \geqslant 2\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.2. Study the recursiveness of the set $A = \{x \in \mathbb{N} : x \in W_x \cap E_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.3. Study the recursiveness of the set

$$B = \{x \mid x \in W_x \cup E_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: We prove that $K \leqslant_m A$. Define

$$g(x,y) = \begin{cases} 1 & x \in K \\ \uparrow & \text{otherwise} \end{cases} = sc_K(x)$$

By the smn theorem, we get a total computable function $s : \mathbb{N} \to \mathbb{N}$, such that $g(x,y) = \varphi_{s(x)}(y)$ and it is easy to see that s can be the reduction function.

Furthermore, B is r.e., in fact

$$sc_B(x) = \mathbf{1}(\mu w.(H(x,x,(w)_2) \vee S(x,(w)_1,x,(w)_2))$$

We therefore conclude that $\bar{A}$ is not r.e. □

Exercise 8.4. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x \subseteq \mathbb{P}\}$, where $\mathbb{P}$ is the set of even numbers, i.e. establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set $\bar{A}$ is not is recursive since $K \leqslant_m \bar{A}$. In fact, consider the function

$$g(x,y) = \begin{cases} 1 & x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

It is computable. Therefore, by the smn theorem, there exists a total computable $s : \mathbb{N} \to \mathbb{N}$, such that $g(x,y) = \varphi_{s(x)}(y)$. Such a function s can be shown to be a reduction function.

In fact, if $x \in K$, we have that $\varphi_{s(x)}(y) = g(x,y) = 1$ for all y. Then $W_{s(x)} = \mathbb{N}$. Therefore $W_{s(x)} \nsubseteq \mathbb{P}$, i.e., $s(x) \in \overline{A}$.

On the other hand, if $x \notin K$, we have that $\varphi_{s(x)}(y) = g(x,y) \uparrow$ for all y. Then $W_{s(x)} = \varnothing$, and therefore $W_{s(x)} \subseteq \mathbb{P}$, then $s(x) \in A$, i.e., $s(x) \notin \overline{A}$.

The set $\bar{A}$ is r.e., in fact

$$sc_{\bar{A}}(x) = \mu w.H(x, 2(w)_1 + 1, (w)_2)$$

Therefore, A is not r.e.

□

Exercise 8.5. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists y, z \in \mathbb{N}.\ z > 1 \ \wedge \ x = y^z\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.6. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \phi_x(y) = y \text{ for infinitely many } y\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.7. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x \subseteq E_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.8. Study the recursiveness of the set $A = \{x \in \mathbb{N} : |W_x| > |E_x|\}$, i.e. establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.9. Study the recursiveness of the set $A = \{x \in \mathbb{N} \mid \varphi_x(y) = x * y \text{ per some } y\}$, that is to say if A e $\bar{A}$are recursive/recursively enumerable.

Solution: The set A is r.e. In fact the semi-characteristic function

$$_A(x) = \mu w.S(x, (w)_1, x * (w)_1, (w)_2)$$

is computable.

It is not is recursive, since $K \leqslant_m A$. In fact, consider the function

$$g(x, y) = \begin{cases} 0 & x \in K \\ \uparrow & \text{otherwise} \end{cases} = \mathbf{0}(sc_K(x))$$

It is computable and thus, by the smn theorem, we deduce that there is a total computable function $s : \mathbf{N} \to \mathbf{N}$ such that, for each $x, y \in \mathbb{N}$,

$$g(x, y) = \varphi_{s(x)}(y)$$

Then s is a reduction function for $K \leqslant_m A$. In fact

- if $x \in K$ then $\varphi_{s(x)}(y) = g(x, y) = 0$ for each $y \in \mathbb{N}$. In particular $\varphi_{s(x)}(0) = 0 = s(x) * 0$. Thus $s(x) \in A$.
- if $x \notin K$ then $\varphi_{s(x)}(y) = g(x, y) \uparrow$ for each $y \in \mathbb{N}$. Therefore surely there is no y such that $\varphi_{s(x)}(y) = x * y$. Thus $s(x) \notin A$.

Finally, since A r.e. and non-recursive, we conclude that $\bar{A}$ is not r.e. and thus not recursive.

□

Exercise 8.10. Study the recursiveness of the set $A = \{x \in \mathbb{N} \mid |W_x \cap E_x| = 1\}$, i.e., establish if A e $\bar{A}$are recursive/recursively enumerable.

Solution: The set A is clearly saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$where $\mathcal{A} = \{f \mid |cod(f) \cap img(f)| = 1\}$. We can deduce, by using Rice-Shapiro's theorem, that A is not r.e. In fact $id \notin \mathcal{A}$ but we can find a finite subfunction $\theta \subseteq id$, defined as follows:

$$\theta(x) = \begin{cases} 0 & \text{if } x = 0 \\ \uparrow & \text{otherwise} \end{cases}$$

such that $cod(\theta) = dom(\theta) = \{0\}$, hence $|cod(\theta) \cap dom(\theta)| = |\{0\}| = 1$. Therefore $\theta \in \mathcal{A}$.

The complement is not r.e. again by Rice-Shapiro's theorem. E.g.,θ, as defined above, is not in $\bar{\mathcal{A}}$, but it admits $\varnothing$ as finite subfunction and $\varnothing \in \bar{\mathcal{A}}$. □

Exercise 8.11. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *strictly increasing* when for each $y, z \in \text{dom}(f)$, if $y < z$ then $f(y) < f(z)$. Study the recursiveness of the set $A = \{x \mid \varphi_x \text{ sharply increasing}\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is clearly saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid f \text{strictly increasing}\}$. We can deduce, by using Rice-Shapiro's theorem, that A is not r.e. In fact $\emptyset \in \mathcal{A}$ and $\emptyset \subseteq id \notin \mathcal{A}$.

The complement is r.e., in fact

$$sc_{\bar{A}}(x) = \mathbf{1}(\mu z.S(x,(z)_1,(z)_2+(z)_3,(z)_4) \wedge S(x,(z)_1+(z)_5+1,(z)_2,(z)_4)$$

Therefore $\bar{A}$ is not recursive. □

Exercise 8.12. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is *almost total* if it is undefined on a finite set of points. Study the recursiveness of the set $A = \{x \mid \varphi_x \text{ almost total}\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.13. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x \cap E_x = \emptyset\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 8.14. Given a set $X \subseteq \mathbb{N}$, we define $X + 1 = \{x + 1 : x \in X\}$. Study the recursiveness of the set $A = \{x \in \mathbb{N} : E_x = W_x + 1\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \mid cod(f) = dom(f) + 1\}$. We can use Rice-Shapiro to show that

- A is not r.e .
  In fact $id \notin \mathcal{A}$ since $cod(id) = \mathbb{N} \neq dom(id) + 1 = \mathbb{N} + 1 = \mathbb{N}\backslash\{0\}$. Moreover, $\emptyset \subseteq id$ and $\emptyset \in \mathcal{A}$ since $cod(\emptyset) = \emptyset = dom(\emptyset) + 1$.
- $\overline{A}$ not r.e .
  In fact, if we define

  $$f(x) = \begin{cases} 1 & \text{if } x \leqslant 1 \\ x & \text{otherwise} \end{cases}$$

  and

  $$\theta(x) = \begin{cases} 1 & \text{if } x = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

  we have that $f \notin \bar{A}$ since $cod(f) = \mathbb{N}\backslash\{0\} = dom(f) + 1 = \mathbb{N} + 1$. Moreover, $\theta \subseteq f$ and $\theta \in \bar{\mathcal{A}}$ since $cod(\theta) = \{1\} = dom(\theta) \neq dom(\theta) + 1$.

□

Exercise 8.15. Let $\mathbb{P}$ be the set of even numbers. Prove that indicated with $A = \{x \in \mathbb{N} : E_x = \mathbb{P}\}$, we have $\bar{K} \leqslant_m A$.

Solution: To obtain the reduction function we can consider

$$f(x,y) = \begin{cases} 2y & \text{if } \neg H(x,x,y) \\ 1 & \text{otherwise} \end{cases}$$

The function f is computable, since it can be written as $f(x,y) = 2\,y\,\bar{sg}(\chi_H(x,x,y)) + \chi_H(x,x,y)$.

Therefore, by the smn theorem, there exists $s : \mathbb{N} \to \mathbb{N}$ computable total, such that $f(x,y) = \varphi_{s(x)}(y)$ for each $x, y \in \mathbb{N}$, which can be used as a reduction function for $\bar{K} \leqslant_m A$. Indeed:

- if $x \in \bar{K}$, then $\chi_H(x, x, y) = 0$ for each y, and therefore $\varphi_{s(x)}(y) = f(x, y) = 2y$ for each y. Thus $E_{s(x)} = \mathbb{P}$ and hence $s(x) \in A$.
- if $x \notin \bar{K}$, or $x \in K$ then there exists y_0 such that $\chi_H(x, x, y) = 1$ for each $y \geqslant y_0$. Therefore $\varphi_{s(x)}(y) = 1$ for $y \geqslant y_0$, thus $1 \in E_{s(x)}$ and therefore $E_{s(x)} \neq \mathbb{P}$. Hence $s(x) \notin A$.

□

Exercise 8.16. Study the recursiveness of the set $\mathbb{A} = \{x \in \mathbb{N} : \varphi_x(x) \downarrow \ \wedge \ \varphi_x(x) < x + 1\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: ($K \leqslant A$) The reduction function can be obtained by considering

$$f(x, y) = \begin{cases} 0 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

Thus A is not recursive. Furthermore, A is r.e. since we can write its semi-characteristic function as follows:

$$sc_A(x) = sg(x + 1 - \varphi_x(x))$$

Finally $\bar{A}$ not r.e., since A r.e. and non-recursive.

□

Exercise 8.17. Study the recursion of the set $A = \{x \in \mathbb{N} : x \in W_x \ \wedge \ \varphi_x(x) = x^2\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: We show that $K \leqslant A$, and thus A is not recursive. Define

$$g(x, y) = \begin{cases} y^2 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

The function $g(x, y)$ is computable, since

$$g(x, y) = y^2 \cdot sc_K(x)$$

Thus by the smn theorem, there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x, y)$$

The function s is a reduction function of K to A. Indeed

- if $x \in K$ then $\varphi_{s(x)}(y) = g(x, y) = y^2$ for each $y \in \mathbb{N}$. Therefore $s(x) \in W_{s(x)} = \mathbb{N}$ and $\varphi_{s(x)}(s(x)) = s(x)^2$. Thus $s(x) \in A$.
- if $x \notin K$ then $\varphi_{s(x)}(y) = g(x, y) \uparrow$ for each $y \in \mathbb{N}$. Therefore $s(x) \notin W_{s(x)} = \varnothing$. Thus $s(x) \notin A$.

Furthermore, A is r.e., since its semi-characteristic function

$$sc_A(x) = \mathbf{1}(\mu w.\ |x^2 - \varphi_x(x)|) = \mathbf{1}(\mu w.\ |x^2 - \Psi_U(x, x)))|$$

is computable. Therefore $\bar{A}$ not r.e. and thus it is not recursive. □

Exercise 8.18. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ \varphi_x(x+3k) \uparrow\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: ($\bar{K} \leqslant A$) The reduction function can be obtained starting from the function

$$f(x,y) = \begin{cases} 0 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

Therefore A not r.e.

($\bar{K} \leqslant \bar{A}$) The reduction function can be obtained starting from the function

$$g(x,y) = \begin{cases} 0 & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

Hence $\bar{A}$ not r.e.

□

Exercise 8.19. Study the recursiveness of the set $A = \{x \in \mathbb{N} : W_x = \overline{E_x}\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated, since $A = \{x : \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \in \mathcal{C} : dom(f) = \overline{cod(f)}\}$.

Using Rice-Shapiro's theorem we can prove that A and $\overline{A}$ are not r.e .:

- A non r.e .
  Observe that no finite function can belong to $\mathcal{A}$ and $\mathcal{A} \neq \varnothing$ (e.g. $sc_{\mathbb{N}-\{1\}}$, the semi-characteristic function of$\mathbb{N} - \{1\}$, is in $\mathcal{A}$)
- $\overline{A}$ is not r.e .
  Note that $sc_{\mathbb{N}-\{1\}} \notin \overline{\mathcal{A}}$, but $\varnothing \in \overline{\mathcal{A}}$.

□

Exercise 8.20. Study the recursiveness of the set

$$B = \{\pi(x,y) \;\; P_x(x) \downarrow \text{ in less than } y \text{steps}\},$$

i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Solution: We have that $B = \{\pi(x,y) \;\; H(x,x,y-1)\}$. Thus B and $\overline{B}$ are recursive. □

Exercise 8.21. Given $A = \{x \mid \varphi_x \text{ is total}\}$, show that $\bar{K} \leqslant_m A$.

Solution: Defines

$$g(x,y) = \begin{cases} y & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

By the smn theorem, we obtain a total computable function $s : \mathbb{N} \to \mathbb{N}$, such that $g(x,y) = \varphi_{s(x)}(y)$ and it is easy to see that s can be the reduction function. □

Exercise 8.22. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \varphi_x(y) = y \text{for infinities} y\}$, that is, say if A and $\bar{A}$ are recursive/recursive enumerable.

Exercise 8.23. Given a subset $X \subseteq \mathbb{N}$ define $F(X) = \{0\} \cup \{y, y+1 \mid y \in X\}$. Studying recursiveness of the set $A = \{x \in \mathbb{N} : W_x = F(E_x)\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated, since $A = \{x : \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \in \mathcal{C} : dom(f) = F(code(f))\}\}$.

Using Rice-Shapiro's theorem we prove that both A and $\overline{A}$ are not r.e .:

- A is not r.e .
  Consider the function
  $$f(x) = \begin{cases} 0 & \text{if } x = 0,1,2 \\ \uparrow & = \text{otherwise} \end{cases}$$
  We have $f \notin \mathcal{A}$, since $dom(f) = \{0,1,2\}e cod(f) = \{0\}$. Thus $F(cod(f)) = \{0,1\} \neq dom(f)$. Moreover consider
  $$\theta(x) = \begin{cases} 0 & \text{if } x = 0,1 \\ \uparrow & = \text{otherwise} \end{cases}$$
  Clearly $\theta \subseteq f$. In addition $dom(\theta) = \{0,1\}e cod(\theta) = \{0\}$. Then $F(cod(\theta)) = \{0,1\} = dom(\theta)$ and therefore $\theta \in \mathcal{A}$. By Rice-Shapiro's theorem, we conclude that A is not r.e.

- $\overline{A}$ is not r.e .
  Note that if θ is the function defined in the previous case, $\theta \notin \overline{\mathcal{A}}$, but the function always undefined $\varnothing \in \overline{\mathcal{A}}$, since $dom(\varnothing) = cod(\varnothing) = \varnothing$ and therefore $F(cod(\varnothing)) = \{0\} \neq dom(\varnothing)$. Thus, summing up $\theta \notin \overline{\mathcal{A}}$, but it admits a finite subfunction, i.e., the function always undefined $\varnothing \in \overline{\mathcal{A}}$. By Rice-Shapiro's theorem, we conclude that $\bar{A}$ is not r.e.

□

Exercise 8.24. Study the recursiveness of the set
$$B = \{x \mid k \cdot (x+1) \in W_x \cap E_x \text{for each } k \in \mathbb{N}\},$$
i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: The set A is not r.e., since $\bar{K} \leqslant_m A$. We prove it by considering
$$g(x,y) = \begin{cases} y & \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$
This is computable and, by using the smn theorem, one can obtain the reduction function.
Also $\bar{A}$ is not r.e., since $\bar{K} \leqslant_m \bar{A}$. The reduction function can be obtained by considering
$$g(x,y) = \begin{cases} y & x \in K \\ \uparrow & \text{otherwise} \end{cases}$$
□

Exercise 8.25. Let $\varnothing$ be the always undefined function. Study the recursiveness of the set $A = \{x \mid \varphi_x = \varnothing\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: The set A is non-recursive, by Rice's theorem, since it is saturated, not empty (the always undefined function is computable) and different from $\mathbb{N}$.
In addition $\bar{A}$ is r.e., since
$$sc_{\bar{A}}(x) = \mathbf{1}(\mu w. H(x,(w)_1,(w)_2)$$
Thus A not r.e. □

Exercise 8.26. Study the recursiveness of the set $A = \{x \ \ \forall y.\text{ if } y + x \in W_x \text{ then } y \leqslant \varphi_x(y+x)\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursive enumerable.

Solution: The set $\bar{A} = \{x \ \ \exists y.\ y + x \in W_x \ \wedge \ y > \varphi_x(y+x)\}$ is not is recursive, since $K \leqslant_m A$. Consider the function

$$g(x,y) = \begin{cases} 0 & x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

It is computable and thus, using the smn theorem, we deduce the existence of a total computable function $s : \mathbb{N} \to \mathbb{N}$, such that $g(x,y) = \varphi_{s(x)}(y)$. The function s can be the reduction function.

In fact, if $x \in K$, we have that $\varphi_{s(x)}(y) = g(x,y) = 0$ for all y. Hence $\varphi_{s(x)}(s(x)+1) = 0 < s(x)+1$, and therefore $s(x) \in \bar{A}$. If, on the other hand, $x \notin K$, we have $s(x) \notin \bar{A}$.

The set $\bar{A}$ is r.e., in fact

$$sc_{\bar{A}}(x) = \mu w.S(x, (w)_1 + x, (w)_1 + (w)_2 + 1, (w)_3)$$

where, intuitively, $(w)_1$ represents the value y we are looking for and the value of the function is required to be $(w)_1 + (w)_2 + 1 > (w)_1$.

Therefore, A is not r.e.

□

Exercise 8.27. Study the recursiveness of the set $A = \{x \mid \varphi_x(y+x) \downarrow \quad \text{for some } y \geqslant 0\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set $A = \{x \mid \varphi_x(y+x) \downarrow \quad \text{for some } y \geqslant 0\}$ is not recursive because $K \leqslant A$. In order to prove this fact, let us consider the function $g : \mathbb{N}^2 \to \mathbb{N}$ defined, by

$$g(x,y) = \begin{cases} 1 & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

The function is computable since $g(x,y) = sc_K(x)$. Hence, by the smn-theorem, there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x,y)$ for all $x, y \in \mathbb{N}$. We next argue that s is a reduction function for $K \leqslant_m A$. In fact

- If $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = 1$ for all $y \in \mathbb{N}$. In particular, $\varphi_{s(x)}(0 + s(x)) \downarrow$. Hence $s(x) \in A$.
- If $x \notin K$ then $\varphi_{s(x)}(y) = g(x,y) \uparrow$ for all $y \in \mathbb{N}$. Hence $\varphi_{s(x)}(y + s(x)) \uparrow$ for all $y \in \mathbb{N}$. Hence $s(x) \notin A$.

The set A is r.e., since it semi-characteristic function

$$sc_A(x) = \mathbf{1}(\mu(y,t).H(x, x+y, t))$$

is computable.

Therefore, $\bar{A}$ is not r.e.

□

Exercise 8.28. Let $X \subseteq \mathbb{N}$ be finite, $X \neq \varnothing$ and define $A_X = \{x \in \mathbb{N} : W_x = E_x \cup X\}$. Study the recursiveness of A, i.e., say if A_X and $\bar{A_X}$ are recursive/recursively enumerable.

Solution: The set A_A is saturated, since $A_X = \{x : \varphi_x \in \mathcal{A}\}$, where $\mathcal{A}_X = \{f \in \mathcal{C} : dom(f) = cod(f) \cup X\}\}$.

Using Rice-Shapiro's theorem we prove that A and $\bar{A}$ are both not r.e .:

- A is not r.e .
  Let $x \in X$ and $y \notin X$ and consider the function

  $$f(x) = \begin{cases} x & \text{if } x \in X \cup \{y\} \\ \uparrow & \text{otherwise} \end{cases}$$

  We have $f \notin \mathcal{A}$, since $dom(f) = X \cup \{y\} \neq X = X \cup \{x\} = X \cup cod(f)$. Moreover, if we consider

$$\theta(x) = \begin{cases} x & \text{if } x \in X \\ \uparrow & \text{otherwise} \end{cases}$$

clearly $\theta \subseteq f$. Note that $dom(\theta) = X = X \cup \{x\} = X \cup cod(\theta)$ and therefore $\theta \in \mathcal{A}$. Thus, by Rice-Shapiro's theorem we conclude that A is not r.e.

- $\bar{A}$ is not r.e .
  Note that if θ is the function defined above, $\theta \notin \overline{\mathcal{A}}$, but the function always undefined $\varnothing \in \overline{\mathcal{A}}$, since $dom(\varnothing) = \varnothing \neq X = cod(\varnothing) \cup X$. Thus, summing up $\theta \notin \overline{\mathcal{A}}$, but it admits a finite subfunction, i.e., the function always undefined $\varnothing \in \overline{\mathcal{A}}$. By Rice-Shapiro's theorem, we conclude that $\bar{A}$ is not r.e.

□

Exercise 8.29. Let $A = \{x \in \mathbb{N} : W_x \cap E_x \ \neq \varnothing\}$. Study the recursiveness of A, i.e., say if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated, since $A = \{x : \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \in \mathcal{C} : dom(f) \cap cod(f) \neq \varnothing\}$. It is not empty (since $\mathbf{1} \in \mathcal{A}$) and it is not the entire $\mathbb{N}$ (since $\varnothing \notin \mathcal{A}$), thus by Rice's theorem A is not recursive. Furthermore,A is r.e. since

$$\begin{aligned} sc_A(x) &= \mathbf{1}(\mu(y,z,t).H(x,y,t) \wedge S(x,z,y,t)) \\ &= \mathbf{1}(\mu w.H(x,(w)_1,(w)_3) \wedge S(x,(w)_2,(w)_1,(w)_3) \end{aligned}$$

Therefore $\bar{A}$ is not is r.e. □

Exercise 8.30. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \forall k \in \mathbb{N}.\ x + k \in W_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: We prove that $\bar{K} \leqslant_m A$, and thus A is not r.e. In order to obtain the reduction function, consider the following computable function

$$g(x,y) = \begin{cases} y & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

and then use the smn theorem.

Also $K \leqslant_m A$. In order to obtain the reduction function, consider the following computable function

$$g(x,y) = \begin{cases} 1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

and again, use the smn theorem. Therefore $\bar{K} \leqslant \bar{A}$ and therefore $\bar{A}$ not r.e. □

Exercise 8.31. A partial function $f : \mathbb{N} \to \mathbb{N}$ is called injective when for each $x, y \in dom(f)$, if $f(x) = f(y)$ then $x = y$. Study the recursiveness of the set $A = \{x \ \ \varphi_x \text{ injective}\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: The set A is clearly saturated, since $A = \{x \mid \varphi_x \in \mathcal{A}\}$, where $\mathcal{A}$ is the set of injective functions. Since $\varnothing \in \mathcal{A}$ and $\mathbf{1} \notin \mathcal{A}$, by Rice's theorem the sets A and $\bar{A}$ are not is recursive. Also $\bar{A}$ is r.e, since

$$sc_A(x) = \mathbf{1}(\mu w.\ (S(x,(w)_1,(w)_3,(w)_4) \wedge S(x,(w)_2,(w)_3,(w)_4) \ \wedge \ (w)_1 \neq (w)_2)).$$

Thus A is not r.e. □

Exercise 8.32. Study the recursiveness of the set $A = \{x \in \mathbb{N} : \exists y \in E_x . \exists z \in W_x .\ x = y * z\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: We show that $K \leqslant A$, thus A is not recursive. In fact, define

$$g(x,y) = \begin{cases} 1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

The function $g(x,y)$ is computable, since

$$g(x,y) =_K (x)$$

So by the SMN theorem, there exists a total computable such function $s : \mathbb{N} \to \mathbb{N}$ such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y)$$

The function s is a reduction function of K to A. Indeed, if $x \in K$, then $\varphi_{s(x)}(y) = y$ for each y, and thus we can take $s(x) \in W_{s(x)}$ and $1 \in E_{s(x)}$ such that $s(x) = s(x) * 1$. Thus $s(x) \in A$. Otherwise, $\varphi_{s(x)} = \varnothing$ and thus it is easy to conclude $s(x) \notin A$.

Furthermore, A is r.e., since

$$sc_A(x) = \mathbf{1}(\mu w.\ S(x, (w)_1, (w)_2, (w)_3) \ \wedge \ (w)_1 * (w)_2 = x)$$

Therefore $\bar{A}$ is not r.e. □

Exercise 8.33. Study the recursiveness of the set $A = \{x \in \mathbb{N} : x \in W_x \ \wedge \ \varphi_x(x) > x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: We show that $K \leqslant A$, thus A is not recursive. Define

$$g(x,y) = \begin{cases} y+1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

The function $g(x,y)$ is computable, since

$$g(x,y) = (y+1) \cdot sc_K(x)$$

So by the SMN theorem, there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y)$$

The function s is a reduction function of K to A. In fact

- if $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = y+1$ for each $y \in \mathbb{N}$. Therefore $s(x) \in W_{s(x)} = \mathbb{N}$e $\varphi_{s(x)}(s(x)) = s(x) + 1 > s(x)$. Therefore $s(x) \in A$.
- if $x \notin K$ then $\varphi_{s(x)}(y) = g(x,y) \uparrow$ for each $y \in \mathbb{N}$. Therefore $s(x) \notin W_{s(x)} = \varnothing$. Thus $s(x) \notin A$.

Furthermore, A is r.e., since its characteristic function

$$sc_A(x) = \mathbf{1}(\mu w.\ (x+1) \dot{-} \varphi_x(x)) = \mathbf{1}(\mu w.\ ((x+1) \dot{-} \Psi_U(x,x)))$$

is computable. Therefore $\bar{A}$ is not r.e., and therefore it is not not even recursive. □

Exercise 8.34. Let f be a total computable function such that $img(f) = \{f(x) : x \in \mathbb{N}\}$ is infinite. Study the recursiveness of the set

$$A = \{x \;\; \exists y \in W_x.\ x < f(y)\},$$

i.e., establish if A e $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is not is recursive since $K \leqslant_m A$. In fact, consider the function

$$g(x,y) = \begin{cases} 1 & x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

It is computable. Therefore for the smn theorem there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$, such that $g(x,y) = \varphi_{s(x)}(y)$. The function s is a reduction function.

In fact, if $x \in K$, we have that $\varphi_{s(x)}(y) = g(x,y) = 1$ for each y. Hence $W_{s(x)} = \mathbb{N}$, and therefore $f(W_{s(x)}) = f(\mathbb{N}) = img(f)$, which is infinite for hypothesis. Thus there certainly exists $z \in f(W_{s(x)})$ such that $x < z$, i.e., there exists $y \in W_{s(x)}$ such that $s(x) < f(y)$. Therefore $s(x) \in A$.

If, on the other hand, $x \notin K$, we have that $\varphi_{s(x)}(y) = g(x,y) = \uparrow$ for each y. Hence $W_{s(x)} = \varnothing$, and therefore, certainly there is no $y \in W_{s(x)}$ such that $s(x) < f(y)$. Thus $s(x) \notin A$.

The set A is r.e., in fact

$$sc_A(x) = \mu w.(H(x,(w)_1,(w)_2) \;\wedge\; x < f((w)_1))$$

Therefore, $\bar{A}$ is not r.e.

□

Exercise 8.35. Study the recursiveness of the set $B = \{x \in \mathbb{N} : x \in E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.36. Study the recursiveness of the set $V = \{x \in \mathbb{N} : W_x \text{ infinity}\}$, i.e., establish if V and $\overline{V}$ are recursive/recursively enumerable.

Exercise 8.37. Study the recursiveness of the set $V = \{x \in \mathbb{N} : \exists y \in W_x.\ \exists k \in \mathbb{N}.\ y = k \cdot x\}$, i.e., establish if V and $\overline{V}$ are recursive/recursive enumerable.

Exercise 8.38. Study the recursiveness of the set $V = \{x \in \mathbb{N} : |W_x| > 1\}$, i.e., establish if V and $\overline{V}$ are recursive/recursive enumerable.

Exercise 8.39. Let P be the set of even numbers and Pr the set of prime numbers. Show that $P \leqslant_m Pr$ and $Pr \leqslant_m P$.

Exercise 8.40. Let $f : \mathbb{N} \to \mathbb{N}$ be a fixed total computable function. Study the recursiveness of the set $B = \{x \in \mathbb{N} \;\; f(x) \in E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: Observe that B is r.e., in fact we can write its semi-characteristic function as follows:

$$sc_B(x) = \mathbf{1}(\mu w.\,(x,(w)_1,f(x),(w)_2))$$

Moreover B is not recursive since $K \leqslant_m B$. In order to obtain the reduction function consider

$$g(x,y) = \begin{cases} y & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

Hence $\bar{B}$ is not r.e. $\square$

Exercise 8.41. Let $f : \mathbb{N} \to \mathbb{N}$ be a fixed total computable function. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ img(f) \cap E_x \neq \varnothing\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable. Please note that $img(f) = \{f(x) \mid x \in \mathbb{N}\}$.

Exercise 8.42. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ E_x \not\supseteq W_x\}$, i.e., establish if B e $\bar{B}$are recursive/recursively enumerable.

Exercise 8.43. Let $B = \{x \mid \forall m \in \mathbb{N}.\ m \cdot x \in W_x\}$. Study the recursiveness of the B set, that is to say if B and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.44. Given $A = \{x \mid \varphi_x \text{ is total}\}$, show that $\bar{K} \leqslant_m A$.

Solution: Define

$$g(x,y) = \begin{cases} y & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

By smn theorem, we obtain a total computable function $s : \mathbb{N} \to \mathbb{N}$, such that $g(x,y) = \varphi_{s(x)}(y)$ and it is easy to see that s can be the reduction function. $\square$

Exercise 8.45. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ \exists y > x.\ y \in E_x\}$, i.e., establish if$B$ and $\bar{B}$ are recursive/recursively enumerable.

Exercise 8.46. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ \forall y > x.\ 2y \in W_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: Observe that B is not r.e. since $\bar{K} \leqslant_m B$. In order to get the reduction function consider

$$g(x,y) = \begin{cases} y & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

Also $\bar{B} = \{x \mid \exists y > x.\ 2y \notin W_x\}$ is not r.e. In order to reduce $\bar{K} \leqslant_m \bar{B}$, the reduction function can be constructed from:

$$g(x,y) = \begin{cases} \uparrow & \text{if } x \notin K \\ 1 & \text{otherwise} \end{cases} = sc_K(x)$$

$\square$

Exercise 8.47. Study the recursiveness of the set $B = \{x \in N : 1 \leqslant |E_x| \leqslant 2\}$, i.e., establish if B e $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is saturated, since it can be expressed as $B = \{x : \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \in \mathcal{C} : 1 \leqslant |cod(f)| \leqslant 2\}$.

Using Rice-Shapiro's theorem, we prove that B and $\bar{B}$ are both not r.e .:

- B is not r.e .
  Note that $id \notin \mathcal{B}$ but there is a finite function
  $$\theta(x) = \begin{cases} 0 & \text{if } x = 0 \\ \uparrow & \text{otherwise} \end{cases}$$
  such that $\theta \subseteq id$ and $\theta \in \mathcal{B}$. Hence by Rice-Shapiro's theorem we conclude that B is not is r.e.

- $\bar{B}$ is not r.e .
  Note that if θ is the function defined in the previous case, $\theta \notin \overline{\mathcal{B}}$, but the function always undefined $\varnothing \in \overline{\mathcal{B}}$. By Rice-Shapiro's theorem we conclude that $\bar{B}$ is not r.e.

□

Exercise 8.48. Study the recursiveness of the set $A = \{x \in \mathbb{N} \mid \mathbb{P} \subseteq W_x\}$, i.e., establish if$A$ and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \mid \mathbb{P} \subseteq dom(f)\}$. We can use Rice-Shapiro's theorem to show that

- A is not r.e .
  In fact $id \in \mathcal{A}$ since $\mathbb{P} \subseteq dom(id) = \mathbb{N}$ and no finite $\theta \subseteq id$ can be in $\mathcal{A}$, since functions in $\mathcal{A}$ necessarily have an infinite domain.

- $\overline{A}$ not r.e .
  In fact, $id \notin \bar{\mathcal{A}}$, and $\varnothing \subseteq id$, $\varnothing \in \bar{\mathcal{A}}$.

□

Exercise 8.49. Study the recursiveness of the set $B = \{x \in \mathbb{N} \;\; \varphi_x(y) = y^2 \text{ for infinitive } y\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: We observe that B is saturated, since $B = \{x \mid \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \mid f(y) = y^2 \text{ for infinite } y\}$. Rice-Shapiro's theorem is used to deduce that both sets are not r.e.

- B is not r.e. because $\mathcal{B}$ contains y^2 and none of its sub-functions finite (it does not contain any finite functions).

- $\bar{B}$ is not r.e. since $\varnothing \in \bar{\mathcal{B}}$ and $\varnothing$ admits as an extension $y^2 \notin \bar{\mathcal{B}}$.

□

Exercise 8.50. Given $X \subseteq \mathbb{N}$, indicate by $2X$ the set $2X = \{2x : x \in X\}$. Study the recursiveness of the set $B = \{x \in \mathbb{N} \;\; 2W_x \subseteq E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: Rice-Shapiro's theorem is used to prove that both sets are not r.e .:

- B is not r.e. because it contains $\varnothing$, but not all functions (e.g. it does not contain $\theta = \{(1, 4)\}$.

- $\bar{B}$ not r.e. since it contains θ, as defined above, but not $\theta' = \{(1, 4), (2, 2)\}$.

□

Exercise 8.51. Study the recursiveness of the set $B = \{x \in \mathbb{N}\ \ W_x \supseteq Pr\}$, where $Pr \subseteq \mathbb{N}$is the set of the prime numbers, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: We use Rice-Shapiro's theorem for proving that both sets are not r.e .:

- B is not r.e. because it does not contain any finite functions and it is not empty (e.g. $id \in \mathcal{B}$, but no finite subfunction of id can be in $\mathcal{B}$).
- $\bar{B}$ is not r.e. since it contains $\varnothing$, but it does not include all functions (e.g. it does not contain id, of which $\varnothing$ is a finite subfunction).

□

Exercise 8.52. Classify the following set from the point of view of recursiveness

$$B = \{\pi(x, y)\ \ P_x \text{ stops on input } x \text{ in more than } y \text{ steps}\},$$

where $\pi : \mathbb{N}^2 \to \mathbb{N}$ is the pair encoding function, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is r.e., but not recursive. In fact

$$B = \{x : x \in K \ \wedge\ \neg H(x, x, y)\}$$

For proving that it is not is recursive, note that $K \leqslant_m B$. In fact, $x \in K$ iff $\pi(x, 0) \in B$. Furthermore, B is r.e. since its semi-characteristic function is computable:

$$sc_B(z) = sc_K(\pi_1(z)) \cdot sc_{\neg H}(\pi_1(z), \pi_1(z), \pi_2(z))$$

Thus $\bar{B}$ non-recursive. □

Exercise 8.53. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is symmetric in the interval $[0, 2k]$ if $dom(f) \supseteq [0, 2k]$ and for each $y \in [0, k]$ we have $f(y) = f(2k - y)$. Study the recursiveness of the set

$$A = \{x \in \mathbb{N} : \exists k > 0.\ \varphi_x \text{ symmetric in } [0, 2k]\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is r.e. In fact:

$$sc_A(x) = \mathbf{1}(\mu h. \forall_{y \leqslant h+1}\ \varphi_x(y) = \varphi_x(2(h + 1) - y))$$

It is not recursive by Rice's theorem. In fact, A is saturated. Moreover, if e_0 ed e_1 are indices for the functions $\varnothing$ and $\mathbf{1}$, respectively, we have that $e_0 \notin A$ and $e_1 \in A$. Hence $A \neq \varnothing, \mathbb{N}$. □

Exercise 8.54. Given $X \subseteq \mathbb{N}$ define $inc(X) = X \cup \{x + 1 : x \in X\}$. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} : inc(W_x) = E_x\}$, i.e. say if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: We have that $B = \{f \mid inc(dom(f)) = cod(f)\}$, thus the set is saturated. Furthermore $\varnothing \in \mathcal{B}$, but $\varnothing \subseteq \mathbf{1}$ and $\mathbf{1} \notin \mathcal{B}$ since $\mathbb{N} = inc(dom(\mathbf{1})) \neq cod(\mathbf{1}) = \{1\}$. Hence, by Rice-Shapiro's theorem, the set B is not r.e.

The function $\theta = \{(0,0)\} \in \bar{\mathcal{B}}$, but $\theta \subseteq id \notin \bar{\mathcal{B}}$, therefore, again by Rice-Shapiro's theorem, also $\bar{\mathcal{B}}$ is not r.e. $\square$

Exercise 8.55. Classify the following set from the point of view of recursiveness

$$B = \{x \ \ \varphi_x(0) \uparrow \ \vee \varphi_x(0) = 0\},$$

i.e., establish if B and $\bar{B}$are recursive/recursively enumerable.

Solution: Observe that B is saturated, the corresponding set of functions can be defined as $\mathcal{B} = \{f : f(0) \uparrow \ \vee f(0) = 0\}$. We have that $\mathbf{1} \notin \mathcal{B}$, while the finite subfunction $\emptyset \in \mathcal{B}$. Thus, by Rice-Shapiro's theorem, B is not r.e. Instead $\bar{B} = \{x : \varphi_x(0) \downarrow \ \wedge \varphi_x(0) \neq 0\}$ is r.e., since $sc_B(x) = \overline{sg}(\varphi_x(0))$ is computable. $\square$

Exercise 8.56. A function $f : \mathbb{N} \to \mathbb{N}$ is said *increasing* when for each $x, y \in dom(f)$, if $x < y$ then $f(x) < f(y)$. Define $B = \{x \in \mathbb{N} : \varphi_x \text{ increasing}\}$ and show that $\overline{K} \leqslant_m B$.

Solution: One can mimic the proof of Rice-Shapiro's theorem and define

$$g(x,y) = \begin{cases} y & \text{if } \neg H(x,x,y) \\ 0 & \text{otherwise} \end{cases}$$

Thus, if $x \in \bar{K}$ then g, seen as a function of y, will be the identity, which is increasing. Otherwise there exists a number of steps y such that $H(x,x,y)$ and therefore from that point onward $g(x,y)$ is constantly equal to 0 and thus not increasing.

More precisely, observe that the function $g(x,y)$ is computable, since

$$g(x,y) = y \cdot \chi_{\neg H}(x,x,y)$$

Thus, by the SMN theorem, there exists a function $s : \mathbb{N} \to \mathbb{N}$ total and computable such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y)$$

The function s is a reduction function of $\bar{K}$ into B. In fact

- If $x \in \bar{K}$ then for every $y \in \mathbb{N}$ the predicate $H(x,x,y)$ is false. Therefore $\varphi_{s(x)}(y) = g(x,y) = y$ for all $y \in \mathbb{N}$. Hence $\varphi_{s(x)}$ is increasing and therefore $s(x) \in B$.
- If $x \notin \bar{K}$ then there exists a $y \in \mathbb{N}$ such that $H(x,x,y)$ holds true, and therefore also $H(x,x,y+1)$ holds. Thus $\varphi_{s(x)}(y) = 0 = \varphi_{s(x)}(y+1)$. Then $\varphi_{s(x)}$ is not increasing and therefore $s(x) \notin B$.

Alternatively, more simply, we can be observe that the function always undefined is increasing and the constant 0 is not. So just define $g(x,y) = sc_K(x)$ (semi-characteristic function of the set K, which is known to be computable since K is r.e.) and then proceed as above. $\square$

Exercise 8.57. Say that a function $f : \mathbb{N} \to \mathbb{N}$ is k-bounded if $\forall x \in dom(f)$ we have $f(x) < k$. For each $k \in \mathbb{N}$, study the recursiveness of the set

$$A_k = \{x \in \mathbb{N} : \varphi_x \ k\text{- bounded}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set $\bar{A}_k$ is r.e. In fact:

$$sc_{\bar{A}_k}(x) = \mathbf{1}(\mu w.S(x, (w)_1, (w)_2 + k, (w)_3))$$

It is not recursive by Rice's theorem. In fact, $\bar{A}_k$ is saturated. Moreover, if e_0 and e_1 are indices for the functions $\varnothing$ and id, we have that $e_0 \notin \bar{A}_k$ and $e_1 \in \bar{A}_k$. Thus $\bar{A}_k \neq \varnothing, \mathbb{N}$ and we conclude that $\bar{A}_k$ is not recursive. Therefore A_k not r.e. □

Exercise 8.58. Classify the following set from the point of view of recursiveness $B = \{x + y : x, y \in \mathbb{N} \ \wedge \ \varphi_x(y) \uparrow\}$, i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is recursive. In fact, let z_0 be the minimum index for the function always undefined. Then, for every $z \geqslant z_0$ we can express z as $z_0 + (z - z_0)$ and we have $\varphi_{z_0}(z - z_0) \uparrow$. Hence $z \in B$. Therefore, if we denote by $\theta = \chi_{B|[0,z_0-1]}$, the finite subfunction of the characteristic function restricted to the interval $[0, z_0 - 1]$, we have

$$\chi_B(z) = \begin{cases} \theta(z) & \text{if } z < z_0 \\ 1 & \text{otherwise} \end{cases}$$

Since θ and the constant 1 are computable, and the predicate $z < z_0$ is decidable, the characteristic function is computable. □

Exercise 8.59. Let f be a total computable function. Classify the following set from the point of view of recursiveness $B_f = \{x \in \mathbb{N} \ \ \varphi_x(y) = f(y) \text{ for infinitives} y\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: Rice-Shapiro's theorem is used for both sets

- B is not r.e. because it contains f and none of its finite subfunctions (since f is total, B does not contain any finite function)
- $\bar{B}$ is not r.e. since $\varnothing \in \bar{B}$ and $\varnothing$ admits $f \notin \bar{B}$ as an extension.

□

Exercise 8.60. Let f be a total computable function, different from the identity. Classify the following set from the point of view of recursiveness $B_f = \{x \in \mathbb{N} \ \ \varphi_x = f \circ \varphi_x\}$, i.e., establish if B_f and $\bar{B}_f$ are recursive/recursively enumerable.

Solution: Observe that B_f is saturated since it can be expressed as $B_f = \{x \mid \varphi_x \in \mathcal{B}_f\}$ where $\mathcal{B}_f = \{g \mid g = f \circ g\}$.

We can use Rice-Shapiro's theorem to show that B_f is not r.e. In fact the identity $id \notin \mathcal{B}_f$ since $id \neq f = f \circ id$. Moreover the function always undefined $\varnothing \in \mathcal{B}_f$ since $\varnothing = f \circ \varnothing$ and clearly $\varnothing \subseteq id$.

Moreover, the complement $\bar{B}_f$ is r.e. In fact, let e be an index for f, i.e., such that $\varphi_e = f$. Then we have that $x \in \bar{B}_f$ iff there is an input z where $v = \varphi_x(z) \downarrow$ and $\varphi_e(v) \neq v$. Hence the semi-characteristic function of $\bar{B}_f$ can be expressed as follows:

$$sc_{B_f}(x) = \mu w.(S(x, (w)_1, (w)_2, (w)_3) \wedge S(e, (w)_2, (w)_4, (w)_3) \wedge (w)_2 \neq (w)_4)$$

□

Exercise 8.61. Study the recursiveness of the set $B = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ k \cdot x \in W_x\}$, i.e. establish whether B and $\overline{B}$ are recursive/recursively enumerable.

Solution: We show that $K \leqslant B$ and therefore B is not recursive. In fact, define

$$g(x,y) = \begin{cases} \varphi_x(x) & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

The function $g(x,y)$ is computable, since

$$g(x,y) = \psi_U(x,x)$$

Hence, by the SMN theorem, we have that there exists a function $s : \mathbb{N} \to \mathbb{N}$ total computable such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y)$$

The function s is a reduction function of K to B.

Furthermore,B is r.e., since

$$sc_B(x) = \mathbf{1}(\mu w.\ H(x,(w)_1 \cdot x,(w)_2))$$

Therefore $\bar{B}$ not r.e. □

Exercise 8.62. Classify from the point of view of recursiveness the set $B = \{x \in \mathbb{N} : \forall k \in \mathbb{N}.\ k+x \in W_x\}$, i.e., establish if B and $\overline{B}$ are recursive/recursively enumerable.

Solution: We show that $\bar{K} \leqslant B$ and therefore B is not r.e. In fact, define

$$g(x,y) = \begin{cases} 0 & \text{if } \neg H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$$

The function $g(x,y)$ is computable, since

$$g(x,y) = \mu z.\chi_H(x,x,y)$$

So by the SMN theorem, we have that there exists a function $s : \mathbb{N} \to \mathbb{N}$ total computable such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y)$$

The function s reduces K to B.

Furthermore, $\bar{B}$ not r.e., since $\bar{K} \leqslant \bar{B}$. In fact, define

$$g(x,y) = \begin{cases} 0 & x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

and proceed as before. □

Exercise 8.63. Classify from the point of view of recursiveness the set $V = \{x \in \mathbb{N} : E_x \text{ infinite}\}$, i.e., establish if V and $\overline{V}$ are recursive/recursively enumerable.

Solution: The set V is saturated since $V = \{x \mid \varphi_x \in \mathcal{A}\}$, dove $\mathcal{A} = \{f \mid cod(f) \text{ infinite}\}$. Then we can use Rice-Shapiro's theorem:

- $id \in \mathcal{A}$, but no finite subfunction of id is in $\mathcal{A}$, hence A is not r.e.;
- $\varnothing \in \bar{\mathcal{A}}$, $\varnothing \subseteq id$, but $id \notin \bar{\mathcal{A}}$, hence $\bar{A}$ is not r.e.

□

Exercise 8.64. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} \;\; x \in W_x \backslash \{0\}\}$, i.e. establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is r.e., since

$$sc_A(x) = \mathbf{1}(\mu w.\ (H(x,(w)_1,(w)_2) \wedge x \neq 0)).$$

and not recursive. In fact, $K \leqslant_m B$. In order to prove this fact consider

$$g(x,y) = \begin{cases} \varphi_x(x) & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

By the smn theorem, since the function is computable, we obtain $s : \mathbb{N} \to \mathbb{N}$, computable and total such that $\varphi_{s(x)}(y) = g(x,y)$. This is almost the reduction function, except for the fact that it might have value 0 for some input. However, it is sufficient to take an index $k \neq 0$ for the function φ_0 and consider:

$$s'(x) = \begin{cases} s(x) & \text{if } s(x) \neq 0 \\ k & \text{otherwise} \end{cases}$$

and we are done. □

Exercise 8.65. Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x \backslash E_x \text{ infinite}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ with $\mathcal{A} = \{f \mid dom(f) \backslash cod(f) \text{ infinite}\}$.
By Rice-Shapiro's theorem:

- A is not r.e., since $\mathbf{1} \in \mathcal{A}$, but no finite subfunction $\theta \subseteq \mathbf{1}$ can belong to $\mathcal{A}$. In fact $dom(\theta)$ is finite and therefore also $dom(\theta) \backslash cod(\theta)$ is finite. Therefore $\theta \notin \mathcal{A}$.
- $\bar{A}$ is not r.e., since $\varnothing \in \bar{\mathcal{A}}$, $\mathbf{1} \notin \bar{\mathcal{A}}$ and $\varnothing \subseteq \mathbf{1}$.

□

Exercise 8.66. Classify the following set from the point of view of recursiveness $B = \{x \in N : |W_x \backslash E_x| \geqslant 2\}$, i.e., establish if B e $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is saturated, since $B = \{x : \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \in \mathcal{C} : |dom(f) \backslash cod(f)| \geqslant 2\}\}$.
Using Rice-Shapiro's theorem we prove that B and $\bar{B}$ are not r.e .:

- B not r.e .
Observe that $f(x) = x - 2 \notin \mathcal{B}$ ($dom(f) = cod(f) = \mathbb{N}$, thus $dom(f) - cod(f) = \varnothing$) but there is a finite subfunction

$$\theta(x) = \begin{cases} 0 & \text{if } x \leqslant 2 \\ \uparrow & \text{otherwise} \end{cases}$$

such that $\theta \subseteq f$ and $\theta \in \mathcal{B}$. By Rice-Shapiro's theorem therefore we conclude that B is not r.e.

- $\bar{B}$ not r.e .
  Note that if θ is the function defined above, then $\theta \notin \overline{\mathcal{B}}$, but the function always undefined $\varnothing \in \overline{\mathcal{B}}$. By Rice-Shapiro's theorem therefore we conclude that $\bar{B}$ is not r.e.

$\square$

Exercise 8.67. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ \forall y \geqslant k.\ \varphi_x(y) \downarrow\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is clearly saturated since it is the set of indexes of functions in $\mathcal{B} = \{f \in \mathcal{C} \mid \exists k \in \mathbb{N}.\ \forall y \geqslant k.\ f(y) \downarrow\}$.

We can conclude that B and $\bar{B}$ are non-r.e. using Rice-Shapiro's theorem. In fact:

- B is not r.e., since $id \in \mathcal{B}$ but obviously no finite subfunction $\theta \subseteq id$ can belong to $\mathcal{B}$ (which does not contain any finite function).
- $\bar{B}$ is not r.e., since $id \notin \bar{\mathcal{B}}$, but there is a finite subfunction $\varnothing \subseteq id$ with $\varnothing \in \mathcal{B}$.

$\square$

Exercise 8.68. Classify the following set from the point of view of recursiveness $B = \{x \in \mathbb{N}\ \ x > 0\ \wedge\ x/2 \notin E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: Observe that $\bar{B}$ is r.e., in fact we can write its semi-characteristic function as follows:

$$sc_{\bar{B}}(x) = \mathbf{1}(\mu w.\, x = 0\ \vee\ S(x,(w)_1, x/2, (w)_2))$$

Moreover $\bar{B}$ is not recursive since $K \leqslant_m \bar{B}$. In order to get the reduction function consider

$$g(x,y) = \begin{cases} y & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

Then, by smn theorem, we have that $g(x,y) = \varphi_{s(x)}(y)$ for some total computable function $s : \mathbb{N} \to \mathbb{N}$. This is almost the reduction function. We need to be sure that when $x \notin K$ then $s(x)$, which is an index of the empty function, is not 0. This can be done by "modifying" function s. More precisely take any index $n_0 > 0$ such that $\varphi_{n_0} = \varnothing$ (there is such n_0 since $\varnothing$ has infinitely many indices). Then define $s'(x) = s(x)$ if $s(x) \neq 0$ and $s(x) = n_0$, otherwise. Then s' is still total and computable, and works as a reduction function.

Hence $\bar{B}$ is not r.e. $\square$

Exercise 8.69. Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} : \forall y \in W_x.\, \exists z \in W_x.\, (y < z)\ \wedge\ (\varphi_x(y) > \varphi_x(z))\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: The set B is saturated, given that $B = \{x : \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \in \mathcal{C} : \forall y \in dom(f).\, \exists z \in dom(f).\ (y < z)\ \wedge\ (f(y) > f(z))\}\}$.

For the complement $\bar{\mathcal{B}} = \{f \mid \exists y \in dom(f). \forall z > y.\ (z \notin dom(f)) \vee (f(y) \leqslant f(z))\}$, we observe that if $f \neq \varnothing$ then we can consider $y \in \{x \in dom(f) \mid f(x) = \min f(\mathbb{N})\}$ and we have that y satisfies the defining condition of $\bar{\mathcal{B}}$. So $\varnothing$ is the only function not in $\bar{\mathcal{B}}$, i.e., $\mathcal{B} = \{\varnothing\}$ and $\bar{\mathcal{B}} = \mathcal{C}\backslash\{\varnothing\}$. Then $\bar{B}$ is r.e. since its semi-characteristic function is $sc_{\bar{B}} = \mathbf{1}(\mu w. H(x,(w)_1,(w)_2))$. Using Rice's theorem, we prove $\bar{B}$ is not recursive, so B not r.e.

This last fact can be deduced by Rice-Shapiro's theorem, noting that $id \notin \mathcal{B}$ but there is a finite function $\varnothing \subseteq id$ such that $\varnothing \in \mathcal{B}$. $\square$

Exercise 8.70. Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} : \forall y \in W_x. \exists z \in W_x. (y < z) \ \wedge \ (\varphi_x(y) < \varphi_x(z))\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: The set B is saturated, given that $B = \{x : \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \in \mathcal{C} : \forall y \in dom(f). \exists z \in dom(f). (y < z) \ \wedge \ (f(y) < f(z))\}\}$.

The set B is not is r.e. by Rice-Shapiro's theorem. In fact, observe that $\mathbf{1} \notin \mathcal{B}$, but $\emptyset \subseteq \mathbf{1}$ and $\emptyset \in \mathcal{B}$.

For the complement $\bar{\mathcal{B}} = \{f \mid \exists y \in dom(f).\forall z \in dom(f).y < z \rightarrow (f(y) \geqslant f(z))\}$, observe that if θ is any finite function,$\theta \neq \emptyset$, $y = \max(dom(\theta))$ clearly satisfies the condition definition of $\bar{\mathcal{B}}$. Hence, it is enough to observe that $id \notin \bar{\mathcal{B}}$ and consider $\theta \subseteq id$,$\theta \neq \emptyset$ noting that $\theta \in \bar{\mathcal{B}}$. □

Exercise 8.71. Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x \cup E_x = \mathbb{N}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ with $\mathcal{A} = \{f \mid dom(f) \cup cod(f) = \mathbb{N}\}$.

By Rice-Shapiro's theorem:

- A is not r.e., since $id \in \mathcal{A}$, but no finite subfunction $\theta \subseteq id$ can belong to $\mathcal{A}$. In fact $dom(\theta)$ is finite and therefore also $cod(\theta)$ is finite. Hence their union $dom(\theta) \cup dom(\theta)$ is again finite, which implies that $dom(\theta) \cup dom(\theta) \neq \mathbb{N}$. Therefore $\theta \notin \mathcal{A}$.
- $\bar{A}$ is not r.e., since $\emptyset \in \bar{\mathcal{A}}$, $id \notin \bar{\mathcal{A}}$ and $\emptyset \subseteq id$.

□

Exercise 8.72. Classify the following set from the point of view of recursiveness

$$B = \{x \mid \exists k \in \mathbb{N}.\ kx \in W_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: We observe that $K \leqslant_m A$. Define

$$g(x, y) = \begin{cases} 1 & x \in K \\ \uparrow & \text{otherwise} \end{cases} = sc_K(x)$$

By smn theorem, we obtain a function $s : \mathbb{N} \rightarrow \mathbb{N}$ which is total and computable, such that $g(x, y) = \varphi_{s(x)}(y)$ and it is easy to see that s can be the reduction function.

Furthermore,A is r.e., in fact

$$sc_A(x) = \mathbf{1}(\mu w.H(x, x \cdot (w)_1, (w)_2))$$

We therefore conclude that $\bar{A}$ is not r.e. □

Exercise 8.73. Given $X, Y \subseteq \mathbb{N}$ define $X + Y = \{x + y \mid x \in X \ \wedge \ y \in Y\}$. Study the recursiveness of the set

$$B = \{x \mid x \in W_x + E_x\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: We observe that $K \leqslant_m A$. Define

$$g(x,y) = \begin{cases} 0 & x \in K \\ \uparrow & \text{otherwise} \end{cases} = \mathbf{0}(sc_K(x))$$

By smn theorem, we obtain a function $s : \mathbb{N} \to \mathbb{N}$ total computable and such that $g(x,y) = \varphi_{s(x)}(y)$. It is easy to see that s can be the reduction function.

Furthermore, B is r.e., in fact

$$sc_B(x) = \mathbf{1}(\mu w.(S((w)_1 + (w)_2, (w)_1, (w)_2, (w)_3))$$

We therefore conclude that $\bar{A}$ is not r.e. □

Exercise 8.74. Classify from the point of view of recursiveness the set $A = \{x \in \mathbb{N} : W_x \cap E_x = \mathbb{N}\}$, i.e., say if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is clearly saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ where $\mathcal{A} = \{f \mid cod(f) \cup img(f) = \mathbb{N}\}$. We can deduce, by using Rice-Shapiro's theorem, that A is not r.e., in fact $id \in \mathcal{A}$ but clearly no finite subfunction $\theta \subseteq id$ can be in $\mathcal{A}$ since $cod(f)$, $img(f)$ are finite and thus $cod(f) \cup img(f) \neq \mathbb{N}$.

The complement is not r.e. again by Rice-Shapiro's theorem. E.g., $id \notin \bar{\mathcal{A}}$, but it admits $\varnothing$ as finite subfunction and $\varnothing \in \bar{\mathcal{A}}$. □

9 Second recursion theorem

Exercise 9.1. State and prove the second recursion theorem.

Exercise 9.2. State the second recursion theorem and use it to prove that K is not is recursive.

Exercise 9.3. State the Second Recursion Theorem and use it for proving that there exists $x \in \mathbb{N}$ such that $\varphi_x(y) = y^x$, for each $y \in \mathbb{N}$.

Exercise 9.4. State the Second Recursion Theorem and use it for proving that there exists $n \in \mathbb{N}$ such that $W_n = E_n = \{x \cdot n : x \in \mathbb{N}\}$.

Exercise 9.5. State the Second Recursion Theorem and use it for proving that $x \in \mathbb{N}$exists such that $\varphi_x(y) = x + y$.

Solution: Define $h(x,y) = x + y$, which is a computable function. By smn theorem there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = h(x,y)$. The second recursion theorem provides a x_0 such that $\varphi_{x_0}(y) = \varphi_{s(x_0)}(y) = h(x_0, y) = x_0 + y$ for all $y \in \mathbb{N}$. □

Exercise 9.6. State the Second Recursion Theorem and use it for proving that there exists $x \in \mathbb{N}$ such that $\varphi_x(y) = x - y$.

Exercise 9.7. State the second recursion theorem and use it for proving that there exists a $n \in \mathbb{N}$ such that φ_n is total and $|E_n| = n$.

Solution: The second recursion theorem states that for each total computable function $h : \mathbb{N} \to \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{h(n)}$.

Consider the function $g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x, y) = rm(x, y)$$

which we know to be computable (with the convention $rm(0, y) = y$). By the smn theorem there exists $h : \mathbb{N} \to \mathbb{N}$ computable total such that $g(x, y) = \varphi_{h(x)}(y)$ and by the second recursion theorem there exists $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{h(n)}$. Therefore

$$\varphi_n(y) = \varphi_{h(n)}(y) = g(n, y) = rm(n, y)$$

If $n \neq 0$, then $E_n = [0, n)$, then $|E_n| = n$, as desired.

But if $n = 0$ things do not work, because $\varphi_n(y) = rm(0, y) = y$. This can be fixed by changing h in a way that the fixed point in 0 is removed. That is, we consider e such that $\varphi_e \neq \varphi_0$, and we define

$$h'(x) = \begin{cases} e & \text{if } x = 0 \\ h(x) & \text{otherwise} \end{cases}$$

Clearly $h'(x) = h(x) * sg(x) + e * \overline{sg}(x)$ is computable and total and then you can reapply the same reasoning first and conclude. □

Exercise 9.8. State the second recursion theorem and use it for proving that the function $\Delta : \mathbb{N} \to \mathbb{N}$, defined by $\Delta(x) = \min\{y : \varphi_y \neq \varphi_x\}$, is not computable.

Solution: Just observe that Δ is total, and by definition, for all x, it holds $\varphi_{\Delta(x)} \neq \varphi_x$. Then, by the second recursion theorem, Δ cannot be computable. □

Exercise 9.9. State the second recursion theorem and use it for proving that, if we indicate by e_0 an index of the function always undefined $\varnothing$ and by e_1 an index of the identity function, the function $h : \mathbb{N} \to \mathbb{N}$, defined by

$$h(x) = \begin{cases} e_0 & \text{if } \varphi_x \text{ is total} \\ e_1 & \text{otherwise} \end{cases}$$

is not computable.

Solution: Observe that h is total. Furthermore $\varphi_x \neq \varphi_{h(x)}$ for each x, since φ_x is total when $\varphi_{h(x)}$ is not. So, by the second recursion theorem, we deduce that h cannot be computable. □

Exercise 9.10. State the Second Recursion Theorem and use it for proving that there exists an index $x \in \mathbb{N}$ such that

$$\varphi_x(y) = \begin{cases} y^2 & \text{if } x \leqslant y \leqslant x + 2 \\ \uparrow & \text{otherwise} \end{cases}$$

Solution: Consider the function

$$f(x, y) = \begin{cases} y^2 & \text{if } x \leqslant y \leqslant x + 2 \\ \uparrow & \text{otherwise} \end{cases}$$

This is clearly computable, hence, by the smn theorem there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $f(x, y) = \varphi_{s(x)}(y)$. Applying the second recursion theorem to s we conclude. □

Exercise 9.11. State the second recursion theorem and use it for proving that the set $C = \{x : 2x \in W_x \cap E_x\}$ is not saturated.

Solution: Define

$$g(x, y) = \begin{cases} 2x & \text{if } y = 2x \\ \uparrow & \text{otherwise} \end{cases}$$

and proceed in the standard way. □

Exercise 9.12. State the second recursion theorem. Use it for proving that the set $C = \{x \in \mathbb{N} \mid x \in E_x\}$ not saturated.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$ there exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

For answering the question, define

$$g(x, y) = x$$

which is a computable function and thus, by smn theorem, there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x, y)$$

By the II recursion theorem there exists an index e such that $\varphi_{s(e)} = \varphi_e$ and then

$$\varphi_e(y) = e$$

Therefore $E_e = \{e\}$ and therefore $e \in C$.

Given any $e' \neq and$ such $\varphi_{e'} = \varphi_e$ one has that $e \notin E_{e'} = E_e$ and therefore $e \notin C$. Therefore C is not is saturated. □

Exercise 9.13. Let e_0 and e_1 be indices for the function always undefined $\varnothing$ and the constant 1, respectively. State the Second Recursion Theorem and use it to prove that the function $g : \mathbb{N} \to \mathbb{N}$ defined as below, is not computable:

$$g(x) = \begin{cases} e_0 & \varphi_x \text{ total} \\ e_1 & \text{otherwise} \end{cases}$$

Solution: The function g is clearly total. If it were computable, for the II Recursion Theorem there would exist $e \in \mathbb{N}$ such that $\varphi_e = \varphi_{g(e)}$. Instead, by definition of g we have that φ_e total iff $\varphi_{g(e)}$ is not total. □

Exercise 9.14. State the second recursion theorem. Prove that, given a function $f : \mathbb{N} \to \mathbb{N}$ total computable injective, the set $C_f = \{x : f(x) \in W_x\}$ is not saturated.

Solution: Define

$$g(x, y) = \begin{cases} f(y) & \text{if } x = f(y) \\ \uparrow & \text{otherwise} \end{cases}$$

By the smn theorem, we obtain a function $s : \mathbb{N} \to \mathbb{N}$ total computable, such that $g(x, y) = \varphi_{s(x)}(y)$ and by the second recursion theorem there exists $e \in \mathbb{N}$ such that $\varphi_e = \varphi_{s(e)}$. Therefore:

$$\varphi_e(y) = \varphi_{s(e)}(y) = g(e, y) = \begin{cases} f(e) & \text{if } x = f(e) \\ \uparrow & \text{otherwise} \end{cases}$$

Thus $e \in C_f$. Now, if we take a different index e such that $\varphi_e = \varphi_{e'}$ we will have that, by injectivity of f, it holds $f(e') \neq f(e)$ and thus $f(e') \notin W_{e'} = W_e = \{f(e)\}$. Hence $e' \notin C_f$. □

Exercise 9.15. State the second recursion theorem. Use it for proving that if C is a set such that $C \leqslant_m \overline{C}$, then C is not saturated.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$ there exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

As for the question, let $C \leqslant_m \bar{C}$ and let f be the reduction function, i.e., $f : \mathbb{N} \to \mathbb{N}$ is computable and total, and satisfies:

$$x \in C \quad \text{iff} f(x) \notin C \tag{1}$$

Since f is computable and total, by the second recursion theorem, there exists e such that

$$\varphi_e = \varphi_{f(e)}. \tag{2}$$

Now if $e \in C$, since C is saturated, from (2) we have that $f(e) \in C$ and this contradicts (1). Similarly if $e \notin C$, we get a contradiction. Thus we conclude that the reduction function cannot exist and therefore C is not saturated. □

Exercise 9.16. State the Second Recursion Theorem and use it for proving that there is an index*and* $\in \mathbb{N}$ such that

$$\varphi_e(y) = \begin{cases} y + e & \text{if } y \text{ multiple of} e \\ \uparrow & \text{otherwise} \end{cases}$$

Solution: Define

$$g(x, y) = \begin{cases} x + y & \text{if } y \text{ multiple of } x \\ \uparrow & \text{otherwise} \end{cases} = (x + y) \cdot \mathbf{1}(\mu z.|z * x - y|)$$

By smn theorem, $g(x, y) = \varphi_{s(x)}(y)$ with s computable total. Then the II recursion theorem can be used to conclude. □

Exercise 9.17. State the second recursion theorem. Use it for proving that every function f which is not total, but undefined only on a single point, i.e. $dom(f) = \mathbb{N}\backslash\{k\}$ for some $k \in \mathbb{N}$, admits a fixed point, i.e., there is $x \neq k$ such that $\varphi_x = \varphi_{f(x)}$.

Solution: Let h be such that $\varphi_h \neq \varphi_k$ and define

$$f'(x) = \begin{cases} f(x) & \text{if } x \neq k \\ h & \text{if } x = k \end{cases}$$

Clearly f' is computable (since f and the constant k are computable, and the predicate $x = k$ is decidable) and total. Therefore for the second recursion theorem there exists $x \in \mathbb{N}$ such that $\varphi_{f'(x)} = \varphi_x$. And by construction $x \neq k$, thus $f'(x) = f(x)$. □

Exercise 9.18. State the Second Recursion Theorem and use it for proving that there is $n \in \mathbb{N}$ such that $W_n = E_n = \{x \cdot n : x \in \mathbb{N}\}$.

Solution: Define

$$g(n, y) = \begin{cases} y & \text{if } y = x \cdot n \\ \uparrow & \text{otherwise} \end{cases}$$

The smn theorem and the second recursion theorem can then be used to conclude.

□

Exercise 9.19. Prove that there exists $n \in \mathbb{N}$ such that $\varphi_n = \varphi_{n+1}$ and also $m \in \mathbb{N}$ such that $\varphi_m \neq \varphi_{m+1}$.

Solution: For the first part, observe that $s(x) = x+1$ is a computable total function and therefore the smn theorem and the second recursion theorem can be used to conclude.

For the second part, if the m index did not exist, all computable functions would coincide, which is clearly not the case. □

Exercise 9.20. State the second recursion theorem. Use it for proving that the set $B = \{x \in \mathbb{N} : \exists k \in \mathbb{N}.\ k \cdot x \in W_x\}$ is not saturated.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$ there exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

Concerning the question, we proceed similarly to the proof of the fact that K is not saturated and find an index e such that $\varphi_e = \{(e, e)\}$. Also, we can assume that $e \neq 0$. In fact, define

$$g(e, x) = \begin{cases} e & \text{if } x = e \\ \uparrow & \text{otherwise} \end{cases}$$

Note that g is computable and therefore by the SMN theorem, we derive the existence of a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for each $e, x \in \mathbb{N}$

$$\varphi_{s(e)}(x) = g(e, x)$$

By the II recursion theorem, there exists an index e such that $\varphi_{s(e)} = \varphi_e$ and then

$$\varphi_e(x) = \begin{cases} e & \text{if } x = e \\ \uparrow & \text{otherwise} \end{cases}$$

We can assume $e \neq 0$ because if it were $e = 0$, it would be sufficient to consider s' such that $s'(0) = e_0$ (index of the function always undefined) and $s'(x) = s(x)$ otherwise, and apply the same reasoning again. The fixed point will certainly be $\neq 0$, since $\varphi_0 \neq \varnothing = \varphi_{e_0} = \varphi_{f(0)}$.

Now, we have that

- $e \in B$, since $e = 1 \cdot and \in W_e = \{e\}$;
- given any index $e' > e$ such that $\varphi_e = \varphi_{e'}$ (it certainly exists, since there are infinite indices for a computable function) we have that $e' \notin B$, since there cannot be a $k \in \mathbb{N}$ such that $k \cdot e' \in W_{e'} = W_e = \{e'\}$. In fact, for $k > 0$ we have that $k \cdot e' > e$ and for $k = 0$, we have $k \cdot e' = 0 \neq e$, by construction.

Thus B not saturated. □

Exercise 9.21. State the second recursion theorem. Use it for proving that the set $C = \{x \in \mathbb{N} : \varphi_x(x) = x^2\}$ is not saturated.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$, there exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

Concerning the question, as in the case of the proof for K we can find an index e such that $\varphi_e = \{(e, e^2)\}$. Then we have $e \in C$, but any other index for the same function is not in C. □

Exercise 9.22. State the second recursion theorem and use it for proving that there is an index k such that $W_k = \{k * i \mid i \in \mathbb{N}\}$.

Solution: Consider the following function

$$g(x, y) = \begin{cases} 0 & \text{if there exists } i \text{ such that } y = x * i \\ \uparrow & \text{otherwise} \end{cases} \qquad = \mu i.|x \cdot i - y|$$

It is computable, hence we can use the smn theorem and the second recursion theorem to conclude. □

Exercise 9.23. State the second recursion theorem. Use it for proving that the set $C = \{x \in \mathbb{N} : [0, x] \subseteq W_x\}$ is not saturated.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$ there exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

Concerning the question, as in the case of the proof for K we can find an index e such that $W_e = [0, e]$ and we can assume that $e \neq 0$. In fact, let us define

$$g(e, x) = \begin{cases} e & \text{if } x \leqslant e \\ \uparrow & \text{otherwise} \end{cases}$$

This is computable and therefore by SMN theorem, we derive the existence of a computable total function $s : \mathbb{N} \to \mathbb{N}$ such that for each $e, x \in \mathbb{N}$

$$\varphi_{s(e)}(x) = g(e, x)$$

By the II recursion theorem there exists an index e such that $\varphi_{s(e)} = \varphi_e$ and then

$$\varphi_e(x) = \begin{cases} e & \text{if } x \leqslant e \\ \uparrow & \text{otherwise} \end{cases}$$

Given any index $e' > e$ such that $\varphi_e = \varphi_{e'}$ (it certainly exists since there are infinite indices for a computable function) we have that $e' \notin C$, since$[0, e'] \nsubseteq [0, e] = W_{e'}$.

Thus C is not saturated. □

Exercise 9.24. State the second recursion theorem and use it for proving that there is an index $n \in \mathbb{N}$ such that $\varphi_{p_n} = \varphi_n$, where p_n is the n-th prime number.

Solution: Just observe that $f(x) = p_x$ is a computable total function and use the second recursion theorem. □

Exercise 9.25. State the second recursion theorem. Use it for proving that there is an index x such that $W_x = \{kx \mid k \in \mathbb{N}\}$.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

For the second part, define a function $g(x, y) = \mu z.|zx - y|$. Note that $dom(\lambda y.g(x, y) = \{kx \mid k \in \mathbb{N}\}$ and then we can use the second recursion theorem to conclude.

$\square$

Exercise 9.26. State the second recursion theorem. Use it for prove that there is an index $e \in \mathbb{N}$ such that $W_e = \{e^n : n \in \mathbb{N}\}$.

Solution: The Second Recursion Theorem states that given a total computable function $h : \mathbb{N} \to \mathbb{N}$ there exists $e \in \mathbb{N}$ such that $\varphi_{h(e)} = \varphi_e$.

Concerning the question, define

$$g(x, y) = \begin{cases} \log_x y & \text{if } y = x^n \text{ for some} n \\ \uparrow & \text{otherwise} \end{cases} = \mu n.\ |y - x^n|$$

It is a computable function and therefore by the smn theorem, we have that there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x, y)$$

By the II recursion theorem there exists an index e such that $\varphi_{s(e)} = \varphi_e$ and then

$$\varphi_e(y) = \begin{cases} \log_e y & \text{if } y = e^n \text{ for some} n \\ \uparrow & \text{otherwise} \end{cases}$$

Therefore $W_e = \{e^n \mid n \in \mathbb{N}\}$. $\square$

Exercise 9.27. Use the second recursion theorem to prove that the following set is not saturated

$$C = \{x \mid W_x = \mathbb{N} \ \wedge \ \varphi_x(0) = x\}.$$

Solution: Consider

$$g(x, y) = x$$

For the smn theorem there exists $s : \mathbb{N} \to \mathbb{N}$ total computable such that $g(x, y) = \varphi_{s(x)}(y)$. By the second recursion theorem there exists e such that $\varphi_e = \varphi_{s(e)}$. Therefore $\varphi_e(y) = \varphi_{s(e)}(y) = e$. In particular $\varphi_e(0) = e$ and clearly $W_e = \mathbb{N}$, then $e \in C$.

Take $e' \neq and$ such $\varphi_{e'} = \varphi_e$. Then we have that $\varphi_{e'}(0) = \varphi_e(0) = and \neq is$. So $is \notin C$.

Therefore C not saturated. $\square$