Computability
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Exercise 1

a. Provide the definition of reducibility, i.e., given sets $A, B \subseteq \mathbb{N}$ define what it means that $A \leqslant_m B$.

b. Show that if A is not recursive and $A \leqslant_m B$ then B is not recursive.

c. Show that if A is recursive then $A \leqslant_m \{1\}$.

Solution:

1. Given sets $A, B \subseteq \mathbb{N}$, we say that $A \leqslant_m B$ if there exists a total computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$, it holds $x \in A$ iff $f(x) \in B$.

2. If A is not recursive and $A \leqslant_m B$ then B is not recursive. In fact, let $f : \mathbb{N} \to \mathbb{N}$ be the reduction function. The characteristic function of A can be written as $\chi_A(x) = \chi_B(f(x))$. If B were recursive, i.e., χ_B computable, then $\chi_A = \chi_B \circ f$ would be computable, i.e., A would be recursive. Hence B is not recursive.

3. Let χ_A be the characteristic function of A, i.e.,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

   If A is recursive, by definition such function is computable and it is total. It is immediate to see that it is a reduction function for $A \leqslant_m \{1\}$ since $x \in A$ iff $\chi_A(x) = 1$ iff $\chi_A(x) \in \{1\}$.

Exercise 2

Is there a non-computable total function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = f(x+1)$ on infinitely many inputs x, i.e., such that the set $\{x \in \mathbb{N} \mid f(x) = f(x+1)\}$ is infinite? Provide an example or show that such a function cannot exist.

Solution: Yes, such a function exists. For instance one can define $f : \mathbb{N} \to \mathbb{N}$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is not a multiple of 3 or } x \notin W_x \\ \varphi_{x/3}(x) + 1 & \text{if } x \text{ is a multiple of 3 and } x \in W_x \end{cases}$$

Observe that

- The function f is total by construction.
- For all n, if we let $x = 3n + 1$, since neither x nor $x + 1$ are multiple of 3, $f(x) = f(x + 1) = 0$.
- The function is not computable, since for all $x \in \mathbb{N}$, $f \neq \varphi_x$. In fact if $\varphi_x(3x) \downarrow$ then $f(3x) = \varphi_x(3x) + 1 \neq \varphi_x(3x)$. If instead, $\varphi_x(3x) \uparrow$ then $f(3x) = 0 \neq \varphi_x(3x)$.

A more elegant, but less immediate solution is to take $f = \chi_K$, the characteristic function of the halting set K, which is total and not computable. It is true but not obvious that $\chi_k(x) = \chi_K(x + 1)$ for infinitely many x. Assume by contradiction that, instead, $D = \{x \mid \chi_K(x) = \chi_k(x + 1)\}$ is finite and let $d = \max D$. This means that for all $x > d$ it holds that $\chi_K(x) \neq \chi_K(x + 1)$ and since χ_K can assume only values 0 and 1, $\chi_K(x + 1) = \overline{sg}(\chi_K(x))$.

Now, let $v_x = \chi_K(x)$ for $x \in \{0, \ldots, d\}$. Moreover, consider the function $g : \mathbb{N} \to \mathbb{N}$ defined by primitive recursion

$$\begin{aligned} g(0) &= v_d \\ g(y + 1) &= \overline{sg}(g(y)) \end{aligned}$$

Then we have that

$$\chi_k(x) = \begin{cases} v_x & \text{if } x \leqslant d \\ f(x \mathbin{\dot{-}} (d + 1)) & \text{otherwise} \end{cases} = \Pi_{i=1}^{k}(v_i \cdot sg(|x - i|) + sg(x \mathbin{\dot{-}} d) f(x \mathbin{\dot{-}} (d + 1))$$

Exercise 3

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is quasi-total if it is undefined on a finite number of inputs, i.e., $\overline{dom(f)}$ is finite. Classify the set $A = \{x \in \mathbb{N} \mid \varphi_x \text{ quasi-total}\}$ from the point of view of recursiveness, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Solution: Observe that A is saturated, since it can be expressed as $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \mid f \text{ quasi-total}\}$.

Hence, by Rice-Shapiro's theorem, we conclude that A and $\bar{A}$ are not r.e., and thus they are not recursive. More in detail:

- A is not r.e.
  The identity $id \in \mathcal{A}$ and for all $\theta \subseteq id$, θ finite, clearly $\theta \notin \mathcal{A}$. In fact, $dom(\theta)$ is finite and thus $\overline{dom(\theta)}$ is infinite and thus θ is not quasi-total. Hence by Rice-Shapiro's theorem we conclude that A is not r.e.

- $\bar{A}$ is not r.e.
  In fact, $id \notin \bar{\mathcal{A}}$, but the always undefined function $\theta = \emptyset \subseteq id$ and $\theta \in \overline{\mathcal{A}}$, since $dom(\theta) = \emptyset$ and thus $\overline{dom(\theta)} = \mathbb{N}$ is infinite. Hence by Rice-Shapiro's theorem we conclude that $\bar{A}$ is not r.e.

Exercise 4

Classify the set $B = \{x \in \mathbb{N} \mid \exists y > 2x.\ y \in E_x\}$ from the point of view of recursiveness, i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is not recursive since $K \leqslant_m B$. In order to prove this fact, let us consider the function $g : \mathbb{N}^2 \to \mathbb{N}$ defined, by

$$g(x, y) = \begin{cases} y & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

The function is computable since $g(x, y) = sc_k(x)$. Hence, by smn-theorem, there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x, y)$ for all $x, y \in \mathbb{N}$. We next argue that s is a reduction function for $K \leqslant_m B$. In fact

- If $x \in K$ then $\varphi_{s(x)}(y) = g(x, y) = y$ for all $y \in \mathbb{N}$. Hence, if we set $y = 2\,s(x) + 1 > 2\,s(x)$ we have $\varphi_{s(x)}(y) = y = 2\,s(x) + 1$. Hence $2\,s(x) + 1 \in E_{s(x)}$ and thus $s(x) \in B$.
- If $x \notin K$ then $\varphi_{s(x)}(y) = g(x, y) =\uparrow$ for all $y \in \mathbb{N}$. Hence $E_{s(x)} = \emptyset$ and therefore there cannot be $y > 2x$ such that $yE_{s(x)}$. Hence $s(x) \notin B$.

The set B is r.e., in fact its semi-characteristic function is

$$sc_B(x) = \mathbf{1}(\mu w.(S(x, (w)_1, x + 1 + (w)_2, (w)_3)),$$

In fact the minimalisation search for a input $(w)_1$ for the machine x, such that in some number $(w)_3$ of steps, the machine stops providing as an output $x+1+(w)_2$ for some $(w)_2$. When $(w)_2$ ranges over the naturals, $x + 1 + (w)_2$ ranges over all values greater then x.

The semi-characteristic function is computable, since it is the minimalisation of computable functions, hence B is r.e.

Since B is r.e. and not recursive, its complement $\bar{B}$ is not r.e. (otherwise both B and $\overline{B}$ would be recursive). Thus $\bar{B}$ is not recursive.

Note: Each exercise contributes with the same number of points (8) to the final grade.