Computability
February 1, 2023

Exercise 1

a. Provide the definition of decidable predicate.

b. Provide the definition of semi-decidable predicate.

c. Show that if predicate $Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ is semi-decidable then also $P(\vec{x}) = \exists y.Q(\vec{x}, y)$ is semi-decidable (do not assume structure and projection theorems). Does the converse hold, i.e., is it the case that if $P(\vec{x}) = \exists y.Q(\vec{x}, y)$ is semi-decidable then $Q(\vec{x}, y)$ is semi-decidable? Provide a proof or a counterexample.

Solution:

1. A predicate $Q(\vec{x}) \subseteq \mathbb{N}^k$ is decidable if the characteristic function $\chi_Q : \mathbb{N}^k \to \mathbb{N}$ defined by

$$\chi_Q(\vec{x}) = \begin{cases} 1 & \text{if } Q(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

   is computable.

2. A predicate $Q(\vec{x}) \subseteq \mathbb{N}^k$ is semi-decidable if the semi-characteristic function $sc_Q : \mathbb{N}^k \to \mathbb{N}$ defined by

$$sc_Q(\vec{x}) = \begin{cases} 1 & \text{if } Q(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases}$$

   is computable.

3. Let $Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ be semi-decidable. Then the semi-characteristic function $sc_Q : \mathbb{N}^{k+1} \to \mathbb{N}$ is computable. Let $e \in \mathbb{N}$ be such that $sc_Q = \varphi_e^{(k+1)}$.

   Then $Q(\vec{x}, y)$ holds iff $\varphi_e^{(k+1)}(\vec{x}, y) = 1$ iff $\varphi_e^{(k+1)}(\vec{x}, y) \downarrow$ iff $\exists t.H^{(k+1)}(e, (\vec{x}, y), t)$.

   Therefore $Q(\vec{x}, y) \equiv \exists t.\, H^{(k+1)}(e, (\vec{x}, y), t)$ and thus

$$P(\vec{x}) \equiv \exists y.\, Q(\vec{x}, y) \equiv \exists y.\exists t.\, H^{(k+1)}(e, (\vec{x}, y), t) \equiv \exists w.\, H^{(k+1)}(e, (\vec{x}, (w)_1), (w)_2)$$

Therefore $sc_P(\vec{x}) = \mathbf{1}(\mu w.\ |\chi_{H^{(k+1)}}(e, (\vec{x}, (w)_1)), (w)_2) - 1|)$ is computable, and thus $P(\vec{x})$ is semi-decidable.

The converse implication does not hold. For instance, consider the predicate $Q(x, y) \equiv$ "$\phi_y(x) \uparrow$". Then $P(x) \equiv \exists y.Q(x, y) \equiv \exists y.\phi_y(x) \uparrow$ is always true, hence decidable. In fact, if e_0 is an index for the always undefined function, for $y = e_0$ clearly $Q(x, y)$ for every $x \in \mathbb{N}$. Instead $Q(x, y) = \phi_y(x) \uparrow$ is not semi-decidable (it is negation of the halting predicate, which is semi-decidable but not decidable).

Exercise 2

Give the definition of the class $\mathcal{PR}$ of primitive recursive functions. Show that the following functions are in $\mathcal{PR}$

1. $isqrt : \mathbb{N} \to \mathbb{N}$ such that $isqrt(x) = \lfloor \sqrt{x} \rfloor$;
2. $lp : \mathbb{N} \to \mathbb{N}$ such that $lp(x)$ is the largest prime divisor of x (Conventionally, $lp(0) = lp(1) = 1$.)

You can assume primitive recursiveness of the basic arithmetic functions seen in the course.

Solution:

1. The basic observation is that $isqrt(x)$ is largest y such that $y^2 \leqslant x$ and in turn this is the smallest y such that $y^2 > x$. In addition, it is immediate to realise that such a y is bounded by x, hence we get

$$isqrt(x) = \mu y < x + 1.\, ((y + 1)^2 > x) = \mu y < x + 1.\, \overline{sg}(((y + 1)^2 \dot{-} x))$$

2. Observe that, for $x > 1$, $lp(x)$ is surely smaller or equal to p_x. Hence one can count the prime divisors of x, restricting the search to $p_1, \ldots, p_x$:

$$count(x) = \Sigma_{i=1}^{x} div(p_i, x)$$

and then $lp(x) = p_{count(x)}$. The function needs to be adjusted for $x = 0$ and $x = 1$, where $count(x) = 0$ and thus $p_{count(x)} = 0$ while $lp(x) = 1$. This is easily done as follows:

$$lp(x) = p_{count(x)} + \overline{sg}(x \dot{-} 1).$$

Since we use only known primitive recursive functions, bounded sum and composition we conclude that lp is primitive recursive.

Alternatively, the idea can be to check explicitly the prime divisors of x, starting from p_x, then p_{x-1} and so on, stopping at the first. In detail, look for the smaller y, call it $i(x)$, such that p_{x-y} is a divisor of x.

$$i(x) = \mu y \leqslant x.\overline{sg}(div(p_{x-y}, x))$$

Then, whenever $x > 1$, $lp(x)$ $p_{x-i(x)}$ and the cases $x \leqslant 1$ must be treated separately as before:

$$lp(x) = p_{x-i(x)} \cdot sg(x \dot{-} 1) + \overline{sg}(x \dot{-} 1).$$

Exercise 3

Classify from the point of view of recursiveness the set $A = \{x \in \mathbb{N} \mid W_x \neq \emptyset \wedge W_x \subseteq E_x\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Solution: Observe that A is saturated, since it can be expressed as $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \mid dom(f) \neq \emptyset \ \wedge \ dom(f) \subseteq dom(f)\}$.

Hence, by Rice-Shapiro's theorem, we conclude that A and $\bar{A}$ are not r.e., and thus they are not recursive. More in detail:

- A is not r.e.
  Consider the predecessor function $pred(x) = x \dot{-} 1$. Then $pred \in \mathcal{A}$ since $dom(pred) = \mathbb{N} = cod(pred)$, hence $dom(pred) \neq \emptyset$ and $dom(pred) \subseteq cod(pred)$. Moreover, consider a generic finite $\theta \subseteq pred$. If $\theta \neq \emptyset$, i.e., θ is not the always undefined function, then it is easy to realise that $dom(\theta) \nsubseteq cod(\theta)$. In fact, if $k = \max(dom(\theta))$ necessarily $k \notin cod(\theta)$ (since $\max(cod(\theta)) = k - 1$). Hence no finite subfunction of $pred$ is in $\mathcal{A}$ and therefore, by Rice-Shapiro, A is not r.e.

- $\bar{A}$ is not r.e.
  In fact, $pred \notin \overline{\mathcal{A}}$ and $\theta = \emptyset \subseteq pred$, $\theta \in \overline{\mathcal{A}}$. Hence by Rice-Shapiro's theorem we conclude that A is not r.e.

Exercise 4

Let $f : \mathbb{N} \to \mathbb{N}$ be some fixed total computable function and for $X \subseteq \mathbb{N}$ define $f(X) = \{f(x) \mid x \in X\}$. Study the recursiveness of the set $B = \{x \mid x \in f(W_x) \cup E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Solution: The set B is not recursive because $K \leqslant B$. In order to prove this fact, let us consider the function $g : \mathbb{N}^2 \to \mathbb{N}$ defined, by

$$g(x, y) = \begin{cases} y & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

The function is computable since $g(x, y) = y \cdot sc_K(x)$. Hence, by the smn-theorem, there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x, y)$ for all $x, y \in \mathbb{N}$. We next argue that s is a reduction function for $K \leqslant_m B$. In fact

- If $x \in K$ then $\varphi_{s(x)}(y) = g(x, y) = y$ for all $y \in \mathbb{N}$. Hence $W_{s(x)} = E_{s(x)} = \mathbb{N}$ and thus $s(x) \in f(W_{s(x)}) \cup E_{s(x)} = f(\mathbb{N}) \cup \mathbb{N} = \mathbb{N}$.
- If $x \notin K$ then $\varphi_{s(x)}(y) = g(x, y) \uparrow$ for all $y \in \mathbb{N}$. Hence $W_{s(x)} = E_{s(x)} = \emptyset$ and thus $s(x) \notin f(W_{s(x)}) \cup E_{s(x)} = f(\emptyset) \cup \emptyset = \emptyset$.

The set B is r.e. In fact $x \in B$ if and only if one of the following conditions hold

- $x \in f(W_x)$, i.e.. there is $z \in W_x$ such that $f(z) = x$ or
- $x \in E_x$, i.e.. there is $z \in W_x$ such that $\varphi_x(z) = x$

Hence the semi-characteristic function of B can be written:

$$sc_B(x) = \mathbf{1}(\mu w.(H(x, (w)_1, (w)_2) \wedge (f((w)_1) = x) \ \vee \ (S(x, (w)_1, x, (w)_2)))$$

and this shows that it is computable.

Therefore, $\bar{B}$ is not r.e. (hence not recursive).

Note: Each exercise contributes with the same number of points (8) to the final grade.