Computability
February 20, 2023

Exercise 1

a. Provide the definition of a saturated (or extensional) set $A \subseteq \mathbb{N}$.

b. State the Second Recursion Theorem.

c. Show that K is not saturated.

Solution:

1. A set $A \subseteq \mathbb{N}$ is saturated whenever, if it includes the index (program) for a computable function, it includes also all the other indexes (programs) for the same function. Formally, for all $x, y \in \mathbb{N}$ if $x \in A$ and $\varphi_x = \varphi_y$ then $y \in A$.

2. The Second Recursion Theorem says that: for all functions $f : \mathbb{N} \to \mathbb{N}$, if f is total and computable then there is $e \in \mathbb{N}$ such that $\varphi_e = \varphi_{f(e)}$.

3. In order to show that K is not saturated, let us first prove that there is an index e such that

$$\varphi_e(y) = \begin{cases} 0 & \text{if } y = e \\ \uparrow & \text{otherwise} \end{cases}$$

In fact, first define $g : \mathbb{N}^2 \to \mathbb{N}$

$$g(x, y) = \begin{cases} 0 & \text{if } y = x \\ \uparrow & \text{otherwise} \end{cases} = \mu z.|y - x|$$

The function is computable, hence by smn-theorem, there exists $s : \mathbb{N} \to \mathbb{N}$ such that

$$\varphi_{s(x)}(y) = g(x, y)$$

By the Second Recursion Theorem there exists e such that $\varphi_e = \varphi_{s(e)}$ and thus:

$$\varphi_e(y) = \varphi_{s(e)}(y) = g(e, y) = \begin{cases} 0 & \text{if } y = e \\ \uparrow & \text{otherwise} \end{cases}$$

as desired.

Clearly $e \in K$ since $\varphi_e(e) = 0$. Now, just take $e' \neq e$ such that $\varphi_{e'} = \varphi_e$ (which exists since there are infinitely many indices for the same computable function). We have $\varphi_{e'}(e') = \varphi_e(e') \uparrow$ and thus $e' \notin K$.

Summing up, $e \in K$, $\varphi_{e'} = \varphi_e$ and $e' \notin K$. Hence K not saturated.

Exercise 2

State the smn-theorem. Show that there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathbb{N}$, $x > 0$ we have $W_{s(x)} = \mathbb{P}$ and $|E_{s(x)}| = 2x$.

Solution:

1. The smn-theorem says that: Given $m, n \geqslant 1$ there is a computable total function $s_{m,n} : \mathbb{N}^{m+1} \to \mathbb{N}$ such that $\forall e \in \mathbb{N},\ \vec{x} \in \mathbb{N}^m, \vec{y} \in \mathbb{N}^n$

$$\varphi_e^{(m+n)}(\vec{x}, \vec{y}) = \varphi^{(n)}_{s_{m,n}(e,\vec{x})}(\vec{y})$$

2. We define $g : \mathbb{N}^2 \to \mathbb{N}$ by

$$g(x,y) = \begin{cases} (y/2) \mod 2x & \text{if } y \in \mathbb{P} \text{ and } x > 0 \\ \uparrow & \text{otherwise} \end{cases}$$

Observe that for a fixed $x > 0$, seen as a function of y, the function g has domain $\mathbb{P}$ and codomain $\{0, 1, \ldots, 2x - 1\}$.

Clearly g is computable since

$$g(x,y) = rm(2x, qt(2,y)) + \mu z.\, (rm(2,y) + \overline{sg}(x))$$

(observe that the term $rm(2,y) + \overline{sg}(x) \neq 0$ and thus its minimalisation is undefined, if and only if y is odd or $x = 0$).

By the smn theorem there is $s : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y) = \begin{cases} (y/2) \mod 2x & \text{if } y \in \mathbb{P} \text{ and } x > 0 \\ \uparrow & \text{otherwise} \end{cases}$$

Hence, as observed above:

- $W_{s(x)} = \mathbb{P}$
- $|E_{s(x)}| = |\{y \mid y < 2x\}| = 2x$

as desired.

Exercise 3

Let $X \subseteq \mathbb{N}$ be a fixed non-empty finite set. Classify from the point of view of recursiveness the set

$$A = \{x \mid E_x \cap X \neq \emptyset\},$$

i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Solution: Observe that A is saturated, since it can be expressed as $A = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \mid cod(f) \cap A \neq \emptyset\}$.

Moreover $A \neq \emptyset, \mathbb{N}$. In fact

- if $e \in \mathbb{N}$ is such $\varphi_e = id$ then $e \in A$, since $X \cap E_e = X \cap \mathbb{N} = X \neq \emptyset$;
- if $e' \in \mathbb{N}$ is such $\varphi_{e'} = \emptyset$ then $e' \notin A$, since $X \cap E_{e'} = X \cap \emptyset = \emptyset$.

Hence by Rice's theorem A is not recursive.

The set A is r.e. In fact $x \in A$ if and only if there is exists an input $y \in \mathbb{N}$ such that $\varphi_x(y) \downarrow$ and $\varphi_x(y) \in X$. The latter condition can be easily checked since X is finite and thus recursive. Hence we can just search for such an input.

Formally the semi-characteristic function of A can be written as:

$$\begin{aligned} sc_A(x) &= \mathbf{1}(\mu w.(S(x,(w)_1,(w)_2,(w)_3) \wedge (w)_2 \in Y)) \\ &= \mathbf{1}(\mu w.(|\chi_S(x,(w)_1,(w)_2,(w)_3) * \chi_Y((w)_2) - 1|)) \end{aligned}$$

and, since S is decidable and X is recursive (since it is finite), this shows that sc_A is computable.

Therefore, $\bar{A}$ is not r.e. (hence not recursive).

Exercise 4

Classify from the point of view of recursiveness the set

$$B = \{x \in \mathbb{N} \mid W_x \neq \emptyset \wedge \min(W_x) > 0\},$$

i.e., establish whether B and $\bar{B}$ are recursive/recursively enumerable.

Solution: Observe that B is saturated, since it can be expressed as $B = \{x \in \mathbb{N} \mid \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \mid dom(f) \neq \emptyset \ \wedge \ \min(dom(f)) > 0\}$.

Hence, by Rice-Shapiro's theorem, we conclude that B and $\bar{B}$ are not r.e., and thus they are not recursive. More in detail:

- B is not r.e.
  Consider the identity function $id(x) = x$. Then $id \notin \mathcal{B}$ since $dom(id) = \mathbb{N}$, hence $\min(dom(id)) = 0$. Moreover, consider the finite function $\theta : \mathbb{N} \to \mathbb{N}$ defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

Clearly $\theta \subseteq id$ and $\min(dom(\theta)) = \min(\{1\}) = 1 > 0$ hence $\theta \in \mathcal{B}$. Therefore, by Rice-Shapiro, B is not r.e.

- $\bar{B}$ is not r.e.
  In fact, if θ is the function defined above, $\theta \notin \overline{\mathcal{B}}$. Moreover $\theta' = \emptyset \subseteq \theta$, $\theta' \in \overline{\mathcal{B}}$. Hence by Rice-Shapiro's theorem we conclude that B is not r.e.

Note: Each exercise contributes with the same number of points (8) to the final grade.