Computability
June 17, 2022

Exercise 1

a. Provide the definition of a decidable predicate.

b. Provide the definition of a semi-decidable predicate.

c. Show that $P(\vec{x})$ is semi-decidable if and only if there exists a decidable predicate $Q(\vec{x}, y)$ such that $P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$.

Solution:

1. A predicated $P(\vec{x}) \subseteq \mathbb{N}^k$ is decidable if the characteristic function $\chi_P : \mathbb{N}^k \to \mathbb{N}$ defined by

$$\chi_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

is computable.

2. A predicated $P(\vec{x}) \subseteq \mathbb{N}^k$ is decidable if the characteristic function $\chi_P : \mathbb{N}^k \to \mathbb{N}$ defined by

$$sc_P(\vec{x}) = \begin{cases} 1 & \text{if } P(\vec{x}) \\ \uparrow & \text{otherwise} \end{cases}$$

is computable.

3. Assume that $P(\vec{x})$ is semi-decidable. Then $\chi_P(\vec{x})$ is computable. Let $e \in \mathbb{N}$ be an index for such function, i.e., $\chi_P = \varphi_e^{(k)}$. Then we have that $P(\vec{x})$ holds iff $sc_P(\vec{x}) = 1$ iff $sc_P(\vec{x}) \downarrow$ iff $\exists y.\ H^{(k)}(e, \vec{x}, y)$. Hence, if we let $Q(\vec{x}, y) = H^{(k)}(e, \vec{x}, y)$, we have

$$P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$$

and $Q(\vec{x}, y)$ is decidable, since $H^{(k)}(e, \vec{x}, y)$ is so.

Assume now that $P(\vec{x}) \equiv \exists y. Q(\vec{x}, y)$ with $Q(\vec{x}, y)$ decidable. Let $\chi_Q : \mathbb{N}^{k+1} \to \mathbb{N}$ be its computable characteristic function.

Then $sc_A(\vec{x}) = \mu y. |\chi_Q(\vec{x}, y) - 1|$ is computable, i.e., $P(\vec{x})$ is semi-decidable.

$$1 \dot{-} sc_{\bar{A}}(x) = \begin{cases} 0 & \text{if } x \in A \\ \uparrow & \text{otherwise} \end{cases}$$

is computable. Let $e_1, e_0 \in \mathbb{N}$ be such that $\varphi_{e_1} = sc_A$ and $\varphi_{e_0} = 1 \dot{-} sc_{\bar{A}}$. Then $\chi_A(x) = (\mu w.S(e_0, x, (w)_1, (w)_2) \vee S(e_1, x, (w)_1, (w)_2))_1$. This is computable and thus A is recursive.

Exercise 2

Define the class of primitive recursive functions. Using only the definition show that the function $f : \mathbb{N} \to \mathbb{N}$ defined below is primitive recursive

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd} \end{cases}$$

Solution: The class of primitive recursive functions is the least class of functions $\mathcal{PR} \subseteq \bigcup_k (\mathbb{N}^k \to \mathbb{N})$ containing the base functions (zero, successor, projections) and and closed under composition and primitive recursion.

In order to show that f is primitive recursive observe that it can be defined as

$$\begin{cases} f(0) & = 1 = succ(0) \\ f(y+1) & = \overline{sg}(y) \end{cases}$$

where $\overline{sg}$ is the complemented sign, which can be defined as

$$\begin{cases} \overline{sg}(0) & = 1 = succ(0) \\ \overline{sg}(y+1) & = 0 \end{cases}$$

Exercise 3

Study the recursiveness of the set $A = \{x \mid \varphi_x(y + x) \downarrow \text{ for some } y \geqslant 0\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set $A = \{x \mid \varphi_x(y + x) \downarrow \text{ for some } y \geqslant 0\}$ is not recursive because $K \leqslant A$. In order to prove this fact, let us consider the function $g : \mathbb{N}^2 \to \mathbb{N}$ defined, by

$$g(x, y) = \begin{cases} 1 & \text{if } x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

The function is computable since $g(x, y) = sc_K(x)$. Hence, by the smn-theorem, there is a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $\varphi_{s(x)}(y) = g(x, y)$ for all $x, y \in \mathbb{N}$. We next argue that s is a reduction function for $K \leqslant_m A$. In fact

- If $x \in K$ then $\varphi_{s(x)}(y) = g(x, y) = 1$ for all $y \in \mathbb{N}$. In particular, $\varphi_{s(x)}(0 + s(x)) \downarrow$. Hence $s(x) \in A$.

- If $x \notin K$ then $\varphi_{s(x)}(y) = g(x, y) \uparrow$ for all $y \in \mathbb{N}$. Hence $\varphi_{s(x)}(y + s(x)) \uparrow$ for all $y \in \mathbb{N}$. Hence $s(x) \notin A$.

The set A is r.e., since it semi-characteristic function

$$sc_A(x) = \mathbf{1}(\mu(y, t).H(x, x + y, t))$$

is computable.

Therefore, $\bar{A}$ is not r.e.

Exercise 4

Let $A = \{x \in \mathbb{N} : W_x \cap E_x \neq \emptyset\}$. Study the recursiveness of A, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated, since $A = \{x : \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \in \mathcal{C} : dom(f) \cap cod(f) \neq \emptyset\}$. It is not empty (since $\mathbf{1} \in \mathcal{A}$) and it is not the entire $\mathbb{N}$ (since $\emptyset \notin \mathcal{A}$), thus by Rice's theorem A is not recursive. Furthermore,A is r.e. since

$$\begin{aligned} sc_A(x) &= \mathbf{1}(\mu(y, z, t).H(x, y, t) \wedge S(x, z, y, t)) \\ &= \mathbf{1}(\mu w.H(x, (w)_1, (w)_3) \wedge S(x, (w)_2, (w)_1, (w)_3) \end{aligned}$$

Therefore $\bar{A}$ is not is r.e.

Note: Each exercise contributes with the same number of points (8) to the final grade.