Computability
February 1, 2023

Exercise 1

a. Provide the definition of decidable predicate.

b. Provide the definition of semi-decidable predicate.

c. Show that if predicate $Q(\vec{x}, y) \subseteq \mathbb{N}^{k+1}$ is semi-decidable then also $P(\vec{x}) = \exists y.Q(\vec{x}, y)$ is semi-decidable (do not assume structure and projection theorems). Does the converse hold, i.e., is it the case that if $P(\vec{x}) = \exists y.Q(\vec{x}, y)$ is semi-decidable then $Q(\vec{x}, y)$ is semi-decidable? Provide a proof or a counterexample.

Exercise 2

Give the definition of the class $\mathcal{PR}$ of primitive recursive functions. Show that the following functions are in $\mathcal{PR}$

1. $isqrt : \mathbb{N} \to \mathbb{N}$ such that $isqrt(x) = \lfloor\sqrt{x}\rfloor$;
2. $lp : \mathbb{N} \to \mathbb{N}$ such that $lp(x)$ is the largest prime divisor of x (Conventionally, $lp(0) = lp(1) = 1$.)

You can assume primitive recursiveness of the basic arithmetic functions seen in the course.

Exercise 3

Classify from the point of view of recursiveness the set $A = \{x \in \mathbb{N} \mid W_x \neq \emptyset \wedge W_x \subseteq E_x\}$, i.e., establish whether A and $\bar{A}$ are recursive/recursively enumerable.

Exercise 4

Let $f : \mathbb{N} \to \mathbb{N}$ be some fixed total computable function and for $X \subseteq \mathbb{N}$ define $f(X) = \{f(x) \mid x \in X\}$. Study the recursiveness of the set $B = \{x \mid x \in f(W_x) \cup E_x\}$, i.e., establish if B and $\bar{B}$ are recursive/recursively enumerable.

Note: Each exercise contributes with the same number of points (8) to the final grade.