Scritto zero

E.1:

Media e varianza per una v.a. reale X

Ricorda: $var(X) = E[X^2] - (E[X])^2$

(i) $P(X=0) = \frac{1}{6}$, $P(X=1) = \frac{1}{3}$, $P(X=2) = \frac{1}{2}$

$$\leadsto \quad E[X] = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} = \underline{\underline{\frac{4}{3}}}$$

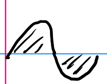
$$E[X^2] = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{3} + 4 \cdot \frac{1}{2} = \frac{7}{3}$$

$$\leadsto \quad var(X) = \frac{7}{3} - \frac{16}{9} = \underline{\underline{\frac{5}{9}}}$$

(ii) $X = \sin(2\pi U)$ con $U \sim Unif(0,1)$

$$\leadsto \quad E[X] = E[\sin(2\pi U)] \qquad | \text{ formula per il valor atteso}$$

$$= \int_{-\infty}^{\infty} \sin(2\pi x) \cdot f_U(x)\, dx \qquad | \; f_U = \mathbb{1}_{(0,1)}$$



$$= \int_0^1 \sin(2\pi x)\, dx$$

$$= \left[-\frac{1}{2\pi} \cos(2\pi x) \right]_{x=0}^{x=1} = 0$$

$$E[X^2] = E[\sin^2(2\pi U)] \overset{\text{come sopra}}{=} \int_0^1 \sin^2(2\pi x)\, dx$$

Ricorda: $\sin^2(x) + \cos^2(x) = 1 \quad \forall x \in \mathbb{R}$

Inoltre, $\int_0^1 \cos^2(2\pi x)\,dx = \int_0^1 \sin^2(2\pi x)\,dx$

$$\left[\begin{array}{lll} \text{Nota:} & \cos(2\pi(x+k)) = \cos(2\pi x), & \forall k \in \mathbb{Z} \\ & \sin(2\pi(x+k)) = \sin(2\pi x) & \\ & & \\ & \sin(x) = \cos(x - \frac{\pi}{2}) & \end{array}\right]$$

$$\int_0^1 \sin^2(2\pi x)\,dx = \int_0^1 (1 - \cos^2(2\pi x))\,dx$$

$$= 1 - \underbrace{\int_0^1 \cos^2(2\pi x)\,dx}_{= \int_0^1 \sin^2(2\pi x)\,dx}$$

$$\rightsquigarrow \quad 2 \cdot \int_0^1 \sin^2(2\pi x)\,dx = 1$$

$$\rightsquigarrow \quad \int_0^1 \sin^2(2\pi x)\,dx = \frac{1}{2}.$$

$$\rightsquigarrow \quad E[X] = 0, \qquad var(X) = E[X^2] = \frac{1}{2}.$$

(iii) $\quad X = e^Y \quad$ con $\quad Y \sim Exp(3)$

$$\rightsquigarrow \quad E[X] = E[e^Y] = \int_{-\infty}^{\infty} e^x \cdot f_Y(x)\,dx$$

$$\mid f_Y(x) = 3 \cdot e^{-3x} \cdot \mathbb{1}_{(0,\infty)}(x)$$

$$= \int_0^\infty e^x \cdot 3 \cdot e^{-3x}\,dx = 3\int_0^\infty e^{-2x}\,dx = \left[-\frac{3}{2} e^{-2x}\right]_{x=0}^{x=\infty} = \underline{\underline{\frac{3}{2}}}.$$

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Formula: Se X è una v.a. assolut. continua con densità f_X,

allora $E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$.

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(iii) cont.: $X = e^Y$

$E[X^2] = E[(e^Y)^2] = E[e^{2Y}]$ | formula

$= \int_{-\infty}^{\infty} e^{2x} \cdot f_Y(x) dx$

$\overset{\text{qui}}{=} \int_{0}^{\infty} e^{2x} \cdot 3 \cdot e^{-3x} dx$

$= 3 \int_{0}^{\infty} e^{-x} dx = \left[-3e^{-x}\right]_{x=0}^{x=\infty} = 3$

$\rightsquigarrow$ $var(X) = E[X^2] - E[X]^2 = 3 - \frac{9}{4} = \underline{\underline{\frac{3}{4}}}$.

E2:

Siano ξ_1, ξ_2, ξ_3 v.a. i.i.d. (indipendenti) con $P(\xi_i = 1) = \frac{1}{2} = P(\xi_i = -1)$.

Poniamo $X \doteq \xi_1 \cdot \xi_2$,

$Y \doteq \xi_1 \cdot (\xi_2 - \xi_3)$.

(i) $E[X] = E[\xi_1 \cdot \xi_2] \overset{\text{indip.}}{=} E[\xi_1] \cdot E[\xi_2]$

Ora, $E[\xi_i] = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0$.

$\rightsquigarrow$ $\underline{\underline{E[X] = 0}}$.

$\rightsquigarrow$ $var(X) = E[X^2] = E[\xi_1^2 \cdot \xi_2^2]$ $\quad | \; \xi_i^2 = 1$ P-q.c.

$= \underline{\underline{1}}$

$E[Y] = E[\xi_1 \cdot (\xi_2 - \xi_3)] \overset{\text{indip.}}{=} \underbrace{E[\xi_1]}_{=0} \cdot E[\xi_2 - \xi_3] = \underline{\underline{0}}$

$\rightsquigarrow$ $var(Y) = E[Y^2] = E[\xi_1^2 \cdot (\xi_2 - \xi_3)^2]$ $\quad | \; \xi_i^2 = 1$ P-q.c.

$= E[(\xi_2 - \xi_3)^2]$

$= E[\xi_2^2] - 2\underbrace{E[\xi_2 \cdot \xi_3]}_{\overset{\text{indip.}}{=} E[\xi_2] \cdot E[\xi_3] = 0} + E[\xi_3^2] = \underline{\underline{2}}$.

$(\bar{m})$ covarianza tra X e Y:

$$\mathrm{cov}(X,Y) = E[(X-E[X])\cdot(Y-E[Y])]$$

| qui: $E[X] = 0 = E[Y]$

$$\rightsquigarrow \quad \mathrm{cov}(X,Y) = E[X\cdot Y]$$

$$= E[\xi_1\cdot\xi_2\cdot\xi_1\cdot(\xi_2-\xi_3)]$$

$$= E[\xi_1^2\cdot\xi_2\cdot(\xi_2-\xi_3)] \qquad | \ \xi_1^2 = 1 \text{ P-q.c.}$$

$$= E[\xi_2\cdot(\xi_2-\xi_3)] = E[\xi_2^2 - \xi_2\cdot\xi_3]$$

$$= 1 - \underbrace{E[\xi_2\cdot\xi_3]}_{\overset{\text{indip}}{=} E[\xi_2]\cdot E[\xi_3] = 0}$$

$$\rightsquigarrow \quad \mathrm{cov}(X,Y) = \underline{\underline{1}}$$

$\rightsquigarrow$ X, Y <u>non</u> indipendenti.

$$Y = c\cdot X \qquad E[X]=0$$

$$\rightsquigarrow \quad \mathrm{cov}(X,Y) = \begin{cases} >0 & \text{se } c>0 \\ 0 & \text{se } c=0 \\ <0 & \text{se } c<0 \end{cases}$$

(iii) distribuzione congiunta di X e Y:

X, Y sono v.a. discrete, quindi basta calcolare la densità congiunta.

$X = \xi_1 \cdot \xi_2$

$Y = \xi_1 \cdot (\xi_2 - \xi_3)$

$$p_{(X,Y)}(x,y) = P(X=x, Y=y).$$

Nota: X a valori in $\{-1, 1\}$, (P-q.c.)
Y " " " $\{-2, 0, 2\}$

$\leadsto$ $P(X=1, Y=-2) = 0,$

$P(X=-1, Y=2) = 0,$

$$P(X=1, Y=0) = \underbrace{P(\xi_1 = \xi_2 = \xi_3)}_{} = \frac{2}{8} = \frac{1}{4},$$

$$= P(\xi_1 = 1, \xi_2 = 1, \xi_3 = 1) + P(\xi_1 = -1, \xi_2 = -1, \xi_3 = -1)$$

$$P(X=-1, Y=0) = P(\xi_1 = -\xi_2, \xi_2 = \xi_3) = \frac{2}{8} = \frac{1}{4}$$

$$P(X=1, Y=2) = P(\xi_1 = \xi_2, \xi_2 = -\xi_3) = \frac{1}{4},$$

$$P(X=-1, Y=-2) = P(\xi_1 = -\xi_2, \xi_2 = -\xi_3) = \frac{1}{4}.$$

E.3:

Siano $X_1, \dots, X_{1000}$ v.a. i.i.d. con $X_i \sim Ber(\frac{1}{500})$

$$S \doteq \sum_{i=1}^{1000} X_i$$

Dare una stima per $N_* \doteq \min\{k \in \mathbb{N} : P(S \leq k) \geq 0{,}98\}$.

a) con Chebyshev:

Nota: $E[X_i] = \frac{1}{500}$, $var(X_i) = \frac{1}{500} \cdot \left(\frac{499}{500}\right)$

$\rightsquigarrow$ $E[S] = 1000 \cdot \frac{1}{500} = 2$,

$$var(S) \underset{\substack{\uparrow \\ \text{indip.}}}{=} 1000 \cdot var(X_i) = \frac{499}{250}$$

Per $k \in \mathbb{N}$: $P(S \leq k) = 1 - P(S > k)$

Ora $P(S > k) = P(S \geq k+1)$ | S a valori in $\mathbb{N}_0$

$= P(S - E[S] \geq k+1-E[S])$ | $E[S] = 2$

$= P(S - E[S] \geq k-1)$

$\leq P(|S - E[S]| \geq k-1)$ | se $k \geq 2$

$$\overset{\text{Chebyshev}}{\leq} \frac{var(S)}{(k-1)^2} = \frac{499}{250} \cdot \frac{1}{(k-1)^2}$$

$K \geq 2$

$\leadsto \quad P(S \leq K) = 1 - P(S > K)$

$$\geq 1 - \frac{499}{250} \cdot \frac{1}{(K-1)^2}$$

Scegliere $K \in \mathbb{N}$ minimo tale che

$$1 - \frac{499}{250} \cdot \frac{1}{(K-1)^2} \geq \frac{49}{50} = 0{,}98$$

$\leadsto \quad \frac{1}{(K-1)^2} \cdot \frac{499}{250} \leq \frac{1}{50}$

$\leadsto \quad (K-1)^2 \geq \frac{499}{5}$

$\leadsto \quad K \geq \sqrt{\frac{499}{5}} + 1$

$\leadsto$ stima per N_* : $\quad N_* = 11.$

b) approssimazione di Poisson :

Nota : $\quad S \sim Bin(1000, \frac{1}{500})$

$\leadsto$ distribuzione di S è vicina alla distribuzione di Poisson di parametro

$$\lambda = 1000 \cdot \frac{1}{500} = 2.$$

$\rightsquigarrow \quad P(S \leq k) \approx F_{Poiss(2)}(k)$

Scegliere $k \in \mathbb{N}$ minimo tale che

$$F_{Poiss(2)}(k) \geq 0{,}98 .$$

tavola
$\rightsquigarrow \quad k \geq 5$

$\rightsquigarrow$ stima per N_*: $N_* = 5$.

c) approssimazione normale:

Ricorda: $E[S] = 2, \quad var(S) = \frac{499}{250}$

$$S = \sum_{i=1}^{1000} X_i \quad , \quad X_i \text{ i.i.d.},$$

$$E[X_i] = \frac{1}{500}, \quad var(X_i) = \frac{1}{500} \cdot \frac{499}{500}$$

$$\bar{S} \doteq \frac{1}{\sqrt{var(S)}}(S - E[S]) = \frac{1}{\sqrt{1000 \cdot var(X_i)}} \cdot \sum_{i=1}^{1000}(X_i - E[X_i])$$

[Nota: $E[\bar{S}] = 0$ e $var(\bar{S}) = 1$]

teorema del limite centrale $\rightsquigarrow$ distribuzione di $\bar{S}$ vicina alla normale standard.

Per $K \in \mathbb{N}$:

$$P(S \leq K) = P(S - E[S] \leq K - E[S])$$

$$= P\Big(\underbrace{\frac{1}{\sqrt{Var(S)}}(S - E[S])}_{= \bar{S}} \leq \frac{K - E[S]}{\sqrt{Var(S)}}\Big)$$

$$\approx \Phi\Big(\frac{K - E[S]}{\sqrt{Var(S)}}\Big)$$

$\uparrow$ funzione di ripartizione della normale standard

Cerchiamo $y \in \mathbb{R}$ minimo tale

$$\Phi(y) \geq 0{,}98$$

$\overset{\text{tavola}}{\rightsquigarrow}$ $y \geq 2{,}06$

Φ crescente

$\rightsquigarrow$ Scegliere $K \in \mathbb{N}$ minimo tale che

$$\frac{K - E[S]}{\sqrt{Var(S)}} \geq 2{,}06$$

| $E[S] = 2$, $\sqrt{Var(S)} = \sqrt{\frac{499}{250}}$

$\rightsquigarrow$ $K \geq \sqrt{\frac{499}{250}} \cdot 2{,}06 + 2$

$\rightsquigarrow$ stima per N_* : $N_* = 5$.

E. 4

Siano X_1, X_2, X_3 v.a. a valori in $\{1, \dots, 6\}$ indipendenti.

Scriviamo $X_i \succ X_j$ se $P(X_i > X_j) > P(X_j > X_i)$.

Trovare distribuzioni marginali per X_1, X_2, X_3 tali che

$$X_1 \succ X_2, \quad X_2 \succ X_3, \quad X_3 \succ X_1.$$

Ansatz: $X_2 \equiv 3$, X_1 a valori in $\{1,3,4\}$,
X_3 a valori in $\{2,3,5\}$.

- Per $X_1 \succ X_2$ dobbiamo avere $P(X_1 = 4) > P(X_1 = 2)$.

Infatti: $P(X_1 > X_2) = P(X_1 > 3) = P(X_1 = 4)$,

$P(X_1 < X_2) = P(X_1 < 3) = P(X_2 = 2)$.

- Per $X_2 \succ X_3$ dobbiamo avere $P(X_3 = 2) > P(X_3 = 5)$.

Infatti: $P(X_2 > X_3) = P(3 > X_3) = P(X_3 = 1)$,

$P(X_2 < X_3) = P(3 < X_3) = P(X_3 = 5)$.

(E.4 cont.)

X_3 a valori in $\{2,3,5\}$

- $$P(X_3 > X_1) = \sum_{z \in \{2,3,5\}} P(z > X_1 \mid X_3 = z) \cdot P(X_3 = z)$$

$$\overset{\text{indip.}}{=} \sum_{z \in \{2,3,5\}} P(z > X_1) \cdot P(X_3 = z) \qquad | X_1, X_3 \text{ indip.}$$

$$= P(X_3 = 5) + P(X_1 = 1) \cdot (\underbrace{P(X_3 = 3) + P(X_3 = 2)}_{= 1 - P(X_3 = 5)}) \qquad | X_1 \text{ a valori in } \{1,3,4\}$$

come sopra

$$P(X_3 < X_1) = \sum_{z \in \{2,3,5\}} P(z < X_1) \cdot P(X_3 = z)$$

$$= P(X_1 = 4) \cdot P(X_3 = 3) + (\underbrace{P(X_1 = 4) + P(X_1 = 3)}_{= 1 - P(X_1 = 1)}) \cdot P(X_3 = 2)$$

Per $X_3 \succ X_1$ dobbiamo quindi avere

$$P(X_3 = 5) + P(X_1 = 1) \cdot (1 - P(X_3 = 5)) > P(X_1 = 4) \cdot P(X_3 = 3) + (1 - P(X_1 = 1)) \cdot P(X_3 = 2)$$

Tutti i vincoli soddisfatti se, ad esempio,

$P(X_2 = 3) = 1$,

$P(X_1 = 1) = \frac{1}{8}$, $P(X_1 = 3) = \frac{5}{8}$, $P(X_1 = 4) = \frac{1}{4}$

$P(X_3 = 2) = \frac{1}{2}$, $P(X_3 = 3) = \frac{1}{16}$, $P(X_3 = 5) = \frac{7}{16}$

Media e varianza di una poissoniana:

Sia $X \sim Poiss(\lambda)$

$\rightsquigarrow$ $P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k \in \mathbb{N}_0$

$$\rightsquigarrow \quad E[X] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \left(\sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} \right)$$

$$= \lambda \cdot e^{-\lambda} \left(\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right)$$

$$= \lambda \cdot e^{-\lambda} \underbrace{\left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)}_{= e^{\lambda}} = \lambda$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \left(\sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{(k-1)!} \right)$$

$$= e^{-\lambda} \cdot \lambda \cdot \underbrace{\left(\sum_{k=1}^{\infty} k \cdot \frac{\lambda^{k-1}}{(k-1)!} \right)}_{= e^{\lambda}(1+\lambda)}$$

grazie al trucco

$$= \lambda(1+\lambda)$$

$$\rightsquigarrow \quad var(X) = E[X^2] - E[X]^2 = \lambda(1+\lambda) - \lambda^2$$

$$= \lambda.$$

Trucco: $\quad f_K(\lambda) \doteq \dfrac{\lambda^K}{(K-1)!}$

$$\rightsquigarrow \quad \frac{d}{d\lambda} f_K(\lambda) = K \cdot \frac{\lambda^{K-1}}{(K-1)!}$$

$$\sum_{K=1}^{\infty} f_K(\lambda) = \sum_{K=1}^{\infty} \frac{\lambda^K}{(K-1)!} = \lambda \cdot \left(\sum_{K=0}^{\infty} \frac{\lambda^K}{K!} \right)$$

$$= \lambda \cdot e^{\lambda}$$

$$\rightsquigarrow \quad \frac{d}{d\lambda} \left(\lambda \cdot e^{\lambda} \right) = e^{\lambda} + \lambda \cdot e^{\lambda}$$

dall'altra parte, $\quad \dfrac{d}{d\lambda} \left(\lambda \cdot e^{\lambda} \right) = \displaystyle\sum_{K=1}^{\infty} \dfrac{d}{d\lambda} f_K(\lambda)$

$$= \sum_{K=1}^{\infty} K \cdot \frac{\lambda^{K-1}}{(K-1)!}$$

$$\rightsquigarrow \quad \sum_{K=1}^{\infty} K \cdot \frac{\lambda^{K-1}}{(K-1)!} = e^{\lambda} \left(1 + \lambda \right)$$