Scritto del 20 luglio 2022

E. 1

Media e varianza di X

(i) X v.a. discreta con

$P(X=-3)=\frac{1}{3}$, $P(X=-1)=\frac{1}{6}$, $P(X=2)=\frac{1}{6}$,

$P(X=6)=\frac{1}{3}$.

$$\Rightarrow \quad E[X] = (-3)\cdot\frac{1}{3} + (-1)\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 6\cdot\frac{1}{3}$$

$$= \frac{-6-1+2+12}{6} = \frac{7}{6}$$

$$E[X^2] = (-3)^2\cdot\frac{1}{3} + (-1)^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{6} + 6^2\cdot\frac{1}{3}$$

$$= \frac{18+1+4+72}{6} = \frac{95}{6}$$

$$\Rightarrow \quad var(X) = E[X^2] - E[X]^2 = \frac{95}{6} - \frac{49}{36} = \frac{521}{36}.$$

(a) $X = U^2$ con $U \sim Unif(0,1)$

Nota: U assol. continua con densità $f_U(x) = 1_{(0,1)}(x)$, $x \in \mathbb{R}$

trasformazione

$$\leadsto E[X] = E[U^2] = \int_{-\infty}^{\infty} x^2 \cdot f_U(x)\,dx$$

$$= \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \underline{\underline{\frac{1}{3}}}. \quad (*)$$

$$E[X^2] = E[U^4] = \int_{-\infty}^{\infty} x^4 \cdot f_U(x)\,dx$$

$$= \int_0^1 x^4\,dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}.$$

$$\leadsto var(X) = E[X^2] - E[X]^2 = \frac{1}{5} - \frac{1}{9}$$

$$= \frac{9-5}{45} = \underline{\underline{\frac{4}{45}}}.$$

(*) Alternativa: $Z \sim Unif(a,b)$

$$\leadsto E[Z] = \frac{b+a}{2}, \quad var(Z) = \frac{(b-a)^2}{12} \leadsto E[Z^2] = \frac{(b-a)^2}{12} + \frac{(b+a)^2}{4}$$

$$= \frac{b^2+ab+a^2}{3} \quad \overset{\text{quindi}}{\leadsto} E[U^2] = \frac{1}{3}.$$

E.1

(iii) $X = 1 - Y$ con $Y \sim Exp(2)$.

Nota: Y assol. continua con densità

$$f_Y(x) = 2 \cdot e^{-2x} \cdot 1_{(0,\infty)}(x), \quad x \in \mathbb{R}.$$

trasformazione $\rightsquigarrow$

$$E[X] = E[1-Y] = \int_{-\infty}^{\infty} (1-x) \cdot f_Y(x)\, dx$$

$$= 2 \int_0^{\infty} (1-x) \cdot e^{-2x}\, dx$$

$$= \underbrace{2 \int_0^{\infty} e^{-2x}\, dx}_{= \left[-e^{-2x}\right]_0^{\infty} = 1} - \underbrace{2 \int_0^{\infty} x \cdot e^{-2x}\, dx}$$

$$\overset{\text{int. parti}}{=} \left[-x \cdot e^{-2x}\right]_0^{\infty} + \int_0^{\infty} e^{-2x}\, dx$$

$$= 0 + \left[-\tfrac{1}{2} e^{-2x}\right]_0^{\infty} = \tfrac{1}{2}$$

$$\rightsquigarrow \quad E[X] = 1 - \tfrac{1}{2} = \underline{\underline{\tfrac{1}{2}}}.$$

$$E[X^2] = E[(1-Y)^2] = \int_{-\infty}^{\infty} (1-x)^2 \cdot f_Y(x)\, dx$$

↓

$$\rightsquigarrow \quad E[X^2] = 2\int_0^\infty (1-x)^2 e^{-2x}\,dx$$

int. per parti

$$= \left[-(1-x)^2 e^{-2x}\right]_0^\infty - \int_0^\infty \left(-2(1-x)\right)\cdot\left(-e^{-2x}\right)dx$$

$$= 1 - \underbrace{2\int_0^\infty (1-x)e^{-2x}\,dx}_{=\frac{1}{2}} \qquad \nearrow \text{sopra}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}.$$

$$\rightsquigarrow \quad var(X) = E[X^2] - E[X]^2 = \frac{1}{2} - \frac{1}{4}$$

$$= \underline{\underline{\frac{1}{4}}}.$$

E.1

(iii) Soluzione alternativa:

$$X = 1 - Y \quad \text{con} \quad Y \sim Exp(2).$$

Noto: Se $Z \sim Exp(\lambda)$, allora

$$E[Z] = \frac{1}{\lambda}, \quad var(Z) = \frac{1}{\lambda^2}, \quad E[Z^2] = \frac{2}{\lambda^2}$$

$$\rightsquigarrow \quad E[Y] = \frac{1}{2}, \quad E[Y^2] = \frac{2}{4} = \frac{1}{2}.$$

$$\rightsquigarrow \quad E[X] = E[1-Y] \overset{\text{lin.}}{=} E[1] - E[Y]$$

$$= 1 - \frac{1}{2} = \frac{1}{2}.$$

$$E[X^2] = E[(1-Y)^2] = E[1 - 2Y + Y^2]$$

$$\overset{\text{lin.}}{=} E[1] - 2E[Y] + E[Y^2] = 1 - 2 \cdot \frac{1}{2} + \frac{1}{2}$$

$$= \frac{1}{2}.$$

$$\rightsquigarrow \quad var(X) = E[X^2] - E[X] = \frac{1}{2} - \frac{1}{4} = \overset{\overset{var(Y)}{\parallel}}{\frac{1}{4}}.$$

Nota: $var(X) = var(1-Y) = var(-Y) = var(Y)$.

E.2

ξ_1, ξ_2, ξ_3 i.i.d. $Ber(\frac{1}{2})$

$X \doteq \xi_1 - \xi_2$, $\quad Y \doteq \xi_3(\xi_1 + \xi_2)$

$\xi_i \sim Ber(\frac{1}{2})$

$\rightsquigarrow E[\xi_i] = \frac{1}{2}$,

$var(\xi_i) = \frac{1}{4}$,

$E[\xi_i^2] = \frac{1}{2}$

(i) Media e varianza di X, Y:

$$E[X] = E[\xi_1 - \xi_2] \overset{lin.}{=} E[\xi_1] - E[\xi_2]$$

$$= \frac{1}{2} - \frac{1}{2} = \underline{\underline{0}}.$$

$$E[X^2] = E[(\xi_1 - \xi_2)^2] = E[\xi_1^2 - 2\xi_1\xi_2 + \xi_2^2]$$

$$\overset{lin.+indip.}{=} \underbrace{E[\xi_1^2]}_{=\frac{1}{2}} - 2\underbrace{E[\xi_1]}_{=\frac{1}{2}} \cdot \underbrace{E[\xi_2]}_{=\frac{1}{2}} + \underbrace{E[\xi_2^2]}_{=\frac{1}{2}}$$

$$= \frac{1}{2}$$

$$\rightsquigarrow var(X) = E[X^2] - E[X]^2 = \frac{1}{2} - 0 = \underline{\underline{\frac{1}{2}}}.$$

Alternativa: $var(X) = var(\xi_1 - \xi_2) \overset{indip.}{=} var(\xi_1) + var(-\xi_2)$

$$= var(\xi_1) + var(\xi_2) = \frac{1}{4} + \frac{1}{4} = \underline{\underline{\frac{1}{2}}}.$$

E.2 (i) cont.

$$E[Y] = E[\xi_3 \cdot (\xi_1 + \xi_2)]$$

$$\overset{\text{indip.+lin.}}{=} \underbrace{E[\xi_3]}_{=\frac{1}{2}} \cdot \left(\underbrace{E[\xi_1]}_{=\frac{1}{2}} + \underbrace{E[\xi_2]}_{=\frac{1}{2}}\right) = \underline{\underline{\frac{1}{2}}}.$$

$$E[Y^2] = E[\xi_3^2 \cdot (\xi_1 + \xi_2)^2]$$

$$\overset{\text{indip.+lin.}}{=} \underbrace{E[\xi_3^2]}_{=\frac{1}{2}} \cdot \left(\underbrace{E[\xi_1^2]}_{=\frac{1}{2}} + 2\underbrace{E[\xi_1]}_{=\frac{1}{2}} \cdot \underbrace{E[\xi_2]}_{=\frac{1}{2}} + \underbrace{E[\xi_2^2]}_{=\frac{1}{2}}\right)$$

$$= \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}.$$

$$\rightsquigarrow \operatorname{var}(Y) = E[Y^2] - E[Y] = \frac{3}{4} - \frac{1}{4} = \underline{\underline{\frac{1}{2}}}.$$

(iii)

Covarianza tra X e Y? Indipendenza?

$$\mathrm{cov}(X,Y) = E[X\cdot Y] - \underbrace{E[X]}_{\overset{(ii)}{=}0}\cdot E[Y]$$

$$\overset{\text{qui}}{=} E[X\cdot Y]$$

$$= E[\xi_3\cdot(\xi_1-\xi_2)\cdot(\xi_1+\xi_2)]$$

$$= E[\xi_3\cdot(\xi_1^2-\xi_2^2)]$$

$$\overset{\text{indip.+lin.}}{=} \underbrace{E[\xi_3]}_{=\frac{1}{2}}\cdot\left(\underbrace{E[\xi_1^2]}_{=\frac{1}{2}}-\underbrace{E[\xi_2^2]}_{=\frac{1}{2}}\right) = 0.$$

$\rightsquigarrow$ X, Y sono incorrelate,

ma <u>non</u> indipendenti.

Ad esempio: $P(X=1) = P(\xi_1=1, \xi_2=0) \overset{\text{ind.}}{=} \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}>0$

$$P(Y=2) = P(\xi_1=1, \xi_2=1, \xi_3=1) \overset{\text{ind.}}{=} \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}>0$$

mentre $P(X=1, Y=2) = P(\emptyset) = 0.$

(III)

Legge congiunta di X e Y:

Con probabilità uno,

X è a valori in $\{-1,0,1\}$,

Y è " " in $\{0,1,2\}$.

Basta quindi calcolare le probabilità

$$P(X=x, Y=y) \quad \text{con} \quad (x,y) \in \{-1,0,1\} \times \{0,1,2\}.$$

- $P(X=-1, Y=0) = P(\xi_1=0, \xi_2=1, \xi_3=0)$
$$\overset{\text{i.i.d.}}{=} \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

- $P(X=-1, Y=1) = P(\xi_1=0, \xi_2=1, \xi_3=1) \overset{\text{i.i.d.}}{=} \frac{1}{8}$

- $P(X=-1, Y=2) = P(\emptyset) = 0.$

($\{X=-1\}$ ~~dovremmo avere~~ $\subseteq \{\xi_1=0\}$ ~~anche~~ ~~[illegible]~~ mentre $\{Y=2\} = \{\xi_1=1, \xi_2=1, \xi_3=1\}$)

- $P(X=0, Y=0) = P(\xi_1=\xi_2=0) + P(\xi_1=\xi_2=1, \xi_3=0)$
$$\overset{\text{i.i.d.}}{=} \frac{1}{4} + \frac{1}{8} = \frac{3}{8}.$$

- $P(X=0, Y=1) = P(\emptyset) = 0$

$\left(\{X=0\} = \{\xi_1 = \xi_2\}, \text{ mentre } \{Y=1\} = \{\xi_3 = 1, \ \xi_1 \neq \xi_2\} \right)$

- $P(X=0, \ Y=2) = P(\xi_1 = 1, \ \xi_2 = 1, \ \xi_3 = 1)$

$$\overset{\text{i.i.d.}}{=} \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

- $P(X=1, \ Y=0) = P(\xi_1 = 1, \ \xi_2 = 0, \ \xi_3 = 0)$

$$\overset{\text{i.i.d.}}{=} \frac{1}{8}.$$

- $P(X=1, \ Y=1) = P(\xi_1 = 1, \ \xi_2 = 0, \ \xi_3 = 1) \overset{\text{i.i.d.}}{=} \frac{1}{8}.$

- $P(X=1, \ Y=2) = P(\emptyset) = 0$

$\left(\{X=1\} = \{\xi_1 = 1, \ \xi_2 = 0\}, \text{ mentre } \{Y=2\} = \{\xi_1 = \xi_2 = \xi_3 = 1\} \right)$

$$\rightsquigarrow \quad P(X=x, Y=y) = \begin{cases} \frac{1}{8} & \text{se } (x,y) \in \{(-1,0), (-1,1), (0,2), (1,0), (1, \\ \frac{3}{8} & \text{se } (x,y) = (0,0), \\ 0 & \text{altrimenti}. \end{cases}$$

E.1

[cf. Scritto zero]

$$S \doteq \sum_{i=1}^{1200} X_i \quad \text{con } X_1, \dots, X_{1200} \text{ i.i.d. } Ber\left(\frac{1}{600}\right)$$

a) $S \sim Bin\left(1200, \frac{1}{600}\right)$, quindi

$$E[S] = 1200 \cdot \frac{1}{600} = 2,$$

$$var(S) = 1200 \cdot \frac{1}{600} \cdot \frac{599}{600} = \frac{599}{300}.$$

Stima per $N \doteq \min\{n \in \mathbb{N} : P(S \leq n) \geq 0.98\}$.

a) mediante disuguaglianza di Chebyshev:

Ricorda: Se Y ha media finita, allora $P(|Y - E[Y]| \geq \varepsilon) \leq \frac{var(Y)}{\varepsilon^2} \quad \forall \varepsilon > 0.$

2) cont. :

Per $n \in \mathbb{N}$ con $n \geq 2$:

$$P(S \leq n) = 1 - P(S > n) = 1 - P(S \geq n+1)$$

Ora $P(S \geq n+1) = P(S - E[S] \geq n+1 - E[S])$

$$\leq P(|S - E[S]| \geq n+1 - E[S]) \qquad (E[S] = 2$$

$$\underset{\text{Chebyshev}}{\leq} \frac{var(S)}{(n+1-E[S])^2}. \qquad (var(S) = \frac{599}{300}$$

$$\rightsquigarrow \quad P(S \geq n+1) \leq \frac{\frac{599}{300}}{(n-1)^2}$$

$$\rightsquigarrow \quad P(S \leq n) \geq 1 - \frac{599}{300(n-1)^2} \overset{!}{\geq} 0.98 = \frac{49}{50}$$

Scegliere $n \in \mathbb{N}$ tale che

$$\frac{599}{300(n-1)^2} \leq \frac{1}{50} \quad \rightsquigarrow \quad (n-1)^2 \geq \frac{599}{6}$$

$$\rightsquigarrow \quad n \geq \underbrace{\sqrt{\frac{599}{6}}}_{\in (9,10)} + 1$$

$$\rightsquigarrow \quad n \geq 11 \quad \overset{\text{stima}}{\rightsquigarrow} \quad \underline{\underline{N = 11}}.$$

E.3

b) approssimazione di Poisson

di parametri 1200, $\frac{1}{600}$

$S \sim Bin(1200, \frac{1}{600})$ distribuzione binomiale ✓

vicina, grazie alla legge dei piccoli numeri,

alla distribuzione di Poisson di parametro

$\lambda = 1200 \cdot \frac{1}{600} = 2.$

$\rightsquigarrow \quad P(S \leq n) \approx F_{Poiss(2)}(n) \overset{!}{\geq} 0.98$

funzione di ripartizione della Poisson di parametro 2

tabelle $\rightsquigarrow \quad n \geq 5 \quad \overset{\text{stima}}{\rightsquigarrow} \quad \underline{\underline{N = 5}}.$

e) approssimazione normale

Grazie al Teorema del limite centrale, abbiamo che la distribuzione di

$$\frac{S-E[S]}{\sqrt{var(S)}}$$ è vicina alla normale standard

$$\leadsto \quad P(S \leq n) = P\left(\frac{S-E[S]}{\sqrt{var(S)}} \leq \frac{n-E[S]}{\sqrt{var(S)}} \right)$$

$$\approx \Phi\left(\frac{n-E[S]}{\sqrt{var(S)}} \right) \overset{!}{\geq} 0{,}98$$

funz. ripartizione $N(0,1)$

$E[S] = 2,$ $var(S) = \frac{599}{300}$

tavola $\leadsto$ $\Phi(y) \geq 0{,}98$ per $y \geq 2{,}06$

Scegliere $n \in \mathbb{N}$ tale che

$$\approx \quad \frac{n-2}{\sqrt{\frac{599}{300}}} \geq 2{,}06$$

$$\leadsto \quad n \geq \underbrace{2{,}06 \cdot \sqrt{\frac{599}{300}}}_{\in (2{,}8,\, 3)} + 2 \quad \overset{n \in \mathbb{N}}{\leadsto} \quad n \geq 5$$

$\overset{\text{stima}}{\leadsto}$ $\underline{\underline{N=5.}}$

E.4

Trovare v.a. non-negative X, Y

tali che $P(X \leq Y \leq 4X) = 1$,

$E[Y] = 2E[X]$,

$E[Y^2] \neq 4E[X^2]$

Ad esempio:

Basta scegliere $X \equiv 1$ (v.a. costante),

$\rightsquigarrow E[X] = 1 = E[X^2]$

Y a valori in $\{1, 3\}$

$$\rightsquigarrow \quad E[Y] = 1 \cdot P(Y=1) + 3 \cdot P(Y=3)$$

$$= P(Y=1) + 3(1 - P(Y=1)) \overset{!}{=} 2E[X] = 2$$

$$\rightsquigarrow \quad P(Y=1) = \frac{1}{2} = P(Y=3)$$

$$\rightsquigarrow \quad E[Y] = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2 \checkmark$$

Dall'altra parte, $E[Y^2] = 1^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{2}$

$$= \frac{10}{2} = 5 \neq 4.$$

$\checkmark$

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