Es. 5 foglio 2

A_k = "nucleo familiare scelto ha esatt. k figli maschi"

B_n = "nucleo familiare scelto e ciò ha ~~...~~ esatt. n tra figlie e..."

$P(B_n) = e^{-\lambda}\frac{\lambda^n}{n!}$, $n \in \mathbb{N}_0$

$P(A_k \mid B_n) = \binom{n}{k}\cdot\left(\frac{1}{2}\right)^n$, $k \in \{0 \dots n\}$

$P(A_k \mid B_n) = 0$ altrimenti

Per $k \in \mathbb{N}_0$ $P(A_k) = \sum_{n=k}^{\infty} P(A_k \mid B_n)\cdot P(B_n) = \sum_{n=k}^{\infty}\binom{n}{k}\left(\frac{1}{2}\right)^n e^{-\lambda}$

$= e^{-\lambda}\cdot\frac{1}{k!}\left(\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}\cdot\left(\frac{1}{2}\right)^n\cdot\frac{\lambda^n}{n!}\right) = \frac{e^{-\lambda}}{k!}\cdot\left(\frac{1}{2}\right)^k\cdot\lambda^k\cdot\left(\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\left(\frac{\lambda}{2}\right)^n\right)$

Siano $C_1, C_2, \dots$ eventi in $(\Omega, \mathcal{F}, P)$ tali che $C_i \cap C_j = \emptyset$ se $i \neq j$

e $P\left(\bigcup_{i\in\mathbb{N}} C_i\right) = 1$, $P(C_i) > 0$ per ogni $i \in \mathbb{N}$.

Allora per $A \in \mathcal{F}$ $P(A) = \sum_{i=1}^{\infty} P(A \mid C_i)\cdot P(C_i)$ (formula delle prob. totali)

$\Rightarrow = e^{\lambda/2} = e^{-\lambda/2}\frac{(\lambda/2)^k}{k!}$.

Es. 6 foglio 2

Siano $p \in (0,1)$, $q \in \left(\frac{1}{2}, 1\right)$

Per $n \in \mathbb{N}$ siano $A_0^k, \dots, A_n^k$ eventi indipendenti in (Ω, P)

tali che $P(A_0)^n = p_0$, $i \in \{1 \dots n\}$

Definiamo v.a. $X_0^n \dots X_n^n$ per ricorsione:

$$X_0^n(\omega) = \begin{cases} 1 & \text{se } \omega \in A_0^n \\ -1 & \text{altrimenti} \end{cases} \qquad X_i^n(\omega) = \begin{cases} X_{i-1}^n(\omega) & \text{se } \omega \in A_i^n \\ -X_{i-1}^n(\omega) & \text{altrimenti} \end{cases}$$

$C_i^n = \{\omega \in \Omega : X_i^n(\omega) = 1\}$

Nota: $C_0^n = A_0^n$

Ora, per $i \in \{1 \dots n\}$ $\quad C_i^n = (A_i^n \cap C_{i-1}^n) \cup ((A_i^n)^c \cap (C_{i-1}^n)^c)$

$\Rightarrow P(C_i^n) = P(A_i^n \cap C_{i-1}^n) + P((A_i^n)^c \cap (C_{i-1}^n)^c)$

$\overset{\text{indipendenza}}{=} P(A_i^n) \cdot P(C_{i-1}^n) + \underbrace{(1 - P(A_i^n))}_{=1-q} \cdot (1 - P(C_{i-1}^n))$

$\Rightarrow P(C_i^n) = q \cdot P(C_{i-1}^n) + 1 - P(C_{i-1}^n) - q + q \cdot P(C_{i-1}^n)$

$= 1 - q + (2q-1) \cdot P(C_{i-1}^n)$

$X_i = a + b \cdot X_{i-1} \qquad X_i = \left(\sum_{l=0}^{i-1} a \cdot b^l\right) + b^i \cdot X_0$

$X_1 = a + b \cdot X_0$

$X_2 = a + b a + b^2 X_0$

$X_3 = a + b \cdot a + b^2 \cdot a + b^3 X_0$

$X_i = a \underbrace{\left(\sum_{l=0}^{i-1} b^l\right)}_{= \frac{1-b^i}{1-b}} + (b^i X_0)$

$\rightarrow X_i = \frac{a(1-b^i)}{1-b} + b^i X_0 \overset{\text{qui}}{=} \frac{(1-q)(1-(2q-1)^i)}{2-2q} + (2q-1)^i X_0$

In particolare

$P(C_i^n) = \frac{1}{2}(1 - \underbrace{(2q-1)^n}_{\in(0,1)}) + \underbrace{(2q-1)^n}_{\in(0,1)} \cdot p_0 \xrightarrow{n\to\infty} \frac{1}{2}$

$P(C_i^n \cap A_0^n) = P(A_i^n \cap C_{i-1}^n \cap A_0^n) + P((A_i^n)^c \cap (C_{i-1})^c \cap A_0^n) = q \cdot P(C_{i-1} \dots$

$+ (1-q) P((C_{i-1}^n)^c \cap A_0^n)$

$\rightarrow P(C_i | A_0^n) = q \cdot P(C_{i-1}^n | A_0^n) + (1-q) P((C_{i-1}^n)^c | A_0^n) = 1 - P(C_{i-1}^n | A_0^n)$

$P(C_0 | A_0^n) = 1$

FOGLIO 2 ES. 7

$Y_1 = (X_1, X_2) \quad Y_2 = (X_2, X_3)$

$P(Y_1 = (1,2)) > 0, \; P(Y_2 = (3,4)) > 0, \; P(Y_1 = (1,2), Y_2 = (3,4)) < 0$

FOGLIO 2 ES. 9

Estrazioni da un'urna senza reinserimento da N palline numerate da 1 a N

X_i = "numero della pallina i-esima estratta"

Distribuzione congiunta di $X_1 \dots X_n$

Modello: $\Omega = \{\sigma : \{1 \dots N\} \to \sigma$ biettiva$\}$ $P = \text{Unif}(\Omega)$ $X_i(\omega) = \omega(i)$

Siano $k_1 \dots k_n \in \{1 \dots N\}$ $P(X_1 = k_1 \dots X_n = k_n) =$

$$= P(\{\omega \in \Omega : \omega(1) = k_n\}) = \begin{cases} \frac{1}{|\Omega|} & \text{se } k_i \neq k_j \text{ per } i \neq j \\ 0 & \text{altrimenti} \end{cases}$$

$$P(X = k) = \sum_{(k_1 \dots k_n) \in (1 \dots N)^{k-1}} P(X_1 = k_1 \; X_{i-1} = k_{-1} \; X_i = k \; X_{i+1} = k_{i+1})$$

$$= \frac{1}{N!} (N-1)! = \frac{1}{N}$$

X_i e X_j sono indipendenti

$P(X_i = 1) = P(X_j = 1) \in (0,1)$

$$P(X_i = 1, X_j = 1) = \begin{cases} 0 & \text{se } i \neq j \\ P(X_i = 1) & \text{se } i = j \end{cases}$$

Sia $\sigma \in S_N$: $(X_{\sigma(1)} \dots X_{\sigma(N)})$

Per $k \dots k_n \in \{1 \dots N\}$: $\{X_{\sigma(1)} = k \dots X_{\sigma(N)} = k_n\}$

Nota: $X_{\sigma(1)} = k \qquad i = \sigma^{-1}(j)$

$= \{X_1 = k_{\sigma^{-1}(1)} \dots X_n = k_{\sigma^{-1}(n)}\}$