Scritto del 27 giugno 2022

E. 1

Media e varianza di X:

(i) $X \sim Unif(-2,-1)$

Formula: se $Y \sim Unif(a,b)$ con $a<b$,

allora $$E[Y] = \frac{b+a}{2},$$

$$var(Y) = \frac{(b-a)^2}{12},$$

quindi qui, con $a=-2$, $b=-1$,

$$E[X] = \frac{-2+(-1)}{2} = \underline{\underline{-\frac{3}{2}}},$$

$$var(X) = \frac{(-1-(-2))^2}{12} = \underline{\underline{\frac{1}{12}}}.$$

In alternativa, calcolo usando la densità

$$f_X(x) = \mathbb{1}_{(-2,-1)}(x), \quad x \in \mathbb{R}.$$

E.1

(ii) X ha funzione di ripartizione

$$F_X(x) = \begin{cases} 0 & \text{se } x < 0, \\ \frac{x^2}{16} & \text{se } x \in [0,4), \\ 1 & \text{se } x \geq 4. \end{cases}$$

Nota: F_X continua e C^1 a tratti

$\rightsquigarrow$ F_X assolutamente continua con densità

$$f_X(x) = F_X'(x) \cdot \mathbb{1}_{\mathbb{R}\setminus\{0,4\}}(x)$$

$$= \frac{x}{8} \cdot \mathbb{1}_{(0,4)}(x), \qquad x \in \mathbb{R}.$$

calcolo $\rightsquigarrow$ valor medio

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_0^4 \frac{x^2}{8}\,dx = \left[\frac{x^3}{24}\right]_0^4 = \frac{64}{24} = \underline{\underline{\frac{8}{3}}}.$$

$$E[X^2] = \int_0^4 x^2 \cdot \frac{x}{8}\,dx = \left[\frac{x^4}{32}\right] = \frac{256}{32} = \underline{\underline{8}}.$$

$\rightsquigarrow$ $var(X) = E[X^2] - E[X]^2 = 8 - \frac{64}{9}$

$$= \underline{\underline{\frac{8}{9}}}.$$

E.1

(iii) $X = e^{\frac{Y^2}{8}}$ con $Y \sim N(0,1)$

$\rightsquigarrow$ Y assol. cont. con densità

$$f_Y(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

$$\rightsquigarrow \quad E[X] = \int_{-\infty}^{\infty} e^{\frac{x^2}{8}} \cdot f_Y(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{x^2}{8}} \cdot e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{3}{8}x^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\cdot\left(\frac{4}{3}\right)}}\, dx$$

$$= \frac{\sqrt{2\pi\left(\frac{4}{3}\right)}}{\sqrt{2\pi}} \underbrace{\frac{1}{\sqrt{2\pi\cdot\left(\frac{4}{3}\right)}} \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\cdot\left(\frac{4}{3}\right)}}\, dx}_{=1}$$

$=1$ poiché integrale della densità della $N(0, \frac{4}{3})$

$$= \frac{\sqrt{2\pi\left(\frac{4}{3}\right)}}{\sqrt{2\pi}} = \underline{\frac{2}{\sqrt{3}}}.$$

Analogamente:

$$E[X^2] = \int_{-\infty}^{\infty} \left(e^{\frac{x^2}{8}}\right)^2 \cdot f_Y(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{\frac{x^2}{4}} \cdot e^{-\frac{x^2}{2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}\,dx$$

$$= \frac{\sqrt{2\pi \cdot 2}}{\sqrt{2\pi}} \cdot \underbrace{\frac{1}{\sqrt{2\pi \cdot 2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2 \cdot 2}}\,dx}_{=1}$$

↗ densità della $N(0,2)$

$$\rightsquigarrow \quad E[X^2] = \sqrt{2}.$$

$$\rightsquigarrow \quad var(X) = E[X^2] - E[X]^2 = \sqrt{2} - \frac{4}{3}$$

E.2

$$X = \xi_1 - \xi_2 \,, \qquad Y = \xi_3 \cdot (\xi_1 + \xi_2) \,,$$

ξ_1, ξ_2, ξ_3 i.i.d. con comune distribuzione $Rad(\frac{1}{2})$.

Nota: - X, Y a valori in $\{-2, 0, 2\}$

poiché ξ_1, ξ_2, ξ_3 a valori in $\{-1, 1\}$ (con probab. uno)

(i) Media e varianza di X, Y:

$$E[X] \overset{\text{Def.}}{=} E[\xi_1 - \xi_2] \overset{\text{lin.}}{=} E[\xi_1] - E[\xi_2] \overset{\xi_1, \xi_2 \text{ ident. distr.}}{=} 0$$

$$\rightsquigarrow \text{var}(X) \overset{\text{poiché}}{=} E[X^2] = E[(\xi_1 - \xi_2)^2]$$

$$\overset{\text{lin.}}{=} \underbrace{E[\xi_1^2]}_{=1} + \underbrace{E[\xi_2^2]}_{=1} - 2\underbrace{E[\xi_1 \cdot \xi_2]}_{\overset{\text{indep.}}{=} E[\xi_1] \cdot E[\xi_2] = 0}$$

poiché $E[\xi] = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$ per $\xi \sim Rad(\frac{1}{2})$

$E[\xi^2] = E[1] = 1$.

$$\rightsquigarrow \text{var}(X) = 2 .$$

Alternativa

E.2 (i) cont.

$$E[Y] = E[\xi_3(\xi_1+\xi_2)] \overset{\text{ind.}}{=} \underbrace{E[\xi_3]}_{=0} \cdot E[\xi_1+\xi_2] \quad \text{(vedi sopra)}$$

$$= \underline{\underline{0}}$$

$$\leadsto \quad var(Y) = E[Y^2] = E[\xi_3^2(\xi_1+\xi_2)^2]$$

$$\overset{\text{indip.}}{=} \underbrace{E[\xi_3^2]}_{=1 \ \text{(v.s.)}} \cdot E[\xi_1^2 + 2\xi_1\xi_2 + \xi_2^2]$$

$$\overset{\text{lin.}}{=} \underbrace{E[\xi_1^2]}_{=1} + 2E[\xi_1 \xi_2] + \underbrace{E[\xi_2^2]}_{=1}$$

$$\overset{\text{indip.}}{=} 1 + \underbrace{2E[\xi_1]\cdot E[\xi_2]}_{=0} + 1$$

$$\leadsto \quad \underline{\underline{var(Y) = 2.}}$$

(ii) $$cov(X,Y) = E[(X-E[X])\cdot(Y-E[Y])]$$

$$\overset{E[X]=0,\ E[Y]=0}{=} E[X\cdot Y] = E[\xi_3\cdot(\xi_1+\xi_2)\cdot(\xi_1-\xi_2)]$$

$$\overset{\text{indip.}}{=} E[\xi_3]\cdot E[\xi_1^2-\xi_2^2] = 0.$$

E.2 (ii) cont.

$\rightsquigarrow$ $\mathrm{cov}(X,Y) = 0$,

$\rightsquigarrow$ X, Y incorrelate, ma

<u>non</u> indipendenti:

Ad esempio: $$P(X=0) = P(\xi_1 = \xi_2)$$

$$= P(\xi_1 = 1, \xi_2 = 1) + P(\xi_1 = -1, \xi_2 = -1)$$

$$\overset{\text{i.i.d.}}{=} \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2} > 0,$$

$$P(Y=0) = P(\xi_1 = -\xi_2)$$

$$= P(\xi_1 = 1, \xi_2 = -1) + P(\xi_1 = -1, \xi_2 = 1)$$

$$= \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2} > 0,$$

mentre $$P(X=0, Y=0) = 0$$

poiché $\{\xi_1 = \xi_2\} \cap \{\xi_1 \neq \xi_2\} = \emptyset$.

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(iii) Legge congiunta di X e Y.

(X, Y) prende valori in $\{-2, 0, 2\}^2$

$\uparrow$ insieme finito

Basta quindi calcolare la densità discreta di (X, Y) ovvero le probabilità

$P(X = x, Y = y)$ con $x, y \in \{-2, 0, 2\}$:

- $P(X = -2, Y = -2) = 0$

poiché $\{X = -2\} = \{\xi_1 = -1, \xi_2 = 1\}$,

$\{Y = -2\} \subseteq \{\xi_1 = \xi_2\}$ e $\{\xi_1 = -1, \xi_2 = 1\} \cap \{\xi_1 = \xi_2\} = \emptyset$;

- $P(X = -2, Y = 0) = P(\xi_1 = -1, \xi_2 = 1) \stackrel{\text{i.i.d.}}{=} \left(\frac{1}{2}\right)^2 = \frac{1}{4}$,

- $P(X = -2, Y = 2) = 0$ (come sopra: $\{Y = 2\} \subseteq \{\xi_1 = \xi_2\}$)

- $P(X = 0, Y = -2) = P(\xi_1 = \xi_2, \xi_3 = -\xi_1)$

$= P(\xi_1 = 1, \xi_3 = -1, \xi_2 = 1) + P(\xi_1 = -1, \xi_3 = 1, \xi_2 = -1)$

$\stackrel{\text{i.i.d.}}{=} \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}$.

E.2 (iii) cont.

- $P(X=0,\ Y=0) = 0$ poiché $\{\xi_1=\xi_2\} \cap \{\xi_1 \neq \xi_2\} = \emptyset$,

- $P(X=0,\ Y=2) = P(\xi_1=\xi_2=\xi_3)$

$= P(\xi_1=1, \xi_2=1, \xi_3=1) + P(\xi_1=-1, \xi_2=-1, \xi_3=-1)$

$\overset{\text{i.i.d.}}{=} \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}$,

- $P(X=2,\ Y=-2) = 0$ (come sopra).

- $P(X=2,\ Y=0) = P(\xi_1=1, \xi_2=-1) \overset{\text{i.i.d.}}{=} \frac{1}{4}$,

- $P(X=2,\ Y=-2) = 0$ (come sopra)

$$\rightsquigarrow \quad P(X=x,\ Y=y) = \begin{cases} \frac{1}{4} & \text{se } (x,y) \in \{(0,-2), (0,2), (-2,0), (2,0)\}, \\ 0 & \text{altrimenti.} \end{cases}$$

$$\rightsquigarrow \quad P_{(X,Y)} = \mathrm{Unif}_{\text{discreta}}\left(\{(0,-2), (0,2), (-2,0), (2,0)\}\right)$$

E.3

Per il procedimento si veda la soluzione dell'Esercizio 3 dello scritto zero.

~~stima tramite Chebyshev~~

Nota: $S \sim Bin\left(900, \frac{1}{300}\right)$

$$\rightsquigarrow \quad E[S] = 900 \cdot \frac{1}{300} = 3,$$

$$var(S) = 900 \cdot \frac{1}{300} \cdot \left(1 - \frac{1}{300}\right) = \frac{299}{100}.$$

2) Stima mediante Chebyshev

$$P(S \leq m) = 1 - P(S \geq m-1) \overset{\text{Chebyshev}}{\geq} 1 - \frac{var(S)}{(m-1)^2} \overset{!}{\geq} \frac{98}{100}$$

$$\rightsquigarrow \quad (m-1)^2 \geq 50 \cdot \frac{299}{100} = \frac{299}{2}$$

$$\rightsquigarrow \quad m \geq \underbrace{\sqrt{\frac{299}{2}}}_{\in (12,13)} + 1 \xrightarrow{m \in \mathbb{N}} m \geq 14$$

$$\rightsquigarrow \quad \underline{M = 14.}$$

E.3

b) Stima tramite l'approssimazione di Poisson:

~~Distrib~~ $S \sim Bin(900, \frac{1}{300})$

distribuzione vicina alla ~~una~~ distribuzione di Poisson

di parametro $\lambda \doteq 900 \cdot \frac{1}{300} = 3$.

Se $Y \sim Poiss(3)$, allora ($\nearrow$ tavola)

$P(Y \leq 6) \approx 0{,}967$, $P(Y \leq 7) \approx 0{,}988$

$\rightsquigarrow m \geq 7$ $\rightsquigarrow$ $\underline{M = 7}$.

c) Stima tramite approssimazione normale:

$$P(S \leq m) = P\left(\frac{S - E[S]}{\sqrt{var(S)}} \leq \frac{m - E[S]}{\sqrt{var(S)}} \right)$$

$$\overset{TLC}{\approx} \Phi\left(\frac{m - E[S]}{\sqrt{var(S)}} \right)$$ | qui: $E[S] = 3$, $\sqrt{var(S)} = \frac{\sqrt{299}}{10}$

$\uparrow$

~~[illegible]~~ ~~[illegible]~~ ~~[illegible]~~
funzione di ripartizione della normale standard

E.3 c) cont.

Dalla tavola: Per $y \geq 2{,}06$ si ha

$\Phi(y) \geq 0{,}98$

$\rightsquigarrow \quad \dfrac{m-3}{\frac{\sqrt{299}}{10}} \geq 2{,}06$

$\rightsquigarrow \quad m \geq 2{,}06 \cdot \dfrac{\sqrt{299}}{10} + 3$

$\rightsquigarrow \quad m > \dfrac{17}{5} + 3 \xrightarrow{m \in \mathbb{N}} m \geq 6{,}4$

$\rightsquigarrow \quad \underline{\underline{M = 7}}$.

E.4 (pescatori padovani)

$X_1, \dots, X_{900}$ i.i.d. con $E[X_1] = \mu$ (>0),

$var(X_1) = \sigma^2$ ~~ignota~~ $\in (0, 60^2]$

↑ ipotesi

$$\bar{S} = \frac{1}{900} \sum_{i=1}^{900} X_i$$

Sia $\delta > 0$.

$$P(\mu \in (\bar{S}-\delta, \bar{S}+\delta))$$

$$= P(\bar{S}-\delta < \mu < \bar{S}+\delta)$$

$$= P(-\delta < \mu - \bar{S} < \delta)$$

$$= P(-\delta < \bar{S} - \mu < \delta)$$

$$= P\left(\frac{-\delta}{\sqrt{var(\bar{S})}} < \frac{\bar{S}-\mu}{\sqrt{var(\bar{S})}} < \frac{\delta}{\sqrt{var(\bar{S})}}\right)$$

nota: $E[\bar{S}] = \mu$, $var(\bar{S}) = \frac{1}{900^2} \cdot 900 \cdot \sigma^2$

$\rightsquigarrow \sqrt{var(\bar{S})} = \frac{\sigma}{30}$

$$\overset{TLC}{\approx} P\left(-\frac{30\delta}{\sigma} < Z < \frac{30\delta}{\sigma}\right) \quad \text{con } Z \sim N(0,1)$$

$$= \Phi\left(\frac{30\delta}{\sigma}\right) - \Phi\left(-\frac{30\delta}{\sigma}\right) = 2\Phi\left(\frac{30\delta}{\sigma}\right) - 1,$$

E4. cont.

dove Φ funzione di ripartizione della normale standard

Vogliamo quindi $P(\mu \in (\bar{S}-\delta, \bar{S}+\delta))$

$$\approx 2\Phi\left(\frac{30\delta}{\sigma}\right) - 1 \overset{!}{\geq} 0{,}95$$

$$\rightsquigarrow \quad \Phi\left(\frac{30\delta}{\sigma}\right) \geq 0{,}975$$

tavola $\rightsquigarrow$ $\frac{30\delta}{\sigma} \geq 1{,}96$

Per ipotesi, $\sigma \leq 60$

$$\rightsquigarrow \quad \frac{30\delta}{60} \leq \frac{30\delta}{\sigma}$$

Scegliamo quindi $\delta > 0$ tale che $\frac{30\delta}{60} = \frac{\delta}{2} \geq 1{,}96$

$$\rightsquigarrow \quad \delta \geq 3{,}92.$$

$\rightsquigarrow$ δ minimale uguale a $3{,}92$.