Scritto 1 del 20/01/2022

E.1

Calcolare media e varianza:

(i) $X \sim Unif(-1,1)$:

X assolutamente continua con densità

$$f_X(x) = \frac{1}{2} \cdot \mathbb{1}_{(-1,1)}(x), \quad x \in \mathbb{R}.$$

$$\rightsquigarrow \quad E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \frac{1}{2} \int_{-1}^{1} x\,dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \right]_{-1}^{1} = \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{1}{2} \right) = \underline{\underline{0}}.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx = \frac{1}{2} \int_{-1}^{1} x^2\,dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3} \right) = \frac{1}{3}.$$

$$\rightsquigarrow \quad var(X) = E[X^2] - E[X]^2 = \underline{\underline{\frac{1}{3}}}.$$

E.1 (cont.)

(ii) X con funzione di ripartizione

$$F_X(x) = \frac{x^2}{4} \cdot 1_{[0,2)}(x) + 1_{[2,\infty)}(x), \quad x \in \mathbb{R}.$$

Nota: F_X continua su $\mathbb{R}$ e C^1 su $\mathbb{R} \setminus \{0,2\}$

$\leadsto$ X assolutamente continua con densità data da

$$f_X(x) = F_X'(x) = \frac{x}{2} \cdot 1_{(0,2)}(x), \quad x \in \mathbb{R}$$

(per $x \in \{0,2\}$ possiamo porre $f_X(x) = 0$).

$$\leadsto \quad E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^2 \frac{x^2}{2}\,dx$$

$$= \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} = \underline{\underline{\frac{4}{3}}}.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_X(dx) = \int_0^2 \frac{x^3}{2}\,dx$$

$$= \left[\frac{x^4}{8}\right]_0^2 = \frac{16}{8} = 2$$

$$\leadsto \quad var(X) = E[X^2] - E[X]^2 = 2 - \frac{16}{9} = \underline{\underline{\frac{2}{9}}}.$$

E.1 (cont.)

(iii) $X = e^Y$ con $Y \sim Exp(4)$

Nota: Y assolutamente continua con densità

$$f_Y(x) = 4 \cdot e^{-4x} \cdot \mathbb{1}_{(0,\infty)}(x), \quad x \in \mathbb{R}.$$

$$\rightsquigarrow \quad E[X] = \int_{-\infty}^{\infty} e^x \cdot f_Y(x)\,dx = \int_0^{\infty} e^x \cdot (4 \cdot e^{-4x})\,dx$$

$$= 4 \int_0^{\infty} e^{-3x}\,dx = 4 \cdot \left[-\frac{1}{3} \cdot e^{-3x} \right]_0^{\infty}$$

$$= 0 + \frac{4}{3} = \underline{\underline{\frac{4}{3}}}.$$

$$E[X^2] = \int_{-\infty}^{\infty} e^{2x} \cdot f_Y(x)\,dx = 4 \cdot \int_0^{\infty} e^{-2x}\,dx$$

$$= 4 \cdot \left[-\frac{1}{2} e^{-2x} \right]_0^{\infty} = \frac{4}{2} = 2.$$

$$\rightsquigarrow \quad var(X) = E[X^2] - E[X]^2 = 2 - \frac{16}{9} = \underline{\underline{\frac{2}{9}}}.$$

E.2

X, Y indipendenti e a valori in $\{0,1\}$;

$$p \doteq P(X=1), \qquad q \doteq P(Y=1)$$

$$\rightsquigarrow \quad P(X=0) = 1-p, \qquad P(Y=0) = 1-q.$$

Per def.,

$$Z = \begin{cases} \frac{1}{2} & \text{se } X=0=Y, \\ 1 & \text{se } (X=1 \text{ e } Y=0) \text{ o } (X=0 \text{ e } Y=1), \\ \frac{4}{5} & \text{se } X=1=Y. \end{cases}$$

$$\text{(i)}\quad E[Z] = \frac{1}{2} \cdot P(X=0, Y=0) + 1 \cdot \big(P(X=1, Y=0) + P(X=0, Y=1)\big)$$

$$+ \frac{4}{5} \cdot P(X=1, Y=1)$$

$$\Big|\ X, Y \text{ indipendenti}$$

$$= \frac{1}{2} P(X=0) \cdot P(Y=0) + \big(P(X=1) \cdot P(Y=0) + P(X=0) + P(Y=1)\big)$$

$$+ \frac{4}{5} P(X=1) \cdot P(Y=1)$$

$$= \frac{1}{2}(1-p)\cdot(1-q) + p(1-q) + q(1-p) + \frac{4}{5} p \cdot q$$

$$= \frac{1}{2}(1+p+q) - \frac{7}{10} \cdot p \cdot q \,. \quad //$$

E.2 (cont.)

($\overline{ii}$) $\quad var(Z) = E[Z^2] - E[Z]^2$.

Sfruttando di nuovo l'indipendenza tra X e Y:

$$E[Z^2] = \frac{1}{4}(1-p)(1-q) + 1\cdot(p(1-q) + q(1-p)) + \frac{16}{25}p\cdot q$$

$$= \frac{1}{4}\cdot(1+3p+3q) - \frac{111}{100}\cdot p\cdot q.$$

$\rightsquigarrow \quad var(Z) = E[Z^2] - E[Z]^2$

$$= \frac{1}{4}(1+3p+3q) - \frac{111}{100}p\cdot q$$

$$- \left(\frac{1}{2}(1+p+q) - \frac{7}{10}p\cdot q\right)^2.$$

[andava bene così].

($\overline{iii}$) $\quad$ Calcolare $E[Z]$ quando $p = \frac{5}{7}$:

$\overset{(i)}{\rightsquigarrow} \quad E[Z] = \frac{1}{2}(1+p+q) - \frac{7}{10}p\cdot q$

$\overset{p=\frac{5}{7}}{\rightsquigarrow}$

$$E[Z] = \frac{1}{2} + \frac{5}{14} + \frac{q}{2} - \underbrace{\frac{7}{10}\cdot\frac{5}{7}}_{=\frac{1}{2}}\cdot q$$

$$= \frac{1}{2} + \frac{5}{14} = \frac{12}{14} = \underline{\underline{\frac{6}{7}}}.$$

E. 3

Per il procedimento si veda la soluzione dello Scritto 0.

Valori qui:

(i) $$m \geq \underbrace{\sqrt{\frac{399 \cdot 100}{160}}}_{} + \frac{3}{2}$$

$$= \sqrt{\frac{399 \cdot 5}{8}} \approx \sqrt{250} \approx 16$$

$\rightsquigarrow \quad m > 17 \quad \rightsquigarrow \quad M_* = 18.$

(ii) $F_{Poiss\left(\frac{5}{2}\right)}(m) \geq 0{,}99$ per $m \geq 7$

$\rightsquigarrow \quad M_* = 7.$

(iii) $\Phi(x) \geq 0{,}99$ per $x \gtrsim 2{,}33$

$$\rightsquigarrow \quad m \geq 2{,}33 \cdot \underbrace{\sqrt{\frac{399}{160}}}_{\approx \sqrt{2{,}5} \approx 1{,}6} + \frac{5}{2}.$$

$\rightsquigarrow \quad m > 6 \quad \rightsquigarrow \quad M_* = 7.$

E.4

Sia ξ v.a. su $(\Omega, \mathcal{F}, P)$ con $\xi \sim \mathrm{Unif}(0,1)$.

Trovare $\Phi : [0,1] \to \mathbb{R}$ tale che

$Y \doteq \Phi(\xi)$ abbia la seguente distribuzione:

$P(Y=1) = \frac{1}{3}, \quad P(Y=2) = \frac{1}{2}, \quad P(Y=4) = \frac{1}{6}.$

Definiamo Φ mediante (ad esempio)

$$\Phi(x) \doteq \begin{cases} 1 & \text{se } x \in [0, \frac{1}{3}), \\ 2 & \text{se } x \in [\frac{1}{3}, \frac{5}{6}), \\ 4 & \text{se } x \in [\frac{5}{6}, 1]. \end{cases} \qquad x \in [0,1].$$

$\leadsto$ Φ ben definita

Con questa definizione:

$$P(Y=1) = P(\Phi(\xi)=1) = P(\xi \in [0, \tfrac{1}{3})) \quad | \ \xi \sim \mathrm{Unif}(0,1)$$

$$= \int_0^{\frac{1}{3}} dx = \frac{1}{3},$$

$$P(Y=2) = P(\Phi(\xi)=2) = P(\xi \in [\tfrac{1}{3}, \tfrac{5}{6})) \quad | \ \xi \sim \mathrm{Unif}(0,1)$$

$$= \int_{\frac{1}{3}}^{\frac{5}{6}} dx = \frac{5}{6} - \frac{2}{6} = \frac{1}{2},$$

e analogamente $P(Y=4) = \frac{1}{6}$. //