

ESERCIZIO TIPO 13

Si trovi una base ortonormale del sottospazio di \mathbb{C}^4

$$V = \left\langle \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}; \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix}; \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} \right\rangle.$$

1^o MODO

[1] Troviamo una base \mathcal{B}_1 di V .

Poniamo

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix}$$

e costruiamo la matrice $\mathbf{A} = (\mathbf{w}_1 \ \mathbf{w}_2 \ \mathbf{w}_3 \ \mathbf{w}_4)$, ossia una matrice tale che $C(\mathbf{A}) = V$.

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & -1 & 0 \\ 0 & -1 & 1 & i \end{pmatrix} \xrightarrow{E_{31}(-i)} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & i \end{pmatrix} \xrightarrow{E_{42}(1)E_{32}(-1)} \\ &\rightarrow \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \xrightarrow{E_{34}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{E_3(-i)} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{U} \end{aligned}$$

Dunque $\mathcal{B}_1 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$ è una base di $C(\mathbf{A}) = V$.

[2] Troviamo una base ortogonale \mathcal{B}_2 di V : poniamo $\mathbf{v}_1 = \mathbf{w}_1, \mathbf{v}_2 = \mathbf{w}_2$ e $\mathbf{v}_3 = \mathbf{w}_4$, e applichiamo l'algoritmo di Gram-Schmidt a $\{\mathbf{v}_1; \mathbf{v}_2; \mathbf{v}_3\}$.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \alpha_{12}\mathbf{u}_1,$$

$$\mathbf{u}_1 \neq \mathbf{0} \implies \alpha_{12} = \frac{(\mathbf{u}_1|\mathbf{v}_2)}{(\mathbf{u}_1|\mathbf{u}_1)}$$

$$(\mathbf{u}_1|\mathbf{v}_2) = \mathbf{u}_1^H \mathbf{v}_2 = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix} = i$$

$$(\mathbf{u}_1|\mathbf{u}_1) = \mathbf{u}_1^H \mathbf{u}_1 = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = 2$$

$$\implies \alpha_{12} = i/2$$

$$\begin{aligned} \mathbf{u}_2 &= \mathbf{v}_2 - \alpha_{12}\mathbf{u}_1 = \\ &= \mathbf{v}_2 - \frac{i}{2}\mathbf{u}_1 = \end{aligned}$$

$$= \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \alpha_{13}\mathbf{u}_1 - \alpha_{23}\mathbf{u}_2,$$

$$\mathbf{u}_1 \neq \mathbf{0} \implies \alpha_{13} = \frac{(\mathbf{u}_1|\mathbf{v}_3)}{(\mathbf{u}_1|\mathbf{u}_1)}$$

$$(\mathbf{u}_1|\mathbf{v}_3) = \mathbf{u}_1^H \mathbf{v}_3 = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = 0$$

$$\implies \alpha_{13} = 0$$

$$\mathbf{u}_2 \neq \mathbf{0} \implies \alpha_{23} = \frac{(\mathbf{u}_2|\mathbf{v}_3)}{(\mathbf{u}_2|\mathbf{u}_2)}$$

$$(\mathbf{u}_2|\mathbf{v}_3) = \mathbf{u}_2^H \mathbf{v}_3 = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = -i$$

$$(\mathbf{u}_2|\mathbf{u}_2) = \mathbf{u}_2^H \mathbf{u}_2 = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{5}{2}$$

$$\implies \alpha_{23} = -\frac{2}{5}i$$

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{v}_3 - \alpha_{13}\mathbf{u}_1 - \alpha_{23}\mathbf{u}_2 = \\
&= \mathbf{v}_3 + \frac{2i}{5}\mathbf{u}_2 = \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} + \frac{2i}{5} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}
\end{aligned}$$

$\mathcal{B}_2 = \{\mathbf{u}_1; \mathbf{u}_2; \mathbf{u}_3\}$, dove

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{2} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix},$$

è una base ortogonale di V .

[3] Troviamo una base ortonormale \mathcal{B} di V , normalizzando gli elementi di \mathcal{B}_2 .

$$\|\mathbf{u}_1\|_2 = \sqrt{(\mathbf{u}_1|\mathbf{u}_1)} = \sqrt{2}$$

$$\|\mathbf{u}_2\|_2 = \sqrt{(\mathbf{u}_2|\mathbf{u}_2)} = \sqrt{5/2}$$

$$\|\mathbf{u}_3\|_2 = \sqrt{(\mathbf{u}_3|\mathbf{u}_3)} = \sqrt{\mathbf{u}_3^H \mathbf{u}_3} = \sqrt{\frac{1}{5} \begin{pmatrix} -1 & -2i & -i & -3i \end{pmatrix} \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}} = \frac{\sqrt{15}}{5}$$

Concludendo: $\mathcal{B} = \{\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|_2}; \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|_2}; \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|_2}\}$, dove

$$\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|_2} = \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix},$$

è una base ortonormale di V .

2^0 MODO

[1] Prima costruiamo un insieme di generatori ortogonale di V : poniamo

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix}$$

e applichiamo l'algoritmo di Gram-Schmidt a $\{\mathbf{v}_1; \mathbf{v}_2; \mathbf{v}_3; \mathbf{v}_4\}$. Otterremo 4 vettori, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, e l'insieme $\{\mathbf{u}_1; \mathbf{u}_2; \mathbf{u}_3; \mathbf{u}_4\}$ sarà un insieme di generatori ortogonale di V .

Per sapere se alcuni degli \mathbf{u}_i saranno nulli, e in tal caso quali, troviamo innanzitutto una forma ridotta di Gauss \mathbf{U} della matrice \mathbf{A} che ha come colonne $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$: le eventuali colonne libere di U corrisponderanno agli \mathbf{u}_i nulli.

$$\begin{aligned} \mathbf{A} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4) &= \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & -1 & 0 \\ 0 & -1 & 1 & i \end{pmatrix} \xrightarrow{E_{31}(-i)} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & i \end{pmatrix} \rightarrow \\ &\xrightarrow{E_{42}(1)E_{32}(-1)} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \xrightarrow{E_{34}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{E_3(-i)} \\ &\longrightarrow \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{U} \end{aligned}$$

Poichè \mathbf{U} ha come unica colonna libera la 3^a, allora applicando l'algoritmo di Gram-Schmidt a $\{\mathbf{v}_1; \mathbf{v}_2; \mathbf{v}_3; \mathbf{v}_4\}$ otterremo $\mathbf{u}_3 = \mathbf{0}$.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \alpha_{12}\mathbf{u}_1, \quad \mathbf{u}_1 \neq \mathbf{0} \implies \alpha_{12} = \frac{(\mathbf{u}_1|\mathbf{v}_2)}{(\mathbf{u}_1|\mathbf{u}_1)}$$

$$(\mathbf{u}_1|\mathbf{v}_2) = \mathbf{u}_1^H \mathbf{v}_2 = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix} = i$$

$$(\mathbf{u}_1|\mathbf{u}_1) = \mathbf{u}_1^H \mathbf{u}_1 = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = 2$$

$$\implies \alpha_{12} = i/2$$

$$\begin{aligned}
\mathbf{u}_2 &= \mathbf{v}_2 - \alpha_{12}\mathbf{u}_1 = \\
&= \mathbf{v}_2 - \frac{i}{2}\mathbf{u}_1 = \\
&= \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}
\end{aligned}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \alpha_{13}\mathbf{u}_1 - \alpha_{23}\mathbf{u}_2,$$

$$\mathbf{u}_1 \neq \mathbf{0} \implies \alpha_{13} = \frac{(\mathbf{u}_1|\mathbf{v}_3)}{(\mathbf{u}_1|\mathbf{u}_1)}$$

$$(\mathbf{u}_1|\mathbf{v}_3) = \mathbf{u}_1^H \mathbf{v}_3 = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} = i$$

$$(\mathbf{u}_1|\mathbf{u}_1) = 2$$

$$\implies \alpha_{13} = \frac{i}{2}$$

$$\mathbf{u}_2 \neq \mathbf{0} \implies \alpha_{23} = \frac{(\mathbf{u}_2|\mathbf{v}_3)}{(\mathbf{u}_2|\mathbf{u}_2)}$$

$$(\mathbf{u}_2|\mathbf{v}_3) = \mathbf{u}_2^H \mathbf{v}_3 = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} =$$

$$= -1 - \frac{1}{2} - 1 = -\frac{5}{2}$$

$$(\mathbf{u}_2|\mathbf{u}_2) = \mathbf{u}_2^H \mathbf{u}_2 = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{5}{2}$$

$$\implies \alpha_{23} = -1$$

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{v}_3 - \alpha_{13}\mathbf{u}_1 - \alpha_{23}\mathbf{u}_2 = \\
&= \mathbf{v}_3 - \frac{i}{2}\mathbf{u}_1 + \mathbf{u}_2 =
\end{aligned}$$

$$= \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{u}_4 = \mathbf{v}_4 - \alpha_{14}\mathbf{u}_1 - \alpha_{24}\mathbf{u}_2 - \alpha_{34}\mathbf{u}_3$$

$$\mathbf{u}_1 \neq \mathbf{0} \implies \alpha_{14} = \frac{(\mathbf{u}_1|\mathbf{v}_4)}{(\mathbf{u}_1|\mathbf{u}_1)}$$

$$(\mathbf{u}_1|\mathbf{v}_4) = \mathbf{u}_1^H \mathbf{v}_4 = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = 0$$

$$\implies \alpha_{14} = 0$$

$$\mathbf{u}_2 \neq \mathbf{0} \implies \alpha_{24} = \frac{(\mathbf{u}_2|\mathbf{v}_4)}{(\mathbf{u}_2|\mathbf{u}_2)}$$

$$(\mathbf{u}_2|\mathbf{v}_4) = \mathbf{u}_2^H \mathbf{v}_4 = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} = -i$$

$$(\mathbf{u}_2|\mathbf{u}_2) = \mathbf{u}_2^H \mathbf{u}_2 = \begin{pmatrix} -\frac{i}{2} & 1 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{5}{2}$$

$$\implies \alpha_{24} = -\frac{2}{5}i$$

$$\mathbf{u}_3 = \mathbf{0} \implies \alpha_{34} = 0 \text{ per def.}$$

$$\begin{aligned} \mathbf{u}_4 &= \mathbf{v}_4 - \alpha_{24}\mathbf{u}_2 = \\ &= \mathbf{v}_4 + \frac{2i}{5}\mathbf{u}_2 = \end{aligned}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix} + \frac{2i}{5} \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}$$

Dunque

$$\left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}; \mathbf{u}_2 = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}; \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u}_4 = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix} \right\}$$

è un insieme di generatori ortogonale di V .

[2] Costruiamo **una base ortogonale di V** togliendo dall'insieme di generatori ortogonale di V trovato al punto [1] gli eventuali \mathbf{u}_i nulli. In questo caso

poniamo:

$$\mathbf{w}_1 = \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \mathbf{u}_2 = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}, \quad \mathbf{w}_3 = \mathbf{u}_4 = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}.$$

L'insieme

$$\left\{ \mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}; \mathbf{w}_2 = \begin{pmatrix} \frac{i}{2} \\ 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}; \mathbf{w}_3 = \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix} \right\}$$

è una base ortogonale di V .

[3] Costruiamo **base ortonormale di V** normalizzando la base ortogonale trovata al punto [2], ossia dividendo ciascun elemento della base ortogonale trovata in [2] per la propria norma euclidea.

Cominciamo con il calcolare la norma euclidea di $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$:

$$\|\mathbf{w}_1\|_2 = \sqrt{(\mathbf{u}_1|\mathbf{u}_1)} = \sqrt{2}$$

$$\|\mathbf{w}_2\|_2 = \sqrt{(\mathbf{u}_2|\mathbf{u}_2)} = \sqrt{5/2}$$

$$\|\mathbf{w}_3\|_2 = \sqrt{(\mathbf{u}_4|\mathbf{u}_4)} = \sqrt{\mathbf{u}_4^H \mathbf{u}_4} = \sqrt{\frac{1}{5} \begin{pmatrix} -1 & -2i & -i & -3i \end{pmatrix} \frac{1}{5} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix}} = \frac{\sqrt{15}}{5}$$

Allora

$$\mathcal{B} = \left\{ \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|_2}; \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|_2}; \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|_2} \right\},$$

dove

$$\frac{\mathbf{w}_1}{\|\mathbf{w}_1\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|_2} = \frac{1}{\sqrt{15}} \begin{pmatrix} -1 \\ 2i \\ i \\ 3i \end{pmatrix},$$

è una base ortonormale di V .