Soluzione esercizi su limiti con sviluppi di Taylor

1) ordine di infinitesimo di $x \sin x - x^2$

$$x \sin x - x^2 = x\left(x - \frac{x^3}{6} + o(x^3)\right) - x^2 = \cancel{x^2} - \frac{x^4}{6} + o(x^4) \cancel{- x^2}$$

$$= -\frac{x^4}{6} + o(x^4) \qquad \text{ORDINE DI INFINITESIMO } 4$$

$$\lim_{x\to 0^+} \frac{x \sin x - x^2}{(1-\cos x)x} = \lim_{x\to 0^+} \frac{-\frac{x^4}{6} + o(x^4)}{\left(\cancel{1} - \cancel{1} + \frac{x^2}{2} + o(x^2)\right)x} =$$

$$= \lim_{x\to 0^+} \frac{x^4\left(-\frac{1}{6} + o(1)\right)}{\frac{x^3}{2} + o(x^3)} = \lim_{x\to 0^+} \frac{\overset{\searrow 0}{\cancel{x^4}}\overbrace{\left(-\frac{1}{6} + o(1)\right)}^{\to -\frac{1}{6}}}{\cancel{x^3}\underbrace{\left(\frac{1}{2} + o(1)\right)}_{\to \frac{1}{2}}} = 0$$

2) $$\lim_{x\to 0^+} \frac{\operatorname{arctg}(x + x^2) - x}{x^\alpha - \sin(x^2)}$$

numeratore $\operatorname{arctg}(x + x^2) = x + x^2 - \frac{1}{3}(x + x^2)^3 + o(x + x^2)^3 =$

$$= x + x^2 + o(x^2)$$

$$\operatorname{arctg}(x + x^2) - x = \cancel{x} + x^2 + o(x^2) \cancel{- x} = x^2 + o(x^2)$$

denominatore $\sin x^2 = x^2 - \frac{1}{6}(x^2)^3 + o(x^2)^3 = x^2 - \frac{1}{6}x^6 + o(x^6)$

$$x^\alpha - \sin x^2 = x^\alpha - x^2 + \frac{1}{6}x^6 + o(x^6) = \begin{cases} \frac{1}{6}x^6 + o(x^6) & \underline{\alpha = 2} \\ -x^2 + o(x^2) & \underline{\alpha > 2} \\ x^\alpha + o(x^\alpha) & \underline{\alpha < 2} \end{cases}$$

$\alpha = 2$

$$\lim_{x\to 0^+} \frac{x^2 + o(x^2)}{\frac{1}{6}x^6 + o(x^6)} = \lim_{x\to 0^+} \frac{\cancel{x^2}(1 + o(1))}{x^{\cancel{6}\,4}\left(\frac{1}{6} + o(1)\right)} = +\infty.$$

$\alpha > 2$

$$\lim_{x\to 0^+} \frac{x^2+o(x^2)}{-x^2+o(x^2)} = \lim_{x\to 0^+} \frac{\cancel{x^2}(1+o(1))}{\cancel{x^2}(-1+o(1))} = -1$$

2

$\alpha < 2$

$$\lim_{x\to 0^+} \frac{x^2+o(x^2)}{x^\alpha+o(x^\alpha)} = \lim_{x\to 0^+} \frac{x^2(1+o(1))}{x^\alpha(1+o(1))} = \lim_{x\to 0^+} x^{2-\alpha}\frac{(1+o(1))}{(1+o(1))} = 0$$

perché $2-\alpha > 0$

Riassumendo $\lim_{x\to 0^+} f(x) = \begin{cases} +\infty & \alpha = 2 \\ -1 & \alpha > 2 \\ 0 & \alpha < 2 \end{cases}$

3 $$\lim_n \frac{e^{-\frac{1}{2n^2}} - \cos\frac{1}{n}}{n^\alpha\left(\operatorname{arctg}\frac{1}{n} - \frac{1}{n}\right)}$$

numeratore $$e^{-\frac{1}{n^2}} = 1 - \frac{1}{2n^2} + \frac{1}{2}\left(-\frac{1}{2n^2}\right)^2 + o\left(\frac{1}{n^4}\right) =$$

$$= 1 - \frac{1}{2n^2} + \frac{1}{8}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

$$\cos\frac{1}{n} = 1 - \frac{1}{2}\frac{1}{n^2} + \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

$$e^{-\frac{1}{2n^2}} - \cos\frac{1}{n} = \cancel{1} - \cancel{\frac{1}{2n^2}} + \frac{1}{8}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) - \cancel{1} + \cancel{\frac{1}{2}\frac{1}{n^2}} - \frac{1}{24}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) =$$

$$= \left(\frac{1}{8} - \frac{1}{24}\right)\frac{1}{n^4} + o\left(\frac{1}{n^4}\right) = \frac{1}{12}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

denominatore $$\operatorname{arctg}\frac{1}{n} = \frac{1}{n} - \frac{1}{3}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$$

$$n^\alpha\left(\operatorname{arctg}\frac{1}{n} - \frac{1}{n}\right) = n^\alpha\left(\cancel{\frac{1}{n}} - \frac{1}{3}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right) - \cancel{\frac{1}{n}}\right) = -\frac{1}{3}\frac{1}{n^{3-\alpha}} + o\left(\frac{1}{n^{3-\alpha}}\right)$$

$$\lim_n \frac{\frac{1}{12}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)}{-\frac{1}{3}\frac{1}{n^{3-\alpha}} + o\left(\frac{1}{n^{3-\alpha}}\right)} = \lim_n \frac{\frac{1}{n^4}\left(\frac{1}{12} + o(1)\right)}{\frac{1}{n^{3-\alpha}}\left(-\frac{1}{3} + o(1)\right)} =$$

$$= \lim_n \frac{n^{3-\alpha}}{n^4}\frac{\left(\frac{1}{12}+o(1)\right)}{\left(-\frac{1}{3}+o(1)\right)} = \lim_n \frac{1}{n^{1+\alpha}}\frac{\left(\frac{1}{12}+o(1)\right)}{\left(-\frac{1}{3}+o(1)\right)} = \begin{cases} -\frac{1}{4} & \alpha = (-1) \\ 0 & \alpha > (-1) \\ & (\alpha+1>0) \\ -\infty & \alpha < -1 \\ & \alpha+1<0 \end{cases}$$

4) $\lim\limits_{x\to 0} \dfrac{5^{1+tg^2 x}-5}{1-\cos x}$

<u>numeratore</u> $5^{1+tg^2 x} = e^{(\lg 5)(1+tg^2 x)} = e^{\lg 5}\, e^{\lg 5\, tg^2 x}$

$$5^{1+tg^2 x}-5 = e^{\lg 5}\left(e^{\lg 5\, tg^2 x}-1\right) = 5\cdot\left(\lg 5\, tg^2 x + o(tg^2 x)\right) =$$

$$= 5\left(\lg 5\, x^2 + o(x^2)\right)$$

<u>denominatore</u> $1-\cos x = \dfrac{x^2}{2} + o(x^2)$

$$\lim_{x\to 0} \frac{5(\lg 5\, x^2 + o(x^2))}{\frac{x^2}{2}+o(x^2)} = \lim_{x\to 0} \frac{\cancel{x^2}(5\lg 5 + o(1))}{\cancel{x^2}(\frac{1}{2}+o(1))} = 10 \lg 5.$$

5) $\lim\limits_{x\to +\infty} x^\alpha - x^2 \lg\left(1+\dfrac{1}{x}\right) = \left(y = \dfrac{1}{x}\right) =$

$$= \lim_{y\to 0^+} \left(\frac{1}{y}\right)^\alpha - \left(\frac{1}{y}\right)^2 \lg(1+y) =$$

$$= \lim_{y\to 0^+} \frac{1}{y^\alpha} - \frac{1}{y^2}\left(y - \frac{y^2}{2} + o(y^2)\right) =$$

$$= \lim_{y\to 0^+} \frac{1}{y^\alpha} - \frac{1}{y} + \frac{1}{2} + o(1) = \begin{cases} \frac{1}{2} & \alpha = 1 \\ +\infty & \alpha > 1 \\ -\infty & \alpha < 1 \end{cases}$$

<u>infatti</u> se $\alpha > 1$ $\quad \dfrac{1}{y^\alpha} - \dfrac{1}{y} = \underbrace{\frac{1}{y^\alpha}}_{\to +\infty}\overbrace{\left(1 - \frac{1}{y^{1-\alpha}}\right)}^{\to 1}$

$\alpha < 1$ $\quad \dfrac{1}{y^\alpha} - \dfrac{1}{y} = \underbrace{\frac{1}{y}}_{+\infty}\underbrace{\left(+\frac{1}{y^{\alpha-1}} - 1\right)}_{\to -1}$

6) $\lim_n n^\alpha \left[\sin\left(\frac{1}{n^2}\right) - \operatorname{arctg}\left(\frac{1}{n^2}\right)\right]$

$$\sin\frac{1}{n^2} = \frac{1}{n^2} - \frac{1}{6}\left(\frac{1}{n^2}\right)^3 + o\left(\frac{1}{n^6}\right) = \frac{1}{n^2} - \frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)$$

$$\operatorname{arctg}\frac{1}{n^2} = \frac{1}{n^2} - \frac{1}{3}\left(\frac{1}{n^2}\right)^3 + o\left(\frac{1}{n^6}\right) = \frac{1}{n^2} - \frac{1}{3}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)$$

$$\lim_n n^\alpha\left[\cancel{\frac{1}{n^2}} - \frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right) - \cancel{\frac{1}{n^2}} + \frac{1}{3}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)\right] =$$

$$= \lim_n n^\alpha\left[\frac{1}{6}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)\right] = \lim_n \frac{n^\alpha}{n^6}\left[\frac{1}{6} + o(1)\right] =$$

$$= \lim_n n^{\alpha-6}\left[\frac{1}{6} + o(1)\right] = \begin{cases} \frac{1}{6} & \alpha = 6 \\ +\infty & \alpha > 6 \\ 0 & \alpha < 6 \end{cases}$$

7) $\lim_n \dfrac{1 + \operatorname{tg}\left(\frac{1}{n^3}\right) - e^{\frac{1}{n^3}}}{n^\alpha\left(e^{\frac{1}{n^2}} - 1\right)}$

<u>numeratore</u> $\operatorname{tg}\left(\frac{1}{n^3}\right) = \frac{1}{n^3} + \frac{1}{3}\left(\frac{1}{n^3}\right)^3 + o\left(\frac{1}{n^9}\right) = \frac{1}{n^3} + \frac{1}{3}\frac{1}{n^9} + o\left(\frac{1}{n^9}\right)$

$$e^{\frac{1}{n^3}} = 1 + \frac{1}{n^3} + \frac{1}{2}\left(\frac{1}{n^3}\right)^2 + o\left(\frac{1}{n^6}\right) = 1 + \frac{1}{n^3} + \frac{1}{2}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)$$

$$1 + \operatorname{tg}\left(\frac{1}{n^3}\right) - e^{\frac{1}{n^3}} = \cancel{1} + \cancel{\frac{1}{n^3}} + \frac{1}{3}\frac{1}{n^9} + o\left(\frac{1}{n^9}\right) - \cancel{1} - \cancel{\frac{1}{n^3}} - \frac{1}{2}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)$$

$$= -\frac{1}{2}\frac{1}{n^6} + o\left(\frac{1}{n^6}\right)$$

<u>denominatore</u> $e^{\frac{1}{n^2}} - 1 = 1 + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right) - 1 = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$

$$n^\alpha\left(e^{\frac{1}{n^2}} - 1\right) = n^\alpha\left(\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right) = n^{\alpha-2}(1 + o(1)).$$

$$\lim_{n} \frac{-\frac{1}{2}\frac{1}{n^6}+o\left(\frac{1}{n^6}\right)}{n^{\alpha-2}\left(1+o(1)\right)} = \lim_{n} \frac{1}{n^{\alpha+4}} \frac{\left(-\frac{1}{2}+o(1)\right)}{\left(1+o(1)\right)} =$$

$$= \begin{cases} -\frac{1}{2} & \alpha=-4 \\ 0 & \alpha>-4 \quad \alpha+4>0 \\ -\infty & \alpha<-4 \quad \alpha+4<0 \end{cases}$$

8) $$\lim_{x\to 0^+} \frac{2^x - \sin(\alpha x) - 1 + x^3}{1-\cos\sqrt{x} - \frac{1}{2}\lg(1+x)}$$

<u>numeratore</u> $2^x = e^{x\lg 2} = 1 + x\lg 2 + \frac{1}{2}x^2\lg^2 2 +$
$+\frac{1}{6}x^3\lg^3 2 + o(x^3)$

$\sin\alpha x = \alpha x - \frac{\alpha^3 x^3}{6} + o(x^3)$

$2^x - \sin(\alpha x) - 1 + x^3 = \cancel{1} + x\lg 2 + \frac{1}{2}x^2\lg^2 2 + \frac{1}{6}x^3\lg^3 2 + o(x^3)$
$-\alpha x + \frac{\alpha^3 x^3}{6} + o(x^3) \cancel{-1} + x^3 =$
$= (\lg 2 - \alpha)\, x + \frac{1}{2}x^2\lg^2 2 + o(x^2)$

<u>denominatore</u> $\cos\sqrt{x} = 1 - \frac{x}{2} + \frac{1}{24}x^2 + o(x^2)$

$\lg(1+x) = x - \frac{x^2}{2} + o(x^2)$

$1-\cos(\sqrt{x}) - \frac{1}{2}\lg(1+x) = \cancel{1}-\cancel{1} + \cancel{\frac{x}{2}} - \frac{x^2}{24} + o(x^2) \cancel{-\frac{1}{2}x} + \frac{1}{4}x^2 +$
$+o(x^2) = \frac{5}{24}x^2 + o(x^2)$

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$$\lim_{x\to 0^+} \frac{(\lg 2-\alpha)x + \frac{1}{2}\lg^2 2\, x^2 + o(x^2)}{\frac{5}{24}x^2 + o(x^2)} =$$

$$= \lim_{x\to 0^+} \frac{\cancel{x}\left[\lg 2-\alpha + \left(\frac{1}{2}\lg^2 2\right)x + o(x)\right]}{x^{\cancel{2}}\left[\frac{5}{24} + o(1)\right]} =$$

$$= \begin{cases} \frac{1}{2}\lg^2 2 \cdot \frac{24}{5} & \lg 2 = \alpha \\ +\infty & \lg 2 > \alpha \\ -\infty & \lg 2 < \alpha \end{cases}$$

9) $$\lim_{x\to 0} \frac{\sqrt[3]{1+x^2} - 1 - \frac{1}{3}\sin(x^2)}{1-\cos\alpha x - x^2}$$

<u>numeratore</u>

$$\sqrt[3]{1+x^2} = (1+x^2)^{\frac{1}{3}} = 1 + \frac{1}{3}(+x^2) + \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)(+x^2)^2 + o(x^4) =$$

$$= 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + o(x^4)$$

$$\sin x^2 = x^2 - \frac{1}{6}(x^2)^3 + o(x^6) = x^2 - \frac{x^6}{6} + o(x^6)$$

$$\sqrt[3]{1-x^2} - 1 - \frac{1}{3}\sin x^2 = \cancel{1} \cancel{+\frac{1}{3}x^2} - \frac{1}{9}x^4 + o(x^4) \cancel{-1} \cancel{-\frac{1}{3}x^2} + \frac{1}{18}x^6 + o(x^6)$$

$$= -\frac{1}{9}x^4 + o(x^4)$$

<u>denominatore</u> $$1-\cos\alpha x - x^2 = 1 - \left(1 - \frac{\alpha^2 x^2}{2} + \frac{\alpha^4 x^4}{24} + o(x^4)\right) - x^2 =$$

$$= \frac{\alpha^2 x^2}{2} - \frac{\alpha^4}{24}x^4 + o(x^4) - x^2 = x^2\left(\frac{\alpha^2}{2} - 1\right) - \frac{\alpha^4}{24}x^4 + o(x^4)$$

$$\lim_{x\to 0} \frac{-\frac{1}{9}x^4 + o(x^4)}{x^2\left(\frac{\alpha^2}{2}-1\right) - \frac{\alpha^4}{24}x^4 + o(x^4)} = \lim_{x\to 0} \frac{x^{\cancel{4}\,2}\left(-\frac{1}{9} + o(1)\right)}{\cancel{x^2}\left(\frac{\alpha^2}{2} - 1 - \frac{\alpha^4}{24}x^2 + o(x^2)\right)}$$

se $\alpha^2 = 2 \iff \alpha = \pm\sqrt{2}$

si ha che $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\cancel{x^2}\left(-\frac{1}{9}+o(1)\right)}{\cancel{x^2}\left(-\frac{4}{24}+o(1)\right)} = \dfrac{-1/9}{-\frac{4}{24}} = \dfrac{2}{3}$

se $\alpha^2 \neq 2$ $\quad \lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{x^2\left(-\frac{1}{2}+o(1)\right)}{\left[\frac{\alpha^2}{2}-1-\frac{\alpha^2}{24}x^4+o(x^4)\right]} = 0$

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10) $\lim_{x\to 0} \dfrac{\lg(1+x)+1-\sqrt{1+2x}}{\sinh x - \lg(1+x)}$

$\lg(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + o(x^3)$

$\sqrt{1+2x} = (1+2x)^{1/2} = 1 + \frac{1}{2}(2x) + \frac{1}{2}\cdot\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)(2x)^2 +$

$+ \frac{1}{3!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)x^3 + o(x^3) = 1 + x - \frac{1}{8}4x^2 +$

$+ \frac{1}{6}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^3 + o(x^3) = 1 + x - \frac{x^2}{2} + \frac{1}{16}x^3 + o(x^3)$

Numeratore $\lg(1+x)+1-\sqrt{1+2x} = \cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + o(x^3) + \cancel{1}$

$-\cancel{1} - \cancel{x} + \cancel{\frac{x^2}{2}} - \frac{x^3}{16} + o(x^3) = \frac{x^3}{3} - \frac{x^3}{16} + o(x^3) = \frac{13}{48}x^3 + o(x^3)$

$\sinh x = x + \dfrac{x^3}{6} + o(x^3)$

denominatore $\sinh x - \lg(1+x) = \cancel{x} + \frac{x^3}{6} + o(x^3) - \cancel{x} + \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$

$= \dfrac{x^2}{2} + o(x^2)$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{x^3\left(\frac{13}{48}+o(1)\right)}{x^2\left(1/2+o(1)\right)} = 0$