$A \to \boxed{\text{SET}} \to$ holds a property

$A \,/\, \bar{A}$ $\to$ RECURSIVE
R.E.

functions

RECURSIVE $\Longleftrightarrow$ COMPUTABLE $\to \chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

NOT RECURSIVE $\Longleftrightarrow$ NOT COMPUTABLE $\to K$

predicates

DECIDABLE $\to \chi_a$

SEMI DECIDABLE $\to SC_a$

PR IR $\Longleftrightarrow$ ACKERMANN (PR UNBOUNDED)

$\downarrow$

μ-OPERATOR

WHY ACKERMANN?

R.E. $\to SC_A = \begin{cases} 1 & x \in A \\ \uparrow & x \notin A \end{cases}$

$x \in \mathbb{N}$, A set

$$A = \{x \in \mathbb{N} \mid dom(f) = \mathbb{N}\}$$

$\left[\begin{array}{l} A \text{ rec./r.e} \\ \bar{A} \end{array}\right]$
- SATURATED
- R.E. (RICE-SHAPIRO)
- RECURSIVE (RICE-THEOREM → NOT χ_A → REC.)

(1) SATURATED?

$$\mathcal{A} = \{f \in A \mid \varphi_x \in \mathcal{A}\} \mid A = \{x \in \mathbb{N} \mid W_x(f) = \mathbb{N}\}$$

(2) RECURSIVE → $\chi_H = \begin{cases} 1 & \text{if } H(x,x,y) \\ \uparrow & \text{otherwise} \end{cases}$

WE STOP

$g(x,y) = \varphi_{s(x)}(y)$

H = TOTAL

$\left[\begin{array}{l} \text{SMN-THEOREM} \rightarrow \text{REDUCTION} \\ \quad H \le_m A \end{array}\right]$ .....

$x \in H$ iff $f(x) \in A$

$x \in H, \exists x \mid$ .....

$x \notin H, \exists x \mid$ .....

$A = \{x \in \mathbb{N} \mid dom(f) = \mathbb{N}\}$

PROPERTY ON ALL SET

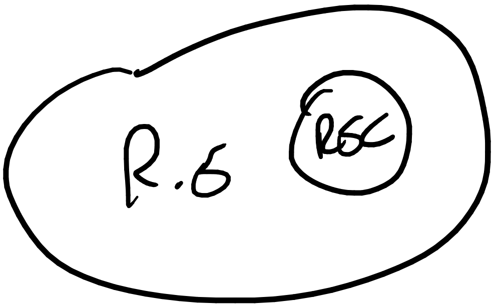
$\exists\, m, n \in \mathbb{N} \mid \varphi_m = \varphi_n,\ m \neq n$

For dummies:

- start to see if set is saturated
  - usually if saturated, then not recursive, but can be r.e.
- if it is saturated, see also if it is r.e. (so, you can write a semicharacteristic function)
- if it is r.e., check if recursive
  - use Rice's Theorem/halting set to prove it's not
  - 98% of the times sets are not recursive – apart from exercise 8.58, usually "cool proof but not real case" exercise to consider for this
- if it is not r.e. use Rice-Shapiro or negation of halting set
  - if not r.e. then it is not recursive

① SATURATED

② R.E / REC.?



$A = \{x \in \mathbb{N} \mid dom(f) = \mathbb{N}\}$

SATURATED => THERE MUST BE TWO PARTICULAR INDICES m, n TO SATISFY A NON-TRIVIAL PROPERTY

↳ DEPENDS ON THE INPUTS

EXTENSIONAL

RICE-SHAPIRO

$\mathcal{C}$ = CLASS OF ALL COMPUTABLE PROGRAMS

Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions. Then if set $A = \{x \mid \phi_x \in \mathcal{A}\}$ is r.e. then

$$\forall f\ (f \in \mathcal{A} \Leftrightarrow \exists \theta \subseteq f, \theta \text{ finite s.t. } \theta \in \mathcal{A})$$

$A \in \mathcal{A} \mid \theta \subset f$ $\quad \theta$ = SUBFUNCTION

- $\exists f \in \mathcal{C}. f \notin \mathcal{A} \wedge \exists \theta \subseteq f$ *finite*, $\theta \in \mathcal{A} \Rightarrow \boxed{\mathcal{A} \text{ not r.e.}}$ → RICE SHAPIRO
- $\exists f \in \mathcal{C}. f \in \mathcal{A} \wedge \forall \theta \subseteq f$ *finite*, $\theta \notin \mathcal{A} \Rightarrow \boxed{\mathcal{A} \text{ not r.e.}}$

$$[\exists f \notin A,\ \exists!\, \theta \in A]$$

↓

THERE IS A FUNCTION NOT IN A
BUT THERE ~~EXIST~~ A <u>SINGLE</u> SUBFUNCTION
INSIDE OF A

$$[\exists f \in A,\ \forall \theta \notin A]$$

THERE IS A FUNCTION IN A
BUT <u>NO FINITE SUBFUNCTION</u> IS IN A

Exercise 8.49. Study the recursiveness of the set $B = \{x \in \mathbb{N} \mid \varphi_x(y) = y^2$ for ~~infinitive~~ INFINITE $y\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

① B → SATURATED?

$\varphi_x = x^{TH}$ COMPONENT OF THE FUNCTION

⇓

UNIVERSAL FUNCTION

Solution: We observe that B is saturated, since $B = \{x \mid \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \mid f(y) = y^2$ for infinite $y\}$. Rice-Shapiro's theorem is used to deduce that both sets are not r.e.

$$B = \{x \mid \varphi_x \in \mathcal{B}\}$$

$$\mathcal{B} = \{f \mid f(y) = y^2 \text{ for infinite } y\}$$

$$\varphi_x(y) \equiv f(y)$$

$$[\varphi_x = \Phi_U(x, y)]$$

$$\left[\begin{array}{l} \exists f \notin A \mid \exists \theta \in A \\ \exists f \in A \mid \forall \theta \notin A \end{array}\right]$$ RICE - SHAPIRO

Exercise 8.49. Study the recursiveness of the set $B = \{x \in \mathbb{N} \ \ \varphi_x(y) = y^2 \text{ for infinitive } y\}$, i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: We observe that B is saturated, since $B = \{x \mid \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \mid f(y) = y^2 \text{ for infinite } y\}$. Rice-Shapiro's theorem is used to deduce that both sets are not r.e.

- B is not r.e. because $\mathcal{B}$ contains y^2 and none of its sub-functions finite (it does not contain any finite functions).
- $\bar{B}$ is not r.e. since $\varnothing \in \bar{\mathcal{B}}$ and $\varnothing$ admits as an extension $y^2 \notin \bar{\mathcal{B}}$.

□

$$\varnothing(x) = \begin{cases} 0 & x \in W_x \\ y & x \notin W_x \end{cases}$$

$$\boxed{\bar{B} = \{\varphi_x(y) \neq y^2 \text{ for infinite } y\}}$$

$$\exists f \notin \mathcal{B},\ \exists \theta \in \mathcal{B}$$

$$\exists f \in \mathcal{B},\ \forall \theta \notin \mathcal{B}$$

RICE-SHAPIRO - NOT R.E.

$$f(x) = \begin{cases} 1 & \text{se } x \leq 2 \\ 0 & \text{altrimenti} \end{cases}$$

$$\theta(x) = \begin{cases} 1 & \text{se } x = 2 \\ 0 & \text{altrimenti} \end{cases}$$

$$\varnothing \in \bar{\mathcal{B}},\ \forall \theta \notin \mathcal{B}$$

SUBFUNCTION (PLOT)

$f(x)$ $\quad \theta \subset f$

$[\theta(x)]$

$$f(x) = id(x)$$

$$id(x) = \begin{cases} x & x \in W_x \\ \uparrow & \end{cases}$$

$$S(x) = \begin{cases} S & x \in W_x \\ \uparrow & \end{cases}$$

$S \subset id$ (IDENTITÀ OTTIENE $f(x) = x$
S OTTIENE SOLO S)

Exercise 8.28. Let $X \subseteq \mathbb{N}$ be finite, $X \neq \emptyset$ and define $A_X = \{x \in \mathbb{N} : W_x = E_x \cup X\}$. Study the recursiveness of A, i.e., say if A_X and $\bar{A}_X$ are recursive/recursively enumerable.

$A_X \Rightarrow$ SATURATED

$A_X = \{x \in \mathbb{N} : \varphi_x \in \mathcal{A}_X\}$ ⇐ TRANSLATE

, SATURO...

VALE SEMPRE IN MODO FINITO?

[$\mathcal{A}_X = \{f \in \mathcal{A} \mid dom(x) = cod(x) \cup X\}$]

ORA, È SATURATO (SATURO). DIMOSTRIAMO RICORSIVO / R.E.

SECONDO TE? — RIC / R.E (?)

R.E $\Rightarrow$ SC (SEMI-CHARACTERISTIC)

NOT R.E $\rightarrow$ RICE-SHAPIRO

OSSERVA $\rightarrow$ ESISTERÀ MAI, COME HAI DETTO TU, UNA FUNZIONE IL CUI DOMINIO È SEMPRE ~~ESATTAMENTE~~ UGUALE AL CODOMINIO?

NO!

[$dom(f) = cod(f) \cup X$]

NOT R.E. $\rightarrow$ USA RICE-SHAPIRO!

⇓

[$X \neq \emptyset, X \neq \mathbb{N}$]

- $\exists f \notin \mathcal{A}_X, \exists \theta \in \mathcal{A}_X$
- $\exists f \in \mathcal{A}_X, \forall \theta \notin \mathcal{A}_X$

$\exists$ una funzione tale per cui $dom(f) = cod(f)$?

<u>SI</u> $\rightarrow$ LA FUNZIONE IDENTITÀ!

id (DEFINITA PER <u>OGNI</u> $n \in \mathbb{N}$)

ES. $f(5) = 5 \mid f(6) = 6$ ....

IL PROBLEMA È X $\{i,j\}$

QUINDI? $\rightarrow$ $\exists x \mid x \neq X$ $f(y)$

$$f(x) = \begin{cases} x & x \in W_x \cup \{y\} \\ \uparrow & \end{cases}$$

$$\theta(x) = \begin{cases} x & x \in W_x \\ \uparrow & \text{otherwise} \end{cases}$$

$\rightarrow$ $\underbrace{(\underbrace{dom(f) \cup cod(f)}_{\theta}) \cup X}_{f}$

$$\bar{A} = \{x \in \mathbb{N} \mid dom(x) \neq cod(x) \cup X\}$$

$\left[\begin{array}{l} \exists f \notin A \mid \exists \theta \in A \\ \exists f \in A \mid \forall \theta \notin A \end{array}\right]$ $\rightarrow$ RICE-SHAPIRO