Esercizio 3. Siano $X_1, X_2, \ldots, X_{1200}$ variabili aleatorie indipendenti ed identicamente distribuite su $(\Omega, \mathcal{F}, \mathbf{P})$ con comune distribuzione di Bernoulli di parametro $1/600$. Poniamo

$$S(\omega) \doteq \sum_{i=1}^{1200} X_i(\omega),\ \omega \in \Omega, \qquad N \doteq \min\{n \in \mathbb{N} : \mathbf{P}(S \leq n) \geq 0.98\}.$$

Sia dia una stima per N in tre modi diversi, usando

a) la disuguaglianza di Chebyshev;

b) l'approssimazione di Poisson;

c) l'approssimazione normale.

$$E[X] = p = \frac{1}{600}$$

$$\mathrm{var}(X) = p(1-p) = \frac{1}{600}\left(1 - \frac{1}{600}\right)$$

$$E[S] = p \cdot S = \frac{1}{600} \cdot 1200 = 2$$

$$\mathrm{var}(S) = p(1-p) = \frac{1}{600}\left(1 - \frac{1}{600}\right) 1200 = \frac{599}{300}$$

(b) POISSON

POISSON $\rightarrow$ BINOMIALE (piccoli numeri)

$$S \sim \mathrm{Bin}(n, k) \sim \mathrm{Bin}\left(1200; \frac{1}{600}\right)$$

$$\lambda = n \cdot k = 1200 \cdot \frac{1}{600} = 2$$

$$P(S \leq n) \geq 0.98$$

TAVOLA

$F_{POISS}(2) \geq 0.98 \rightsquigarrow n \geq 5$

SORPASSATO IN X

$N_x = 5$

Cumulative probability, Poisson distribution

λ	$x=0$	1	2	3	4	5	6	7	8	9
0.02	0.980	1.000								
0.04	0.961	0.999	1.000							
0.06	0.942	0.998	1.000							
0.08	0.923	0.997	1.000							
0.10	0.905	0.995	1.000							
0.15	0.861	0.990	1.000	1.000						
0.20	0.819	0.983	0.999	1.000						
0.25	0.779	0.974	0.998	1.000						
0.30	0.741	0.963	0.996	1.000						
0.35	0.705	0.951	0.995	1.000						
0.40	0.670	0.938	0.992	0.999	1.000					
0.45	0.638	0.925	0.989	0.999	1.000					
0.50	0.607	0.910	0.986	0.998	1.000					
0.55	0.577	0.894	0.982	0.998	1.000					
0.60	0.549	0.878	0.977	0.997	1.000					
0.65	0.522	0.861	0.972	0.996	0.999	1.000				
0.70	0.497	0.844	0.966	0.994	0.999	1.000				
0.75	0.472	0.827	0.960	0.993	0.999	1.000				
0.80	0.449	0.809	0.953	0.991	0.999	1.000				
0.85	0.427	0.791	0.945	0.989	0.998	1.000				
0.90	0.407	0.772	0.937	0.987	0.998	1.000				
0.95	0.387	0.754	0.929	0.984	0.997	1.000				
1.00	0.368	0.736	0.920	0.981	0.996	0.999	1.000			
1.1	0.333	0.699	0.900	0.974	0.995	0.999	1.000			
1.2	0.301	0.663	0.879	0.966	0.992	0.999	1.000			
1.3	0.273	0.627	0.857	0.957	0.989	0.998	1.000			
1.4	0.247	0.592	0.834	0.946	0.986	0.997	0.999	1.000		
1.5	0.223	0.558	0.809	0.934	0.981	0.996	0.999	1.000		
1.6	0.202	0.525	0.783	0.921	0.976	[illegible]	[illegible]	1.000		
1.7	0.183	0.493	0.757	0.907	0.970	[illegible]	0.998	1.000		
1.8	0.165	0.463	0.731	0.891	0.964	0.990	0.997	0.999	1.000	
1.9	0.150	0.434	0.704	0.875	0.956	0.987	0.997	0.999	1.000	
2.0	0.135	0.406	0.677	0.857	0.947	0.983	0.996	0.999	1.000	
2.2	0.111	0.355	0.623	0.819	0.927	0.975	0.993	0.998	1.000	
2.4	0.091	0.308	0.570	0.779	0.904	0.964	0.988	0.997	0.999	1.000
2.6	0.074	0.267	0.518	0.736	0.877	0.951	0.983	0.995	0.999	1.000

$x \geq 5$

NORMALE STANDARD → TEOREMA LIMITE CENTRALE

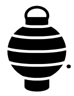
$P(S \leq n)$

$$\left(\frac{S - E[S]}{\sqrt{var(S)}}\right) \Rightarrow P\left(\frac{S - E[S]}{\sqrt{var(S)}}\right) \leq \left(\frac{n - E[S]}{\sqrt{var(S)}}\right)$$

S

$$\Phi\left(\frac{n - E[S]}{\sqrt{Var(S)}}\right) \geq 0{,}98$$

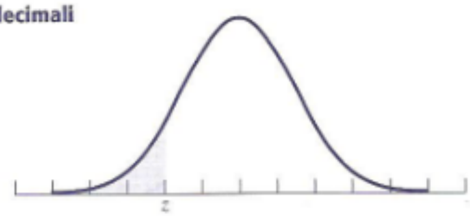
$E[S] = 2$ $\quad Var(S) = \frac{599}{300}$

LANTERNA USNOG 

TAVOLA

$$\Phi(y) \geq 0{,}98 \longrightarrow 2{,}06$$

Approssimazione a due decimali



P(Z < z)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.979[illegible]	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.984[illegible]	[illegible]	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964

-0,06

$$n \in \mathbb{N} \wedge \frac{n - \overset{2}{E[S]}}{\sqrt{var(S)}} \geq 2.06$$

$$\frac{n - 2}{\sqrt{\frac{599}{300}}} \geq 2.06 \quad \Rightarrow \quad n \geq 5$$

$\uparrow$

TROVARE N

CHEBYSHEV

$\downarrow$

$$P(|Y - E[Y]| \geq \varepsilon) \leq \frac{var(Y)}{\varepsilon^2}$$

$\downarrow$

$$P(S \le n) = 1 - P(S > n) =$$

$$1 - P(S \ge n+1)$$

$$P(S \ge n+1) = P\big(S - E[S] \ge n+1 - \underbrace{E[S]}_{2}\big)$$

$$\underbrace{P\big(|S - E[S]| \ge \overset{n+1}{n-1}\big)}_{\text{CHEBYSHEV}} \le \frac{Var(S)}{(n+1-\underbrace{E[S]}_{2})^2}$$

CHEBYSHEV

$$\underbrace{P(S \ge n+1)}_{\text{ORIGINALE} \;=\; \text{COMPLEMENTARE}} \le \frac{\frac{589}{300}}{(n-1)^2}$$

ORIGINALE

= COMPLEMENTARE

$$P(S \le n) \ge 1 - \frac{589}{300} \cdot \frac{1}{(n-1)^2}$$

$$1 - \frac{599}{300} \cdot \frac{1}{(n-1)^2} \geq 0.98$$

$\Downarrow$ RISOLVI PER N

RISULTATO $\Rightarrow$ $n \geq 9.10+1$

$n \geq 10.10$

$n \geq 11 \rightarrow N = 11$

Esercizio 4. Si trovino variabili aleatorie non-negative X e Y tali che

$$\mathbf{P}(X \leq Y \leq 4X) = 1, \quad \mathbf{E}[Y] = 2\,\mathbf{E}[X], \quad \mathbf{E}[Y^2] \neq 4\,\mathbf{E}[X^2].$$

$X = 1 \quad P(1 \leq y \leq 4) = 1$

$E[Y] = \underbrace{2\,E[X]}_{2 \cdot 1} = 2$

$\underbrace{E[Y^2]}_{4} \neq \underbrace{4\,E[X^2]}_{4}$

$X \doteq 1 \quad \sim Y\{1;3\}$

DISCRETE

$$E[Y] = \sum_{i=1}^{m} x_i \cdot p(x_i)$$

SOMMA DISCRETO

$$= 1 \cdot \underbrace{P(Y=1)}_{1/2} + 3 \cdot \underbrace{P(Y=3)}_{\frac{1}{2}}$$

$$! = 2 \; E[X] = 2$$

$$= 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = \frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$$

$$E[Y^2] = \underbrace{(1)^2}_{y^2} \cdot \underbrace{\frac{1}{2}}_{p} + \underbrace{(3)^2}_{y^2} \cdot \underbrace{\frac{1}{2}}_{p}$$

$$= \frac{10}{2} = 5 \neq 4$$

ES. 1

$$\sum_{i=1}^{n} x_i \Rightarrow -3\cdot\frac{1}{3} - 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 6\cdot\frac{1}{3}$$

$$= \frac{7}{6} = E[X]$$

$$E[X^2] = (-3)^2\cdot\frac{1}{3} + (-1)^2\cdot\frac{1}{6} + (2)^2\cdot\frac{1}{6} + (6)^2\cdot\frac{1}{3} = \frac{85}{6}$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2$$

$$= \frac{85}{6} - \frac{49}{36} = \frac{521}{36}$$

$$X = U^2 \sim \underline{\mathrm{Unif}}(0;1)$$

$$\mathbb{1}_{(0,1)} = \frac{1}{b-a} = \frac{1}{1-0}$$

ü

1

$$E[X] = E[U^2] =$$

$$\int_{-\infty}^{\infty} x \cdot f_U(x)\,dx$$

$$Y \sim U^2 \Rightarrow \int_{-\infty}^{\infty} x^2 \cdot \underbrace{f_U(x)}_{1}\,dx$$

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$$

$$E[X^2] = E[U^4] \Rightarrow \int_{-\infty}^{\infty} x^4 \cdot f_U(x)\,dx$$

$$= \int_0^1 x^4\,dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = \frac{4}{45}$$

$$X = 1 - Y \rightsquigarrow Y \sim \text{Exp}(\overset{\lambda}{2})$$

$$f_Y = \lambda \cdot e^{-\lambda x} = 2 \cdot e^{-2x} \cdot \mathbb{1}_{[0,\infty)}(x)$$

DENSITÀ

$$E[X] = E[1 - Y] = \int_{-\infty}^{\infty} (1-x) \cdot f_Y(x)\, dx$$

$$= \int_0^{\infty} 2e^{-2x}(1-x)\, dx$$

$$= 2\int_0^{\infty} \underbrace{e^{-2x}}_{f} \underbrace{(1-x)}_{g}\, dx$$

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

SHIT HAPPENS...

$$= \underbrace{2\int_0^\infty e^{-2x}\,dx}_{= \left[-e^{-2x}\right]_0^\infty = 1} - \underbrace{2\int_0^\infty x\cdot e^{-2x}\,dx}$$

$$\overset{\text{int. parti}}{=} \left[-x\cdot e^{-2x}\right]_0^\infty + \int_0^\infty e^{-2x}\,dx$$

$$= 0 + \left[-\tfrac{1}{2} e^{-2x}\right]_0^\infty = \frac{1}{2}$$

$$\leadsto \quad E[X] = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$E[X^2] = E[(1-Y)^2] = \int_{-\infty}^{\infty} (1-x)^2 \cdot f_Y(x)\,dx$$

↓

(iii) cont.

$$\leadsto \quad E[X^2] = 2\int_0^\infty (1-x)^2 e^{-2x}\,dx$$

$$\overset{\text{int. per parti}}{=} \left[-(1-x)^2 e^{-2x}\right]_0^\infty - \int_0^\infty \left(-2(1-x)\right)\cdot\left(-e^{-2x}\right)dx$$

$$= 1 - \underbrace{2\int_0^\infty (1-x)e^{-2x}\,dx}_{=\frac{1}{2}}$$

↗ sopra

$$= 1 - \frac{1}{2} = \frac{1}{2}.$$

$$\leadsto \quad var(X) = E[X^2] - E[X]^2 = \frac{1}{2} - \frac{1}{4}$$