1. Si considerino le seguenti matrici su $\mathbb{C}$:

$$A = \begin{pmatrix} 1 & -1 \\ i & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3i \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 \\ -2 & -2 \end{pmatrix} \quad D = \begin{pmatrix} 0 & -1 \\ 1+i & 2 \end{pmatrix}$$

Si determinino le seguenti matrici:

(a) (CD)A

(b) $A^T B$

(c) $3A(B - D^T)$

(d) $(4B - C)^T - DC$

1 (a)

$$(CD)A = \left(\begin{pmatrix} 3 & 4 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1+i & 2 \end{pmatrix} \right) \begin{pmatrix} 1 & -1 \\ i & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 4+4i & 5 \\ -2-2i & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ i & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 4+9i & -\frac{3}{2}-4i \\ -2-4i & 1+2i \end{pmatrix}$$

(b) $$A^T B = \begin{pmatrix} 1 & i \\ -1 & \frac{i}{2} \end{pmatrix} \begin{pmatrix} 0 & 3i \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 3i \\ -\frac{i}{2} & -3i \end{pmatrix}$$

(c) $$3A(B-D^T) = \begin{pmatrix} 3 & -3 \\ 3i & \frac{3}{2} \end{pmatrix} \left(\begin{pmatrix} 0 & 3i \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1+i \\ -1 & 2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 3 & -3 \\ 3i & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 & -1+2i \\ 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3+6i \\ 0 & -9-3i \end{pmatrix}$$

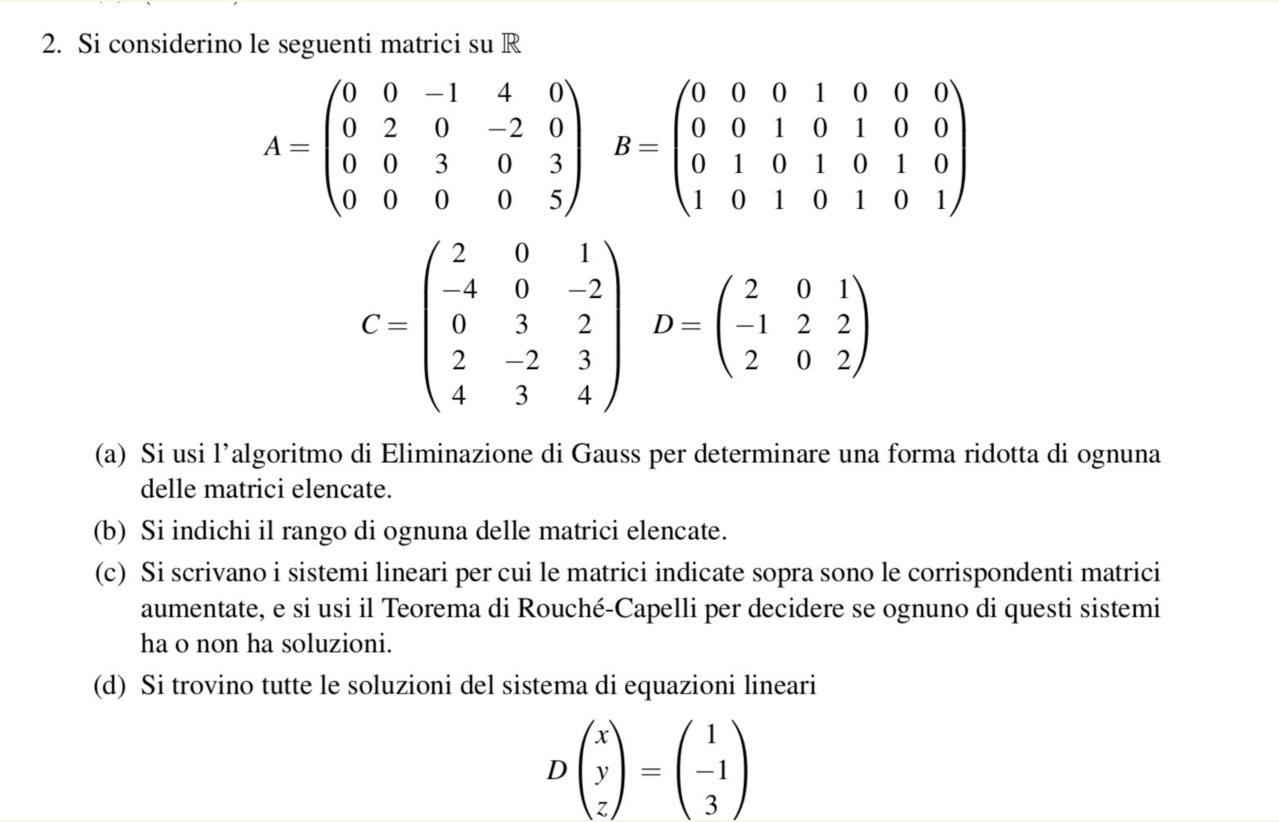
(d)

$$(4B - C)^T - DC = \left(\begin{pmatrix} 0 & 12i \\ -4 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ -2 & -2 \end{pmatrix} \right)^T - \begin{pmatrix} 0 & -1 \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4+12i \\ -2 & 2 \end{pmatrix}^T - \begin{pmatrix} 2 & 2 \\ -1+3i & 4i \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -2 \\ -4+12i & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ -1+3i & 4i \end{pmatrix}$$

$$= \begin{pmatrix} -5 & -4 \\ -3+9i & 2-4i \end{pmatrix}$$



2. Si considerino le seguenti matrici su $\mathbb{R}$

$$A = \begin{pmatrix} 0 & 0 & -1 & 4 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 & 1 \\ -4 & 0 & -2 \\ 0 & 3 & 2 \\ 2 & -2 & 3 \\ 4 & 3 & 4 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 2 \\ 2 & 0 & 2 \end{pmatrix}$$

(a) Si usi l'algoritmo di Eliminazione di Gauss per determinare una forma ridotta di ognuna delle matrici elencate.

(b) Si indichi il rango di ognuna delle matrici elencate.

(c) Si scrivano i sistemi lineari per cui le matrici indicate sopra sono le corrispondenti matrici aumentate, e si usi il Teorema di Rouché-Capelli per decidere se ognuno di questi sistemi ha o non ha soluzioni.

(d) Si trovino tutte le soluzioni del sistema di equazioni lineari

$$D \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$$

2(a) + (b)

$$A = \begin{pmatrix} 0 & 0 & -1 & 4 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \overset{E_{12}}{\sim} \begin{pmatrix} 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & -1 & 4 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \overset{\substack{E_1(\frac{1}{2}) \\ E_2(-1)}}{\sim} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

$$\overset{E_{32}(-3)}{\sim} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 12 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \underset{E_4(\frac{1}{5})}{\overset{E_3(\frac{1}{12})}{\sim}} \begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \operatorname{rk} A = 4$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \overset{E_{14}}{\underset{E_{23}}{\sim}} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

B''

$$\operatorname{rk} B = 4$$

$$C = \begin{pmatrix} 2 & 0 & 1 \\ -4 & 0 & -2 \\ 0 & 3 & 2 \\ 2 & -2 & 3 \\ 4 & 3 & 4 \end{pmatrix} \overset{E_1(\frac{1}{2})}{\sim} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ -4 & 0 & -2 \\ 0 & 3 & 2 \\ 2 & -2 & 3 \\ 4 & 3 & 4 \end{pmatrix} \overset{E_{21}(4)}{\underset{\substack{E_{41}(-2) \\ E_{51}(-4)}}{\sim}} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & -2 & 2 \\ 0 & 3 & 2 \end{pmatrix} \overset{E_{24}}{\sim} \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{E_2(\frac{1}{3})}{\sim}\begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{2}{3}\\ 0 & 3 & 2\\ 0 & -2 & 2\\ 0 & 0 & 0\end{pmatrix}\underset{E_{42}(2)}{\overset{E_{32}(-3)}{\sim}}\begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{2}{3}\\ 0 & 0 & 0\\ 0 & 0 & \frac{10}{3}\\ 0 & 0 & 0\end{pmatrix}\underset{E_3(\frac{3}{10})}{\overset{E_{34}}{\sim}}\begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{2}{3}\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\quad \operatorname{rk} C = 3.$$

$$D=\begin{pmatrix}2 & 0 & 1\\ -1 & 2 & 2\\ 2 & 0 & 2\end{pmatrix}\overset{E_1(\frac{1}{2})}{\sim}\begin{pmatrix}1 & 0 & \frac{1}{2}\\ -1 & 2 & 2\\ 2 & 0 & 2\end{pmatrix}\underset{E_{31}(-2)}{\overset{E_{21}(1)}{\sim}}\begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 2 & \frac{5}{2}\\ 0 & 0 & 1\end{pmatrix}\overset{E_2(\frac{1}{2})}{\sim}\begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{5}{4}\\ 0 & 0 & 1\end{pmatrix}\operatorname{rk} D = 3$$

(b)

$$A=\left(\underbrace{\begin{matrix}0 & 0 & -1 & 4\\ 0 & 2 & 0 & -2\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0\end{matrix}}\left|\begin{matrix}0\\ 0\\ 3\\ 5\end{matrix}\right.\right)\sim \dots \sim\left(\begin{matrix}0 & 1 & 0 & -1\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{matrix}\left|\begin{matrix}0\\ 0\\ \frac{1}{4}\\ 1\end{matrix}\right.\right)$$

A'

$\mathrm{rk}A' < \mathrm{rk}A$ quindi il sistema non ammette soluzione

$$B = \left(\begin{array}{c|c} \underbrace{\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{matrix}}_{B'} & \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \end{array}\right) \sim \left(\begin{array}{c|c} \overbrace{\underbrace{\begin{matrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}}_{\mathrm{rk}\,4}}^{\mathrm{rk}\,4} & \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} \end{array}\right)$$

Abbiamo che $\mathrm{rk}B = \mathrm{rk}B' < \#$ colonne di B'

Quindi il sistema ammette infinite soluzione.

$$C = \left(\underbrace{\begin{matrix} 2 & 0 \\ -4 & 0 \\ 0 & 3 \\ 2 & -2 \\ 4 & 3 \end{matrix}}_{C'} \middle| \begin{matrix} 1 \\ -2 \\ 2 \\ 3 \\ 4 \end{matrix} \right) \sim \ldots \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

Quindi $rkC' < rkC$ e non esiste soluzione

$$D = \left(\underbrace{\begin{matrix} 2 & 0 \\ -1 & 2 \\ 2 & 0 \end{matrix}}_{D'} \middle| \begin{matrix} 1 \\ 2 \\ 2 \end{matrix} \right) \sim \ldots \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{array} \right)$$

Quindi $rkD' < rkD$ e non esiste soluzione

(d)

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ -1 & 2 & 2 & -1 \\ 2 & 0 & 2 & 3 \end{array}\right) \overset{E_1(\frac{1}{2})}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ -1 & 2 & 2 & -1 \\ 2 & 0 & 2 & 3 \end{array}\right) \underset{E_3(-2)}{\overset{E_{21}(1)}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & \frac{5}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 2 \end{array}\right) \overset{E_2(\frac{1}{2})}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\rightsquigarrow \begin{cases} x_1 + \frac{1}{2}x_3 = \frac{1}{2} \\ x_2 + \frac{5}{4}x_3 = \frac{-1}{4} \\ x_3 = 2 \end{cases} \rightsquigarrow \begin{cases} x_1 = \frac{1}{2} - 1 = -\frac{1}{2} \\ x_2 = \frac{-1}{4} - \frac{10}{4} = \frac{-11}{4} \\ x_3 = 2 \end{cases}$$

3. Per ogni parametro t in $\mathbb{R}$ si consideri la matrice

$$A_t := \begin{pmatrix} 1 & t & -1 & t \\ 2 & 2t & -1 & 3t+1 \\ 1 & -1 & -t & 2 \end{pmatrix}$$

(a) Si calcoli il rango di A_t quando $t = -1$.

(b) Si calcoli il rango di A_t per ogni valore di t.

(c) Supponiamo che la matrice A_t sia la matrice aumentata di un sistema lineare su $\mathbb{R}$. Per quali valori di t il sistema avrà soluzione?

(a)

$$A_t = \begin{pmatrix} 1 & t & -1 & t \\ 2 & 2t & -1 & 3t+1 \\ 1 & -1 & -t & 2 \end{pmatrix} \underset{E_{31}(-1)}{\overset{E_{21}(-2)}{\sim}} \begin{pmatrix} 1 & t & -1 & t \\ 0 & 0 & 0 & t+1 \\ 0 & -1-t & -t+1 & 2-t \end{pmatrix} \overset{E_{23}}{\sim} \begin{pmatrix} 1 & t & -1 & t \\ 0 & -1-t & -t+1 & 2-t \\ 0 & 0 & 0 & t+1 \end{pmatrix} = B_t$$

$$\operatorname{rk} A_{-1} = \operatorname{rk} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \operatorname{rk} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2$$

(b)

$$B_t = \begin{pmatrix} 1 & t & -1 & t \\ 0 & -1-t & -t+1 & 2-t \\ 0 & 0 & 0 & t+1 \end{pmatrix} \overset{t\neq -1}{\sim} \begin{pmatrix} 1 & t & -1 & t \\ 0 & 1 & \frac{-t+2}{-1-t} & \frac{2-t}{-1-t} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{rk}\, A_t = \begin{cases} 2 & \text{se } t=-1 \\ 3 & \text{se } t\neq -1 \end{cases}$$

(c)

$$\boxed{t\neq -1} \quad A_t = \left(\underbrace{\begin{matrix} 1 & t & -1 \\ 2 & 2t & -2 \\ 1 & -1 & -t \end{matrix}}_{A_t'} \left| \begin{matrix} t \\ 3t+1 \\ 2 \end{matrix} \right. \right) \sim \dots \sim \left(\begin{matrix} 1 & t & -1 \\ 0 & 1 & \frac{-t+2}{-1-t} \\ 0 & 0 & 0 \end{matrix} \left| \begin{matrix} t \\ \frac{2-t}{-1-t} \\ 1 \end{matrix} \right. \right)$$

$\text{rk}\, A_t' < \text{rk}\, A_t \implies$ il sistema non ammette soluzione

$\boxed{t=-1}$

$$A_{-1} = \left(\underbrace{\begin{matrix} 1 & -1 & -1 \\ 2 & -2 & -1 \\ 1 & -1 & 1 \end{matrix}}_{A'_{-1}} \middle| \begin{matrix} -1 \\ -2 \\ 2 \end{matrix} \right) \sim \dots \sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$\mathrm{rk}A'_{-1} = \mathrm{rk}A_{-1} < \#$ col. di $A'_{-1} \Longrightarrow$ il sistema ammette infinite soluzione

4. Si risolva la seguente equazione nell'insieme di numeri complessi: $x^4 + 1 = 0$.

Troviamo le radici quarte di -1 :

$$-1 = \cos\pi + i\sin\pi$$

Quindi le radici sono

$$z_0 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$z_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

$$z_2 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$$

$$z_3 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$$

5. Si mostri che la trasposta A^T e l'inversa A^{-1} coincidono per $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ per ogni θ.

$A^T = A^{-1}$ se e solo se $A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$A^T A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$