Lessons touched by this meeting according to schedule:

- 18. 09/12/2024
  - Reducibility and r.e. sets [§9.1.3]
  - Alternative characterisation of r.e. sets [§7.2.3, §7.2.7]
  - Introduction to Rice-Shapiro's theorem [§7.2.16]
- 19. 10/12/2024
  - Rice-Shapiro's theorem. Proof, examples, counterexample to the converse implication [§7.2.16]

Still on predicates (logical implications):

Structure theorem (Structure of semi-decidable predicates)

$P(\vec{x})$ semi-decidable $\Leftrightarrow$ there is $Q(t,\vec{x}) \subseteq \mathbb{N}^{k+1}$ decidable predicate s.t. $P(\vec{x}) = \exists t. Q(t,\vec{x})$

(in words: the predicate P is a generalization of a decidable predicate that is computed over multiple points. In general, existentially quantifying transforms a decidable predicate into a semidecidable one)

Definition (projection theorem) – closure by existential quantification

Let $P(x,\vec{y}) \subseteq \mathbb{N}^{k+1}$ semi-decidable

Then $R(\vec{y}) \equiv \exists x. P(x,\vec{y})$ is semi-decidable

What does recursively enumerable mean?

- There is a countable number of steps for which a function is computable
- Consequence: etymology theorem

Proposition: Let $A \subseteq \mathbb{N}$ be a set

A r.e. iff $A = \emptyset$ or ($A = img(f)$ $f: \mathbb{N} \to \mathbb{N}$ total computable)

$\Rightarrow$

Proposition: Let $A \subseteq \mathbb{N}$

A r.e. iff $A = dom(f)$, f computable

This is used logically in parallel with Rice's theorem:

THEOREM 14.6 (Rice's theorem). *Let $A \in \mathbb{N}, A \neq \emptyset, A \neq \mathbb{N}$ be saturated. Then it is not recursive.*

This is one of the two ways to prove if a set is recursive; otherwise, for the previous one, use a set for which both domain/codomain are not computable by definition → K, the halting set!

The Rice-Shapiro theorem helps in proving that a set is not r.e.; in particular, the theorem says that any observation made about computable functions can be done by testing the values of the functions at finitel many arguments.

More precisely, the only properties of the behavior of programs which can be semi-decidable are the "finitary properties" (properties which depend on the behaviour on a finite number/amount of inputs).

one finite number of inputs

The Rice-Shapiro theorem is a fundamental result in computability theory that characterizes the conditions under which a set of computable functions can be recursively enumerable

It states that for a set of computable functions A ⊆ C, if the corresponding set of indices A = {x | ϕ_x ∈ A} is r.e., then for any function f, f ∈ A if and only if there exists a finite subfunction θ ⊆ f such that θ ∈ A.

Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of computable functions. Then if set $A = \{x \mid \phi_x \in \mathcal{A}\}$ is r.e. then

$$\forall f\ (f \in \mathcal{A} \Leftrightarrow \exists \theta \subseteq f, \theta \text{ finite s.t. } \theta \in \mathcal{A})$$

What does it mean in practice?

It's used to show that sets are not r.e. – alternatively (in a few cases), it can be shown a set is not r.e. using a reduction from the complement of the halting set $\overline{K}$. Usually, it's the easiest way. If a set is not r.e., then it is also not recursive (given r.e. it's a superset of recursive).

How to use it

Generally, it can be used in two ways:

- $\exists f \in \mathcal{C}. f \notin \mathcal{A} \land \exists \theta \subseteq f\ finite, \theta \in \mathcal{A} \Rightarrow A\ not\ r.e.$
- $\exists f \in \mathcal{C}. f \in \mathcal{A} \land \forall \theta \subseteq f\ finite, \theta \notin \mathcal{A} \Rightarrow A\ not\ r.e.$

1. A ⊆ C denotes a set of computable functions.
2. A = {x | ϕ_x ∈ A} represents the set of indices corresponding to the functions in A, where ϕ_x is the function computed by the x^{th} program in an enumeration of computable functions.
3. f ∈ A means that the function f belongs to the set A.
4. θ ⊆ f denotes that θ is a subfunction of f, i.e., dom(θ) ⊆ dom(f) and for all x ∈ dom(θ), θ(x) = f(x). In other words, θ agrees with f wherever θ is defined.
5. θ ∈ A signifies that the subfunction θ belongs to the set A.

First, the set <u>has to</u> be saturated (which means there is a set of computable functions holding the exercise property, just writing dom/cod in place of W/E). We can use here functions like:

- id, which is the identity function, always defined for every natural number
  - This one is *usually* used to show $id \in \mathcal{A}$ and $\exists \theta \notin \mathcal{A}$ (or viceversa) is not r.e.
  - So, this is a function and also requires the use of subfunctions
- $\emptyset$, which is not the empty set, but a function with empty domain, called "always undefined function"
  - This one is usually used as a subfunction (so first you need to have a function, such as the identity or something), so like $f \in \mathcal{A}$ *but* $\emptyset \notin \mathcal{A}$ (or viceversa) so A is not r.e.
- Other times, constant functions fit the bill (like $\mathbf{0}, \mathbf{1}$) or just create custom functions/subfunctions

* How do we use it? We use it to show that sets are not r.e.

1. $\exists f \quad f \notin \mathcal{A}$ and $\exists \vartheta \subseteq f$ ϑ finite $\vartheta \in \mathcal{A} \Rightarrow A$ not r.e
2. $\exists f \quad f \in \mathcal{A}$ and $\forall \vartheta \subseteq f$ ϑ finite $\vartheta \notin \mathcal{A} \Rightarrow A$ not r.e.

Let's go now see many examples:

Exercise 8.48. Study the recursiveness of the set $A = \{x \in \mathbb{N} \mid \mathbb{P} \subseteq W_x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$, where $\mathcal{A} = \{f \mid \mathbb{P} \subseteq dom(f)\}$. We can use Rice-Shapiro's theorem to show that

- A is not r.e .
  In fact $id \in \mathcal{A}$ since $\mathbb{P} \subseteq dom(id) = \mathbb{N}$ and no finite $\theta \subseteq id$ can be in $\mathcal{A}$, since functions in $\mathcal{A}$ necessarily have an infinite domain.
- $\overline{A}$ not r.e .
  In fact, $id \notin \bar{\mathcal{A}}$, and $\varnothing \subseteq id$, $\varnothing \in \bar{\mathcal{A}}$.

□

Exercise 8.65. Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x \backslash E_x \text{ infinite}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursively enumerable.

Solution: The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ with $\mathcal{A} = \{f \mid dom(f)\backslash cod(f) \text{ infinite}\}$. By Rice-Shapiro's theorem:

- A is not r.e., since $\mathbf{1} \in \mathcal{A}$, but no finite subfunction $\theta \subseteq \mathbf{1}$ can belong to $\mathcal{A}$. In fact $dom(\theta)$ is finite and therefore also $dom(\theta)\backslash cod(\theta)$ is finite. Therefore $\theta \notin \mathcal{A}$.
- $\bar{A}$ is not r.e., since $\varnothing \in \bar{\mathcal{A}}$, $\mathbf{1} \notin \bar{\mathcal{A}}$ and $\varnothing \subseteq \mathbf{1}$.

□

Exercise 8.70. Classify the following set from the point of view of recursiveness

$$B = \{x \in \mathbb{N} : \forall y \in W_x. \exists z \in W_x. (y < z) \ \wedge \ (\varphi_x(y) < \varphi_x(z))\},$$

i.e., establish if B and $\bar{B}$ are recursive/recursive enumerable.

Solution: The set B is saturated, given that $B = \{x : \varphi_x \in \mathcal{B}\}$, where $\mathcal{B} = \{f \in \mathcal{C} : \forall y \in dom(f). \exists z \in dom(f). (y < z) \ \wedge \ (f(y) < f(z))\}\}$.

The set B is not is r.e. by Rice-Shapiro's theorem. In fact, observe that $\mathbf{1} \notin \mathcal{B}$, but $\varnothing \subseteq \mathbf{1}$ and $\varnothing \in \mathcal{B}$.

For the complement $\bar{\mathcal{B}} = \{f \mid \exists y \in dom(f). \forall z \in dom(f). y < z \rightarrow (f(y) \geqslant f(z))\}$, observe that if θ is any finite function,$\theta \neq \varnothing$, $y = \max(dom(\theta))$ clearly satisfies the condition definition of $\bar{\mathcal{B}}$. Hence, it is enough to observe that $id \notin \bar{\mathcal{B}}$ and consider $\theta \subseteq id$,$\theta \neq \varnothing$ noting that $\theta \in \bar{\mathcal{B}}$. □

Exercise 8.71. Classify the following set from the point of view of recursiveness

$$A = \{x \mid W_x \cup E_x = \mathbb{N}\},$$

i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: The set A is saturated since $A = \{x \mid \varphi_x \in \mathcal{A}\}$ with $\mathcal{A} = \{f \mid dom(f) \cup cod(f) = \mathbb{N}\}$.

By Rice-Shapiro's theorem:

- A is not r.e., since $id \in \mathcal{A}$, but no finite subfunction $\theta \subseteq id$ can belong to $\mathcal{A}$. In fact $dom(\theta)$ is finite and therefore also $cod(\theta)$ is finite. Hence their union $dom(\theta) \cup dom(\theta)$ is again finite, which implies that $dom(\theta) \cup dom(\theta) \neq \mathbb{N}$. Therefore $\theta \notin \mathcal{A}$.
- $\bar{A}$ is not r.e., since $\varnothing \in \bar{\mathcal{A}}$, $id \notin \bar{\mathcal{A}}$ and $\varnothing \subseteq id$.

□

Other similar recursiveness exercises:

Exercise 8.15. Let $\mathbb{P}$ be the set of even numbers. Prove that indicated with $A = \{x \in \mathbb{N} : E_x = \mathbb{P}\}$, we have $\bar{K} \leqslant_m A$.

Solution: To obtain the reduction function we can consider

$$f(x, y) = \begin{cases} 2y & \text{if } \neg H(x, x, y) \\ 1 & \text{otherwise} \end{cases}$$

The function f is computable, since it can be written as $f(x, y) = 2\, y\, \bar{sg}(\chi_H(x, x, y)) + \chi_H(x, x, y)$.

Therefore, by the smn theorem, there exists $s : \mathbb{N} \rightarrow \mathbb{N}$ computable total, such that $f(x, y) = \varphi_{s(x)}(y)$ for each $x, y \in \mathbb{N}$, which can be used as a reduction function for $\bar{K} \leqslant_m A$. Indeed:

- if $x \in \bar{K}$, then $\chi_H(x, x, y) = 0$ for each y, and therefore $\varphi_{s(x)}(y) = f(x, y) = 2y$ for each y. Thus $E_{s(x)} = \mathbb{P}$ and hence $s(x) \in A$.
- if $x \notin \bar{K}$, or $x \in K$ then there exists y_0 such that $\chi_H(x, x, y) = 1$ for each $y \geqslant y_0$. Therefore $\varphi_{s(x)}(y) = 1$ for $y \geqslant y_0$, thus $1 \in E_{s(x)}$ and therefore $E_{s(x)} \neq \mathbb{P}$. Hence $s(x) \notin A$.

□

Smn-theorem:

Exercise: there exists a function $s : \mathbb{N} \to \mathbb{N}$ total computable

$$|W_{s(x)}| = 2^x \qquad |E_x| = x+1$$

Define

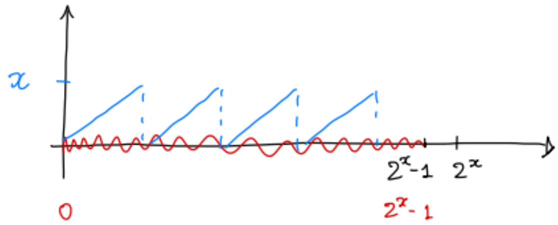
$$g : \mathbb{N}^2 \to \mathbb{N}$$

$$g(x, y) = \begin{cases} rem(x+1, y) & y < 2^x \\ \uparrow & \text{otherwise} \end{cases}$$

$$= rem(x+1, y) + \mu w.\ \overline{sg}(2^x \dot{-} y)$$

$\hookrightarrow 0$ if $y < 2^w$

$\hookrightarrow \uparrow$ otherwise



By smn there exists $s : \mathbb{N} \to \mathbb{N}$ such that

$$\varphi_{s(x)}(y) = g(x,y) \quad \forall x, y$$

Conclude by showing that s is the desired function

① $|W_{s(x)}| = 2^x$

② $|E_{s(x)}| = x + 1$

① $W_{s(x)} = \{ y \mid \varphi_{s(x)}(y)\downarrow \}$

$= \{ y \mid g(x,y)\downarrow \} = \{ y \mid y < 2^x \} = [0, 2^x)$

$\Rightarrow |W_{s(x)}| = 2^x$

② $E_{s(x)} = \{ \varphi_{s(x)}(y) \mid y \in W_{s(x)} \}$

$= \{ rem(x+1, y) \mid y \in [0, 2^x) \}$

$= [0, x+1)$

$\Rightarrow |E_{s(x)}| = |[0, x+1)| = x+1$

Exercise 8.33. Study the recursiveness of the set $A = \{x \in \mathbb{N} : x \in W_x \ \wedge\ \varphi_x(x) > x\}$, i.e., establish if A and $\bar{A}$ are recursive/recursive enumerable.

Solution: We show that $K \leqslant A$, thus A is not recursive. Define

$$g(x,y) = \begin{cases} y+1 & \text{if } x \in K \\ \uparrow & \text{otherwise} \end{cases}$$

The function $g(x,y)$ is computable, since

$$g(x,y) = (y+1) \cdot sc_K(x)$$

So by the SMN theorem, there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that for each $x, y \in \mathbb{N}$

$$\varphi_{s(x)}(y) = g(x,y)$$

The function s is a reduction function of K to A. In fact

- if $x \in K$ then $\varphi_{s(x)}(y) = g(x,y) = y+1$ for each $y \in \mathbb{N}$. Therefore $s(x) \in W_{s(x)} = \mathbb{N}$e $\varphi_{s(x)}(s(x)) = s(x)+1 > s(x)$. Therefore $s(x) \in A$.
- if $x \notin K$ then $\varphi_{s(x)}(y) = g(x,y) \uparrow$ for each $y \in \mathbb{N}$. Therefore $s(x) \notin W_{s(x)} = \varnothing$. Thus $s(x) \notin A$.

Furthermore, A is r.e., since its characteristic function

$$sc_A(x) = \mathbf{1}(\mu w.\ (x+1) \dot{-} \varphi_x(x)) = \mathbf{1}(\mu w.\ ((x+1) \dot{-} \Psi_U(x,x)))$$

Exercise 7.4. Let $A \subseteq \mathbb{N}$ be a set and let $f : \mathbb{N} \to \mathbb{N}$ be a computable function. Prove that if A is r.e. then $f(A) = \{y \in \mathbb{N} \mid \exists x \in A.\ y = f(x)\}$ is r.e. Is the converse also true? That is, from $f(A)$ r.e. can we deduce that A is r.e.?

Solution: Let e, e' be such that $f = \varphi_e$ and $sc_A = \varphi_{e'}$. Then

$$sc_{f(A)}(y) = \mathbf{1}(\mu w.\ H(e', (w)_1, (w)_2) \ \wedge\ S(e, (w)_1, y, (w)_3))$$

hence $f(A)$ is r.e. The converse is not true. For example $\mathbf{1}(\bar{K}) = \{1\}$ is r.e., but $\bar{K}$ is not r.e. □

Exercise 7.5. Let A be a recursive set and $f : \mathbb{N} \to \mathbb{N}$ be a total computable function. Is it true, in general, that $f(A)$ is r.e.? Is it true that $f(A)$ is recursive? Justify your answers with a proof or counterexample.

Solution: We have that $f(A)$ is r.e. since

$$sc_{f(A)}(x) = \mathbf{1}(\mu z.|f(z) - y|)$$

However, $f(A)$ is not recursive. For example, consider the function defined as follows. Take $a \in K$ and define:

$$\begin{aligned} f(x) &= \begin{cases} (x)_1 & \text{if } H((x)_1, (x)_1, (x)_2) \\ a & \text{otherwise} \end{cases} \\ &= (x)_1 \cdot \chi_H((x)_1, (x)_1, (x)_2) + a \cdot \bar{sg}(\chi_H((x)_1, (x)_1, (x)_2)) \end{aligned}$$

computable and total. Moreover $f(\mathbb{N}) = K$. □

Exercise 2

State the s-m-n theorem and use it to prove that there exists a total computable function $s : \mathbb{N} \to \mathbb{N}$ such that $|W_{s(x)} \setminus E_{s(x)}| = 2x$.

Solution: We can define, for instance,

$$f(x, y) = y + 2x$$

which is computable. Seen as a function of y, it has domain $\mathbb{N}$ and as codomain $\{y \mid y \geqslant 2x\}$, hence the set diffence is $\{y \mid y < 2x\}$, which has cardinality $2x$. Then one can use the smn theorem to get function $s(x)$.

The s-m-n theorem states that given m, n ≥ 1, there is a computable total function $s_{m,n} : N^{m+1} \to N$ such that for all $e \in N$, $\vec{x} \in N^m$, $\vec{y} \in N^n$:

$\phi_e^{(m+n)}(\vec{x},\vec{y}) = \phi_{s_{m,n}(e,\vec{x})}^{(n)}(\vec{y})$

We want to use the s-m-n theorem to prove there exists a total computable function $s : N \to N$ such that $|W_{s(x)}| = 2x$.

Define the function $f : N^2 \to N$ as:

$f(x,y) = y + 2x$

This function is clearly computable:

$f(x,y) = y + 2x$

$= y + \mu z.(z=2x)$

$= y + \mu z.(\exists w.(w \cdot 2=z) \wedge w=x)$

which shows f is computable since addition, equality checking, and bounded minimization are computable operations.

Viewed as a function of y with x as a parameter, f has domain N and codomain $\{y \mid y \geq 2x\}$, so the set difference $\{y \mid y < 2x\}$ has cardinality 2x.

By the s-m-n theorem, there exists a total computable function $s : N \to N$ such that for all $x,y \in N$:

$\phi_{s(x)}(y) = f(x,y) = y + 2x$

This function s satisfies the required property. For any $x \in N$:

$|W_{s(x)}| = |\{y \mid \phi_{s(x)}(y)\downarrow\}|$

$= |\{y \mid f(x,y)\downarrow\}|$

$= |N|$

$= \infty$

$|E_{s(x)}| = |\{\phi_{s(x)}(y) \mid y \in W_{s(x)}\}|$

$= |\{y + 2x \mid y \in N\}|$

$= |\{y \mid y \geq 2x\}|$

$= |N| - |\{y \mid y < 2x\}|$

$= \infty - 2x$

$= 2x$

Therefore, the total computable function s obtained from the s-m-n theorem satisfies $|W_{s(x)}| = \infty$ and $|E_{s(x)}| = 2x$ for all $x \in N$, proving the desired result.