Es: trovare l'int. generale di

$$y' = 8xy$$

Sol: Osserviamo che se $b(x) = 0$ allora

$y' = a(x)y \Rightarrow y = c\,e^{A(x)}$

Qui $a(x) = 8x \Rightarrow A(x) = 4x^2$

$\Rightarrow y(x) = c\,e^{4x^2}$

Ulteriore domanda: se vogliamo $y(1) = 3$ serve che

$$3 = c\,e^{4\cdot 1^2} = 3 = c\,e^4 \Rightarrow c = 3e^{-4}$$

Es: trovare l'integrale generale di

$$y' = \frac{x}{x^2+1}\,y$$

Sol: $a(x) = \dfrac{x}{x^2+1}$

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\log(x^2+1) + C \Rightarrow A(x) = \frac{1}{2}\log(x^2+1)$$

$\Rightarrow y(x) = c\,e^{\frac{1}{2}\log(x^2+1)}$

Notiamo che

$$e^{\frac{1}{2}\log(x^2+1)} = e^{\log((x^2+1)^{1/2})} = e^{\log(\sqrt{x^2+1})} = \sqrt{x^2+1}$$

$\Rightarrow y(x) = c\sqrt{x^2+1}$

Es: trovare l'int. generale di

$$y' = (\cot g\, x)\, y$$

nell'intervallo $]0,\pi[$

Sol: Ricordiamo che

$$\cot g x = \frac{\cos x}{\sin x} \Rightarrow \int \cot g x = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + c$$

In $]0,\pi[$, $\sin x > 0 \Rightarrow \log|\sin x| = \log(\sin x)$ in $(0,\pi)$

Prendiamo

$$A(x) = \log(\sin x)$$

$$\Rightarrow \quad e^{A(x)} = e^{\log(\sin x)} = \sin x$$

$$\Rightarrow \quad y(x) = c \sin x$$

Verifichiamo $\quad y'(x) = c\cos x = c\frac{\cos x}{\sin x} \sin x = c \cot g x \sin x$

$$= c \cot g x\, y \quad \checkmark$$

Ex: Risolvere

$$y' = -\frac{2}{x} y + \frac{\sin 4x}{x^2}$$

Sol: Se

$$y' = a(x)\, y + b(x)$$

allora

$$y(x) = e^{\int a(x)dx}\left(\int e^{-\int a(x)dx} b(x)\, dx + c\right)$$

In questo caso $a(x) = -\frac{2}{x} \Rightarrow \int a(x)dx = \int -\frac{2}{x} dx = -2\log|x| + c$

$$A(x) = -2\log|x| = \log|x|^{-2} = \log\frac{1}{x^2}$$

$$e^{A(x)} = e^{\log \frac{1}{x^2}} = \frac{1}{x^2}$$

$$e^{-A(x)} = e^{-\log \frac{1}{x^2}} = e^{\log\left(\frac{1}{x^2}\right)^{-1}} = x^2$$

$$\Rightarrow \int e^{-\int a(x)\,dx}\, b(x)\,dx = \int x^2 \frac{\sin 4x}{x^2}\,dx = \int \sin 4x\,dx$$

$$= \frac{1}{4}\int 4 \sin 4x = -\frac{1}{4}\cos 4x + c$$

$$\Rightarrow y(x) = \frac{1}{x^2}\left(-\frac{1}{4}\cos 4x + c\right)$$

Es: risolvere

$$y' - \frac{1}{x}y = x + \frac{1}{x}$$

Sol

$$y' = \frac{1}{x}y + \left(x + \frac{1}{x}\right)$$

$$\hookrightarrow b(x) = x + \frac{1}{x}$$

$$a(x) = \frac{1}{x} \Rightarrow A(x) = \log|x|$$

$$\Rightarrow e^{A(x)} = |x| \quad , \quad e^{-A(x)} = \frac{1}{|x|}$$

$$y(x) = e^{A(x)}\left(\int e^{-A(x)}\, b(x)\,dx + c\right)$$

$$= |x|\left(\int \frac{1}{|x|}\left(x + \frac{1}{x}\right)dx + c\right)$$

$\Rightarrow$ Risolviamo per $x>0$ e per $x<0$ (separatamente)

Se $x > 0$: $y(x) = x\left(\int \frac{1}{x}\left(x + \frac{1}{x}\right) dx + c\right)$

$$= x\left(\int 1 + \frac{1}{x^2}\, dx + c\right)$$

$$= x\left(x - \frac{1}{x} + c\right) = x^2 - 1 + cx$$

Se $x < 0$: $y(x) = -x\left(\int -\frac{1}{x}\left(x + \frac{1}{x}\right) dx + c\right)$

$$= -x\left(\int\left(-1 - \frac{1}{x^2}\right) dx + c\right)$$

$$= -x\left(-x + \frac{1}{x} + c\right) = x^2 - 1 - cx$$

$\Rightarrow$ stessa forma (c o $-c$ non importa)

Es: risolvere

$$y' + y\cos x - \frac{1}{2}\sin 2x = 0$$

Sol: $y' = \underbrace{-\cos x}_{a(x) = -\cos x}\, y + \underbrace{\frac{1}{2}\sin 2x}_{b(x) = \frac{1}{2}\sin 2x}$

$\Downarrow$

$\int(-\cos x)\,dx = -\sin x + c$

$A(x) = -\sin x$, $e^{A(x)} = e^{-\sin x}$, $e^{-A(x)} = e^{\sin x}$

$$\Rightarrow y(x) = e^{-\sin x}\left(\int e^{\sin x}\, \frac{1}{2}\sin 2x\, dx + c\right)$$

$$= e^{-\sin x}\left(\int e^{\sin x}\, \frac{1}{2} 2\sin x \cos x\, dx + c\right)$$

$$= e^{-\sin x}\left(\int e^{\sin x} \sin x\, d\sin x + c\right)$$

$$\int e^t t\, dt = e^t t - \int e^t\, dt = e^t t - e^t + c$$
$$= e^t (t-1) + c$$

$$\Rightarrow y(x) = e^{-\sin x}\left(e^{\sin x}(\sin x - 1) + c\right)$$

$$= \sin x - 1 + c\, e^{-\sin x}$$

Es: Risolvere il Problema di Cauchy

$$\begin{cases} y' + \dfrac{2x}{1+x^2}\, y = \dfrac{\cot x}{1+x^2} & \text{(i)} \\ y\left(\frac{\pi}{2}\right) = 4 & \text{(ii)} \end{cases}$$

Sol: Notiamo che $\cot x$ è definita $x \in \mathbb{R} \setminus \pi\mathbb{Z}$

Riscriviamo (i) come

$$y' = \underbrace{-\frac{2x}{1+x^2}}_{a(x)}\, y + \underbrace{\frac{\cot x}{1+x^2}}_{b(x)}$$

$$\int a(x)\,dx = \int \frac{-2x}{1+x^2}\,dx = -\log(1+x^2) + c$$

$$\Rightarrow A(x) = -\log(1+x^2) \Rightarrow e^{A(x)} = \frac{1}{1+x^2}, \quad e^{-A(x)} = 1+x^2$$

$$\int e^{-A(x)} b(x)\,dx = \int (1+x^2)\cdot \frac{\cot x}{1+x^2}\,dx = \int \frac{\cos x}{\sin x} = \log|\sin x| + c$$

Consideriamo l'intervallo $]0, \pi[$

$$y(x) = \frac{1}{1+x^2}\left(\log|\sin x| + c\right)$$

$$= \frac{1}{1+x^2}\left(\log(\sin x) + c\right) \qquad \sin x > 0 \text{ in }]0,\pi[$$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{1+\frac{\pi^2}{4}}\left(\log\underbrace{\left(\sin\frac{\pi}{2}\right)}_{=1} + c\right) = \frac{1}{1+\frac{\pi^2}{4}}\left(c \right)$$

$$\| \quad 4$$

$$\Rightarrow \quad 4 \cdot \left(1+\frac{\pi^2}{4}\right) = c \quad \Rightarrow \quad c = 4+\pi^2$$

Es: Risolvere

$$\begin{cases} y' = xy + |x| & \text{(i)} \\ y(1) = -1 & \text{(ii)} \end{cases}$$

Sol:

$$y' = xy + |x|$$

$$a(x) = x \quad , \quad b(x) = |x|$$

$$\int a(x)\,dx = \int x\,dx = \frac{1}{2}x^2 + c \quad , \quad A(x) = \frac{1}{2}x^2$$

$$e^{-A(x)} = e^{-\frac{1}{2}x^2} \quad , \quad e^{A(x)} = e^{\frac{1}{2}x^2}$$

$$\int e^{-A(x)} b(x)\,dx = \int e^{-\frac{1}{2}x^2}|x|\,dx = ?$$

Troviamo una primitiva

$$\int_0^x |t|\,e^{-\frac{t^2}{2}}\,dt = \begin{cases} \int_0^x t\,e^{-t^2/2}\,dt = -e^{-t^2/2}\Big|_0^x = -e^{-x^2/2}+1 & x>0 \\ -\int_0^x t\,e^{-t^2/2}\,dt = e^{-t^2/2}\Big|_0^x = e^{-x^2/2}-1 & x<0 \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} -e^{-x^2/2}+1 & x \geq 0 \\ e^{-x^2/2}-1 & x<0 \end{cases}$$

$$\Rightarrow \quad y(x) = e^{x^2/2}g(x) + c\,e^{x^2/2}$$

$$= e^{x^2/2}\left(-\operatorname{sgn}(x)\right)\left(e^{-x^2/2}-1\right) + c\,e^{x^2/2}$$

$$-1 = y(1) = e^{1/2}(-1)\left(e^{-1/2}-1\right) + c\,e^{1/2}$$

$$= -1 + e^{1/2} + c\,e^{1/2} \Rightarrow e^{1/2} + c\,e^{1/2} = 0$$

$$\Rightarrow c = -1$$

$$\Rightarrow y(x) = \begin{cases} -1 & x \geq 0 \\ 1 - 2e^{-x^2/2} & x < 0 \end{cases}$$

Es: risolvere $y'' - 2y' + 5y = 0$.

Sol: l'eq. caratt. è

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\Rightarrow \Delta < 0 \quad e \quad \lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i$$

$$\Rightarrow y(t) = c_1 e^{t} \cos(2t) + c_2 e^{t} \sin(2t), \quad c_1, c_2 \in \mathbb{R}$$

Es: risolvere $y'' - y = 0$.

Sol: l'eq. caratt. è

$$\lambda^2 - 1 = 0$$

$$\Rightarrow \Delta > 0 \qquad \lambda_1 = 1, \quad \lambda_2 = -1$$

$$\Rightarrow y(t) = c_1 e^{t} + c_2 e^{-t}, \qquad c_1, c_2 \in \mathbb{R}$$

Es: risolvere

$$y'' - 2y' + y = 0$$

Sol: l'eq. caratteristica è

$$\lambda^2 - 2\lambda + 1 = 0$$

$\Rightarrow \quad \Delta = 0 \qquad \tilde{\lambda} = 1$

$\Rightarrow \quad y(t) = c_1 e^t + c_2 t e^t \quad , \quad c_1, c_2 \in \mathbb{R}$

Es: Risolvere

$$(P.C.) \begin{cases} y' = \cos t \cos^2 y & (ED) \\ y(0) = 0 \end{cases}$$

Sol: $b(y) = \cos^2 y \Rightarrow b(\bar{y}) = 0 \Leftrightarrow \bar{y} = \frac{\pi}{2} + k\pi \quad , \quad k \in \mathbb{Z}$

$\Rightarrow y = \frac{\pi}{2} + k\pi \quad , \quad k \in \mathbb{Z}$ non possono essere sol di (P.C.)

La sol di PC deve essere t.c. $y(t) \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$

$$(ED) \Leftrightarrow \quad \frac{y'(t)}{\cos^2 y(t)} = \cos t$$

$$\Rightarrow \quad \int \frac{y'(t)}{\cos^2 y(t)} \, dt = \int \cos t \, dt$$

$$\Rightarrow \quad \left(\int \frac{1}{\cos^2 s} \, ds \right)_{s = y(t)} = \sin t + c$$

$$\Rightarrow \quad \tan(y(t)) = \sin t + c$$

$\Rightarrow \quad y(t) = \arctan(\sin t + c)$

Voglio

$y(0) = 0 \quad \Rightarrow \quad 0 = \arctan(\sin 0 + c)$

$\Rightarrow \quad c = 0$

$\Rightarrow$ sol è $\quad y(t) = \arctan(\sin t)$

<u>Es</u>: risolvere

$$\begin{cases} y' = \dfrac{ty}{2\log y} & \text{(ED)} \\ y(0) = e^{-1} \end{cases}$$

<u>Sol</u>: $\quad b(y) = \dfrac{y}{\log y} \quad \Rightarrow \quad b(\bar{y}) \neq 0$

$$\text{(ED)} \Leftrightarrow \quad \frac{\log y}{y} y' = \frac{t}{2}$$

$$\Rightarrow \quad \int \frac{\log y(t)}{y(t)} y'(t) \, dt = \int \frac{t}{2} \, dt$$

$$\parallel$$

$$\left(\int \frac{\log s}{s} \, ds \right)_{s = y(t)} = \frac{1}{4} t^2 + c$$

$$\int \frac{\log s}{s} \, ds = \frac{1}{2} \log^2 s + c$$

$$\Rightarrow \quad \frac{1}{2} \log^2 y(t) = \frac{1}{4} t^2 + c$$

$$\Rightarrow \quad \log^2 y(t) = \frac{1}{2} t^2 + \overbrace{2C}^{\rightsquigarrow C'}$$

$$\Rightarrow \quad \log y(t) = \pm \sqrt{\frac{1}{2} t^2 + C'}$$

Vo agl. $y(0) = e^{-1}$

$$\Rightarrow \quad \log y(0) = \log\left(e^{-1}\right) \Rightarrow \pm \sqrt{\frac{1}{2} 0^2 + C'} = -1$$

$$\Rightarrow - \sqrt{C'} = -1$$

$$\Rightarrow C' = 1$$

$$\Rightarrow \quad \log y(t) = - \sqrt{\frac{1}{2} t^2 + 1}$$

$$y(t) = e^{-\sqrt{\frac{1}{2} t^2 + 1}}$$