Soluzioni esercizi sui limiti

1. $$\lim_{x\to+\infty} \frac{5x^4+x^3+1}{3^{2x}+5^x} = \lim_{x\to+\infty} \frac{x^4\overbrace{\left(5+\frac{1}{x}+\frac{1}{x^4}\right)}^{\to 5}}{9^x\underbrace{\left(1+\left(\frac{5}{9}\right)^x\right)}_{\to 1}} = 0$$

per confronto infiniti
(x^4 è infinito di ordine minore di 9^x)

2. $$\lim_{x\to-\infty} \frac{5x^4+x^3+1}{3^{2x}+5^x} = \lim_{x\to-\infty} \frac{\overbrace{x^4}^{\to+\infty}\overbrace{\left(5+\frac{1}{x}+\frac{1}{x^4}\right)}^{\to 5}}{\underbrace{3^{2x}+5^x}_{\to 0^+}} = +\infty$$

3. $$\lim_{x\to 0^+} 4x^4 \lg(x^5+x^2) = \lim_{x\to 0^+} 4x^4 \lg[\underbrace{x^2(1+x^3)}_{\to 0}] =$$

$$= \lim_{x\to 0^+} \underbrace{4x^2}_{\to 0}\, \underbrace{\frac{1}{1+x^3}}_{\to 1}\, \underbrace{x^2(1+x^3)\lg(x^2(1+x^3))}_{\to 0} = 0$$

$\left(\lim_{y\to 0^+} y \lg y = 0\right)$

4. $$\lim_{x\to 0^+} x^{x\lg x} = \lim_{x\to 0^+} e^{x\lg x \lg x} =$$

$$= \lim_{x\to 0^+} e^{\overbrace{x(\lg x)^2}^{\to 0}} = 1$$

$$\boxed{\lim_{x\to 0^+} x(\lg x)^2 = 0^+}$$

5. $\lim_{x \to 0^+} |\lg x|^{1/x} = \lim_{x \to 0^+} \lg\left(\frac{1}{x}\right)^{\frac{1}{x}}$

per $x<1$
$\lg x < 0$
$|\lg x| = -\lg x =$
$= \lg \frac{1}{x}$

$= \lim_{y \to +\infty} (\lg y)^y = \lim_{y \to +\infty} e^{y \lg(\lg y)} = +\infty$

②

6. $\lim_{x \to +\infty} \frac{\lg x^5 + \sqrt{x} + 2x}{x + \underbrace{\operatorname{arctg} x}_{\text{limitato}} + \underbrace{\sin x}_{\text{limitato}}} =$

$= \lim_{x \to +\infty} \frac{\cancel{x}\left[\frac{\lg x^5}{x} + \frac{1}{\sqrt{x}} + 2\right]}{\cancel{x}\left[1 + \frac{\operatorname{arctg} x}{x} + \frac{\sin x}{x}\right]} = 2$

$\frac{\lg x^5}{x} \to 0$, $\frac{1}{\sqrt{x}} \to 0$, $\to 2$

$\frac{\operatorname{arctg} x}{x} \to 0$, $\frac{\sin x}{x} \to 0$, $\to 1$

7. $\lim_{x \to +\infty} (x + \sqrt{x}) \sin\left(\frac{5}{x}\right)$

$\frac{5}{\cancel{x}} = y$

$y = \frac{5}{x} \Rightarrow x = \frac{5}{y}$

$= \lim_{y \to 0^+} \left(\frac{5}{y} + \frac{\sqrt{5}}{\sqrt{y}}\right) \sin y =$

$= \lim_{y \to 0^+} \underbrace{(5 + \sqrt{5}\sqrt{y})}_{\to 5} \frac{\sin y}{y} = 5$

$\sqrt{5}\sqrt{y} \to 0$, $\frac{\sin y}{y} \to 1$

8. $\lim_{x \to \pi^-} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{1 - \cos^2 x} =$

$1 - \cos^2 x = \sin^2$

$$= \lim_{x\to\pi^-} \frac{\sqrt{(1+\sin x)} - \sqrt{1-\sin x}}{\sin^2 x} \cdot \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}} =$$

$$= \lim_{x\to\pi^-} \frac{\cancel{1}+\sin x-\cancel{1}+\sin x}{\sin^2 x} \frac{1}{\sqrt{1+\sin x}+\sqrt{1-\sin x}} =$$

$$= \lim_{x\to\pi^-} \frac{2}{\underbrace{\sin x}_{0^+}} \frac{1}{\underbrace{\sqrt{1+\sin x}+\sqrt{1-\sin x}}_{2}} = +\infty$$

9. $$\lim_{x\to-\infty} \left(\sqrt{3x^2-x}-\sqrt{3x^2+x+1}\right) \cdot \frac{\sqrt{3x^2-x}+\sqrt{3x^2+x+1}}{\sqrt{3x^2-x}+\sqrt{3x^2+x+1}} =$$

$$= \lim_{x\to-\infty} \frac{3x^2-x-(3x^2+x+1)}{\sqrt{3x^2-x}+\sqrt{3x^2+x+1}} = \lim_{x\to-\infty} \frac{-2x-1}{\sqrt{3x^2-x}+\sqrt{3x^2+x+1}}$$

$$= \lim_{x\to-\infty} \frac{-2x-1}{\sqrt{x^2}\sqrt{3-\frac{1}{x}}+\sqrt{x^2}\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = \lim_{x\to-\infty} \frac{-2x-1}{-x\left[\sqrt{3-\frac{1}{x}}+\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}\right]}$$

$\boxed{\sqrt{x^2} = -x \text{ dato che } x<0 \,!}$

$$= \lim_{x\to-\infty} \frac{\cancel{-x}\overbrace{\left(+2+\frac{1}{x}\right)}^{\to 2}}{\cancel{-x}\left[\underbrace{\sqrt{3-\frac{1}{x}}}_{\sqrt{3}}+\underbrace{\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}_{\sqrt{3}}\right]} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

10. $$\lim_{x\to+\infty} \frac{1+3\sin x - x\sin 2x}{x^2-1} =$$

$$= \lim_{x\to+\infty} \frac{\cancel{x}\left(\frac{1}{x} + \frac{3\sin x}{x} - \sin 2x\right)}{x^{\cancel{2}}\left(1-\frac{1}{x}\right)} = 0$$ (4)

$\downarrow$ limitato

$x \downarrow +\infty$, $\left(1-\frac{1}{x}\right) \downarrow 1$

11. $$\lim_{x\to+\infty} \frac{\lg(\lg x)}{1+\lg x} = \lim_{y\to+\infty} \frac{\lg(y)}{1+y} = 0$$ per gerarchie infiniti

$y = \lg x$

12. $$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} =$$

$$= \lim_{x\to 0} \frac{1+x-(1-x)}{x\left[\sqrt{1+x}+\sqrt{1-x}\right]} = \lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}(\underbrace{\sqrt{1+x}+\sqrt{1-x}}_{\downarrow\ 2})} = 1$$

13. $$\lim_{x\to 0} \frac{\sin(\operatorname{tg} x)}{\sin x} = \lim_{x\to 0} \frac{\sin(\operatorname{tg} x)}{(\operatorname{tg} x)} \cdot \frac{\operatorname{tg} x}{x} \cdot \frac{x}{\sin x}$$

$$= 1$$

$\downarrow 1$, $\downarrow 1$, $\downarrow 1$

$$\lim_{y\to 0} \frac{\sin y}{y} = 1$$

$y = \operatorname{tg} x$

14. $$\lim_{x\to\sqrt{2}} \frac{x-\sqrt{2}}{\sin(x^2-2)} = \lim_{x\to\sqrt{2}} \frac{x^2-2}{\sin(x^2-2)} \cdot \frac{1}{x+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$\downarrow 1$, $\downarrow 2\sqrt{2}$

$$\lim_{y\to 0} \frac{y}{\sin y} = 1$$

15. $\lim_{x\to-\infty} \underbrace{\sqrt{x^2+3x-2}}_{+\infty} \underbrace{+x}_{-\infty}$! $= \lim_{x\to-\infty} \left(\sqrt{x^2+3x-2}+\cancel{x}\right)\frac{\left(\sqrt{x^2+3x-2}-\cancel{x}\right)}{\left(\sqrt{x^2+3x-2}-x\right)} =$

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$$= \lim_{x\to-\infty} \frac{\cancel{x^2}+3x-2-\cancel{x^2}}{\sqrt{x^2+3x-2}-x} = \lim_{x\to-\infty} \frac{x\left(3-\frac{2}{x}\right)}{\sqrt{x^2}\sqrt{1+\frac{3}{x}-\frac{2}{x^2}}-x} =$$

$\sqrt{x^2} = -x$!!

$\sqrt{x^2}=-x$

$$= \lim_{x\to-\infty} \frac{\cancel{x}\overbrace{\left(3-\frac{2}{x}\right)}^{\to 3}}{-\cancel{x}\underbrace{\left[\sqrt{1+\frac{3}{x}-\frac{2}{x^2}}+1\right]}_{\to 2}} = -\frac{3}{2}$$

16. $\lim_{x\to+\infty} 3^{x+1}-3^{\sqrt{x^2+1}} = \lim_{x\to+\infty} 3^{x+1}\left[1-3^{\sqrt{x^2+1}-x-1}\right] =$

$$\lim_{x\to+\infty} \sqrt{x^2+1}-x-1 = \lim_{x\to+\infty} \frac{\left(\sqrt{x^2+1}-x-1\right)\left(\sqrt{x^2+1}+x+1\right)}{\sqrt{x^2+1}+x+1} =$$

$$= \lim_{x\to+\infty} \frac{x^2+1-(x+1)^2}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}+x+1} = \lim_{x\to+\infty} \frac{-2\cancel{x}}{\cancel{x}\underbrace{\left[\sqrt{1+\frac{1}{x^2}}+1+\frac{1}{x}\right]}_{\downarrow 2}} = -1$$

$\sqrt{x^2}=x$!

Quindi $\lim_{x\to+\infty} \left(1-3^{\sqrt{x^2+1}-x-1}\right) = 1-3^{-1} =$

$$= 1-\frac{1}{3} = \frac{2}{3}$$

$$\lim_{x\to+\infty} 3^{x+1}\left(1-3^{\sqrt{x^2+1}-x-1}\right) = +\infty.$$

17. ~~$\lim_{x\to+\infty}$ ...~~ $a>0$

$$\lim_{x\to+\infty} \frac{a^x+4^x+\sinh x}{7^x+2^x\sin(e^x)} = \lim_{x\to+\infty} \frac{a^x+4^x+\frac{e^x-e^{-x}}{2}}{7^x+2^x\sin(e^x)} =$$

$$\frac{e^{-x}}{a^x}=(ea)^{-x}\to 0$$

① caso $a>4$

$$= \lim_{x\to+\infty} \frac{a^x\left[1+\left(\frac{4}{a}\right)^x+\frac{1}{2}\left(\frac{e}{a}\right)^x-\frac{1}{2}(ae)^{-x}\right]}{7^x\left[1+\left(\frac{2}{7}\right)^x\sin(e^x)\right]} =$$

$$= \lim_{x\to+\infty} \left(\frac{a}{7}\right)^x \left[\frac{\overbrace{\left[1+\left(\frac{4}{a}\right)^x+\frac{1}{2}\left(\frac{e}{a}\right)^x-\frac{1}{2}(ae)^{-x}\right]}^{\nearrow 1}}{\underbrace{1+\left(\frac{2}{7}\right)^x\sin(e^x)}_{\searrow 1}}\right]$$

$a>7$ $\left(\frac{a}{7}\right)^x\to+\infty$ $\quad \frac{4}{a}<1 \quad \frac{e}{a}<1$

DUNQUE $\underline{a>7 \quad \lim_{x\to+\infty}(\quad) = +\infty}$

$a=7 \quad \lim_{x\to+\infty}(\quad) = 1$

$4<a<7 \quad \lim_{x\to+\infty}(\quad) = 0$ dato che $\left(\frac{a}{7}\right)^x\to 0$

② caso $a=4$

$$= \lim_{x\to+\infty} \frac{4^x\overbrace{\left(2+\left(\frac{e}{4}\right)^x\frac{1}{2}-(4e)^{-x}\frac{1}{2}\right)}^{\nearrow 2}}{7^x\underbrace{\left(1+\left(\frac{2}{7}\right)^x\sin e^x\right)}_{\searrow 1}} = 0$$

$$\frac{4^x}{7^x}\to 0$$

(3) caso $a<4$

$$= \lim_{x\to+\infty} \frac{4^x\left(\overbrace{\frac{a^x}{4^x}+1+\frac{e^x}{2\,4^x}-\frac{e^{-x}}{2\,4^x}}^{\nearrow 1}\right)}{7^x\left(\underbrace{1+\left(\frac{2}{7}\right)^2 \sin e^x}_{\searrow 1}\right)} = 0$$

$\frac{4^x}{7^x} \searrow 0$

Riassumendo
- $a>7$ limite è $+\infty$
- $a=7$ limite è 1
- $a<7$ limite è 0.