Soluzione esercizi sugli integrali

Es 1

(1) $\int x^3 e^{-x} dx =$ per parti $= x^3(-e^{-x}) + \int 3x^2(-e^{-x})dx =$
$= -x^3e^{-x} + \int 3x^2 e^{-x} dx =$ per parti $= -x^3e^{-x} + 3x^2(-e^{-x}) +$
$- \int 6x(-e^{-x})dx = -x^3e^{-x} - 3x^2e^{-x} + \int 6xe^{-x}dx =$ per parti $=$
$= -x^3e^{-x} - 3x^2e^{-x} + 6x(-e^{-x}) - \int 6(-e^{-x})dx =$
$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} + \int 6e^{-x}dx = -x^3e^{-x} - 3x^2e^{-x} +$
$- 6xe^{-x} - 6e^{-x} + c.$

(2) $\int x^2 \sin x \, dx =$ per parti $= x^2(-\cos x) - \int 2x(-\cos x)dx =$
$= -x^2\cos x + \int 2x\cos x \, dx =$ per parti $= -x^2\cos x + 2x(\sin x) +$
$- \int 2\sin x \, dx = -x^2\cos x + 2x\sin x + 2\cos x + c$

(3) $\int \arcsin x \, dx =$ per parti $= x\arcsin x - \int x\frac{1}{\sqrt{1-x^2}}dx =$
$=$ sostituzione $\begin{matrix} y = 1-x^2 \\ dy = -2x\,dx \end{matrix} = x\arcsin x + \frac{1}{2}\int\frac{1}{\sqrt{y}}dy =$
$= x\arcsin x + \frac{1}{2}\frac{1}{1-\frac{1}{2}}y^{1-\frac{1}{2}} + c = x\arcsin x + \sqrt{1-x^2} + c$

(4) $\int x \lg^2 x \, dx =$ per parti $= \frac{x^2}{2}\lg^2 x - \int \frac{x^2}{2}\cdot 2\lg x\frac{1}{x}dx =$
$= \frac{x^2}{2}\lg^2 x - \int x\lg x \, dx =$ per parti $= \frac{x^2}{2}\lg^2 x - \frac{x^2}{2}\lg x +$
$+ \int \frac{x^2}{2}\frac{1}{\cancel{x}}dx = \frac{x^2}{2}\lg^2 x - \frac{x^2}{2}\lg x + \frac{1}{4}x^2 + c.$

(5) $\int x \arctan x \, dx =$ per parti $= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\frac{1}{1+x^2}dx$
$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int\frac{x^2+1-1}{x^2+1}dx = \frac{x^2}{2}\arctan x - \frac{1}{2}\int 1\,dx +$
$+ \frac{1}{2}\int\frac{1}{1+x^2}dx = \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + c$

(6) $\int \frac{\cos x}{1+\sin^2 x}\,dx = \underset{y=\sin x \quad dy=\cos x\,dx}{\text{sostituzione}} = \int \frac{1}{1+y^2}\,dy =$

$= \operatorname{arctg} y + c = \operatorname{arctg}(\sin x) + c.$

(7) $\int \frac{e^{1/x}}{x^3}\,dx = \underset{y=\frac{1}{x} \quad dy=-\frac{1}{x^2}dx}{\text{sostituzione}} = -\int e^y \cdot y\,dy = \text{per parti} =$

$= -ye^y + \int e^y\,dy = -ye^y + e^y + c = -\frac{1}{x}e^{\frac{1}{x}} + e^{1/x} + c$

(8) $\int \frac{2x+1}{x^2+6x+9}\,dx = \int \frac{2x+6-5}{x^2+6x+9}\,dx = \int \frac{2x+6}{x^2+6x+9}\,dx - 5\int \frac{1}{(x+3)^2}\,dx$

$= \lg(x^2+6x+9) + 5\,\frac{1}{x+3} + c.$

(9) $\int \frac{x+1}{x^2+2}\,dx = \frac{1}{2}\int \frac{2x}{x^2+2}\,dx + \int \frac{1}{x^2+2}\,dx = \frac{1}{2}\lg(x^2+2) +$

$+ \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) + c$

(10) $\int \frac{x}{x^2-4}\,dx = \frac{1}{2}\int \frac{2x}{x^2-4}\,dx = \frac{1}{2}\lg|x^2-4| + c$

(11) $\int \frac{e^x}{e^{2x}-3e^x+2}\,dx = \underset{y=e^x \quad dy=e^x dx}{\text{sostituz}} = \int \frac{1}{y^2-3y+2}\,dy =$

$\frac{A}{y-1} + \frac{B}{y-2} = \frac{1}{y^2-3y+2} \Rightarrow \begin{matrix} A=-1 \\ B=1 \end{matrix}$

$= \int -\frac{1}{y-1}\,dy + \int \frac{+1}{y-2}\,dy = -\lg|y-1| + \lg|y-2| + c =$

$= \lg\frac{|y-2|}{|y-1|} + c = \lg\frac{|e^x-2|}{|e^x-1|} + c$

(12) $\int \frac{1}{x(\lg^2 x+3)}\,dx = \text{sost} \; \begin{matrix} y=\lg x \\ dy=\frac{1}{x}dx \end{matrix} = \int \frac{1}{y^2+3}\,dy =$

$= \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{y}{\sqrt{3}}\right) + c = \frac{1}{\sqrt{3}}\operatorname{arctg}\left(\frac{\lg x}{\sqrt{3}}\right) + c.$

Es. 2

(1) $\int_{-2}^{0} |2t+2| \operatorname{arctg} t \, dt$ notiamo che $|2t+2| = \begin{cases} 2t+2 & \text{se } t > -1 \\ -2t-2 & \text{se } t < -1 \end{cases}$

$= -\int_{-2}^{-1} (2t+2) \operatorname{arctg} t \, dt + \int_{-1}^{0} (2t+2) \operatorname{arctg} t \, dt$

$\int (2t+2) \operatorname{arctg} t \, dt = \text{per parti} = (t^2+2t) \operatorname{arctg} t - \int \frac{t^2+2t}{t^2+1} dt$

$= (t^2+2t) \operatorname{arctg} t - \int \frac{t^2+1}{t^2+1} dt - \int \frac{2t}{t^2+1} dt + \int \frac{1}{t^2+1} dt =$

$= (t^2+2t) \operatorname{arctg} t - t - \lg(t^2+1) + \operatorname{arctg} t + c =$

$= (t+1)^2 \operatorname{arctg} t - t - \lg(t^2+1) + c$

$\int_{-2}^{0} |2t+2| \operatorname{arctg} t \, dt = -\left[(t+1)^2 \operatorname{arctg} t - t - \lg(1+t^2)\right]_{-2}^{-1} +$

$+ \left[(t+1)^2 \operatorname{arctg} t - t - \lg(t^2+1)\right]_{-1}^{0} = -\left[1 - \lg 2 - \operatorname{arctg}(-2) +\right.$

$\left. -2 + \lg 5\right] + \left[-1 + \lg 2\right] = -\cancel{1} + \lg 2 - \operatorname{arctg} 2 + \cancel{2}$

$- \lg 5 - \cancel{1} + \lg 2 = 2\lg 2 - \operatorname{arctg} 2 - \lg 5.$

(2) $\int_{-1}^{1} \frac{1}{x^2-4} dx$ $\int \frac{1}{x^2-4} dx = \int \frac{1}{4} \frac{1}{x-2} dx - \int \frac{1}{4} \frac{1}{x+2} dx =$

$= \frac{1}{4} \lg|x-2| - \frac{1}{4} \lg|x+2| + c =$

$= \frac{1}{4} \lg \frac{|x-2|}{|x+2|} + c.$

$\int_{-1}^{1} \frac{1}{x^2-4} dx = \left[\frac{1}{4} \lg \frac{|x-2|}{|x+2|}\right]_{-1}^{1} = \frac{1}{4} \lg\left(\frac{1}{3}\right) - \frac{1}{4} \lg 3 = -\frac{1}{2} \lg 3$

(3) $\int_{-1}^{1} \frac{1}{x^2 - 2|x| + 2x + 4} dx = \int_{-1}^{0} \frac{1}{(x^2+2x+2x+4)} dx + \int_{0}^{1} \frac{1}{x^2 \cancel{-2x} \cancel{+2x} + 4} dx$

$= \int_{-1}^{0} \frac{1}{(x+2)^2} dx + \int_{0}^{1} \frac{1}{x^2+4} dx =$

$$= \left[-\frac{1}{x+2}\right]_{-1}^{0} + \left[\frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)\right]_{0}^{1} = -\frac{1}{2} + 1 + \frac{1}{2}\operatorname{arctg}\frac{1}{2}.$$

4) $$\int_{-\pi}^{\pi} x^2 |\sin x|\, dx = \int_{-\pi}^{0} x^2(-\sin x)\, dx + \int_{0}^{\pi} x^2 \sin x\, dx =$$

$$= -\int_{-\pi}^{0} x^2 \sin x\, dx + \int_{0}^{\pi} x^2 \sin x\, dx = 2\int_{0}^{\pi} x^2 \sin x\, dx$$

$$\int x^2 \sin x\, dx = \text{per parti} = -x^2\cos x + \int 2x\cos x\, dx =$$

$$= -x^2\cos x + 2x\sin x - \int 2\sin x\, dx = -x^2\cos x + 2x\sin x$$

$$+ 2\cos x + c.$$

$$2\int_{0}^{\pi} x^2 \sin x\, dx = 2\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_{0}^{\pi} =$$

$$= 2\left(-\pi^2\cos\pi \cancel{+ 2\pi\sin\pi} + 2\cos\pi - 2\cos 0\right) =$$

$$= 2(\pi^2 - 4).$$

5) $$\int_{e}^{e^3} \lg x\, dx = \text{per parti} = x \lg x \Big|_{e}^{e^3} - \int_{e}^{e^3} \cancel{x}\frac{1}{\cancel{x}}\, dx =$$

$$= e^3 \lg e^3 - e\lg e - [x]_{e}^{e^3} = 3e^3 \cancel{- e} - e^3 \cancel{+ e} = 2e^3$$

6) $$\int_{0}^{\pi/2} \operatorname{arctg} x\, dx = \text{per parti} = x \operatorname{arctg} x \Big|_{0}^{\pi/2} - \int_{0}^{\pi/2} \frac{x}{x^2+1}\, dx$$

$$= \frac{\pi}{2}\operatorname{arctg}\frac{\pi}{2} - \frac{1}{2}\left[\lg(x^2+1)\right]_{0}^{\pi/2} = \frac{\pi}{2}\operatorname{arctg}\frac{\pi}{2} - \frac{1}{2}\lg\left(1+\frac{\pi^2}{4}\right).$$

7) $$\int_{\pi/6}^{\pi/4} \frac{1}{\operatorname{tg} x \lg(\sin x)}\, dx = \cancel{\text{sostituzione}} \int_{\pi/6}^{\pi/4} \frac{\cos x}{\sin x \lg(\sin x)}\, dx$$

$= \text{sostituzione } y = \sin x \quad dy = \cos x\, dx$

$\sin \pi/6 = \frac{1}{2} \quad \sin\frac{\pi}{4} = \frac{1}{2}\sqrt{2}$

$$= \int_{1/2}^{1/2\sqrt{2}} \frac{1}{y \lg y}\, dy = \text{sost } z = \lg y \quad dz = \frac{1}{y}\, dy =$$

Notiamo che $\lg\left(\frac{1}{2}\right) = -\lg 2 \quad \lg\frac{\sqrt{2}}{2} = -\lg\sqrt{2}$ (5)

$$= \int_{-\lg 2}^{-\lg\sqrt{2}} \frac{1}{z}\,dz = \Big[\lg|z|\Big]_{-\lg 2}^{-\lg\sqrt{2}} = \lg\lg\sqrt{2} - \lg\lg 2 =$$

$$= \lg\left(\frac{\lg\sqrt{2}}{\lg 2}\right).$$

(8) $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx =$ sost. $y = e^{-x}$ $dy = -e^{-x}dx$

$$= -\int_1^{1/e} \frac{1}{\sqrt{1-y^2}}\,dy = \int_{1/e}^{1} \frac{1}{\sqrt{1-y^2}}\,dy = \Big[\arcsin y\Big]_{1/e}^{1} =$$

$$= \pi/2 - \arcsin\left(\frac{1}{e}\right).$$

(9) $\displaystyle\int_{\sqrt{3}}^{2} x\sqrt{x^2-3}\,dx =$ sost. $y = x^2-3$, $dy = 2x\,dx$ $=$

$$= \int_0^1 \frac{1}{2}\sqrt{y}\,dy = \frac{1}{2}\left[\frac{1}{1+\frac{1}{2}}\, y^{1+\frac{1}{2}}\right]_0^1 = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$$

(10) $\displaystyle\int_1^2 \frac{\sin\frac{1}{x}}{x^3}\,dx =$ sostituzione $y = \frac{1}{x}$, $dy = -\frac{1}{x^2}\,dx$

$$= -\int_1^{1/2} y\sin y\,dy = \int_{1/2}^{1} y\sin y\,dy = \text{per parti} =$$

$$= -y\cos y\Big|_{1/2}^{1} + \int_{1/2}^{1}\cos y\,dy = -\cos 1 + \frac{1}{2}\cos\frac{1}{2} + \Big[\sin y\Big]_{1/2}^{1} =$$

$$= -\cos 1 + \frac{1}{2}\cos\frac{1}{2} + \sin 1 - \sin\frac{1}{2}.$$

(11)

11 $\displaystyle\int_1^e \frac{\lg x}{x(\lg x+3)}\,dx =$ sost. $y = \lg x$, $dy = \frac{1}{x}\,dx$ $=$

$$= \int_{\lg 1}^{\lg e} \frac{y}{y+3}\,dy = \int_0^1 \left(1 - \frac{3}{y+3}\right) dy =$$

$$= \Big[y - 3\lg|y+3|\Big]_0^1 = 1 - 3\lg 4 + 3\lg 3 = 1 + 3\lg\left(\frac{3}{4}\right)$$

12 $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{\cos^2 x + \cos x - 2}\,dx =$ sost $y = \cos x$, $dy = -\sin x\,dx$

$$= -\int_{1/\sqrt{2}}^{0} \frac{dy}{y^2+y-2} = \int_0^{1/\sqrt{2}} \frac{1}{y^2+y-2}\,dy =$$

$$\frac{A}{y-1} + \frac{B}{y+2} = \frac{1}{y^2+y-2} \qquad A = 1/3, \quad B = -1/3$$

$$= \frac{1}{3}\int_0^{1/\sqrt{2}} \frac{1}{y-1}\,dy - \frac{1}{3}\int_0^{1/\sqrt{2}} \frac{1}{y+2}\,dy = \frac{1}{3}\Big[\lg|y-1|\Big]_0^{1/\sqrt{2}} +$$

$$-\frac{1}{3}\Big[\lg|y+2|\Big]_0^{1/\sqrt{2}} = \frac{1}{3}\lg\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\lg\left(2 + \frac{1}{\sqrt{2}}\right) + \frac{1}{3}\lg 2$$