Supplement to Solutions of HW7

Problem 4.8. Show that the set $N \times N \times N$ is countable, where N is the set of natural numbers.

Alternative Proof: We have seen that the set $N \times N$ is countable, thus there exists a bijection (one-to-one correspondence) $f : N \times N \to N$. Define $g : N \times N \times N \to N$ as $g(x, y, z) = f(f(x, y), z)$, then g is a bijection, because the composition of bijections is a bijection. Thus, $N \times N \times N$ is countable.

Problem 4.11. Show that the set $INFINITE_{PDA}$ is decidable.

Comment: The useless variables in the grammar should removed before converting to Chomsky NF.

Problem 4.13. Let $A = \{\langle R, S\rangle \mid R$ and S are regular expressions and $L(R) \subseteq L(S)\}$. Show A is decidable.

Proof: Since $X \subseteq Y$ iff $X \cap \overline{Y} = \emptyset$ and we have a decider S for E_{DFA} (the emptyness of problems of DFAs), we can construct a TM M that decides A as follows:

$M =$ "On input $\langle R, S\rangle$ where R and S are regular expressions:
 1. Construct a DFA X such that $L(X) = L(R) \cap \overline{L(S)}$;
 Run the decider S for E_{DFA} on input $\langle X\rangle$ and return the output of S."

Since X can be constructed in a finite number of steps and S exists, M is a decider. Hence, A is decidable.

Problem 4.21. Let $L = \{\langle D\rangle \mid D$ is a DFA that accepts ^{R} whenever it accepts $\}$. Show that S is decidable.

Proof: We construct a TM T that decides L, using the decider S for EQ_{DFA} (the equivalence problem of DFAs).

$T =$ "On input $\langle D\rangle$ where D is a DFA:
 Construct DFA E that recognizes $L^R(D)$, the reverse of $L(D)$;
 Run S for EQ_{DFA} on the input $\langle D, E\rangle$ and return the output of S.

This decider works by accepting only DFA $\langle D\rangle$ where $L(D) = L^R(D)$ (D accepts both a word and its reverse).

Problem 4.24. Let $USELESS_{PDA} = \{\langle P\rangle \mid P$ is a PDA that has useless states $\}$. Since we have a decider S for E_{CFG} (the emptyness problem for CFGs), we use S to construct a decider T for $USELESS_{PDA}$.

$T =$ "On input $\langle P\rangle$ where P is a PDA:
 For each state p of P,
 Construct PDA P' from P by replacing the definition of $\delta(p, a, b)$ by
 $\delta(p, a, \epsilon) = \{(p, \epsilon)\}$ for any $a \in \Sigma$ and $b \in \Gamma$, and

using p as the only accepting state.
Convert P' into equvialent CFG G;
Run S for E_{PDA} on the input $\langle G \rangle$;
If S accepts, *accept*.
At the end of loop, *reject*.

In T, we treat in turn each p as the only accepting state. The decider T works because if p is uselss, then it will not be reached and no strings can be accepted if p is the only accepting state.

Problem 4.29. Let $C_{CFG} = \{\langle G, k \rangle \mid G$ is a CFG and $L(G)$ contains exactly k strings where $k \geq 0$ or $k = \infty$ $\}$.

Proof: Since we have a decider M for $\mathit{INFINITE}_{CFG} = \{\langle G \rangle \mid G$ is a CFG and $L(G)$ is an infinite language$\}$, we use M to show a decider T for C_{CFG}. The following

T "On input $\langle G, k \rangle$, where G is a CFG and k is a natural number:
1. Check if $L(G)$ is infinite using decider M.
   If $L(G)$ is infinite, *accept* when $k = \infty$ and *reject* when $k \neq \infty$.
   If $L(G)$ is finite, *reject* when $k = \infty$ and continue when $k \neq \infty$.
2. Convert G into Chomsky NF and derive from the start variable of G all the terminal strings, and count them. If the number of distinct termimal strings is equal to k, accept; otherwise *reject*."

The above TM T decides C_{CFG} by first chekcing if $L(G)$ is infinite or not using decider M. If $L(G)$ is finite and $k \neq \infty$, we simplify G into CNF and derive all the strings in $L(G)$ and count them. Since G is simplified and $L(G)$ is finite, the derivation cannot have infinite sequences. If $L(G)$ has exactly k strings, we accept. T will terminate after a finite number of steps. Thus, C_{CFG} is decidable.