A.6.1 An Oscillating Sum

If we plug in some values for $f(n)$, we notice that the following closed-form expression can represent it:

$$f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$$

Proof

Base Case

In order to prove that our closer-form expression holds, we first consider out base case at $n = 0$. In this case expression holds since:

$$f(0) = (-1)^0 \cdot \sum_0^0 (-1)^0 \cdot (-1)^0 \cdot 0 =$$

$$= \left\lfloor \frac{0+1}{2} \right\rfloor$$

$$= \left\lfloor \frac{1}{2} \right\rfloor$$

$$= 0$$

Inductive Step

Assume the hypothesis holds for (all) $n \leq N$, where $N \in \mathbb{N}$. We will show the hypothesis also holds for $n + 1$.

<u>Case A</u>) If n is **even**, $f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = \frac{n}{2}$. This is true because, as we are working with natural numbers and n is **even**, we know for sure that $\exists o \in \mathbb{N} : 2 \cdot o = n$. Therefore, $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2} = o$.

Therefore, for our expression to hold we assume that our induction hypothesis $f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ holds, and we prove that $f(n+1) = \left\lfloor \frac{(n+1)+1}{2} \right\rfloor$:

$$f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor \rightarrow f(n+1) = \left\lfloor \frac{(n+1)+1}{2} \right\rfloor.$$

$$f(n+1) = (-1)^k \cdot (\sum_{k=0}^{n+1} (-1)^k \cdot k)$$

$$= (-1)^{n+1} \cdot ((\sum_{k=0}^{n} (-1)^k \cdot k) + ((-1)^{n+1} \cdot (n+1)))$$

$$= (-1) \cdot (-1)^n \cdot (\sum_{k=0}^{n} (-1)^k \cdot k)) + ((-1)^{n+1} \cdot (-1)^{n+1} \cdot (n+1))$$

From the expression above, we know that $(-1)^{n+1} \cdot (-1)^{n+1} = 1$, regardless of the value of n. We can easily prove this by observing that n is **odd**, $(-1)^{n+1} = 1$ and therefore $(-1)^{n+1} \cdot (-1)^{n+1} = 1$; on the other hand, if n is **even**, $(-1)^{n+1} = -1$ and therefore $(-1)^{n+1} \cdot (-1)^{n+1} = 1$.

With that in mind, we can simplify the closed form:

$$= (-1) \cdot f(n) + (n+1)$$

$$= -\left\lfloor \frac{n+1}{2} \right\rfloor + (n+1) \qquad [f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor, \text{ induction hypothesis}]$$

$$= -\frac{n}{2} + (n+1)$$

$$= \frac{(n+1)+1}{2}$$

If n is even, $(n+1)+1$ is also even, so the floor is equal to the expression.

$$= \left\lfloor \frac{(n+1)+1}{2} \right\rfloor \text{ QED!}$$

<u>Case B</u>) If n is **odd**, we know that $n+1$ is **even** and therefore $f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n+1}{2}$ (same logic of $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2} = o$ applies). For this case, we also assume that our induction hypothesis $f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor$ holds, and we prove that $f(n+1) = \left\lfloor \frac{(n+1)+1}{2} \right\rfloor$:

$$f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor \rightarrow f(n+1) = \left\lfloor \frac{(n+1)+1}{2} \right\rfloor$$

$$f(n+1) = (-1)^k \cdot (\sum_{k=0}^{n+1} (-1)^k \cdot k)$$

$$= (-1)^{n+1} \cdot ((\sum_{k=0}^{n} (-1)^k \cdot k) + ((-1)^{n+1} \cdot (n+1)))$$

$$= (-1) \cdot (-1)^n \cdot (\sum_{k=0}^{n} (-1)^k \cdot k)) + ((-1)^{n+1} \cdot (-1)^{n+1} \cdot (n+1))$$

$$= (-1) \cdot f(n) + (n+1)$$

$$= -\left\lfloor \frac{n+1}{2} \right\rfloor + (n+1) \quad [f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor \text{, induction hypothesis]}$$

$$= -\frac{n+1}{2} + (n+1)$$

$$= \frac{n+1}{2}$$

$$= \left\lfloor \frac{(n+1)+1}{2} \right\rfloor \text{ QED!}$$

We know this because, if n is odd, $(n+1)+1$ is also odd, so the floor is $\left\lfloor \frac{(n+1)+1}{2} \right\rfloor$, which is equal to $\left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor$. As we also know that $n+1$ is **even** and therefore $f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n+1}{2}$, and that $\left\lfloor \frac{1}{2} \right\rfloor = 0$, we get that $\frac{n+1}{2} = \left\lfloor \frac{(n+1)+1}{2} \right\rfloor$.

Hence, our closed form expression holds.

A.6.2 An Approximate Sum

In order to prove that $\sum_{k=1}^{n} k^2 \cdot 2^k = \theta(n^2 \cdot 2^n)$, we have to prove that $\exists c_1, c_2, N : c_1 > 0 \wedge c_2 > 0 \wedge n > N$ such that $c_1 \cdot |f(n)| \leq g(n) \leq c_2 \cdot |f(n)|$, which means that $\sum_{k=1}^{n} k^2 \cdot 2^k = \Omega(n^2 \cdot 2^n)$ and $\sum_{k=1}^{n} k^2 \cdot 2^k = O(n^2 \cdot 2^n)$.

To prove that $\sum_{k=1}^{n} k^2 \cdot 2^k = \Omega(n^2 \cdot 2^n)$, we let $c_1 = 1$. If we plug in the value, we get:

$$\sum_{k=1}^{n} k^2 \cdot 2^k = \Omega(n^2 \cdot 2^n)$$

$$\sum_{k=1}^{n} k^2 \cdot 2^k \geq c_1 \cdot (n^2 \cdot 2^n)$$

$$\sum_{k=1}^{n} k^2 \cdot 2^k \geq 1 \cdot n^2 \cdot 2^n$$

$$(\sum_{k=1}^{n-1} k^2 \cdot 2^k) \cdot (n^2 \cdot 2^n) \geq 1 \cdot n^2 \cdot 2^n$$

As the last term of the summation by itself is already equal to the right-hand expression, we prove that $\sum_{k=1}^{n} k^2 \cdot 2^k = \Omega(n^2 \cdot 2^n)$.

Now to prove that $\sum_{k=1}^{n} k^2 \cdot 2^k = O(n^2 \cdot 2^n)$, we first rearrange the expression to compare it to $f(n)$.

$$\sum_{k=1}^{n} k^2 \cdot 2^k \leq \sum_{k=1}^{n} n^2 \cdot 2^k \quad [k^2 \leq n^2]$$

$$= \sum_{k=1}^{n} k^2 \cdot 2^k \leq n^2 \cdot \sum_{k=1}^{n} 2^k$$

$$= \sum_{k=1}^{n} k^2 \cdot 2^k \leq n^2 \cdot \frac{2^{n+1} - 1}{2 - 1}$$

As we're dealing with big-O, we can take $2^{n+1} - 1$ as the upper-bound of $\frac{2^{n+1} - 1}{2 - 1}$. As a result, we simplify the expression to:

$$= \sum_{k=1}^{n} k^2 \cdot 2^k \leq n^2 \cdot 2^{n+1}$$

$$= \sum_{k=1}^{n} k^2 \cdot 2^k \leq 2 \cdot (n^2 \cdot 2^n)$$

Hence, if we let $c_2 = 2$, we prove that $\sum_{k=1}^{n} k^2 \cdot 2^k = O(n^2 \cdot 2^n)$, and, as a consequence, we get that $\sum_{k=1}^{n} k^2 \cdot 2^k = \theta(n^2 \cdot 2^n)$.

A.6.3 A Stretched Function

From the prompt, we know that if $f(n) = O(n)$, $\exists c_1, n_1$ where if $n \geq n_1$, then $f(n) \leq c_1 \cdot n$.

Moreover, if $g(n) = O(n)$, $\exists c_2, n_2$ where if $n \geq n_2$, then $g(n) \leq c_2 \cdot n$.

Now, to prove that $f(g(n)) = O(n)$,

$\exists c_3, n_3, m : (c_3 = c_1 \cdot c_2)$, $(m = max(f(g(n)), \text{when } g(n) < n_1) : (n > n_3) \wedge (n_3 > n_2) \wedge (n_3 > \frac{m}{c1 \cdot c2})$

There are two cases we have to analyze:

<u>Case A</u>) If $g(n) > n_1 \wedge n > n_2$, then:

$$f(g(n)) \leq c_1 \cdot g(n) \leq c_1 \cdot c_2 \cdot n \quad [\, f(n) \leq c_1 \cdot n \,]$$

$$f(g(n)) \leq c_1 \cdot g(n) \leq c_3 \cdot n$$

Therefore, in Case A, $f(g(n)) = O(n)$ holds.

<u>Case B</u>) If $g(n) < n_1$, we have the following bounded range:

$$g(n) \in [0, n_1]$$

Therefore, $f(g(n))$ is also in a bounded range. From our variable definition, we know that $m = max(f(g(n))$ for this bounded range and that $m < n_3 \cdot c_1 \cdot c2$. Having in mind that and the fact that $f(g(n))$ has this bounded range, we can write:

$$f(g(n)) \leq m < c_1 \cdot c_2 \cdot n_3$$

As $n > n_3$, we can rewrite it as:

$$f(g(n)) \leq m < c_1 \cdot c_2 \cdot n_3 < c_1 \cdot c_2 \cdot n$$

$$f(g(n)) \leq m < c_1 \cdot c_2 \cdot n_3 < c_3 \cdot n \qquad [c_1 \cdot c_2 = c_3]$$

Hence, $f(g(n)) = O(n)$ holds for Case B, and therefore, it is true for all cases.