HomeWork 1
Convex Optimization
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Gabriel Singer

Exercice 1

1. Let $(u, v, \lambda) \in \Omega^2 \times [0,1]$ where $\Omega := \{x \in \mathbb{R}^n \quad x_i \in [\alpha_i, \beta_i] \quad \forall i \in [1,n]\}$.
   Let $i \in [1,n]$, we have
   $$(\lambda u + (1-\lambda)v)_i = \lambda u_i + (1-\lambda)v_i \in [\lambda\alpha_i + (1-\lambda)\alpha_i, \lambda\beta_i + (1-\lambda)\beta_i] = [\alpha_i, \beta_i].$$
   This show that Ω is convex.

2. Let $(u, v, \lambda) \in \Omega^2 \times [0,1]$ where $\Omega := \left\{x \in \mathbb{R}^2_+ \quad x_1x_2 \geq 1\right\}$
   $$\begin{aligned}(\lambda u + (1-\lambda)v)_1(\lambda u + (1-\lambda)v)_2 &= (\lambda u_1 + (1-\lambda)v_1)(\lambda u_2 + (1-\lambda)v_2)\\ &= \lambda^2 u_1u_2 + \lambda(1-\lambda)u_1v_2 + \lambda(1-\lambda)u_2v_1 + (1-\lambda)^2 v_1v_2\\ &\geq \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)(u_1v_2 + u_2v_1)\end{aligned}$$
   Since $u, v \in \Omega$ and $\lambda \in [0,1]$ we have that
   $$\lambda(1-\lambda)(u_1v_2 + u_2v_1) \geq \lambda(1-\lambda)(\frac{v_2}{v_1} + \frac{v_1}{v_2}) \geq \frac{1}{4}(\frac{v_2}{v_1} + \frac{v_1}{v_2}). \tag{1}$$
   Let $g : x \mapsto x + \frac{1}{x}$ define on $]0, \infty[$ $g \in C^\infty(]0,\infty[, \mathbb{R})$ and
   $$\forall x > 0 \quad g''(x) = 2x^{-3} > 0$$
   which implies that g is convex and since $g'(x) = 0 \implies x = 1$ by convexity of g we have that $g(x) \geq g(1) = 2$. This mean that form (1) we have
   $$\lambda(1-\lambda)(u_1v_2 + u_2v_1) \geq \frac{1}{2}$$
   Thus
   $$(\lambda u + (1-\lambda)v)_1(\lambda u + (1-\lambda)v)_2 \geq \lambda^2 + (1-\lambda)^2 + \frac{1}{2} \geq 1$$
   Where we have used that
   $$\lambda^2 + (1-\lambda)^2 \geq \left(\frac{1}{2}\right)^2 + (1 - \frac{1}{2})^2 = \frac{1}{2}.$$
   We have shown that Ω is convex.

3. Let $<>$ denote the usual scalar product on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ fixed. Let $y \in S$,
   $$||x - x_0||_2 \leq ||x - y||_2 \iff 2 < x, y - x_0 > \leq ||y||^2 - |x_0|^2$$

which shows that

$$\Omega := \{x \in \mathbb{R}^n \quad abs|x - x_0|_2 \leq ||x-y||_2 \quad \forall y \in S\} = \bigcap_{y \in S} \phi^{-1}\left( ]-\infty; \frac{||y||^2 - |x_0|^2}{2}] \right)$$

With

$$\phi : x \mapsto < x, y - x_0 >$$

since ϕ is linear, ϕ is convex and $]-\infty; \frac{||y||^2 - |x_0|^2}{2}]$ is convex, thus Ω is convex.

4. Let

$$\Omega := \{x \in \mathbb{R}^n \quad dist(x, S) \leq dist(x, T)\}.$$

Here the goal is to show that the set is not convex. Let $T := \{x \in \mathbb{R}^n \quad ||x|| \leq 1\}$ and $S := \{x \in \mathbb{R}^n \quad 1 \leq ||x|| \leq 2\}$.
Let $x \in \mathbb{R}^n \setminus int(T)$ we have that

$$dist(x, T) = dist(x, \partial T) \tag{2}$$

and since $\partial T = \mathbb{S}_n(0, 1) \subset S$ we have that

$$dist(x, S) \leq dist(x, \partial T) \tag{3}$$

Combining (2) and (3) we have that

$$dist(x, S) \leq dist(x, T).$$

So we have shown that:

$$\Omega = \mathbb{R}^n \setminus T.$$

But $\mathbb{R}^n \setminus T$ is not convex because one can take :

$$x = e_1 \quad y = -e_1 \quad \lambda = \frac{1}{2} \implies \lambda x + (1-\lambda) y = 0 \notin \Omega.$$

Where e_1 is the first vector of the canonical base of $\mathbb{R}^n$.

5. Let $\{x \quad x + S_2 \subset S_1\}$, with S_1 which is convex.
We have that

$$\Omega = \{x, \quad \forall y \in S_2 \quad x + y \subset S_1\} = \bigcap_{y \in S_2} \{x, \quad x + y \in S_1\}$$

Let's show that, since S_1 is convex then $S_y := \{x, \quad x + y \in S_1\}$ is convex and thus Ω is convex.
Let $y \in S_2$ and $(u, v, \lambda) \in S_y^2 \times [0, 1]$ we have that $u + y \in S_1$ and $v + y \in S_1$

$$\lambda u + (1-\lambda) v + y = \lambda \underbrace{(u+y)}_{\in S_1} + (1-\lambda) \underbrace{(v+y)}_{\in S_1} \underbrace{\in}_{S_1 \text{is convex}} S_1$$

Thus S_y is convex.

Exercice 2 First, all the function in this exercices are C^∞ so their Hessian matrix exists. Moreover, to prove that the functions f are convex I use the following characterisation of $S_n^+(\mathbb{R})$:

Lemma 1. *Let $n \in \mathbb{N}$ we have that $A \in S_n(\mathbb{R})$ is semi-definite positive if and only if all the eigenvalues of A are positive.*

Let $X = (x_1, ..x_n) \in \mathbb{R}^n$ and $Sp(A) = (\lambda_i)_{1\leq i\leq n}$ we have that

$$X^T AX = \sum_{1\leq i\leq n} \lambda_i x_i^2$$

which is positive (for any X) if and only if for any $i = 1, ..., n$ $\lambda_i \geq 0$.

1. Since f is a polynomial function of degres 1 it is clear that

$$Hessf = 0$$

and if $X \in \mathbb{R}^2$ and $(x_1, x_2) \in \mathbb{R}^2_{++}$ we have that

$$X^T Hess(f)(x_1, x_2)X = 0 \geq 0$$

By the second order condition, f is convex on $\mathbb{R}^2_{++}$.

2. For any $(x_1, x_2) \in dom(f)$ we have

$$Hess(f)(x_1, x_2) = \begin{pmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_2^2 x_1^2} \\ \frac{1}{x_2^2 x_1^2} & \frac{2}{x_2^3 x_1} \end{pmatrix} = \frac{1}{x_1^2 x_2} \begin{pmatrix} \frac{2}{x_1} & \frac{1}{x_2} \\ \frac{1}{x_2} & \frac{2}{x_1} \end{pmatrix}$$

Its characteristic polynomial is equal to

$$\Lambda^2 - (2\Lambda)/(x_1^3 x_2) - (2\Lambda)/(x_1 x_2^3) + 3/(x_1^4 x_2^4) = (\Lambda - \mu_1(x_1, x_2)\,(\Lambda - \mu_2(x_1, x_2))$$

with

$$\mu_{1,2} = \frac{x_1^2 + x_2^2 \pm \sqrt{x_1^4 - x_2^2 x_1^2 + x_2^4}}{x_1^3 x_2^3}.$$

μ_2 is clearly positive and for m_1 we have,

$$-x_1^2 x_2^2 \geq -2x_1^2 x_2^2 \implies x_1^4 - x_2^2 x_1^2 + x_2^4 \geq (x_1^2 - x_2^2)^2,$$

it follows that :

$$\mu_1 \geq \frac{x_1^2 + x_2^2 - \left|x_1^2 - x_2^2\right|}{x_1^3 x_2^3}.$$

But, $x_1^2 + x_2^2 - \left|x_1^2 - x_2^2\right| = 2x_2^2 \geq 0 \quad \text{or} \quad 2x_1^2 \geq 0.$
So

$$\mu_1 \geq 0.$$

Thus, by lemma (3) the hessian of f is semi-definite positive this implies (by the second order theorem) that f is convex on its domain.

3. Let $f : (x_1, x_2) \mapsto \frac{x_1}{x_2}$ defined on $\mathbb{R}^2_{++}$. To prove that f is not convex I us the lemma (3)
Let's $(x_1, x_2) \in \mathbb{R}^2_{++}$ we have that

$$Hess(f)(x_1, x_2) = \begin{pmatrix} 0 & \frac{-1}{x_2^2} \\ \frac{-1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

If we denote by $\lambda(x_1, x_2)$ an eigenvalue of

$$Hess(f)(x_1, x_2)$$

since,

$$det(Hess(f)(x_1 x_2) = \frac{-1}{x_2^2} < 0$$

so there exists $(x_1^\star, x_2^\star) \in \mathbb{R}^2_{++}$ such that $\lambda(x_1^\star, x_2^\star) < 0$. By lemma we get that $Hess(f)(x_1^\star, x_2^\star)$ is not semi definite positive so by the second order condition we conclude that f is not convex.

However, f is quasi-convex. In fact its domain is clearly convex. Let's show that for any $\alpha \in \mathbb{R}$, $S_\alpha := \{x \in dom(f) \quad f(x) \leq \alpha\}$ is convex.

For $\alpha < 0$ we have S_α is empty and if $\alpha \geq 0$

$$S_\alpha = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} \quad x_1 - \alpha x_2 \leq 0\right\}$$

which is clearly convex (half plane).

4. If $\alpha = 0$ $f = x_2$ which is convex and (thus) quasi-convex. Same thing for $\alpha = 1$ by replacing the role of x_1 and x_2.

   Let $\alpha \in]0, 1[$, a direct computation gives :

$$\forall (x_1, x_2) \in dom(f) \quad Hessf(x_1, x_2) = (1-\alpha)\alpha x_1^\alpha x_2^{-\alpha} \begin{pmatrix} -x_1^{-2} x_2 & x_1^{-1} \\ x_1^{-1} & -x_2^{-1} \end{pmatrix}$$

   here we remark that

$$\forall (x_1, x_2) \in dom(f) \quad det(hess(f)(x_1, x_2) = \left((1-\alpha)\alpha x_1^\alpha x_2^{-\alpha}\right)^2 \left(x_1^{-2} - x_2^{-2}\right)$$

   so we can choose $x_1^\star, x_2^\star \in dom(f)$ such that

$$det(hess(f)(x_1^\star, x_\star^2) < 0$$

   which implies that one of the two eigenvalue of

$$hess(f)(x_1^\star, x_2^\star)$$

   is negative so by the lemma (3) and the second order condition we have that f is not convex. However f is quasi-convex. In fact, $dom(f)$ is clearly convex and for any $\beta \in \mathbb{R}$

$$S_\beta = \left\{x \quad x_1^\alpha x_2^{1-\alpha} \leq \beta\right\} = \left\{x \quad x_2 \leq \beta^{\frac{1}{1-\alpha}} x_1^{-\frac{\alpha}{1-\alpha}}\right\}$$

   if $\beta \leq 0$ then S_β is empty (and thus convex) and if $\beta > 0$ since the function

$$\omega_\alpha : t \mapsto \beta^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}}$$

   is clearly convex ($\omega_\alpha''(t) \geq 0$, $\forall t > 0$) then

$$S_\beta = \left\{(x_1, x_2) \in \mathbb{R}^2_{++} \quad x_2 \leq \omega_\alpha(x_1)\right\}$$

   is convex. Thus f is quasi convex for $\alpha \in]0, 1[$.

Exercice 3

1. Let $V \in M_n(\mathbb{R})$ and $g : t \mapsto Tr\left((X + tV)^{-1}\right)$ First we have that (since $X \in S_n^{++}$ there exists a base B such that

$$Mat_B(I_n + tX^{\frac{-1}{2}} V X^{\frac{-1}{2}})) = Diag(1 + t\lambda_i) \tag{4}$$

   where (λ_i) are the eigenvalues of

$$X^{\frac{-1}{2}} V X^{\frac{-1}{2}}$$

. This shows us first that

$$I_n + tX^{\frac{-1}{2}}VX^{\frac{-1}{2}} \in GL_n(\mathbb{R})$$

since,

$$det(I_n + tX^{\frac{-1}{2}}VX^{\frac{-1}{2}})) = \prod_{1\le i\le n} 1 + t\lambda_i \neq 0$$

so thank's to (4) and using the fact that $(AB)^{-1} = B^{-1}A^{-1}$ and that $tr(AB) = tr(BA)$ we find :

$$g(t) = Tr\left((Diag(1+t\lambda_i))^{-1} X^{-1} \right) = \sum_{1\le i\le n} \frac{(X^{-1})_{i,i}}{1+t\lambda_i}$$

Let $1 \le i \le n$, since $X \in S_n^{++}$ we
have that $X_{ii} > 0 \implies X_{ii}^{-1} = (X_{ii})^{-1} > 0$ and plus

$$\left(\frac{X_{ii}^{-1}}{1+\lambda_i t}\right)'' = \frac{2X_{ii}^{-1}\lambda_i^2}{(1+\lambda_i t)^3} \ge 0$$

Thus, g is convex as a finite sum of convex function.

2. I really can't find.

3. This follows from the min-max theorem for singular values. Let $i \in [1, n]$ by the min max theorem we have that

$$\sigma_i(X) = \max_{dim(U)=i} \quad \min_{c\in U \quad |c|=1} ||Xc||_2 \,.$$

It suffices to show that for any subspace U such that $dim(U) = i$ the function

$$X \mapsto \quad \min_{c\in U \quad |c|=1} ||Xc||_2$$

is convex. To do so it suffices to show that for any $c \in U$ such that $|c| = 1$

$$X \mapsto ||Xc||_2$$

is convex which is clearly the case since the norme is convex and $X \mapsto Xc$ is linear.

This shows that for any $1 \le i \le n$, σ_i is convex, it implies that

$$\phi = \sum_{1\le i\le n} \sigma_i$$

is convex.

I have no idea of how to show that ϕ is a supremum...

Exercice 4 We have to show that :

1. That the interior of K_{m+} is not empty. Since

$$Int(K_{m+}) = \{x \in \mathbb{R}^n \quad x_n > ... > x_0 > 0\}$$

we can choose

$$U := (e^k)_{1\le k\le n} \in Int(K_{m+}).$$

2. K_{m+} is closed. Let $x \in \mathbb{R}^n$ and $(x_p) \in K_{m+}^{\mathbb{N}}$ such that

$$\lim_{p \to \infty} ||x_p - x|| = 0.$$

By definition of $(x_p)_p$ we have that for any $p \in \mathbb{N}$

$$x_{p,n} \geq x_{p,n-1} \geq ... \geq 0. \tag{5}$$

The convergence in norm implies the convergence coordinate by coordinate by taking the limit in p in (5) we have

$$x_n \geq x_{n-1} \geq ... \geq 0.$$

which is

$$x \in K_{m+}$$

K_{m+} is closed.

3. For any $u = (\alpha, \beta) \neq 0$ we define $D_u := \{x \in \mathbb{R}^n, \quad x = \lambda u, \quad \forall \lambda \in \mathbb{R}\}$. Assume that there exists an $u = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n_\star$ such that

$$D_u \subset K_{m+} \tag{6}$$

it follows that

$$\forall \lambda \in \mathbb{R} \quad \lambda x_n \geq ... \geq 0 \tag{7}$$

Choose $j := \min k \quad \alpha_k \neq 0$, such j exists because $u \neq 0$, by (6) it follows in particular that for $\lambda = -\alpha_j$ the equation (7) reads:

$$\alpha_j^2 = 0.$$

Which is not possible by the definition of j. So there is no line in K_{m+}.

$$K_{m+} \quad \text{is a proper cone.}$$

Let determines the proper cone.

$$K_{m+}^\star := \left\{y \in \mathbb{R}^n \quad y^T x \geq 0 \quad \forall x \in K_{m+}\right\} = \left\{y \in \mathbb{R}^n \quad \sum_{1 \leq i \leq n} y_i x_i \geq 0 \quad \forall x \in K_{m+}\right\}.$$

What is clear is that $\mathbb{R}^n_+ \subset K^\star_{m+}$ However I am not able to say more, even if I take $n = 2$.

Exercice 5

1. Let's show that $f^\star$ is the indicator function of the set

$$\left\{y \geq 0 \quad \sum_{1 \leq i \leq n} y_i = 1\right\}.$$

First, if $y < 0$ then define :

$$\forall p \geq 0 \quad x_p = (-e^p, 0, 0, ..., 0)$$

We have

$$\max_{1 \leq i \leq n} x_i = 0 \implies y^T x_p - \max_{1 \leq i \leq n} x_i = -y_1 e^p \to +\infty.$$

Thus if $y < 0$

$$f^\star(y) = \infty.$$

If $y \geq 0$ and $\sum_{1\leq i\leq n} y_i = 1$ then there is l such that $y_l \neq 0$ by choosing $x_l = (0, 0, \underbrace{1}_{l} ..0)$ one has that $f^\star(y) = 0$. If $y \geq 0$ and $\sum_{1\leq i\leq n} y_i \neq 1$ we have too cases : if $\sum_{1\leq i\leq n} y_i = 0 \implies y = 0$ then you choose $x_p = (-p, ..., 0)$ so

$$y^T x_p - \max_{1\leq i\leq n} x_i = p \to \infty.$$

if $y \neq 0$ you have an $1 \leq l \leq n$ such that $y_l \neq 0$ so define

$$x_p = (0, ..., \underbrace{e^p}_{l} .., 0) \implies y^T x_p - \max_{1\leq i\leq n} x_i = e^p \left(\sum_{1\leq i\leq n} y_i - 1 \right) \to \infty.$$

2. Let $y > 1$ we notice that

$$y^T x - f(x) > \sum_{i=r+1}^{n} x_i$$

so we can construct a sequence $(x_p)_p$ such that

$$\lim_{p\to\infty} y^T x_p - f(x_p) = \infty$$

Then, assume that

$$\sum_{1\leq i\leq n} y_i = r$$

One has :

$$y^T x - \sum_{1\leq i\leq r} x_{[i]} = r - \sum_{1\leq i\leq r} x_{[i]} \tag{8}$$

define $x_r = (\underbrace{1, 1, ..., 1}_{r}, 0...0)$ put it into (8) and you get

$$f^\star(y) = 0.$$

We have show that $f^\star$ is the indicator function of the set

$$\Omega_r = \left\{ y \leq 1 \quad \sum_{1\leq i\leq n} y_i = r \right\}$$