Existence of optimal shapes with irregular boundaries for heat diffusion

Anna Rozanova-Pierrat* Gabriel Singer †

Abstract. We consider a heat problem with discontinuous diffusion coefficient and discontinuous transmission boundary conditions with a resistance coefficient. Firstly we prove the well-posedness of the problem on a very large class of domain. Then we show the existence of such an optimal shape in the classes of bounded Lipschitz domains and in bounded (ε,∞)-domains with possibly fractal boundaries that can have parts of any nonuniform Hausdorff dimension greater than or equal to $n-1$ and less than n.

Keywords: : heat, diffusion, shape optimization, trace and extension, rough.

1 Introduction

The problem of the most efficient heat transfer is a problem connected with new technologies, such as the cooling of microprocessors. The natural assumption is that the more exchange surface area there is, the better the heat transfer. That is why in recent years the mathematical and physical study of fractals has been growing.

When we think about irregular boundary we are mostly interested in fractals which were first introduced by Mandelbrot [23]. Fractals are mathematical objects quite different from Lipschitzian sets, their dimension d is not an integer (for the Von Koch snowflake $d = \frac{\log(4)}{\log(3)} \approx 1.26$) and they can have an infinite perimeter and a finite area. This makes them very interesting for maximising heat exchange. To study the physical phenomena and therefore the equations defined on the fractals we need, from a mathematical point of view to extend the usual notions of functional analysis and in particular trace operators and extensions which is done in [1, 19].

Since the famous works of de Gennes [6], it is known that the shape of the radiator is important for the speed of the diffusive heat transfer. In fact, he proposes to transform the problem of heat propagation by an exclusively geometric problem. In which the amount of heat propagated is proportional to the total perimeter, [27, 23]. For a surface of Minkowski dimension d the perimeter is proportional to a constant high at power d according to in [27]. Which shows the interest of this type of ensemble for the problem of diffusion.

They showed in [27], for a particular case of prefractal configuration that the irregularity of the boundary significantly accelerates the heat transfer across its boundary.

Moreover, in the case of perfect isolation, de Gennes argued that for small times, the heat quantity is proportional to the volume of the interior of the Ω boundary, see [2].

Many physicists have been interested in the question of heat propagation, and have proposed new methods to quantify this. In [5], they use thermochemical liquid crystals as speed and temperature tracers. Or in [24], where they mathematically study the model of a heat exchanger containing an incompressible fluid. One can also think of [9, 28] , who in the context of electrochemistry, are interested in the propagation of current through an irregularly shaped electrode.

Shape optimisation is a branch of control theory. It consists of finding a geometric structure that minimises a given functional, in our case the energy functional, (30), (4.3),

*CentraleSupélec, Université Paris-Saclay, France (correspondence, anna.rozanova-pierrat@centralesupelec.fr).

†CentraleSupélec, Université Paris-Saclay, France (correspondence, gabriel.singer@centralesupelec.fr).

(1). The optimum is sought in a class of domains that verify constraints on the perimeter and/or the volume, see [10]. These constraints give the admissible sets the property of compactness, which is the main interest, and make the functional continuous. This ensures the existence of an optimum. An introduction to the transmission problem can be found in [13]. Actually, transitory diffusion from an irregular surface has been already studied in the context of Brownian motion near fractal surfaces [20]. We refer to [3, 18, 25, 4] for other transmission problem with irregular boundaries. For example, Capitanelli studied in [3] the asymptotic behavior of a solution in dimension 2 of an elliptic problem with conditions on prefractal boundaries.

More specifically shape optimization for the heat equation has already been studied in dimension 2 in [11] and [29]. In both papers, the studies are restricted to dimension 2 and do not propose quite the same problem. In [29] they consider the problem of minimizing the time average performance and the focus is on the "turnpike" phenomenon, i.e. the convergence of the solutions of the time-dependent problem to the solutions of the stationary problem and they minimize on sets whose number of components of the complement is bounded. When in [11] it is the shape of the heat source that they want to optimize.

In our case we fix a heat source in a domain noted Ω_+ and we try to optimize the heat exchange by varying the shape of the boundary $\partial\Omega_+$. In the sense that there exists a positive measure μ on the boundary of the optimal shape $\partial\Omega_{opt}$ equivalent to the usual $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$ such that $\mathcal{H}^{N-1}(\partial\Omega_{opt}) \le \mu(\partial\Omega_{opt})$ and the weak solution of the corresponding weak problem realizes the infimum of the corresponding functional energy.

To do so, we study a linear heat equation system $\Omega = \Omega_+$ and $\Omega_- = \mathbb{R}^N \cap \Omega_+^c$ characterized by heat diffusion coefficient (distinct) D_+ and D_-, which implies the discontinuity of the metric on $\partial\Omega$, see [2]. On the boundary $\partial\Omega$ lives a continuous positive function $0 < \lambda \le \infty$, which describes the resistivity to heat exchange through the boundary.

What can be written formally:

$$\begin{array}{l}
\partial_t u_\pm - D_\pm \Delta u_\pm = 0, \quad x \in \Omega_\pm, t > 0, \\
u_+|_{t=0} = 1, \quad u_-|_{t=0} = 0, \\
D_- \frac{\partial u_-}{\partial n}\Big|_{\partial\Omega} = \lambda(x)\,(u_- - u_+)|_{\partial\Omega}\,, \\
D_+ \frac{\partial u_+}{\partial n}\Big|_{\partial\Omega} = D_- \frac{\partial u_-}{\partial n}\Big|_{\partial\Omega}.
\end{array}$$

The minimization is done with respect to the "hot" domain Ω and its boundary $\partial\Omega$ which is varied in an admissible set (5), (39). The shapes which answer the problem are called "optimal shapes".

Firstly, the optimal shapes belong to the classes of Lipschitz domains contained in a larger domain D and having the ε-cone property with the same $\varepsilon > 0$, a definition is given in 4. D.Chenais proved that for an open Ω such as $\partial\Omega$ is bounded, ϵ-cone properties is equivalent to have Lipschitz boundary. One can found a proof in [10], Theorem 2.4.7. The notions of convergence for domains discussed in this situation are the Hausdorff convergence, the convergence in the sense of characteristic functions, and the convergence in the sense of compacts, for further details see [10]. Sets that have the ε-cone property are compact for the three previous types of convergence.

Then the optimal shapes are (ε, ∞)-domains,see [14], with additional conditions on the boundary, 39. They are also compact for the three types of convergence.

The article is divided into the following parts: We first present the main notations in section 2. Then in the section 3 , we establish a well-posedness result and a prior estimate, depending only on time T, Theorem 3, Theorem 3.

Using the method in [21] we prove in section 4, for the stationary and time-dependent case, the existence of an optimal shape over a class of bounded Lipschitz domains, Theorem 8 and Theorem 9.

Employing results from [12] and [22] we prove in 5 the existence of an optimal shape in a larger class of bounded uniform domains which then realizes the minimum of the energy over this class, Theorem 11.

2 Main notations

We introduce the $W^{1,2}$-extensions domains. For a given $n \in \mathbb{N}$ a domain $\Omega \subset \mathbb{R}^n$ is a $W^{1,2}$-extension if there is a bounded linear extension operator $E : W^{1,2}(\Omega) \mapsto W^{1,2}(\mathbb{R}^n)$ [14, 26]. That is, there exists a constant $C > 0$ independent of $u \in W^{1,2}(\Omega)$, such as:

$$\|Eu\|_{W^{1,2}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,2}(\Omega)} .$$

Definition 1. *((ε, δ)-**domain** [14, 16, 30]) An open connected subset Ω of $\mathbb{R}^n$ is an (ε, δ)-domain, $\varepsilon > 0$, $0 < \delta \leq \infty$, if whenever $x, y \in \Omega$ and $|x - y| < \delta$, there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying*

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ *and*
2. $d(z, \partial\Omega) \geq \varepsilon |x - z| \frac{|y-z|}{|x-y|}$ *for* $z \in \gamma$.

For $\delta = +\infty$ it is possible to avoid the local character of this definition and in this case Ω is said to be an (ε, ∞)-domain. By [14], Theorem 1 any (ε, ∞)-domain is an $W^{1,2}$-extension. The second condition prohibits the boundary to collapse into thin structures.

Definition 2. *For a given $d > 0$ a Borel measure μ_F on $\mathbb{R}^n$ with $F = supp(\mu)$ is said to be upper-d regular if there is a constant $c_d > 0$ such as*

$$\mu(B(x, r)) \leq c_d r^d, \quad x \in F, \quad 0 < r \leq 1. \tag{1}$$

Thanks to the results in [30, 17, 19, 2, 1] we generalize the trace operator to cases of more irregular boundaries to sets without a fixed dimension [15].

Definition 3. *For an extension domain Ω of $\mathbb{R}^n$ and μ a measure such as $supp(\mu) = \Omega$. The trace operator* Tr *is defined for $u \in H^1(\Omega)$ by*

$$\operatorname{Tr} u(x) = \lim_{r \to 0} \frac{1}{\lambda^n (\Omega \cap B(r, x))} \int_{\Omega \cap B(r,x)} u(y) d\lambda^n, \quad \mu - a.e.$$

Where λ^n denotes the n-dimensional Lebesgue measure.

Let us denote by $B(\partial\Omega, \mu) = \operatorname{Im}\left(\operatorname{Tr}\left(W^{1,2}(\Omega)\right)\right)$ for a boundary $\partial\Omega$ defined as the support of a measure μ.

Theorem 1 (Theorem of Trace). *Let $\Omega \subset \mathbb{R}^n$ be a bounded $W^{1,2}$ -extension domain. Suppose that μ is a Borel measure with supp $\mu = \partial\Omega$ and such that (1) holds with some $n - 2 < d \leq n$. There are a compact linear operator $\operatorname{Tr} : W^{1,2}(\Omega) \to L^2(\partial\Omega, \mu)$ and a constant $C > 0$, depending only on n, d and c_d, such that*

$$\| \operatorname{Tr} f\|_{L^2(\partial\Omega,\mu)} \leq C\|f\|_{W^{1,2}(\Omega)}, \quad f \in W^{1,2}(\Omega).$$

Endowed with the norm

$$\|\varphi\|_{\operatorname{Tr}(W^{1,2}(\Omega))} := \inf\left\{\|g\|_{W^{1,2}(\Omega)} \mid \varphi = \operatorname{Tr} g\right\}$$

the image $\operatorname{Tr}\left(W^{1,2}(\Omega)\right)$ becomes a Hilbert space. The embedding

$$\operatorname{Tr}\left(W^{1,2}(\Omega)\right) \subset L^2\left(\partial\Omega, \mu_{\partial\Omega}\right)$$

is compact.

There is a linear operator $\mathrm{H} : \operatorname{Tr}\left(\mathrm{W}^{1,2}(\Omega)\right) \to W^{1,2}(\Omega)$ of norm one such that $\operatorname{Tr}(H\varphi) = \varphi$ for all $\varphi \in \operatorname{Tr}\left(W^{1,2}(\Omega)\right)$.

Proposition 1 (The external normal derivative). *Let $\Omega \subset \mathbb{R}^n$ be a bounded $W^{1,2}$ -extension domain. Let $B = B(\partial\Omega, \mu) = \operatorname{Tr}(W^{1,2}(\Omega))$ and $B' = B'(\partial\Omega, \mu)$ his dual. Then it holds*

$$\left\langle \frac{\partial u}{\partial x}, \operatorname{Tr}(v) \right\rangle_{B',B} = \int_\Omega v\Delta u dx + \int_\Omega \nabla v \nabla u dx,$$

with $(u, v) \in H^1(\Omega) \times H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. Where the external normal derivative is a continuous linear form on $B(\partial\Omega, \mu)$.

3 Weak formulation

In this section we establish and discuss the weak formulation of our problem. We introduce the class of admissible domain as all the $\Omega \subset \mathbb{R}^n$, such as Ω is a bounded $W^{1,2}$ -extension with supp $\mu = \partial\Omega$ for μ a Borel measure and such that (1) holds with some $n-2 < d \leq n$.

Let Ω be an admissible domain. Thanks to the definition above of the normal derivative we can proceed as usual and therefore get the weak formulation of our propagation problem, it is firstly done in [2].

The stationary case is the simplest case to study, so we start with this one.

In order to have a problem well defined we proceed as in [2]. We introduce the Hilbert space

$$V(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) / f_{|\Omega_+} \in H^1(\Omega_+), f_{|\Omega_-} \in H^1(\Omega_-) \right\}. \tag{2}$$

$V(\mathbb{R}^n)$ is endowed with the following norm:

$$\|u\|^2_{V(\mathbb{R}^n)} = \int_{\Omega_+} |\nabla u_+|^2 \, dx + \int_{\Omega_-} |\nabla u_-|^2 \, dx + \int_{\Omega_+ \cup \Omega_-} u^2 dx.$$

Theorem 2. *Let Ω be an admissible domain, $u(x,0) = u_0 \in L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$. We assume that λ is a continuous function on $\partial\Omega$. There is a unique u in $V(\mathbb{R}^n)$ such as:*

$$\forall v \in V(\mathbb{R}^n) \quad a(u,v) = \langle f, v \rangle_{L^2(\mathbb{R}^n)}. \tag{3}$$

With

$$a(u,v) = \int_{\Omega_+} \nabla u_+ \nabla v_+ dx + \int_{\Omega_-} \nabla u_- \nabla v_- dx + \int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_+ - u_-) \operatorname{Tr}(v_+ - v_-) d\mu, \tag{4}$$

Plus there is a unique $u \in V(\mathbb{R}^n)$ such as

$$\forall v \in V(\mathbb{R}^n) \quad a_\infty(u,v) = \langle f, v \rangle_{L^2(\mathbb{R}^n)}. \tag{5}$$

Where

$$a_\infty(u,v) = \int_{\Omega_+} \nabla u_+ \nabla v_+ dx + \int_{\Omega_-} \nabla u_- \nabla v_- dx. \tag{6}$$

Proof. The continuity of all applications in play comes from the continuity of the trace and the Cauchy-Schwarz inequality. As for the coercivity we have that

$$a(u,u) \geq \|u\|^2_{V(\mathbb{R}^n)} - \|u\|^2_{L^2(\mathbb{R}^n)}$$

and the Lax-Milgram's theorem conclude the proof. □

Let us now turn to the more complex model for which we also have a well-posed problem. Let $V(\mathbb{R}^n)$ be defined by the equation (2).

We endow $V(\mathbb{R}^n)$ with the norm

$$\|u\|^2_{V(\mathbb{R}^n)} = D_+ \int_{\Omega_+} |\nabla u_+|^2 \, dx + D_- \int_{\Omega_-} |\nabla u_-|^2 \, dx + \int_{\Omega_+ \cup \Omega_-} u^2 dx, \tag{7}$$

it is compactly injected in $L^2(\mathbb{R}^n)$, Section 2.1 in [2]. This is a very strong property for the future. This variational formulation has been first discovered in [2].

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be an admissible domain. and $T \in \mathbb{R}^+_*$, $f \in L^2([0,T], L^2(\mathbb{R}^n))$, and $\lambda \in C(\partial\Omega)$. Let $D_+ \geq 0$ and $D_- \geq 0$ be the two distinct diffusive coefficients and*

$$a(u,v,t) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ dx + D_- \int_{\Omega_-} \nabla u_- \nabla v_- dx + \int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_+ - u_-) \operatorname{Tr}(v_+ - v_-) d\mu. \tag{8}$$

1. *The weak formulation is the following. Find $u \in V(\mathbb{R}^n)$ such as $\forall v \in V(\mathbb{R}^n)$ and $u(x,0) = u_0 \in L^2(\mathbb{R}^n)$.*

$$\frac{\mathrm{d}\langle u, v\rangle_{L^2(\mathbb{R}^n)}}{\mathrm{d}t} + a(u,v,t) = \langle f, v\rangle_{L^2(\mathbb{R}^n)}. \tag{9}$$

2. *There is a unique $u \in L^2([0,T], V(\mathbb{R}^n))$ such as u solves (9).*

3. *Moreover for u a solution of (9) there is a constant C positive depending only on T such as*

$$\|u(t)\|^2_{L^2(([0,T]),V)} = \int_0^T \|u(t)\|^2_V \, dt \leq C\left(\|u_0\|^2_{L^2(\mathbb{R}^n)} + \|f(t)\|^2_{L^2(([0,T]),L^2(\mathbb{R}^n))}\right). \tag{10}$$

To prove this theorem we use the Galerkin's method, see page 353 of [7]. This method consists to build a sequence $(u_m)_{m\in\mathbb{N}}$ solution of our problem on sub finite dimensional subspace $V_m = Span(w_k)_{0\leq k\leq m}$. With the aim of passing to the limit, such as in a sense that we will specify, $u = \lim_{m\to\infty} u_m$ is a solution of our weak formulation on V.

Proof. Proof of point 1) Let $v \in V(\mathbb{R}^n)$: We multiply both side of strong formulation by v and it follows that

$$\int_{\mathbb{R}^n} (\partial_t u_\pm v_\pm - D_\pm \Delta u_\pm v_\pm)\, dx = \int_{\mathbb{R}^n} f v dx.$$

Since $\mathbb{R}^n = \Omega_+ \cup \Omega_-$ we split the previous equation into Ω_+ and Ω_- thus

$$\frac{\mathrm{d}\langle u, v\rangle_{L^2(\mathbb{R}^n)}}{\mathrm{d}t} - D_+ \int_{\Omega_+} \Delta u_+ v_+ dx - D_- \int_{\Omega_-} \Delta u_- v_- dx = \langle f, v\rangle_{L^2(\mathbb{R}^n)}. \quad (*)$$

Then we apply the Green's formula to $D_+ \int_{\Omega_+} \Delta u_+ v_+ dx$ and then to $-D_- \int_{\Omega_-} \Delta u_- v_- dx$, and by the linearity of the trace we get

$$\begin{aligned} &D_+ \int_{\Omega_+} \Delta u_+ v_+ dx - D_- \int_{\Omega_-} \Delta u_- v_- dx \\ &= D_+ \int_{\Omega_+} \nabla v_+ \nabla u_+ dx + \langle \lambda(x) \operatorname{Tr}(u_+ - u_-), \operatorname{Tr}(v_+ - v_-)\rangle_{B',B} + D_- \int_{\Omega_-} \nabla v_- \nabla u_- dx. \end{aligned} \tag{11}$$

Finally by replacing (11) in $(*)$ we get: $\forall v \in V(\mathbb{R}^n)$

$$\begin{aligned} &\frac{\mathrm{d}\langle u, v\rangle_{L^2(\mathbb{R}^n)}}{\mathrm{d}t} + D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ dx + D_- \int_{\Omega_-} \nabla u_- \nabla v_- dx + \\ &\qquad \int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_+ - u_-) \operatorname{Tr}(v_+ - v_-) d\mu = \langle f, v\rangle_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Proof of point 2, which is largely inspired by [8]. We replace $\frac{\mathrm{d}}{\mathrm{d}t}$ by $'$ and $\langle\rangle$ by $(,)$.

Let $(w_k)_{k\in\mathbb{N}}$ be an orthogonal base of V and $L^2(\mathbb{R}^n)$. Let's fix $T \geq 0$ and $m \in \mathbb{N}$, we define $V_m = Span\,(w_k)_{0\leq k\leq m}$. Without loss of generality we assume, by Gramm-Schmidt's theorem that for any $0 \leq k \leq m$ $\|w_k\|_{L^2(\mathbb{R}^n)} = 1$.
Moreover we assume that $u_m(t)$ has the following form $u_m(t) := \sum_{k=1}^m d_m^k(t) w_k$. One has $d_m^k(t) = (u_m(t), w_k) \quad (0 \leq t \leq T, k = 1, \ldots, m)$ we obtain $d_m^k(0) = (u(x, t=0), w_k) \quad (k = 1, \ldots, m)$. The following estimate is one of the most important of this section because we use it to prove that the sequence of extension is uniformly bounded in m.

One can see that by triangular inequality and Bessel:

$$\|u_m(0)\|_{L^2(\mathbb{R}^n)} = \left\| \sum_{0\le k\le m} (w_k, u_0) w_k \right\|_{L^2(\mathbb{R}^n)} \le \sum_{0\le k\le m} |(w_k, u_0)| \, \|w_k\|_{L^2(\mathbb{R}^n)} \le \|u_0\|_{L^2(\mathbb{R}^n)} . \quad (12)$$

We assume that: $(u_m', w_k) + a\,(u_m, w_k; t) = 0 \quad (0 \le t \le T, k = 1, .., m)$. By using the linearity of the derivative in time and the linearity of $a(., ., .)$ We discover that for all $0 \le j \le n$

$$\sum_{k=1}^{m} d_m^k(t) a(w_k, w_j, t) + {d'_m}^j(t) = f^j(t). \quad (13)$$

With $f^j(t) := (f(t), w_j)$

Hence (9) with $d_m^j(0) = (u_0, w_j)$ becomes a system of m ordinary differential equations and we know there exists a unique function absolutely continuous $d_m(t) = (d_m^1, d_m^2, ..., d_m^m)$.

Therefore $u_m(t) := \sum_{k=1}^m d_m^k(t) w_k$ solves $(u_m', w_k) + a(u_m, w_k, t) = 0$.

We will now need an estimation independent of m.

Firstly summing from 0 to m $(u_m', d_m^k w_k) + a(u_m, d_m^k w_k, t) = (f, d_m^k w_k)$. we obtain:

$$(u_m', u_m) + a(u_m, u_m, t) = (f, u_m). \quad (14)$$

Thank's to

$$a(u_m, u_m, t) \ge \|u_m\|_V^2 - \|u_m\|_{L^2(\mathbb{R}^n)}^2 . \quad (15)$$

and

$$|(f, u_m)| \le \frac{1}{2} \|f\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|u_m\|_{L^2(\mathbb{R}^n)}^2 \quad (16)$$

We first notice that by adding $\frac{1}{2} \frac{\mathrm{d}\|u_m\|_{L^2(\mathbb{R}^n)}^2}{\mathrm{d}t}$ on both sides of (15) and using (16) we get:

$$\frac{1}{2} \frac{\mathrm{d}\,\|u_m\|_{L^2(\mathbb{R}^n)}^2}{\mathrm{d}t} - \|u_m\|_{L^2(\mathbb{R}^n)}^2 + \|u_m\|_V^2 \le \frac{1}{2} \|f\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|u_m\|_{L^2(\mathbb{R}^n)}^2 \quad (17)$$

which leads us to

$$\frac{\mathrm{d}\,\|u_m\|_{L^2(\mathbb{R}^n)}^2}{\mathrm{d}t} \le 3 \|u_m\|_{L^2(\mathbb{R}^n)}^2 + \|f\|_{L^2(\mathbb{R}^n)}^2 .$$

Thank's to the differential form of Gronwall's inequality

$$\|u_m\|_{L^2(\mathbb{R}^n)}^2 \le e^{3t} \left(\|u_m(0)\|_{L^2(\mathbb{R}^n)}^2 + C_2 \int_0^t \|f(s)\|_{L^2(\mathbb{R}^n)}^2 \, ds \right) \le e^{3t} \left(\|u_0\|_{L^2(\mathbb{R}^n)}^2 + C_2 \int_0^t \|f(s)\|_{L^2(\mathbb{R}^n)}^2 \, ds \right) \quad (18)$$

with C_2 depending only on $0 \le t \le T$.

This leads you to:

$$\max_{0\le t\le T} (\|u_m(t)\|_{L^2(\mathbb{R}^n)}^2) \le \max(e^{3T}, C_2 e^{3T}) \left(\|u_0\|_{L^2(\mathbb{R}^n)}^2 + \|f(t)\|_{L^2([0,T], L^2(\mathbb{R}^n))}^2 \, dt \right) . \quad (19)$$

Returning to (17) and integrating from 0 to T and employ the inequality above (19):

$$\|u_m(t)\|_{L^2(([0,T]), V)}^2 = \int_0^T \|u_m(t)\|_V^2 \, dt \le C \left(\|u_0\|_{L^2(\mathbb{R}^n)}^2 + \|f(t)\|_{L^2(([0,T]), L^2(\mathbb{R}^n))}^2 \right) . \quad (20)$$

The uniform estimate (12) ensures that $C \geq 0$ doesn't depend on m and the Gronwall's lemma that C depends only on T. We will now show that u'_m is bounded.

To do this fix any $v = v^1 + v^2$ in V such as $v^1 \in Span\,(w_k)_{k\in\mathbb{N}}$ and $(v^2, u_m) = 0$ and $\|v\|_V \leq 1$ for all m. By the continuity of a on $V \times V$ there is a constant $\theta \geq 0$ independent of n such as $\left|a(u_m, v^1, t)\right| \leq \theta \left\|u_m(t)\right\|_V \left\|v^1\right\|_V$,

$$|(u'_m, v)| = \left|(u'_m, v^1)\right| = \left|(f, v^1) - a(u_m, v^1, t)\right| \leq C\left(\|f\|_{L^2(\mathbb{R}^n)} + \|u_m(t)\|_V\right).$$

Integrating once again from 0 to T and you find:

$$\|u'_m(t)\|^2_{L^2([0,T],V')} = \int_0^T \|u'_m(t)\|^2_{V'}\,dt \leq C\left(\|u_0\|^2_{L^2(\mathbb{R}^n)} + \|f(t)\|^2_{L^2([0,T],L^2(\mathbb{R}^n))}\right) \tag{21}$$

Once again thanks to (12) and Gronwall's lemma we can ensure that C doesn't depend on m.

Hereafter we are able to prove the existence and uniqueness.

Thanks to (20) and (21) , $(u_m)_{m\in\mathbb{N}}$ and $(u'_m)_{m\in\mathbb{N}}$ are bounded. Since $L^2([0,T], V)$ and $L^2([0,T], V'))$ are reflexive $(u_m)_{m\in\mathbb{N}}$ and $(u'_m)_{m\in\mathbb{N}}$ converge weakly to x in $L^2([0,T], V)$ and y in $L^2([0,T], V'))$.

Let's now fix an integer N and any $v \in C^1([0,T], V)$ such as

$$v(t) := \sum_{k=1}^{N} d^k(t) w_k \tag{22}$$

where $t \mapsto d^k(t)$ are smooth function.

After noticing that for any v of the form (22) and any m, $(u'_m, v) + a(u_m, v, t) = (f(t), v)$ you integrate from 0 to T and pass to the weak limit to finally get the result :

$$\int_0^T (y, v) + a(x, v, t)dt = \int_0^T (f(t), v)dt,$$

which became by density of the form (22) in $L^2([0,T], V)$:

$$\forall v \in V \quad (y, v) + a(x, v, t) = (f(t), v).$$

By the uniqueness we find $x = u$ and $y = u'$

We now have to show that the initial value coincides.

One can notice that

$$\forall v \in C^1([0,T]; V), \int_0^T -(v', u) + a(u, v, t)dt = \int_0^T (f, v)dt + (u(0), v(0)) - (u(T), v(T)). \tag{23}$$

Similarly,

$$\int_0^T -(v', u_m) + a(u_m, v, t)dt = \int_0^T (f, v)dt + (u_m(0), v(0)) - (u_m(T), v(T)).$$

We employ once again the weak limit of the derivative term to find

$$\int_0^T -(v', u) + a(u, v, t)dt = \int_0^T (f, v)dt + (u_0, v(0)) - (u(T), v(T)). \tag{24}$$

Since $u_m(0) \to u_0$ in $L^2(\mathbb{R}^n)$. We deduce by comparing (23) to (24) that $u(0) = u_0$.

This shows the existence of a weak solution to (9).

For the uniqueness we assume that there is u_1 and u_2 solutions of our problem. Let's set $w = u_1 - u_2$ and $w(x,0) = 0$. It leads us to $(u_1 - u_2, v) + a(u_1 - u_2, v, t) = 0 = (w, v) + a(w, v, t)$ and we find that

$$\frac{1}{2}\frac{\mathrm{d}\,\|w\|^2_{L^2(\mathbb{R}^n)}}{\mathrm{d}t} \leq \frac{3}{2}\,\|w\|^2_{L^2(\mathbb{R}^n)}\,.$$

Since $w(x,0) = 0$ by the Gronwall's lemma it follows $w = 0$. Which conclude the uniqueness. Finally we find a unique solution to our weak problem. Now for the estimate you pass to the limit in (20) and you get a constant C positive independent of m such as (10) holds. □

We know that in the classical case of the heat equation we have convergence in L^2 norm when the time tends to infinity. Do we have the same thing in our case? Yes the solution of the time-dependent problem converges for the strong L^2-topology to the solution of the stationary problem when time increases infinitely.

To prove the following theorem we use the spectral decomposition of our solution. It follows :

Theorem 4. *Let $u : (x,t) \mapsto u(x,t)$ a solution of (9) and $u^* : x \mapsto u^*(x)$ a solution of (3) and any $f \in L^2([0,T], \mathbb{R}^n)$. We have:*

$$\lim_{t\to\infty} \|u(x,t)\|_{L^2(\mathbb{R}^n)} = \|u^*(x)\|_{L^2(\mathbb{R}^n)}\,. \tag{25}$$

Proof. We principally use spectral decomposition method.
Let $d = u - u^*$, is solution in the weak sense of :

$$\begin{cases} \frac{\partial d_\pm}{\partial t} = D_\pm \Delta d_\pm & x \in \Omega_\pm \quad t > 0, \\ \quad d(x,0) = 0 & x \in \Omega_\pm \\ \frac{\partial d_+}{\partial n} = \frac{\partial d_-}{\partial n} = 0 & x \in \partial\Omega_+ \end{cases} \tag{26}$$

We know, [8], that there is an orthogonal basis $(\phi_k)_{k\in\mathbb{N}}$ associated to the weak formulation of (26) in $L^2(\mathbb{R}^n)$ and a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < ... < \lambda_k < ...$
Thus $d(x,t)$ can be written in the basis of the eigenfunctions as $d(x,t) = \sum_{k\in\mathbb{N}} v_k e^{-\lambda_k t}\phi_k(x)$ for any $(x,t) \in \mathbb{R}^n \times (0,+\infty)$. It follows that :

$$\begin{aligned} \|d(x,t)\|^2_{L^2(\mathbb{R}^n)} &= \left\| \sum_{k\in\mathbb{N}} v_k e^{-\lambda_k t}\phi_k(x) \right\|^2_{L^2(\mathbb{R}^n)} \\ &= \sum_{k\in\mathbb{N}} \langle e^{-\lambda_k t} v_k \phi_k(x), e^{-\lambda_k t} v_k \phi_k(x) \rangle_{L^2(\mathbb{R}^n)} \\ &= \sum_{k\in\mathbb{N}} \langle e^{-2\lambda_k t} v_k \phi_k(x), v_k \phi_k(x) \rangle_{L^2(\mathbb{R}^n)} \\ &\leq e^{-2\lambda_1 t}\,\|d(x,0)\|^2_{L^2(\mathbb{R}^n)} \xrightarrow{t\to\infty} 0. \end{aligned}$$

□

Assume now that we have isolated the boundary $\partial\Omega$ perfectly, we find ourselves in the case $\lambda = \infty$ and we find that

Theorem 5. *Let Ω be an admissible domain, $f \in L^2([0,T], L^2(\mathbb{R}^n))$, for $\lambda = \infty$ and $u(x,0) = u_0 \in L^2(\mathbb{R}^n)$.*

1. *The weak formulation is: find $u \in V(\mathbb{R}^n)$ such as $\forall v \in V(\mathbb{R}^n)$*

$$\frac{\mathrm{d}\langle u, v\rangle_{L^2(\mathbb{R}^n)}}{\mathrm{d}t} + a_\infty(u,v,t) = \langle f, v\rangle_{L^2(\mathbb{R}^n)}. \tag{27}$$

Where

$$a_\infty(u,v,t) = D_+ \int_{\Omega_+} \nabla u_+ \nabla v_+ dx + D_- \int_{\Omega_-} \nabla u_- \nabla v_- dx. \tag{28}$$

2. *There is a unique u in $L^2([0,T], V(\mathbb{R}^n))$ such as:* $\forall v \in V(\mathbb{R}^n)$

$$\frac{\mathrm{d}\langle u, v\rangle_{L^2(\mathbb{R}^n)}}{\mathrm{d}t} + a_\infty(u, v, t) = \langle f, v\rangle_{L^2(\mathbb{R}^n)}. \tag{29}$$

3. *Moreover there is a constant $C > 0$ depending only on T such as*

$$\|u(t)\|^2_{L^2(([0,T]),V(\mathbb{R}^n)} \leq C\left(\|u_0\|^2_{L^2(\mathbb{R}^n)} + \|f(t)\|^2_{L^2(([0,T]),L^2(\mathbb{R}^n))}\right).$$

Now that we know that our weak formulations have a single relatively regular solution. We move on to shape optimisation.

4 Shape Optimization

Let N be the dimension of the space. Then we fix and open $D = \Omega_+ \cup \Omega_-$ with $\Omega_- = D \cap \Omega_+^c$. The aim is to find the optimal shape that removes the heat from Ω_+ to Ω_- as quickly as possible. To do this we study the following energy functional:

$$J(\Omega_+, G, \varepsilon, u_+, u_-, T) = \int_0^T \int_{\Omega_+} |\nabla u_+|^2 \, dxdt + \int_0^T \int_{\Omega_+} |u_+|^2 \, dxdt + \int_0^T \int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_+ - u_-)^2 d\mu dt, \tag{30}$$

with μ an abstract measure verifying (1) and $T > 0$, a given time, $\beta = \int_\Omega d\lambda^N$ a given volume and $G = \Omega_1 \setminus \bar{\Omega_0}$ are fixed. This is typically the kind of constraint that meets industrial needs.
This functional represents the total energy of Ω_+. We will minimize it on two types of sets. On the one hand the uniformly length bounded Lipschitzian sets and on the other hand the admissible sets.

One can notice that actually:

$$J(\Omega_+, G, \varepsilon, u_+, u_-, T) = \|u_+\|^2_{L^2([0,T],H^1(\Omega_+))} + \left\|\sqrt{\lambda(x)} \operatorname{Tr}(u_+ - u_-)\right\|^2_{L^2([0,T],L^2(\partial\Omega_+),\mu)}.$$

4.1 Optimal shape with Lipschitz boundary

In this part we show the existence of optimal Lipschitz boundary shapes which minimize the energy functional.

We assume that every optimal shapes are included in a fixed open set D.

Definition 4. *We say that a set $\Omega \subset \mathbb{R}^N$ verifies the ε-cone property if for a ε>0 fixed and whenever $x \in \partial\Omega$ there is a vector ζ with norm 1 , such as for all $y \in \bar{\Omega} \cap B(x, \varepsilon)$: $\{z \in \mathbb{R}^n, (z - y, \zeta) > cos(\varepsilon) \|z - y\|$ and $0 < \|z - y\| < \varepsilon\} \subset \Omega$.*
Let $\Theta(D, \varepsilon) = \{\Omega \subset D, \Omega \ \varepsilon - cone\}$ at a given time $T \geq 0$ fixed.

Moreover we impose some regularity on $\partial\Omega$. We want that for any $\Omega \in \Theta(D, \varepsilon)$ there exists a fixed $\hat{c} > 0$

$$\int_{\partial\Omega \cap B(x,r)} d\mathcal{H}^{N-1} \leq \hat{c}(r) \quad \forall x \in \partial\Omega. \tag{31}$$

Where $B(x, r)$ is the Euclidian ball centered in x with radius r.

Definition 5. *We fixe the time $T \geq 0$. We define the admissible class of set as follow.*
Let $\varepsilon > 0$, $M_0 > 0$ and $\hat{c} > 0$ be fixed constant.

$U_{ad}(G, D_0, M_0, \varepsilon, \hat{c}) =$

$$\left\{\Omega \in \Theta(D, \varepsilon) \mid M_0 \leq \int_{\partial\Omega} d\mathcal{H}^{N-1} \leq M(\hat{c}) \; \forall x \in \partial\Omega, \partial\Omega \subset \bar{G} \;, Vol(D_0) = \int_{\Omega} d\lambda^N\right\}$$

where λ^N *is the N-dimensional Lebesgue measure and* $G = D_1 \setminus \overline{D_0}$.

Theorem 6. *Let* $\varepsilon > 0$ *be fixed.* $\Theta(D, \varepsilon) = \{\Omega \subset D, \Omega \; \varepsilon - cone\}$ *is compact with respect to the convergence in the sense of characteristic functions, the convergence in the sense of Hausdorff and the convergence in the sense of compacts. For any sequence* $\Omega_n \to \Omega$ *of sets in this class converging in all three senses, their boundaries* $\partial\Omega_n \to \partial\Omega$ *and also their closures* $\Omega_n \to \Omega$ *converge in the Hausdorff sense see Theorem 2.4.10 [10] for further details.*

Slight modifications of the proof of lemma 3.1 in [21] gives us the following theorem.

Theorem 7. *Let the parameters* $\varepsilon, D_0, \hat{c}, M_0$ *be fixed in definition 5.*
$U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ *is compact for the three modes of convergence of theorem 6.*

Lemma 1. *Let the parameters* $M_0, \varepsilon, \hat{c}$ *be defined in definition 5.*
If $(\Omega_n)_{n\in\mathbb{N}^*} \subset U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$, *then there exists a sub sequence* $(\Omega_{n_k})_{k\in\mathbb{N}} \subset (\Omega_n)_{n\in\mathbb{N}^*}$ *and a domain* $\Omega \in U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ *such that* $(\Omega_{n_k})_{k\in N}$ *converges to* Ω *with respect to these three types of convergences and in addition* $\bar{\Omega}_{n_k}$ *and* $\partial\Omega_{n_k}$ *converge in the sense of Hausdorff, respectively, to* $\bar{\Omega}$ *and* $\partial\Omega$.

Proof. This is an immediate consequence of theorem 6. □

The previous lemma is very useful for shape optimisation as it ensures the existence of a convergent sequence of shapes and that the limit stay in $U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$. Furthermore some slight modifications to the proof of Lemma 3.1 in [21] give us the following lemma.

Lemma 2. *Let the parameters* $G, \varepsilon, D_0, \hat{c}, M_0$ *be defined in 5.*
Let $(\Omega_n)_{n\in\mathbb{N}^*} \subset U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ *be a sequence converging to* Ω *in* $U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ *in the sense of Lemma 1. Then there exists a sub-sequence* $(\Omega_{n_k})_{k\in\mathbb{N}}$ *with boundaries* $(\partial\Omega_{n_k})_{k\in\mathbb{N}^*}$, *and there exists a positive Radon measure* μ^* *with support on* $\partial\Omega$, *equivalent to the (N-1)-dimensional Hausdorff measure, such that:*

1. $\forall\psi \in C(\bar{D}) \quad \int_{\partial\Omega_{n_k}} \psi d\mathcal{H}^{N-1} \to \int_{\partial\Omega} \psi d\mu^*$.
2. $\int_{\partial\Omega} d\mathcal{H}^{N-1} \leq \int_{\partial\Omega} d\mu^*$ *or equivalently* $\mu^*(\partial\Omega) \geq \mathcal{H}^{N-1}(\partial\Omega)$.

4.2 Stationary case and time-dependant case

Let $(\Omega_n)_{n\in\mathbb{N}^*} \subset U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ be a sequence converging to Ω_+ in $U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ in the sense of Lemma 1. Where the parameters $G, \varepsilon, D_0, \hat{c}, M_0$ are defined in 5.

4.2.1 Stationary case

For the optimization purpose we introduce and fix D an open set which contains each Ω_n for all n. Such as $D = \Omega_+ \cup \Omega_-$ with $\Omega_- = D \cap \Omega_+^c$. Plus we denoting by $V(D)$ the space of all function f defined on D such as $f_{|\Omega+} \in H^1(\Omega^+)$ and $f_{|\Omega-} \in H^1(\Omega^-)$.
The energy functional becomes:

1. $J(\Omega_+, G, \varepsilon, u_+, u_-) = \int_{\Omega_+} |\nabla u_+|^2 dx + \int_{\Omega_+} |u_+|^2 dx + \int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_+ - u_-)^2 d\mu$ for $\lambda < \infty$.
2. $J(\Omega_+, G, \varepsilon, u_+, u_-) = \int_{\Omega_+} |\nabla u_+|^2 dx + \int_{\Omega_+} |u_+|^2 dx$ for $\lambda = \infty$.

In order to pass to the limit, we have to find uniform estimate in n. Indeed, as $H^1(D)$ is a reflexive space, it is sufficient that $(\|Eu_n^+\|_{H^1(D)})_{n\in\mathbb{N}}$ is bounded to be able to extract a sequence which converges weakly in $H^1(D)$.
To do so we consider u_n^+ and u_n^- be the weak solution of (3) on Ω_n^+ respectively Ω_n^-.

There exist constants $D_1 \geq 0$ and $D_2 \geq 0$ independent of n such as $\|Eu_n^+\|_{H^1(D)} \leq D_1$ and $\|Eu_n^-\|_{H^1(D)} \leq D_2$. Where E is the extension operator $E : H^1(\Omega_n^\pm) \to H^1(D)$. In fact by 1:

On one hand there exists two constants $C_1 > 0$ and $C_2 > 0$ independent of n such as $\|u_n^+\|_{H^1(\Omega_n^+)} \leq C_1$ and $\|u_n^-\|_{H^1(\Omega_n^-)} \leq C_2$.

On the other hand by the continuity of E on $H^1(\Omega_n^+)$ there is $a_1 > 0$ independent of n such as for all n :

$$\left\|Eu_n^+\right\|_{H^1(D)} \leq a_1 \left\|u_n^+\right\|_{H^1(\Omega_n^+)} \leq a_1 C_1.$$

We put $D_1 = a_1 C_1$. We proceed in the same way for u_- on Ω_n^-.

This proposition is very crucial because it gives us two weak limits: u_+^* and u_-^* such as $Eu_n^+ \xrightarrow[n\to\infty]{} u_+^*$ and $Eu_n^- \xrightarrow[n\to\infty]{} u_-^*$.

Since $H^1(D)$ is compactly injected in $L^2(D)$ we finally get the strong $L^2(D)$ convergence, what can be written as $Eu_n^+ \xrightarrow[n\to\infty]{L^2(D)} u_+^*, Eu_n^- \xrightarrow[n\to\infty]{L^2(D)} u_-^*$. We also remark that by the continuity of $\mathrm{Tr} : H^1(D) \to L^2(D)$ one has $\lim_{n\to\infty} \mathrm{Tr}(Eu_n^\pm) = \mathrm{Tr}(\lim_{n\to\infty} Eu_n^\pm) = \mathrm{Tr}(u_\pm^*)$. We now show that variationnal problem on each Ω_n converge to the one on $\Omega \subset D$.

Let $N \in \mathbb{N}^*$ be the dimension of the space. Let $V(D)$ be defined above. Let $(\Omega_n)_{n\in\mathbb{N}}$ a sequence of set with Lipschitz boundary.

We assume that

$$u_n \xrightarrow[n\to\infty]{V(D)} u^*.$$

Then for all v in V

$$a(u_n, v) \xrightarrow[n\to\infty]{} a(u^*, v).$$

Where $a(.,.)$ is defined in (4).

Let $u_n \xrightarrow[n\to\infty]{V} u^*$. It means that we have $u_+^* \in H^1(D)$ and $u_-^* \in H^1(D)$ such as: $u_n^+ \xrightarrow[n\to\infty]{H^1(D)} u_+^*$
and $u_n^- \xrightarrow[n\to\infty]{H^1(D)} u_-^*$. Then we consider the difference $|a(u_n, v) - a(u^*, v)|$:

$$\begin{aligned}
|a(u_n, v) - a(u^*, v)| \leq{}& \left|(\nabla u_n^+, \nabla v^+)_{L^2(\Omega_n^+)} - (\nabla u_+^*, \nabla v^+)_{L^2(\Omega^+)}\right| + \\
& \left|(u_n^+, v^+)_{L^2(\Omega_n^+)} - (u_+^*, v^+)_{L^2(\Omega^+)}\right| + \\
& \left|(\nabla u_n^-, \nabla v^-)_{L^2(\Omega_n^-)} - (\nabla u_-^*, \nabla v^-)_{L^2(\Omega^-)}\right| + \\
& \left|(u_n^-, v^-)_{L^2(\Omega_n^-)} - (u_-^*, v^-)_{L^2(\Omega^-)}\right| + \\
& \left|(\lambda(x)\,\mathrm{Tr}(u_n^+ - u_n^-), \mathrm{Tr}(v^+ - v^-))_{L^2(\partial\Omega_n^+, \mathcal{H}^{N-1})} - (\lambda(x)\,\mathrm{Tr}(u_+^* - u_-^*), \mathrm{Tr}(v^+ - v^-))_{L^2(\partial\Omega^+, \mu^*)}\right|.
\end{aligned} \tag{32}$$

As $H^1(D)$ is compactly injected in $L^2(D)$ one has $\nabla u_n^+ \xrightarrow[n\to\infty]{H^1(D)} \nabla u_+^* \Rightarrow \nabla u_n^+ \xrightarrow[n\to\infty]{L^2(D)} \nabla u_+^*$.
We directly have by the convergence in $L^2(D)$ of χ_{Ω_n} to χ_Ω that $\chi_{\Omega_n} \nabla v^+ \xrightarrow[n\to\infty]{L^2(D)} \chi_\Omega \nabla v^+$,

which with $\bigtriangledown u_n^+ \xrightarrow[n\to\infty]{H^1} \bigtriangledown u_+^*$ gives,

$$
\begin{aligned}
&(\bigtriangledown u_n^+, \bigtriangledown v^+)_{L^2(\Omega_n^+)} - (\bigtriangledown u_*^+, \bigtriangledown v^+)_{L^2(\Omega^+)} \\
&\qquad\qquad = (\bigtriangledown u_n^+, \chi_{\Omega_n} \bigtriangledown v^+)_{L^2(D)} - (\bigtriangledown u_*^+, \chi_{\Omega} \bigtriangledown v^+)_{L^2(D)} \xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

We similarly show that: $\left|(u_n^+, \chi_{\Omega_n} v^+)_{L^2(\Omega_n^+)} - (u_*^+, \chi_{\Omega} v^+)_{L^2(\Omega^+)}\right|$
$\left|(\bigtriangledown u_n^-, \chi_{\Omega_n^c \cap D} \bigtriangledown v^-)_{L^2(\Omega_n^-)} - (\bigtriangledown u_*^-, \chi_{\Omega^c \cap D} \bigtriangledown v^-)_{L^2(\Omega^-)}\right|$ and also
$\left|(u_n^-, \chi_{\Omega_n^c \cap D} v^-)_{L^2(\Omega_n^-)} - (u_*^-, \chi_{\Omega^c \cap D} v^-)_{L^2(\Omega^-)}\right|$ tend to 0 as n tend to $+\infty$.
It remains to show that the last term tends to 0 as n tends to ∞. One can notice that

$$
\left|(\lambda(x)\,\mathrm{Tr}(u_n^+ - u_n^-), \mathrm{Tr}(v^+ - v^-))_{L^2(\partial\Omega_n^+)} - (\lambda(x)\,\mathrm{Tr}(u_*^+ - u_*^-), \mathrm{Tr}(v^+ - v^-))_{L^2(\partial\Omega^+)}\right|
$$

is less or equal than

$$
\|\lambda\|_{L^\infty(D)} \left|(\mathrm{Tr}(u_n^+ - u_n^-), \mathrm{Tr}(v^+ - v^-))_{L^2(\partial\Omega_n^+)} - (\mathrm{Tr}(u_*^+ - u_*^-), \mathrm{Tr}(v^+ - v^-))_{L^2(\partial\Omega^+)}\right|.
$$

Since $C(D) \cap H^1(D)$ is dense in $H^1(D)$ we have:

$$
\forall v \in H^1(D) \int_{\partial\Omega_n} \lambda(x)\,\mathrm{Tr}(w_n)\,\mathrm{Tr}(v) d\mathcal{H}^{N-1} \xrightarrow[n\to\infty]{} \int_{\partial\Omega} \lambda(x)\,\mathrm{Tr}(w)\,\mathrm{Tr}(v) d\mu^*, \tag{33}
$$

as soon as $w_n \rightharpoonup w$ in $H^1(D)$ and $\lambda \in C(\partial D)$.
The reader can find a proof of (33) in [12], Theorem 3.2.
Finally thanks to (33) each terms of (32) tend to 0 as n tends to ∞.

Hence taking $w_n^+ = Eu_n^+ \in H^1(D)$ and $w_n^- = Eu_n^- \in H^1(D)$ we know by the previous section that they weakly converge to $u_+^\star$ respectively $u_-^\star$,that we writte $Eu_n \rightharpoonup u^*$ in $V(D)$. Therefore, $\forall v \in V(D)$, $a(Eu_n, v) \xrightarrow[n\to\infty]{} a(u^*, v)$.

By the uniqueness of the weak solution for (9) on Ω , $u^*_{|\Omega} = u(\Omega, \mu^*)$.

Now that we know that our variational problem converges, we can state the main shape optimization theorem.

Theorem 8. *Let D_0 and D_1 fixed domains such as $D_0 \subset \Omega \subset D_1 \subset D$ and D_0 be a domain with a Lipschitz boundary ∂D_0 of bounded length and the parameters $\varepsilon, \Omega_0, \hat{c}, M_0$ be defined in (5). Let $U_{ad}(D, G, \varepsilon, D_0, \hat{c}, M_0)$ be defined in definition (5). We set $N \in \mathbb{N}^\star$ as the dimension of $\mathbb{R}^N$.*

For the objective function $J(\Omega, G, \varepsilon, u_+, u_-)$, defined in (1) or (2) and constructed with the weak solution of (3) (respectively (5)) for some fixed resistivity function $\lambda \in C(\partial\Omega)$, there exists $\Omega_{opt} \in U_{ad}(D, D_0, \varepsilon, \hat{c}, G, M_0)$, and there exists a finite valued (N-1)-dimensional positive measure μ^* on its boundary $\partial\Omega_{opt}$ equivalent to its Hausdorff measure $\mathcal{H}^{N-1}$ such that

$$
\begin{aligned}
&\int_{\partial\Omega_{opt}} d\mu^* \geq \int_{\partial\Omega_{opt}} d\mathcal{H}^{N-1}, \\
&\qquad J\left(\Omega_{opt}, u\left(\Omega_{opt}, \mu^*\right), \mathcal{H}^{N-1}\right) \leq \inf_{\Omega \in U_{ad}(D_0, \varepsilon, \hat{c}, G, M_0)} J(\Omega, u(\Omega, \mathcal{H}^{N-1}), \mathcal{H}^{N-1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = J\left(\Omega_{opt}, u\left(\Omega_{opt}, \mu^*\right), \mu^*\right).
\end{aligned}
$$

Proof. We treat the case $\lambda < \infty$.
Let $(\Omega_n)_{n\in\mathbb{N}^*} \subset U_{ad}(G, \varepsilon, D_0, c, M, M_0)$ be a minimizing sequence of the functional $J(\Omega)$. (It exists because $J(\Omega) \geq 0$).

Let us consider now the solutions $(u_n)_{n\in\mathbb{N}}$ of (3) on $(\Omega_n)_{n\in\mathbb{N}}$.

We will show that: $\chi_{\Omega_n} Eu_n$ converge strongly to $\chi_\Omega Eu$ in $H^1(\Omega)$.

The weak convergence of $\chi_{\Omega_n} Eu_n$ and the compact injection of $H^1(D)$ in $L^2(D)$ implies the strong convergence in $L^2(D)$ of $\chi_{\Omega_n} Eu_n$ to $\chi_\Omega Eu$.

Let's now show the convergence of norm of $\chi_{\Omega_n} Eu_n$ then we will have the weak convergence and convergence of norms in $L^2(D)$ which implies the strong convergence of $\chi_{\Omega_n} Eu_n$ to $\chi_\Omega Eu$.

The main point is to obtain the convergence in norm of the gradient, which is only possible because we consider a solution of the weak formulation.

As u_n is solution of the variational problem (3) we can write that:

$$a(u_n, v) = \int_{\Omega_n^+} \nabla u_n^+ \nabla v_+ + \int_{\Omega_n^-} \nabla u_n^- \nabla v_- + \int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_n^+ - u_n^-) \operatorname{Tr}(v_+ - v_-) d\mathcal{H}^{N-1} = 0$$

$$\Rightarrow \int_{\Omega_n^+} \nabla u_n^+ \nabla v_+ + \int_{\Omega_n^-} \nabla u_n^- \nabla v_- = -\int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_n^+ - u_n^-) \operatorname{Tr}(v_+ - v_-) d\mathcal{H}^{N-1}.$$

By (33) the sequence $\left(-\int_{\partial\Omega} \lambda(x) \operatorname{Tr}(u_n^+ - u_n^-) \operatorname{Tr}(v_+ - v_-) d\mathcal{H}^{N-1}\right)_{n\in\mathbb{N}}$ converge therefore $\|\nabla \chi_{\Omega_n} Eu_n\|^2_{L^2(D)} \xrightarrow[n\to\infty]{} \|\nabla \chi_\Omega Eu\|^2_{L^2(D)}$. This enables us to write that

$$\begin{aligned}\lim_{n\to\infty} \|\chi_{\Omega_n} Eu_n\|^2_{H^1(D)} &= \lim_{n\to\infty} \|\chi_{\Omega_n} Eu_n\|^2_{L^2(D)} + \|\nabla \chi_{\Omega_n} Eu_n\|^2_{L^2(D)} \\ &= \|\chi_\Omega Eu\|^2_{L^2(D)} + \|\nabla \chi_\Omega Eu\|^2_{L^2(D)} = \|u\|^2_{H^1(\Omega)}.\end{aligned}$$

We have the weak convergence and convergence of norms in $H^1(D)$ thereby strong convergence of $\chi_{\Omega_n} Eu_n$ to $\chi_\Omega Eu$ in $H^1(\Omega)$. Hence

$$\lim_{n\to\infty} J(\Omega_n, u_n(\Omega_n, \mathcal{H}^{N-1}), \mathcal{H}^{N-1}) = J(\Omega, u(\Omega, \mu^*), \mu^*).$$

Let's take $\Omega_{opt} = \Omega$ and $\mu_{\Omega_{opt}} = \mu^*$ and it's complete the proof. The case $\lambda = \infty$ is proven by the same way. □

Having dealt with the time-independent case, we turn to the more complex time-dependent case. For this case we have to prove, as before, that the variational problem converges and that there is indeed an optimal form.

4.3 Time dependent case.

Let's fix D an open set and $\lambda \leq \infty$. The functionals we considere here are: for $\lambda = +\infty$

$$J(\Omega_+, G, \varepsilon, u_+, u_-, T) = \|u_+\|^2_{L^2([0,T], H^1(\Omega_+))}, \tag{34}$$

or the one defined in (30).

Such as $\Omega_- = D \cap \Omega_+^c$ and we define the sequence of sets $(\Omega_n^-)_{n\in\mathbb{N}} := \left(D \cap \Omega_{n+}^c\right)_{n\in\mathbb{N}}$.

Before showing that variationnal problem converge, we need uniform estimates on n.

Let (u_m) be a sequence of solution of (9) on Ω_m and $V(D), V'(D)$ as defined above in section (4.2.1). There is $C > 0$ independent of m, such as $\forall m \in \mathbb{N}$:

$$u_m \in L^2([0,T], V(D)) \cap L^\infty([0,T], V(D)) \quad u_m' \in L^2([0,T], V'(D))$$

$$\begin{aligned}&\|u_m(t)\|^2_{L^2([0,T],V(D))} + \|u_m'(t)\|^2_{L^2([0,T],V'(D))} \\ &\qquad + \sup_{0\leq t\leq T} \|u_m(t)\|^2_{V(D)} \leq C\left(\|u_0\|^2_{V(D)} + \|f\|^2_{L^2([0,T],L^2(D))}\right).\end{aligned} \tag{35}$$

Let $m \in \mathbb{N}$ be fixed. We consider an approximation $(u_m^l)_{l\in\mathbb{N}}$ of $u_m \in V(\Omega_m)$. According to (20) and (21) exactly as in the Galerkin's method we get, for any $l \in \mathbb{N}$

$$\left\|u_m^l(t)\right\|^2_{L^2([0,T],V(D))} \leq C_0 \left(\|u_0\|^2_{V(D)} + \|f\|^2_{L^2([0,T],L^2(D))}\right)$$

and

$$\left\|(u_m^l)'(t)\right\|^2_{L^2([0,T],V'(D))} \leq C_1 \left(\|u_0\|^2_{V(D)} + \|f\|^2_{L^2([0,T],L^2(D))}\right).$$

Since the constants $C_1 > 0$ and $C_2 > 0$ depends only on T we find the estimates for the first two terms when $l \to \infty$.
Plus by coercivity of the bilinear form a we deduce the estimate for $\sup_{0\leq t\leq T} \|u_m(t)\|^2_{V(D)}$.

Actually by using the method in [7] we can prove that any solution of (9), belongs to $C(\mathbb{R}_t^+, L^2(\mathbb{R}^n))$. In exactly the same way as in the stationary case, we start by showing the convergence of the variational formulation. The theorem (4.3) allows us to write that the extension operator are uniformly bounded and therefore that they weak converge in $L^2([0,T], V(D))$.

Let $t \geq 0$ and $(w_n)_{n\in\mathbb{N}}$ in $V(D)^{\mathbb{N}}$ such as $w_n \xrightarrow[n\to\infty]{V(D)} w$.
We define for any given $v \in V(D)$ $F^n(w_n, v, t) = \frac{\mathrm{d}(w_n,v)_{L^2(D)}}{\mathrm{d}t} + a(w_n, v, t)$ and $F(w, v, t) = \frac{\mathrm{d}(w,v)_{L^2(D)}}{\mathrm{d}t} + a(w, v, t)$ with $a(.,.,t)$ is defined in (8). We claim that:

$$\lim_{n\to\infty} F^n(w_n, v, t) = F(w, v, t).$$

In fact, thanks to the stationary case we immediately have the convergence of $(a(w_n, v, t))_{n\in\mathbb{N}}$ to $a(w, v, t)$ as $n \to \infty$ and $t \geq 0$ is fixed.
In addtion to that the continuity of $L : u \mapsto \frac{\mathrm{d}\langle u,v\rangle}{\mathrm{d}t}$ for any given $v \in V(D)$ is obtained by the compact injection of $V(D)$ in $L^2(D)$.

Let $(w_n)_{n\in\mathbb{N}} \in V(D)^{\mathbb{N}}$ such as $w_n \xrightarrow[n\to\infty]{V(D)} w$ since the strong convergence implies the weak convergence in $V(D)$ of $(w_n)_{n\in\mathbb{N}}$ which gives us the weak convergence of $(w'_n)_{n\in\mathbb{N}}$ to w' in $V(D)$. Since $V(D)$ is compactly injected in $L^2(D)$. One has that: $w'_n \xrightarrow[n\to\infty]{L^2(D)} w'$. Hence by the continuity of the scalar product on $L^2(D)$ we finally have

$$\forall\, (w_n)_{n\in\mathbb{N}} \in V(D)^{\mathbb{N}} \quad \text{such as} \quad w_n \xrightarrow[n\to\infty]{V(D)} w \Rightarrow \lim_{n\to\infty} L(w_n) = L(w).$$

We conclude that for all v in $V(D)$ and $t \geq 0$ fixed: $\lim_{n\to\infty} F^n(w_n, v, t) = F(w, v, t)$.

Then we put $w_n = Eu_n \in V(D)$ and we apply the previous theorem, estimate (4.3) assures us that the weak limit u^* of $(w_n = Eu_n)_{n\in\mathbb{N}}$ exists. Consequently

$$\lim_{n\to\infty} F^n(Eu_n, v, t) = F(u^\star, v, t)$$

and by the uniquness of the limit we conclude that $u^*_{|\Omega} = u(\Omega, \mu^\star)$. Knowing that we find the following theorem.

Theorem 9. *Let D_0 and D_1 fixed such as $D_0 \subset \Omega \subset D_1 \subset D$ and D_0 be a domain with a Lipschitz boundary ∂D_0 of bounded length and $T \geq 0$. Let $U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ be defined in definition (5). For the objective function $J(\Omega, G, \varepsilon, u_+, u_-, T)$, defined in (30) or (4.3) and constructed with the weak solution of (9) (respectively (2)), for some fixed resistivity function $\lambda \in C(\partial\Omega)$.*

There exists $\Omega_{opt} \in U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$, and there exists a finite valued N-1-dimensional positive measure μ^* on its boundary $\partial\Omega_{opt}$ equivalent to its Hausdorff measure $\mathcal{H}^{N-1}$ such that

$$\int_{\partial\Omega_{opt}} d\mu^* \geq \int_{\partial\Omega_{opt}} d\mathcal{H}^{N-1},$$

and

$$J\left(\Omega_{opt}^{+}, u\left(\Omega_{opt}^{+}, \mu^{*}\right), \mathcal{H}^{N-1}\right) \leq \inf_{\Omega \in U_{ad}(G,D_0,M_0,\varepsilon,\hat{c})} J(\Omega^{+}, u(\Omega^{+}, \mathcal{H}^{N-1}), \mathcal{H}^{N-1}) = J\left(\Omega_{opt}^{+}, u\left(\Omega_{opt}^{+}, \mu^{*}\right), \mu^{*}\right).$$

Proof. Let $(\Omega_n)_{n\in\mathbb{N}^*} \subset U_{ad}(G, D_0, M_0, \varepsilon, \hat{c})$ be a minimizing sequence of the functional $J(\Omega)$. (It exists because $J(\Omega) \geq 0$).

Let us consider now the solutions $(u_n)_{n\in\mathbb{N}}$ of (9) on $(\Omega_n)_{n\in\mathbb{N}}$. It suffices to prove that: $\chi_{\Omega_n} Eu_n$ converge strongly to $\chi_\Omega Eu$ in $L^2([0,T], V(D))$, to get the existence.

According to the uniform estimate in proposition (4.3) the sequence $\chi_{\Omega_n} Eu_n$ is bounded in $L^2([0,T], V(D))$ thus the sequence converges weakly in $L^2([0,T], V(D))$ to $\chi_\Omega Eu$.

Where $E : V(\Omega_n) \to V(D)$ is the extension in space. Let's now show the convergence of norm $L^2([0,T], V(D))$ of $\chi_{\Omega_n} Eu_n$ then we will have the weak convergence and convergence of norms which implies the strong convergence of $\chi_{\Omega_n} Eu_n$ to $\chi_\Omega Eu$.

The important point is to show the convergence in norm of the gradient, which is only possible because we consider a solution of the weak formulation.

As $Eu_n \in V(D)$ is solution of the variational problem we have

$$D_+ \int_{\Omega_n^+} \nabla Eu_n^+ \nabla v_+ dx + D_- \int_{\Omega_n^-} \nabla Eu_n^- \nabla v_- dx = - (Eu_n', v) - \int_{\partial\Omega_n} \lambda(x) \operatorname{Tr}(Eu_n^+ - Eu_n^-) \operatorname{Tr}(v_+ - v_-) d\mathcal{H}^{N-1}.$$

Thanks to the proposition 4.3 $\left(-(Eu_n', v) - \int_{\partial\Omega_n} \lambda(x) \operatorname{Tr}(Eu_n^+ - Eu_n^-) \operatorname{Tr}(v_+ - v_-) d\mathcal{H}^{N-1}\right)_{n\in\mathbb{N}}$ is convergent. Consequently,

$$\lim_{n\to\infty} D_+ \left\|\nabla \chi_{\Omega_n} Eu_n^+\right\|_{L^2(D)}^2 + D_- \left\|\nabla \chi_{\Omega_n^-} Eu_n^-\right\|_{L^2(D)}^2 = D_+ \left\|\nabla \chi_{\Omega^+} Eu^+\right\|_{L^2(D)}^2 + D_- \left\|\nabla \chi_{\Omega^-} Eu^-\right\|_{L^2(D)}^2. \tag{36}$$

Since $V(D)$ is compactly injected in $L^2(D)$ we know that:

$$\left\|\chi_{\Omega_n^+} Eu_n^+\right\|_{L^2(D)}^2 + \left\|\chi_{\Omega_n^-} Eu_n^-\right\|_{L^2(D)}^2 \xrightarrow[n\to\infty]{} \left\|\chi_{\Omega^-} Eu^-\right\|_{L^2(D)}^2 + \left\|\chi_{\Omega^+} Eu^+\right\|_{L^2(D)}^2.$$

Therefore:

$$\int_0^T \left\|\chi_{\Omega_n^+} Eu_n^+\right\|_{L^2(D)}^2 + \left\|\chi_{\Omega_n^-} Eu_n^-\right\|_{L^2(D)}^2 dt \xrightarrow[n\to\infty]{} \int_0^T \left\|\chi_{\Omega^-} Eu^-\right\|_{L^2(D)}^2 + \left\|\chi_{\Omega^+} Eu^+\right\|_{L^2(D)}^2 dt. \tag{37}$$

We immediately found:

$$\lim_{n\to\infty} \left\|\chi_{\Omega_n^+} Eu_n\right\|_{L^2([0,T],V(D))}^2 = \lim_{n\to\infty} \int_0^T \left\|\chi_{\Omega_n^+} Eu_n\right\|_{V(D)}^2 dt$$

$$= \lim_{n\to\infty} \int_0^T D_+ \left\|\nabla \chi_{\Omega_n^+} Eu_n^+\right\|_{L^2(D)}^2 + D_- \left\|\nabla \chi_{\Omega_n^-} Eu_n^-\right\|_{L^2(D)}^2 + \left\|\chi_{\Omega_n^+} Eu_n^+\right\|_{L^2(D)}^2 + \left\|\chi_{\Omega_n^-} Eu_n^-\right\|_{L^2(D)}^2 dt$$

$$= \int_0^T D_+ \left\|\nabla \chi_{\Omega^+} Eu^+\right\|_{L^2(D)}^2 dt + \int_0^T \left\|\chi_{\Omega^-} Eu^-\right\|_{L^2(D)}^2 dt + \int_0^T \left\|\chi_{\Omega^+} Eu^+\right\|_{L^2(D)}^2 dt + \int_0^T D_- \left\|\nabla \chi_{\Omega^-} Eu^-\right\|_{L^2(D)}^2 dt$$

$$= \|\chi_\Omega E u\|^2_{L^2([0,T],V(D))}.$$

We have the weak convergence and convergence of norms in $L^2([0,T],V(D))$ which implies the strong convergence $\chi_{\Omega_n^+} E u_n$ to $\chi_\Omega E u$ in $L^2([0,T],V(D))$.

$$\chi_{\Omega_n^+} E u_n \xrightarrow[n\to\infty]{L^2([0,T],V(D))} \chi_\Omega E u.$$

Which means, as J is an equivalent norm on $V(D)$ that:

$$\lim_{n\to\infty} J(\Omega_n^+, u_n(\Omega_n^+, \mathcal{H}^{N-1}), \mathcal{H}^{N-1}) = J(\Omega^+, u(\Omega^+, \mu^*), \mu^*).$$

Let's take $\Omega^+_{opt} = \Omega^+$ and $\mu_{\Omega^+_{opt}} = \mu^*$ to conclude the existence of an optimal shape.

But we get, by Lemma 2: $\int_{\partial\Omega^+_{opt}} d\mu^* \geq \int_{\partial\Omega^+_{opt}} d\mathcal{H}^{N-1}$ which gives us immediately

$$J\left(\Omega^+_{opt}, u\left(\Omega^+_{opt}, \mu^*\right), \mu^*\right) \geq J\left(\Omega^+_{opt}, u\left(\Omega^+_{opt}, \mu^*\right), \mathcal{H}^{N-1}\right).$$

Finally we have proved the existence of an optimal shape among the class of Lipschitz sets in the time dependent case.

You notice that there is never a single optimal form for our problem.

□

5 Optimal shape with Non-Lipschitz boundary

We now relax the constraint on Ω. Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $n \in \mathbb{N}^\star$. Let $\varepsilon > 0$ we denote by $O(\varepsilon, D)$ the collection of $(\varepsilon,\infty) - domains$ $\Omega \subset D$.

We assume that the measures on the boundary satisfy (1) and in addition to that for a Borel measure μ with $\partial\Omega := \operatorname{supp}\mu$, and an exponents $0 \leq s \leq n$, and constant $c_s > 0$

$$\mu(\overline{B(x,r)}) \geq \overline{c_s} r^s, \quad x \in \partial\Omega, \quad 0 < r \leq 1. \tag{38}$$

Now let D_0 be a non-empty Lipschitz domain and a subset of D

The important point is to have the compactness, for the three modes of convergence, of the set on which we minimize. Given $n-1 \leq s < n$ and $0 \leq d \leq s$ and $\overline{c_s} \geq 0$, $c_d \geq 0$, we define the class of sets on which we minimize functional (30) as follow:

$$U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d) =$$
$$\left\{(\Omega, \mu_{\partial\Omega}) \mid \Omega \in O(D,\varepsilon), \partial\Omega \subset \bar{G}, Vol(D_0) = \int_\Omega d\lambda^n, \partial\Omega = supp(\mu_{\partial\Omega}) \quad \text{satisfies} \quad (1),(38)\right\}. \tag{39}$$

We find the same kind of condition on the measure in the article [12, 2]. Using Theorem 3 in [12] one prove:

Theorem 10. *Assume that the parameters are fixed in Definition (1) and (38), then*

1. *The class $U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$ of admissible domains is compact in the Hausdorff sense, in the sense of characteristic functions, in the sense of compacts, and in the sense of weak convergence of the boundary volumes.*

2. *If for a sequence $(\Omega_m)_{m\in\mathbb{N}}$ of shape admissible domains the boundary volume converge weakly, it follows that $(\Omega_m)_{m\in\mathbb{N}}$ converge in the Hausdorff sense, in the sense of characteristic function and in the sense of compact.*

Using Theorem 10 one notice that $U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$ is compact with respect to Hausdorff convergence, characteristic functions convergence, compact convergence and weak measure convergence on the boundary. Slight modification of estimate (4.3) gives us the uniform extensions operator estimates for all $(\Omega, \mu) \in U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$ and the convergence of the weak formulation is obtained exactly as for (4.3). Putting all these argument together we obtain the following theorem.

Theorem 11. *Let D_0 and D_1 fixed such as $D_0 \subset \Omega \subset D_1 \subset D$ and D_0 be a domain with a Lipschitz boundary ∂D_0 of bounded length.*

Let $U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$ be defined in definition (39). For the objective function J, defined in (30), (4.3) and constructed with the weak solution of (9) (respectively (2)).

There exists $(\Omega_{opt}, \mu_{opt}) \in U_{ad}(D, D_0, \varepsilon, s, d, \bar{c}_s, c_d)$,

$$J\left(\Omega_{opt}^{+}, u\left(\Omega_{opt}^{+}, \mu_{opt}\right), \mu_{opt}\right) = \min_{(\Omega,\mu)\in U_{ad}(D,D_0,\varepsilon,s,d,\bar{c}_s,c_d)} J(\Omega^{+}, \mu_{\partial\Omega}, u(\Omega^{+}, \mu_{\partial\Omega})).$$

We notice that here the minimum is reached for this set class and that once again we have no information about the uniqueness of the minimizer.

To conclude this study we are interested in the Mosco convergence of functional energies. Using the same argument as in the proof of Theorem 6.3 in [12], we obtain:

Theorem 12. *Let $D \subset \mathbb{R}^n$ be a bounded Lipshitz domain and $\varepsilon > 0$. Let $\Omega_m \subset D$ be uniformly bounded (ε,∞)-domains and μ_m finite Borel measures all satisfying (1) with $n-1 \le d \le n$ and the same constant. For each m let $J(\Omega_m, \mu_m)$ be as in (30) in place of Ω μ.*

If $\Omega_m \to \Omega$ in the sense of Hausdorff and characteristic functions and μ_m converge weakly to μ. Then

$$\lim_{m\to\infty} J(\Omega_m, \mu_m) = J(\Omega, \mu), \tag{40}$$

in the sense of Mosco.

References

[1] K. Arfi and A. Rozanova Pierrat, *Dirichlet-to-Neumann or Poincaré-Steklov operator on fractals described by d-sets*, Discrete and Continuous Dynamical Systems - S, 12 (2019), pp. 1–26.

[2] C. Bardos, D. Grebenkov, and A. Rozanova Pierrat, *Short-time heat diffusion in compact domains with discontinuous transmission boundary conditions*, Math. Models Methods Appl. Sci., 26 (2016), pp. 59–110.

[3] R. Capitanelli, *Asymptotics for mixed Dirichlet–Robin problems in irregular domains*, Journal of Mathematical Analysis and Applications, 362 (2010), pp. 450–459.

[4] R. Capitanelli, *Robin boundary condition on scale irregular fractals*, Communications on Pure Applied Analysis, 9 (2010), pp. 1221–1234.

[5] D. Dabiri, *Digital particle image thermometry/velocimetry: A review. exp*, Experiments in Fluids, 46 (2009), pp. 191–241.

[6] P.-G. de Gennes, *Physique des surfaces et des interfaces*, C. R. Acad. Sc. série II, 295 (1982), pp. 1061–1064.

[7] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, 1994.

[8] C. Even, S. Russ, V. Repain, P. Pieranski, and B. Sapoval, *Localizations in Fractal Drums: An Experimental Study*, Phys. Rev. Lett., 83 (1999), pp. 726–729.

[9] M. Filoche and B. Sapoval, *Transfer across random versus deterministic fractal interfaces*, Phys. Rev. Lett., 84 (2000), pp. 5776–5779.

[10] A. Henrot and M. Pierre, *Variation et optimization de formes. Une analyse géométrique*, Springer, 2005.

[11] A. Henrot and J. Sokolowski, *Shape Optimization Problem for Heat Equation*, Research Report RR-3185, INRIA, 1997.

[12] M. Hinz, A. Rozanova Pierrat, and A. Teplyaev, *Non-Lipschitz uniform domain shape optimization in linear acoustics*, Preprint, hal-02919526, 2020.

[13] H. P. Huy and E. Sanchez-Palencia, *Phénomènes de transmission à travers des couches minces de conductivitéélevée*, Journal of Mathematical Analysis and Applications, 47 (1974), pp. 284–309.

[14] P. W. Jones, *Quasi onformal mappings and extendability of functions in Sobolev spaces*, Acta Mathematica, 147 (1981), pp. 71–88.

[15] A. Jonsson, *Besov spaces on closed sets by means of atomic decomposition*, Complex Variables and Elliptic Equations, 54 (2009), pp. 585–611.

[16] A. Jonsson and H. Wallin, *Function spaces on subsets of* $\mathbb{R}^n$, Math. Reports 2, Part 1, Harwood Acad. Publ. London, 1984.

[17] ——, *Boundary value problems and brownian motion on fractals*, Chaos, Solitons & Fractals, 8 (1997), pp. 191–205.

[18] M. Lancia, *On some second order transmission problems*, Arabian Journal for Science and Engineering, 29 (2004).

[19] M. R. Lancia, *A Transmission Problem with a Fractal Interface*, Zeitschrift für Analysis und ihre Anwendungen, 21 (2002), pp. 113–133.

[20] P. Levitz, D. S. Grebenkov, M. Zinsmeister, K. M. Kolwankar, and B. Sapoval, *Brownian flights over a fractal nest and first-passage statistics on irregular surfaces*, Phys. Rev. Lett., 96 (2006), p. 180601.

[21] F. Magoulès, T. P. Kieu Nguyen, P. Omnes, and A. Rozanova Pierrat, *Optimal Absorption of Acoustic Waves by a Boundary*, SIAM Journal on Control and Optimization, 59 (2021), pp. 561–583.

[22] F. Magoulès, A. Rozanova Pierrat, and M. Rynkovskaya, *Existence of an optimal shape in the architecture applications*, In preparation, (2020).

[23] B. B. Mandelbrot, *The fractal geometry of nature*, Henry Holt and Company, Juvenile Nonfiction, 1983.

[24] F. Marcotte, C. R. Doering, J.-L. Thiffeault, and W. R. Young, *Optimal heat transfer and optimal exit times*, SIAM Journal on Applied Mathematics, 78 (2018), p. 591–608.

[25] U. Mosco and M. A. Vivaldi, *Variational problems with fractal layers*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl, p. 2003.

[26] L. G. Rogers, *Degree-independent Sobolev extension on locally uniform domains*, Journal of Functional Analysis, 235 (2006), pp. 619–665.

[27] A. Rozanova-Pierrat, D. S. Grebenkov, and B. Sapoval, *Faster diffusion across an irregular boundary*, Phys. Rev. Lett., 108 (2012), p. 240602.

[28] B. Sapoval, *General formulation of laplacian transfer across irregular surfaces*, Phys. Rev. Lett., 73 (1994), pp. 3314–3316.

[29] E. Trélat, C. Zhang, and E. Zuazua, *Optimal shape design for 2D heat equations in large time*, Pure and Applied Functional Analysis, 3 (2018), pp. 255–269.

[30] H. Wallin, *The trace to the boundary of Sobolev spaces on a snowflake*, Manuscripta Math, 73 (1991), pp. 117–125.