Nassau County Interscholastic Mathematics League

Contest #4 Answers must be integers from 0 to 999, inclusive. 2016 – 2017

Calculators are allowed.

**Time: 10 minutes**

19) Two positive integers differ by 10 and their squares differ by 200. Compute the smaller of the two positive integers.

20) If, for all positive integers $k$, $f(k) - f(k-1) = (-1)^k \cdot k$ and $f(0) = 0$, compute the remainder when $f(2016)$ is divided by 1000.

---

**Time: 10 minutes**

21) If $a$ and $b$ are primes that differ by 995, and $c$ and $d$ are non-negative integral powers of 2 that differ by 4095, compute the remainder when the sum of $a, b, c$, and $d$ is divided by 1000.

22) In $\Delta ABC$, $AC = 8$, $BC = 15$, and $AB = 17$. Point $M$ is the midpoint of $\overline{AB}$ and a circle with diameter $\overline{CM}$ intersects $\overline{CB}$ at points $C$ and $D$. Compute $MD$.

---

**Time: 10 minutes**

23) Compute the product of the roots of $|x-3|^2 - 12|x-3| + 35 = 0$.

24) Compute the number of positive integers less than 1000 that are divisible by neither 2 nor 3 nor 5.

## Solutions for Contest #4

19) Suppose $x - y = 10$ and $x^2 - y^2 = 200$. Thus, $(x + y)(x - y) = (x + y) \cdot 10 = 200 \rightarrow x + y = 20$. Therefore, $(x, y) = (15,5)$ and the smaller of the two integers is **5**.

20) In the given equation, substitute for $k = 1, 2, 3, 4, \ldots, 2016$ to yield $f(1) - f(0) = -1$; $f(2) - f(1) = 2$; $f(3) - f(2) = -3$; $f(4) - f(3) = 4$; … $f(2016) - f(2015) = 2016$ (positive because 2016 is even). Now add the resulting equations to yield $f(2016) - f(0) = (-1 + 2) + (-3 + 4) + (-5 + 6) + \cdots + (-2015 + 2016) = 1008 \cdot 1 = 1008$. The required remainder is **8.**

21) If two primes differ by an odd number, they cannot both be odd. Therefore, the primes must be 2 and 997. If two integral powers of 2 differ by the odd number 4095, one of them must be $2^0 = 1$ and the other $2^{12} = 4096$. When the sum of 1, 2, 997, and 4096 is divided by 1000, the remainder is **96.**

22) Since the sides of the triangle satisfy the hypothesis of the converse of the Pythagorean Theorem, $\Delta ABC$ is a right triangle with right angle $C$. Since arc $CDM$ is a semicircle, $\measuredangle MDC$ is a right angle. Therefore, $\overline{MD} \parallel \overline{AC}$ and since point $M$ is the midpoint of $\overline{AB}$, $MD = \frac{1}{2}AC = \mathbf{4}$.

23) Let $a = |x - 3|$. Then, $a^2 - 12a + 35 = 0 \rightarrow (a - 7)(a - 5) = 0 \rightarrow a = 7$ or $a = 5 \rightarrow |x - 3| = 7$ or $5 \rightarrow x = 10, -4, 8$, or $-2$. The product of the roots is **640**.

24) Begin the solution in the same manner as the similar problem in the previous contest. Let $A$ be the set of all positive integers less than 1000 which are divisible by 2. Then, $A = \{2,4,6, \ldots, 998\}$ and $n(A) = \frac{998}{2} = 499$. Let $B$ be the set of all positive integers less than 1000 which are divisible by 3. Then, $B = \{3,6,9, \ldots, 999\}$ and $n(B) = \frac{999}{3} = 333$. Let $A \cap B$ be the set of all positive integers less than 1000 which are divisible by 6. Then $A \cap B = \{6,12,18, \ldots, 996\}$ and $n(A \cap B) = \frac{996}{6} = 166$. Let $C$ be the set of all positive integers less than 1000 which are divisible by 5. Then $C = \{5, 10, 15, \ldots, 995\}$ and $n(C) = \frac{995}{5} = 199$. Consider $A \cap C$, $B \cap C$, and $A \cap B \cap C$. $A \cap C$ is the set of all positive integers less than 1000 which are divisible by 10. $A \cap C = \{10, 20, 30, \ldots, 990\}$ and $n(A \cap C) = \frac{990}{10} = 99$. $B \cap C$ is the set of all positive integers less than 1000 which are divisible by 15. Then $B \cap C = \{15, 30, 45, \ldots, 990\}$ and $n(B \cap C) = \frac{990}{15} = 66$. $A \cap B \cap C$ is the set of all positive integers less than 1000 which are divisible by 30. Then $A \cap B \cap C = \{30, 60, 90, \ldots, 990\}$ and $n(A \cap B \cap C) = \frac{990}{30} = 33$. By the principle of inclusion/exclusion, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C) = 499 + 333 + 199 - 166 - 66 - 99 + 33 = 733$. Therefore, the number of integers not divisible by 2 or 3 or 5 is $999 - 733 = \mathbf{266}$.

Note that the Venn diagram to the right concisely shows this result: $999 - (267 + 133 + 134 + 66 + 33 + 33 + 67) = \mathbf{266}$.