Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999, inclusive. 2016 – 2017

No calculators are allowed.

Time: 10 minutes

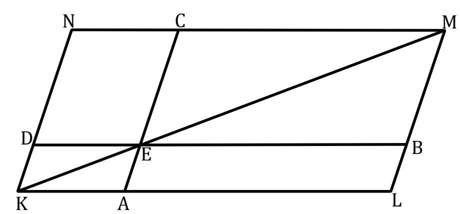
13) The product $4096 \cdot 2187$ may be re-written as $2^a \cdot 3^b$. Compute $a + b$.

14) In base b notation, 165_b represents an integer which is the product of exactly two distinct primes. Compute the value of b where $13 \leq b \leq 30$.

Time: 10 minutes

15) One thousand pounds of potatoes is 99% water. After some of the water evaporates, what remains is 98% water. Compute the number of pounds of water that evaporated.

16) In parallelogram *KLMN*, points *A, B, C*, and *D* are chosen on sides $\overline{KL}$, $\overline{LM}$, $\overline{MN}$, and $\overline{NK}$ respectively, so that $\overline{AC} \parallel \overline{KN}$ and $\overline{BD} \parallel \overline{KL}$. Segments $\overline{AC}$ and $\overline{BD}$ intersect at point E on diagonal $\overline{KM}$. If the ratio $\frac{KE}{EM} = \frac{2}{5}$ and the ratio of the area of parallelogram $BEAL$ to the area of parallelogram $DECN$ is an integer, compute the integer.



Time: 10 minutes

17) A train leaves New York City for Washington D.C. at 9 AM traveling at the constant rate of 50 mph on one track. Another train leaves Washington D.C. for New York City at 9 AM traveling at the constant rate of 70 mph on a track parallel and next to the track of the first train. Compute the number of miles the two trains are apart from each other one-half hour before the front of the trains pass each other.

18) Compute the number of positive integers less than 1000 that are neither divisible by 2 nor divisible by 3.

Solutions for Contest #3

13) The sum of the digits of 2187 is divisible by 3 and the sum of the digits of 4096 is not. Therefore, 2187 is divisible by 3 and 4096 is not. These facts help to see that $4096 \cdot 2187 = 2^{12} \cdot 3^7$. Thus, the required sum is $12 + 7 = \mathbf{19}$.

14) Because the digit 6 appears in the base b number, $b \geq 7$. Note that if b is odd, then $165_b = 1b^2 + 6b^1 + 5b^0 = b^2 + 6b + 5 = (b+5)(b+1)$ will be the product of two even numbers, each greater than 2 and therefore, will be a product of more than 2 distinct primes. So, we need consider only even values of b such that $13 \leq b \leq 30$. At this point, trial and error yields $(18+5)(18+1) = (23)(19)$. Thus, $b = \mathbf{18}$.

15) Suppose x pounds of water evaporated. Originally, there were 990 pounds of water and 10 pounds of potatoes. Later, there were $990 - x$ pounds of water and still 10 pounds of potatoes. So, $\frac{990-x}{1000-x} = 0.98 \rightarrow 990 - x = 980 - 0.98x \rightarrow 10 = 0.02x \rightarrow x = \mathbf{500}$.

16) Let $KE = 2x$ and $EM = 5x$. Note that $\Delta KED \sim \Delta MEB$ and $\Delta KEA \sim \Delta MEC$. So, $\frac{DE}{EB} = \frac{2}{5} = \frac{AE}{EC}$. Let $DE = 2y, EB = 5y, AE = 2z$ and $EC = 5z$. Also, $\measuredangle DEC \cong \measuredangle AEB$, so the required ratio of the area of parallelogram $BEAL$ to the area of parallelogram $DECN = \frac{2z \cdot 5y \cdot \sin \measuredangle AEB}{2y \cdot 5z \cdot \sin \measuredangle DEC} = \mathbf{1}$.

17) The distance between the trains is decreasing at a rate of 120 mph. So, one-half hour before they pass each other they will be **60** miles apart.
Alternatively, if d is the distance between the two cities, and t is the number of hours it takes for the two trains to pass each other, then $50t + 70t = d \longrightarrow 120t = d$. When the trains stop $\frac{1}{2}$ hour before they pass each other, their total distance traveled is $50\left(t - \frac{1}{2}\right) + 70\left(t - \frac{1}{2}\right) = 120t - 60 = d - 60$. Thus, the trains are **60** miles apart at that time.

18) The principle of inclusion/exclusion applies to this problem: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Of course, if $n(A \cap B)$ were not subtracted from the sum, it would be counted twice. Let A be the set of all positive integers less than 1000 which are divisible by 2. Then $A = \{2,4,6, \ldots, 998\}$ and $n(A) = \frac{998}{2} = 499$. Let B be the set of all positive integers less than 1000 which are divisible by 3. Then, $B = \{3,6,9, \ldots, 999\}$ and $n(B) = \frac{999}{3} = 333$. Let $A \cap B$ be the set of all positive integers less than 1000 which are divisible by 6. Then $A \cap B = \{6,12,18, \ldots, 996\}$ and $n(A \cap B) = \frac{996}{6} = 166$. So, $n(A \cup B) = 499 + 333 - 166 = 666$. The set $A \cup B$ contains all the positive integers less than 1000 which are divisible by 2 or by 3. Therefore, the number of positive integers less than 1000 which are not divisible by 2 or by 3 is $999 - 666 = \mathbf{333}$.
Note: It is not a coincidence that the number of positive integers that are not divisible by 2 or 3 is the same as the number of positive integers that are divisible by 3. Can you prove it?