Nassau County Interscholastic Mathematics League
Contest #4 Answers must be integers from 0 to 999 inclusive. 2012 – 2013
Calculators are allowed.

**Time: 10 minutes**

1. If $10 \le n \le 99$, find the integer $n$ such that $n$ divided by the sum of its digits is a minimum.

2. Quadrilateral $ABCD$ is cyclic (that is, it has a circumscribed circle).
If $AB = AD = 5$, and $CB = CD = 10$, compute $BD^2$.

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**Time: 10 minutes**

3. In Priscilla's Lottery, you and Priscilla each pick 5 distinct positive integers between 1 and 50, inclusive. You win if your five numbers match the ones that are picked by Priscilla. In Quincy's Lottery, you and Quincy each pick 5 distinct positive integers between 1 and 50, inclusive and then randomly designate one as special. You win if your special integer is the same as Quincy's and your other four integers are the same as Quincy's other four. Let $p$ be the probability that you win Priscilla's Lottery and let $q$ be the probability that you win Quincy's Lottery. Compute $p/q$.

4. When ice melts to water, the volume of the water is 90% of the volume of the ice. A glass containing ice and water is filled to the top. When all the ice melts, the glass is 96% full. What per cent of the original volume was ice?

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**Time: 10 minutes**

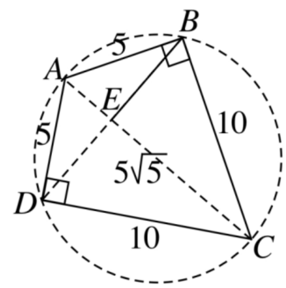
5. In equilateral triangle $ABC$ with $AB = 8$, points $P$ and $Q$ are chosen on side $\overline{AB}$ so that $AP = BQ = 2$. Similarly, points $R$ and $S$ are chosen on side $\overline{BC}$ so that $BR = CS = 2$, and points $T$ and $U$ are chosen on side $\overline{CA}$ so that $CT = AU = 2$. If the area of hexagon $PQRSTU = H$, find $H^2$.

6. Given that $\sum_{k=1}^{89} \sin^2 k^\circ = m/n$, where $m$ and $n$ are relatively prime positive integers, find $m + n$.

## Solutions for Contest #4

1. Represent the tens' digit and units' digit of a two-digit number by $a$ and $b$, respectively, and denote the desired quotient by $Q$. Then $Q = (10a + b)/(a + b) = 1 + (9a)/(a + b)$. Thus, $Q$ is minimum when $b = 9$. When $b = 9$, $Q - 1 = (9a)/(a + 9) = 9/(1 + 9/a)$. Thus, $Q$ is minimum when $1 + 9/a$ is maximum or when $a = 1$. Therefore, the minimum value of $Q$ is $19/(1 + 9) = 19/10$ and $n = 19$.
Alternate solution: $\frac{10a+b}{a+b} = \frac{10a+10b}{a+b} - \frac{9b}{a+b} = 10 - \frac{9b}{a+b}$ is a minimum when $a = 1$ and $b = 9 \rightarrow n = 19$.

2. Notice that $\overleftrightarrow{AC}$ is a symmetry line for the quadrilateral and its circumscribed circle. Therefore, $\overline{AC}$ is a diameter of the circle. Use the Pythagorean Theorem in $\Delta ABC$ to find that $AC = 5\sqrt{5}$. The symmetry implies that $BD = 2 \cdot BE$, where point $E$ is the foot of an altitude of $\Delta ABC$ from vertex $B$. Because $\overline{BE}$ is a height of $\Delta ABC$, conclude that $5 \cdot 10 = 2|ABC| = 5\sqrt{5}.\cdot BE$. Thus, $BE = 2\sqrt{5}.$, and so $BD = 4\sqrt{5} = \sqrt{80}$. Thus, $BD^2 = 80$.

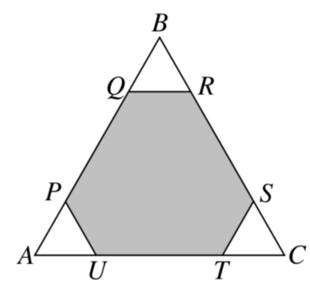



Alternate Solution 1: Use the Law of Cosines in $\Delta BAD$ and in $\Delta BCD$. Note that $\cos\measuredangle BCD = -\cos\measuredangle BAD$.
So, $BD^2 = 50 - 50\cos\measuredangle BAD$ and $BD^2 = 200 + 200\cos\measuredangle BAD \rightarrow BD^2 = 80$.
Alternate Solution 2: Use Ptolemy's Theorem: $AC \cdot BD = AB \cdot CD + AD \cdot BC$ together with the fact that $AC = 5\sqrt{5}$. from the first solution.

3. Since $p = \frac{1}{\binom{50}{5}}$ and $= \frac{1}{5} \cdot \frac{1}{\binom{50}{5}}$, $\frac{p}{q} = 5$.

4. Let $I$ and $W$ be the fractions of the volume of the glass that were ice and water, respectively, originally. Then $I + W = 1$. Let $I'$and $W'$ be the fractions of the volume of the glass that are ice and water, respectively, after the ice melts. Then $I' + W' = 96$. But $I' = 0$, and $W' = W + .9I$. Thus, $W + .9I = .96$. Subtract the last equation from the first to find that $.1I = .04$, so $I = .4$. Hence, 40% of the original volume was ice.

5. Triangles $APU$, $BQR$, and $CST$ are equilateral, and the area of each of them is 1/16 that of $\Delta ABC$. Thus, $H$ is $13/16$ the area of $\Delta ABC$. Use the formula $K = (s^2\sqrt{3})/4$ to find that the area of $\Delta ABC$ is $16\sqrt{3}$. Then $H = 13\sqrt{3}$, so $H^2 = 507$.




Alternate solution: Draw $\overline{RU}$. Using $30° - 60° - 90°$ triangles, we can calculate the height of trapezoid $URQP$ to be $\sqrt{3}$ and the height of trapezoid $TSRU$ to be $2\sqrt{3}$. The required area is the sum of the areas of the trapezoids: $\frac{1}{2} \cdot (4 + 6) \cdot \sqrt{3} + \frac{1}{2} \cdot (2 + 6) \cdot 2\sqrt{3} = 13\sqrt{3}$.

6. Let $S = ((\sin 1°)^2 + (\sin 2°)^2 + \cdots + (\sin 43°)^2 + (\sin 44°)^2) + (\sin 45°)^2 + ((\sin 46°)^2 + (\sin 47°)^2 + \cdots + (\sin 89°)^2)$. $\rightarrow$
$S = ((\sin 1°)^2 + (\sin 2°)^2 + \cdots + (\sin 43°)^2 + (\sin 44°)^2) + (\sin 45°)^2 + ((\cos 44°)^2 + (\cos 43°)^2 + \cdots + (\cos 1°)^2)$. Re-arrange to get
$S = ((\sin 1°)^2 + (\cos 1°)^2) + ((\sin 2°)^2 + (\cos 2°)^2) + \cdots$
$+((\sin 44°)^2 + (\cos 44°)^2) + (\sin 45°)^2 = 44 + \frac{1}{2} = \frac{89}{2}$. So, $m + n = 91$.