Nassau County Interscholastic Mathematics League

Contest #5 Answers must be integers from 0 to 999, inclusive. 2016 – 2017

No calculators are allowed.

Time: 10 minutes

25) Compute the value of x that satisfies $8^8 + 8^8 + 8^8 + 8^8 + 8^8 + 8^8 + 8^8 + 8^8 = 2^x$.

26) The three-digit base 10 number 1A3 is written 2017 times consecutively. The resulting integer 1A31A31A3 … 1A3 is a multiple of 11. Compute A.

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Time: 10 minutes

27) Points with coordinates $(2, c)$ and $(c, 23)$ are on a line perpendicular to the line whose equation is $3x + 4y = 12$. Compute c.

28) In acute ΔABC, $AB = 45$, and $BC = 60$. Points M and N are midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Angle bisector $\overline{BT}$ intersects $\overline{MN}$ in point P and intersects $\overline{AC}$ in point T. Compute the integer value of $\frac{MP}{PN}$.

Time: 10 minutes

29) If, for all positive integers k, $f(k+4) = f(k) + 12$ and $f(0) = 0$, compute the remainder when $f(2016)$ is divided by 1000.

30) Maxine tosses a single, fair, six-sided die x times. She wins if her throw is either 5 or 6 at least one time in those x tosses. Compute the minimum number x so that Maxine's chance of winning is greater than $\frac{9}{10}$.

Solutions for Contest #5

25) Re-write the given equation as $8 \cdot 8^8 = 2^x \rightarrow 8^9 = 2^x \rightarrow (2^3)^9 = 2^x \rightarrow 2^{27} = 2^x \rightarrow$ $x = \mathbf{27}$.

26) Consider the six-digit base 10 number 1A31A3 that appears 1008 times in the given integer with the digits 1A3 appended at the far right of the number. Use the test for divisibility by 11 and note that $(1 + 3 + \text{A}) - (\text{A} + 1 + 3) = 0$. Therefore, each of the above six-digit base 10 numbers is divisible by 11 and so is the entire number up to, but not including, the three rightmost digits. For the final three digits to be divisible by 11, $1 + 3 - \text{A} = 0$, so A = **4.**

27) The equation of the given line in slope-intercept form is $y = -\frac{3}{4}x + 3$. The slope of the given line is therefore $-\frac{3}{4}$. Since the lines are perpendicular, their slopes are negative reciprocals. Thus, $\frac{23-c}{c-2} = \frac{4}{3} \longrightarrow 4c - 8 = 69 - 3c \longrightarrow 7c = 77 \longrightarrow c = \mathbf{11}$.

28) Since point M is the midpoint of $\overline{AB}$, $AM = MB = 22.5$. Since points M and N are midpoints of $\overline{AB}$ and $\overline{AC}$ respectively, $\overline{MN} \parallel \overline{BC}$ and $MN = \frac{1}{2}BC = 30$. Since $\overline{MN} \parallel \overline{BC}$, $\measuredangle MPB \cong \measuredangle TBC$. But, since $\overline{BT}$ is an angle bisector, $\measuredangle TBA \cong \measuredangle TBC$. Therefore $\measuredangle MPB \cong \measuredangle TBA$ and ΔMPB is isosceles. So, $MP = 22.5$ and $PN = 7.5$ and the required ratio expressed as an integer is $\frac{22.5}{7.5} = \mathbf{3}$.
Note: Can you prove the general result $\frac{MP}{PN} = \frac{AB}{(BC-AB)}$ for every triangle defined as in this question?

29) Let $k = 0, 4, 8, 12, \ldots \rightarrow f(4) = 12, \; f(8) = 24, \; f(12) = 36, \ldots$. Observe that since $f(0) = 0$, f is the linear function $f(k) = 3k$. Thus, $f(2016) = 3 \cdot 2016 = 6048$. The required remainder is **48.**

30) If $x = 1$, the probability of winning on the first throw, $P(W_1) = \frac{1}{3}$.
If $x = 2$, the probability of winning on the first throw, or not winning on the first throw and winning on the second throw is $P(W_1)$ or $P(\text{not } W_1 \text{ and } W_2) = \frac{1}{3} + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{5}{9} < \frac{9}{10}$.
If $x = 3$, we want to win on at least one of the first three throws. So,
$P(W_1)$ or $P(\text{not } W_1 \text{ and } W_2)$ or $P(\text{not } W_1 \text{ and not } W_2 \text{ and } W_3) = \frac{5}{9} + \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{19}{27} < \frac{9}{10}$.
If $x = 4$, $\frac{19}{27} + \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) = \frac{65}{81} < \frac{9}{10}$. If $x = 5$, $\frac{65}{81} + \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) = \frac{211}{243} < \frac{9}{10}$.
Finally, if $x = 6$, $\frac{211}{243} + \left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right) = \frac{665}{729} > \frac{9}{10}$. Thus, the minimum required value of x is **6.**
Alternatively, the required probability of at least one win in x throws can be expressed as $1 - \left(\frac{2}{3}\right)^x$, the complement of no wins in x throws. Try small values of x. Since $1 - \left(\frac{2}{3}\right)^5 \approx 0.87 < 0.9$, and $1 - \left(\frac{2}{3}\right)^6 \approx 0.91 > 0.9$, the minimum required value of x is **6.**