Nassau County Interscholastic Mathematics League

Contest #3 Answers must be integers from 0 to 999 inclusive. 2012 – 2013

No calculators are allowed.

Time: 10 minutes

1. A rectangular field measures 60 yards by 80 yards. Farrel starts at one corner of the field and heads directly to the opposite corner at an average rate of 35 yards per minute. Kevin starts at the same corner and the same time as Farrel does, but he travels to the opposite corner along the edges of the field. Compute the number of yards per minute Kevin must average to arrive at the opposite corner at the same time Farrel does.

2. For how many real numbers x is it true that $(x+3)^8 = (x^2-9)^4$?

Time: 10 minutes

3. In $\Delta ABC, m\measuredangle A = 54°, m\measuredangle B = 59°$ and point O is the center of the circumscribed circle. Compute $m\measuredangle OAC$ in degrees.

4. If $\tan\left(\frac{\pi}{4} - x\right) + \cot\left(\frac{\pi}{4} - x\right) = 4$, compute $\cot^2 x$.

Time: 10 minutes

5. For how many positive integers n does 100 leave a remainder of 4 when 100 is divided by n?

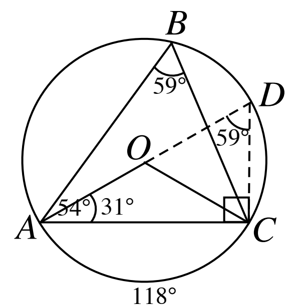
6. If $\log_x 4 = \log_{3x} 324$, then find x^2.

Solutions for Contest #3

1. Apply the Pythagorean Theorem to find that Farrel must travel 100 yards. Because the ratio of Kevin's distance to Farrel's distance is 140/100 = 7/5, the ratio of Kevin's average velocity to Farrel's must also be 7/5. Thus, Kevin's average velocity must be 7/5 of 35 or 49 yards per minute.

2. The given equation is equivalent to $(x+3)^8 = (x+3)^4(x-3)^4$, which is satisfied when $x+3=0$, that is, when $x = -3$. If $x \neq -3$, divide both sides by $(x+3)^4$ to get $(x+3)^4 = (x-3)^4$, which is satisfied if and only if $x+3 = -(x-3)$, that is, when $x=0$. Thus, there are two solutions to the given equation, namely 0 and -3.

3. Extend $\overline{AO}$ to form diameter $\overline{AD}$. Then $\measuredangle ADC$ and $\measuredangle B$ intercept the same arc (arc AC), so $m\measuredangle ADC = m\measuredangle B = 59°$. Also, $\measuredangle ACD$ is a right angle because it is inscribed in a semicircle. Thus, $m\measuredangle OAC = 90° - 59° = 31°$.
<u>Alternate solution</u>: Draw congruent radii $\overline{OA}$ and $\overline{OC}$. Since $m\measuredangle B = 59°, m$ (arc AC) $= m\measuredangle AOC = 118°$. So, $m\measuredangle OAC = \frac{1}{2}(180° - 118°) = 31°$.



4. Use the formula for the tangent of a difference of two angles to get $\tan\left(\frac{\pi}{4} - x\right) = \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x} = \frac{1-\tan x}{1+\tan x}$, so $\cot\left(\frac{\pi}{4} - x\right) = \frac{1+\tan x}{1-\tan x}$.
Thus, the given equation is equivalent to $\frac{1-\tan x}{1+\tan x} + \frac{1+\tan x}{1-\tan x} = 4$.

Clear fractions to obtain $2 + 2(\tan x)^2 = 4 - 4(\tan x)^2$.

Then $(\tan x)^2 = \frac{1}{3}$. So $(\cot x)^2 = 3$.

5. The given conditions imply that $n > 4$ and that 100 is 4 more than a multiple of n, that is, 96 is a multiple of n. But 96 has 12 divisors, and 8 of them are greater than 4. So, there are 8 of the desired integers.

6. Let $y = \log_x 4 = \log_{3x} 324$. Then $4 = x^y$ and $324 = (3x)^y = 3^y \cdot x^y$. Divide the equations to get $324/4 = 81 = 3^y$. So, $y = 4$. Hence, $x^4 = 4$, and $x^2 = 2$.