Nassau County Interscholastic Mathematics League

Team Contest   Answers must be integers from 0 to 999, inclusive.   2016 – 2017

Calculators are allowed.

Time: 40 minutes

31) The three interior angles of a triangle each measure a positive integral number of degrees. If the measure of one of these angles is 60, compute the greatest possible difference between the measures of the other two angles.

32) Compute the sum of the real roots of $\sqrt[3]{x^2-2x+1}-7\sqrt[3]{x-1}+10=0$.

33) Compute the greatest prime factor of $2^{2017}-2^{2009}$.

34) Circle O is tangent to $\overline{RS}$, a side of equilateral ΔRST at its midpoint, M. All other points of circle O lie outside ΔRST. Circle O is also tangent to the circle circumscribed about ΔRST, and $RS=4$. In simplest form, the radius of circle O is $\frac{\sqrt{a}}{b}$. Compute a^b.

35) If Melanie walks to school and returns home by bus, the entire round trip takes her 2 hours. If she takes the bus both ways, it takes her 20 minutes. Compute the number of minutes it would have taken her if she walked both ways.

36) If $x^3+x^2+x=-1$, compute x^{2016}.

37) The lengths of two sides of a triangle are 12 and 15 and the length of the angle bisector of their included angle is 10. Compute the length of the third side of the triangle.

38) In ΔABC, $AB=27$, $BC=27$, and $CA=18$. Point P is interior to ΔABC and its distance from $\overline{AB}$ is $2\sqrt{2}$ and its distance from $\overline{BC}$ is also $2\sqrt{2}$. If the distance between point P and $\overline{AC}$ is expressed in simplest form as $p\sqrt{q}$, compute $p+q$.

39) When $x^{2017}-49x^{1008}+1$ is divided by x^2-1, the remainder is expressible as $ax+b$. Compute $a-b$.

40) The Nassau Ice Cream Math League (NICML) Store sells 9 different flavors of ice cream. When the number of ways your coach can choose 5 ice cream cups, not necessarily of different flavors, is divided by 1000, compute the remainder.

Solutions for Team Contest

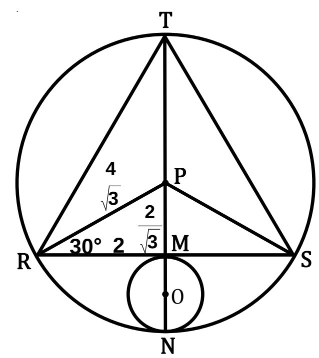
31) The measures of the two remaining angles have a sum of 120. Since the measures are positive integers, the smaller of these two angles has a measure of 1 and the larger has a measure of 119. The greatest difference is $119 - 1 = \mathbf{118}$.

32) If $a = \sqrt[3]{x-1}$, then $\sqrt[3]{x^2 - 2x + 1} = \sqrt[3]{(x-1)^2} = a^2$ and $a^2 - 7a + 10 = (a-5)(a-2) = 0$. So, $a = 5$ or $a = 2 \longrightarrow \sqrt[3]{x-1} = 5$ or $2 \rightarrow x - 1 = 125$ or $8 \rightarrow x = 126$ or 9. The required sum is **135.**

33) $2^{2017} - 2^{2009} = 2^{2009}(2^8 - 1) = 2^{2009} \cdot 255 = 2^{2009} \cdot 5 \cdot 51 = 2^{2009} \cdot 5 \cdot 3 \cdot 17.$

The required greatest prime factor of the given number is **17.**

34) Let point P be the center of the circumscribed circle. $RM = 2$, and since ΔPRM is a 30-60-90 triangle, $PM = \frac{2}{\sqrt{3}}$, and the radius of the circumscribed circle, $PR = \frac{4}{\sqrt{3}}$. The line through points O and M contains point P and contains the common point of tangency, N, of the two circles. Then, $PMON = PR = \frac{4}{\sqrt{3}}$. So, the radius of circle O, $MO = \frac{1}{2}(PN - PM) = \frac{1}{2}\left(\frac{4}{\sqrt{3}} - \frac{2}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Thus, $a = b = 3$ and the required power is **27.**



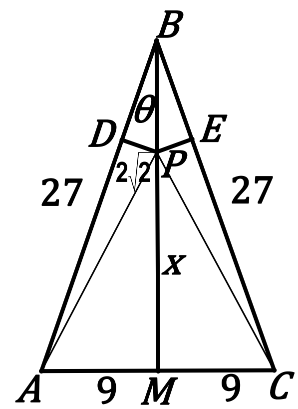
35) Let $d =$ distance in miles between home and school, $w =$ rate in miles per hour at which Melanie walked to school, and $b =$ rate in miles per hour at which the bus traveled. The time it takes Melanie to walk to school is $\frac{d}{w}$ and the time it takes her to bus to school is $\frac{d}{b}$. Then, $\frac{d}{w} + \frac{d}{b} = 2$ and $\frac{2d}{b} = \frac{1}{3}$. So, $\frac{d}{b} = \frac{1}{6}$. Therefore, $\frac{d}{w} = \frac{11}{6}$ and $\frac{2d}{w} = \frac{11}{3}$ hours or **220** minutes.
Alternatively, let $b =$ the number of minutes by bus one way, and let $w =$ the number of minutes to walk one way. Since $b = 10$ and $b + w = 120$, then $w = 110$ and $2w = \mathbf{220}$.

36) Re-write the given equation as $x^3 + x^2 + x + 1 = 0$. Factoring by grouping yields $(x+1)(x^2+1) = 0 \longrightarrow x + 1 = 0$ or $x^2 + 1 = 0 \longrightarrow x = -1$ or $x = \pm i$. Any of these three roots raised to the 2016th power equals **1**. Alternatively, by inspection, $x = -1$ is a root of $x^3 + x^2 + x + 1 = 0$. Thus, $(-1)^{2016} = \mathbf{1}$.

37) Let $\overline{CD}$ be the angle bisector in ΔABC, with point D in $\overline{AB}$, $AC = 12, CB = 15$, and $CD = 10$. A result of the angle bisector in a triangle theorem is $\frac{AD}{DB} = \frac{4}{5}$. Let $AD = 4x, DB = 5x$, and $m\measuredangle ACD = m\measuredangle BCD = \theta$. Use the Law of Cosines in the two smaller triangles to yield: $16x^2 = 144 + 100 - 240\cos\theta$ and $25x^2 = 100 + 225 - 300\cos\theta$. Solve each for $\cos\theta$ and set the results equal to obtain: $\frac{244 - 16x^2}{240} = \frac{325 - 25x^2}{300} \rightarrow x = 2$. Thus, $AB = AD + DB = 4x + 5x = \mathbf{18}$.

Alternatively, let $m\measuredangle ACD = m\measuredangle BCD = \theta$. Since the sum of the areas of the two smaller triangles equals the area of the larger triangle, $\frac{1}{2}(12)(10)\sin\theta + \frac{1}{2}(15)(10)\sin\theta = \frac{1}{2}(12)(15)\sin 2\theta \longrightarrow 135\sin\theta = 90(2\sin\theta\cos\theta)$. Dividing by $180\sin\theta$ yields $\cos\theta = \frac{3}{4}$. Using a double-angle identity, $\cos 2\theta = 2\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}$. So, by the Law of Cosines in ΔABC, $AB^2 = 144 + 225 - 360\left(\frac{1}{8}\right) = 324$. Thus, $AB =$ **18.**

38) Draw line segments from point P to the three vertices of ΔABC. The sum of the areas of the three triangles thus formed equals the area of ΔABC. The latter area is obtained with Heron's formula. Let x be the required distance from point P to $\overline{AC}$. Then, $\frac{1}{2}\cdot 27\cdot 2\sqrt{2} + \frac{1}{2}\cdot 27\cdot 2\sqrt{2} + \frac{1}{2}\cdot 18x = \sqrt{36\cdot 9\cdot 9\cdot 18} = 162\sqrt{2} \rightarrow 54\sqrt{2} + 9x = 162\sqrt{2} \rightarrow 9x = 108\sqrt{2} \rightarrow x = 12\sqrt{2}$. So, the required sum is **14.**
Alternatively, in isosceles ΔABC, point P lies on altitude and median $\overline{BM}$. Let points D and E be the feet of the perpendiculars from point P to $\overline{AB}$ and $\overline{CB}$, respectively, and let $m\measuredangle ABM = \theta$.
In ΔABM, $\sin\theta = \frac{9}{27} = \frac{1}{3}$, and in ΔDBP, $\sin\theta = \frac{2\sqrt{2}}{BP}$. So, $\frac{1}{3} = \frac{2\sqrt{2}}{BP} \longrightarrow BP = 6\sqrt{2}$. By the Pythagorean Theorem in ΔABM, $BM = \sqrt{27^2 - 9^2} = \sqrt{648} = 18\sqrt{2}$. Thus, the required distance, $PM = 18\sqrt{2} - 6\sqrt{2} = 12\sqrt{2}$, and the required sum is **14.**



39) We can express the given division as $x^{2017} - 49x^{1008} + 1 = Q(x)(x^2 - 1) + ax + b$, where $Q(x)$ is the quotient and $ax + b$ is the remainder. Then let $x = 1$: $1 - 49 + 1 = a + b \longrightarrow a + b = -47$. Now let $x = -1$: $-1 - 49 + 1 = -a + b \longrightarrow -a + b = -49$. Solve this system of equations to get $a = 1$ and $b = -48$. Thus, $a - b =$ **49**.

40) Suppose the flavors are A, B, C, D, E, F, G, H, and I. A choice of 5 flavors can be depicted as listing the flavors in alphabetical order. For example, ACCEI means one cup each of A, E, and I and two cups of C. Draw slashes to separate the A's from the B's, the B's from the C's, … , and the H's from the I's: A//CC//E////I. Notice that there are 13 symbols, of which 5 are letters and 8 are slashes. Instead, begin with 13 pound (#) symbols. Erase 8 of them and replace them with slashes. For example, the result BDGGH is depicted by /#//#///##/#/. The number of ways of choosing 5 ice cream cups out of 9 flavors is ${}_{13}C_5 = 1287$. Thus, the required remainder is **287.**