QF620 – G1

Stochastic Modelling in Finance

AY 21/22 Term 1

Project Report

Submitted by:

BRANDON LEE ZI CONG

CAI SHENG SIANG LEON

CHONG CHIN CHUAN KENNETH

NICOLE LIM YING XI

TAN GUAN WEI GABRIEL

Part I (Analytical Option Formulae):

We have derived and implemented the following models to value European a) Vanilla call/put options, b) Digital cash-or-nothing call/put options and c) Digital asset-or-nothing call/put options in Python:

1. Black-Scholes model
2. Bachelier model
3. Black76 model
4. Displaced-Diffusion model

Insights:

- Based on the formulae shown below, we can see that regardless of the models being used, the value of vanilla option is equals to the value of digital asset-or-nothing option minus the value of the digital cash-or-nothing option.

1. Black-Scholes model:

The Black-Scholes formula for a **vanilla call option** is given by

$$C(S, K, r, \sigma, T) \;=\; S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where $d_1 = \frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ $\qquad d_2 = d_1 - \sigma\sqrt{T}$

The Black-Scholes formula for a European **cash or nothing call option** is given by

$$C(A, K, T, \sigma, S, r) \;=\; e^{-rT}A\Phi(d_1 - \sigma\sqrt{T})$$

where $d_1 = \frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and:

- K = Strike Value
- A = Payable Amount
- t = at time t
- T = Expiry Date

The Black-Scholes formula for a European **asset or nothing call option** is given by

$$C(K, T, \sigma, S, r) \;=\; S\Phi(d_1)$$

where $d_1 = \frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

2. Bachelier model:

The Bachelier formula for a **vanilla call option** is given by

$$C(S, K, r, \sigma, T) = e^{-rT}\left((S_0 - K)\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)\right)$$

The Bachelier formula for a **European cash or nothing call option** is given by

$$C(A,S,K,r,\sigma,T) = e^{-rT} A\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)$$

The Bachelier formula for a **European digital asset or nothing call option** is given by

$$C(S,K,r,\sigma,T) = e^{-rT}\left(S_0\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\Phi\left(\frac{S_0 - K}{\sigma\sqrt{T}}\right)\right)$$

3. Black76 model:

The Black76 formula for a **vanilla call option** is given by

$$C(F,K,\sigma,T,r) = D(0,T)[F_0\Phi(d_1) - K\Phi(d_2)]$$

$$D(0,T) = e^{-rT}$$

where $d_1 = \frac{\log\frac{F_0}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$ $\quad d_2 = d_1 - \sigma\sqrt{T}$

The Black76 formula for a **European cash or nothing call option** is given by

$$C(A,F,K,\sigma,T,r) = D(0,T)[A\Phi(d_2)]$$

$$D(0,T) = e^{-rT}$$

where $d_1 = \frac{\log\frac{F_0}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$ $\quad d_2 = d_1 - \sigma\sqrt{T}$

The Black76 formula for a **European asset or nothing call option** is given by

$$C(F,K,\sigma,T,r) = D(0,T)[F_0\Phi(d_1)]$$

where $d_1 = \frac{\log\frac{F_0}{K} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$

4. Displaced-Diffusion model:

The Displaced Diffusion model formula for a **vanilla call option** is given by

$$C(F,K,\sigma,T,\beta,r) = D(0,T)\left[\left(\frac{F_0}{\beta}\right)\Phi(d_1) - \left(K + \frac{1-\beta}{\beta}F_0\right)\Phi(d_2)\right]$$

$$D(0,T) = e^{-rT}$$

where $d_1 = \frac{\log\frac{F_0/\beta}{K + \frac{1-\beta}{\beta}F_0} + \frac{1}{2}\beta^2\sigma^2 T}{\beta\sigma\sqrt{T}}$ $\quad d_2 = d_1 - \beta\sigma\sqrt{T}$

The Displaced Diffusion model formula for a **European cash or nothing call option** is given by

$$C(A,F,K,\sigma,T,\beta,r) = D(0,T)\left[\left(A + \frac{1-\beta}{\beta}F_0\right)\Phi(d_2)\right]$$

$$D(0,T) = e^{-rT}$$

where $d_1 = \frac{\log\frac{F_0/\beta}{K + \frac{1-\beta}{\beta}F_0} + \frac{1}{2}\beta^2\sigma^2 T}{\beta\sigma\sqrt{T}}$ $\quad d_2 = d_1 - \beta\sigma\sqrt{T}$

The Displaced Diffusion model formula for a **European asset or nothing call option** is given by

$$C(F,K,\sigma,T,\beta,r) = D(0,T)\frac{F_0}{\beta}\Phi(d_1)$$

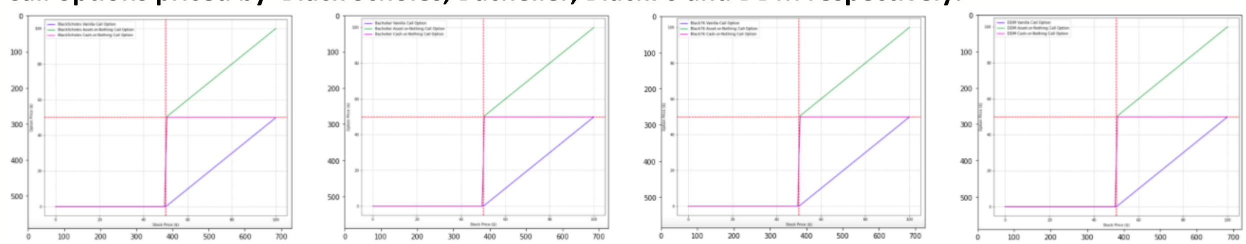
$D(0,T) = e^{-rT}$

where $d_1 = \dfrac{\log\frac{F_0/\beta}{K+\frac{1-\beta}{\beta}F_0}+\frac{1}{2}\beta^2\sigma^2 T}{\beta\sigma\sqrt{T}}$

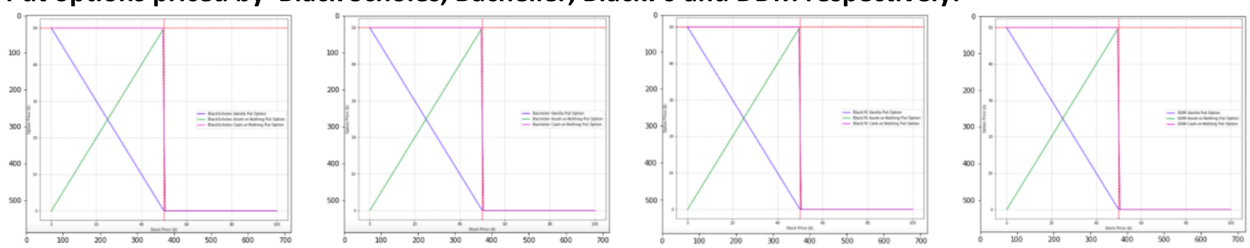
To validate that the models derived in Part 1 are correct, we plotted the payoff function for each type of the European options as shown below.

These are the parameters used to implement the models: $T = 0.01, \sigma = 0.03, K = 50, r = 0.212862 * 0.01, S =$ np.linspace(0, 100, 100) and $\beta = 1$ (for DDM)

Call options priced by Black Scholes, Bachelier, Black76 and DDM respectively:



Put options priced by Black Scholes, Bachelier, Black76 and DDM respectively:



Part II (Model Calibration):

SPX (European options) and SPY (American options) datasets were given, containing the call and put option prices (bid & offer) over 3 maturities: 17, 45 and 80 days respectively.

The objective is to calibrate the Displaced-Diffusion model derived in part 1 and SABR model to match the SPX and SPY option prices.

The SABR Model formula is given by

$$\{\sigma_{SABR}(F_0, K, \alpha, \beta, \rho, \nu) = \frac{\alpha}{(F_0K)^{\frac{1-\beta}{2}}(1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right)+\cdots} * \frac{z}{x(z)} * (1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(F_0K)^{\frac{1-\beta}{2}}}+\frac{2-3\rho^2}{24}\nu^2\right]T + \cdots$$

where $z = \frac{\nu}{\alpha}(F_0K)^{(1-\beta)/2}\log\left(\frac{F_0}{K}\right)$ and $x(z) = \log\left[\frac{\sqrt{1-2\rho z+z^2}+z+\rho}{1-\rho}\right]$

Steps and methods used to derive the calibrated parameters for DDM and SABR model:

1. Derive Market Implied Volatility observed from raw data

Market Implied Volatility is derived using the scipy.optimize.brentq method available in Python.

For American options (SPY), we cannot use the analytical Black Scholes model formula directly because American options can be exercised before its maturity. Therefore, the Binomial Tree model is used to price the SPY options through the "backward induction" method. The Early Exercise Premium is calculated by subtracting the SPY options priced using the Black Scholes model from the price calculated using the Binomial Tree model. Thereafter, the calculated Early Exercise Premium will be subtracted from the Mid Price of each SPY options, and thus we can use these adjusted Mid Prices to derived implied volatilities the same way as SPX options in the subsequent steps described below (i.e. Mid Price = (Best Bid - Best Offer)/2 - Early Exercise Premium).

i.e. Lambda function (applied to each data point) = Mid Price - Option Price (calculated using Black Scholes Model with σ unknown), where the brentq method will find the $\sigma_{implied\ vol}$ that will make the lambda function = 0.

Calibration of DDM and SABR Models are done using the scipy.optimize.least_squares method

2. Calculate Implied Volatility using DDM

The objective is to find the optimal or calibrated σ and β parameter values that give the minimum error between the Market Mid Price and the Option Price calculated using DDM.

The calibrated σ and β parameters are then used to calculate the option prices using DDM and these calculated option prices will in turn be used to derive the implied volatility $(\sigma_{implied\ vol})$ using the same method described in Step 1.

Calibrated σ and β parameters for SPX are:
For Maturity = 17 days: $\sigma = 0.183$ and $\beta = 0.098$
For Maturity = 45 days: $\sigma = 0.195$ and $\beta = 0.095$
For Maturity = 80 days: $\sigma = 0.202$ and $\beta = 0.091$

Calibrated σ and β parameters for SPY are:
For Maturity = 17 days: $\sigma = 0.194$ and $\beta = 0.097$
For Maturity = 45 days: $\sigma = 0.195$ and $\beta = 0.094$
For Maturity = 80 days: $\sigma = 0.203$ and $\beta = 0.091$

3. Calculate Implied Volatility using SABR Model (fixed $\beta = 0.7$)

The objective is to find the optimal or calibrated α, ρ and ν parameter values that give the minimum error between the Market Implied Volatility and the Volatility calculated using SABR model.

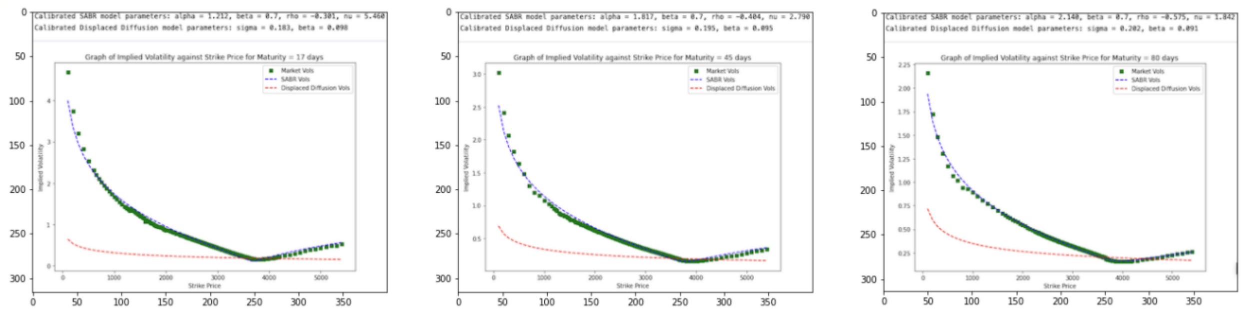
The calibrated α, ρ and ν parameters are then used to calculate the implied volatility $(\sigma_{implied\ vol})$ using SABR model.

Calibrated α, ρ and ν parameters for SPX are:

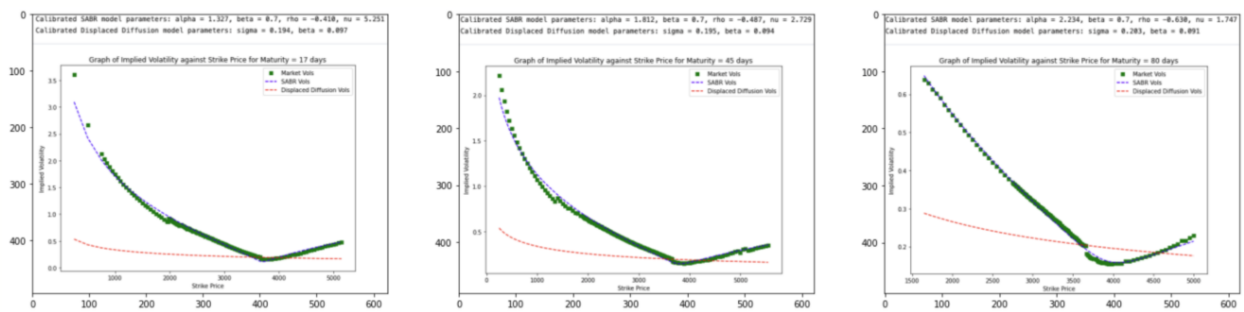
For Maturity = 17 days: $\alpha = 1.212, \rho = -0.301$ and $\nu = 5.410$
For Maturity = 45 days: $\alpha = 1.817, \rho = -0.404$ and $\nu = 2.790$
For Maturity = 80 days: $\alpha = 2.410, \rho = -0.575$ and $\nu = 1.842$

Calibrated α, ρ and ν parameters for SPY are:
For Maturity = 17 days: $\alpha = 1.327, \rho = -0.410$ and $\nu = 5.251$
For Maturity = 45 days: $\alpha = 1.812, \rho = -0.487$ and $\nu = 2.729$
For Maturity = 80 days: $\alpha = 2.234, \rho = -0.630$ and $\nu = 1.747$

Below are the graphical results obtained by plotting the calculated Implied Volatility (of SPX) against Strike Price over 3 different maturities.

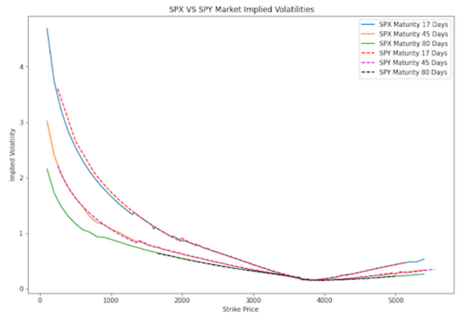


Below are the graphical results obtained by plotting the calculated Implied Volatility (of SPY) against Strike Price over 3 different maturities.



From the graphs above, both SPX and SPY, it can be observed that the implied volatility calculated using SABR model fit very closely to the market implied volatility across all maturities. As the SPX and SPY market implied volatilities are very steep, DDM is not expected to fit well as it does not have sufficient degree of freedom to fit to market implied volatilities.

It can also be observed that the market implied volatilities for both SPX and SPY options are very similar to each other for their respective maturities. Generally, the volatility smile is steepest for short expiries and is flatter for longer expiries and these can be observed from the graph below.

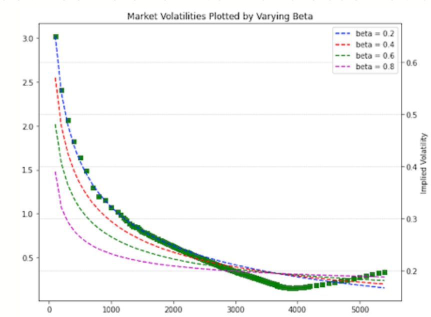


(For the illustration above, the SPY options' Strike Price, Mid Price and Stock Price have been scaled to match the SPX options for ease of comparion.)

Discussion on how changing $\boldsymbol{\beta}$ in DDM and $\boldsymbol{\rho}, \boldsymbol{\nu}$ in the SABR model affect the shape of the implied volatility smile.

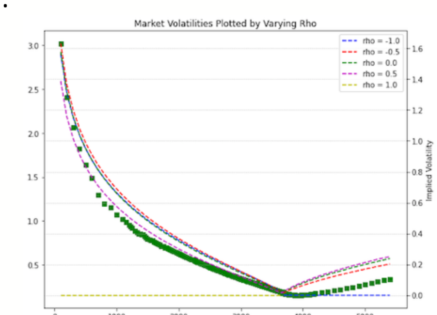
In this part of the discussion, I have selected the SPX options with maturity = 45 days to observe that effects of varying the β parameter in DDM and ρ and ν in the SABR model (i.e. 45days-SPX SABR calibrated parameters: $\alpha = 1.817, \rho = -0.404, \nu = 2.790$ and DDM calibrated parameters: $\sigma = 1.950, \beta = 0.095$).

Fixed $\sigma = 0.195$ while varying β between 0 and 1 to make observations:



Observation: As β for DDM increases, implied volatility smile for DDM becomes steeper and it seem to become a better fit to the market as β increases. However, fit is still not sufficiently accurate as DDM does not have sufficient degree of freedom to fit to market implied volatilities.

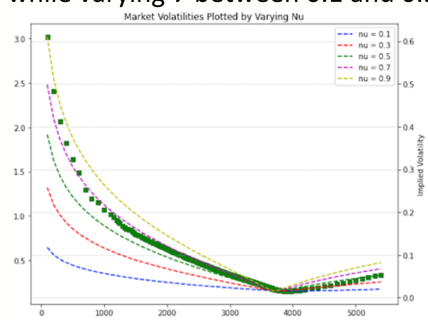
Fixed $\alpha = 1.817$ and $\nu = 2.790$ while varying ρ between -1 and 1 to make observations (ρ is proportional to the skewness of stock returns):



Observations: As ρ increases in the SABR model, the slope of the implied volatility smile will decrease. (Or conversely, the more negative ρ is, the steeper it becomes.)

- Positive correlation between stock and volatility is associated with positive skew in return distribution.
- Negative correlation between stock and volatility is associated with negative skew in return distribution. Negative correlation increases the price of out-of-the-money put options and decreases the price of out-of-the-money call options.

Fixed $\alpha = 1.817$ and $\rho = -0.404$ while varying ν between 0.1 and 0.9 to make observations:



Observation: As ν increases, the curvature of the implied volatility smile will become more pronounced. Increasing ν has the effect of increasing the kurtosis of return. When $\nu = 0$, volatility will be deterministic. Larger ν increases the price of out-of-the-money call and put options (more fat tail), the implied volatility smile will become more symmetrical.

Part III (Static Replication):

Given the **payoff function:** $h(S_T) = S_T^{\frac{1}{3}} + 1.5\log(S_T) + 10.0$, we have determined on 1-Dec-2020 the price of the exotic European derivative expiring on 15-Jan-2021 according to

1. Black-Scholes model,
2. Bachelier model, and
3. Static replication of European payoff using the SABR model calibrated in Part II.

Insights:

- Using the Black-Scholes model, Bachelier model or static replication according to SABR model to price the exotic European derivative yield us the same result of approximately **$37.71**.

1. **Black-Scholes model:**

σ_{LN} is constant and determined by the at-the-money option (i.e. S = K). Assuming that we are evaluating an exotic European derivative using SPX as underlying, S = 3662.45 on 1-Dec-2020, sigma at K = 3660 (nearest strike price) is 0.197 (as per Part 2).

According to Black-Scholes, $S_T = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma W_T}$

Substituting above into Payoff function,

$$h(S_T) = S_0^{\frac{1}{3}} e^{\left(\left(r-\frac{1}{2}\sigma^2\right)T+\sigma W_T\right)\frac{1}{3}} + 1.5\log(S_0) + 1.5(r - \frac{\sigma^2}{2})T + 1.5\sigma W_T + 10.0$$

$$\Rightarrow \mathbb{E}^*[h(S_T)] = \left[S_0 e^{rT-\frac{\sigma^2 T}{3}}\right]^{\frac{1}{3}} + 1.5\log(S_0) + 1.5\left(r - \frac{\sigma^2}{2}\right)\mathrm{T} + 10.0$$

$V_0 = e^{-rT}\mathbb{E}^* [h(S_T)]$
The price of the derivative is **$37.70** (rounded to 2 decimal places).

2. **Bachelier model:**

According to Bachelier, $S_T = S_0 + \sigma W_T$
Substituting above into Payoff function, $h(S_T) = (S_0 + \sigma W_T)^{\frac{1}{3}} + 1.5\log(S_0 + \sigma W_T) + 10.0$

Similar to Black-Scholes model, we set $\sigma_N = \sigma_{LN}$ as determined by the at-the-money option (i.e. S = K). Assuming that we are evaluating an exotic European derivative using SPX as underlying, S = 3662.45 on 1-Dec-2020, sigma at K = 3660 (nearest strike price) is 0.197 (as per Part 2).

Using numerical methods, the price of the derivative is **$37.71** (rounded to 2 decimal places).

3. Static replication of European payoff (using SABR):
Payoff function:

$h(S_T) = S_T^{\frac{1}{3}} + 1.5\log(S_T) + 10.0$
$\Rightarrow h(F) = F^{\frac{1}{3}} + 1.5 * \log(F) + 10.0$, where $F = S_0 e^{rT}$
$h'(S_T) = \dfrac{1}{3}S_T^{-\frac{2}{3}} + 1.5\dfrac{1}{S_T}$
$h''(S_T) = -\dfrac{2}{9}S_T^{-\frac{5}{3}} - 1.5S_T^{-2} \Rightarrow h''(K) = -\dfrac{2}{9}K^{-\frac{5}{3}} - 1.5K^{-2}$

To value the exotic European derivative, we substituted the σ_{SABR} calibrated from Part II into the Black-Scholes formula. Refer to PartII.ipynb for implementation. The price of the derivative is **$37.72** (2 d.p.).

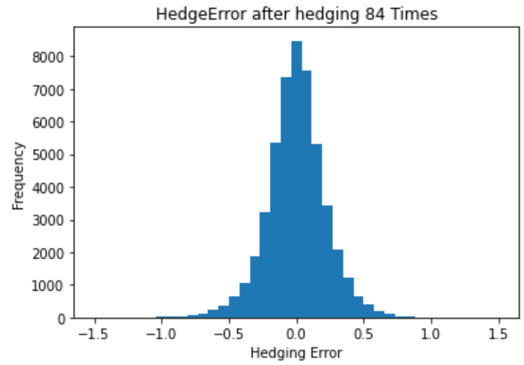
Part IV (Dynamic Hedging):

Black-Scholes introduced the notion of dynamic delta hedging - by executing delta hedges instantaneously. We ensured that our portfolio is delta neutral, and consequently hedged the exposure of our call position using the underlying stock and the risk-free bond.

The hedged portfolio is $V_t = \phi_t S_t + \psi_t B_t$, where $\phi_t = \Delta_t = \frac{\partial C}{\partial S} = \Phi\left(\frac{\log\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$ and

$\psi_t B_t = -Ke^{-rT}\Phi\left(\frac{\log\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

We have worked out the hedging error of the dynamic delta hedging strategy by comparing the replicated position based on ϕ and ψ with the final call option payoff at maturity. Using 50,000 paths in our simulation, we plot the histogram of the hedging error for N = 21 and N = 84 as shown below.



From the histograms, we are able to determine that the average final profit/loss is close to zero and that the distribution of final profit/loss is similar to a normal distribution regardless of the frequency of hedging being 21 or 84.

In addition, we are also able to observe that as the frequency of hedging increases, the standard deviation of the final profit/loss decreases, which leads to more values being concentrated around the mean and the distribution having thinner tails.