Project — Part I (Analytical Option Formulae)

Consider the following European options:

- Vanilla call/put
- Digital cash-or-nothing call/put
- Digital asset-or-nothing call/put

Derive and implement the following models to value these options in Python:

1. Black-Scholes model
2. Bachelier model
3. Black76 model
4. Displaced-diffusion model

Project — Part II (Model Calibration)

On 1-Dec-2020, the S&P500 (SPX) index value was 3662.45, while the SPDR S&P500 Exchange Traded Fund (SPY) stock price was 366.02. The call and put option prices (bid & offer) over 3 maturities are provided in the spreadsheet:

- `SPX_options.csv`
- `SPY_options.csv`

The discount rate on this day is in the file: `zero_rates_20201201.csv`.

Calibrate the following models to match the option prices:

1. Displaced-diffusion model
2. SABR model (fix $\beta = 0.7$)

Plot the fitted implied volatility smile against the market data.

Report the model parameters:

1. σ, β
2. α, ρ, ν

And discuss how does change β in the displaced-diffusion model and ρ, ν in the SABR model affect the shape of the implied volatility smile.

Project — Part III (Static Replication)

Suppose on 1-Dec-2020, we need to evaluate an exotic European derivative expiring on 15-Jan-2021 which pays:

1. Payoff function:

$$S_T^{1/3} + 1.5 \times \log(S_T) + 10.0$$

2. "Model-free" integrated variance:

$$\sigma_{\mathsf{MF}}^2 T = \mathbb{E}\left[\int_0^T \sigma_t^2 \; dt\right]$$

Determine the price of these 2 derivative contracts if we use:

1. Black-Scholes model (what σ should we use?)
2. Bachelier model (what σ should we use?)
3. Static-replication of European payoff (using the SABR model calibrated in the previous question)

Project — Part IV (Dynamic Hedging)

Suppose $S_0 = \$100$, $\sigma = 0.2$, $r = 5\%$, $T = \frac{1}{12}$ year, i.e. 1 month, and $K = \$100$. Use a Black-Scholes model to simulate the stock price. Suppose we sell this at-the-money call option, and we hedge N times during the life of the call option. Assume there are 21 trading days over the month.

The dynamic hedging strategy for an option is

$$C_t = \phi_t S_t - \psi_t B_t,$$

where

$$\phi_t = \frac{\partial C}{\partial S} = \Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right),$$

and

$$\psi_t B_t = -Ke^{-r(T-t)}\Phi\left(\frac{\log\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

Project — Part IV (Dynamic Hedging)

Work out the hedging error of the dynamic delta hedging strategy by comparing the replicated position based on ϕ and ψ with the final call option payoff at maturity.

Use 50,000 paths in your simulation, and plot the histogram of the hedging error for $N = 21$ and $N = 84$.

Reference: `http://pricing.free.fr/docs/when_you_cannot_hedge.pdf`

Project Report

Deadline: 7-Nov-21 (Sunday) noon.

Please submit

- Project report (no more than 10 pages, including title page and appendix)
- Python codes (1 file for each part, 4 files overall)