Degree correlations

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Agenda

- Last class we discussed:
  - Growth and Preferential Attachment
  - The Barabási-Albert Model
  - The Bianconi-Barabási Model
  - Directions and weights

Agenda

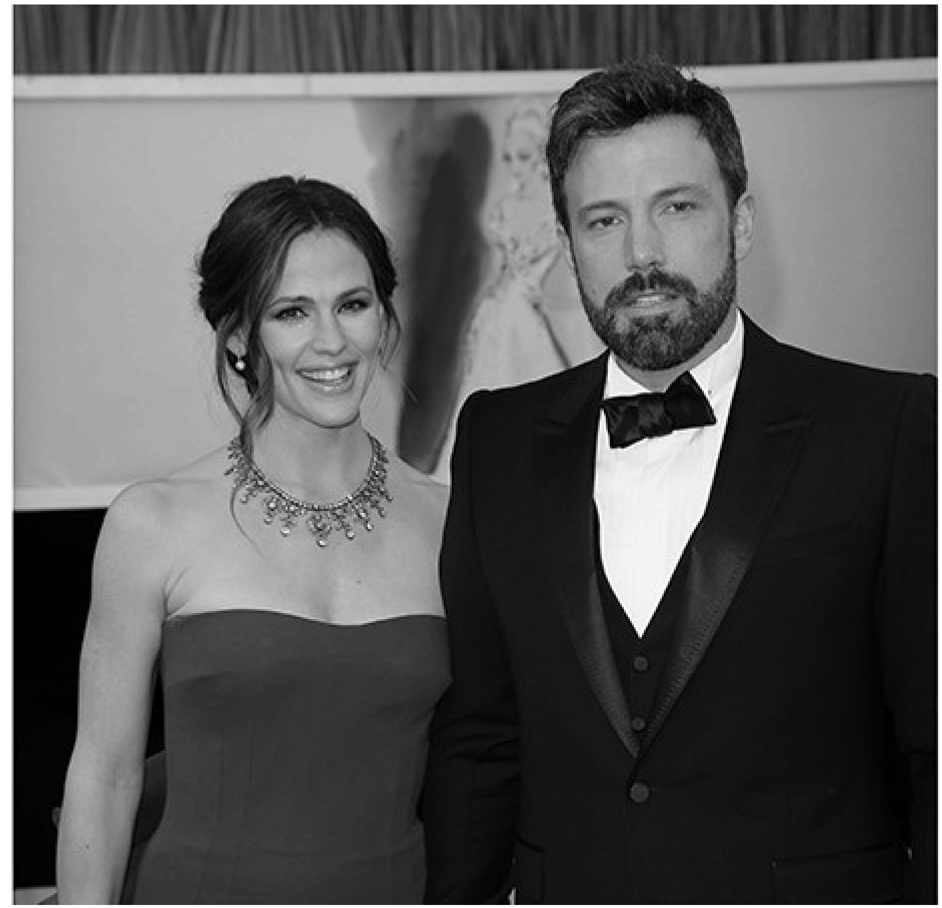
- Today:
  - Discussion about the homework
  - Degree correlations
  - Assortativity
  - Centrality measures

Discussion

Degree correlations

What is the true chance that a celebrity marries another celebrity?

Clue: in social networks, hubs tend to have ties to other hubs.



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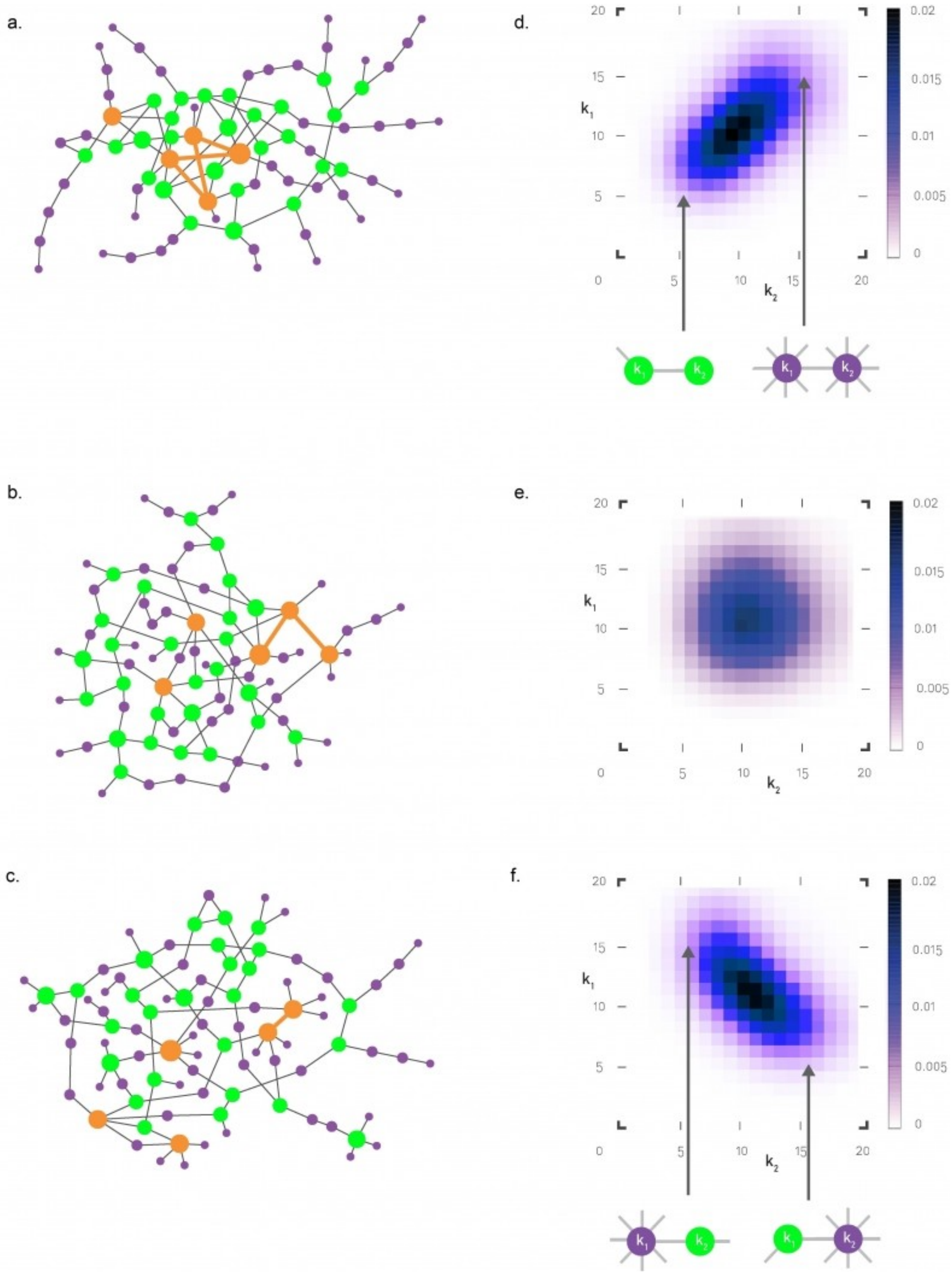
Assortativity and Disassortativity



Same degree distr. (Poisson)

$N = 1,000 \quad \langle k \rangle = 10$

a,d) Assortative network

b,e) Neutral network

$$p_{kk'} = \frac{kk'}{2L}$$

c,f) Disassortative network

(hub-and-spoke character)

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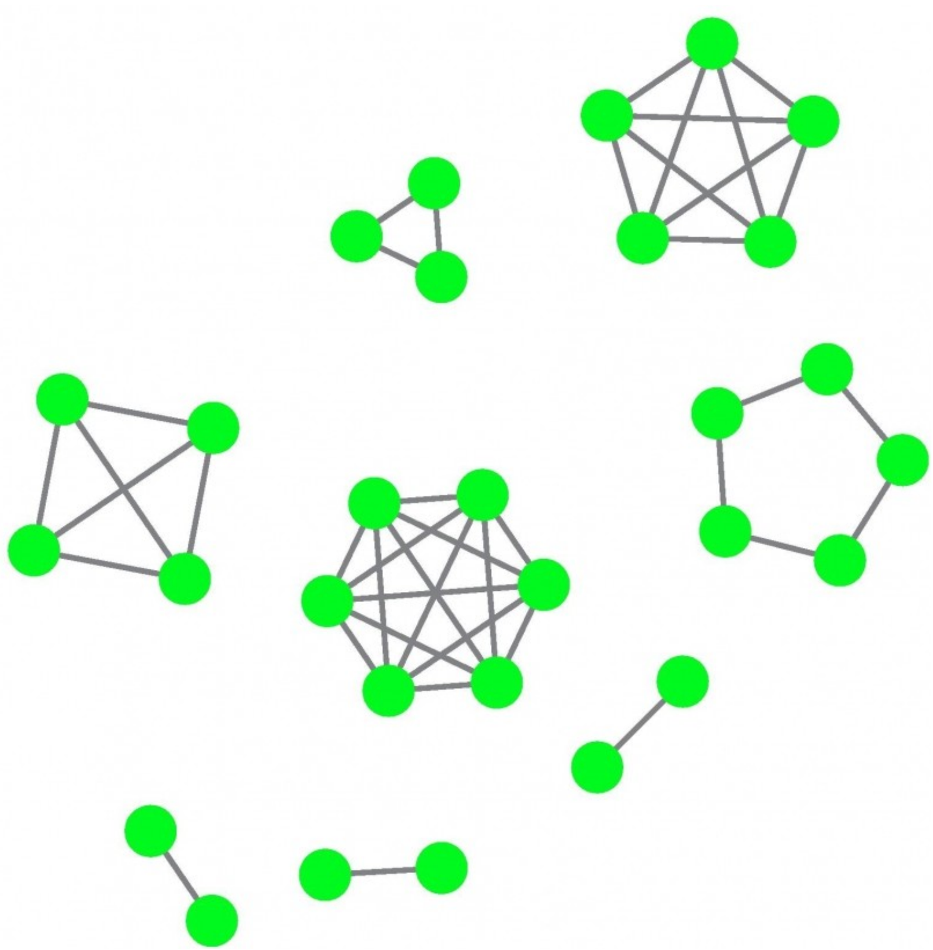
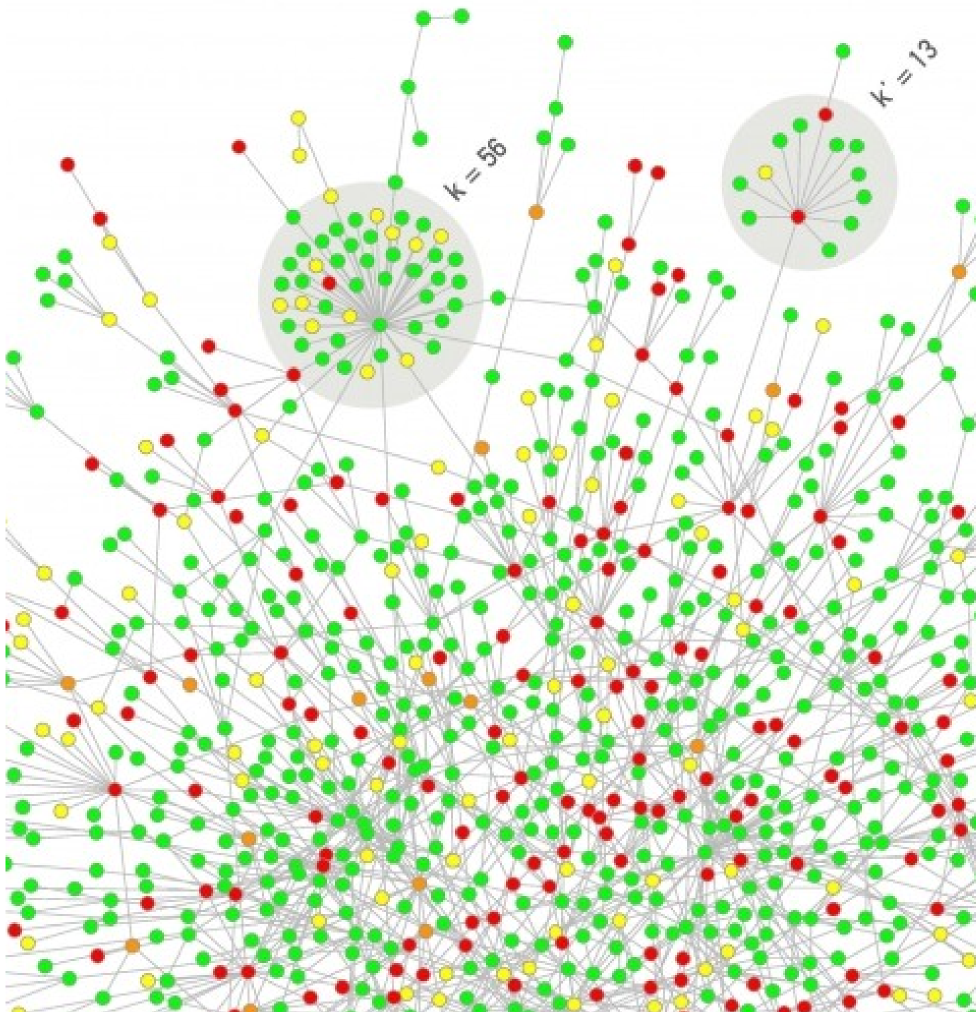
Degree correlations

Perfect Assortativity



Hub-and-spoke



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Assortativity and Disassortativity

Degree correlation matrix:

$$\sum_{i,j} e_{ij} = 1$$

Probability that a randomly selected node has a degree-k node at its end:

$$q_k = \frac{k p_k}{\langle k \rangle}$$

Then

$$\sum_{j} e_{ij} = q_i$$

In neutral networks, we expect:

$$e_{ij} = q_i q_j \tag{1}$$

A network displays degree correlations when e_{ij} deviates from the random expectation (1).

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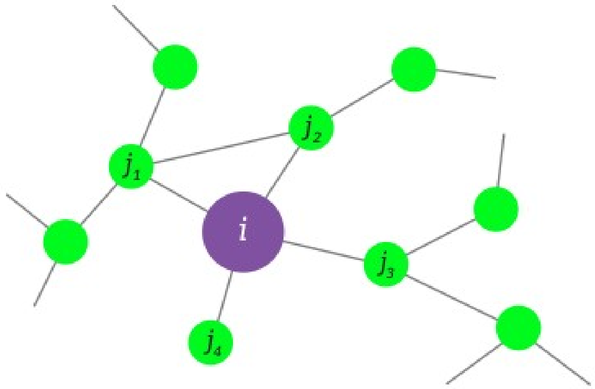
Measuring Degree Correlations

Average degree of *i*'s neighbors:

$$k_{nn}(k_i) = \frac{1}{k_i} \sum_{j=1}^{N} A_{ij} k_j$$



The degree correlation function:

$$k_{nn}(k) = \sum_{k'} k' P(k'|k)$$

is the average degree of the neighbors of all degree-k nodes, with $P(k'|k)$ being the conditional probability that following a link of a k-degree node we reach a degree-k' node.

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Degree correlations

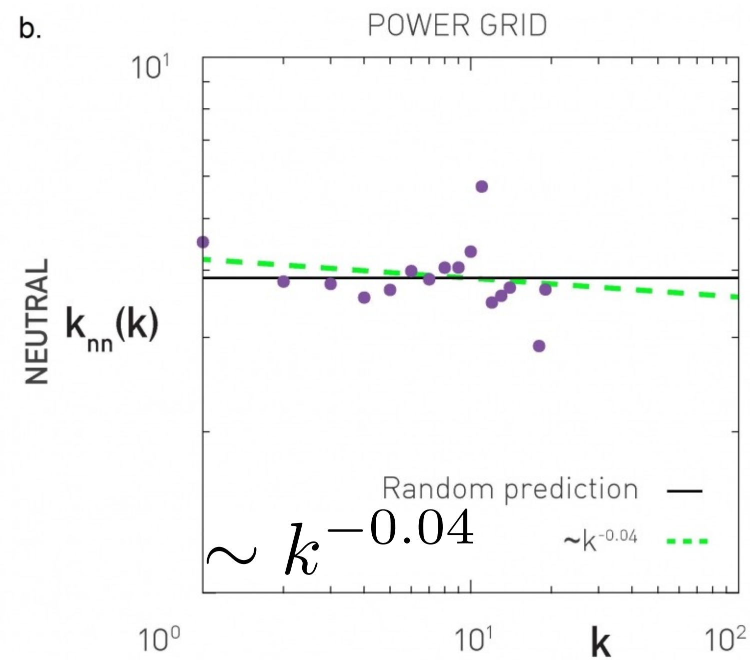
Neutral network: $k_{nn}(k)$ does not depend on specific k values:

$$k_{nn}(k) = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

which gives a horizontal line on a $k \times k_{nn}$ plot.

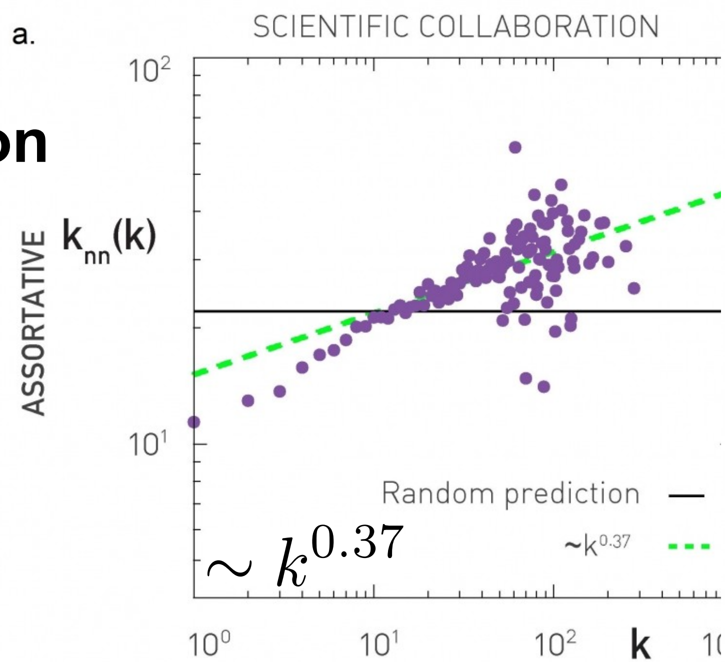
Assortative: $k_{nn}(k)$ increases with k.

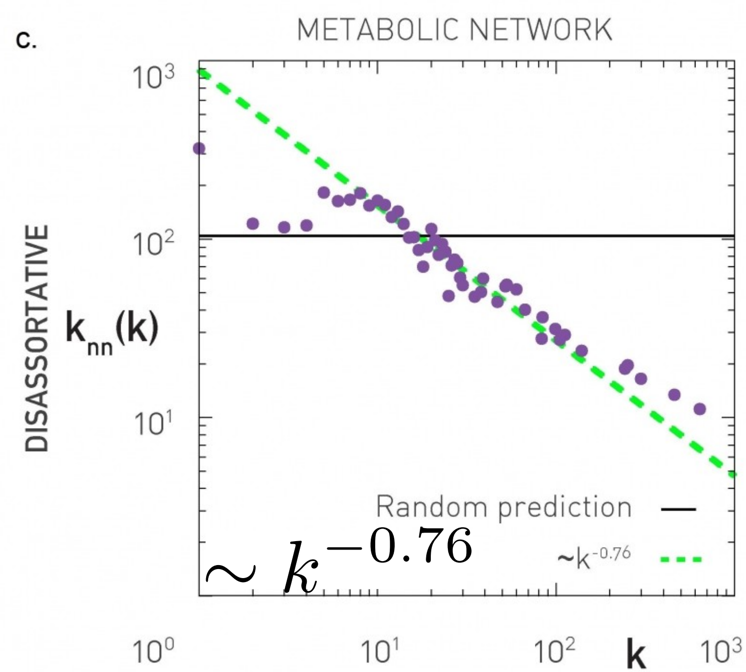
Disassortative: $k_{nn}(k)$ decreases with k.



Degree correlation function:

$$k_{nn}(k) = ak^{\mu}$$





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Degree correlations

Degree correlation coefficient

$$r = \sum_{jk} \frac{jk(e_{jk} - q_j q_k)}{\sigma^2}$$

with

$$\sigma^2 = \sum_k k^2 q_k - \left[\sum_k k q_k\right]^2$$

in which $-1 \leq r \leq 1$.

Neutral: $r = 0$
Assortative: $r > 0$
Disassortative: $r < 0$

It boils down to the Pearson correlation coefficient

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Structural disassortativity

Structural disassortativity: the expected number of links between k and k':

$$E_{kk'} = e_{kk'} \langle k \rangle N$$

may be higher than 1, which is not possible in simple networks. This happens when $k_{max} > k_s$, with k_s being the so-called structural cutoff, that depends on *N* as follows:

$$k_s(N) \sim (\langle k \rangle N)^{1/2}$$

Example: consider scale-free networks, with $k_{max} \sim N^{\frac{1}{\gamma-1}}$:

- For $\gamma \geq 3$: there is no structural cutoff, because $k_{max} < k_s$.
- For $\gamma < 3$: k_{max} could be higher than k_s, which results in hubs with fewer links between themselves, a phenomenon called *structural disassortativity*.

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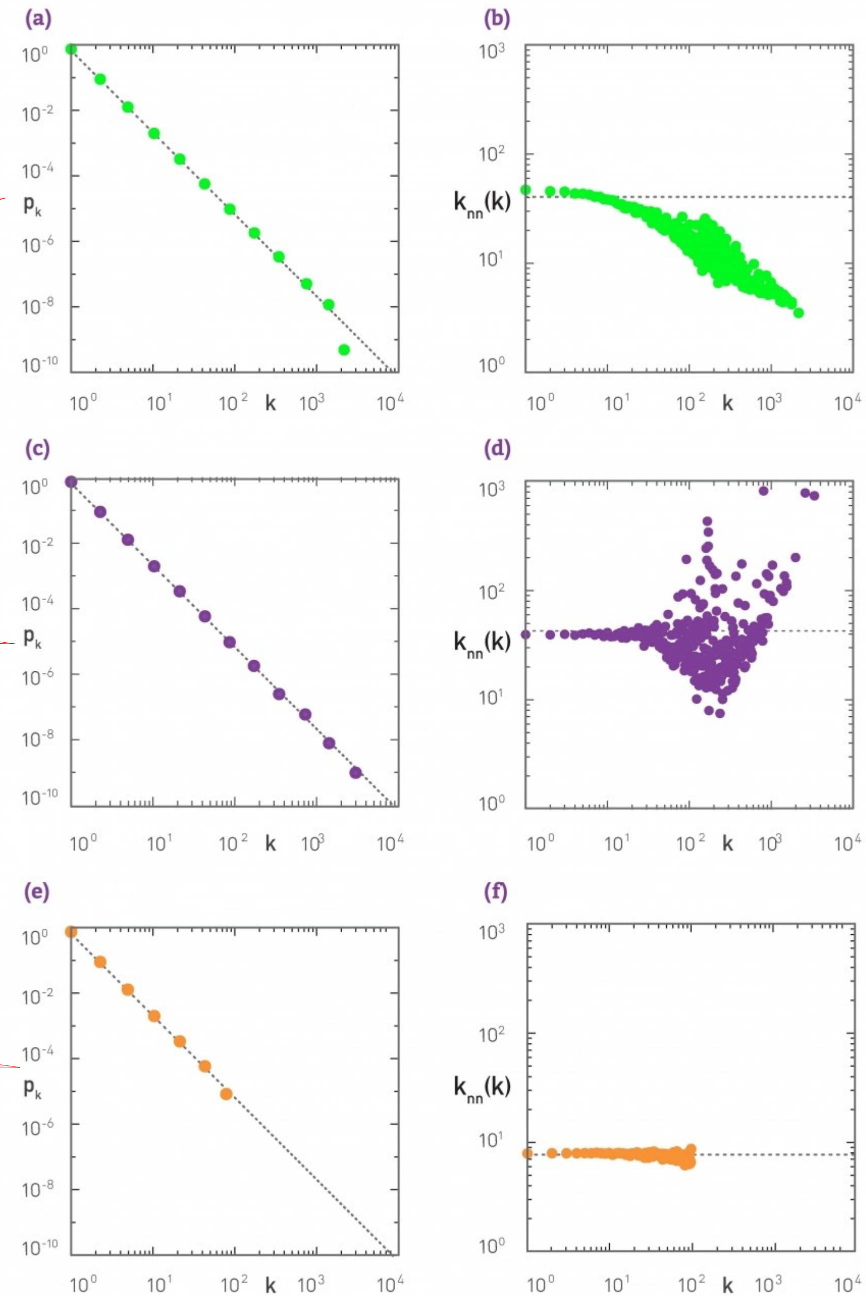
Structural disassortativity

Scale-free networks generated with the Configuration Model, with: $N = 10,000$ and $\gamma = 2.5$.

Structural disassortativity

Allowing multiple links between two nodes

Removing all nodes with $k \geq k_s \simeq 100$



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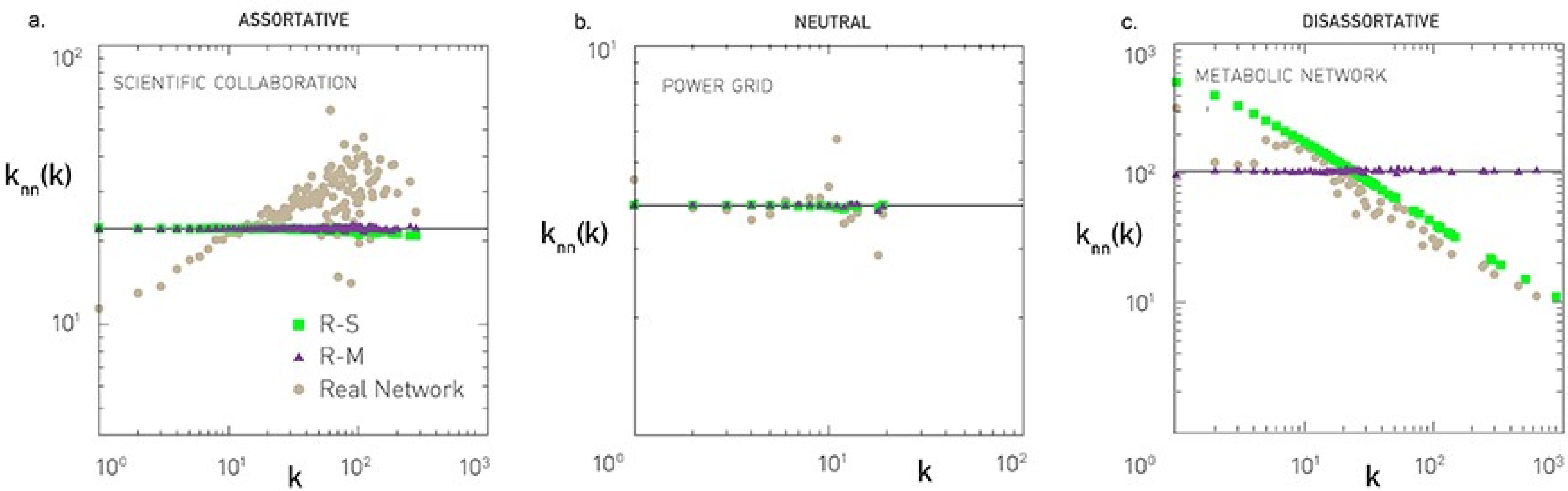
Degree correlations

How to properly check the degree correlations of a real network?

- *Degree Preserving Randomization with Simple Links (R-S).* If $k_{nn}(k)$ and $k_{nn}^{R-S}(k)$ are indistinguishable, then the correlations are all structural, and fully explained by the degree distribution. However, if $k_{nn}(k)$ presents degree correlations and $k_{nn}^{R-S}(k)$ does not, then there is some process that generates the observed degree correlations.
- *Degree Preserving Randomization with Multiple Links (R-M).*



- *Scale-free networks can induce disassortativity in simple networks.*

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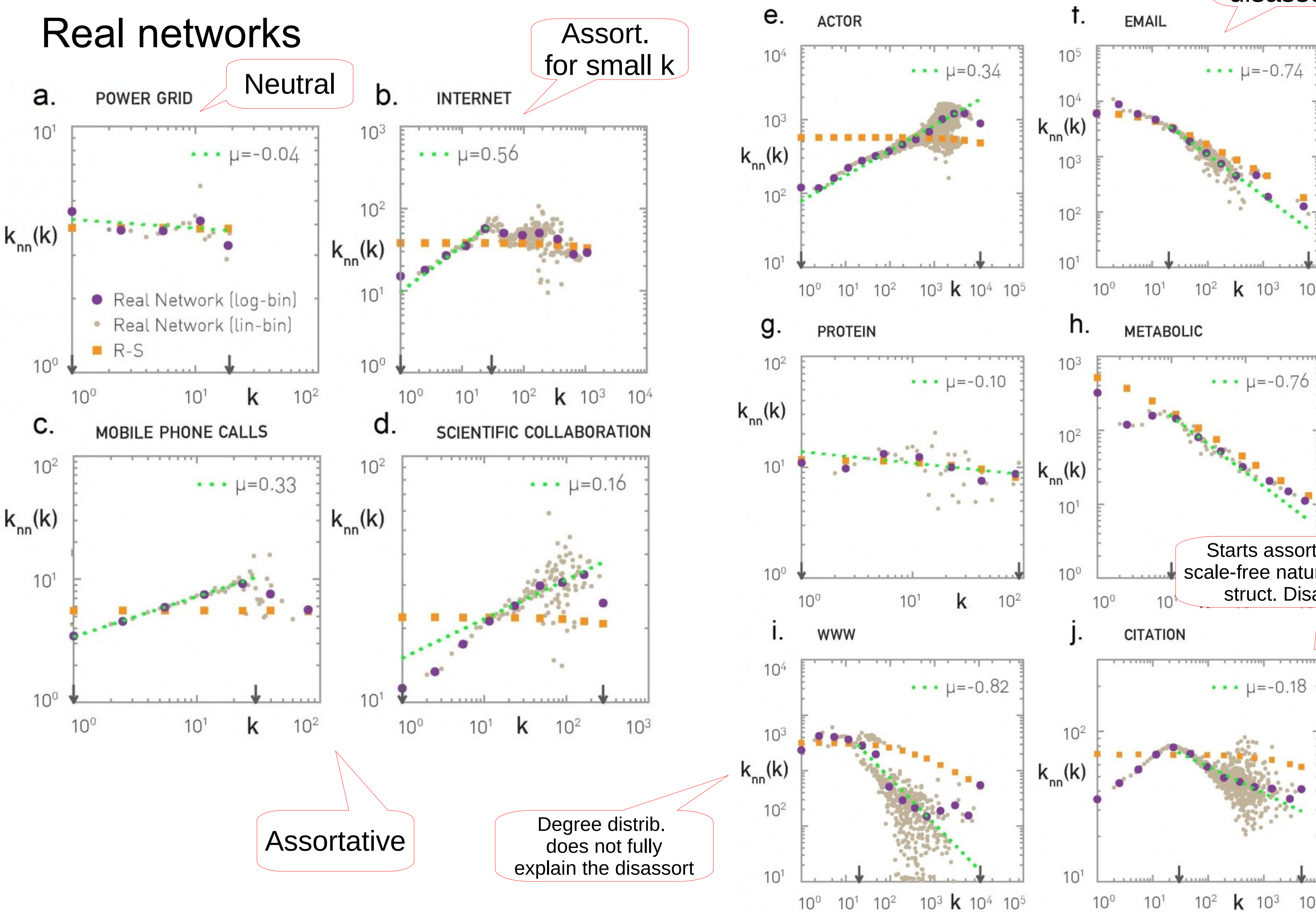
Degree correlations

Structural disassort.

Real networks

Neutral

Assort. for small k



Starts assort. but the scale-free nature induces struct. Disassort.

Assortative

Degree distrib. does not fully explain the disassort

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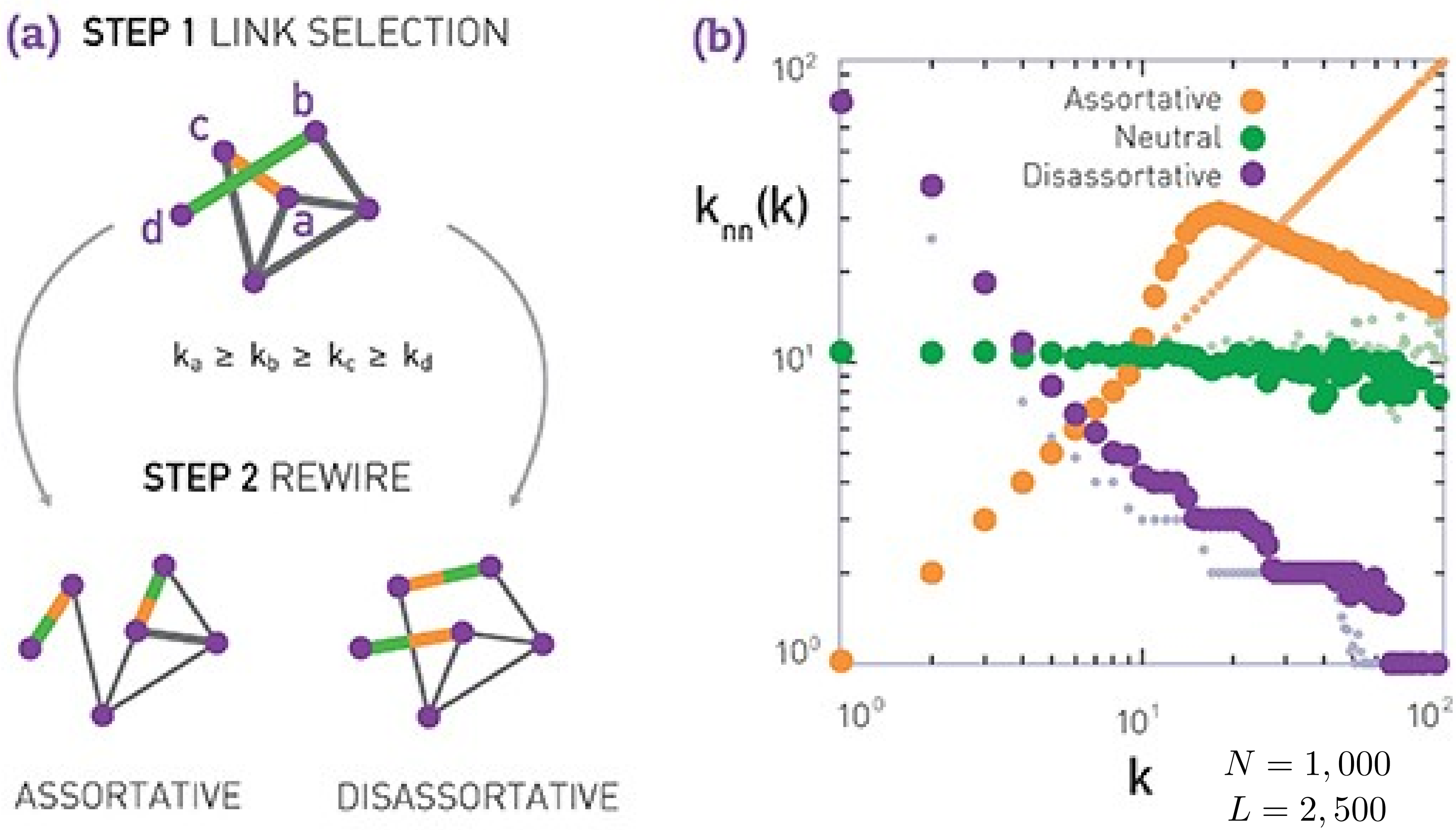
Creating degree correlated networks

Creating degree correlated networks: Xulvi-Brunet & Sokolov Algorithm



$N = 1,000$
$L = 2,500$
$\gamma = 3$

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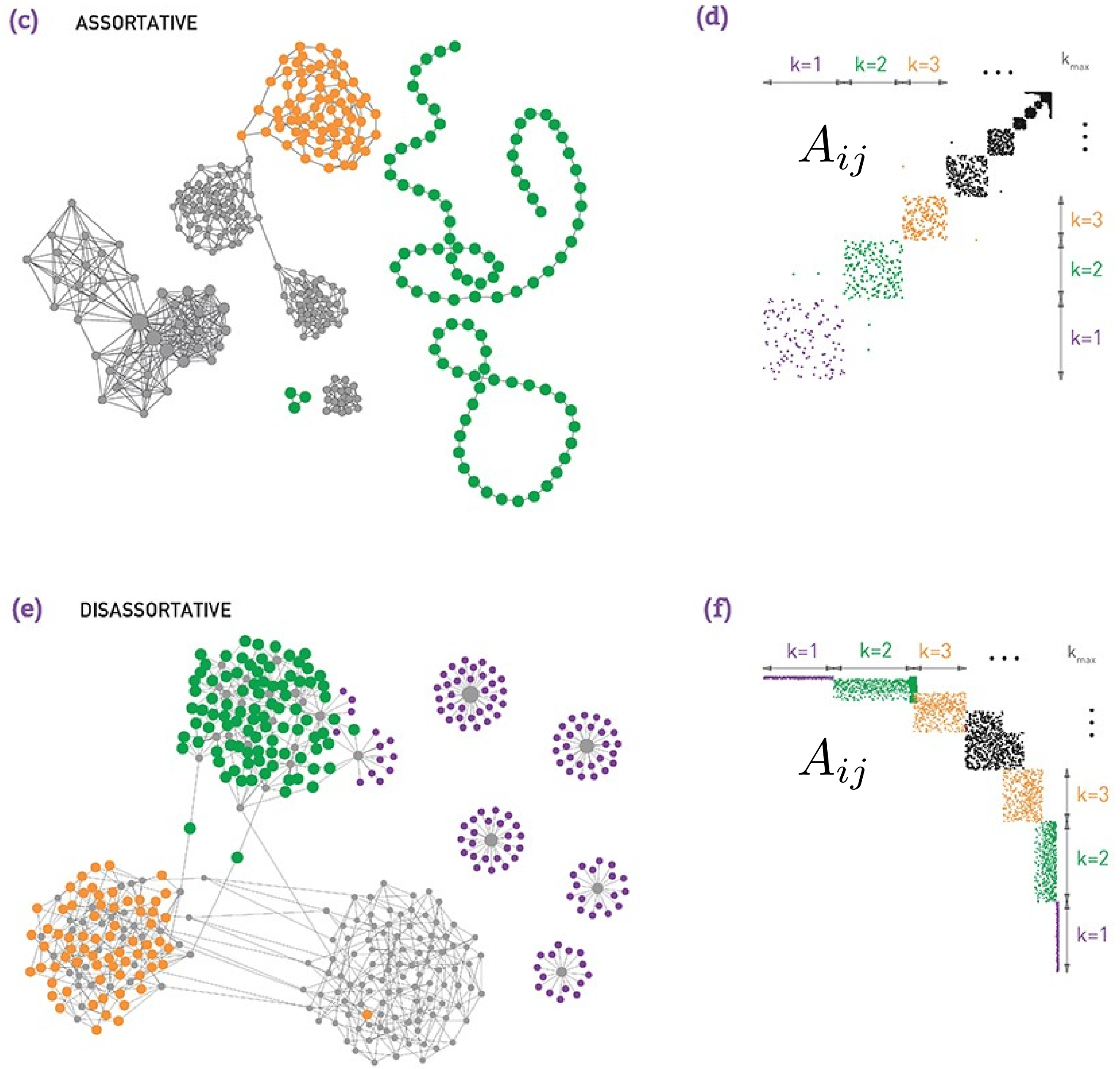
Creating degree correlated networks

Creating degree correlated networks: Xulvi-Brunet & Sokolov Algorithm



$$N = 1,000$$
$$L = 2,500$$
$$\gamma = 3$$

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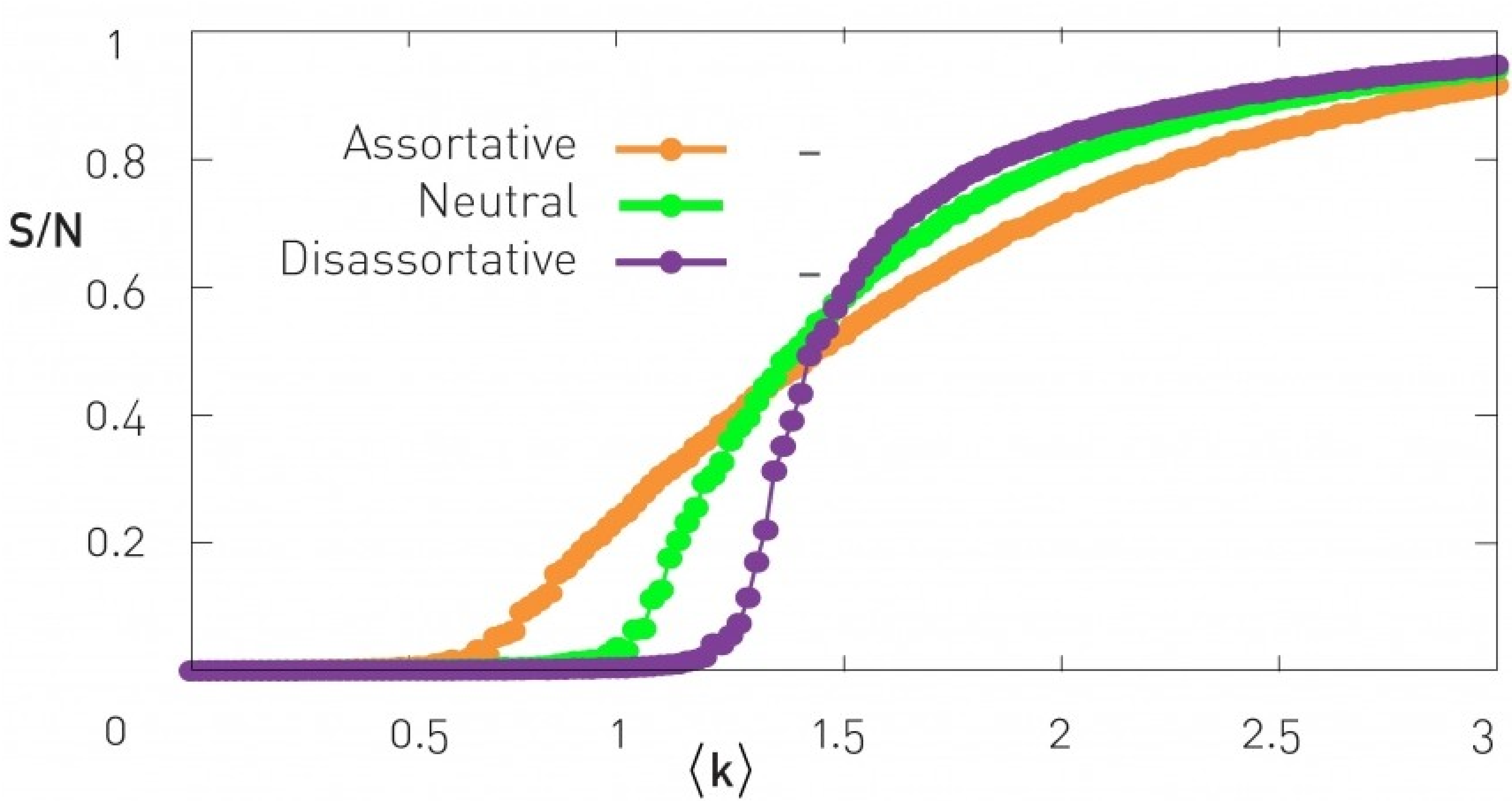
The impact of degree correlations

Size of the giant component for varying $\langle k \rangle$:



Could you explain what is going on?

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The impact of degree correlations

- **Epidemics:** The high degree nodes of assortative social networks form a “reservoir” for diseases, sustaining even when the network is not dense (on average) for the virus to persist.
- **Network robustness:** The removal of hubs in disassortative networks causes more damage than in assortative.
- **Synchronization:** Degree correlations influence a system’s stability against stimuli and perturbations as well as the synchronization of oscillators placed in a network.
- **Vertex cover problem:** fundamental impact on the vertex cover problem.
- **Control:** Degree correlations impact our ability to control a network, altering the number of input signals one needs to achieve full control.

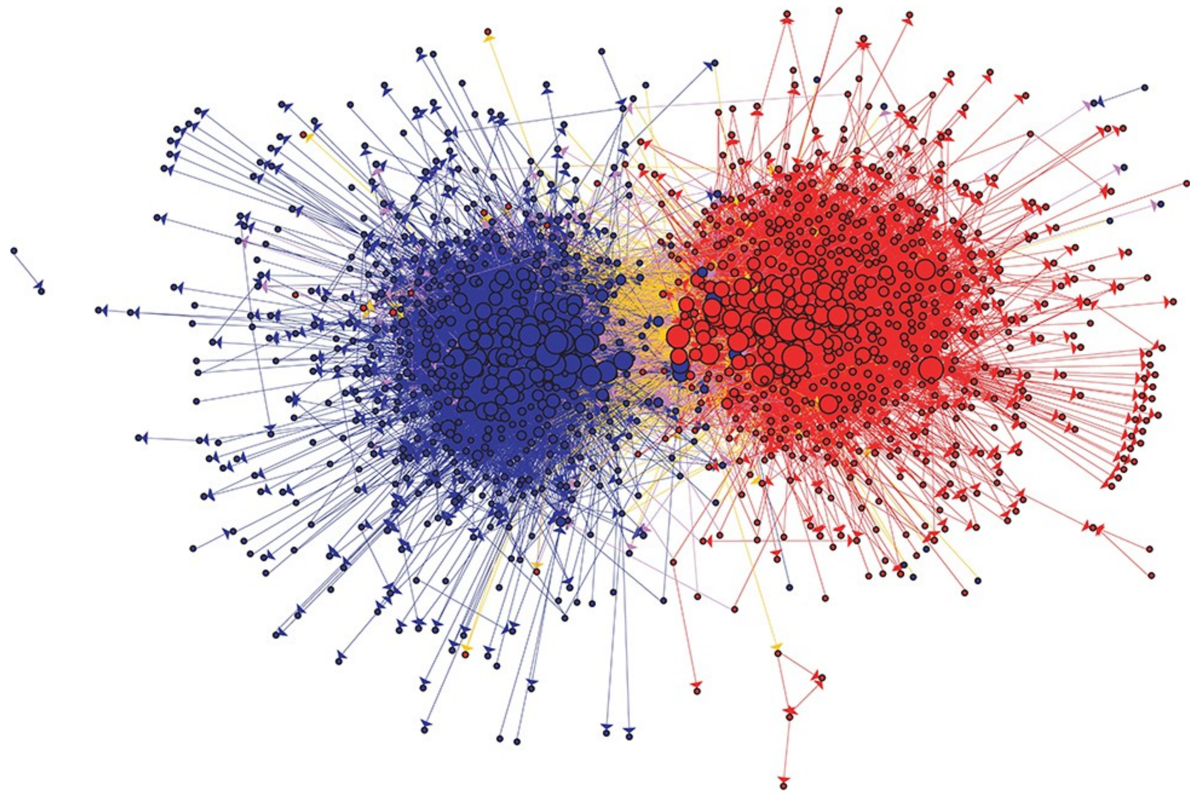
Assortative mixing

Properties of nodes could also match:

- Assortative mating: individuals usually date or marry individuals that are similar to them in terms of income, education, etc.
- Disassortative mixing: in sexual networks the links are in most cases between nodes with different gender; the baker does not sell bread to other bakers, etc.

The network behind the US political blogosphere illustrates the presence of assortative mixing.

Blue: liberal.
Red: conservative.



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Assortative mixing

Networkx:
r = nx.degree_assortativity_coefficient(G)

At a Glance: Degree Correlations

Degree Correlation Matrix e_{ij}

Neutral networks:

$$e_{ij} = q_i q_i = \frac{k_i p_{k_i} k_j p_{k_j}}{\langle k \rangle^2}$$

Degree Correlation Function

$$k_{nn}(k) = \sum_{k'} k' p(k' | k)$$

Neutral networks:

$$k_{nn}(k) = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

Scaling Hypothesis

$$k_{nn}(k) \sim k^{\mu}$$

$\mu > 0$: Assortative
$\mu = 0$: Neutral
$\mu < 0$: Disassortative

Degree Correlation Coefficient

$$r = \sum_{jk} \frac{jk(e_{jk} - q_j q_k)}{\sigma^2}$$

$r > 0$: Assortative
$r = 0$: Neutral
$r < 0$: Disassortative

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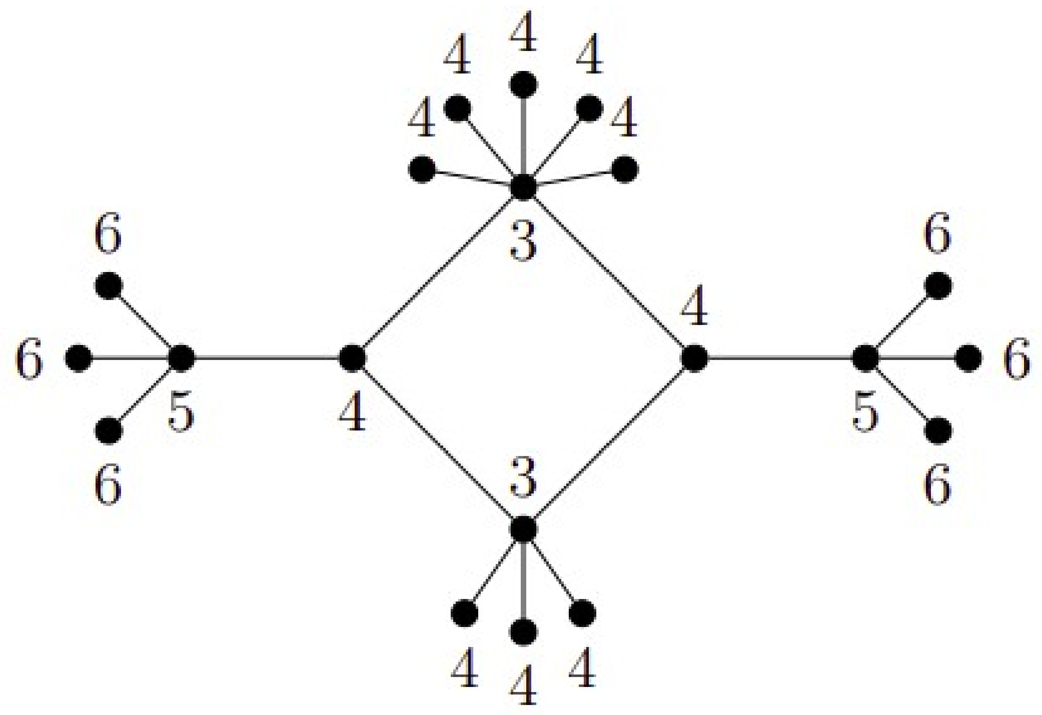
Centrality

Eccentricity of node *i*: highest distance from *i* to any other node.

$$E(i) = \max_{j \in V - \{i\}} d(i,j)$$

It reminds the closeness measure, but considers only the max(d).



Matjaž Krnc, Jean-Sébastien Sereni, Ristě Skrekovski, Zelealem Yilma. Eccentricity of Networks with Structural Constraints. Discussiones Mathematicae Graph Theory, University of Zielona Góra, 2020, 40 (4), pp.1141-1162. hal-01385481v2.

Centrality

Eigenvector centrality:

$$x(t) = A^t x(0)$$

for *t* iterations.

Initial condition: $x(0) = [x_1(0)\ x_2(0)\ \cdots\ x_N(0)]$

First iteration: $x(1) = Ax(0)$
Second iteration: $x(2) = Ax(1)$
Normalize the vector each iteration: $x \leftarrow \dfrac{x}{|x|}$

The vector converges after a certain number of iterations so that $Ax = \lambda x$,
In which λ is the highest eigenvalue of A.

Katz centrality (directed networks):

$$x_i = \alpha \sum_{j=1}^{N} A_{ij} x_j + \beta$$

with $\alpha < 1/\lambda$ to ensure convergence and $\beta = 1$ (usually).

- Big alpha leads to divergence; Small alpha makes beta the unique governing parameter.
- Important nodes spread importance equally.

PageRank centrality (directed networks):

$$x_i = \alpha \sum_{j=1}^{N} A_{ij} \frac{x_j}{k_j^{out}} + \beta$$

With $\beta = (1 - \alpha)/N$.

- Out degree to ponder importance

Personalized PageRank (directed networks):

$$x_i = \alpha \sum_{j=1}^{N} A_{ij} \frac{x_j}{k_j^{out}} + \beta$$

Considering a reference node i*. Thus,

$\beta = (1 - \alpha)/N$, If i = i*

$\beta = 0$, otherwise