On Laplace approximation

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1. The Laplace approximation in the univariate case

The aim is to approximate a univariate probability density function $g(x)$ using a normal density.

Denote the logarithm of $g(x)$ as $log\ g(x)$, and expand $log\ g(x)$ around its mode $\hat{x}$, we have:

$log\ g(x) \approx log\ g(\hat{x}) + \frac{d}{dx} log\ g(x)|_{x=\hat{x}}(x - \hat{x}) + \frac{1}{2}\frac{d^2}{dx^2} log\ g(x)|_{x=\hat{x}}(x - \hat{x})^2$

$= log\ g(\hat{x}) + \frac{1}{2}\frac{d^2}{dx^2} log\ g(x)|_{x=\hat{x}}(x - \hat{x})^2\ (\because \frac{d}{dx} log\ g(x)|_{x=\hat{x}} = 0)$

Let $1/\hat{\sigma}^2 = -\frac{d^2}{dx^2} log\ g(x)|_{x=\hat{x}}$, then: $log\ g(x) \approx log\ g(\hat{x}) - \frac{1}{2}\frac{1}{\hat{\sigma}^2}(x - \hat{x})^2\ - (1)$

Take exponentiation on both sides of (1), we get: $g(x) \approx g(\hat{x}) e^{-\frac{1}{2\hat{\sigma}^2}(x-\hat{x})^2} \propto N(\hat{x}, \hat{\sigma}^2)\ - (2)$

which implies that $g(x)$ can be approximated by a normal density $N(\hat{x}, \hat{\sigma}^2)$), where $\hat{x}$ satisfies $\frac{d}{dx} log\ g(x)|_{x=\hat{x}} = 0$ and $\hat{\sigma}^2 = -1/\frac{d^2}{dx^2} log\ g(x)|_{x=\hat{x}}$

2. An example

Consider a chi square density with k degrees of freedom $p(x; k) = \frac{1}{c} x^{\frac{k}{2}-1} e^{-\frac{1}{2}x}$, where c is a constant not involving x

$log\ p(x) \stackrel{c}{=} (\frac{k}{2} - 1) log x - \frac{x}{2}$

$\frac{d}{dx} log\ p(x) \stackrel{set}{=} 0 => \hat{x} = k - 2$

$\frac{d^2}{dx^2} log\ p(x)|_{x=\hat{x}} = -\frac{k/2-1}{x^2}|_{x=\hat{x}} = -\frac{1}{2(k-2)}$

Therefore, according to (2), $p(x)$ can be approximated by $N(\hat{x} = k - 2, \hat{\sigma}^2 = 2(k - 2))$

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par(mfrow=c(2,2))

k= 3
x= seq(from= 0, to= 10, by= 0.1)
y= dchisq(x= x, df= k)

plot(x= x, y= y, type= 'l', ylim= c(0, 0.3), ylab= "density", xlab= "x")
lines(x= x, y= dnorm(x= x, mean= k-2, sd= sqrt(2*(k-2))),lty= 2)
legend("topright", legend= c(expression(paste(chi^2,"(3)")),
                   "normal(2,4)"),
       lty= c(1,2), bty= 'n')



k= 6
x= seq(from= 0, to= 10, by= 0.1)
y= dchisq(x= x, df= k)

plot(x= x, y= y, type= 'l', ylim= c(0, 0.15), ylab= "density", xlab= "x")
lines(x= x, y= dnorm(x= x, mean= k-2, sd= sqrt(2*(k-2))),lty= 2)
legend("topright", legend= c(expression(paste(chi^2,"(6)")),
       "normal(4,8)"), lty= c(1,2), bty= 'n')




k= 10
x= seq(from= 0, to= 20, by= 0.1)
y= dchisq(x= x, df= k)

plot(x= x, y= y, type= 'l', ylim= c(0, 0.10), ylab= "density", xlab= "x")
lines(x= x, y= dnorm(x= x, mean= k-2, sd= sqrt(2*(k-2))),lty= 2)
legend("topright", legend= c(expression(paste(chi^2,"(10)")),
       "normal(8,16)"), lty= c(1,2), bty= 'n')




k= 20
x= seq(from= 0, to= 40, by= 0.1)
y= dchisq(x= x, df= k)

plot(x= x, y= y, type= 'l', ylim= c(0, 0.10), ylab= "density", xlab= "x")
lines(x= x, y= dnorm(x= x, mean= k-2, sd= sqrt(2*(k-2))),lty= 2)
legend("topright", legend= c(expression(paste(chi^2,"(20)")),
                             "normal(18,36)"), lty= c(1,2), bty= 'n')
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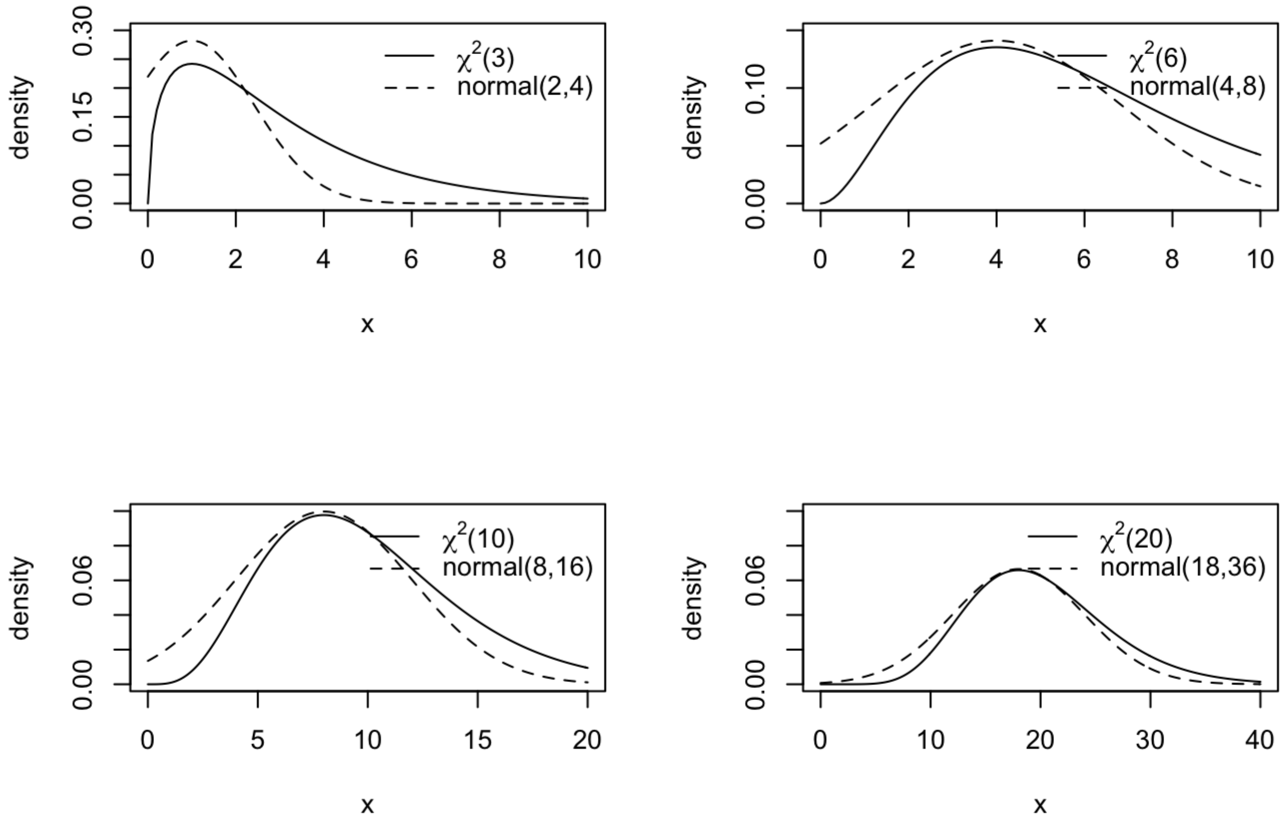


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par(mfrow=c(1,1))
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It can be seen that as degrees of freedom of the chi square distribution increase, the approximation by a normal density appears to be better.