The Mathematics Of a Heartbeat

Francis E.K. Osei
oseifek@gmail.com

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Have you ever wondered about the intricate workings of the human heart? It's a fascinating organ that beats tirelessly to keep us alive, and it turns out that there's a lot of math involved in its rhythmic pumping. From the electrical impulses that initiate each beat to the complex patterns of blood flow throughout the body, the mathematics of a heartbeat are truly awe-inspiring.

At its most basic level, the heart can be thought of as a pump that moves blood through a network of vessels. But there's so much more to it than that. The heart's electrical system ensures that each beat happens at the right time and in the right order, while the pressure changes within the heart and blood vessels require intricate calculations to understand.

In this essay, we'll explore the fascinating mathematics behind the heart's beating, delving into topics such as cardiac cycle timing, blood flow dynamics, and the complex geometry of the heart's chambers. By the end, you'll have a newfound appreciation for the amazing machine that keeps us all ticking.

The circulatory system is made up of the heart and the blood vessels which form the cardiovascular system and the blood. The main function of the circulatory system is to transport essential substances such as oxygen and nutrients throughout the body while removing waste products such as carbon dioxide. This process is achieved with blood as a medium. The heart is a muscular pump that keeps blood flowing through the vessels, where the vessels carry blood to tissues throughout the body and unload oxygen and nutrients necessary for the survival of the tissues and carry the blood back to the heart to be carried to the lungs to offload the carbon dioxide. The beating of the heart drives this process. During one heartbeat, two things happen:

1. blood is carried to the lungs from the body system and returned to the heart
2. blood is carried to the body system, heart and also the lungs itself from the lungs

This implies there are two types of circulation

1. Pulmonary circulation: where blood is carried to the lungs from the body system and returned to the heart mainly for gaseous exchange
2. Systemic circulation: blood is carried to the body system, heart and also the lungs itself from the lungs

Blood in the pulmonary circulation is the deoxygenated blood and oxygenated blood for systemic circulation. The heart is divided into four chambers. The Left and Right Atria and Ventricles. They are illustrated in the diagram below. This division is based on anatomical analysis.

The heart can also be divided into two sides, the left and right sides. Each consists of two of these chambers. The left side of the heart, composed of the left atrium and the left ventricle and the right side of the heart, is composed of the right atrium and the right ventricle. The left side of the heart is responsible for systemic circulation and predictably, the right side is responsible for pulmonary

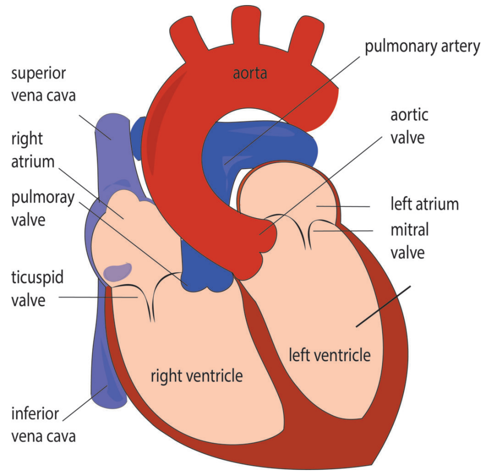

Figure 1: Anatomy of the heart.

circulation.

Circulation of blood throughout the body is driven by the rhythmic relaxation and contraction of the heart muscles, controlled by the electrical system of the heart. This process is what is referred to as the heartbeat. The sounds the heart makes are a consequence of the opening and closure of the heart valves.

The heart beats independently of the nervous system. Instead, the electrical system of the heart is made up of specialized muscle cells that generate nerve impulses which drive the heartbeat. The specialized cells include the sinoatrial and atrioventricular nodes and the bundle of His and perkunje fibres which arises from the bundle of His. This is shown in figure 2 below.

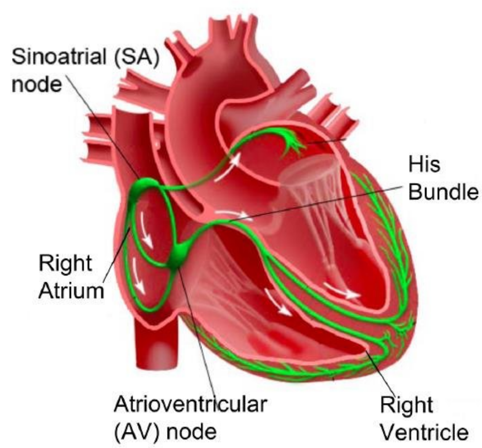

Figure 2: Electrical system of the heart.

The specialized muscle cells are muscle cells that have lost their ability to contract. These cells generate, transmit and stimulate the cells that can contract to generate the heartbeat.

The generation of electrical impulses starts from the sinoatrial node (SA node). After the SA node fires, the electrical impulse is transmitted in two directions.

1. Across the atria
2. down the atria to the atrioventricular node

This makes the atria contract. An important point to know is that the movement of the electrical signals is restricted by an insulation system arising from the fibrous skeleton of the heart. As a result, the impulse from the SA node doesn't cause the ventricles to contract. The contraction of the ventricles is caused by the impulse from the AV node and some of the impulse from the SA node - the impulse from the SA node passes through the AV node before it can get to the ventricles. The impulse from the AV node travels through the bundle of His and the purkinje fibres and then causes the ventricles to contract.

Generally, whenever muscles contract, they shorten in length. This implies the heart volume will reduce. This also implies blood should leave the chamber because there isn't enough room for the blood to stay in the chamber and also the force of contraction causes the blood to also leave the heart under pressure. After some time, the electrical impulse decays and the muscle relaxes - they regain their original length. This in turn increases the volume of the chamber and allows blood to fill the chamber. In the case of filling, blood from the atria to the ventricles, because of gravity and difference in pressure, blood starts "leaking" into the ventricles before the contraction comes in to drive the full volume of blood down to the ventricles.

Now, with this information, we describe our mathematical model and use it to describe how the electrical activity coordinates with the muscular activity to pump blood to the necessary parts of the body. Utilizing the power of mathematics to examine the beauty of a heartbeat! We are using mathematics as a tool for thinking, not as a source of reality, as Henri Poincaré famously stated. Whilst they may not be a perfect match for the precise specifics of physiology, our mathematical models help us understand the amazing rhythm of the heartbeat in a whole new manner. Who would have thought that the simple subject of mathematics might aid us in comprehending the intricacy of the human body? It seems as though we have cracked the code to existence itself. And even while our models might not be a perfect representation of reality, they open our eyes to the marvel of how our hearts work. So let's embrace the power of math and marvel at the magic of the heartbeat. Who knows what other mysteries we can uncover with the tools of science and imagination! Without further ado, let's keep the heart beating!

Muscle contraction is a response to an electrical impulse reaching the muscle fibre. When the impulse reaches the muscle cell, it suddenly reduces in size or length. In the heart, it is the muscle contraction that drives blood out of the heart into blood vessels. An electric impulse is a sudden change in voltage in an electrical system. The SA node produces these impulses and as these impulses travel along the neurone, the muscles are stimulated to contract in response to the voltage V, it receives. This is some kind of Hooke's law stuff going on here. Anyways, if we zoom into what is going on in a single muscle cell, the cell is stimulated to contract at a certain voltage. This is the only numerical value that will cause the cell to contract. We denote this V_{max}. As soon as the contraction starts, the voltage decays to the lowest voltage V_o. Then it grows to V_{max} and the cycle repeats itself again. It behaves like a sinusoidal process. One thing about this is that this process occurs so fast that, almost almost all the cells (about millions of cells) being involved almost respond at approximately the same time. The time interval between the first cell to be affected and even the last cell to be affected is very very very tiny. Thus, we can assume the whole muscle is one cell. Now, we need to guess which fits this description. A sine waveform or a cosine waveform.

Contraction occurs at V_{max} and as V $\to 0$, the muscle relaxes - returns to its original dimensions till it gets to V = V_0. Then, the muscle stays at the same original dimensions till V = V_{max} again.

Below the line V = 0, it is the ventricle's turn to undergo the same process. So the atria have no electrical stuff going on in it. So, between this time interval, the ventricles undergo this very same process but at a higher V_{max} because of the numerous neurones due to their large muscles. If we measure time with respect to the dynamics of the voltage, then, at the start of the process V = 0. So (V, t) = (0, 0). Physiologically, V must build up to V_{max} and back to V = 0 as time increases. This behaviour fits the waveform of the graph of a sine function. So we write

$$V(t) = V_{max} \sin(2\pi f t).$$

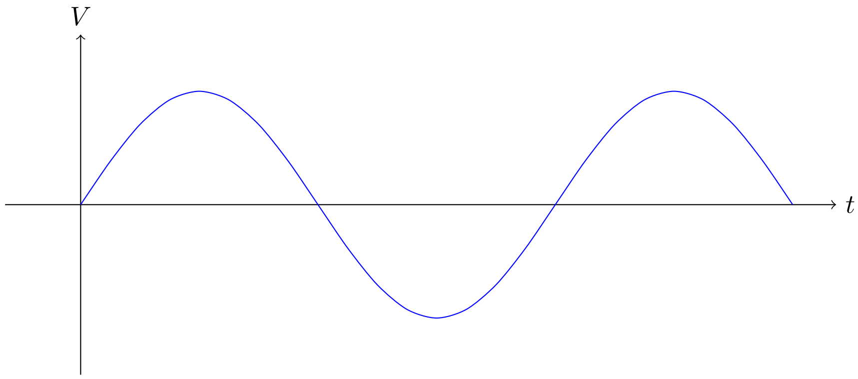

Figure 3: Voltage dynamics in the atria.

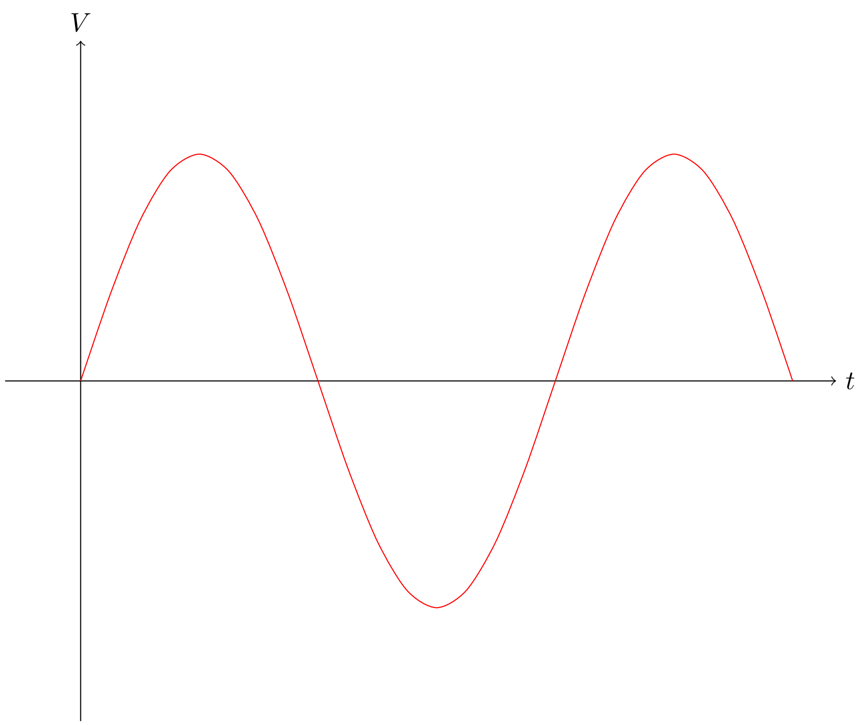

Figure 4: Voltage dynamics in the ventricles.

At this point, we've described how the voltages change as the heart beats. Now, we see how this voltage is converted to force to drive blood out of the heart and into respective chambers. In chemistry, there are some materials - piezoelectric materials. These materials respond to mechanical force by generating voltage. The heart behaves the exact opposite of the 'piezos'. They respond to a voltage by generating a force. This implies, the force generated by the material depends on the voltage. This voltage is the V_{max} discussed earlier. So we have $F \propto V_{max}$. The higher the value of V_{max}, the higher the force produced. This is what actually happens in the heart. Remember, we are taking the cells as one. So, summing up all the cells, we have more cells in the ventricles than the atria and hence, more voltage in the ventricles than the atria which results in a larger force of contraction in the ventricles than the atria. To get rid of the proportionality sign, we recall an equation from physics, F = qV where the charge is the charge carried in the nerves to the muscles. From Hooke's law, the force applied to an elastic material is directly proportional to its deformation - the changes in its dimensions. Mathematically, F $\propto$ change in dimensions. We can say that the voltage generates a change in length of the heart muscle and which communicates the force. If we imagine the muscle as a sheet of paper rolled onto a cylinder, as V_{max} is reached, the size of the circle decreases. In so doing, the pressure inside increases and forces the blood out of the chamber.

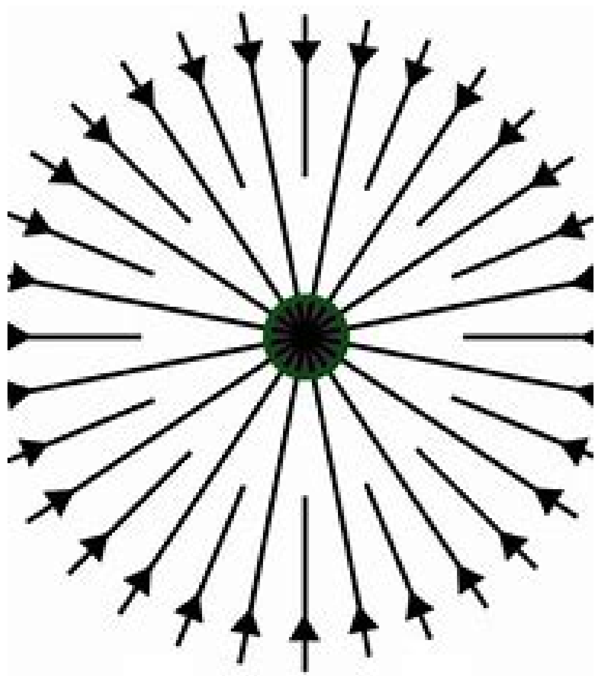

Figure 5: Pressure building up in the heart chamber.

This can also imply the force is acting in all directions on the surface of the circle. So, the force acting on the surface generates pressure. This can be written mathematically as

$$P = \frac{F}{A}.$$

Where F = $\nabla \cdot$ F and A is the surface area of the circle A_{surf}. The heart is a 3d object, not a 2d object Hence, calculus is needed to get the equation for the surface area of a 3D object characterized by a function F. Depending on the function F, this equation might be rather complicated. In general, the surface area of a 3D object may be calculated by integrating the item's cross-sectional area over its whole surface. The cross-sectional area of an item is its area measured along a plane perpendicular to its normal vector and passing through the object at each point along the plane. The surface integral formula is one method that might be used. The formula is as follows:

$$A_{surf} = \int\int \sqrt{1 + |\nabla F|^2}\, \mathrm{d}A$$

So, the equation for the Pressure becomes

$$P = \frac{\nabla \cdot F}{\int\int \sqrt{1 + |\nabla F|^2}\, \mathrm{d}A}$$

As the heart receives blood from the body and the lungs, all these mechanisms work together to pump blood out of the heart. To keep us alive, this process never stops. Blood carries nutrition and oxygen to the different tissues while also removing waste materials and carbon dioxide (CO_2) from the tissues. Lastly, using the previous explanation as a foundation, we explore how blood is transported throughout the heart.

Initially, at (V, t) = (0, 0), all four chambers of the heart are relaxed but the AV valves are open. Without any effort from the heart, blood flows passively into the ventricles through these valves. This blood accounts for 70 percent of the blood it needs to be supplied with. Next, the SA node fires - remember at V = V_{max}. This excites the heart muscles and the atria contract (note both Left and right). This adds to what is in the ventricles the remaining supply of 30 percent of blood. Then the atria relax. Figure 3 shows the dynamics of V for this process. After the atria complete its job, the AV node also fires at V = V_{max} but for figure 4. But because, the ventricles have a larger wall and hence, more neurones than in the atria, it contracts to produce a larger force than the atria and consequentially, more pressure. This blood to the lungs and body through the aorta and the pulmonary trunk respectively.

In conclusion, the electrical system of the heart controls the rhythmic relaxation and contraction of the cardiac muscles, which constitute the heartbeat. The cardiac cycle is the time between one full contraction and the complete relaxation of all four heart chambers. This causes the blood to circulate throughout the body, supplying different tissues with oxygen and nutrients and removing carbon dioxide (CO_2) so that it may be evacuated from the body through the lungs. The human heart is one of the body's most vital organs. We must take care of our cardiovascular system seriously. To maintain the health of our cardiovascular system, we must eat and drink sensibly. Not to be overlooked is the fact that exercise aids in keeping a healthy cardiovascular system.

> "The heart is the engine of the body, and it's vital that we keep it healthy. Simple lifestyle changes like exercise and a healthy diet can go a long way in preventing heart disease."
>
> *- Dr. Sanjay Gupta, Neurosurgeon and Medical Correspondent*

References

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[2] James Stewart. *CALCULUS Early Transcendentals.* Publisher Cengage Learning, 2015.

[3] The images of fig 1 and 2 .

[4] The images of fig 5 .