Theoretical Computer Science Exam, July 8, 2022

The exam consists of **4 exercises**. Available time: 1 hr 30 min.; you may write your answers in Italian, if you wish.

All exercises must be addressed for the written exam to be considered sufficient.

F.NAME …………………………….. L.NAME …………………………………… PERSON CODE …………………

Exercise 1 (8 pts)

Consider the language

$$L = \{ a^{2n+1} b^{3n-1} \mid n > 0 \}$$

For instance, strings *aaabb*, a^5b^5, $a^7b^8 \in L$.

a) Write a grammar, of the least general type, that generates language L. For this grammar, write a derivation of the string a^5b^5.

b) Design an automaton, of the least powerful type, that accepts language L. Write a computation of the automaton for the string *aaabb* and one for the string a^5b^5.

Exercise 2 (8 pts)

Consider a subprogram, called *ratios*(*in*, *nIn*, *out*), having three parameters: an array of integers *in*, its size *nIn*, an array of integers *out*. The input parameters are: *in* and *nIn*, whereas the array *out* is an output parameter. The two arrays can be assumed to have the same size (i.e., number of components), greater than 1; all the elements of array *in* are assumed to be strictly positive and strictly increasing; every element of the array *in*, except the first one, is a multiple of the preceding one. For instance, an acceptable input array *in* is the following [3, 9, 18, 54, 270].

After execution of the subprogram the first element of array *in* is also stored, in the first position, into the array *out*; furthermore, every element of array *out* in the other positions is equal to the value of the element having the same position in array *in* divided by the previous element in array *in*. For instance, the specification is satisfied by the following values: *nIn*=5, *in*=[3, 9, 18, 54, 270], *out*=[3, 3, 2, 3, 5].

Please answer the following questions:

1. write pre- and post-conditions specifying the requirements of the above described procedure; assume that indices of the arrays start at 0;
2. for each of the following input/output data say if the procedure *ratios* satisfies the specification if it computes the given value of the *out* array when it receives the given input parameter values; please provide suitable justifications for your answers.

   a) *nIn*=5, *in*=[2, 7, 9, 11, 6], *out*=[2, 9, 7, 13, 8]

   b) *nIn*=5, *in*=[2, 6, 12, 48, 96], *out*=[2, 3, 2, 3, 2]

   c) *nIn*=5, *in*=[2, 6, 12, 48, 96], *out*=[2, 3, 2, 4, 2]

Exercise 3 (8 pts)

Let us consider a fixed finite set S of natural numbers.

1. Is the set S', that includes exactly the natural numbers that are multiple of all elements of S, decidable?
2. Is the set S'' of Turing machines that, starting the computation with an empty input tape, *do not* print all the numbers that belong to S decidable?
3. Is the set S'' of Turing machines that, starting the computation with an empty input tape, *do not* print all the numbers of S semidecidable?

Please provide precise, detailed explanations for your answers.

Exercise 4 (8 pts)

The following C-like program reads from the input a positive integer value n and writes the first n powers of 2, that is, 2, 2^2, … 2^n.

```
int n, ev;
read(n);
ev = 1;
for ( i from 1 to n ) {
   ev = 2 * ev;
   write(ev);
}
```

Assuming that the algorithm, after a standard translation, is executed on a RAM machine, evaluate its time and space complexity class using as parameter the ***value*** of input n, adopting both the uniform and the logarithmic cost criteria.

In addition, evaluate the same complexity figures using as parameter the input ***size***, that is, the length x of the string that encodes, in a standard way, the value of n.

Solutions

Exercise 1

Consider the language

$$L = \{ a^{2n+1} b^{3n-1} \mid n > 0 \}$$

For instance, strings *aaabb*, a^5b^5, $a^7b^8 \in L$.

a) Write a grammar, of the least general type, that generates language L. For this grammar, write a derivation of the string a^5b^5.

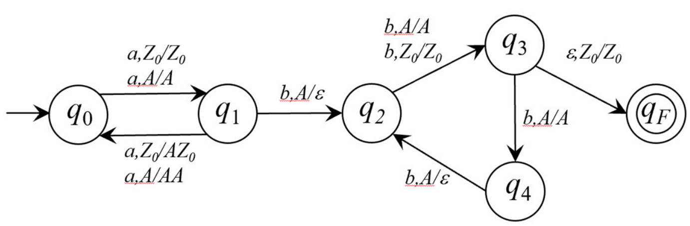
The language is clearly contect-free.

$S \rightarrow aa\ S\ bbb \mid aaabb$

$S \Rightarrow aa\ S\ bbb \Rightarrow aa\ aaabb\ bbb$

b) Design an automaton, of the least powerful type, that accepts language L. Write a computation of the automaton for the string *aaabb* and one for the string a^5b^5.

A deterministic pushdown automaton suffices



$<q_0, aaabb, Z_0> \vdash < q_1, aabb, Z_0> \vdash < q_0, abb, AZ_0> \vdash < q_1, bb, AZ_0> \vdash < q_2, b, Z_0> \vdash < q_3, \varepsilon, Z_0>$
$\vdash < q_F, \varepsilon, Z_0>$

$<q_0, aaaaabbbbb, Z_0> \vdash < q_1, aaaabbbbb, Z_0> \vdash < q_0, aaabbbbb, AZ_0> \vdash < q_1, aabbbbb, AZ_0> \vdash$
$< q_0, abbbbb, AAZ_0> \vdash < q_1, bbbbb, AAZ_0> \vdash < q_2, bbbb, AZ_0> \vdash < q_3, bbb, AZ_0> \vdash < q_4, bb, AZ_0> \vdash$
$< q_2, b, Z_0> \vdash < q_3, \varepsilon, Z_0> \vdash < q_F, \varepsilon, Z_0>$

Exercise 2

1. write pre- and post-conditions specifying the requirements of the above described procedure; assume that indices of the arrays start at 0;

precondition

$nIn > 1 \wedge a[0] > 0 \wedge \ \forall\, i \,(\, 0 < i < nIn \rightarrow \exists k \,(\, k > 1 \ \wedge\ a[i] = k \cdot a[i\text{-}1] \ ) \ )$

postcondition

$out[0] = in[0] \wedge \ \forall\, i \,(\, 0 < i < nIn \rightarrow out[i] = in[i] \,/\, in[i\text{-}1] \ \ )$

2. for each of the following input/output data say if the procedure *ratios* satisfies the specification if it computes the given value of the *out* array when it receives the given input parameter values; please provide suitable justifications for your answers.

   a) *nIn*=5, *in*=[2, 7, 9, 11, 6], *out*=[2, 9, 7, 13, 8] yes, precondition is not satisfied

   b) *nIn*=5, *in*=[2, 6, 12, 48, 96], *out*=[2, 3, 2, 3, 2] no, precondition is satisfied, postcondition is not, because $48 = 4{\cdot}12$

   c) *nIn*=5, *in*=[2, 6, 12, 48, 96], *out*=[2, 3, 2, 4, 2] yes, both precondition and postcondition are satisfied

Exercise 3

Let us consider a fixed finite set *S* of natural numbers.

1. Is the set *S* ', that includes exactly the natural numbers that are multiple of all elements of *S*, decidable?

   Yes, because for any natural number it is possible to determine, by a simple procedure that always terminates, if it is multiple of all numbers in *S*.

2. Is the set *S* " of Turing machines that, starting the computation with an empty input tape, *do not* print all the numbers that belong to *S* decidable?

   No, as a consequence of the Rice theorem.

3. Is the set *S* " of Turing machines that, starting the computation with an empty input tape, *do not* print all the numbers of *S* semidecidable?

   First, notice that *S* " is not decidable. Then notice that the set ¬*S* ", the complement of *S* ", is the set of the TM that print all the elements of *S* (and possibly other ones). The set ¬*S* " is semidecidable: any given TM can be executed with an empty input tape: if it eventually prints all elements of *S* then the given TM is established to belong to the set ¬*S* ". Otherwise the simulation never terminates and no answer is provided. Therefore, since the set ¬*S* " is semidecidable, the set the set *S* " cannot be semidecidable, otherwise it would also be decidable.

Exercise 4

Let us assume that the variables of this program are stored in the RAM memory as follows

n $\Rightarrow$ M[1]
ev $\Rightarrow$ M[2]

With the constant cost criterion the time complexity class is $\Theta(n)$, because the loop is executed n times; the space complexity class is $\Theta(1)$, because a constant amount of memory is used.

With the logarithmic cost criterion, the space complexity class is $\Theta(\log 2^n) = \Theta(n)$ because the value 2^n must be stored and no value greater than 2^n is ever computed nor stored.

Concerning the time complexity class, the dominating factor in the complexity estimation derives from the execution of the code fragment **`ev = 2*ev`**. This will be translated into a sequence of RAM instructions as shown here below, where the cost of each instruction, at the i-th iteration of the loop, is also reported.

LOAD 2	$\log 2 + \log 2^{i-1}$
MUL =2	$\log 2 + \log 2 + \log 2^{i-1}$
STORE 2	$\log 2 + \log 2^{i}$

The overall cost is therefore $T(n) \approx \sum_{i=1}^{n}(k + 3 \log 2^{i}) \approx k\,n + 3/2\ n\,(n+1)$

so that $T(n) \in \Theta(n^2)$.

If we adopt as parameter the input ***size***, that is, the length x of the string that encodes, in a standard way, the value of n, then we have $x = \log n$, hence $n = 2^x$, and the following figures:

With the constant criterion, space is $\Theta(1)$ and time is $\Theta(2^x)$.

With the logarithmic criterion, space is $\Theta(2^x)$ and time is $\Theta(2^{2x})$.