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Super-leading logarithms in non-global observables in QCD?

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Abstract: We reconsider the calculation of a non-global QCD observable and find the possible breakdown of QCD coherence. This breakdown arises as a result of wide angle soft gluon emission developing a sensitivity to emission at small angles and it leads to the appearance of super-leading logarithms. We use the 'gaps between jets' cross-section as a concrete example and illustrate that the new logarithms are intimately connected with the presence of Coulomb gluon contributions. We present some rough estimates of their potential phenomenological significance.

Contents

1. Introduction

The summation of single logarithmic effects in QCD observables arising as a consequence of 'wide angle' soft gluon emission has a long history [1]–[22], with the discovery of non-global logarithms providing a recent highlight [8][9]. In this paper, we wish to report the possible emergence of a new class of 'super-leading' logarithms which could arise in general non-global observables. We refer to the logarithms as super-leading since they are formally more important than the 'leading logarithmic' summations that have hitherto been performed. The fact that these new contributions first arise at quite a high order in the perturbative expansion in processes involving at least four external coloured particles and are subleading in the number of colours N may account for their lying undiscovered until now. Their origin is related to the non-Abelian Coulomb phase terms which are present in the colour evolution.

At the present stage in our understanding, we are not able to claim strictly to have proven the existence of super-leading logarithms and the corresponding breakdown of QCD coherence that such logarithms would imply. As we shall see, the superleading logarithms emerge under the assumption that successive emissions can be ordered in transverse momentum and we have not proven this[1]. We do however wish to stress that the failure of k_T ordering would itself be of significant interest.

[1]We expect similar logarithms to emerge using other ordering variables although it is possible that the coefficient of the superleading logarithm may differ.

The paper is organized as follows. Throughout we shall focus upon one particular non-global observable, namely the 'gaps between jets' cross-section, although our conclusions are clearly more general. This is the cross-section for producing a pair of high transverse momentum jets (Q) with a restriction on the transverse momentum of any additional jets radiated in between the two leading jets, i.e. $k_T < Q_0$ for emissions in the gap region. This process has been much studied in the literature [10]–[14] and has been measured experimentally [23]–[27]. In the following section we explain how to sum the logarithms which arise as a result of soft gluon corrections to the hard scattering. We explain how the non-global nature of the observable affects the summation and in particular how it necessitates the summation over real and virtual soft gluon emissions outside of the region between the jets. We organize our calculation in terms of the number of gluons which lie outside of the gap region and compute the contribution to the cross-section which arises from one emission (real or virtual) outside of the gap. In Section 3, we uncover the super-leading logarithmic structure. We show that it is intimately connected with the imaginary ($i\pi$) terms which are present in the soft gluon evolution due to the exchange of Coulomb gluons, and that it is subleading in N. We show also that the super-leading logarithms can be seen to arise as a result of the breakdown of the 'plus prescription' in the evolution of radiation which is collinear with either of the incoming parton legs above the scale Q_0 because, although the real and virtual parts (at $z < 1$ and $z = 1$ respectively) are equal and opposite, their subsequent evolution down to the scale Q_0 is not. In Section 4 we present some numerical results.

2. Some history

In the original calculations of the gaps between jets cross-section [10][11], all those terms $\sim \alpha_s^n \ln^n(Q/Q_0)$ that can be obtained by dressing the primary $2 \to 2$ scattering in all possible ways with soft virtual gluons were summed. The restriction to soft gluons implies the use of the eikonal approximation and greatly simplifies the calculations. For the validity of the eikonal approximation, it is assumed that all collinear radiation can be summed inclusively and hence that any collinear logarithms in Q/Q_0 can be absorbed into the incoming parton density functions. We shall later question this assumption but for now we assume its validity.

We shall focus our attention upon quark-quark scattering: the colour structure is simpler and all of the key ideas are present. In this case, the re-summed scattering amplitude can be written[2]

$$\mathbf{M}(Q_0) = \exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\,\mathbf{\Gamma}\right)\mathbf{M}(Q), \tag{2.1}$$

where [3][7]

$$\mathbf{\Gamma} = \begin{pmatrix} \frac{N^2-1}{4N}\rho(Y,\Delta y) & \frac{N^2-1}{4N^2}i\pi \\ i\pi & -\frac{1}{N}i\pi + \frac{N}{2}Y + \frac{N^2-1}{4N}\rho(Y,\Delta y) \end{pmatrix} \tag{2.2}$$

[2]We neglect the running of the strong coupling throughout this paper although it is straightforward to re-instate it.

is the matrix which tells us how to attach a soft gluon to a primary four-quark hard scattering. $\mathbf{\Gamma}$ is defined in the t-channel (singlet-octet) basis where

$$\sigma = \mathbf{M}^\dagger \mathbf{S}_V \mathbf{M} \tag{2.3}$$

is the scattering cross-section and

$$\mathbf{M} = \begin{pmatrix} M^{(1)} \\ M^{(8)} \end{pmatrix} \text{ and } \mathbf{S}_V = \begin{pmatrix} N^2 & 0 \\ 0 & \frac{N^2-1}{4} \end{pmatrix}. \tag{2.4}$$

In Eq.(2.2), Y is the size of the rapidity region over which emission with $k_T > Q_0$ is vetoed and Δy is the distance between the two jet centres (in the most commonly used event definition, $\Delta y = Y + 2R$ where R is the radius of the jet cone) and

$$\rho(Y, \Delta y) = \log \frac{\sinh(\Delta y/2 + Y/2)}{\sinh(\Delta y/2 - Y/2)} - Y. \tag{2.5}$$

In this basis we have

$$\mathbf{M}(Q) \equiv \mathbf{M}_0 = \sqrt{\frac{4}{N^2-1}} \begin{pmatrix} 0 \\ \sqrt{\sigma_{\mathrm{born}}} \end{pmatrix}. \tag{2.6}$$

Equation (2.2) quantifies the effect of adding a soft and virtual gluon in all possible ways to a four-quark matrix element. Within the eikonal approximation one obtains contributions from two distinct regions of the loop integral: the first, sometimes denoted the 'eikonal gluon' contribution [19], comes from the pole at $k^2 = 0$. The residue of this pole is identical, but with opposite sign, to the phase space integral for the emission of a real soft gluon. In particular, it makes sense to ascribe definite values of rapidity, azimuth and transverse momentum to the eikonal virtual gluon. Of these, Eq.(2.2) includes only those whose rapidity lies within the gap region (i.e. within the region between the two hard jets), the contributions from outside the gap region cancelling with corresponding real emission contributions. The second region of the loop integral, sometimes denoted the 'Coulomb gluon' contribution [19, 28, 29], comes from a pole at which one of the emitting partons is on-shell, which pinches the contour integrals at the point at which the gluon's positive and negative light-cone momenta are zero, corresponding to a space-like gluon with only transverse momentum. Its residue is purely imaginary and only non-zero if both the emitting partons are in the final state or both in the initial state, giving the $i\pi$ terms in the evolution matrix. Strictly speaking, for the Coulomb gluons, the region of k_T below Q_0 must be included, however this region only contributes a pure phase which cancels in observables, as shown explicitly in [19].

At first sight, one may suppose that the evolution just described correctly captures all of the leading logarithms. This would indeed be so if it were the case that the contributions arising from real gluon emission always cancel with a corresponding virtual emission. In this case, the only region in phase-space where the real-virtual cancellation would not occur would be the region where real emissions are forbidden, i.e. within the gap region and with transverse momentum above Q_0. This may seem a straightforward consequence of the Bloch-Nordsieck Theorem however it is not.

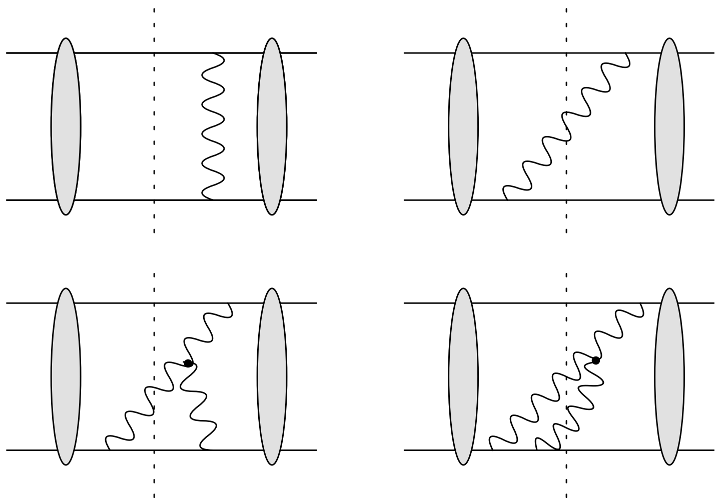

Figure 1: Illustrating the cancellation (and miscancellation) of soft gluon corrections.

Although it is true that the real and virtual contributions cancel exactly at the cross-section level in the case where we dress a hard scattering amplitude with a single soft gluon, it is not true that the cancellation survives subsequent dressing with additional soft gluons. This is illustrated in Fig.1. The upper two panes contain typical diagrams where a soft gluon dresses a $2 \to 2$ hard scattering. At the cross-section level such single soft gluon corrections exactly cancel each other since all cuts through a particular uncut diagram sum to zero. Now consider the lower two panes. To capture the leading logarithms we assume that it is appropriate to order strongly the transverse momenta of successive gluon emissions as one moves away from the hard scatter. If we first consider a real gluon emission above Q_0 then it must lie outside of the gap region. We should then consider virtual corrections to this five parton amplitude. Bloch-Nordsieck guarantees only that it is true that those virtual corrections which lie outside the gap region in rapidity, or have transverse momentum below Q_0, will be exactly cancelled by the corresponding real emission graphs. Virtual corrections to the five-parton amplitude which lie above Q_0 and are within the gap region have nothing to cancel against, for their corresponding real emissions are forbidden by the definition of the observable. These virtual corrections embody the fact that any emission outside of the gap region is forbidden from radiating back into the gap with $k_T > Q_0$. Thus we see that the non-global nature of the observable has prevented the soft gluon cancellation which is necessary in order that Eq.(2.1) should be the whole story.

It is therefore necessary to include the emission of any number of soft gluons outside the gap region (real and virtual) dressed with any number of virtual gluons within the gap region; all gluons having transverse momentum above Q_0. Clearly it is a formidable challenge to sum all of the leading logarithms, mainly because of the complicated colour structure of an amplitude with a large number of final state gluons. Progress has been made, working within the large N approximation [13, 14]. In fact a great deal of interest has been generated [30]–[32] by the fact that, in this large N limit, the evolution equation for the out-of-gap gluons [17] maps onto the Kovchegov equation for non-linear corrections to the BFKL equation [33]–[35]. Here, however, we prefer to keep the exact colour structure but instead we only compute the cross-section for one gluon outside of the gap region. This can be viewed as the first term in an expansion in the number of out-of-gap gluons.

2.1 One emission outside of the gap

Thus motivated, we now compute the cross-section for emitting one soft gluon outside of the gap region dressed with any number of virtual gluons. There are two new ingredients compared to the four-parton case:

1. We need to consider the emission of a real gluon off any one of the four external quarks. The corresponding five parton amplitude needs four colour basis states and hence the action of emitting a real gluon from the four-quark amplitude will be described using a 4x2 matrix, $\varepsilon^\mu \mathbf{D}_\mu$, where ε^μ denotes the gluon polarisation vector.
2. We need also to determine the 4x4 matrix $\mathbf{\Lambda}$ which acts on the five particle amplitude in order to account for the dressing with a virtual soft gluon.

The real emission contribution is obtained from the four-quark amplitude $\mathbf{M}$ via[3]

$$\mathbf{M}_R^\mu(k_T) = \mathbf{D}^\mu \mathbf{M}(k_T) \tag{2.7}$$

with

$$\mathbf{D}^\mu = \begin{pmatrix} \frac{1}{2}(-h_1^\mu - h_2^\mu + h_3^\mu + h_4^\mu) & \frac{1}{4N}(-h_1^\mu - h_2^\mu + h_3^\mu + h_4^\mu) \\ 0 & \frac{1}{2}(-h_1^\mu - h_2^\mu + h_3^\mu + h_4^\mu) \\ \frac{1}{2}(-h_1^\mu + h_2^\mu + h_3^\mu - h_4^\mu) & \frac{1}{4N}(h_1^\mu - h_2^\mu - h_3^\mu + h_4^\mu) \\ 0 & \frac{1}{2}(-h_1^\mu + h_2^\mu - h_3^\mu + h_4^\mu) \end{pmatrix} \tag{2.8}$$

and the eikonal factors are

$$h_i^\mu = \frac{1}{2} k_T \frac{p_i^\mu}{p_i \cdot k}, \tag{2.9}$$

where k is the gluon's four-momentum and p_i are the external quark momenta. In particular, we choose

$$\begin{aligned} p_1 &= \frac{\sqrt{s}}{2}\,(1;0,0,1)\,, \\ p_2 &= \frac{\sqrt{s}}{2}\,(1;0,0,-1)\,, \\ p_3 &= Q\,(\cosh(\Delta y/2);0,1,\sinh(\Delta y/2))\,, \\ p_4 &= Q\,(\cosh(\Delta y/2);0,-1,-\sinh(\Delta y/2))\,, \\ k &= k_T\,(\cosh y;\sin\phi,\cos\phi,\sinh y)\,. \end{aligned} \tag{2.10}$$

[3]For notational convenience, we suppress the dependence on the rapidity and azimuth of the emitted gluon.

Eq.(2.8) is defined in the 't-channel' basis, i.e. the four basis vectors for the process $q_i q_j \rightarrow q_k q_l g_a$ are

$$\mathbf{C}_1 = T^a_{ki}\delta_{lj} + T^a_{lj}\delta_{ki}, \tag{2.11}$$
$$\mathbf{C}_2 = T^b_{ki}T^c_{lj}\ d^{abc}, \tag{2.12}$$
$$\mathbf{C}_3 = T^a_{ki}\delta_{lj} - T^a_{lj}\delta_{ki}, \tag{2.13}$$
$$\mathbf{C}_4 = T^b_{ki}T^c_{lj}\ if^{abc}. \tag{2.14}$$

The cross-section for one real gluon emission off the four-quark amplitude $\mathbf{M}$ is then given by[4]

$$\sigma_R = -\frac{2\alpha_s}{\pi}\int\frac{dk_T}{k_T}\int\frac{dy\ d\phi}{2\pi}(\mathbf{M}^\dagger\mathbf{D}^\dagger_\mu\mathbf{S}_R\mathbf{D}^\mu\mathbf{M}) \tag{2.15}$$

where

$$\mathbf{S}_R = \begin{pmatrix} N(N^2-1) & 0 & 0 & 0 \\ 0 & \frac{1}{4N}(N^2-1)(N^2-4) & 0 & 0 \\ 0 & 0 & N(N^2-1) & 0 \\ 0 & 0 & 0 & \frac{1}{4}N(N^2-1) \end{pmatrix}. \tag{2.16}$$

One can readily check that the single soft gluon cancellation is assured since[5]

$$\int_{\text{gap}}\frac{dy\ d\phi}{2\pi}\mathbf{D}^\dagger_\mu\mathbf{S}_R\mathbf{D}^\mu + \mathbf{\Gamma}^\dagger\mathbf{S}_V + \mathbf{S}_V\mathbf{\Gamma} = \mathbf{0}. \tag{2.17}$$

The subsequent evolution of this five parton amplitude is determined by $\mathbf{\Lambda}$:

$$\mathbf{M}_R(Q_0) = \exp\left(-\frac{2\alpha_s}{\pi}\int_{Q_0}^{k_T}\frac{dk'_T}{k'_T}\ \mathbf{\Lambda}\right)\mathbf{M}_R(k_T), \tag{2.18}$$

[4]The minus sign arises after the sum over gluon polarisations using $\sum\varepsilon^*_\mu\varepsilon_\nu = -g_{\mu\nu}$.

[5]The cancellation occurs already at the level of the integrand.

where the evolution matrix was computed in [22] to be

$$
\begin{aligned}
\mathbf{\Lambda} = &\begin{pmatrix}
\frac{N}{4}(Y-i\pi)+\frac{1}{2N}i\pi & \left(\frac{1}{4}-\frac{1}{N^2}\right)i\pi & -\frac{N}{4}s_y Y & 0 \\
i\pi & \frac{N}{4}\left(2Y-i\pi\right)-\frac{3}{2N}i\pi & 0 & 0 \\
-\frac{N}{4}s_y Y & 0 & \frac{N}{4}(Y-i\pi)-\frac{1}{2N}i\pi & -\frac{1}{4}i\pi \\
0 & 0 & -i\pi & \frac{N}{4}\left(2Y-i\pi\right)-\frac{1}{2N}i\pi
\end{pmatrix} \\
&+\begin{pmatrix}
N & 0 & 0 & 0 \\
0 & N & 0 & 0 \\
0 & 0 & N & 0 \\
0 & 0 & 0 & N
\end{pmatrix}\frac{1}{4}\rho(Y,2\left|y\right|) \\
&+\begin{pmatrix}
C_F & 0 & 0 & 0 \\
0 & C_F & 0 & 0 \\
0 & 0 & C_F & 0 \\
0 & 0 & 0 & C_F
\end{pmatrix}\frac{1}{2}\rho(Y,\Delta y) \\
&+\begin{pmatrix}
-\frac{N}{4} & 0 & -\frac{N}{4}s_y & \frac{1}{4}s_y \\
0 & -\frac{N}{4} & 0 & \frac{N}{4}s_y \\
-\frac{N}{4}s_y & 0 & -\frac{N}{4} & -\frac{1}{4} \\
s_y & \left(\frac{N}{4}-\frac{1}{N}\right)s_y & -1 & -\frac{N}{4}
\end{pmatrix}\frac{1}{2}\lambda
\end{aligned}
\tag{2.19}
$$

with

$$
\lambda = \frac{1}{2}\log\frac{\cosh(\Delta y/2+|y|+Y)-s_y\cos(\phi)}{\cosh\left(\Delta y/2+|y|-Y\right)-s_y\cos(\phi)}-Y, \tag{2.20}
$$

$$
s_y = \mathrm{sgn}(y). \tag{2.21}
$$

We now have the machinery to state the all-orders cross-section for one gluon outside of the gap. For the real emission we have

$$
\begin{aligned}
\sigma_R = -\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\int\limits_{\mathrm{out}}\frac{dy\ d\phi}{2\pi} \\
\mathbf{M}_0^\dagger \exp\left(-\frac{2\alpha_s}{\pi}\int\limits_{k_T}^{Q}\frac{dk_T'}{k_T'}\mathbf{\Gamma}^\dagger\right)\mathbf{D}_\mu^\dagger \exp\left(-\frac{2\alpha_s}{\pi}\int\limits_{Q_0}^{k_T}\frac{dk_T'}{k_T'}\mathbf{\Lambda}^\dagger\right)\mathbf{S}_R \\
\exp\left(-\frac{2\alpha_s}{\pi}\int\limits_{Q_0}^{k_T}\frac{dk_T'}{k_T'}\mathbf{\Lambda}\right)\mathbf{D}^\mu \exp\left(-\frac{2\alpha_s}{\pi}\int\limits_{k_T}^{Q}\frac{dk_T'}{k_T'}\mathbf{\Gamma}\right)\mathbf{M}_0
\end{aligned}
\tag{2.22}
$$

and for a virtual emission

$$\sigma_V = -\frac{2\alpha_s}{\pi} \int_{Q_0}^{Q} \frac{dk_T}{k_T} \int\limits_{\text{out}} \frac{dy\; d\phi}{2\pi}$$
$$\left[\mathbf{M}_0^\dagger \exp\left(-\frac{2\alpha_s}{\pi} \int\limits_{Q_0}^{Q} \frac{dk'_T}{k'_T} \mathbf{\Gamma}^\dagger \right) \mathbf{S}_V \right.$$
$$\left. \exp\left(-\frac{2\alpha_s}{\pi} \int\limits_{Q_0}^{k_T} \frac{dk'_T}{k'_T} \mathbf{\Gamma} \right) \boldsymbol{\gamma} \exp\left(-\frac{2\alpha_s}{\pi} \int\limits_{k_T}^{Q} \frac{dk'_T}{k'_T} \mathbf{\Gamma} \right) \mathbf{M}_0 + \text{c.c.} \right] \tag{2.23}$$

where the matrix $\boldsymbol{\gamma}$ adds the virtual soft gluon which is to lie outside of the gap. It differs from $\mathbf{\Gamma}$ in the fact that the rapidity integral is left undone and in that it is purely real since the imaginary Coulomb terms have already been entirely accounted for by the evolution matrix $\mathbf{\Gamma}$. We have that

$$\boldsymbol{\gamma} = \frac{1}{2} \begin{pmatrix} \frac{N^2-1}{2N}\left(\omega_{13}+\omega_{24}\right) & \frac{N^2-1}{4N^2}\left(-\omega_{12}-\omega_{34}+\omega_{14}+\omega_{23}\right) \\ -\omega_{12}-\omega_{34}+\omega_{14}+\omega_{23} & \frac{N}{2}(\omega_{14}+\omega_{23}) - \frac{1}{2N}\left(\omega_{13}+\omega_{24}\right) \\ & \quad +\frac{1}{N}\left(\omega_{12}+\omega_{34}-\omega_{14}-\omega_{23}\right) \end{pmatrix} \tag{2.24}$$

where

$$\omega_{ij} \equiv 2h_i \cdot h_j = \frac{1}{2} k_T^2 \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}. \tag{2.25}$$

3. Super-leading logarithms

In the next section we shall present some numerical results obtained by evaluating the sum of equations (2.22) and (2.23) but first we shall take a closer look at the singularity structure of each. We expect both to contain divergences in the formal limit that the out-of-gap gluon becomes collinear with any of the external quarks and we might suppose that these divergences always cancel. Such cancellations are to be expected as a result of QCD coherence which informs us that large angle soft gluon emission should not be able to resolve emissions at small angles. Let us first explore emissions that are collinear with an outgoing quark.

3.1 Final state collinear emission

We state the result first: emissions collinear to an outgoing quark do cancel between the real and virtual corrections. To see this it is better to shift to a colour basis in which the evolution matrix $\mathbf{\Lambda}$ is block diagonal. The relevant results are summarized in Appendix A. Let's consider the particular case in which the emission is collinear with p_3 (i.e. $y > 0$). In this case, Eq.(2.24) simplifies to

$$\boldsymbol{\gamma} \rightarrow \frac{N^2-1}{4N}\, \omega_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $\omega_3 = \omega_{13} = \omega_{23} = \omega_{34}$ are the collinear divergent eikonal factors. Similarly, $\mathbf{D}^\mu$ can be much simplified by keeping only those terms that will induce the collinear divergence, i.e.

$$\mathbf{D}^\mu = \sqrt{\frac{N^2-1}{2N}}(h_3^\mu - h^\mu)\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.2}$$

where we have taken $h_1 = h_2 = h_4 = h$. Using the fact that in the collinear limit we should take $\phi = 0$, $y = \Delta y/2$ and hence $\lambda = \rho(Y, 2\,|y|) = \rho(Y, \Delta y)$, the evolution is described by

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 + \frac{N+1}{4}\rho(Y,\Delta y) & 0 & \mathbf{0} \\ 0 & \lambda_2 + \frac{N-1}{4}\rho(Y,\Delta y) & \\ \mathbf{0} & & \mathbf{\Gamma} \end{pmatrix}, \tag{3.3}$$

where the upper left block is unimportant for the evolution because of the structure of $\mathbf{D}^\mu$ (see Eq.(3.9) for the definition of λ_i). The final ingredient is the matrix $\mathbf{S}_R$ which has the property that its bottom right-hand entries coincide with the matrix $\mathbf{S}_V$, i.e.

$$\mathbf{S}_R = \begin{pmatrix} \frac{N^2}{2}\frac{N+1}{N+2} & 0 & \mathbf{0} \\ 0 & \frac{N^2}{2}\frac{N-1}{N-2} & \\ \mathbf{0} & & \mathbf{S}_V \end{pmatrix}. \tag{3.4}$$

Again the upper left block is not important for the argument here. Hence in this collinear limit, the evolution of the five-parton amplitude collapses into the evolution of the four-parton amplitude and we are guaranteed a complete cancellation between the real and virtual emissions, i.e. since $(h_3 - h)^2 = -\omega_3$ it follows that

$$\mathbf{D}^{\mu\dagger}(\mathbf{\Lambda}^\dagger)^{n-m}\mathbf{S}_R\mathbf{\Lambda}^m\mathbf{D}_\mu + (\mathbf{\Gamma}^\dagger)^{n-m}\mathbf{S}_V\mathbf{\Gamma}^m\boldsymbol{\gamma} + \boldsymbol{\gamma}^\dagger(\mathbf{\Gamma}^\dagger)^{n-m}\mathbf{S}_V\mathbf{\Gamma}^m = \mathbf{0}. \tag{3.5}$$

3.2 Initial state collinear emission

Now we turn our attention to the case where the out-of-gap gluon is collinear with an incoming quark. It is perhaps worth recalling that by ‘collinear’ we mean that the rapidity is tending to infinity and $k_T > Q_0$. Arbitrarily, we choose the emission to be collinear with p_1 (i.e. $y > 0$). Now

$$\boldsymbol{\gamma} \to \frac{N^2-1}{4N}\,\omega_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.6}$$

where $\omega_1 = \omega_{12} = \omega_{13} = \omega_{14}$ are the collinear divergent eikonal factors in this case. The real emission matrix is not so simple this time:

$$\mathbf{D}^\mu = \sqrt{\frac{N^2-1}{2N}}(h^\mu - h_1^\mu)\begin{pmatrix} 0 & \frac{1}{2}\frac{N+2}{N+1} \\ 0 & \frac{1}{2}\frac{N-2}{N-1} \\ 1 & 0 \\ 0 & -\frac{1}{N^2-1} \end{pmatrix}. \tag{3.7}$$

The evolution matrix $\mathbf{\Lambda}$ is slightly different too since $\phi = 0$, $y \to \infty$ in this limit and hence $\lambda = \rho(Y, 2\,|y|) = 0$, i.e.

$$\boldsymbol{\Lambda} = \begin{pmatrix} \begin{matrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma} \end{pmatrix}, \tag{3.8}$$

where

$$\begin{aligned} \lambda_1 &= \frac{NY}{2} + \frac{N-1}{2N} i\pi + \frac{N^2-1}{4N}\rho(Y,\Delta y), \\ \lambda_2 &= \frac{NY}{2} - \frac{N+1}{2N} i\pi + \frac{N^2-1}{4N}\rho(Y,\Delta y). \end{aligned} \tag{3.9}$$

Now because of the form of $\mathbf{D}^\mu$ the upper left blocks of both $\boldsymbol{\Lambda}$ and $\mathbf{S}_R$ play a role. Clearly any cancellation between the real and virtual parts is going to occur only for particular forms of these blocks. Remarkably, the miscancellation lies wholly in the hands of the $i\pi$ terms in the evolution matrices, for if we artificially switch these terms off one finds that

$$\boldsymbol{\Lambda} \underset{i\pi\to 0}{=} \frac{NY}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{N^2-1}{4N}\rho(Y,\Delta y)\mathbf{1} \tag{3.10}$$

and

$$\boldsymbol{\Gamma} \underset{i\pi\to 0}{=} \frac{NY}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{N^2-1}{4N}\rho(Y,\Delta y)\mathbf{1}. \tag{3.11}$$

This diagonal and real structure is sufficient for the cancellation to operate, i.e.

$$\mathbf{D}^{\mu\dagger}(\boldsymbol{\Lambda}^\dagger)^{n-m}\mathbf{S}_R\boldsymbol{\Lambda}^m\mathbf{D}_\mu + (\boldsymbol{\Gamma}^\dagger)^{n-m}\mathbf{S}_V\boldsymbol{\Gamma}^m\boldsymbol{\gamma} + \boldsymbol{\gamma}^\dagger(\boldsymbol{\Gamma}^\dagger)^{n-m}\mathbf{S}_V\boldsymbol{\Gamma}^m \underset{i\pi\to 0}{=} \mathbf{0}. \tag{3.12}$$

The $i\pi$ terms in the evolution arising from Coulomb gluons generally destroy the cancellation between real and virtual emissions in the case that the out-of-gap gluon is collinear with one of the incoming partons. In more familiar terms, we appear to have discovered that the 'plus prescription' employed in the splitting functions for collinear evolution fails for emissions with transverse momentum above Q_0. It is particularly interesting that the miscancellation occurs only once one includes the imaginary parts in the evolution matrices.

As it stands we have a divergence arising from the integral over the rapidity of the out-of-gap gluon:

$$\int_{\text{out}} \frac{dy\ d\phi}{2\pi}\ \omega_1 \sim y_{\max} - \frac{Y}{2}. \tag{3.13}$$

In the soft approximation the integral is divergent which is the signal that we need to go beyond the soft approximation when considering these emissions. Strictly speaking we ought to work in the collinear (but not soft) approximation which means that the integral over rapidity ought to be replaced by

$$\int d^2k_T \int_{\text{out}} dy \quad \left.\frac{d\sigma}{dy d^2k_T}\right|_{\text{soft}} \to \int d^2k_T \left[\int^{y_{\max}} dy \quad \left.\frac{d\sigma}{dy d^2k_T}\right|_{\text{soft}} + \int_{y_{\max}}^{\infty} dy \quad \left.\frac{d\sigma}{dy d^2k_T}\right|_{\text{collinear}} \right]. \tag{3.14}$$

In this equation $y_{\max}$ is a matching point between the regions in which the soft and collinear approximations are used. If $y_{\max}$ is in the region in which both approximations are valid the dependence on it should cancel in the sum of the two terms. Now we know that

$$\int_{y_{\max}}^{\infty} dy \;\; \left.\frac{d\sigma}{dyd^2k_T}\right|_{\text{collinear}} = \int_{y_{\max}}^{\infty} dy \;\; \left(\left.\frac{d\sigma_{\text{R}}}{dyd^2k_T}\right|_{\text{collinear}} + \left.\frac{d\sigma_{\text{V}}}{dyd^2k_T}\right|_{\text{collinear}} \right) \tag{3.15}$$

where the contribution due to real gluon emission is

$$\begin{aligned}\int_{y_{\max}}^{\infty} dy \;\; \left.\frac{d\sigma_{\text{R}}}{dyd^2k_T}\right|_{\text{collinear}} &= \int_0^{1-\delta} dz\frac{1}{2}\left(\frac{1+z^2}{1-z}\right)\frac{q(x/z,\mu^2)}{q(x,\mu^2)}A_{\text{R}} \\ &= \int_0^{1-\delta} dz\frac{1}{2}\left(\frac{1+z^2}{1-z}\right)\left(\frac{q(x/z,\mu^2)}{q(x,\mu^2)}-1\right)A_{\text{R}} + \int_0^{1-\delta} dz\frac{1}{2}\frac{1+z^2}{1-z}A_{\text{R}}\end{aligned} \tag{3.16}$$

and the contribution due to virtual gluon emission is

$$\int_{y_{\max}}^{\infty} dy \;\; \left.\frac{d\sigma_{\text{V}}}{dyd^2k_T}\right|_{\text{collinear}} = \int_0^{1-\delta} dz\frac{1}{2}\left(\frac{1+z^2}{1-z}\right)A_{\text{V}}. \tag{3.17}$$

In Eq.(3.16), $q(x,\mu^2)$ is the parton distribution function for a quark in a hadron at scale μ^2 and momentum fraction x. The factors A_{R} and A_{V} contain the z independent factors which describe the soft gluon evolution and the upper limit on the z integral is fixed since we require $y > y_{\max}$[6]:

$$\delta \approx \frac{k_T}{Q}\exp\left(y_{\max} - \frac{\Delta y}{2}\right). \tag{3.18}$$

We have already established that $A_{\text{R}} + A_{\text{V}} \neq 0$ due to Coulomb gluon contributions to the evolution. If it were the case that $A_{\text{R}} + A_{\text{V}} = 0$ then the virtual emission contribution would cancel identically with the corresponding term in the real emission contribution leaving behind a term regularised by the 'plus prescription' (since we can safely take $\delta \to 0$ in the first term of Eq.(3.16)). This term could then be absorbed into the evolution of the incoming quark parton distribution function by choosing the factorisation scale to equal the jet scale Q.

The miscancellation therefore induces an additional contribution of the form

$$\int_0^{1-\delta} dz\frac{1}{2}\left(\frac{1+z^2}{1-z}\right)(A_{\text{R}}+A_{\text{V}}) = \ln\left(\frac{1}{\delta}\right)(A_{\text{R}}+A_{\text{V}}) + \text{subleading} \tag{3.19}$$

$$\approx \left(-y_{\max} + \frac{\Delta y}{2} + \ln\left(\frac{Q}{k_T}\right)\right)(A_{\text{R}}+A_{\text{V}}). \tag{3.20}$$

[6] The approximation arises since we assume for simplicity that Δy is large and δ is small. This approximation does not affect the leading behaviour and can easily be made exact if necessary.

Provided we stay within the soft-collinear region in which both the soft and collinear approximations are valid, the $y_{\max}$ dependence will cancel with that coming from the soft contribution in Eq.(3.14) leaving only the logarithm. The leading effect of treating properly the collinear region is therefore simply to introduce an effective upper limit to the integration over rapidity in Eq.(3.13). More precisely, we can therefore estimate the leading behaviour simply by setting $y_{\max} = \Delta y/2 + \ln(Q/k_T)$ in the soft integral, effectively including the entire soft-collinear region. We are left with

$$\frac{2\alpha_s}{\pi}\int_{Q_0}^{Q}\frac{dk_T}{k_T}\int\limits_{Y/2}^{\ln(Q/k_T)+\Delta y/2}\frac{dy\;d\phi}{2\pi}\,\omega_1 = \frac{2\alpha_s}{\pi}\frac{1}{2}\ln^2(Q/Q_0) + \text{subleading}. \tag{3.21}$$

This is the super-leading logarithm: the failure of the 'plus prescription' has resulted in the generation of an extra collinear logarithm. The implications for the gaps-between-jets cross-section are clear: collinear logarithms can be summed into the parton density functions only up to scale Q_0 and the logarithms in Q/Q_0 from further collinear evolution must be handled separately. Moreover, since we now have a source of double logarithms, the calculation of the single logarithmic series necessarily requires knowledge of the two-loop evolution matrices [36].

Indeed we appear to have uncovered a breakdown of QCD coherence: radiation at large angles does appear to be sensitive to radiation at low angles. However this striking conclusion was arrived at under the assumption that it is correct to order successive emissions in transverse momentum. Coherence indicates that one does not need to take too much care over the ordering variable, e.g. k_T, E and k_T^2/E are all equally good ordering variables but the super-leading logarithms arise counter to the expectations of coherence and in particular as a result of radiation which is both soft and collinear. It is therefore required to prove the validity of k_T ordering before we can claim without doubt the emergence of super-leading logarithms or confirm their size.

We note that the super-leading logarithm makes its appearance at the lowest possible order in the perturbative expansion, i.e. at order α_s^4 relative to the Born cross-section. More explicitly, the $O(\alpha_s)$ and $O(\alpha_s^2)$ corrections to the Born cross-section simply never involve more than one $i\pi$ term and hence any $i\pi$ terms must cancel since the cross-section is real. The first candidate order at which two factors of $i\pi$ can appear is therefore $O(\alpha_s^3)$. However, the addition of the gluon with the lowest k_T can never generate a net factor of $i\pi$ since any such factors must cancel between the two diagrams where the lowest k_T gluon lies either side of the cut. Hence we anticipate that the first super-leading logarithm makes a contribution

$$\sigma \sim \sigma_0\left(\frac{2\alpha_s}{\pi}\right)^4\ln^5\left(\frac{Q}{Q_0}\right)\pi^2 Y. \tag{3.22}$$

Note that for each factor of π we pay a price in colour (the leading contribution in colour goes like $(\alpha_s N)^n$). The factor of Y is from the rapidity volume of the in-gap gluon. To be a little more explicit, we now expand in α_s.

The lowest order contribution for a single emission outside of the gap (with $y > 0$) is

$$\sigma_{1,\mathrm{LO}} = \sigma_0 \left(\frac{2\alpha_s}{\pi}\right)^2 \frac{1}{8} \ln^2\left(\frac{Q}{Q_0}\right) \Bigg\{ 2Y \int\limits_{\mathrm{out}} dy \; \frac{d\phi}{2\pi}\, \omega_{24}$$
$$- \int\limits_{\mathrm{out}} dy \; \frac{d\phi}{2\pi} \rho(Y, 2\,|y|) \; \left[\left(N^2-2\right)(\omega_{23}+\omega_{14}) - \omega_{13} + 2(\omega_{12}+\omega_{34}) - \omega_{24}\right]$$
$$+ \int\limits_{\mathrm{out}} dy \; \frac{d\phi}{2\pi} \lambda \left[\left(N^2-2\right)\omega_{23} - \omega_{13} + 2\omega_{34}\right] \Bigg\}. \tag{3.23}$$

The dominant contributions at large enough Y come from emissions close to the edge of the gap. To see this we note that at large Y the integral over ω_{24} vanishes as $\exp(-2Y)$ and so the only significant contribution arises from the terms proportional to ρ and λ which are dominated by the region around $y = Y/2$.

The lowest order contribution that contains a super-leading logarithm is

$$\sigma_{1,\mathrm{SLL}} = -\sigma_0 \left(\frac{2\alpha_s}{\pi}\right)^4 \ln^5\left(\frac{Q}{Q_0}\right) \pi^2 Y \; \frac{(3N^2-4)}{480}. \tag{3.24}$$

Subsequent terms alternate in sign and are $\sim \alpha_s^n L^{n+1} \pi^2 N^2 Y (NY)^{n-4}$ for large N and Y.

Since we have only considered one emission out of the gap region, we should convince ourselves that there is no possibility that the new collinear logarithm cancels with a similar contribution from two (or more) emissions outside of the gap. We here present an argument which confirms that the lowest order (in α_s) super-leading logarithm has nothing to cancel against. As we have seen, this contribution occurs at order α_s^4 relative to the Born cross-section. We know that the gluon with the smallest k_T does not give any $i\pi$ term and we know that there is an exact cancellation if this gluon is outside of the gap. We also know that there is an exact cancellation if all $i\pi$ terms are zero. Since the cross-section is real, we must have an even number of $i\pi$ terms, which can in this lowest order case only be two. Pulling all this together, we therefore have four gluons of which the lowest k_T gluon must lie inside the gap and two are Coulomb gluons. Therefore we can only have, at most, one gluon outside of the gap.

Thus, we have shown that at order α_s^4 all contributions are of the type 'zero gluons outside the gap' or 'one gluon outside the gap' and we have explicitly computed these and know that there is no cancellation.

As we have already shown, the miscancellation is specifically related to the exchange of Coulomb gluons, since with the resulting $i\pi$ terms set to zero cancellation is restored. It is worth recalling the special role of Coulomb gluons in the proofs of factorization by Collins, Soper and Sterman [1],[37],[38]. They consider the exchange of potentially factorization-breaking soft gluons and show that the eikonal gluons cancel in the sum over cuts through a given diagram, while some Coulomb gluon terms remain uncancelled. Only after summing over all diagrams in which a Coulomb gluon is exchanged, in particular including diagrams in which it is attached to the hadron remnants, can the corresponding contribution be shown to cancel. In our case, since we consider a high-p_t process (the exchanged gluons we are interested in populate the strongly-ordered region $k_T \gg Q_0$ where we assume $Q_0 \gg \Lambda_{\rm QCD}$), emission from the hadron remnants is irrelevant (power-suppressed) and hence we have no *a priori* reason to assume that the Coulomb phase terms will cancel. It must be checked explicitly and in our case they do not.

4. Numerical results

In the following figures[7], we have computed the out-of-gap cross-section obtained by summing Eq.(2.22) and Eq.(2.23) each evaluated in the super-leading (soft and collinear) approximation. This amounts to setting all the $\omega_{ij} = 0$ except $\omega_{12} = \omega_{13} = \omega_{14} = 1$ (in the case $y > 0$) and $\rho(Y, 2\,|y|) = \lambda = 0$. In addition, the integral over rapidity is performed over an interval of size $\ln(Q/k_T)$ and we multiply by a factor of 2 to account for the possibility that the out-of-gap gluon can be either side of the gap. We refer to the cross-section thus computed as 'SLL' in all of the plots since it contains the complete super-leading contribution. For comparison, we also compute the sum of Eq.(2.22) and Eq.(2.23) without making the collinear approximation. In this case the integral over y is over the region $Y/2 < |y| < \Delta y/2 + \ln(Q/k_T)$ and we take $R = 1$. These cross-sections are labelled 'all' in the plots and they necessarily include a partial summation of the single logarithmic terms as well as the super-leading terms. Throughout we keep the strong coupling fixed at $\alpha_S = 0.15$ and our cross-sections are usually normalized to the fully resummed cross-section corresponding to zero gluons outside of the gap region, i.e. as determined by Eq.(2.1).

The plots in Fig.2 show the cross-section dependence upon $L = \ln(Q^2/Q_0^2)$ at two different values of Y whilst the dependence upon Y at two different values of L is illustrated in Fig.3. It seems that while the out-of-gap cross-section is not dominant anywhere it is also not negligible. This is of course already known: the non-global logarithms are generally significant. We can also see from these plots that the super-leading series is generally small relative to the 'all' result for $L \lesssim 4$, which indicates that the single logarithms are phenomenologically much more important than the formally super-leading logs at these values of L. Of course one should remember that our calculations are for the emission of one gluon outside the gap region and the full super-leading series requires the computation of any number of such gluons.

From a more theoretical perspective it is interesting to take a look at the cross-sections out to larger values of L and Y. In Fig.4 we show the cross-section out to large values of L. It is immediately striking that the cross-section asymptotes to a constant value, which implies that the out-of-gap cross-section is directly proportional to the in-gap cross-section at large L with a Y dependent prefactor. In Fig.5 we show how the large L behaviour of the cross-section varies with Y.

Fig.6 shows the dependence of the cross-section out to large Y. Note that this time we have normalized the cross-section by the square of the in-gap cross-section (and have set the Born cross-section equal to unity). The cross-section again saturates at large enough Y. Fig.7 shows how the large Y behaviour varies with L. We consider the fact that $\sigma_1 \sim -\sigma_0^2$

[7] We generically write the in-gap cross-section as σ_0 and the out-of-gap cross-section σ_1.

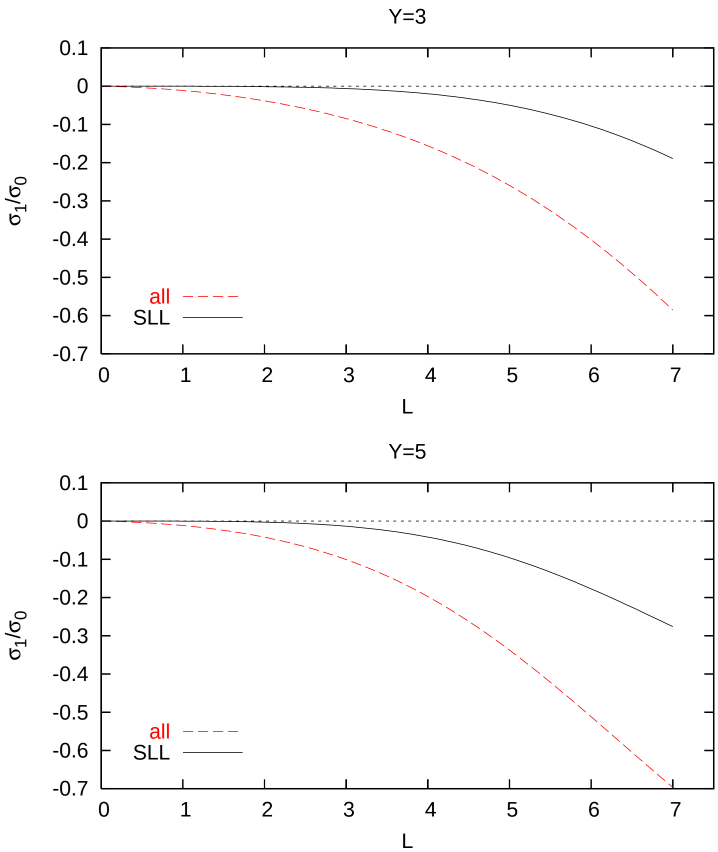

Figure 2: L dependence of the out-of-gap cross-section (normalized to the in-gap cross-section) at two different values of Y.

at large Y to reflect the deeper link which is known to exist between QCD dynamics in non-global observables and small-x physics where such non-linear effects lie behind the phenomenon of parton saturation [30]–[35].

5. Conclusions

Conventional calculations of non-global observables assume that emission well away from the region in which the observable is calculated cancels. When starting this work, we aimed to check this assumption for one of the simplest non-global observables in hadron–hadron collisions, the gaps-between-jets cross-section, by explicitly calculating the all-orders contribution from configurations with one gluon outside the gap region. Based on the pioneering work of Dasgupta and Salam, we expected to find additional contributions from emission *just* outside the gap. Physically, the probability that such radiation is not accompanied by additional nearby radiation reduces the gap cross-section, giving rise to additional towers of leading logarithms; the so-called non-global logs. We indeed found such a contribution, illustrated in Eq.(3.23).

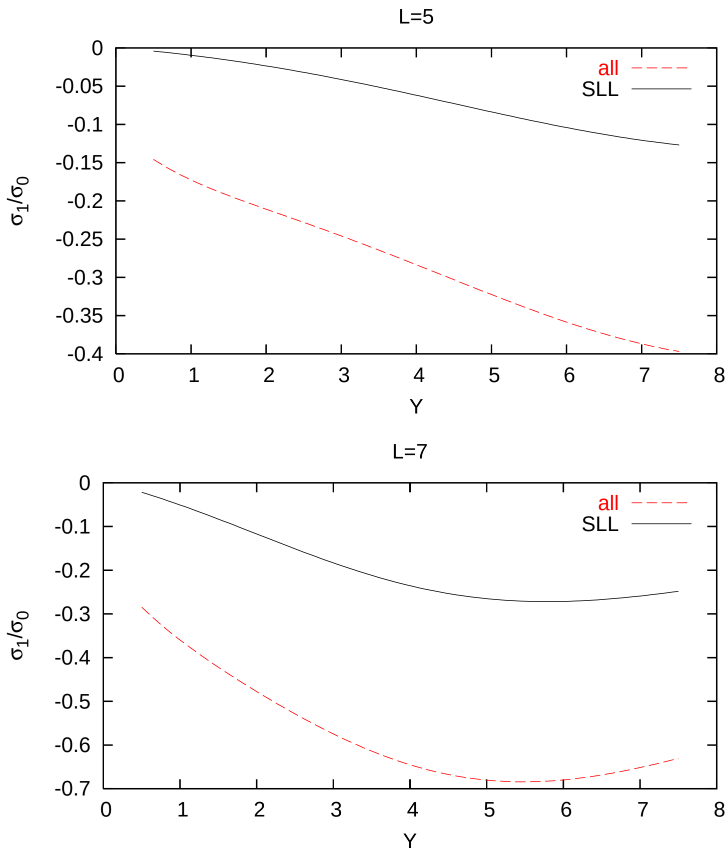

Figure 3: Y dependence of the out-of-gap cross-section (normalized to the in-gap cross-section) at two different values of L.

However, when calculating the evolution of five-parton configurations produced by real radiation outside the gap, we found a mismatch between it and the evolution of the four-parton configurations corresponding to virtual emission. This can be traced to the Coulomb phase terms (the imaginary parts of the loop integrals) coming from singularities that pinch the contour integral at the point $k_+ = k_- = 0$. As illustrated in Eq. (3.12), if these terms are artificially set to zero, the mismatch vanishes. However, keeping these terms, a mismatch remains, even for emission arbitrarily far away from the gap region. Integrating over phase space results in a new *superleading* logarithm, formally more important than any so-called leading logarithm previously included, as illustrated in Eq.(3.24). Our conclusions are subject to the caveat that we have assumed the validity of transverse momentum ordering for successive soft gluon emissions.

Although from our numerical results it may appear that the phenomenological impact of this formally-dominant effect is modest for L and Y values of interest, we should recall that we have only calculated the contribution from one gluon outside the gap. Having identified such a contribution, it is clearly necessary to examine how contributions from

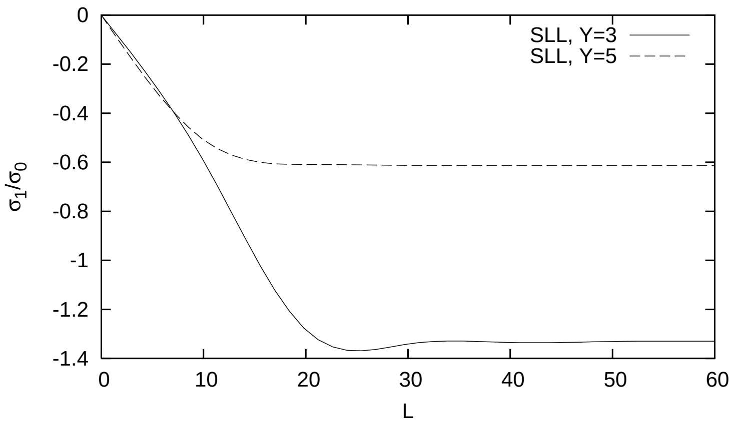

Figure 4: L dependence of the out-of-gap cross-section (normalized to the in-gap cross-section) at two different values of Y and plotted out to very large L.

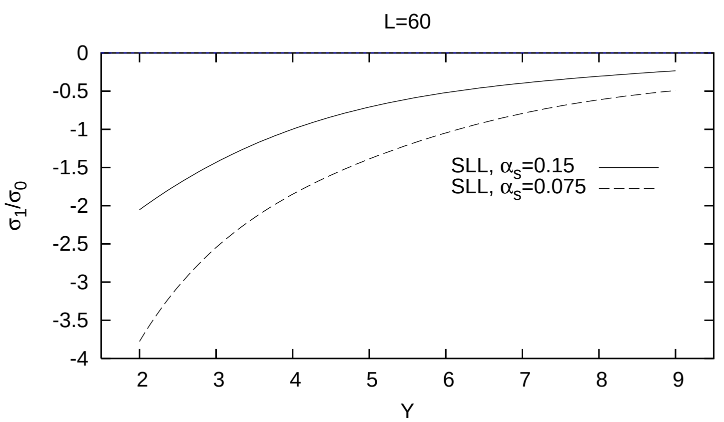

Figure 5: The Y dependence of the large L behaviour of the out-of-gap cross-section normalized to the in-gap cross-section at two different values of α_s.

arbitrary numbers of gluons outside the gap will contribute. In fact we see no reason why the argument at the end of Section 3 should not hold for the leading such contribution and expect that at the nth order of perturbation theory the leading contribution will come from $n-3$ gluons outside the gap, resulting in a term $\sim \alpha_s^n L^{2n-3} \pi^2 Y$. Calculating such contributions analytically seems a formidable challenge without a deeper understanding of the colour evolution of multi-parton systems.

We close this paper with a remark about the more theoretical interest of our result. One can view the gaps-between-jets process as a look at the pomeron loop in QCD, since one sums over radiation outside the gap (corresponding to a cut pomeron) and forbids it inside the gap (corresponding to one pomeron either side of the cut). In [16] we calculated the conventional gap-between-jets cross-section in the high energy limit and showed that

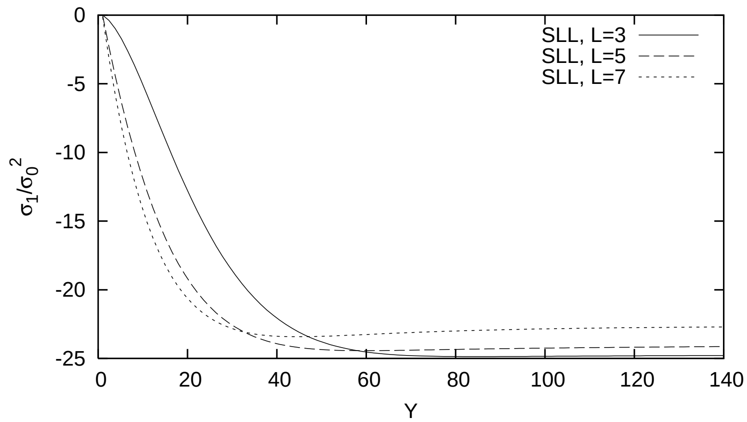

Figure 6: Y dependence of the out-of-gap cross-section normalized to the square of the in-gap cross-section at three different values of L and plotted out to very large Y.

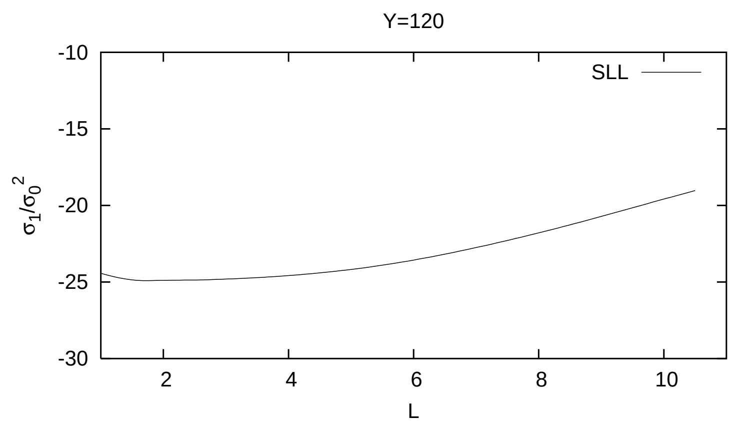

Figure 7: The L dependence of the large Y behaviour of the out-of-gap cross-section normalized to the square of the in-gap cross-section

it is equivalent to the BFKL result in the region in which both are valid. In this paper, we noted that in the high-energy (large Y) limit the cross section for one emission outside the gap is proportional to the square of the conventional gap cross-section, offering a tantalizing clue to the structure of higher orders. A deeper understanding of this connection would almost certainly open new avenues to understanding non-global observables.

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A. The block diagonal basis

Here we present the action of shifting from the t-channel colour basis to the block diagonal basis. We first exploit the fact that we can add any imaginary multiple of the unit matrix to the evolution matrices without affecting any observables in order to introduce

$$\mathbf{\Lambda}' = \mathbf{\Lambda} + \frac{N}{4} i\pi \mathbf{1}. \tag{A.1}$$

The required block diagonalization of $\mathbf{\Lambda}'$ is effected by

$$\mathbf{R} = \sqrt{\frac{N}{2(N^2-1)}} \begin{pmatrix} \frac{1}{2}s_y & -\frac{1}{2}s_y & s_y & \frac{1}{2N}s_y \\ \frac{N}{N+2}s_y & \frac{N}{N-2}s_y & 0 & s_y \\ -\frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2N} \\ 1 & 1 & 0 & -1 \end{pmatrix}. \tag{A.2}$$

The real emission matrix then transforms to

$$\begin{aligned} &\mathbf{D}^\mu \to \mathbf{R}^{-1}\mathbf{D}^\mu = \\ &\frac{1+s_y}{2}\sqrt{\frac{N^2-1}{2N}} \begin{pmatrix} \frac{N+2}{N+1}(h_4^\mu - h_2^\mu) & \frac{1}{2}\frac{N+2}{N+1}(h_4^\mu - h_1^\mu + \frac{1}{N}h_2^\mu - \frac{1}{N}h_4^\mu) \\ \frac{N-2}{N-1}(h_2^\mu - h_4^\mu) & \frac{1}{2}\frac{N-2}{N-1}(h_4^\mu - h_1^\mu + \frac{1}{N}h_4^\mu - \frac{1}{N}h_2^\mu) \\ h_3^\mu - h_1^\mu & \frac{1}{2N}(h_4^\mu - h_2^\mu) \\ \frac{2N}{N^2-1}(h_4^\mu - h_2^\mu) & \frac{1}{N^2-1}\left(h_1^\mu - h_3^\mu + N^2\left(h_3^\mu - h_2^\mu\right) + 2(h_2^\mu - h_4^\mu)\right) \end{pmatrix} \\ &-\frac{1-s_y}{2}\sqrt{\frac{N^2-1}{2N}} \begin{pmatrix} \frac{N+2}{N+1}(h_3^\mu - h_1^\mu) & \frac{1}{2}\frac{N+2}{N+1}(h_3^\mu - h_2^\mu + \frac{1}{N}h_1^\mu - \frac{1}{N}h_3^\mu) \\ \frac{N-2}{N-1}(h_1^\mu - h_3^\mu) & \frac{1}{2}\frac{N-2}{N-1}(h_3^\mu - h_2^\mu + \frac{1}{N}h_3^\mu - \frac{1}{N}h_1^\mu) \\ h_4^\mu - h_2^\mu & \frac{1}{2N}(h_3^\mu - h_1^\mu) \\ \frac{2N}{N^2-1}(h_3^\mu - h_1^\mu) & \frac{1}{N^2-1}\left(h_2^\mu - h_4^\mu + N^2\left(h_4^\mu - h_1^\mu\right) + 2(h_1^\mu - h_3^\mu)\right) \end{pmatrix} \end{aligned} \tag{A.3}$$

and the evolution matrix becomes

$$\begin{aligned} \mathbf{\Lambda} \to \mathbf{R}^{-1}\mathbf{\Lambda}'\mathbf{R} = &\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \frac{N^2-1}{4N}\rho(Y,\Delta y) & \frac{N^2-1}{4N^2}i\pi \\ 0 & 0 & i\pi & -\frac{1}{N}i\pi + \frac{N}{2}Y + \frac{N^2-1}{4N}\rho(Y,\Delta y) \end{pmatrix} \\ &+ \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ 0 & 0 & 0 & N \end{pmatrix} \frac{1}{4}\rho(Y, 2\,|y|) \\ &+ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -N & 0 \\ 0 & 0 & 0 & -N \end{pmatrix} \frac{\lambda}{4} \end{aligned} \tag{A.4}$$

where the λ_i are specified in Eq.(3.9).

Finally, the colour matrix transforms to

$$\mathbf{S}_R \to \mathbf{R}^\dagger \mathbf{S}_R\, \mathbf{R} = \begin{pmatrix} \frac{N^2}{2}\frac{N+1}{N+2} & 0 & 0 & 0 \\ 0 & \frac{N^2}{2}\frac{N-1}{N-2} & 0 & 0 \\ 0 & 0 & N^2 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(N^2-1) \end{pmatrix}. \tag{A.5}$$

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