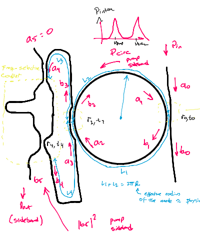
MATLAB model

Friday, March 12, 2021 7:24 AM



$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} r e^{i\varphi_0} & t e^{i\theta_0} \\ t e^{i\theta_1} & r e^{i\varphi_1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} r_0 & t_0 \\ t_1 & r_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

i.e. multiply $\begin{bmatrix} r_0 & t_0 \\ t_1 & r_1 \end{bmatrix}$ by this matrix to the right side $\rightarrow \begin{bmatrix} e^{-i\theta_1} & 0 \\ 0 & e^{-i\theta_0} \end{bmatrix}$

Then $\rightarrow \begin{bmatrix} r e^{i(\varphi_0-\theta_1)} & t \\ t & r e^{i(\varphi_1-\theta_0)} \end{bmatrix}$

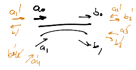
$\varphi_1 - \theta_0$ $\rightarrow$ $-r_0^*$

(i) $\theta_0 + \theta_1 = \varphi_0 + \varphi_1 + (2n+1)\pi \; ; \; n \in \mathbb{Z}$ (lossless/reciprocal network)

(ii) $r^2 + t^2 = 1$ (Lossless condition)

→ We can always change the reference plane of one of the ports such that $\theta_0 = \theta_1 = 0$. Then, from (i) $e^{i\varphi_1} = -e^{-i\varphi_0}$ and $r_1 = -r_0^*$

$$\therefore \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} r_0 & t_0 \\ t_0 & -r_0^* \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad \text{where } t_0 \in \mathbb{R} \text{ and } r_0 = |r_0| e^{i\varphi_0} = r e^{i\varphi_0} \in \mathbb{C}$$



$$\begin{bmatrix} b_1' \\ b_2' \\ b_3' \\ b_4' \end{bmatrix} = \begin{bmatrix} 0 & r_{21} & t_{31} & 0 \\ r_{21} & 0 & 0 & t_{24} \\ t_{31} & 0 & 0 & r_{34} \\ 0 & t_{24} & r_{34} & 0 \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \end{bmatrix}$$

$a_2' = a_3' = 0$

$$\begin{bmatrix} b_2' \\ b_3' \end{bmatrix} = \begin{bmatrix} r_{21} & t_{24} \\ t_{31} & r_{34} \end{bmatrix} \begin{bmatrix} a_1' \\ a_4' \end{bmatrix}$$

$$\begin{cases} r_{21} t_{24}^* + t_{31} r_{34}^* = 0 \; ; \\ r_{21} t_{31}^* + t_{24} r_{34}^* = 0 \; ; \end{cases}$$

$r_{21} = |r| e^{i\varphi_1}$; $r_{34} = |r| e^{i\varphi_2}$

$t_{24} = |t| e^{i\theta_1}$; $t_{31} = |t| e^{i\theta_2}$

$$\begin{cases} e^{i(\varphi_1-\theta_1)} + e^{i(\theta_2-\varphi_2)} = 0 \\ e^{i(\varphi_1-\theta_2)} + e^{i(\theta_1-\varphi_2)} = 0 \end{cases}$$

$$\therefore \begin{cases} \varphi_1 - \theta_1 = (2n+1)\pi + \theta_2 - \varphi_2 \\ \varphi_1 - \theta_2 = (2m+1)\pi + \theta_1 - \varphi_2 \end{cases}$$

$$\begin{cases} \varphi_1 + \varphi_2 = \theta_1 + \theta_2 + (2n+1)\pi \\ \varphi_1 + \varphi_2 = \theta_1 + \theta_2 + (2m+1)\pi \end{cases}$$

$$\begin{bmatrix} e^{-i\theta_1'} & & & \\ & e^{-i\theta_2'} & & \\ & & e^{-i\theta_3'} & \\ & & & e^{-i\theta_4'} \end{bmatrix} \begin{bmatrix} 0 & r_{21} & t_{31} & 0 \\ r_{21} & 0 & 0 & t_{24} \\ t_{31} & 0 & 0 & r_{34} \\ 0 & t_{24} & r_{34} & 0 \end{bmatrix} \begin{bmatrix} e^{-i\theta_1'} & & & \\ & e^{-i\theta_2'} & & \\ & & e^{-i\theta_3'} & \\ & & & e^{-i\theta_4'} \end{bmatrix}$$

Phase shift due to reference plane convenient. change

Now θ's are shifts

$$= \begin{bmatrix} e^{-i\theta_1'} & & & \\ & e^{-i\theta_2'} & & \\ & & e^{-i\theta_3'} & \\ & & & e^{-i\theta_4'} \end{bmatrix} \begin{bmatrix} 0 & r_{21} e^{-i\theta_2'} & t_{31} e^{-i\theta_3'} & 0 \\ r_{21} e^{-i\theta_1'} & 0 & 0 & t_{24} e^{-i\theta_4'} \\ t_{31} e^{-i\theta_1'} & 0 & 0 & r_{34} e^{-i\theta_4'} \\ 0 & t_{24} e^{-i\theta_2'} & r_{34} e^{-i\theta_3'} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & r_{21} e^{-i(\theta_2'+\theta_1')} & t_{31} e^{-i(\theta_3'+\theta_1')} & 0 \\ r_{21} e^{-i(\theta_1'+\theta_2')} & 0 & 0 & t_{24} e^{-i(\theta_4'+\theta_2')} \\ t_{31} e^{-i(\theta_1'+\theta_3')} & 0 & 0 & r_{34} e^{-i(\theta_4'+\theta_3')} \\ 0 & t_{24} e^{-i(\theta_2'+\theta_4')} & r_{34} e^{-i(\theta_3'+\theta_4')} & 0 \end{bmatrix}$$

$\therefore$ $\theta_1' + \theta_3' = \angle t_{31} \therefore t_{31} e^{-i(\theta_1'+\theta_3')} = |t_{31}| = t$

$\theta_4' + \theta_2' = \angle t_{24} \therefore t_{24} e^{-i(\theta_2'+\theta_4')} = |t_{24}| = t$

for $r_2 = 1$

$M = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$P = \begin{bmatrix} 0 & e^{-i\alpha L_2} & 0 & 0 \\ e^{-i\alpha L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} r_{21} e^{-i(\theta_1'+\theta_2')} & t \\ t & r_{34} e^{-i(\theta_3'+\theta_4')} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\begin{aligned} \theta_1' + \theta_3' &= \theta_2 \\ \theta_4' + \theta_2' &= \theta_1 \end{aligned} \Rightarrow \theta_1' + \theta_2' + \theta_3' + \theta_4' = \theta_1 + \theta_2$$

$\therefore \theta_1' + \theta_2' = -(\theta_3' + \theta_4') + \theta_1 + \theta_2$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} r\, e^{i(\theta_3'+\theta_4')} \underbrace{e^{-i(\theta_1+\theta_2)} e^{i\varphi_1}}_{\varphi_1 - (\theta_1+\theta_2) = -\varphi_2 + (2m+1)\pi} & t \\ t & r\, e^{-i(\theta_3'+\theta_4')} e^{i\varphi_2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \underbrace{r\, e^{i(\theta_3'+\theta_4')} e^{-i(\theta_1+\theta_2)} e^{i\varphi_1}}_{\hat{r}} & t \\ t & -r\, e^{-i(\theta_3'+\theta_4')} e^{i(\theta_1+\theta_2)} e^{-i\varphi_1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} \hat{r} & t \\ t & -\hat{r}^* \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$-10\log\left(e^{-2\beta'' \cdot 1\,m}\right) = Loss(1m)\,[dB]$

$\therefore -2\beta''(1m)\log e = -\frac{Loss}{10}$

$\therefore \beta'' = \frac{1}{2\log e}\frac{Loss(1m)}{10(1m)}$

$\beta'' = \frac{loss\,[dB/m]}{20\log e}$

$10\log\left(e^{-2\beta''(1m)}\right) = 10\log\left(e^{-\frac{loss}{10\log e}}\right)$

$= -\frac{loss}{\log e}\log e$ ✓

$$Q = -\frac{2\pi m}{\ln \alpha^2} = -\frac{2\pi \frac{L}{\lambda_0} n_{eff}}{\ln e^{-2\beta'' L}} = -\frac{2\pi \frac{L}{\lambda_0} n_{eff}}{-2\beta'' L} = \frac{\pi\, n_{eff}}{\beta''\lambda_0}$$

Coupling point equations:

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} r_0 & t_0 & 0 & 0 & 0 & 0 \\ t_0 & -r_0^* & 0 & 0 & 0 & 0 \\ 0 & 0 & r_2 & t_2 & 0 & 0 \\ 0 & 0 & t_2 & -r_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 & r_4 & t_4 \\ 0 & 0 & 0 & 0 & t_5 & r_5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

← $a_5 = 0$ in our case.

This is the Mach-Zender-like Coupler: we don't write it as $\begin{bmatrix} r_4 & t_4 \\ t_4 & -r_4^* \end{bmatrix}$ because losses might not be negligible.

Parameters: $\begin{cases} \text{Real} : & t_0, t_2 \\ \text{Complex} : & r_0, r_2, r_4, t_4, r_5, t_5 \end{cases}$; All wavelenght dependent (especially r_4, t_4, r_5, t_5) the rest might be considered constant for pump and sideband.

Propagation equations.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 & e^{-i\beta L_2} & 0 & 0 \\ e^{-i\beta L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\beta L_4} \\ 0 & 0 & e^{-i\beta L_3} & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \; ; \; \beta = \beta' - i\beta'' = \frac{\omega}{c}\left(n_{eff} - i\kappa\right)$$

β: wavelength-dependent. n_{eff}: mode's effective refractive index. κ: mode's absorption index. In theory, wavelenght-dependent but perhaps can be assumed constant for pump and sideband.

Solution:

Setting

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} ; \; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} ; \; P = \begin{bmatrix} 0 & e^{-i\beta L_2} & 0 & 0 \\ e^{-i\beta L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\beta L_4} \\ 0 & 0 & e^{-i\beta L_3} & 0 \end{bmatrix} , \; M = \begin{bmatrix} -r_0^* & 0 & 0 & 0 \\ 0 & r_2 & t_2 & 0 \\ 0 & t_2 & -r_2^* & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix}$$

then,

$$A = PB, \quad B = MA + t_0 a_0 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{A_0} \therefore A = PMA + PA_0 t_0$$

excitation ($a_0 = 1$ to normalize to 1W)

$\therefore (I - PM)A = PA_0 t_0$ ← Identity matrix

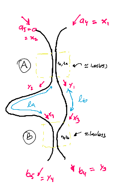
$$\therefore \boxed{A = (I - PM)^{-1} t_0 P A_0} \quad 1$$

Once A is known, we can know B by doing $B = MA$ ∴

~~$B = M(I-PM)^{-1} P A_0$~~ 2 → $B = P^{-1}A$

Finally, $\begin{aligned} b_0 &= r_0 a_0 + t_0 a_1 \\ b_5 &= t_5 a_4 + r_5 \underset{0}{a_5} \end{aligned}$ and everything is solved. 3

What about $\begin{bmatrix} r_4 & t_4 \\ t_5 & r_5 \end{bmatrix}$?



$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} r_A & t_A \\ t_A & -r_A^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} r_B & t_B \\ t_B & -r_B^* \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} e^{-i\beta_b l_b} & 0 \\ 0 & e^{-i\beta_a l_a} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \; \therefore$$

$\beta_b = \frac{\omega}{c}\left(n_{eff} - i\kappa_b\right) = \beta_b' - i\beta_b''$

$\beta_a = \frac{\omega}{c}\left(n_{eff} - i\kappa_a\right) = \beta_a' - i\beta_a''$

↖ we let the loss-per-wavelenght of branch l_a to be different than that of branch l_b to account for the possible larger losses of l_a due to strongest waveguide bending.

$$\text{Sol:} \rightarrow \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} r_B & t_B \\ t_B & -r_B^* \end{bmatrix} \begin{bmatrix} e^{-i\beta_b l_b} & 0 \\ 0 & e^{-i\beta_a l_a} \end{bmatrix} \begin{bmatrix} r_A & t_A \\ t_A & -r_A^* \end{bmatrix}}_{\begin{bmatrix} r_4 & t_4 \\ t_5 & r_5 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad 4$$

—▷ Nominal (desired) values

* M-Z Coupler: $\begin{cases} t_B = t_A = 1/\sqrt{2} \\ |r_b| = |r_A| = 1/\sqrt{2} \; ; \text{ The phases of } r_a \text{ and } r_b \text{ are irrelevant.} \\ \therefore r_b = \frac{1}{\sqrt{2}} e^{i\varphi_b} \; ; \; r_a = \frac{1}{\sqrt{2}} e^{i\varphi_a} \end{cases}$

* Resonance condition: $\begin{cases} \boxed{e^{-i\beta'(L_1+L_2)} r_2 \left(-r_0^*\right) = e^{-i2m\pi}} \; ; \; m \in \mathbb{Z} \\ \therefore -\frac{\omega}{c} n_{eff}(L_1+L_2) + \varphi_2 - \varphi_0 + \pi = -2m\pi \\ \therefore \boxed{\frac{\omega}{c} n_{eff}(\underbrace{L_1+L_2}_{2\pi R}) = \varphi_2 - \varphi_0 + (2m+1)\pi} \\ \rightarrow 2\pi R \rightarrow \text{ We want } FSR = \left(\frac{2\pi R}{c} n_{eff}\right)^{-1} = 500\,GHz = \frac{\omega_{THz}}{2\pi} \end{cases}$

$r_2 = |r_2| e^{i\varphi_2}$, $-r_0^* = |r_0| e^{-i\varphi_0 + i\pi}$

For $\omega = \omega_{pump} \rightarrow m = m_{pump}$

For $\omega = \omega_{sideband} \rightarrow m = m_{sideband} = m_{pump} + 1$

Note: Probably it is not mandatory the for the Coupler to resonate

* Coupling strength $|r_0|$: $\begin{cases} \text{It can be tuned numerically in Matlab to reach critical coupling.} \\ |r_0| \text{ should be bounded as:} \\ \underbrace{e^{-\beta''(L_1+L_2+L_3+L_4)} |r_4|}_{\text{If } |r_2| \rightarrow 0} \leq |r_0| \leq \underbrace{e^{-\beta''(L_1+L_2)}}_{\text{If } |r_2| \rightarrow 1} \end{cases}$

* Coupling Strength $|\Gamma_2|$:

Depends on how much we desire to increase the sideband's linewidth

$$Q_c = \frac{-2\pi m}{2\ln|r|} = \frac{\omega}{\Delta\omega_c}$$

due to coupling

So, if we want to increase $\Delta\omega_c$ by a factor K, we need to reduce $|\Gamma_2|$ by a factor $\left(|\Gamma_2|\Big|_{c.c}\right)^K$, where $|\Gamma_2|\Big|_{c.c} \simeq e^{-\beta''(L_1+L_2)}$