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HW 1 Optimization

2.12

a) $\{x\in\mathbb{R}^n / \alpha \le a^T x \le \beta\} = \underbrace{\{x\in\mathbb{R}^n / \alpha \le a^T x\}}_{\text{halfspace, hence convex}} \cap \underbrace{\{x\in\mathbb{R}^n / a^T x \le \beta\}}_{\text{halfspace, hence convex}}$

The set is the intersection of convex sets, it is thus convex

b) Let $\varphi_i : \begin{cases} \mathbb{R}^n \to \mathbb{R} \\ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to x_i \end{cases}$

φ_i is linear and $[\alpha_i ; \beta_i]$ is a convex set in $\mathbb{R}$.

Thus $\varphi_i^{-1}([\alpha_i, \beta_i])$ is convex in $\mathbb{R}^n$

Hence $\{x\in\mathbb{R}^n / \forall i \in [\![1,n]\!], x_i \in [\alpha_i, \beta_i]\} = \bigcap_{i=1}^{n} \varphi_i^{-1}([\alpha_i, \beta_i])$

is the intersection of convex sets. It is thus convex.

c) $\{x\in\mathbb{R}^n / a_1^T x \le b_1 , a_2^T x \le b_2\} = \underbrace{\{x\in\mathbb{R}^n / a_1^T x \le b_1\}}_{\text{halfspace, thus convex}} \cap \underbrace{\{x\in\mathbb{R}^n / a_2^T x \le b_2\}}_{\text{halfspace, thus convex}}$

the set is the intersection of convex sets. It is thus convex

d) Let $y\in S$. Note that

$$\|x-x_0\|_2 \le \|x-y\|_2 \iff \|x-x_0\|_2^2 \le \|x-y\|_2^2$$

$$\iff \|x\|_2^2 - 2x^T x_0 + \|x_0\|_2^2 \le \|x\|_2^2 - 2x^T y + \|y\|_2^2$$

$$\iff 2x^T(y-x_0) \le \|y\|_2^2 - \|x_0\|_2^2$$

Hence $A_y := \{x\in\mathbb{R}^n / \|x-x_0\|_2 \le \|x-y\|_2\}$ is a halfspace, hence a convex set

The set in the question can be written as $\bigcap_{y\in S} A_y$, it is convex as an intersection of convex sets

e) This set is <u>not necessarily convex</u> :

consider $n = 1$, $S = \{-1, 1\}$ and $T = \{0\}$

Then $\{x \in \mathbb{R} / d(x,S) \leq d(x,T)\} =]-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, +\infty[$

is not convex.

f) Let $x, y \in \mathbb{R}^n$ such that $x + S_2 \subset S_1$ and $\lambda \in [0,1]$

$y + S_2 \subset S_1$

Let us prove $[\lambda x + (1-\lambda) y] + S_2 \subset S_1$

Consider $z \in S_2$

$$\lambda x + (1-\lambda) y + z = \lambda \underbrace{(x+z)}_{\substack{\in x+S_2 \\ \in S_1}} + (1-\lambda) \underbrace{(y+z)}_{\substack{\in y+S_2 \\ \in S_1}}$$

$\in S_1$ because S_1 is convex

Hence $[\lambda x + (1-\lambda) y] + S_2 \subset S_1$

thus <u>the set in question is convex</u>

g) Let us prove that the set in question is a ball or a hyperplane.

Let us first derive an alternate characterization of a $\| \ \|_2$ - ball:

$$\|x - x_0\|_2^2 \leq r^2 \iff \|x\|_2^2 - 2x^T x_0 + \|x_0\|_2^2 - r^2 \leq 0 \qquad (*)$$

Note that $\|x-a\|_2^2 \leq \theta^2 \|x-b\|_2^2 \iff \|x\|_2^2 (1-\theta^2) - 2x^T(a - \theta^2 b) + \|a\|_2^2 - \theta^2 \|b\|_2^2 \leq 0$

<u>If $\theta = 1$</u> : this rewrites as $-2x^T(a - \theta^2 b) \leq \theta^2 \|b\|_2^2 - \|a\|_2^2$

which is <u>a halfspace, hence a convex set</u>

<u>If $\theta < 1$</u> : $\|x-a\|_2^2 \leq \theta^2 \|x-b\|_2^2 \iff \|x\|_2^2 - 2x^T\left(\dfrac{a - \theta^2 b}{1-\theta^2}\right) + \dfrac{\|a\|_2^2 - \theta^2\|b\|_2^2}{1-\theta^2} \leq 0$

Comparing with $(*)$, this is <u>a ball with center</u> $\dfrac{a - \theta^2 b}{1-\theta^2}$

and radius $\sqrt{\left\| \dfrac{a-\theta^2 b}{1-\theta^2} \right\|_2^2 - \dfrac{\|a\|_2^2 - \theta^2\|b\|_2^2}{1-\theta^2}}$, <u>provided the quantity</u>

<u>inside the square root is ≥ 0</u>. Let us prove that this is the case :

$$\left\| \frac{a - \theta^2 b}{1-\theta^2} \right\|^2 - \frac{\|a\|^2 - \theta^2 \|b\|^2}{1-\theta^2} = \frac{1}{(1-\theta^2)^2} \left(\|a\|^2 - 2\theta^2 a^T b + \theta^4 \|b\|^2 - (1-\theta^2)(\|a\|^2 - \theta \|b\|^2) \right)$$

$$= \frac{1}{(1-\theta^2)^2} \left(\theta^2 \|a\|^2 - 2\theta^2 a^T b + \|b\|^2 \theta^2 \right)$$

$$= \frac{\theta^2}{(1-\theta^2)^2} \|a-b\|^2 \geq 0 .$$

This proves that the radius of the ball is $\frac{\theta}{1-\theta^2} \|a-b\|$

3.21

a) Let $i \in [\![1,k]\!]$. $\varphi_i : x \rightarrow \| A^{(i)} x - b^{(i)} \|$ is convex as the norm of an affine function. Thus $f = \max_{1 \leq i \leq k} \varphi_i$ is convex as a maximum of k convex functions.

b) It is clear that $f : x \longrightarrow \max_{\substack{J \subset [\![1,n]\!] \\ |J| = r}} \sum_{j \in J} |x_j|$

And for fixed $J \subset [\![1,n]\!]$ with $|J| = r$, $x \longrightarrow \sum_{j \in J} |x_j|$

is easily seen to be convex (using the definition of convexity and the triangle inequality)

Thus f is convex as a maximum of a finite number of convex functions ($\binom{n}{r}$ precisely).

3.32

a) Let $x, y \in I$ and $\lambda \in [0,1]$

By convexity of f and g, $f((1-\lambda)x+\lambda y) \leq (1-\lambda) f(x) + \lambda f(y)$

$g((1-\lambda)x+\lambda y) \leq (1-\lambda) g(x) + \lambda g(y)$

Since f and g are positive, we may **multiply** both inequalities to get

$$f((1-\lambda)x+\lambda y)\, g((1-\lambda)x+\lambda y) \leq [(1-\lambda)f(x)+\lambda f(y)][(1-\lambda)g(x)+\lambda g(y)]$$

$$= (1-\lambda)^2 f(x)g(x) + (1-\lambda)\lambda [f(x)g(y)+f(y)g(x)] + \lambda^2 f(y)g(y)$$

$$= (1-\lambda) f(x)g(x) + \lambda f(y)g(y) + (1-\lambda)[-\lambda f(x)g(x) - \lambda f(y)g(y) + \lambda(f(x)g(y)+f(y)g(x))]$$

$$= (1-\lambda) f(x)g(x) + \lambda f(y)g(y) + \lambda(1-\lambda)[f(x)g(y)+f(y)g(x) - f(x)g(x) - f(y)g(y)]$$

$$= (1-\lambda) f(x)g(x) + \lambda f(y)g(y) + \lambda(1-\lambda)\,(f(x)-f(y))(g(y)-g(x))$$

Since f and g are monotonic in the same direction,

$f(x)-f(y)$ and $g(y)-g(x)$ have opposite signs, that is

$[f(x)-f(y)][g(y)-g(x)] \leq 0$

Hence $f((1-\lambda)x+\lambda y)\, g((1-\lambda)x+\lambda y) \leq (1-\lambda) f(x)g(x) + \lambda f(y)g(y)$

That is, fg is convex

b) The computation is similar to a)

the only difference is that $(f(x)-f(y))(g(y)-g(x)) \geq 0$

c) Since g is positive and nonincreasing, $\frac{1}{g}$ is positive and nondecreasing.

By a), $f \times \frac{1}{g}$ is convex, ie $\frac{f}{g}$ is convex

3.36

a) Let us prove that $\boxed{\begin{array}{l} \cdot\ \text{dom } f^* = \{ y \in \mathbb{R}^n \,/\, \forall i \in [\![1,n]\!],\ y_i \geq 0 \text{ and } \sum_{i=1}^n y_i = 1 \} \\ \cdot\ f^* = 0 \end{array}}$

1 Consider $y \in \mathbb{R}^n$ such that $x \to y^T x - f(x)$ is bounded above.

- For $t > 0$, let $x_k = -t$ and $x_i = 0$ for $i \neq k$

  Then $y^T x - f(x) = -y_k t$ which must remain bounded above as $t \to +\infty$

  Thus $y_k \geq 0$ (and this holds for any $k \in [\![1,n]\!]$)

- For $t > 0$, let $x_1 = \cdots = x_n = t$

  Then $y^T x - f(x) = t\left(\sum_{i=1}^n y_i - 1\right)$ which must be bounded above as $t \to +\infty$

  Hence $\sum_{i=1}^n y_i - 1 \leq 0$, that $\sum_{i=1}^n y_i \leq 1$.

2 Consider $y \in \mathbb{R}^n$ such that $y \geq 0$ and $\sum_{i=1}^n y_i = 1$

Let $x \in \mathbb{R}^n$ and $i_0 \in \underset{i \in [\![1,n]\!]}{\text{argmin}}\ x_i$

Then $y^T x - f(x) = \sum_{k \neq i_0} x_k y_k + (y_{i_0} - 1) x_{i_0} \leq \sum_{k \neq i_0} x_{i_0} y_k + (y_{i_0} - 1) x_{i_0}$

$$= x_{i_0}\left(\sum_{i=1}^n y_i - 1\right)$$

$$= 0$$

This upper bound is attained when $x = 0$, hence it's the maximum

Hence $f^* = 0$.

b) Let us prove that $\boxed{\begin{array}{l} -\ \text{dom } f^* = \{ y \in \mathbb{R}^n \,/\, \forall i \in [\![1,n]\!],\ y_i \in [0,1] \text{ and } \sum_{i=1}^n y_i = r \} \\ -\ f^* = 0 \end{array}}$

- If $r = n$: then $y^T x - f(x) = \sum_{i=1}^n (y_i - 1) x_i$

  for t consider $x_k = t$ and $x_i = 0$ for $i \neq k$.

  then $y^T x - f(x) = (y_k - 1) t$. This must remain bounded above as $t \to \pm\infty$, hence $y_k = 1$, and this holds for all k

  Conversely, consider $y \in \mathbb{R}^n$ such that $0 \leq y \leq 1$ and $\sum_{i=1}^n y_i = n$

  this implies $\forall i \in [\![1,n]\!]$, $y_i = 1$, thus $y^T x - f(x) = 0\ \forall x$, hence

$\mathrm{dom}\, f^* = \left\{ \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\} = \left\{ y \in \mathbb{R}^n \,/\, \forall i \in [\![1, n]\!],\ y_i \in [0,1] \text{ and } \sum_{i=1}^{n} y_i = n \right\}$

and $f^* = 0$.

- If $r < n$:

$\boxed{C}$ For $t > 0$, consider $x_k = -t$ and $x_i = 0$ for $i \neq k$

then $y^T x - f(x) = -y_k t$ which must remain bounded above as $t \to +\infty$

Thus $y_k \geq 0$, and this holds for all k

For $t > 0$, consider $x_k = t$ and $x_i = 0$ for $i \neq k$

then $y^T x - f(x) = t(y_k - 1)$, hence $y_k \leq 1$ and this holds for all k.

For $t \in \mathbb{R}$, consider $x_1 = \dots = x_k = t$ then

$y^T x - f(x) = t\left(\sum_{i=1}^{n} y_i - r\right)$ which must remain bounded above as $t \to \pm\infty$

Hence $\sum_{i=1}^{n} y_i = r$

$\boxed{D}$ Let $x \in \mathbb{R}^n$ and y such that $0 \leq y \leq 1$ and $\sum_{i=1}^{n} y_i = r$

$$y^T x - f(x) = \sum_{i=1}^{r} x_{[i]} (y_{[i]} - 1) + \sum_{i=r+1}^{n} x_{[i]} y_{[i]}$$

$$\leq x_{[r]} \Bigg[\underbrace{\sum_{i=1}^{r} (y_{[i]} - 1) + \sum_{i=r+1}^{n} y_{[i]}}_{= \sum_{i=1}^{n} y_i - r \; = 0} \Bigg]$$

$$\leq 0$$

The bound is attained when $x = 0$, it is thus the maximum of $x \to x^T y - f(x)$

Hence $f^* = 0$.

c) I assume something stronger than what's stated in the problem: for each k, there exists x such that $\forall i \neq k$, $a_k x + b_k > a_i x + b_i$

I will say "a_j dominates all the a_i at x" if $\forall i \neq j$, $a_j x + b_j > a_i x + b_i$

By the **redundancy** assumption, we must have $a_i \neq a_j$ for $i \neq j$.

Note that a_j dominates at x $\iff \begin{cases} \forall i < j, & x > \dfrac{b_i - b_j}{a_j - a_i} \\ \forall i > j, & x < \dfrac{b_i - b_j}{a_j - a_i} \end{cases}$

So a_m dominates $\iff x > \max\limits_{i<m} \dfrac{b_i - b_m}{a_m - a_i}$

a_{m-1} must dominate at some x, that is $x < \dfrac{b_m - b_{m-1}}{a_{m-1} - a_m}$

$$x > \frac{b_i - b_{m-1}}{a_{m-1} - a_i} \quad \forall i < m-1 \quad (*)$$

So a_{m-1} dominates over $\left] \max\limits_{i<m-1} \dfrac{b_i - b_{m-1}}{a_{m-1} - a_i}, \dfrac{b_m - b_{m-1}}{a_{m-1} - a_m} \right[$.

It remains to prove that a_m dominates over $\left] \dfrac{b_m - b_{m-1}}{a_{m-1} - a_m}, \infty \right[$

which is equivalent to $\max\limits_{i<m} \dfrac{b_i - b_m}{a_m - a_i} = \dfrac{b_m - b_{m-1}}{a_{m-1} - a_m}$

If this were not the case, **some** a_j with $j < m-1$ would dominate at $x > \dfrac{b_m - b_{m-1}}{a_{m-1} - a_m}$. This implies $x < \dfrac{b_{m-1} - b_j}{a_j - a_{m-1}}$, a contradiction with $(*)$.

In a similar fashion, one proves that $\max\limits_{i<m-1} \dfrac{b_i - b_{m-1}}{a_{m-1} - a_i} = \dfrac{b_{m-1} - b_{m-2}}{a_{m-2} - a_{m-1}}$

and that
- $\forall j \in [\![2, m-1]\!]$, a_j **dominates on** $\left] \dfrac{b_j - b_{j-1}}{a_{j-1} - a_j}, \dfrac{b_{j+1} - b_j}{a_j - a_{j+1}} \right[$
- a_1 dominates on $\left] -\infty, \dfrac{b_2 - b_1}{a_1 - a_2} \right[$

Hence f is piecewise affine (and continuous) over $\left] -\infty, \dfrac{b_2 - b_1}{a_1 - a_2} \right[, \underbrace{\left] \dfrac{b_j - b_{j-1}}{a_{j-1} - a_j}, \dfrac{b_{j+1} - b_j}{a_j - a_{j+1}} \right[}_{j \in [\![2, m-1]\!]}, \left] \dfrac{b_m - b_{m-1}}{a_{m-1} - a_m}, \infty \right[$

Let us prove that $\boxed{\begin{array}{l} - \operatorname{dom} f^* = [a_1, a_m] \\ - f^* \text{ is piecewise affine} \end{array}}$

$\boxed{C}$ For x sufficiently big, $yx - f(x) = yx - (a_m x + b_m)$

$$= (y - a_m)x - b_m$$

which must remain bounded above as $x \to +\infty$,

hence $y - a_m \leq 0$, that is $y \leq a_m$.

For x sufficiently small, one gets $a_1 \leq y$

$\boxed{D}$ Consider $y \in [a_1, a_m]$

$\varphi : x \to yx - f(x)$ is piecewise affine and **continuous**, with successive slopes $y - a_1, y - a_2, \dots, y - a_m$

An affine piecewise continuous function increases as long as its slope is ≥ 0, hence φ increases as long as $y - a_i \geq 0$, hence φ reaches its maximum at the breakpoint between a_i and a_{i+1} (where i **is** such that $a_i \leq y < a_{i+1}$), ie at $\dfrac{b_{i+1} - b_i}{a_i - a_{i+1}}$

Hence $f^*(y) = (y - a_i) \dfrac{b_{i+1} - b_i}{a_i - a_{i+1}} - b_i$ for all $y \in [a_i, a_{i+1}[$

Hence f^* is piecewise affine **continuous on each** $]a_i, a_{i+1}[$

d) <u>If $p > 1$</u>, let us prove that $\boxed{\begin{array}{l} \operatorname{dom} f^* = \mathbb{R} \\ f^*(y) = \begin{cases} (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases} \end{array}}$

Consider any $y \in \mathbb{R}$ and note that $yx - x^p \xrightarrow[x \to \infty]{} -\infty$,

hence $\varphi \begin{cases} \mathbb{R}_+^* \to \mathbb{R} \\ x \to yx - x^p \end{cases}$ is bounded above

Thus $\operatorname{dom} f^* = \mathbb{R}$

Note that $\varphi'(x) = y - px^{p-1}$

$$\varphi'(x) \leq 0 \iff y \leq px^{p-1} \iff \begin{cases} \text{True no matter } x \text{ if } y \leq 0 \\ x \geq \left(\frac{y}{p}\right)^{\frac{1}{p-1}} \text{ if } y > 0 \end{cases}$$

Hence $f^*(y) = \begin{cases} \varphi(0) & \text{if } y \leq 0 \\ \varphi\left(\left(\frac{y}{p}\right)^{\frac{1}{p-1}}\right) & \text{if } y > 0 \end{cases}$

$$f^*(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y > 0 \end{cases}$$

If $p < 0$, let us prove
$$\boxed{\begin{array}{l} \text{dom } f^* = \mathbb{R}_- \\ f^* : y \to (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}} \end{array}}$$

C Consider $y \in \mathbb{R}$ and note that $yx - f(x) = yx - \frac{1}{x^{-p}}$

For this to remain bounded above as $x \to \infty$, one must have $y \leq 0$

⊃ Let $y \leq 0$ and $\varphi : x \to yx - x^p$

$\varphi'(x) = y - px^{p-1}$

$\varphi'(x) \leq 0 \iff y \leq px^{p-1} \iff \frac{y}{p} \geq x^{p-1} \iff x \geq \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

Hence $f^*(y) = (p-1)\left(\frac{y}{p}\right)^{\frac{p}{p-1}}$.

e) Let us prove
$$\boxed{\begin{array}{l} \text{dom } f^* = \left\{ y \in \mathbb{R}^n \,/\, \forall i \in [\![1,n]\!], y_i < 0 \text{ and } \left(\prod_{i=1}^{n}(-y_i)\right)^{1/n} \geq \frac{1}{n} \right\} \\ f^* = 0 \end{array}}$$

C For $t > 0$, let $x_k = t$ and $x_i = 1$ for $i \neq k$

Note that $y^T x - f(x) = t y_k - \sum_{i \neq k} y_i + t^{1/n}$

For this to remain bounded above as $t \to +\infty$, one must have $y_k < 0$, and this holds for every k.

Besides, for $t > 0$, let $x_k = -\frac{t}{y_k}$

Then $y^T x - f(x) = t\left(\prod_{i=1}^{m}\left(-\frac{1}{y_i}\right)^{1/m} - m\right)$

This must remain bounded above as $t \to +\infty$, hence

$$\left[\prod_{i=1}^{m}\left(-\frac{1}{y_i}\right)\right]^{1/m} \leq m$$

ie $$\left[\prod_{i=1}^{m}(-y_i)\right]^{1/m} \geq \frac{1}{m}.$$

$\boxed{\supseteq}$ Consider y such that $y < 0$ and $\left[\prod_{i=1}^{m}(-y_i)\right]^{1/m} \geq \frac{1}{m}$. (*)

By AM-GM, $\left(\prod_{i=1}^{m} x_i(-y_i)\right)^{1/m} \leq \frac{1}{m}\sum_{i=1}^{m} x_i(-y_i)$

Hence $-m\left(\prod_{i=1}^{m} x_i(-y_i)\right)^{1/m} \geq \sum_{i=1}^{m} x_i y_i$

Thus
$$\begin{aligned} y^T x - f(x) &= \sum_{i=1}^{m} x_i y_i + \left(\prod_{i=1}^{m} x_i\right)^{1/m} \\ &\leq -m\left(\prod_{i=1}^{m} x_i(-y_i)\right)^{1/m} + \left(\prod_{i=1}^{m} x_i\right)^{1/m} \\ &= \left(\prod_{i=1}^{m} x_i\right)^{1/m}\Bigg[\underbrace{1 - m\left(\prod_{i=1}^{m}(-y_i)\right)^{1/m}}_{\leq 0 \text{ by } (*)}\Bigg] \\ &\leq 0. \end{aligned}$$

For $x_1 = \dots = x_m = \varepsilon > 0$, $y^T x - f(x) = \varepsilon\left(\sum_{i=1}^{m} y_i + 1\right) \xrightarrow[\varepsilon \to 0]{} 0$

Hence the bound is tight : $f^*(y) = 0$.

f) Let us prove
$$\boxed{\begin{aligned} &\operatorname{dom} f^* = \{(y,u) \in \mathbb{R}^n \times \mathbb{R} \,/\, u < 0 \text{ and } \|y\| < -u\} \\ &f^*(y,u) = -2 + \log 4 - \log(y^T y - u^2) \end{aligned}}$$

$\boxed{\subseteq}$ Consider $(y,u) \in \operatorname{dom} f^*$

- $y^T x + ut + \log(t^2 - x^T x)$ must remain bounded above as $t \to +\infty$

  This implies $u < 0$ (if $u = 0$, the log takes over)

- Let $x = \lambda y$ with $\lambda > 0$ and $t = \lambda\|y\| + \varepsilon$ with $\varepsilon > 0$

Then $y^T x + ut + \log(t^2 - x^T x) = \lambda \|y\| (\|y\| + u) + \log(2\varepsilon \lambda \|y\| + \varepsilon^2)$

This must remain bounded above as $\lambda \to \infty$, hence $\|y\| + u < 0$

(if $\|y\| + u = 0$, the log takes **over**)

$\boxed{D}$ Consider (y, u) with $u < 0$ and $\|y\| < -u$

$$y^T x + ut + \log(t^2 - x^T x) \leq \|y\| \|x\| + ut + \log(t^2) + \log\left(1 - \frac{\|x\|^2}{t^2}\right)$$

$$\leq t \underbrace{(\|y\| + u)}_{<0} + 2 \log t$$

$$\leq C \text{ where } C \text{ is a constant.}$$

Even better, $t(\|y\| + u) + 2 \log t \underset{t \to \infty}{\longrightarrow} -\infty$

Therefore, $\varphi : (x, t) \to y^T x + ut + \log(t^2 - x^T x)$ is coercive, thus reaches a global maximum. This maximum clearly lies in the interior of the domain, thus may be found with 1^{st} order methods:

$$\nabla_x \varphi(x, t) = y - \frac{2x}{t^2 - \|x\|^2}$$

$$\nabla_t \varphi(x, t) = u + \frac{2t}{t^2 - \|x\|^2}$$

hence critical points are given by

$$\begin{cases} y = \dfrac{2x}{t^2 - \|x\|^2} \\ 2t = -u(t^2 - \|x\|^2) \end{cases}$$

One gets $\|x\| = \sqrt{\frac{2t}{u} + t^2}$, hence $\|y\| = \dfrac{2\sqrt{\frac{2t}{u} + t^2}}{-\frac{2t}{u}}$, **which** yields

$$t = -\frac{2u}{u^2 - \|y\|^2}, \text{ and then } x = \frac{2y}{u^2 - \|y\|^2}$$

Hence $f^*(y, u) = \varphi(x, t) = -2 + \log 4 - \log(\|y\|^2 - u^2)$