Optiver Prove It: Episode 1

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1. Context and notation

The video containing the problem statement is available here: https://www.youtube.com/watch?v=fmtscKsfMTg.

We consider movement on the $n \times n$ grid $G = \{(i,j) : 0 \leq i, j \leq n\}$, where $n \geq 1$ an integer. We let $V_0, \dots, V_{2n}$ denote our successive positions on the grid, and $W_0, \dots, W_{2n}$ the positions of our friend. Note that each V_t is a G-valued random vector with $V_0 = (0,0)$ and $V_{2n} = (n,n)$, and similarly for W_t with $W_0 = (n,n)$ and $W_{2n} = (0,0)$. Since our tosses are independent of our friend's, $(V_0, \dots, V_{2n}) \perp\!\!\!\perp (W_0, \dots, W_{2n})$.

Let also $X_1, \dots, X_{2n}$ denote random variables modelling our tosses at each timestep: tossing heads means that we go one step to the right if possible. The X_t are independent and uniformly distributed on $\{H, T\}$.

2. Probability of meeting

Let p_n denote the probability of meeting. A meeting happens necessarily at time n, hence

$$p_n = \mathbb{P}(V_n = W_n) = \sum_{i=0}^{n} \mathbb{P}\big(V_n = W_n = (i, n-i)\big) = \sum_{i=0}^{n} \mathbb{P}\big(V_n = (i, n-i)\big)\mathbb{P}\big(W_n = (i, n-i)\big),$$

where the last equality follows from independence. By symmetry of the grid, V_n and W_n follow the same distribution. Moreover, the event $\{V_n = (i, n-i)\}$ happens when we have obtained exactly i heads among the first n tosses. Therefore,

$$p_n = \sum_{i=0}^{n} \mathbb{P}\big(V_n = (i, n-i)\big)^2 = \sum_{i=0}^{n} \mathbb{P}\left(\sum_{t=1}^{n} \mathbb{1}_{X_t = H} = i\right)^2.$$

Noting that $\sum_{t=1}^{n} \mathbb{1}_{X_t = H}$ has binomial distribution $B(n, \frac{1}{2})$, we obtain

$$p_n = \sum_{i=0}^{n} \left(\binom{n}{i} \frac{1}{2^n}\right)^2 = \frac{1}{2^{2n}} \sum_{i=0}^{n} \binom{n}{i}^2 = \frac{1}{2^{2n}} \binom{2n}{n},$$

where the last equality follows from Vandermonde's identity.

3. Exercise 1

3.1. Solution 1

Since $p_n = \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2}$, Stirling's estimate for the factorial yields

$$p_n \sim \frac{1}{\sqrt{\pi n}},$$

and in particular $\lim_{n\to\infty} p_n = 0$.

3.2. Solution 2

We give an alternate argument that does not involve Stirling's formula. Repeated integration by parts shows that $p_n = \frac{2}{\pi} \int_0^{\pi/2} \cos^{2n}(x)dx$.

Next, we prove the estimate $\cos x \leq e^{-x^2/2}$ for $x \in [0, \pi/2]$. Since $\tan' = 1 + \tan^2$ is increasing over $[0, \pi/2)$, tan is convex over this interval, thus $\tan(x) \geq x$ for $x \in [0, \pi/2)$. The function $f : x \mapsto e^{x^2/2} \cos(x)$ has derivative $f'(x) = e^{x^2/2} \cos(x)(x - \tan x) \leq 0$, thus f is nonincreasing and $f(x) \leq f(0) = 1$, which rewrites as $\cos x \leq e^{-x^2/2}$.

Consequently $p_n \leq \frac{2}{\pi} \int_0^{\pi/2} e^{-nx^2} dx \leq \frac{2}{\pi} \int_0^{+\infty} e^{-nx^2} dx = \frac{1}{\sqrt{\pi n}}$ and

$$\lim_{n\to\infty} p_n = 0.$$

4. Exercise 2

Straightforward algebra shows that

$$\frac{(n+1)p_{n+1}}{np_n} = \frac{2n+1}{2n} > 1.$$

Thus the sequence $(np_n)_{n\geq 1}$ is increasing and $p_n > \frac{p_1}{n} = \frac{1}{2n}$ for each $n \geq 2$.

The Cauchy–Schwarz inequality provides the bound

$$\sqrt{\sum_{i=0}^{n} \binom{n}{i}^2} \sqrt{\sum_{i=0}^{n} 1} > \sum_{i=0}^{n} \binom{n}{i} \cdot 1,$$

which turns into

$$p_n > \frac{1}{n+1}$$

for every $n \geq 1$.

Optiver's original claim that $p_n > \frac{1}{n}$ is true if and only if $n \geq 4$. Indeed, if $n \geq 4$ then

$$p_n \geq \frac{4p_4}{n} = \frac{35}{32n} > \frac{1}{n}.$$

However, if $n \leq 3$ then $p_n \leq \frac{3p_3}{n} = \frac{15}{16n} < \frac{1}{n}$.