Optiver Prove It: Episode 3

Gabriel Romon[1]

[1] *e-mail:* gabriel.s.romon@gmail.com

The video containing the problem statement is available here: https://www.youtube.com/watch?v=y2mjE8RDPxg.

1. Exercise 1: Optimal number of 6-sided dice

1.1. Probability of getting a single Six

Let $n \geq 1$ denote the number of dice, and let $X_1, \ldots, X_n$ be i.i.d. random variables uniformly distributed over $\{1, \ldots, 6\}$.

We denote by p_n the probability of getting a single 6 after throwing the n dice. Note that

$$\begin{aligned} p_n &= \mathbb{P}(\text{only one Six among } X_1, \ldots, X_n) \\ &\overset{(i)}{=} n\mathbb{P}\left(\{X_1 = 6\} \cap \bigcap_{i=2}^{n} \{X_i \neq 6\}\right) \\ &\overset{(ii)}{=} n\mathbb{P}(X_1 = 6) \prod_{i=2}^{n} \mathbb{P}(X_i \neq 6) \\ &= n \cdot \frac{1}{6}\left(\frac{5}{6}\right)^{n-1} \\ &= \frac{n5^{n-1}}{6^n}, \end{aligned}$$

where (i) follows from symmetry and (ii) from independence.

1.2. Optimizing over the number of dice

In what follows we let S denote the set of optimal values for n.

We observe that

$$\frac{p_{n+1}}{p_n} = \frac{n+1}{n} \cdot \frac{5}{6},$$

hence $\frac{p_{n+1}}{p_n} > 1 \iff n < 5 \iff n \leq 4$ and similarly $\frac{p_{n+1}}{p_n} < 1 \iff n > 5 \iff n \geq 6$. Therefore $p_1 < \ldots < p_4 < p_5$ and $p_6 > p_7 > p_8 \ldots$

Consequently, if $n \notin \{5, 6\}$, then n is suboptimal. In other words, we have the inclusion $S \subset \{5, 6\}$. Since $p_5 = \left(\frac{5}{6}\right)^5 = p_6$, we conclude that $S = \{5, 6\}$.

2. Exercise 2: Optimal number of m-sided dice with a target of r Sixes

To avoid trivialities we assume that $m \geq 6$.

Again, let $n \geq 1$ denote the number of dice, and let $X_1, \dots, X_n$ be i.i.d. random variables uniformly distributed over $\{1, \dots, m\}$.

We denote by p_n the probability of getting r Sixes after throwing the n dice. If $n < r$, then clearly $p_n = 0$. Otherwise, if $n \geq r$, note that

$$\begin{aligned} p_n &= \mathbb{P}(\text{exactly } r \text{ Sixes among } X_1, \dots, X_n) \\ &= \mathbb{P}\left(\sum_{i=1}^{n} \mathbb{1}_{X_i = 6} = r\right) \\ &= \binom{n}{r} \frac{1}{m^r} \left(1 - \frac{1}{m}\right)^{n-r}. \end{aligned}$$

In what follows we let S denote the set of optimal values for n.

2.1. The case $r = 0$

When $r = 0$, we have $p_n = \binom{n}{0} \frac{1}{m^0} \left(1 - \frac{1}{m}\right)^{n-0} = \left(1 - \frac{1}{m}\right)^n$, which is strictly decreasing in n. Thus $n = 1$ is optimal and $S = \{1\}$.

2.2. The case $r \geq 1$

For each n such that $n \geq r$, we have

$$\frac{p_{n+1}}{p_n} = \frac{n+1}{n+1-r} \left(1 - \frac{1}{m}\right). \tag{1}$$

Furthermore, some algebra shows that

$$\frac{p_{n+1}}{p_n} < 1 \iff n > rm - 1 \iff n \geq rm$$

and

$$\frac{p_{n+1}}{p_n} > 1 \iff n < rm - 1 \iff n \leq rm - 2.$$

Besides, $rm - 2 \geq r \iff r(m-1) \geq 2$, and this last inequality is true since $r \geq 1$ and $m \geq 6$. As a consequence, we have the following inequalities:

$$p_r < \dots < p_{(rm-2)+1} \quad \text{and} \quad p_{rm} > p_{rm+1} > \dots,$$

from which we derive the inclusion $S \subset \{rm - 1, rm\}$.

Lastly, by Equation (1),

$$\frac{p_{rm}}{p_{rm-1}} = \frac{rm}{rm - r} \left(1 - \frac{1}{m}\right) = 1,$$

hence $p_{rm-1} = p_{rm}$ and $S = \{rm - 1, rm\}$.

2.3. Conclusion

To summarize, the set of optimal values for n is

$$\begin{cases} S = \{1\} & \text{if } r = 1, \\ S = \{rm - 1, rm\} & \text{if } r \geq 2. \end{cases}$$