Optiver Prove It: Episode 2

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1. Context and notation

The video containing the problem statement is available here: https://www.youtube.com/watch?v=u76c4QDHXME.

We consider the random walk on $\{0, \dots, n\}$ where $n \geq 1$. When tossing the coin, we denote by $p \in (0, 1)$ the probability of getting heads (i.e., moving to the right). The probability of getting tails is $q = 1 - p$.

2. Expected number of steps

For each $k \in \{0, \dots, n\}$, we let e_k denote the expected number of steps needed to reach the friend's house located at state n, when starting from state k. We are looking for the value of e_0 and we already know that $e_n = 0$ and $e_0 = 1 + e_1$.

Assume that we have reached state $k \in \{1, \dots, n-1\}$. Conditioning on the outcome of the coin toss, we obtain $e_k = 1 + pe_{k+1} + qe_{k-1}$.

In other words, we have the following nonhomogeneous linear recurrence relation:

$$e_k = \frac{1}{p} e_{k-1} - \frac{q}{p} e_{k-2} - 1 \quad \text{for each } k \in \{2, \dots, n\} \tag{1}$$

with boundary conditions

$$e_0 = 1 + e_1, \quad e_n = 0. \tag{2}$$

The corresponding characteristic polynomial is $X^2 - \frac{1}{p}X + \frac{q}{p}$, which has discriminant $\big((1-2p)/p\big)^2 \geq 0$ and roots $\frac{1 \pm |1-2p|}{2p}$. The roots are distinct if and only if $p \neq \frac{1}{2}$.

2.1. *The case where $p = \frac{1}{2}$*

In that case, the only root of the characteristic polynomial is 1 and solutions of the homogeneous recurrence relation

$$f_k = 2f_{k-1} - f_{k-2} \quad \text{for each } k \in \{2, \dots, n\} \tag{3}$$

have the form $f_k = \lambda + \beta k$ for some real numbers λ, β. Besides, we observe that $g_k = k - k^2$ is a solution of (1).

By the theory of linear recurrence relations (see, e.g., (Balakrishnan, 1996, Chapter 3)), every solution of (1) is then of the form $e_k = \lambda + \beta k + k - k^2$. We find that the boundary conditions (2) are verified with $\lambda = n^2$ and $\beta = -1$, so that the only solution to both (1) and (2) is

$$e_k = n^2 - k^2 \quad \text{for each } k \in \{0, \dots, n\},$$

and in particular we obtain $e_0 = n^2$, as conjectured in the video.

2.2. The case where $p \neq \frac{1}{2}$

In that case, the distinct roots of the characteristic polynomial are $\frac{1-p}{p}$ and 1. Solutions of the homogeneous recurrence relation

$$f_k = \frac{1}{p} f_{k-1} - \frac{q}{p} f_{k-2} \quad \text{for each } k \in \{2, \ldots, n\} \tag{4}$$

have the form $f_k = \lambda \left(\frac{1-p}{p}\right)^k + \beta$ for some real numbers λ, β. Besides, we observe that $g_k = \frac{1}{1-2p} k$ is a solution of (1).

Every solution of (1) is then of the form $e_k = \lambda \left(\frac{1-p}{p}\right)^k + \beta + \frac{1}{1-2p} k$. We find that the boundary conditions (2) are verified with $\lambda = -\frac{2p(1-p)}{(1-2p)^2}$ and $\beta = \frac{2p(1-p)}{(1-2p)^2} \left(\frac{1-p}{p}\right)^n - \frac{1}{1-2p} n$, so that the only solution to both (1) and (2) is

$$e_k = \frac{2p(1-p)}{(1-2p)^2} \left[\left(\frac{1-p}{p}\right)^n - \left(\frac{1-p}{p}\right)^k \right] - \frac{1}{1-2p}(n-k) \quad \text{for each } k \in \{0, \ldots, n\},$$

and in particular we obtain

$$e_0 = \frac{2p(1-p)}{(1-2p)^2} \left[\left(\frac{1-p}{p}\right)^n - 1 \right] - \frac{1}{1-2p} n.$$

When $p = \frac{1}{3}$ (so that the coin is twice as likely to land on tails, as considered in the video), the expected number of steps is

$$e_0 = 2^{n+2} - 3n - 4.$$

References

BALAKRISHNAN, V. K. (1996). *Introductory discrete mathematics.* Dover Publications, Inc., Mineola, NY Corrected reprint of the 1991 original. MR1402469