Exploring Quantum Process Calculi

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October 7, 2022

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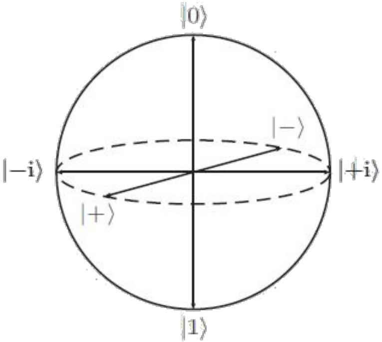
Qubits

The simplest quantum system is a **qubit**. Like a bit, a qubit has two separate states, $|0\rangle$ and $|1\rangle$.
A qubit can also be in a linear combination of states, knows as a **superposition**.

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



Measurements

A qubit in superposition cannot be directly observed, because when it get **measured**, it **decays** in one of its basis states.

$$M_{01}\left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right) \begin{array}{l} \nearrow \left|0\right\rangle \text{ with probability } |\alpha|^2 \\ \searrow \left|1\right\rangle \text{ with probability } |\beta|^2 \end{array}$$

For example, when measuring the state $\left|+\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$, we get either $\left|0\right\rangle$ or $\left|1\right\rangle$ with the same probability.

No Cloning theorem

No-Cloning

Quantum information cannot be **duplicated**. That is, given a qubit q_1 in state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

it is impossible to prepare a qubit q_2 in the same state, without destroying the information in q_1.

This means that, contrary to classical bits, qubits cannot be freely copied, stored or broadcasted to multiple receivers.

Linear qCCS: Type System

lqCCS

Linear qCCS (lqCCS) is an asynchronous value-passing calculus, equipped with a **linear type system** to regulate quantum communication. It features parallelism, non-determinism and quantum operations.

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The type system is needed to prevent **duplication** of quantum information: after receiving a qubit, a process must send it **exactly once**.

$$c?q.H(q).c!q \qquad a?q.\big(b!q \parallel c!q\big) \qquad c?q.H(q).\mathbf{0}$$

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In lqCCS, the programmer must **explicitly** describe what happens to each qubit

$$P' = c?q.X(q).c!q \qquad Q' = c?q.H(q).c!q$$

$$P'' = c?q.X(q).disc(q) \qquad Q'' = c?q.H(q).disc(q)$$

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$$\langle\, |0\rangle\,, H(q).c!q \parallel c?x.M_{01}[x \triangleright y].disc(x)\rangle$$
$$\downarrow$$
$$\langle\, |+\rangle\,, c!q \parallel c?x.M_{01}[x \triangleright y].disc(x)\rangle$$
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Linear qCCS: Bisimilarity via barbs and contexts

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- Context: a "program" with a hole, like $[-] \parallel R$. We compare P and Q "inside" this context, i.e. we study $P \parallel R$ and $Q \parallel R$.

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The two processes seen before

$$P' = c?q.X(q).c!q \qquad Q' = c?q.H(q).c!q$$

are not bisimilar, because there is a context R which tells them apart

$$R = c?q.M_{01}[q \triangleright x].\,(disc(q) \parallel \textbf{if } x = 0 \textbf{ then } a!0 \textbf{ else } b!0)$$

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$$\mathcal{C} = \langle\, |+\rangle\,, M_{01}[q \rhd x].c!q \rangle \to \langle\, |0\rangle\,, c!q \rangle\ {}_{\frac{1}{2}}\!\oplus \langle\, |1\rangle\,, c!q \rangle$$

$$\mathcal{C}' = \langle\, |0\rangle\,, M_{\pm}[q \rhd x].c!q \rangle \to \langle\, |+\rangle\,, c!q \rangle\ {}_{\frac{1}{2}}\!\oplus \langle\, |-\rangle\,, c!q \rangle$$

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According to the usual notion of **probabilistic bisimilarity**, two distributions are bisimilar if they assign the same probability to bisimilar processes. In our example, the two configurations are **not** bisimilar, because they evolve in two non-bisimilar distributions.

Indistinguishable distributions

According to quantum mechanics, it's impossible to distinguish the two sources S and S':

$$S \text{ emits a qubit } |0\rangle \text{ or } |1\rangle \text{ with the same probability}$$
$$S' \text{ emits a qubit } |+\rangle \text{ or } |-\rangle \text{ with the same probability}$$

Suppose we receive a qubit from either S or S', and measure it in the 01 basis. If we measure a qubit from S, it would result in either $|0\rangle$ or $|1\rangle$. If we measure a qubit from S', a $|+\rangle$ qubit would decay in either $|0\rangle$ or $|1\rangle$, and a $|-\rangle$ qubit would decay in either $|0\rangle$ or $|1\rangle$ as well.

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Inadequacy of Probabilistic Bisimilarity

The usual notion of probabilistic bisimilarity is too fine, when comparing distributions of quantum configurations.

Solution I: Quantum Bisimilarity

Quantum Bisimilarity

We introduce an *equivalence* relation

$$\equiv \subseteq \mathfrak{D}(Conf) \times \mathfrak{D}(Conf)$$

and a new notion of **quantum bisimilarity**. Two processes are quantum bisimilar if they evolve in distributions that are bisimilar **up to equivalence**.

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Minimal Quantum Process Algebra

Solution II: mQPA

We introduce a new calculus, equipped with a minimal set of features: communication, non-determinism and quantum measurement.
In mQPA, the transitions are of the form

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In mQPA, the previous example can be rewritten as

$$(S \ {}_{|0\rangle\langle 0|}\boxplus \ S')(|+\rangle) = S \ {}_{\frac{1}{2}}\oplus \ S'$$
$$(S \ {}_{|+\rangle\langle +|}\boxplus \ S')(|0\rangle) = S \ {}_{\frac{1}{2}}\oplus \ S'$$

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- **lqCCS**, an asynchronous linear calculus inspired by qCCS. It rephrases the syntax e semantics of previous calculi in a more standard formalism, and allows to compare different notions of bisimilarity.
- Even though quantum systems exhibit a probabilistic behaviour, **probabilistic bisimilarity** is not really well suited for the quantum setting.
- **mQPA**, a minimal calculus, pursuing a foundational approach on which are the dynamics and observable properties of quantum systems.

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- We will investigate the relation between **mQPA** semantics and the usual, configuration-based semantics, together with their respective bisimilarities.

Thank you for your attention!