Exploring Quantum Process Calculi

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Qubits

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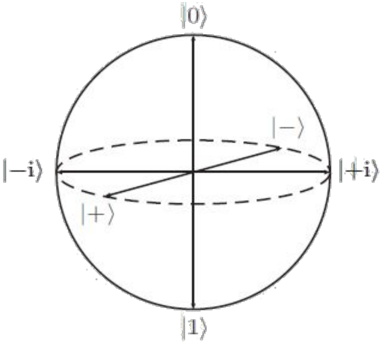
The simplest quantum system is a **qubit**. Like a bit, a qubit has two separate states, $|0\rangle$ and $|1\rangle$.
A qubit can also be in a linear combination of states, knows as a **superposition**.

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



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M_{01}\left(|+\rangle \right)
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Interestingly, the same happens when we measure $|0\rangle$ in the $+-$ basis, we get either the outcome $|+\rangle$ either the outcome $|-\rangle$ with the same probability 0.5.

No Cloning theorem

No-Cloning

Quantum information cannot be **duplicated**. That is, given a qubit q_1 in state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

it is impossible to prepare a qubit q_2 in the same state, without destroying the information in q_1.

This means that, contrary to classical bits, qubits cannot be freely copied, stored or broadcasted to multiple receivers.

Linear qCCS: Type System

lqCCS

Linear qCCS (lqCCS) is an asynchronous value-passing calculus, equipped with a **linear type system** to regulate quantum communication. It features parallelism, non-determinism and quantum operations.

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The type system is needed to prevent **duplication** of quantum information: after receiving a qubit, a process must send it **exactly once**.

$$c?q.H(q).c!q \qquad a?q.\big(b!q \parallel c!q\big) \qquad c?q.H(q).\mathbf{0}$$

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$$P = c?q.X(q).\mathbf{0} \qquad Q = c?q.H(q).\mathbf{0}$$

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Are P and Q **observationally equivalent**? It depends on what happens to q, different calculi followed different assumptions.

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In lqCCS, the programmer must **explicitly** describe what happens to each qubit

$$P' = c?q.X(q).c!q \qquad Q' = c?q.H(q).c!q$$

$$P'' = c?q.X(q).disc(q) \qquad Q'' = c?q.H(q).disc(q)$$

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$$\langle\, |0\rangle\,, H(q).c!q \parallel c?x.M_{01}[x \rhd y].disc(x) \rangle$$
$$\downarrow$$
$$\langle\, |+\rangle\,, c!q \parallel c?x.M_{01}[x \rhd y].disc(x) \rangle$$
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$$\langle\, |+\rangle\,, M_{01}[q \rhd y].disc(q) \rangle$$
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$$\langle\, |0\rangle\,, disc(q) \rangle \;{}_{\frac{1}{2}}\!\oplus \langle\, |1\rangle\,, disc(q) \rangle$$

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- Context: a "program" with a hole, like $[-] \parallel R$. We compare P and Q "inside" this context, i.e. we study $P \parallel R$ and $Q \parallel R$.

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The two processes seen before

$$P' = c?q.X(q).c!q \qquad Q' = c?q.H(q).c!q$$

are not bisimilar, because there is a context R which tells them apart

$$R = c?q.M_{01}[q \triangleright x].(disc(q) \parallel \mathbf{if}\ x = 0\ \mathbf{then}\ a!0\ \mathbf{else}\ b!0)$$

Probabilistic Bisimilarity

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Consider the two configurations

$$\mathcal{C} = \langle\, |+\rangle\, , M_{01}[q \triangleright x].c!q \rangle \rightarrow \langle\, |0\rangle\, , c!q \rangle\ {}_{\frac{1}{2}}\!\oplus \langle\, |1\rangle\, , c!q \rangle$$
$$\mathcal{C}' = \langle\, |0\rangle\, , M_{\pm}[q \triangleright x].c!q \rangle \rightarrow \langle\, |+\rangle\, , c!q \rangle\ {}_{\frac{1}{2}}\!\oplus \langle\, |-\rangle\, , c!q \rangle$$

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According to the usual notion of **probabilistic bisimilarity**, two distributions are bisimilar if they assign the same probability to bisimilar processes. In our example, the two configurations are **not** bisimilar, because they evolve in two non-bisimilar distributions.

Indistinguishable distributions

According to quantum mechanics, it's impossible to distinguish $\mathcal{C}$ and $\mathcal{C}'$, if we treat them as two sources of qubits:

$\mathcal{C}$ emits a qubit $|0\rangle$ or $|1\rangle$ with the same probability 0.5

$\mathcal{C}'$ emits a qubit $|+\rangle$ or $|-\rangle$ with the same probability 0.5

Suppose we receive a qubit from either $\mathcal{C}$ or $\mathcal{C}'$, and measure it in the 01 basis. If we measure a qubit from $\mathcal{C}$, it would result in either $|0\rangle$ or $|1\rangle$. If we measure a qubit from $\mathcal{C}'$, a $|+\rangle$ qubit would decay in either $|0\rangle$ or $|1\rangle$, and a $|-\rangle$ qubit would decay in either $|0\rangle$ or $|1\rangle$ as well.

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Inadequacy of Probabilistic Bisimilarity

The usual notion of probabilistic bisimilarity is too fine, when comparing distributions of quantum configurations.

Solution I: Quantum Bisimilarity

Quantum Bisimilarity

We introduce an **equivalence** relation

$$\equiv \subseteq \mathfrak{D}(Conf) \times \mathfrak{D}(Conf)$$

and a new notion of **quantum bisimilarity**. Two processes are quantum bisimilar if they evolve in distributions that are bisimilar **up to equivalence**.

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Solution II: mQPA

We introduce a new calculus, equipped with a minimal set of features: communication, non-determinism and quantum measurement.
In mQPA, the transitions are of the form

$$\rightarrow \subseteq S \times \mathfrak{D}(S)^{\mathcal{H}}$$

where $\mathfrak{D}(S)^{\mathcal{H}}$ is the set of **quantum distributions**. The probabilistic observable behaviour is **parametric** with respect to an input quantum state.

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In mQPA, the previous example can be rewritten as

$$(S \;{}_{|0\rangle\langle 0|}\boxplus\; S')(|+\rangle) = S \;{}_{\frac{1}{2}}\oplus\; S' \qquad (S \;{}_{|+\rangle\langle +|}\boxplus\; S')(|0\rangle) = S \;{}_{\frac{1}{2}}\oplus\; S'$$

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- Even though quantum systems exhibit a probabilistic behaviour, **probabilistic bisimilarity** is not really well suited for the quantum setting.
- **Quantum bisimilarity** relaxes the conditions of probabilistic bisimilarity, better representing quantum systems.
- **mQPA**, a minimal calculus, pursuing a foundational, where the probabilistic classical behaviour is parametric with respect to quantum values.

Future work

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- From weak transitions, it is possible to define reachability, temporal logics and **model checking**.
- Saturated bisimilarity can be cumbersome to prove, it will be interesting to explore how the existing **proof techniques** adapt to the quantum setting.
- We will investigate the relation between **mQPA** semantics and the usual, configuration-based semantics.

Thank you for your attention!