NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2021-2022

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

April/May 2022 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **SECTION A & SECTION B** and comprises **SIX (6)** pages.
2. **COMPULSORY** to answer **ALL** questions in both sections.
3. Marks for each question are as indicated.
4. This is a **RESTRICTED OPEN BOOK** examination. One double-sided A4 size reference sheet of paper is allowed.

SECTION A

1 (a) If $p(x,y)$ satisfies the partial differential equation

$$\frac{\partial}{\partial x}(x^2\frac{\partial p}{\partial x})+x^2\frac{\partial^2 p}{\partial y^2}=0 \text{ for } x\neq 0,$$

and if $u(x,y)=xp(x,y)$, show that $u(x,y)$ is a solution of the two-dimensional Laplace's equation.

(6 marks)

(b) Apply the method of separation of variables on the partial differential equation

$$2\frac{\partial \psi}{\partial x}+\frac{\partial}{\partial y}(e^{x+y}\frac{\partial \psi}{\partial y})=\psi$$

to obtain a pair of ordinary differential equations having an arbitrary constant in them. (Note. Let $\psi(x,y)=X(x)Y(y)$. Do <u>not</u> attempt to solve the ordinary differential equations.)

(6 marks)

Note: Question 1 continues on page 2.

(c) Consider the boundary value problem defined by the partial differential equation

$$9\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0 \text{ for } 0<x<2 \text{ and } 0<y<6,$$

and the boundary conditions

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0}=0 \quad\text{and}\quad \left.\frac{\partial \phi}{\partial x}\right|_{x=2}=0 \quad\text{for}\quad 0<y<6,$$
$$\phi(x,0)=0 \quad\text{and}\quad \phi(x,6)=x \quad\text{for}\quad 0<x<2,$$

where $\phi(x,y)$ is the unknown function to be determined and x and y are Cartesian coordinates of points in space.

(i) Verify by direct substitution that

$$\phi(x,y)=A_0 y+B_0+\sum_{n=1}^{\infty}(A_n e^{3n\pi y/2}+B_n e^{-3n\pi y/2})\cos(\frac{n\pi x}{2})$$

satisfies the partial differential equation in the boundary value problem above for arbitrary constant coefficients A_n and B_n (for $n=0,1,2,3,\cdots$).

(6 marks)

(ii) Verify that the boundary conditions on the sides $x=0$ and $x=2$ of the rectangular solution domain are satisfied by the function $\phi(x,y)$ given in part (i) above.

(3 marks)

(iii) Choose B_0 and express A_n in terms of B_n (for $n=1,2,3,\cdots$) in such a way that the boundary condition on the side $y=0$ of the rectangular solution domain is satisfied.

(3 marks)

(iv) Use parts (i), (ii) and (iii) to solve the boundary value problem above.
(Note. You may use: $\int x\cos(px)dx=\frac{1}{p^2}(\cos(px)+px\sin(px))+C.$)

(6 marks)

2 (a) Figure 1 shows a truss member AB of length $L = 1\text{m}$ pin-joined to a rigid support at B and a spring of stiffness $k = 5\times10^5$ N/m at A. The other end of the spring is connected to a roller (at C) that can roll freely in the vertical direction. The truss member has an axial rigidity of $EA = 10^6$ Nm^2. A concentrated force P acts in the downward direction at A. The spring always remains horizontal because of the roller support at C.

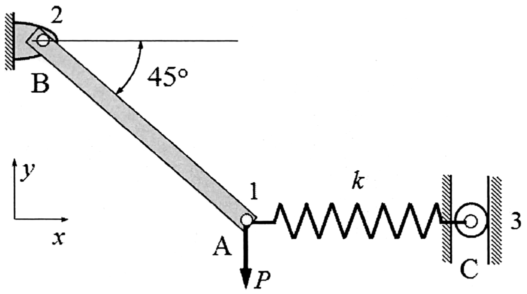

Figure 1

Model the structure using a *truss element* to represent the truss member and a *spring element* to represent the spring. Take node 1 as the reference node for measuring the angle of inclination of the truss element.

Write the element stiffness matrices and assemble them to obtain the global equilibrium equations, and thereby solve the equations for the vertical deflection at A in terms of P.

(12 marks)

(b) Assuming a quadratic trial solution in the form $\tilde{u} = c_0 + c_1 x + c_2 x^2$, solve the following BVP by Galerkin's method of weighted residuals:

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = 5, \qquad 0 < x < 1$$

$$u = 0 \quad \text{at } x = 0$$

$$u = 0 \quad \text{at } x = 1$$

Ensure that the trial solution satisfies the admissibility requirements before attempting to solve the BVP.

(8 marks)

SECTION B

3 (a) Figure 2 shows a finite element assemblage of four weightless springs of same stiffness k. Nodes 1, 3 and 4 are connected to rigid supports as shown. The rigid slab is weightless.

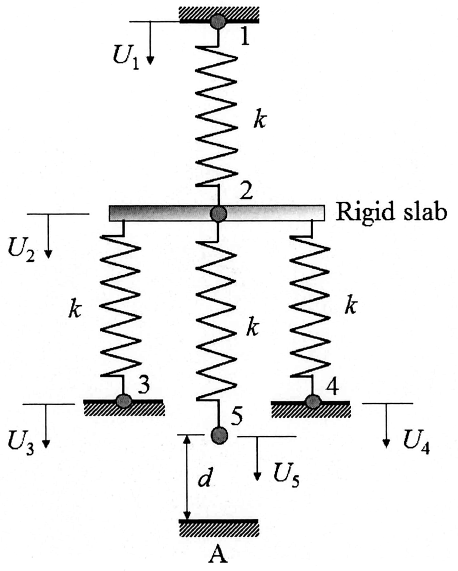

Figure 2

It is intended to pull node 5 downward by a distance d (as shown) so as to attach node 5 to the rigid support at A.

Obtain the global equilibrium equations by assembling the element stiffness matrices, and thereby solve the equations for the downward pulling force that is needed at node 5.

(8 marks)

(b) Figure 3 shows a tapered bar AB of length 2m. The mass per unit length $\rho(x)$, which is called the linear mass density, varies with x coordinate in the form $\rho(x) = 100(2 - x)$. The bar rotates in the xy-plane about end A with a constant angular velocity of $\omega = 5$ rad/s. It is intended to model the bar using a single 3-node bar element as shown. The centrifugal force per unit length due to rotation is $\rho(x)x\omega^2$. Ignore acceleration due to gravity.

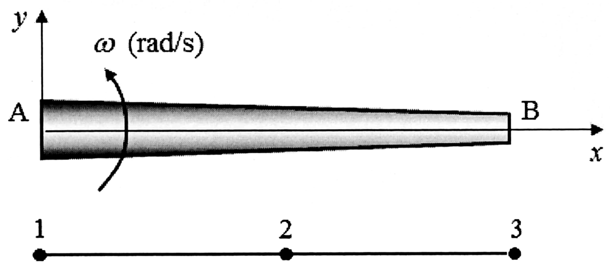

Figure 3

Determine the equivalent nodal force at node 2 using the formula $f_2 \equiv \int_0^L N_2\, T\, dx$.

(7 marks)

4 (a) The Fokker-Planck equation is important in the study of dynamics. Consider the one-dimensional Fokker–Planck equation in the unknown function $\phi(x,t)$, with a constant drift coefficient μ and a positive constant diffusion coefficient σ, as given by

$$\frac{\partial \phi}{\partial t} = \mu \frac{\partial \phi}{\partial x} + \sigma \frac{\partial^2 \phi}{\partial x^2}.$$

Use the uniform mesh in Figure 4, and apply an implicit time scheme with a uniform time step Δt for temporal discretization and an upwind differencing scheme (central differencing discretization for the diffusion term and upwind differencing scheme for the convection term) with a uniform space step Δx for spatial discretization, to derive the discretized equation of the standard form $a_P \phi_p = a_E \phi_E + a_W \phi_W + a_P^0 \phi_p^0 + S_u$, at only the internal node for each of the following cases.

(i) Positive constant drift coefficient ($\mu > 0$).

(10 marks)

(ii) Negative constant drift coefficient ($\mu < 0$).

(10 marks)

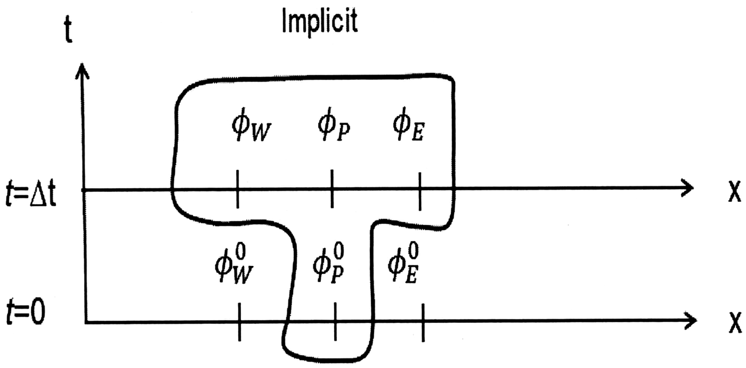

Figure 4

Note: Question 4 continues on page 6.

(b) (i) Rewrite the system of equations given by

$$\left.\begin{aligned} \phi_1 + 2\phi_2 &= 3 \\ 2\phi_1 - \phi_2 &= 1 \end{aligned}\right\}$$

in such a way that it satisfies the Scarborough criterion. (Check that your rewritten system satisfies the criterion.)

(7 marks)

(ii) Solve iteratively your rewritten system by using both the Jacobi iteration method and the Gauss-Seidel iteration method. Take the initial guess as $\phi_1 = \phi_2 = 0$ and consider only the first three iterations (excluding the initial guess). For each of the methods, compare your results with the exact solution $\phi_1 = \phi_2 = 1$.

(8 marks)

End of Paper

MA3004 MATHEMATICAL METHODS IN ENGINEERING

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.