Part (A1): Energy Methods

T1: Principle of virtual displacements ($\delta W = 0$) *For static mechanisms w/o springs; Gives force needed for equilibrium.*

1. Define coordinates along directions of forces. Mark virtual displacements δx and δy in the positive directions.
2. Express total virtual work as $F\delta x + W\delta y = 0$. Using geometry, express δx in terms of δy (or vice versa)
   a. Use $\delta x = \frac{dx}{dy}\delta y$. Alternatively, define both δx and δy in terms of δθ.
3. Substitute expressions back into Eqn in (2), then eliminate all virtual displacement terms

T2: Principle of virtual work ($\delta W = \delta U$) *For syst. w/ springs; Gives force expression.*

1. Define coordinates along directions of forces. Mark virtual displacements δx and δy in the positive directions.
   a. (IMPT) Use fixed point as datum. δx is positive in the same direction as x. If δx and F are in opp. directions, then $\delta W = -F\delta x$
2. Express total virtual work as $F\ \delta x + W\ \delta y = F\ \delta e = ke\ \delta e$
3. Using geometric relations, express all virtual displacements in terms of δθ (or any other method)
   a. Spring extension e = final length – initial length

T2: Principle of virtual complementary work ($\delta W^* = \delta U^*$) *For syst. w/ springs; Gives displacement expression.*

1. δW^* refers to the virtual complementary work done by all the external forces; δU^* refers to virtual complementary strain energy of all the deformable bodies
2. Express total virtual complementary work as $\Delta_x\ \delta F_x + \Delta_y\ \delta F_y = e_1\ \delta F_1 = \frac{F_1}{k_1}\ \delta F_1$
3. Eliminate all δF terms from the LHS except at the point to be determined
4. Use separate FBDs for equilibrium of real forces AND virtual forces respectively. Express everything in terms of δF_x, then eliminate all δF_x terms to obtain expression for real displacement Δ_x.

T3/4: Unit Load Method ($1 \cdot \Delta_x = \delta U^*$) *For statically determinate/indeterminate structures, elastically yielding supports, impact load etc.*

1. Similar method to PVCW, but apply one virtual load at the point of interest x, along the direction of displacement to be found. Can also be a unit moment ($1 \cdot \theta_x = \delta U^*$) to find angular displacement at x.
2. Spring + non-spring expressions for virtual strain energy δU^*: *(can use GC to solve/check!)*

$$1 \cdot \Delta_i = \delta U^* = \int_0^L \frac{Pp}{EA}dx + \int_0^L \frac{Mm}{EI}dx + \int_0^L \frac{Qq}{GA}dx + \int_0^L \frac{Tt}{GJ}dx + \frac{Ff}{k_{spring}}$$

3. In FBDs, +ve moment direction can be assigned arbitrarily, as long as consistent in real and virtual FBDs.
4. M = real moment expression, m = virtual moment expression. Either in terms of dS or $dS = Rd\theta$.
5. For statically indeterminate, can use $\Delta_x = 0$ at the support, to obtain an additional equilibrium equation
6. **Impact Load:** first determine static deflection ΔS (using unit load method, etc.) by treating the load as a gradually applied load. Then, to determine the maximum deflection: $\Delta_{dyn} = \Delta_s\left(1 + \sqrt{1 + \frac{2h}{\Delta_s}}\right)$

Trig Identities

$$sin(A \pm B) = sinAcosB \pm sinBcosA$$
$$cos(A \pm B) = cosAcosB \mp sinAsinB$$

$$tan(A \pm B) = \frac{tanA \pm tanB}{1 \mp tanAtanB}$$

Degree of indeterminacy = $r - n$
R = unknown support reactions (force/moments)
N = useful static equilibrium equations
unit load provides extra equation

Part (A2): Fracture Mechanics & Fatigue

T5/6: Fracture (Brittle LEFM; Plastic Zone Correction; Ductile) ***IMPT: crack length a** = "half-crack" length*

a) Griffith's Criterion. $\boldsymbol{G = \frac{\pi\sigma^2 a}{E}k}$ (k=1 for thin plate, plane stress; k = $(1-v^2)$ for thick block, plane strain)

- G = strain energy release rate = energy absorbed by crack surfaces when the crack grows by unit length.
- $G_C = \frac{\pi\sigma_f^2 a_c}{E}$ = critical strain energy release rate = material property; higher value means harder for cracks to propagate = high fracture resistance. **Units kJ/m² or kN/m**

b) Irwin's Theory. Stress Intensity Factor $\boldsymbol{\sigma_{tip} = \frac{K_I}{\sqrt{2\pi r}}}$ *(r = distance from crack tip, K_I = stress intensity factor)*

- $\mathbf{K_I = Y\sigma\sqrt{\pi a}}$ *(σ = applied nominal stress, Y = geometric constant = 1 for centre crack in infinitely large plate)*
- $\mathbf{K_{IC} = Y\sigma_f\sqrt{\pi a_c}}$ *(σ$_f$ = fracture stress; a$_c$ = critical crack length;* Fracture condition: $K_I > K_{IC}$)
    - K_{IC} is a property of the material, will be given. **Units MPa m$^{1/2}$**. Aka Fracture Toughness Equation.
- Rectangular Plate: $Y = \left(\frac{W}{\pi a}\tan\frac{\pi a}{W}\right)^{\frac{1}{2}}$ (w = plate width, parallel to crack

c) Combination: Relationship between G_C and K_{IC}: $\left(\frac{EG_c}{k}\right)^{\frac{1}{2}} = K_{IC}$ *(k=1 for plane stress; k=(1-v²) for plain strain)*

Experimental Determination of K_{IC}: (Compact Tension / Bend Test) $\boldsymbol{K_Q = \frac{F_Q}{B\sqrt{W}} f\left(\frac{a}{w}\right)}$ -- *(f $\left(\frac{a}{w}\right)$ table provided, F_Q=F at 95% line)*

- Conditions to take $K_Q = K_{IC}$: ***(1)*** $\frac{F_{max}}{F_Q} < 1.1$ and ***(2a)*** $B, a, (w-a) \geq 2.5\left(\frac{K_Q}{\sigma_Y}\right)^2$ and ***(2b)*** $0.45 < \left(\frac{a}{w}\right) < 0.55$

Plastic Zone Correction: $K_{I,corr.} = Y\sigma\sqrt{\pi(a + r_p)}$; plastic zone radius $r_p = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_y}\right)^2$ *(plane stress)*, $r_p = \frac{1}{6\pi}\left(\frac{K_I}{\sigma_y}\right)^2$ *(plane strain)*

- *Increases the crack length by accounting for the plastic zone around the crack tip, i.e., more energy is stored in the same crack length*
- ***Decreases the tolerable defect size****, and increases calculated fracture toughness for identical cracking conditions*
- *No longer valid when plastic zone is large relative to crack size, or when plastic zone near free edge*

Crack Tip Opening Displacement (COD): $\delta_c = \frac{K_{IC}^2}{\lambda\sigma_Y E}$ (plane stress); $\delta_c = \frac{K_{IC}^2(1-v)}{\lambda\sigma_Y E}$ (plane strain); Usually $\lambda = 1$

J-Integral: $J = \frac{2U}{B(W-a)}$ *(U = Strain energy = area under force-displacement curve; B = Specimen thickness; W = Specimen width; a = Crack length)*

- Critical J value $J_C = G_C$; can use $K_{IC} = (EJ_c)^{\frac{1}{2}}$

T6/7 Fatigue

Modified Endurance Limit $S_e = \frac{S_e' C_{size} C_{load} C_{surface}}{K_f}$ ***(TBC: MECH393 L7 says $\sigma' = K_f\sigma_{nom}$)***

- $S_e = 0.5S_u(UTS)$ for steels with $S_u < 690MPa$
- $S_e = 0.4S_u(UTS)$ for aluminium alloys with $S_u < 131MPa$ and copper alloys with $S_u < 96.5MPa$
- $C_{load} = 1.0\ for\ bending; 0.7\ for\ axial;\ 0.577\ for\ torsional$ (for multiple loading, choose highest)
- $C_{size} = 1.0\ for\ axial; 1.0\ if\ d \leq 8mm, 1.189d^{-0.097}\ if\ 8mm < d \leq 250mm$ (for torsional/bending)
- $C_{surface}$ graph will be provided

Find mean and alternating stress values using $\boldsymbol{\sigma = \frac{My}{I}}$ and $\boldsymbol{\tau = \frac{Tc}{J}}$ $I_{circular} = \frac{\pi d^4}{64}, I_{rectangular} = \frac{bh^3}{12}, J = \frac{\pi d^4}{32}$

Stress Ratio $= \frac{\sigma_{min}}{\sigma_{max}}$

Stress Concentration $K_t = \sigma_{max}/\sigma_{norm}$

Notch Sensitivity $q = \frac{K_f - 1}{K_t - 1}$ (Note $K_f < K_t$ always!)

Goodman: $\frac{\sigma_a \times SF}{S_e} + \frac{\sigma_m}{S_u} = 1$ Gerber: $\frac{\sigma_a \times SF}{S_e} + \left(\frac{\sigma_m}{S_u}\right)^2 = 1$ Soderberg: $\frac{\sigma_a \times SF}{S_e} + \frac{\sigma_m}{S_y} = 1$

Shot Peening: Shot peening is a cold working process used to produce a compressive residual stress layer and modify mechanical properties of metals. It entails impacting a surface with shot (round metallic, glass, or ceramic particles) with force sufficient to create plastic deformation.

Multi-axial loading:

Calculate effective mean and alternating stresses, then apply Goodman/Gerber/Soderberg similarly

- Effective Uniaxial Stress Amplitude: $\overrightarrow{\sigma_a} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{xa} - \sigma_{ya})^2 + (\sigma_{ya} - \sigma_{za})^2 + (\sigma_{za} - \sigma_{xa})^2 + 6(\tau_{xya}^2 + \tau_{yz}^2 + \tau_{zxa}^2)}$
- Effective Uniaxial Mean Stress: $\overrightarrow{\sigma_m} = \sigma_{xm} + \sigma_{ym} + \sigma_{zm}$

Miner's Rule for Cumulative Fatigue Damage: $\sum n_i/N_i = 1$ *(n = cycles operated @ stress i, N = cycles to failures @ stress i)*

Fatigue Crack Growth Rate $\frac{da}{dN} = C\Delta K^m = C(K_{max} - K_{min})^m = C(YS_r\sqrt{\pi a})^m$ **(Unit: MPa/m$^{0.5}$)**

- c, m = constants related to material etc. (typical c value: x$10^{-11\sim-12}$); S_r = stress range $\sigma_{max} - \sigma_{min}$ **(Unit: MPa)**

Fatigue Life Formula $\boxed{N_f = \frac{2}{C(YS_R)^m \pi^{m/2}(2-m)}\left(a_f^{1-m/2} - a_0^{1-m/2}\right)}$ (a_0; a_f = initial/final crack size, in m; N_f = cycles to failure)

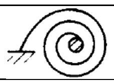
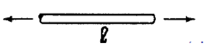
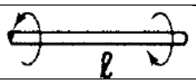
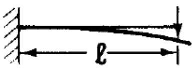
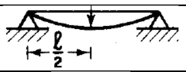
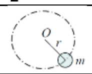
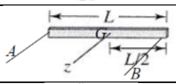
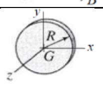
- *Tip: Save this formula in GC, with constant m!*
- Determine final crack size a_f from fracture toughness equation $\mathbf{K_{IC}} = \boldsymbol{Y\sigma_f\sqrt{2\pi a_c}}$

Part (b): Vibrations

T8/9: Undamped & Damped Free Vibration

NOTE: For vertical vibrations, static forces (e.g. m_1g) should be cancelled out at SEP by spring forces. Draw static FBD, then its terms should cancel themselves in the dynamic EOM.

IMPT: Always keep +ve direction consistent, after marking fixed reference point (usually take SEP as reference)

Cantilever beams as effective spring-mass system with $k = \frac{F}{\Delta}$		
	$k = \frac{EI}{l}$, (Torsional stiffness)	I = moment of inertia of cross-sectional area; l = total length
	$k = \frac{EA}{l}$	A = cross-sectional area
	$k = \frac{GJ}{l}$	J = polar moment of inertia of cross-sectional area
	$k = \frac{3EI}{l^3}$	
	$k = \frac{48EI}{l^3}$	
	$J_o = mr^2$ (for point mass)	
	$J_z = \frac{1}{12}mL^2$	$J_A = J_B = \frac{1}{3}mL^2$
	$J_z = \frac{1}{2}mR^2$	$J_x = J_y = \frac{1}{4}mR^2$
Parallel Axis Theorem: $J = J_{CG} + Md^2$		

$$m\ddot{x} + c\dot{x} + kx = 0$$

Overall system's inertia, damping and spring constants may not be equal to the individual elements.

$$\omega_n = \sqrt{k/m}$$

$$c_c = 2\sqrt{km} = 2m\omega_n \text{ (unit: Ns m}^{-1}\text{)}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$\boldsymbol{\zeta > 1}$: $x(t) = C_1e^{r_1t} + C_2e^{r_2t}$

- $r_{1,2} = -\zeta - \omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$\boldsymbol{\zeta = 1}$: $x(t) = (C_1 + C_2t)e^{-\omega_nt}$

$\boldsymbol{\zeta < 1}$: $x(t) = e^{-\zeta\omega_nt}(A\cos\omega_dt + B\sin\omega_dt)$

$$= X\,e^{-\zeta\omega_nt}\sin(\omega_dt + \phi)$$

- Find C1, C2, X, ϕ from initial conditions
- Equations written for rectilinear; Swap m, c, k for J_0, c_θ, k_θ to obtain rotational.
- No damping: $\zeta = 0$

Logarithmic Decrement over n cycle: $\delta = \frac{1}{n-1}\ln\frac{x_1}{x_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$; Damping Ratio $\zeta = \sqrt{\frac{\delta^2}{4\pi^2+\delta^2}}$

T10/11: Forced Vibration of 1-DoF System

Spring-Mass-Damper System under Harmonic Force Excitation

EOM: $m\ddot{x} + c\dot{x} + kx = F_0\sin\omega t$

- Amplitude $X = \frac{F_0}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}} = \frac{F_0/k}{\sqrt{(1-(\omega/\omega_n)^2)^2+(2\zeta\omega/\omega_n)^2}}$, Phase $\phi = \tan^{-1}\frac{c\omega}{k-M\omega^2} = \tan^{-1}\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}$
  - Magnification Factor: $\frac{X}{F_0/k} = \frac{1}{\sqrt{(1-(\omega/\omega_n)^2)^2+(2\zeta\omega/\omega_n)^2}}$; At resonance $\omega = \omega_n$, $\frac{X_{res}}{F_0/k} = \frac{1}{2\zeta}$ or $X_{res} = \frac{F_0}{c\omega_n}$
  - At forced resonance, $\omega = \omega_n$ (unrelated to damped natural frequency, ω_d)

Spring-Mass (+ Damper) System with Rotating Unbalance

NOTE: Remove c term for systems without damping. (e.g. phase = 0)

EOM: $M\ddot{x} + c\dot{x} + kx = me\omega^2\sin\omega t$ *(i.e. replace F with $me\omega^2$, the centripetal force)*

(e = radial distance of eccentric mass m; M = total mass)

- SS Solution: $x_p = X\sin(\omega t - \phi)$
- Amplitude $X = \frac{me\omega^2}{\sqrt{(k-M\omega^2)^2+(c\omega)^2}}$; Phase $\phi = \tan^{-1}\frac{c\omega}{k-M\omega^2} = \tan^{-1}\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}$
    - Magnification Factor: $\frac{MX}{me} = \frac{(\omega/\omega_n)^2}{\sqrt{(1-(\omega/\omega_n)^2)^2+(2\zeta\omega/\omega_n)^2}}$

Harmonic **Base Excitation** of Spring-Mass (+ Damper) System:

NOTE: Without damper, substitute $c = \alpha = 0$

EOM: $m\ddot{x} + c\dot{x} + kx = F_{eq}\sin(\omega t + \alpha)$ where $F_{eq} = Y\sqrt{k^2 + (c\omega)^2}$

- SS Solution: $x = X\sin(\omega t + \alpha - \phi)$
- Amplitude $X = \frac{F_{eq}}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}$
- "Maximum vertical excursion" = 2 x amplitude

T11: Transient Vibration of 1-DoF

Modelling Drop Test without Damping

Use Initial Conditions to solve for constants in EOM (e.g. $x(0) = 0$, $v_0 = \sqrt{2gh}$)

General Solution: $x(t) = \sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{v_0}{\omega_n}\right)^2}\sin(\omega_n t + \phi) + \frac{mg}{k}$ where $\phi = \tan^{-1}\frac{-mg/k}{v_0/\omega_n}$

NOTE: Solving $\sin\theta = C$ *(see T11 Q4)*: $\theta = \arcsin(C) + 2n\pi$ or $\theta = \pi - \arcsin(C) + 2n\pi$

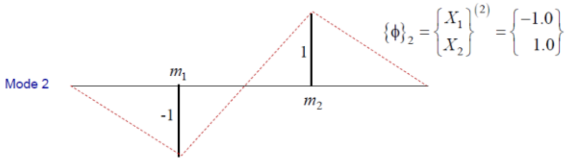
T12: Free/Forced Vibration of 2-DoF

(don't memorize; must understand and derive, possible to have many situations)

2-DoFs can have one x and one θ: Just use the same method with solution $\binom{\theta}{x} = \binom{\Theta}{X}\sin\omega t$

Natural Frequencies and Modeshapes

1. **Find natural frequencies**. Write out EOMs in matrix form. Substitute in harmonic solution form $x = X\sin\omega t$ and its derivatives, then eliminate constant $\sin\omega t$ term. To find ω_{n1} and ω_{n2}, determinant of the system matrix must be zero.
2. **Find Modeshapes**. Substitute expression for ω_{ni} into any EOM. Amplitude Ratio = X_1/X_2 or even Θ/X
    a. modeshape vector: $\frac{X_1}{X_2} = -1$, then $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

    b. modeshape plot:

Using Cramer's Rule to find vibration amplitude (solve Matrix EOM in forced vibration)

To solve X1, replace the first column of the system matrix with the force vector. Find its determinant, then divide it by the determinant of the system matrix to obtain X1. (NOTE: Use GC to solve/check!)