NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2023-2024

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

April/May 2024 Time Allowed: $2\frac{1}{2}$ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **FIVE (5)** pages.
2. Answer **ALL** questions.
3. Marks for each question are as indicated.
4. This is **RESTRICTED OPEN BOOK EXAMINATION**. You are allowed to bring into the examination hall one double-sided A4-size reference sheet with texts handwritten or typed on the A4 paper or one restricted material as instructed by the examiner(s) without any attachments (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).

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1 (a) A partial differential equation in $u(x, y, t)$ is given by

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + b\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right),$$

where a and b are given constants. If the function $w(x, y, t)$ is related to $u(x, y, t)$ by

$$w(x, y, t) = e^{bu(x,y,t)/a},$$

derive a partial differential equation in $w(x, y, t)$.

(10 marks)

(b) Consider the initial-boundary value problem defined by the partial differential equation

$$\frac{\partial^2 \phi}{\partial t^2} + 2\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \quad \text{for } 0 < x < 1 \text{ and } t > 0,$$

and the conditions

$$\phi(x, 0) = 0 \text{ and } \left.\frac{\partial \phi}{\partial t}\right|_{t=0} = 1 \text{ for } 0 < x < 1,$$

$$\phi(0, t) = 0 \text{ and } \phi(1, t) = 0 \text{ for } t > 0,$$

where ϕ is a function of the Cartesian coordinate x and the time coordinate t.

Note: Question 1 continues on page 2.

(i) Show by direct substitution that

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{-t}\left(A_n \sin(t\sqrt{n^2\pi^2 - 1}) + B_n \cos(t\sqrt{n^2\pi^2 - 1})\right) \sin(n\pi x)$$

is a solution of the governing partial differential equation of the initial-boundary value problem, no matter what the constant coefficients A_n and B_n are.

(6 marks)

(ii) Verify that the boundary conditions of the initial-boundary value problem are satisfied by the series solution in part (i).

(3 marks)

(iii) Use part (i) to solve the initial-boundary value problem, that is, find the constant coefficients A_n and B_n in the series such that the initial conditions are satisfied.

(6 marks)

2. Figure 1 shows a uniform beam AB and uniform bar BC pin-jointed together at B. The length of the beam is 1 m and its flexural rigidity (*EI*) is 3×10^5 Nm^2. The axial rigidity (*EA*) of the beam is 4×10^5 N. The length of the bar is 0.8 m and its axial rigidity (*EA*) is 3.2×10^5 N. The beam carries a distributed load of intensity $w = 2\times10^5$ N/m as shown. Pin-joint B is pulled down vertically (by applying a vertical downward force of unknown magnitude *P*) so that the vertical displacement at B is 0.1 m (which includes the displacement caused by the load *w*). Assume that the displacements due to the self-weight of the beam and bar are negligibly small and hence can be ignored.

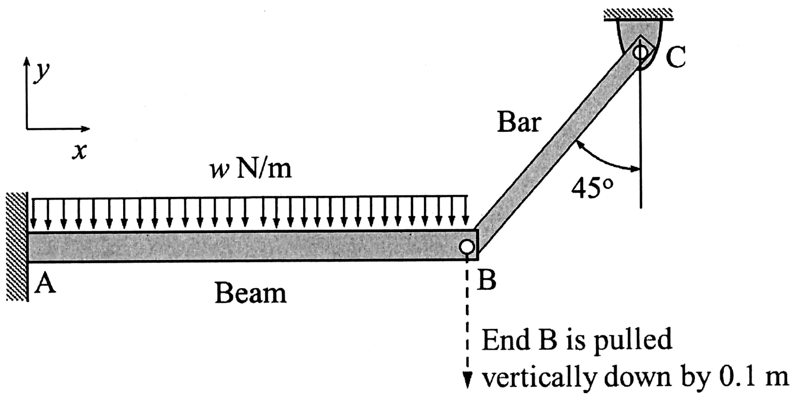

Figure 1

Consider modelling the beam as a *beam cum bar element* and the bar as a *truss element*.

Note: Question 2 continues on page 3.

(a) Draw the finite element model of the structure labelling all the node numbers, element numbers, nodal displacements/rotations, and nodal forces/moments. Use generalised symbols (Q_1, Q_2, Q_3, etc) for labelling the displacements/rotations and (F_1, F_2, F_3, etc) for labelling the forces/moments.

(5 marks)

(b) Lump the distributed load (w) acting on beam AB into equivalent nodal loads using appropriate formula. Show the lumped loads on a sketch of beam AB. List all the displacement boundary conditions for the problem.

(5 marks)

(c) Write down the element stiffness matrices and label their rows and columns. Assemble the element stiffness matrices and loads, and thereby obtain the global equilibrium equations in matrix form. Apply the boundary conditions by crossing out appropriate rows/columns. Solve the reduced system for the horizontal displacement of pin B and the corresponding cross-sectional rotation of the beam at B.

(10 marks)

(d) Using your solution for part (c), determine the magnitude of force P that is necessary to be applied at B so as to cause the vertical displacement of 0.1 m at B.

(5 marks)

3 (a) Figure 2 shows a spring assembly with stiffness values denoted as k_1, k_2 and k_3. The horizontal springs always remain horizontal and vertical springs always remain vertical because of roller supports at nodes 3, 4, 5 and 6. A force of P is acting on node 1 as shown. It is intended to solve this problem using the *principle of minimum potential energy*. Denote the horizontal displacements of nodes 1 and 2 as U_1 and U_2, and the corresponding vertical displacements as V_1 and V_2, respectively.

(i) Apply the principle of minimum potential energy to derive the equilibrium equations for the spring assembly in terms P, U_1, U_2, V_1, V_2, k_1, k_2 and k_3, and write them in matrix form.

(8 marks)

(ii) Solve the system of equations that you obtained in part (i) for U_1, U_2, V_1, and V_2.

(2 marks)

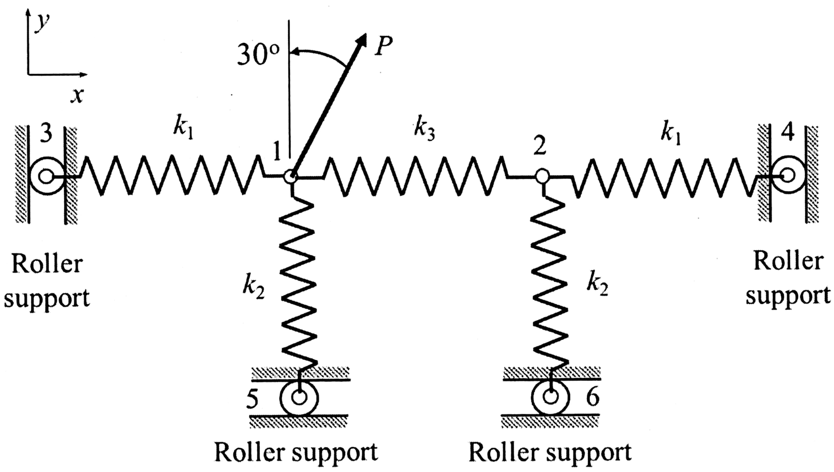

Figure 2

Note: Question 3 continues on page 4.

(b) Figure 3 shows a thin rectangular plane sheet ABCD with length AB = 1.5 m, breadth BC = 1 m and thickness 0.01 m (into the plane of the paper) subjected to concentrated as well as distributed loadings as indicated. There are two holes of 0.25 m diameter symmetrically placed with a spacing of 0.75 m between their centres as shown. It is intended to carry out a 2D finite element analysis of this thin sheet.

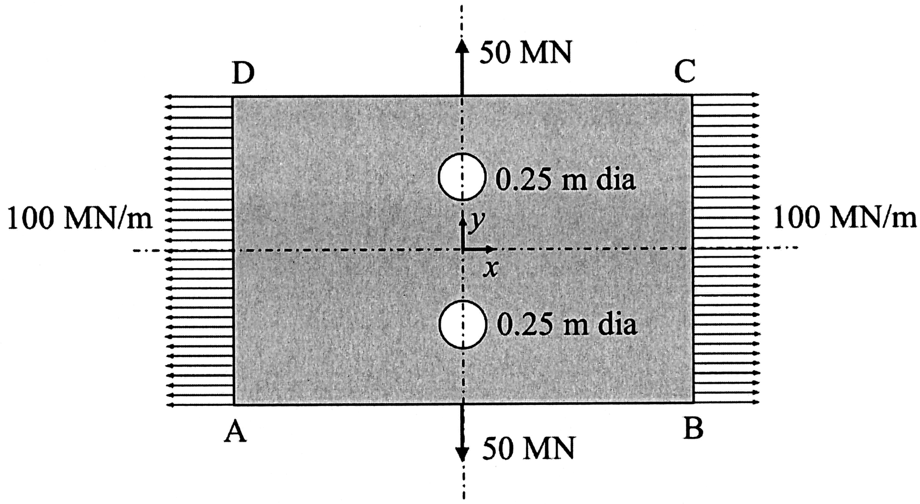

Figure 3

(i) Select (from the list given below) two types of finite element that are suitable for the 2D finite element analysis of the above sheet:

3-node triangular plane strain element

4-node quadrilateral plane strain element

3-node triangular plane stress element

4-node quadrilateral plane stress element

4-node tetrahedron plane stress element

8-node hexahedron plane stress element

(2 marks)

(ii) For the finite element modelling of the above problem, it is intended to exploit symmetry. Draw a neat sketch of that part of the geometry you would consider for the purpose. On the sketch, indicate all the boundary conditions and loads to be applied. Also indicate the region where you need to use a finer mesh.

(3 marks)

4(a) A fin is an extended surface that increases the rate of heat transfer to the environment by increasing convection. The rate of energy stored by the thermal capacity of a fin with uniform cross-sectional area can be modeled using the unsteady-state fin equation:

$$\frac{\partial T}{\partial t} - D\frac{\partial^2 T}{\partial x^2} + mT = 0,$$

Note: Question 4 continues on page 5.

where $D>0$ and $m>0$ are known constants, and T is a function of the time coordinate t and the spatial coordinate x between the boundary points A and B.

Use the computational grid in Figure 4 together with an **implicit time scheme** of a uniform time step δt to discretise the unsteady-state fin equation, that is, to derive equations of the form $a_P T_p = a_E T_E + a_W T_W + a_P^0 T_p^0 + S_u$ for the nodes stated in (i), (ii), and (iii) below.

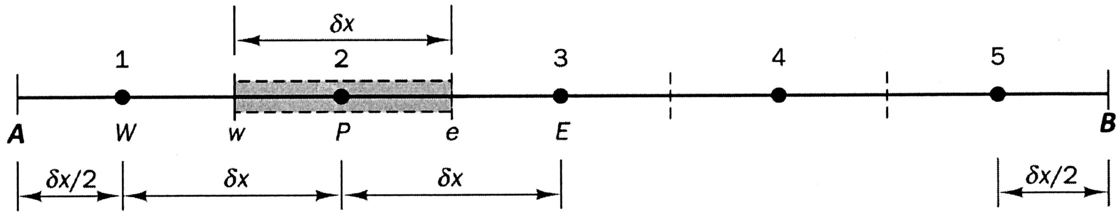

Figure 4

(i) At all internal nodes labeled 2, 3, and 4.

(10 marks)

(ii) At node 1, if the boundary condition at A is given by $T|_A = 1$.

(6 marks)

(iii) At node 5, if the boundary condition at B is given by $\left.\dfrac{\partial \ln T}{\partial x}\right|_B = 1$.

(4 marks)

(b) (i) Rearrange the three linear equations given by

$$\begin{cases} -x+y+2z=-3 \\ 2x+y-z=3 \\ x-3y+z=0 \end{cases}$$

to ensure convergence of the iterative method used for solving the equations. Find the first three iterations of the **Jacobi iterative method** using the starting values $x=0$, $y=0$, and $z=0$.

(8 marks)

(ii) Rearrange the two linear equations given by

$$\begin{cases} x+y=0 \\ 2x-y=3 \end{cases}$$

to ensure convergence of the iterative method used for solving the equations. Find the first three iterations of the **Gauss-Seidel iterative method** using the starting values $x=0$ and $y=0$.

(7 marks)

END OF PAPER

MA3004 MATHEMATICAL METHODS IN ENGINEERING

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.