NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2022-2023

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

November/December 2022 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **FIVE (5)** pages.
2. Answer **ALL** questions.
3. Marks for each question are as indicated.
4. This is a **RESTRICTED OPEN-BOOK** examination. One double-sided A4-size reference sheet with texts handwritten or typed on the A4 paper (no sticky notes/post-it notes on the reference sheet) is allowed.

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1 (a) By letting $T(x,y) = X(x)Y(y)$, apply the method of separation of variables on the partial differential equation

$$\frac{\partial^2 T}{\partial x^2} + (1 + x^4)\frac{\partial^2 T}{\partial y^2} = (y^2 + 3 + (y^2+3)x^4)T$$

to obtain a pair of ordinary differential equations containing an arbitrary constant.

(6 marks)

(b) Consider the boundary value problem defined by

$$25\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 10 \text{ for } 0 < x < 5 \text{ and } 0 < y < 1,$$

and the boundary conditions

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = \left.\frac{\partial \phi}{\partial x}\right|_{x=5} = 0 \text{ for } 0 < y < 1,$$

$$\left.\frac{\partial \phi}{\partial y}\right|_{y=0} = 0 \text{ for } 0 < x < 5,$$

$$\phi(x,1) + \left.\frac{\partial \phi}{\partial y}\right|_{y=1} = \begin{cases} 25 & \text{for} \quad 0 < x < \frac{5}{2}, \\ 15 & \text{for} \quad \frac{5}{2} < x < 5. \end{cases}$$

Note: Question 1 continues on page 2.

(i) Verify that

$$\phi(x,y) = 5y^2 + A_0 + \sum_{n=1}^{\infty} (A_n e^{n\pi y} + B_n e^{-n\pi y}) \cos\left(\frac{n\pi x}{5}\right),$$

is a solution of the partial differential equation in the boundary value problem. Note that A_0, A_n and B_n ($n = 1,2,3,\ldots$) are arbitrary constant coefficients.

(6 marks)

(ii) Show that the boundary conditions at $x = 0$ and $x = 5$ (for $0 < y < 1$) are automatically satisfied by $\phi(x,y)$ given in part (i).

(3 marks)

(iii) Express A_n in terms of B_n for $n = 1,2,3,\ldots$, in such a way that the boundary condition at $y = 0$ (for $0 < x < 5$) is satisfied by $\phi(x,y)$ given in part (i).

(3 marks)

(iv) Use parts (i), (ii) and (iii) to solve the boundary value problem stated above.

(7 marks)

2. Figure 1 shows a uniform beam AB of length L = 1 m and flexural rigidity $EI = 10^5$ Nm^2 pin-jointed to a vertical spring of stiffness $k = 10^7$ N/m at B. The other end of the spring is pin-jointed to a support at C. The beam has an axial rigidity $EA = 5\times10^6$ N. A concentrated load P = 100 kN acts at B as shown. The beam carries a distributed load of intensity w = 1 MN/m. Ignore the weights of beam and spring. Consider small displacements.

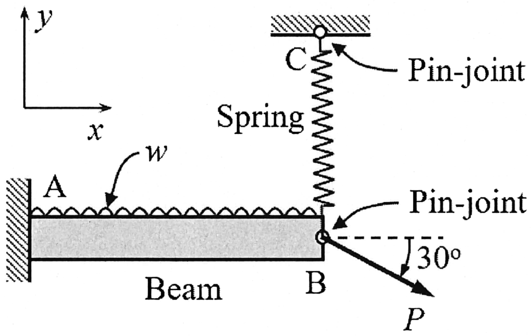

Figure 1

Note that under the given loads, the beam will not only undergo bending deformation but also axial deformation. It is intended to solve this problem using finite element method by modelling the beam as a *beam cum bar element* (so that bending as well as axial deformations can be captured) and the spring as a *spring element*. Show three significant places for all the results you calculate.

Note: Question 2 continues on page 3.

(a) Draw the finite element model of the structure with the symbols for the following clearly marked: Node numbers, element numbers, nodal displacements/rotations and nodal forces/moments. Use generalised symbols (Q_1, Q_2, Q_3, etc) for denoting the displacements/rotations and symbols (F_1, F_2, F_3, etc) for forces/moments.

(4 marks)

(b) Convert the distributed load w into equivalent nodal loads by using appropriate formula. Indicate all nodal forces, moments and reactions on a neat sketch of Figure 1, and thereby write down the expressions for F_1, F_2, F_3, etc).

(6 marks)

(c) Write down all the element stiffness matrices and label their rows and columns.

(5 marks)

(d) Assemble the element matrices to obtain global equilibrium equations, and apply the boundary conditions to obtain the reduced system of equations. (The reduced system of equations need not be solved.)

(10 marks)

3 (a) Figure 2 shows a three-layer composite wall of a furnace with the thermal conductivity values (k_T) for the layers indicated therein. The temperature of inner surface is 1800°C and that of the outer surface is 30°C. It is intended to carry out finite element analysis of this furnace by modelling it using three 2-node heat conduction elements. Consider heat flow through an area of 1 m^2 for the analysis. Determine the temperature at the interfaces between outer layer and middle layer, and inner layer and middle layer. Also determine the heat flowing though the composite wall. Show three significant places for all the results you calculate.

(10 marks)

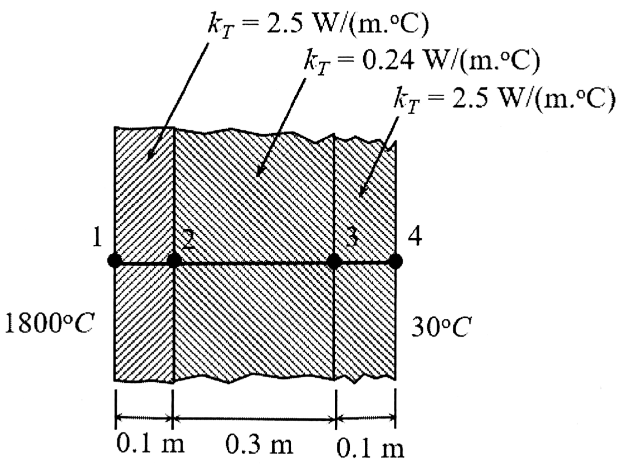

Figure 2

Note: Question 3 continues on page 4.

(b) Figure 3 shows an annular disc of inner diameter 0.3 m and outer diameter 0.5 m and thickness 0.05 m subjected to four forces of magnitude 50 kN as shown. It is intended to perform 2D finite element analysis using plane stress finite elements so as to determine the von-Mises stress at the inner surface of the disc.

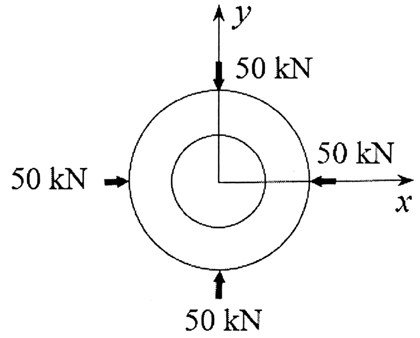

Figure 3

(i) Name two types of 2D finite elements that can be used for this analysis.

(2 marks)

(ii) Sketch a typical finite element mesh (using quadrilateral elements) exploiting the symmetry of the disc and show all the boundary conditions.

(3 marks)

4 (a) Consider the one-dimensional Euler-type advection equation in $\phi(x,t)$ given by

$$\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial x}\big(F(\phi)\big)=0 \quad \text{with} \quad F(\phi)=\phi+x.$$

Apply the finite volume method to the uniform mesh shown in Figures 4 and 5, together with an explicit time scheme of a uniform time step Δt for temporal discretization and an upwind differencing scheme of a uniform space step Δx for spatial discretization, to derive discretized equations of the form $a_P\phi_P = a_W^0\phi_W^0 + a_P^0\phi_P^0 + a_E^0\phi_E^0 + S_u$ for:

(i) the internal nodes 2, 3 and 4,

(10 marks)

(ii) node 1, if the boundary condition at A is given by $\phi\big|_A = 0$,

(5 marks)

(iii) node 5, if the boundary condition at B is given by $\left.\left(\dfrac{1}{\phi}\dfrac{\partial \phi}{\partial x}\right)\right|_B = 1$.

(5 marks)

Note: Question 4 continues on page 5.
Figures 4 & 5 appears on page 5.

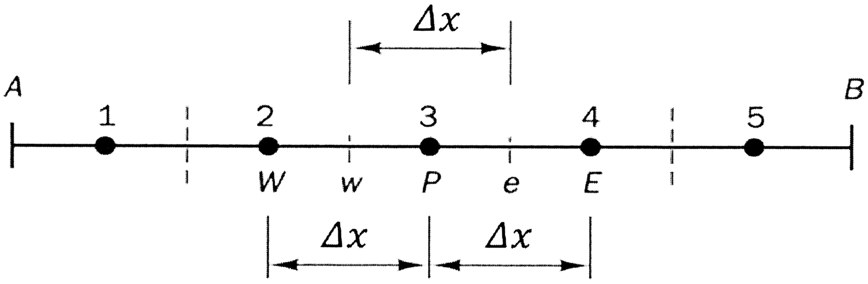

Figure 4

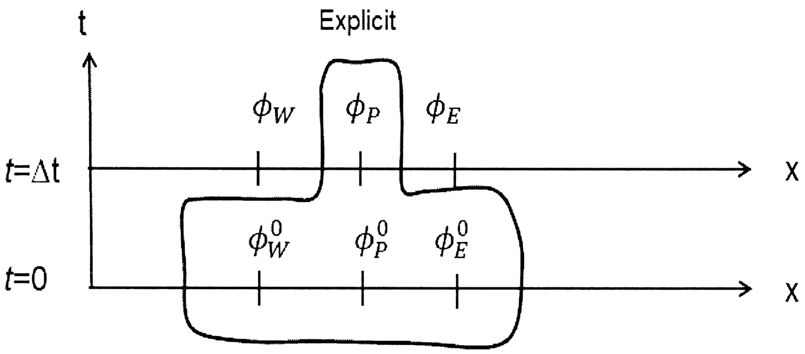

Figure 5

(b) Consider the system of linear equations given by

$$\begin{cases} 2x_1 + x_3 = 1 \\ -x_2 + 2x_3 = 1 \\ -x_1 + 2x_2 - 2x_3 = 0 \end{cases}$$

(i) Rewrite the above system in a form that ensures convergence in the Jacobi and the Gauss-Seidel iteration methods.

(5 marks)

(ii) Use the Jacobi iteration method to solve approximately your rewritten system of equations in part (i). Take $x_1 = 0$, $x_2 = -1$ and $x_3 = 1$ as your initial approximation and give only the first three iterations.

(5 marks)

(iii) Repeat part (ii) using the Gauss-Seidel iteration method.

(5 marks)

END OF PAPER

MA3004 MATHEMATICAL METHODS IN ENGINEERING

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.