NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2021-2022

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

November/December 2021 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **SECTION A & SECTION B** and comprises **FIVE (5)** pages.
2. **COMPULSORY** to answer **ALL** questions in both sections.
3. Marks for each question are as indicated.
4. This is a **RESTRICTED OPEN BOOK** examination. One double-sided A4 size reference sheet of paper is allowed.

SECTION A

1 (a) Find the values of the constants a and b if $T(x,y)=\left(x^3+axy^2+bx^2y-y^3\right)^{1/3}$ is a solution of the partial differential equation

$$\frac{\partial}{\partial x}(T^2\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(T^2\frac{\partial T}{\partial y})=0.$$

(6 marks)

(b) Apply the method of separation of variables on the partial differential equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+(x-y)(x+y)\psi=0$$

to obtain a pair of ordinary differential equations having an arbitrary constant in them. (Let $\psi(x,y)=X(x)Y(y)$. Do not attempt to solve the ordinary differential equations.)

(6 marks)

Note: Question 1 continues on page 2.

(c) Consider the boundary value problem defined by the partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4 \text{ for } 0 < x < 1, 0 < y < 2,$$

and the boundary conditions

$$\left.\begin{aligned} \left.\frac{\partial \phi}{\partial x}\right|_{x=0} &= 0 \\ \phi(1, y) &= y^2 \end{aligned}\right\} \text{ for } 0 < y < 2,$$

$$\left.\begin{aligned} \phi(x, 0) &= x^2 \\ \phi(x, 2) &= x^2 + 4 \end{aligned}\right\} \text{ for } 0 < x < 1$$

The unknown function to be determined is $\phi(x, y)$, where x and y are the Cartesian coordinates.

(i) Verify by direct substitution that

$$\phi(x, y) = x^2 + y^2 + \sum_{n=1}^{\infty} (A_n e^{n\pi x/2} + B_n e^{-n\pi x/2}) \sin(\frac{n\pi y}{2})$$

satisfies the partial differential equation in the boundary value problem. (Assume that the coefficients A_n and B_n are constants and the series converges.)

(6 marks)

(ii) Identify all the boundary conditions that are satisfied by the solution in part (i) for arbitrary constants A_n and B_n.

(3 marks)

(iii) Use the first boundary condition (that is, the one on $x = 0$) to express A_n in terms of B_n.

(3 marks)

(iv) Use parts (i), (ii) and (iii) to solve the boundary value problem.

(6 marks)

2. Figure 1 shows a stepped beam ABC of length $2L$ fixed to rigid walls at A and C. The flexural rigidity of portion AB is $2EI$ and that of BC is EI. The beam is subjected to two distributed loads as shown. It is intended to model each of portions AB and BC using a single beam element.

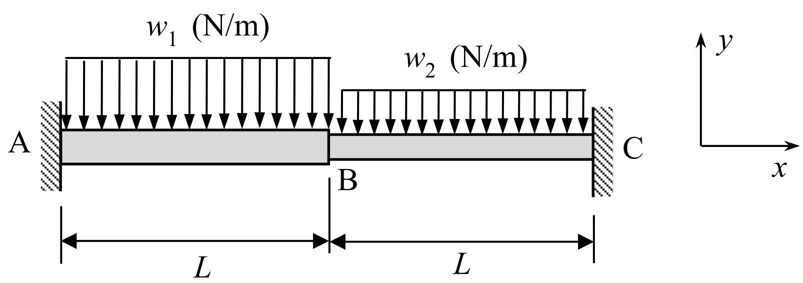

Figure 1

(a) Draw a finite element model of the beam showing element numbers, node numbers and symbols for nodal displacements/rotations and nodal forces/ moments. List all the boundary conditions.

(5 marks)

(b) Convert the distributed loads into equivalent lumped nodal loads using the formula $\left\{\mathbf{f}_q^e\right\} = \left[qL/2 \quad qL^2/12 \quad qL/2 \quad -qL^2/12\right]^T$ where q is the transverse load intensity (N/m) along y direction and show the nodal loads pictorially on a neat sketch of beam elements.

(5 marks)

(c) Write all element matrices and load vectors and assemble them to obtain the global equilibrium equations. Apply the boundary conditions to obtain a reduced system of equations.

(5 marks)

(d) For L = 1 m, EI = 10000 Nm^2, w_1 = 10000 N/m and w_2 = 5000 N/m, solve the reduced system of equations for the vertical deflection and cross sectional rotation at B.

(5 marks)

SECTION B

3 (a) A cantilever bar of length L and cross sectional area A is subjected to a distributed axial load of intensity cx^2 (N/m) where c is a constant and a concentrated load P is applied at the free end of the beam as shown in Figure 2. The strain energy (U) stored in the bar is given by

$$U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$$

where $u \equiv u(x)$ is the axial displacement and E is the Young's modulus of the bar material.

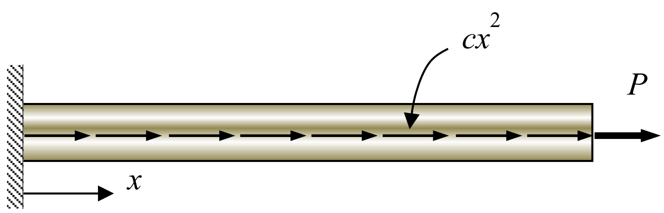

Figure 2

Solve this problem for the axial displacement by Rayleigh-Ritz method using a linear polynomial trial solution in the form $\tilde{u}(x) = a_0 + a_1 x$. Ensure the trial solution satisfies all admissibility criteria before attempting to solve the problem.

(8 marks)

(b) Figure 3 shows an assemblage of four spring elements with loads 1 N and 2 N acting on nodes 2 and 4, respectively. The stiffness values of the springs are given as k_1 = 1 N/m, k_2 = 2 N/m and k_3 = 3 N/m. The nodal displacements (in metre) have already been determined to be U_1 = 0, U_2 = 3, U_3 = 3.5 and U_4 = 4.1667.

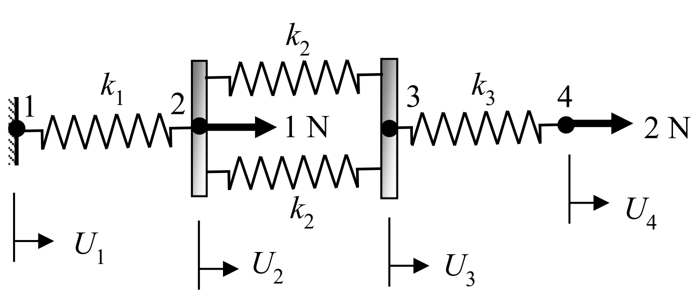

Figure 3

Determine the internal forces in the springs and represent all the forces pictorially on a neat sketch of springs.

(7 marks)

4 (a) The one-dimensional steady convection–diffusion of a property ϕ is governed by the ordinary differential equation $\frac{d}{dx}\left(\frac{d\phi}{dx}+\phi\right)=\phi-1$ for x between the boundaries A and B. Apply the upwind difference scheme (central difference discretization of diffusion term and upwind difference discretization of convection term) together with the space step δx. Use the mesh shown in Figure 4 below to write the discretized equations in standard form $a_P\phi_p = a_E\phi_E + a_W\phi_W + S_u$ at the nodes stated in parts (i), (ii) and (iii).

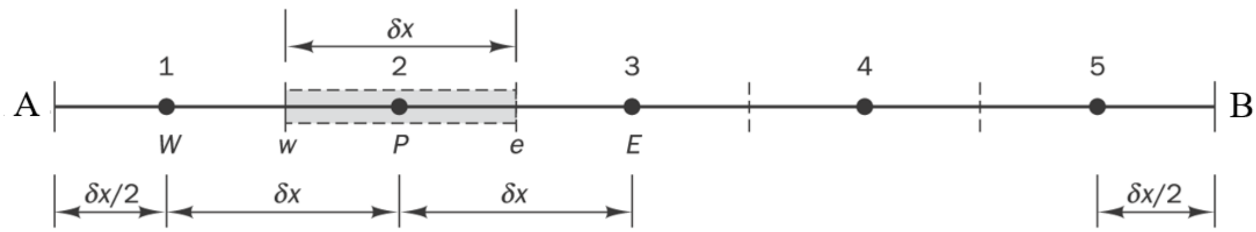

Figure 4

(i) At all the internal nodes labelled 2, 3 and 4.

(10 marks)

(ii) At node 1 if the boundary condition at A is given by $\phi|_A = 0$.

(5 marks)

(iii) At node 5 if the boundary condition at B is given by $\left.\left(\frac{d\phi}{dx}+\phi\right)\right|_B = 0$.

(5 marks)

(b) (i) Use the Scarborough criterion to analyse the iterative solutions of the following set of two linear algebraic equations in two unknowns. (Do not change the order of the equations.) Are the iterative solutions converging or diverging?

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(3 marks)

(ii) Use both the Jacobi iterative method and the Gauss-Seidel iterative method to solve the above equations. Take $\phi_1 = \phi_2 = 0$ as your initial guess. Give only the first <u>THREE</u> iterations, excluding the initial guess.

(10 marks)

(iii) Briefly explain how the iterative solutions may be affected if you rearrange the order of the equations above.

(2 marks)

END OF PAPER