NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2022-2023

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

April/May 2023 Time Allowed: $2\frac{1}{2}$ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **SIX** (6) pages.

2. Answer **ALL** questions.

3. Marks for each question are as indicated.

4. This is a **RESTRICTED OPEN-BOOK** examination. One double-sided A4-size reference sheet with texts handwritten or typed on the A4 paper (no sticky notes/post-it notes on the reference sheet) is allowed.

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1 (a) Find all real constants a and b such that $\phi(x, y) = e^{-by}\cos(x + ay)$ is a solution of the partial differential equation

$$13\frac{\partial^2\phi}{\partial x^2} + 4\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2} = 0$$

for all x and y.

(10 marks)

(b) Consider the boundary value problem that requires solving the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 \text{ for } 0 < x < 2, 0 < y < 3,$$

subject to the boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \text{ and } \left.\frac{\partial u}{\partial x}\right|_{x=2} = 10 \text{ for } 0 < y < 3,$$
$$u(x, 0) = 0 \text{ and } u(x, 3) = 9 \text{ for } 0 < x < 2.$$

NOTE: Question 1 continues on page 2.

(i) Verify by direct substitution that

$$u(x,y) = y^2 + \sum_{n=1}^{\infty} A_n(e^{n\pi x/3} + e^{-n\pi x/3})\sin\left(\frac{n\pi y}{3}\right)$$

satisfies the partial differential equation in the boundary value problem stated above for arbitrary constant coefficients A_n.

(5 marks)

(ii) Identify all the boundary conditions that are satisfied by the series solution given in part (b)(i), no matter what the constant coefficients A_n are.

(5 marks)

(iii) Use the series solution in part (b)(i) to solve the boundary value problem.

(5 marks)

2. Figure 1 shows an assemblage of two bars (AB and AC) and a spring (AD), connected to three supports labelled B, C and D. The two bars have the same length $L = 1$ m. The axial rigidity (EA) of bar AB is 2×10^5 N and that of bar AC is 4×10^5 N. The spring has a stiffness $k = 9\times10^5$ N/m. The connections at A, B, C and D are all pin-jointed. A vertical load P = 12 kN is applied at A as shown. Ignore the weights of the bars and spring. It is intended to model the bars using *truss elements* and the spring using *spring element*.

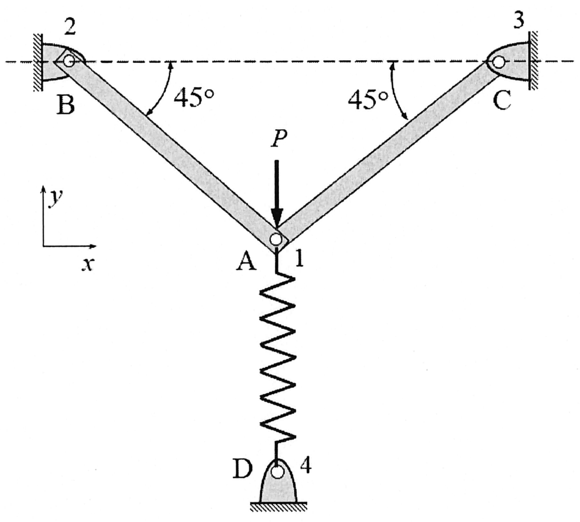


Figure 1

NOTE: Question 2 continues on page 3.

(a) Draw the finite element model of the structure clearly marking node numbers, element numbers, nodal displacements/rotations and nodal forces/moments. Use generalised symbols (Q_1, Q_2, Q_3, etc) for denoting the displacements/rotations and notations (F_1, F_2, F_3, etc) for forces/moments.

(5 marks)

(b) Take node 1 as the reference node for measuring the angle of inclination of truss elements. Write down the element stiffness matrices for all the three elements, and label their rows and columns.

(5 marks)

(c) Assemble the element matrices to obtain global equilibrium equations, apply the boundary conditions to obtain the reduced system of equations, and solve the reduced system for horizontal and vertical displacements at point A.

(10 marks)

(d) Reconsider the global equilibrium equations that you obtained in part (c). We now apply a horizontal force (Q) toward right at A in addition to the vertical force P (= 12 kN). What should be the magnitude of force Q so that the horizontal displacement at A is zero?

(5 marks)

3 (a) Figure 2 shows a system of five springs subjected to a downward force of 2 kN at node 5. All the springs have the same stiffness of 1000 N/m and the same weight of 100 N. Slabs 1 and 2 are rigid and have the same weight of 500 N each. Nodes 1 and 3 are fixed to rigid supports and hence have zero displacements. Include the weight of springs and slabs in the finite element modelling of the system.

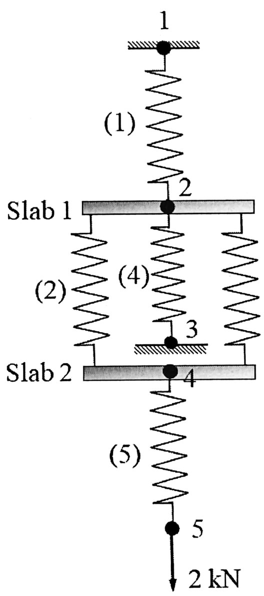


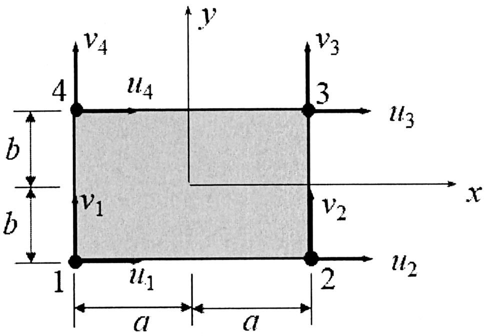
Figure 2

NOTE: Question 3 continues on page 4.

Derive the global equilibrium equations for the system in terms of displacements of nodes 2, 4 and 5 using the *principle of minimum total potential energy* and write them in matrix form. You are not required to solve the equations. (As the displacements at nodes 1 and 3 are zero, they can be substituted in the total potential energy expression even before applying the principle of minimum total potential energy.)

(5 marks)

(b) The shape functions for a 4-node bilinear rectangular element with dimension $a = 0.2$ m and $b = 0.1$ m are as shown below.



$$N_1 = \frac{(a-x)(b-y)}{4ab}, \quad N_2 = \frac{(a+x)(b-y)}{4ab}$$

$$N_3 = \frac{(a+x)(b+y)}{4ab}, \quad N_4 = \frac{(a-x)(b+y)}{4ab}$$

Using the nodal displacements ($u_1 = 0$, $u_2 = 0.1$, $u_3 = 0.1$, $u_4 = 0$, $v_1 = 0$, $v_2 = 0$, $v_3 = 0.015$, $v_4 = 0.015$) in metre, derive the expression for the *element displacement interpolation* for the horizontal displacement u using the shape functions, and write the resulting expression for u in the simplest form. The expression for the vertical displacement v is <u>*not*</u> required.

(5 marks)

(c) Figure 3 below shows a spanner that is used to tighten a hexagonal nut. It is intended to carry out a stress analysis of the spanner using 2-D plane triangular elements. The hexagonal nut (shown in dotted lines) is for reference only and is not to be included in the finite element analysis.

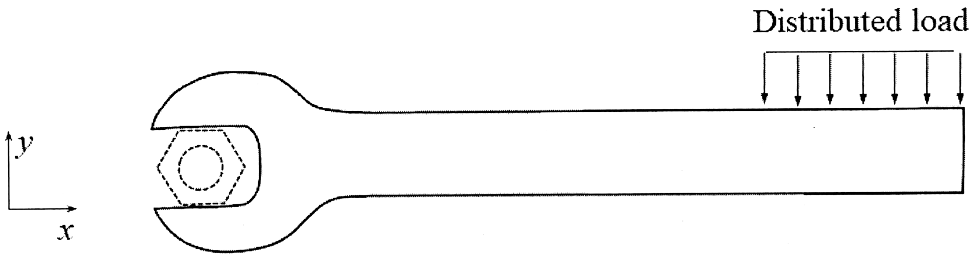


Figure 3

(i) On a sketch of the spanner, indicate all the regions where a finer mesh is needed to capture the stresses accurately. Remember that a finer mesh is needed in the regions of stress concentration.

(3 marks)

(ii) On a sketch of the spanner, show the displacement boundary conditions (that need to be applied) symbolically.

(2 marks)

4 (a) A predator-prey ecosystem with spatial inhomogeneity may be modeled by using a pair of coupled linearized and normalized reaction-diffusion equations of the form

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2} + U - \alpha V \\ \dfrac{\partial V}{\partial t} = \dfrac{\partial^2 V}{\partial x^2} - V + \beta U \end{cases}$$

where α and β are given positive constants, and $U(x,t)$ and $V(x,t)$ are the population sizes of the prey (for example, rabbits) and the predator (for example, foxes), respectively, at location x and time t.

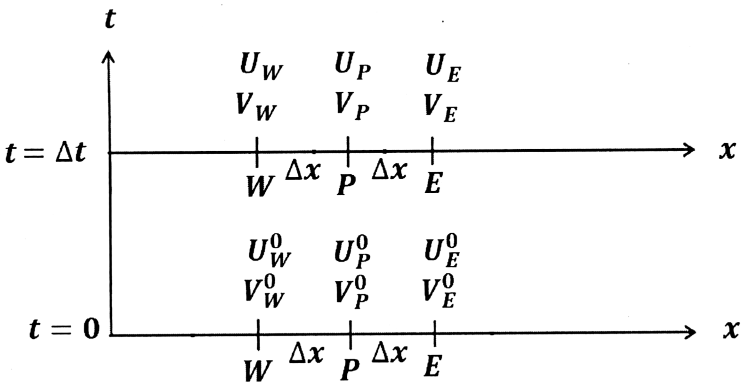


Figure 4

Discretize the pair of partial differential equations above on the mesh in Figure 4 by using:

(i) an **explicit time scheme** to derive equations that are of the form given by $a_P U_p = a_E^0 U_E^0 + a_P^0 U_P^0 + a_W^0 U_W^0 + b_P^0 V_P^0$ and $c_P V_p = c_E^0 V_E^0 + c_P^0 V_P^0 + c_W^0 V_W^0 + d_P^0 U_P^0$,
(10 marks)

(ii) an **implicit time scheme** to derive equations that are of the form given by $a_P U_p + b_P V_p = a_E U_E + a_W U_W + a_P^0 U_p^0$ and $c_P V_p + d_P U_p = c_E V_E + c_W V_W + c_P^0 V_p^0$.
(10 marks)

NOTE: Question 4 continues on page 6.

(b) Consider the system of linear algebraic equations given by

$$\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + x_2 = 2 \end{cases}$$

(i) Explain why a divergent solution is expected if the **Gauss-Seidel** iteration method is applied directly to solve the system above.

(5 marks)

(ii) Rearrange the equations above to ensure that the **Gauss-Seidel** iteration method gives a convergent solution.

(5 marks)

(iii) Using the **Gauss-Seidel** iteration method on the system with the rearranged equations in part (b)(ii) above. Taking $x_1 = 0$ and $x_2 = 0$ as initial guesses, give only the first three iterations of the solution.

(5 marks)

END OF PAPER

MA3004 MATHEMATICAL METHODS IN ENGINEERING

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.