NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2023-2024

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

Nov/Dec 2023 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **FIVE (5)** pages.

2. Answer **ALL** questions.

3. Marks for each question are as indicated.

4. This is **Restricted Open Book** Examination. You are allowed to bring into the examination hall one double-sided A4-size reference sheet with texts handwritten or typed on the A4 paper or one restricted material as instructed by the examiner(s) without any attachments (e.g. sticky notes, post-it notes, gluing or stapling of additional papers).

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1 (a) Find the values of the constants b and c such that

$$u(x,t) = -\frac{2}{9}(x+bt)(cx+t) + \sin(x+bt) + \sin(cx+t)$$

is a solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} - 2\frac{\partial^2 u}{\partial t^2} = -1$$

for all x and t.

(10 marks)

(b) Consider the boundary value problem that requires solving the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} = 8 \text{ for } 0 < x < 1, 0 < y < 2,$$

subject to the boundary conditions

$$u(x,0) = 0 \text{ and } u(x,2) = 2 \text{ for } 0 < x < 1,$$
$$u(0,y) = y^2 - y + 2 \text{ and } u(1,y) = y^2 - y + 1 \text{ for } 0 < y < 2.$$

(i) Verify by direct substitution that

$$u(x,y) = y(y-1) + \sum_{n=1}^{\infty}(A_n e^{n\pi x} + B_n e^{-n\pi x})\sin\left(\frac{n\pi y}{2}\right)$$

satisfies the partial differential equation governing the boundary value problem stated above for arbitrary constant coefficients A_n and B_n $(n = 1, 2, 3, \ldots)$.

(5 marks)

NOTE: Question 1 continues on page 2.

(ii) Check that the boundary conditions at where $y = 0$ and $y = 2$ are satisfied by the series solution given in part (b)(i), no matter what the coefficients A_n and B_n
$(n = 1, 2, 3, \ \dots)$ are.

(4 marks)

(iii) Obtain a pair of simultaneous equations for A_n and B_n such that the boundary conditions at where $x = 0$ and $x = 1$ are satisfied. (You do not have to solve the equations for A_n and B_n.)

(6 marks)

2. Figure 1 shows a uniform beam AB and a uniform bar BC pin-jointed together at B. The beam is of length $L = 1$ m and flexural rigidity $EI = 2{\times}10^5$ Nm^2, and the bar is of length $l = 0.6$ m and axial rigidity $EA = 3{\times}10^5$ N. The beam is subjected to a distributed load of intensity $w = 10^5$ N/m as shown. A concentrated horizontal load $P = 100$ kN is acting toward left at B. Ignore the self-weight of the structure and assume the displacements of the structure are small.

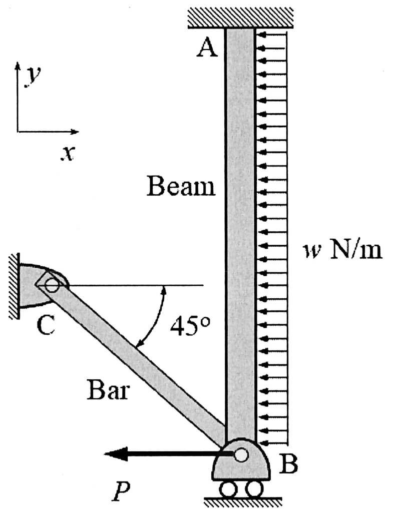

Figure 1

It is intended to solve this problem using the finite element method by modelling the beam as a *beam element* and the bar as a *truss element*.

(a) Draw the finite element model of the structure labelling all the node numbers, element numbers, nodal displacements/rotations, and nodal forces/moments. Use generalised symbols (Q_1, Q_2, Q_3, etc) for labelling the displacements/rotations and (F_1, F_2, F_3, etc) for labelling the forces/moments.

(5 marks)

NOTE: Question 2 continues on page 3.

(b) Write down all the element stiffness matrices and label their rows and columns, and convert the distributed loads into equivalent nodal loads using appropriate formula.

(5 marks)

(c) Assemble the element stiffness matrices and loads, and thereby write the global equilibrium equations with labels for all rows and columns.

(5 marks)

(d) Apply the boundary conditions and thereby obtain the reduced system of equations.

(5 marks)

(e) Solve the reduced system obtained in part (d) for the horizontal displacement of pin B and cross-sectional rotation of the beam at B.

(5 marks)

3 (a) Figure 2 shows a stepped rod ABD of Young's modulus E. The segments AB and BD have the same length L, but different cross-sectional areas A_1 and A_2, respectively. The spring has a stiffness of k. A horizontal force P acts at end D as shown. It is intended to solve this problem using the *principle of minimum potential energy*. For this purpose, you are required to model each of segments AB and BD as a *bar element*, and the spring as a *spring element*. The axial displacements at A, B, C and D are denoted as U_1, U_2, U_3, and U_4, respectively.

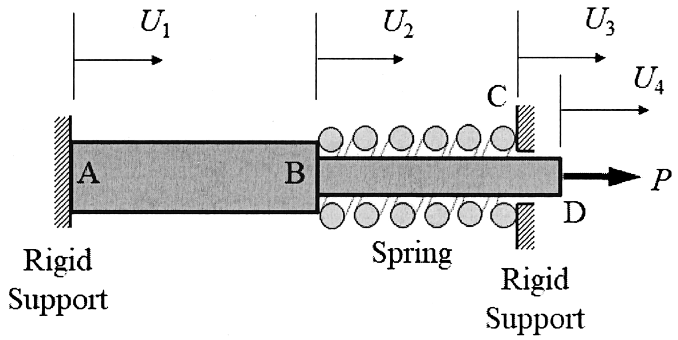

Figure 2

(i) Obtain an expression for the total potential energy of the structure in terms of U_1, U_2, U_3, U_4, P, E, L, k, A_1 and A_2. Do not apply the support boundary conditions

($U_1 = 0$ and $U_3 = 0$) as yet. You need to do it only in part (ii).

(2 marks)

NOTE: Question 3 continues on page 4.

(ii) Apply the principle of minimum potential energy to derive the global equilibrium equations, write them in matrix form, apply the boundary conditions, and solve them for the nodal displacements in terms of P, E, L, k, A_1 and A_2.

(8 marks)

(b) Figure 3 shows a planar structure where all the structural members are of uniform cross-section and interconnected with adjacent members by pin-joints. The arrows show the various loads acting on the structure. The supports at A and E are rigid. It is intended to model this structure using 1-D finite elements such as truss elements and beam elements. Assume the weights of all the members are too small as compared to the loads applied and hence are negligible.

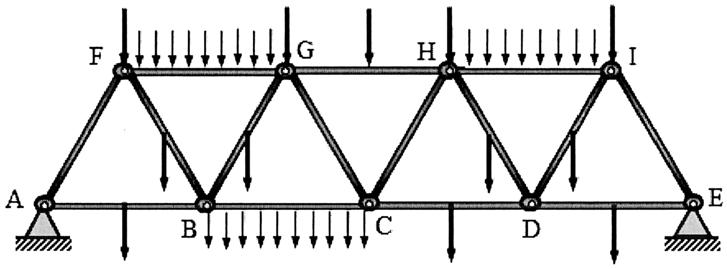

Figure 3

(i) Identify all the members that can be modelled as truss elements.

(2 marks)

(ii) If the weights of the members are not negligible (as compared to the loads applied), what would be your new answer to the question in part (i)? Give reasons.

(3 marks)

4 (a) Consider the one-dimensional non-steady heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) - CT + B,$$

where the temperature T is a function of the space variable x and time t. Assuming that the coefficients κ (thermal diffusivity), B and C are positive constants, discretize the above equation over a uniform time step Δt and a uniform space step Δx by using the following schemes.

(i) An **implicit time scheme** of the form $a_P T_p = a_E T_E + a_W T_W + a_P^0 T_p^0 + S_u$.

(10 marks)

(ii) An **explicit time scheme** of the form $a_P T_p = a_E^0 T_E^0 + a_P^0 T_P^0 + a_W^0 T_W^0 + S_u$.

(6 marks)

NOTE: Question 4 continues on page 5.

(b) Derive a **stability condition** involving Δt, Δx, $\kappa > 0$ and $C > 0$ such that the **explicit time scheme** that you obtained in part (ii) above is stable.

(4 marks)

(c) Consider the system of linear equations

$$\begin{cases} x_1 - 3x_2 + x_4 = 0 \\ x_1 - 3x_3 + x_4 = 0 \\ 2x_1 + 3x_4 = 1 \\ 3x_1 + 2x_4 = -1 \end{cases}$$

(i) Rearrange the above equations to ensure convergence if the Jacobi method or the Gauss-Seidel method is used to solve the rearranged system iteratively.

(5 marks)

(ii) Apply the **Jacobi method**, with $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$ as initial guesses, to solve the rearranged equations in part (i) above. Give only the first three iterative solutions.

(5 marks)

(iii) Repeat part (ii) by using the **Gauss-Seidel method**.

(5 marks)

END OF PAPER

MA3004 MATHEMATICAL METHODS IN ENGINEERING

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.