Describing the behavior of a complex LCR circuit

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Abstract—

A mathematical model has been developed to characterize the behavior of a series LCR circuit. The model predicts that the natural frequency f_0 of the circuit is independent of the resistance R and that the quality factor Q varies inversely proportional to R. However, experimental data has shown that while the prediction of f_0 is in agreement with the observed results, the rest of the model does not accurately describe the behavior of the circuit. This discrepancy suggests that the series LCR circuit in question cannot be fully described by the mathematical model for series LCR circuits.

I. Introduction

A driven damped harmonic oscillator refers to an engineered system in which an object vibrates with a specific amplitude and frequency, powered by an external energy source [1]. The magnitude of the oscillations depends on the frequency of the driving force, and reaches a maximum when the drive frequency equals the system's natural frequency, a phenomenon referred to as resonance [2]. One example of such a system is an electrical circuit that comprises an alternating current (AC) source, a resistor, a capacitor, and an inductor, commonly known as an LCR circuit. The purpose of this study is to evaluate a particular configuration of an LCR circuit, shown in Fig. 1, and compare the obtained results with the mathematical model that describes the behavior of a series LCR circuit, as outlined in the theoretical section, as well as with an idealised version of the experiment. This analysis aims to provide a more comprehensive understanding of an LCR circuit and the impact that each of the components has on the overall performance of the circuit.

II. THEORY

In AC electrical systems, each component exhibits resistance to the flow of electrical current. This resistance is quantified by the impedance, which, due to the frequency-dependent contributions of capacitance and inductance, is a function of the frequency of the driving source [3].

The natural frequency of an LCR, ω_0 , circuit is obtained when the overall impedance of the capacitor and the inductor is a minimum and, for the case when the capacitor and the inductor are in parallel it is given by the equation:

$$\omega_0 = 1/\sqrt{LC} \tag{1}$$

Where L is the inductance of the inductor and C is the capacitance of the capacitor. In this case the impedance in the circuit is given only by the resistance of the resistor, R. Note it is independent of R itself.

The amplitude of a driven oscillator depends on the frequency, ω, according to equation (2) [4].

$$A_{(\omega)} = \frac{A_0}{\sqrt{\omega^2 - {\omega_0}^2 + \frac{R}{L}\omega}} \tag{2}$$

Where A_0 is the amplitude of the AC source.

A curve whose abscissas are the frequencies lying near to and on both sides of the natural frequency of a vibrating system and whose ordinates are the corresponding amplitudes is called a resonant curve [5].

The driver and the oscillator are not always in phase. The phase difference, ϕ, is described by equation (3) [4].

$$\tan^{-1}\left(\frac{-\frac{R}{L}\omega}{{\omega_0}^2 - \omega^2}\right) \tag{3}$$

When the circuit is driven at ω_0, for the reasons mentioned before the amplitude reaches its peak, given by equation (4), and the phase difference becomes $-\frac{\pi}{2}$ rad.

$$A_{(\omega_0)} = A_0 R\sqrt{\frac{L}{C}} \tag{4}$$

A curve whose abscissas are the frequencies lying near to and on both sides of the natural frequency of a vibrating system and whose ordinates are the corresponding phase difference is called a phase curve [5].

The bandwidth Δω is defined as the difference of the frequencies at which the amplitude A is $\frac{1}{\sqrt{2}}$ of $A_{(\omega_0)}$ and for the case of a series LCR circuit it is given by (5) [4].

$$\Delta\omega = \frac{R}{L} \tag{5}$$

An important property of oscillating systems is the Q factor, which quantifies the strength of damping in the system [6]. It is defined as the ratio of the energy stored in the system and the energy dissipated during one complete oscillation and it characterises the width of the resonance curve [4]. It is described by:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{6}$$

It is important to note that when the term "frequency" is used in this context, it refers to angular frequency. The relationship between frequency, f, and angular frequency, ω, can be expressed as follows:

$$\omega = 2\pi f \tag{7}$$

III. METHODOLOGY

A. Setup

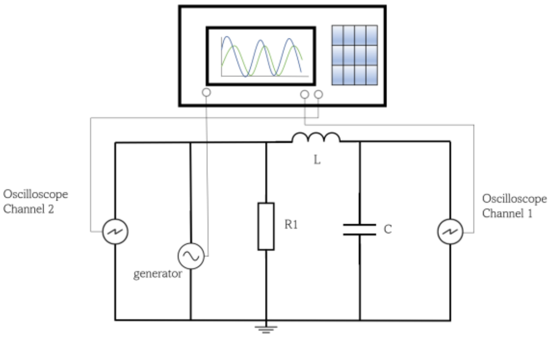

Fig.1: sketch of the setup of the experiment including the circuit schematic. Numerical values of the components of the circuit are displayed in Table I.

TABLE I
NUMERICAL VALUES OF THE COMPONENTS OF THE CIRCUIT

Quantity	Value	SI[a] units
$R1$	1 ± 0.01	Ω
$R2$	2 ± 0.02	Ω
$R3$	3 ± 0.03	Ω
L	$1 \pm 0.1 \times 10^{-3}$	H
C	$1 \pm 0.1 \times 10^{-7}$	F

Numerical values for the circuit components shown in Fig. 1.

The experiment was set up on a breadboard according to the circuit schematic shown in Fig. 1. The Rohde & Schwarz RTB2004 oscilloscope both for driving the circuit and taking measurements. The oscilloscope was set to measure the peak-to-peak voltage across the capacitor in Channel 1 and across the generator in channel 2 as well as the phase difference between the 2 signals. To minimise the impact of noise on the measurements, AC coupling was enabled for both Channel 1 and Channel 2, and the bandwidth of the filter was reduced to 20 MHz. The measurement statistics were activated to display the mean, standard deviation, and number of measurements of all quantities, and the averaging was set to 100, which, after some preliminary measurements, was determined to be the best compromise between reducing high-frequency noise and maintaining measurement speed.

B. Strategy

The aim of the data acquisition was to gather data to construct an accurate resonance and phase curve. In order to get a good curve, it is necessary to gather more data around the natural frequency, f_0, and to obtain data for a broad range of frequencies. To achieve this, preliminary measurements were conducted to estimate the natural frequency, and data was collected at the frequencies specified in Table II.

The oscilloscope was configured to generate a 2V peak-to-peak sinusoidal wave starting at a frequency of 8000 Hz. After the wave count reached approximately 500, the data was collected. This process was repeated for all 33 frequencies. Subsequently, the resistor R1 was substituted with R2 and then R3, and the same data acquisition method was applied for each resistor.

TABLE II
MEASUREMENT FREQUENCIES

Δf (KHz)	f (KHz)
1.0	8.0, 9.0, 10.0, 11.0, 12.0, 13.0
0.5	13.5, 14.0, 14.5, 15.0, 15.5
0.1	16.0, 16.1, 16.2, 16.3, 16.4, 16.5, 16.6, 16.7, 16.8, 16.9, 17.0
0.5	17.5, 18.0, 18.5, 19.0, 19.5
1.0	20.0, 21.0, 22.0, 23.0, 24.0, 25.0

Frequencies at which measurements were made.

IV. RESULTS

A. Data analysis

The experimental data were processed using a Python program that employs the Numpy and Scipy libraries. The program was utilized to interpolate the amplitude data using the function described by equation (2) and the phase data using equation (3). Subsequently, the fit was utilized to calculate the values of f0 and Q. The program generated a plot of the resonance and phase curves using the interpolated data of all resistors.

B. Simulations

In order to evaluate the validity of the experimental results, an LTspice simulation was created based on the circuit depicted in Figure 1 [7]. The simulation data was processed in a similar program to the one used to analyse the experimental data, which then calculated the expected values of f_0 and Q, and generated a resonance and phase curve plot for all resistor values. Upon examination of the output from the simulation's data analysis, it was evident that the acquired data did not reflect the ideal circuit depicted in Fig. 1, but rather a more complex circuit that included additional capacitance and resistance in parallel introduced by the oscilloscope, and an additional resistance in series introduced by the function generator and the inductor. This circuit is depicted in Fig. 2. The values of the additional components are listed in the specification sheets of the oscilloscope and the inductor and are presented in Table III [8], [9].

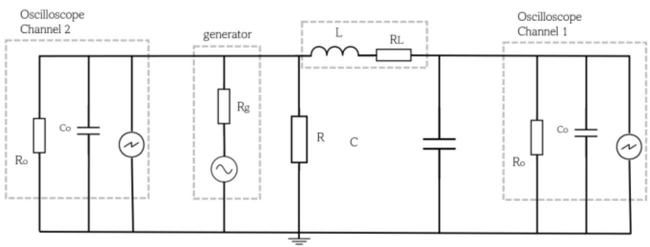

Fig.2: circuit schematic of the non-ideal circuit describing the experiment setup.

TABLE III
NUMERICAL VALUES OF THE COMPONENTS OF THE CIRCUIT

Quantity	Value	SI[a] units
R_O	1×10^{6}	Ω
C_O	9×10^{-12}	F
R_L	3.63	Ω
R_G	50	Ω

Numerical values for the additional circuit components shown in Fig. 2 which are note outlined in Table II.

C. Results

The plots generated from the data analysis of the experimental data, the ideal circuit simulation data, and the non-ideal circuit simulation data are presented for comparison in Fig. 3. The values of f_0 and Q calculated from the experimental data, the non-ideal circuit simulation data, and those calculated based on the mathematical model described in the theory section, using equation (7), (6) and (1), are presented in Table IV.

TABLE IV
RESULTS

Resistor	Quantity	Experimental	Theoretical	Non-idea circuit simulatio
$R1$	f_0	16656.96 ± 0.05	16000 ± 1000	15902.3
$R1$	Q	15.4 ± 0.1	100 ± 12	21.6
$R2$	f_0	16639.08 ± 0.01	16000 ± 1000	15902.3
$R2$	Q	13.0 ± 0.1	50 ± 6	17.9
$R3$	f_0	16614.2 ± 0.01	16000 ± 1000	15902.3
$R3$	Q	11.8 ± 0.1	33 ± 4	15.4

Numerical results produced by the data analysis program.

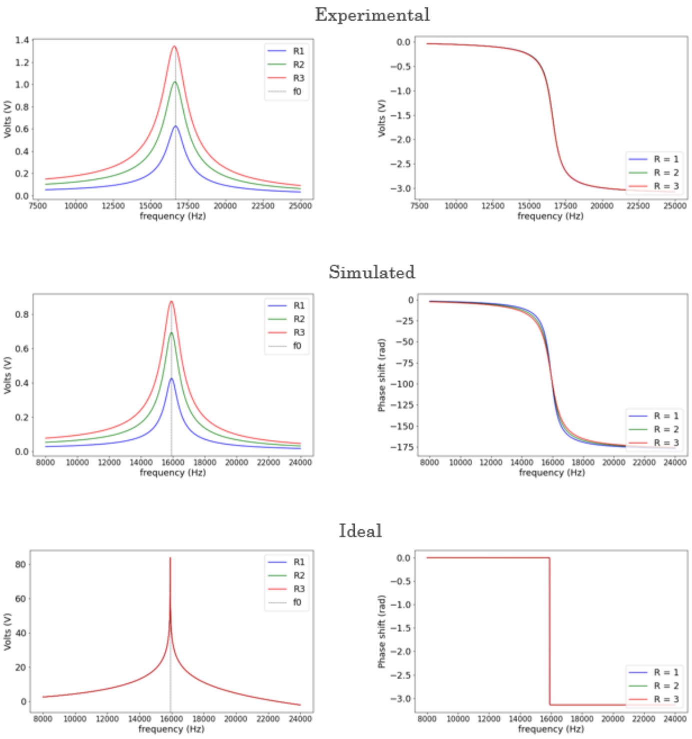

Fig.3: Plots of resonance and phase curves for the 3 resistors generated by using the experimental data, the ideal circuit simulation data, and the non-ideal circuit simulation data.

V. UNCERTAINTIES

The uncertainties of the resistance values for the resistors were calculated based on the 1% tolerance specified on the resistors. The uncertainties of the inductance of the inductor and the capacitance of the capacitor were determined using the 10% tolerance values reported in their respective datasheets [9], [10]. The uncertainties in the experimental results were taken into account by considering the uncertainties of the measurements, which were quantified by calculating the standard error using the standard deviation and wave count. These uncertainties were incorporated into the Python program by implementing a curve fit that accounts for the uncertainties in the data. The combined uncertainties were calculated using standard error propagation rules for functions with multiple variables [11]. It is worth noting that there are additional sources of systematic uncertainties, such as parasitic resistance in the wires and coaxial cables used to make the connections in the circuit, that were not possible to quantify. The uncertainties associated with the output impedance R_G, the oscilloscope impedance R_O, or the inductor's resistance $R_L$were not considered as they were not relevant to the objective of the experiment and were only used for circuit simulation purposes.

VI. DISCUSSION

In the ideal circuit simulation, the amplitude at the resonant frequency approaches infinity, as the overall impedance in the circuit line comprising only the inductor and the capacitor approaches zero, resulting in a short circuit. This observation serves as evidence that equation (2) does not accurately describe the resonance curve. The convergence of the amplitude at the resonant frequency to a finite value is due to the presence of resistance in the inductor, as indicated by the resonance curve of the non-ideal circuit simulation. A similar argument can be made for the inaccuracy of equation (3). The calculated value of f_0 using experimental data and the prediction of f_0 based on the mathematical model described in the theory section agree within one standard error. Confirmation of this arises with both the f_0 value calculated using the ideal and the non-ideal circuit simulation's data. Thus equation (1) appears to be correct. This is not surprising considering what is stated in the theory section that as long as the capacitor and the inductor are in the same circuit line that formula holds. The expected uncertainty in the theoretical $f_0$is large, due to the large $\pm$10% tolerance given for C and L. In order to reduce the expected uncertainty, the quantities could be previously measured. The calculated value of the quality factor Q for all resistors using experimental data and the predicted values based on equation (6) were found to be consistent within more than five standard errors. This discrepancy suggests that equation (6) is an inaccurate description of the quality factor in the LCR circuit depicted in Fig. 2. A more complex equation, with the form $Q \propto \frac{1}{R^{\alpha}}$, where $\alpha < 1$ is required to replace equation (6). This relationship holds for both the experimental data and the non-ideal circuit simulation data.

VII. CONCLUSION

The mathematical model used to describe the behavior of the series LCR circuit proposes that f_0 is independent of the resistance of R and that $Q \propto \frac{1}{R}$,. The results of the experiments conducted reveal that the model's prediction for f_0 is in agreement with the experimental data within 1 standard error, while the prediction for Q is not. The experimental results suggest that $Q \propto \frac{1}{R^{\alpha}}$, where $\alpha < 1$. The results also indicate that the equations used to describe the amplitude and phase difference in a series LCR circuit are not applicable to the LCR circuit depicted in Fig. 2.

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