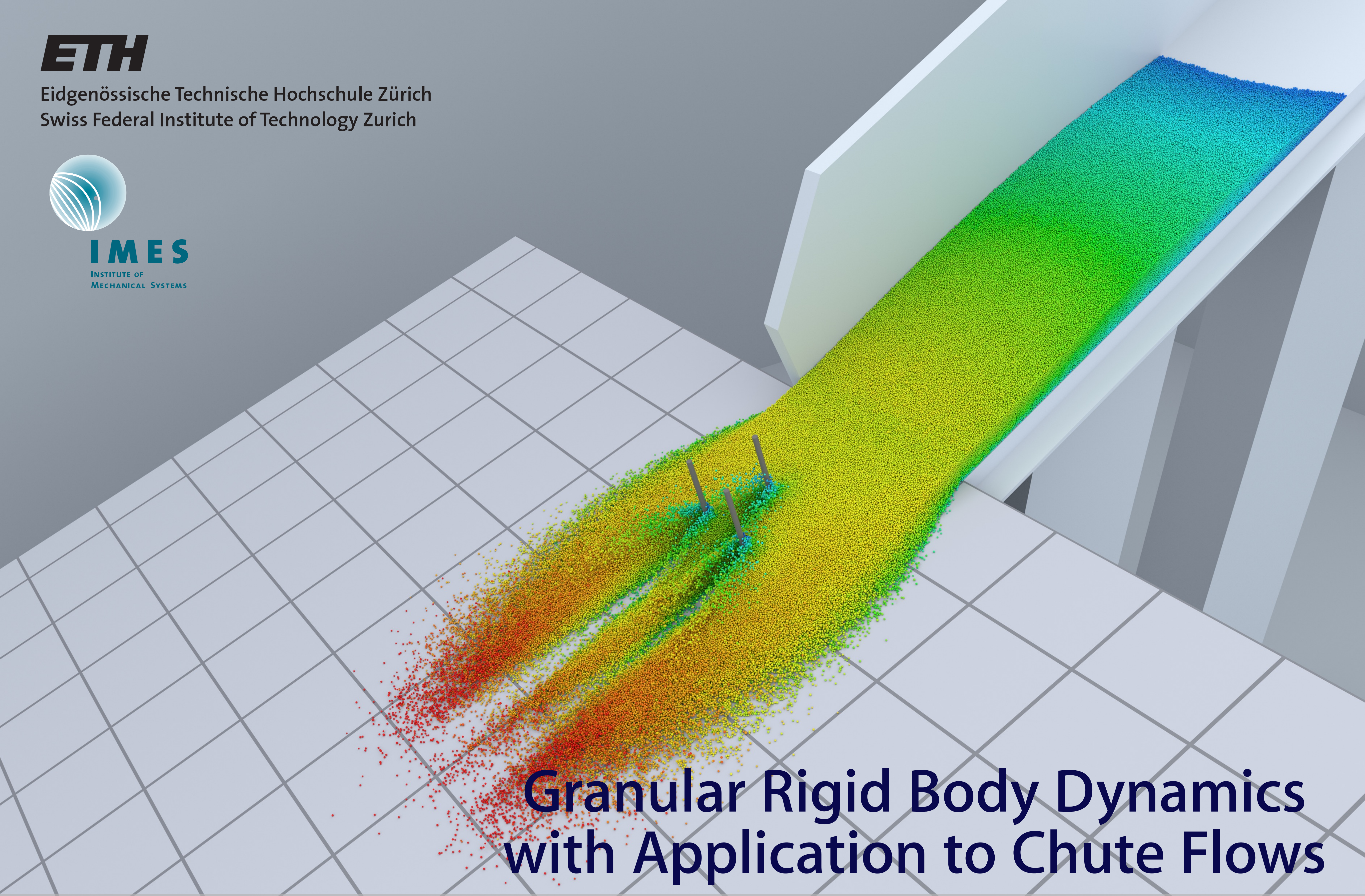
ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

IMES
Institute of
Mechanical Systems



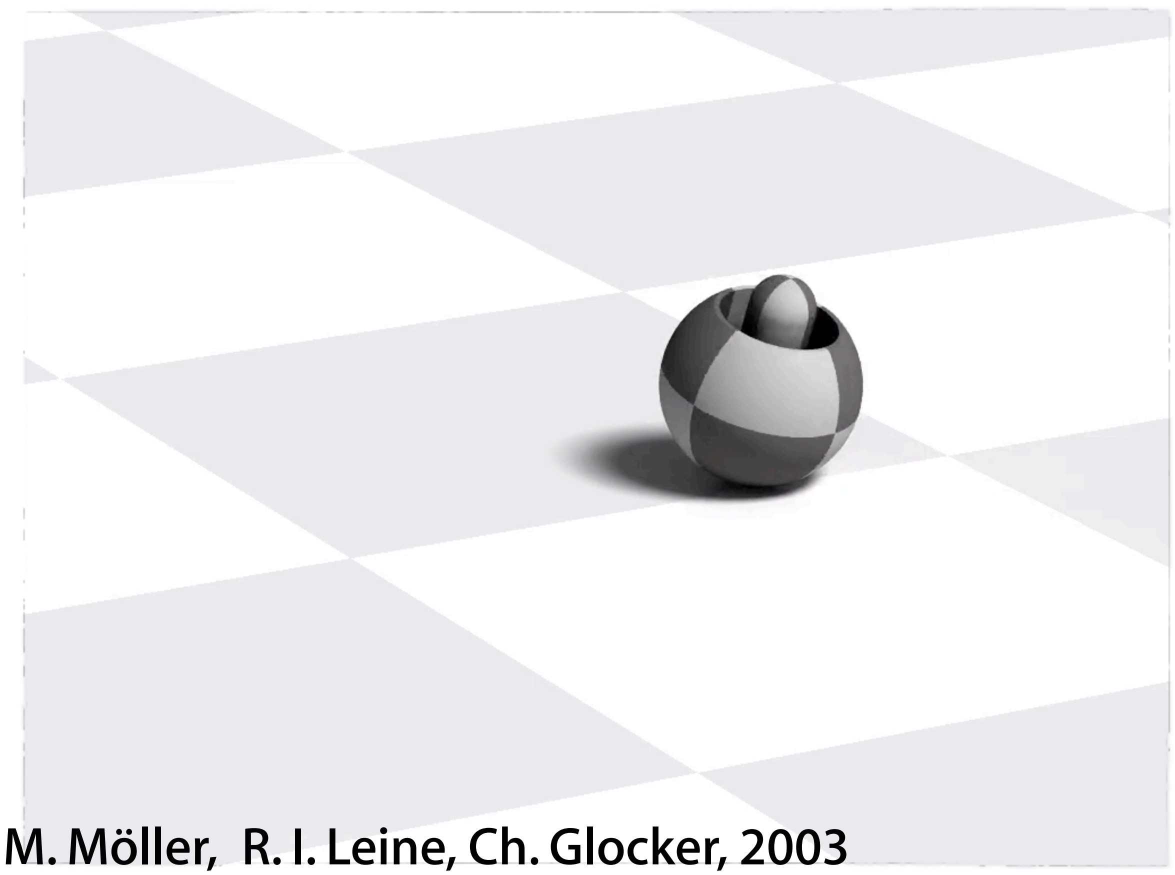
Granular Rigid Body Dynamics with Application to Chute Flows

MSc ETH ME Gabriel Nützi

Examiner: Prof. Dr. Ch. Glocker
Co-Examiner: Prof. Dr. J. Dual, Prof. Dr. R. Leine

Mechanics

Computer Science



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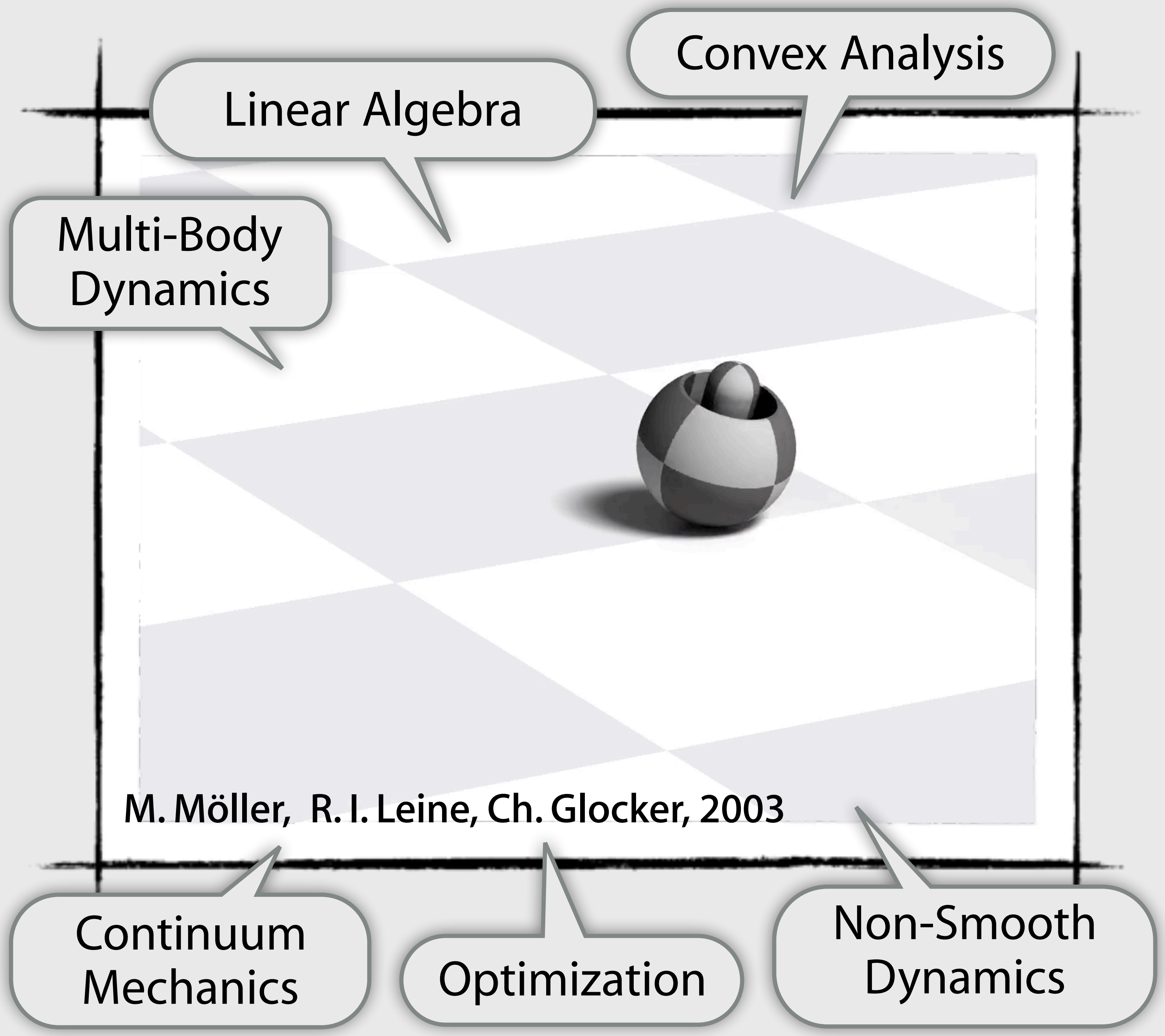
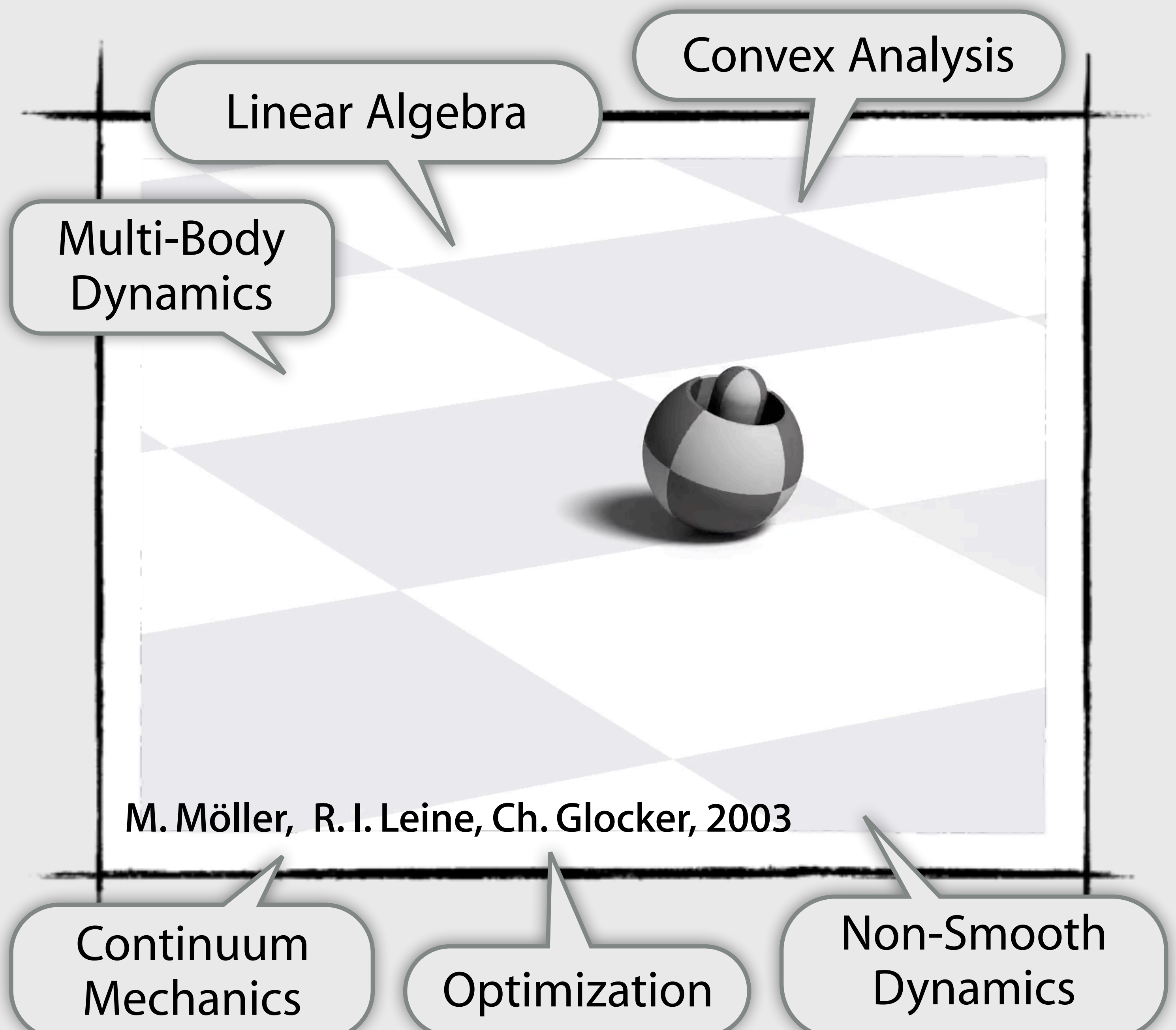
Mechanics

Computer Science



Mechanics



Computer Science

```
#include <unistd.h>
float o=0.075,h=1.5,T,r,O,l,I;int
_,L=80,s=3200;main(){for(;s%L||
(h-=o,T= -2),s;4 -(r=O*O)<(l=I*I)|
++ _==L&&write(1,(--s%L?_<L?--_
%6:6:7)+"World! \n",
1)&&(O=I=l=_=r=0,T+=o /2))O=I*2*O
+h,I=l+T-r;}
```

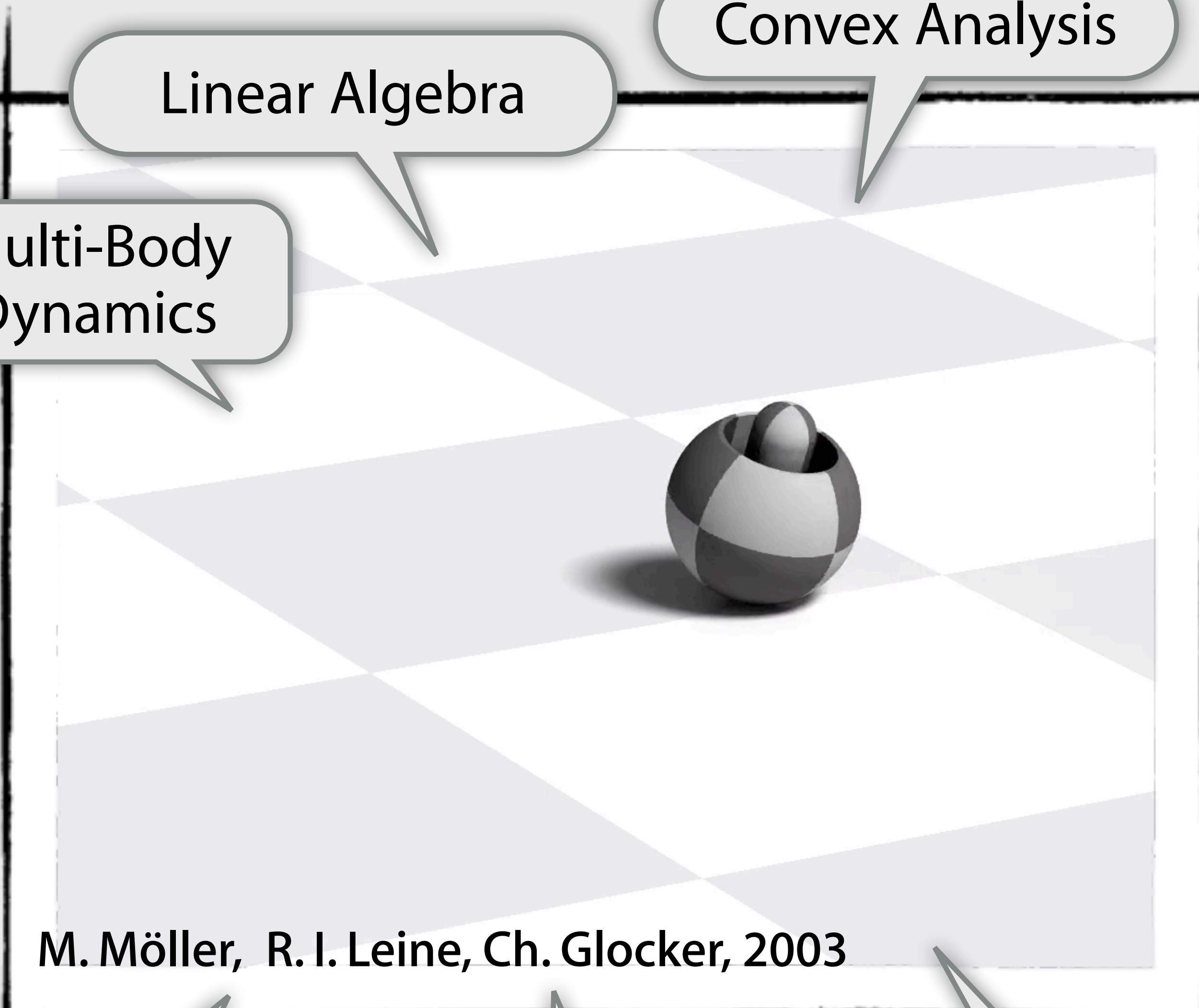
Algorithm (noun.)
Word used by programmers when they don't want to explain what they did.

Mechanics

Linear Algebra

Convex Analysis

Multi-Body Dynamics



M. Möller, R. I. Leine, Ch. Glocker, 2003

Continuum Mechanics

Optimization

Non-Smooth Dynamics

Computer Science

Algorithms

Logic & Data

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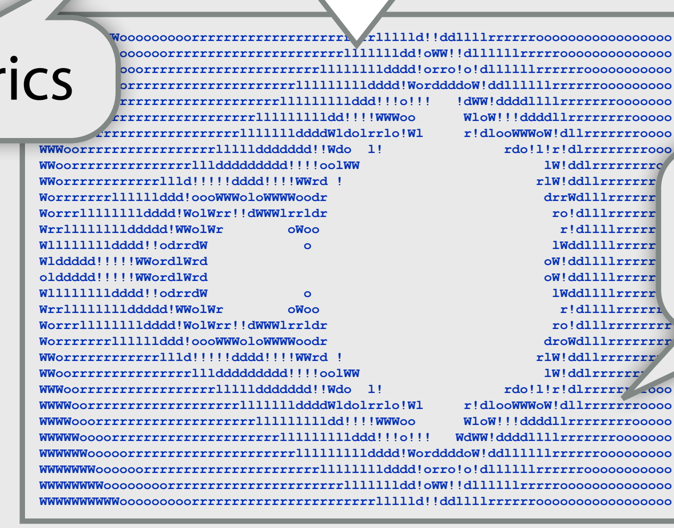
Numerics



Computer Graphics

Algorithm (noun.)
Word used by programmers when they don't want to explain what they did.

Mechanics

Computer Science

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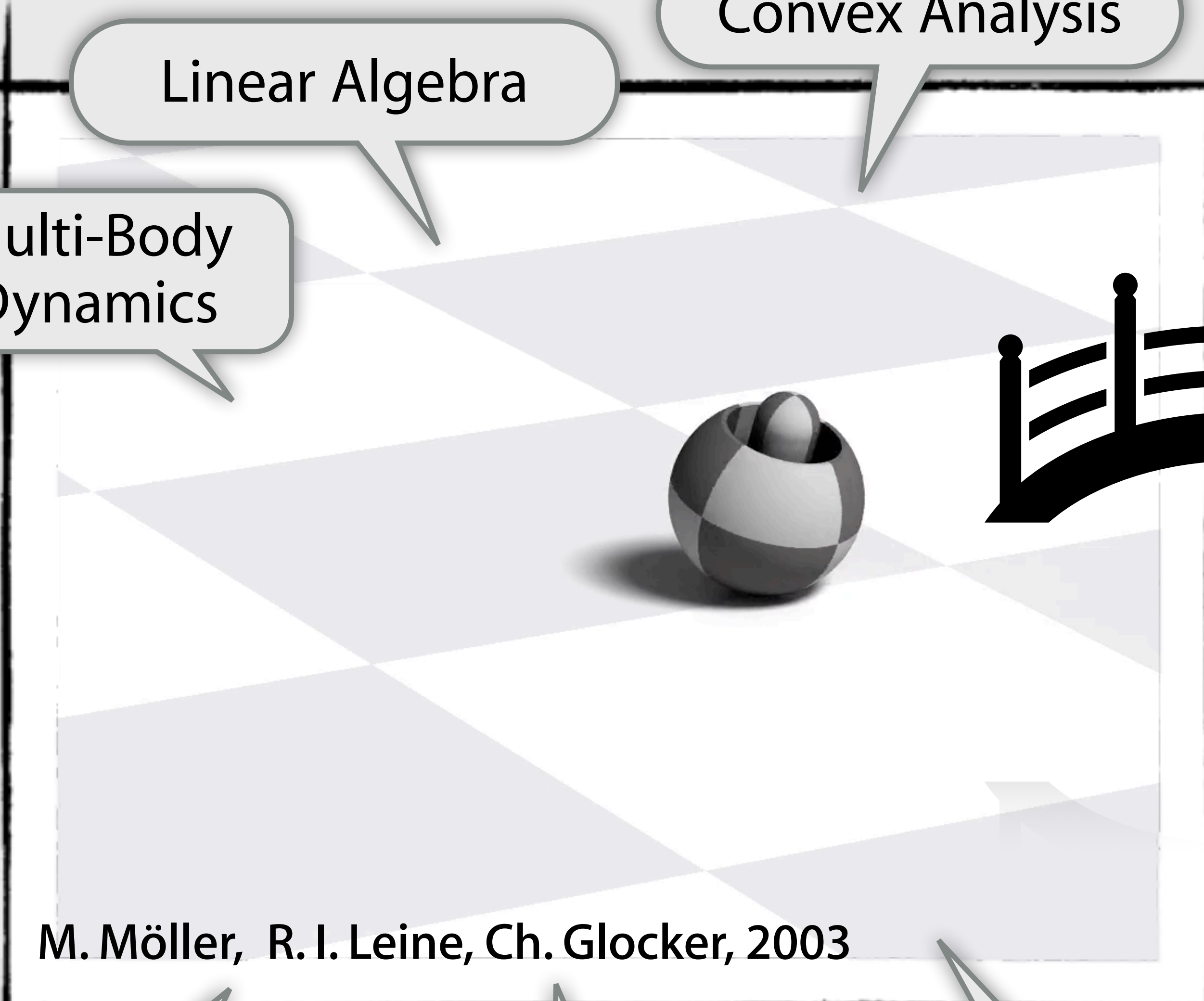
Convex Analysis

Multi-Body Dynamics



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Continuum Mechanics

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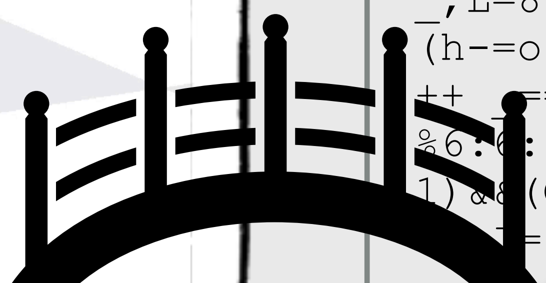
Non-Smooth Dynamics



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Numerics

Computer Graphics

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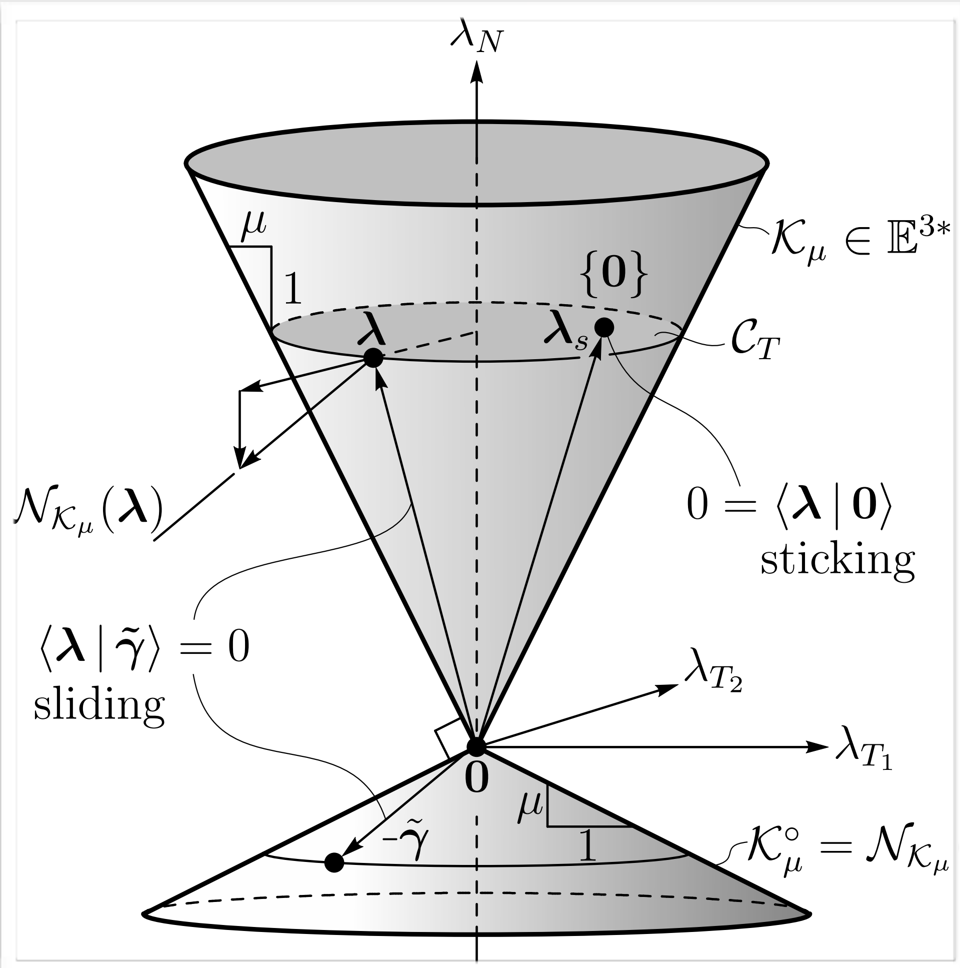
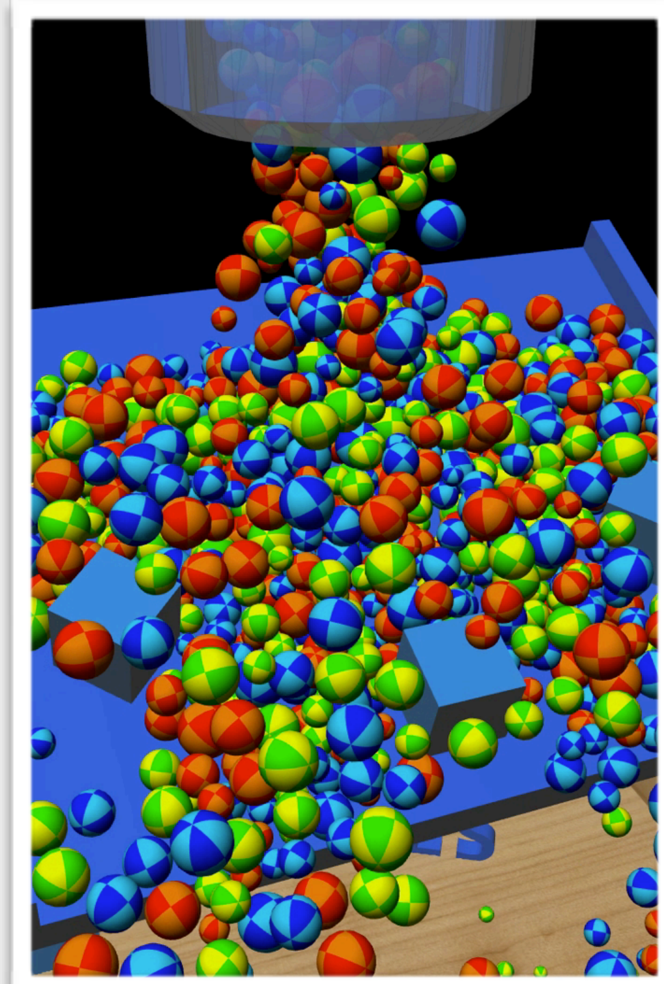
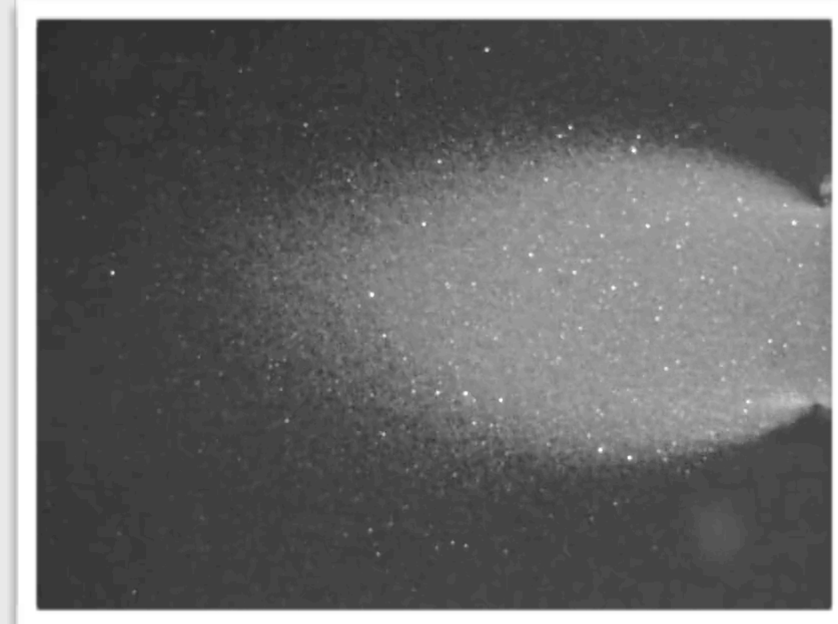
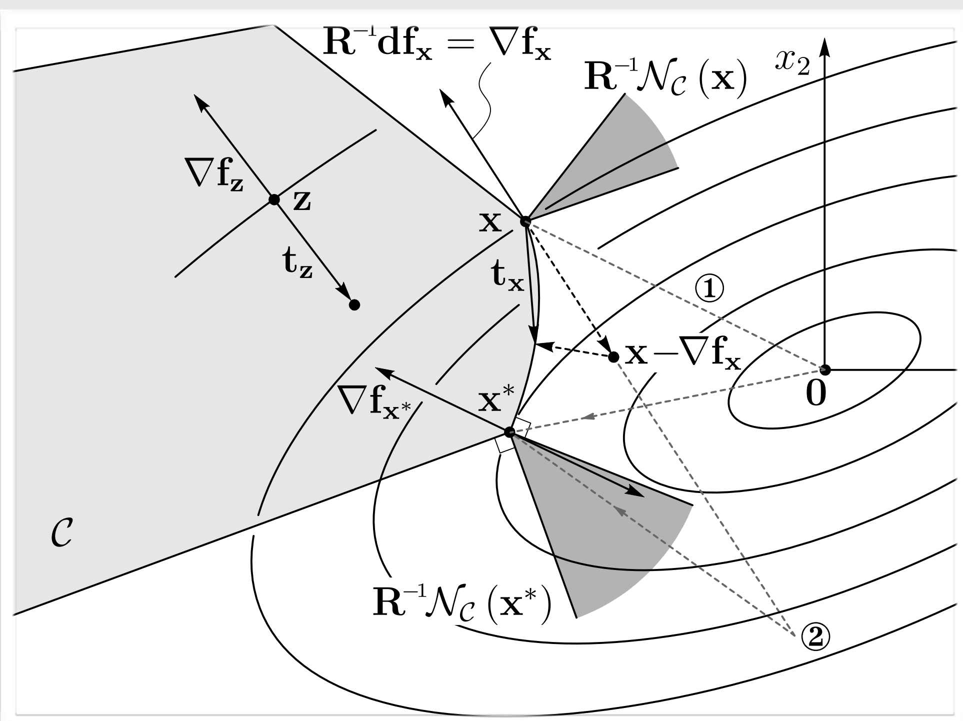
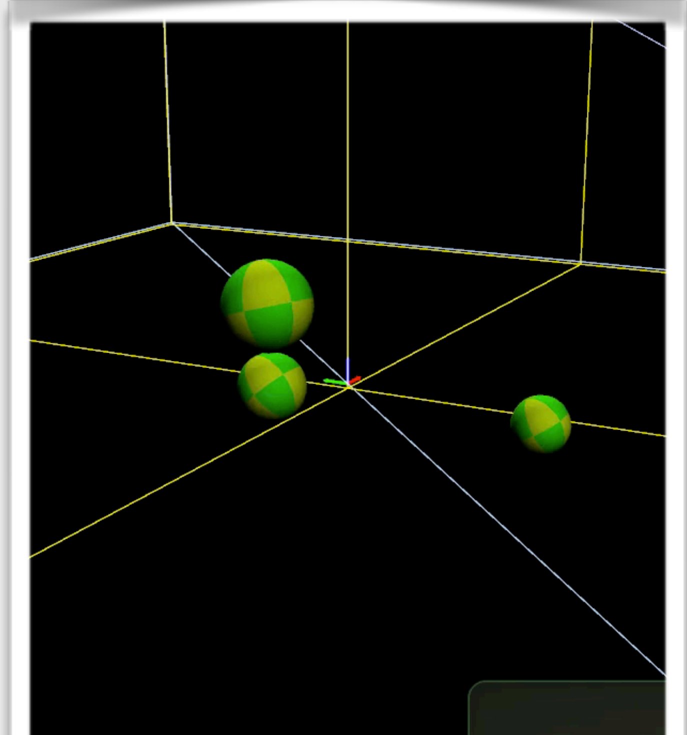
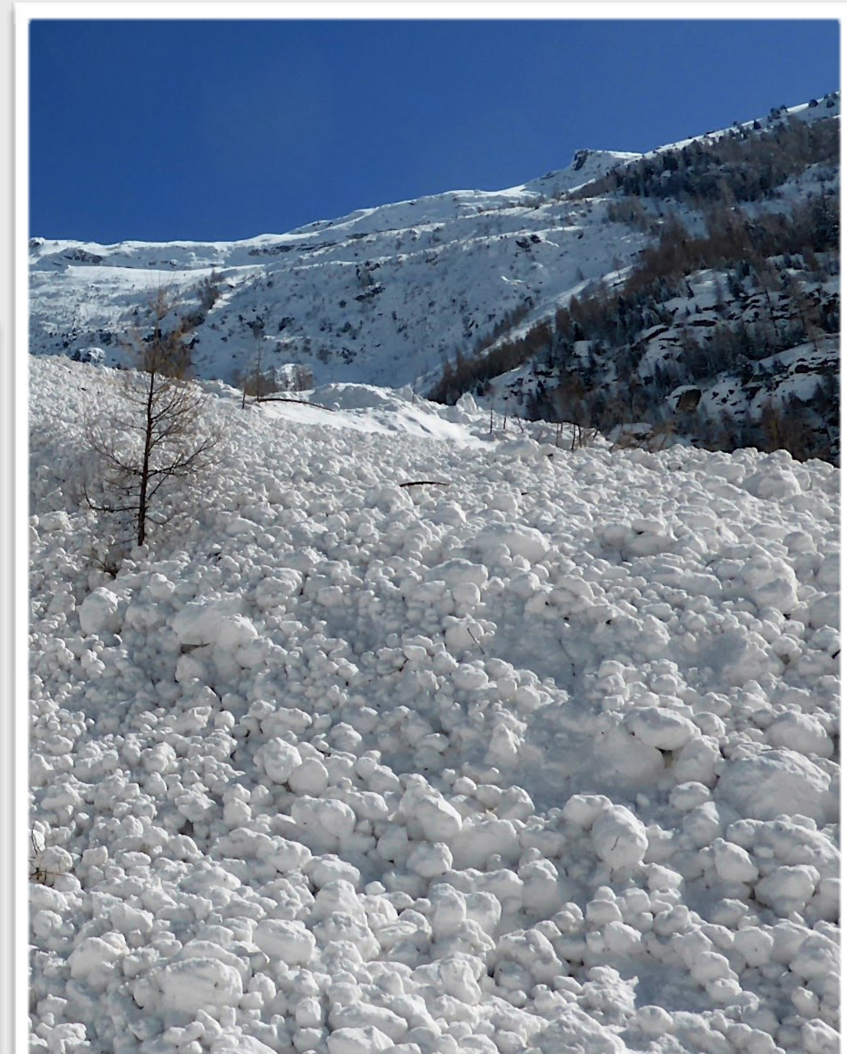

λ_N
μ
1
$\{0\}$
$\mathcal{K}_\mu \in \mathbb{E}^{3*}$
$\boldsymbol{\lambda}$
$\boldsymbol{\lambda}_s$
$\mathcal{C}_T$
$\mathcal{N}_{\mathcal{K}_\mu}(\boldsymbol{\lambda})$
$0 = \langle \boldsymbol{\lambda} \,|\, \mathbf{0} \rangle$
sticking
$\langle \boldsymbol{\lambda} \,|\, \tilde{\boldsymbol{\gamma}} \rangle = 0$
sliding
λ_{T_2}
λ_{T_1}
$\mathbf{0}$
$-\tilde{\boldsymbol{\gamma}}$
$\mathcal{K}^\circ_\mu = \mathcal{N}_{\mathcal{K}_\mu}$

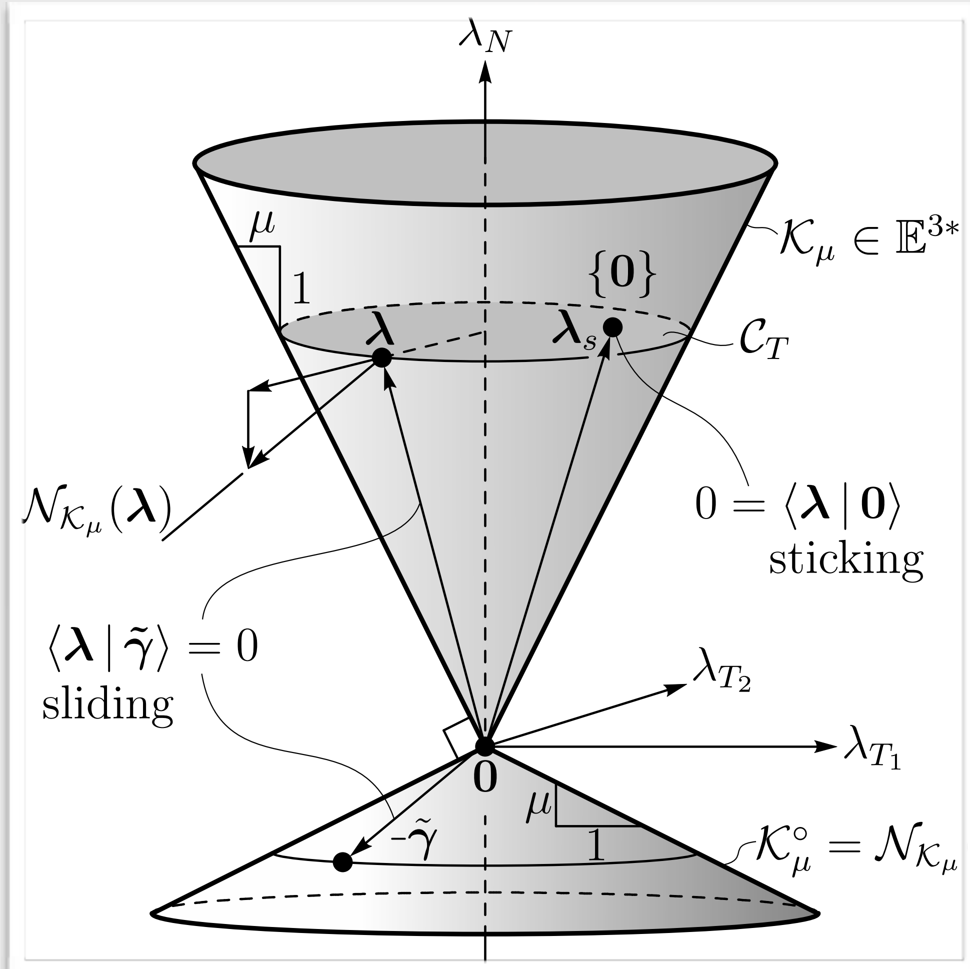


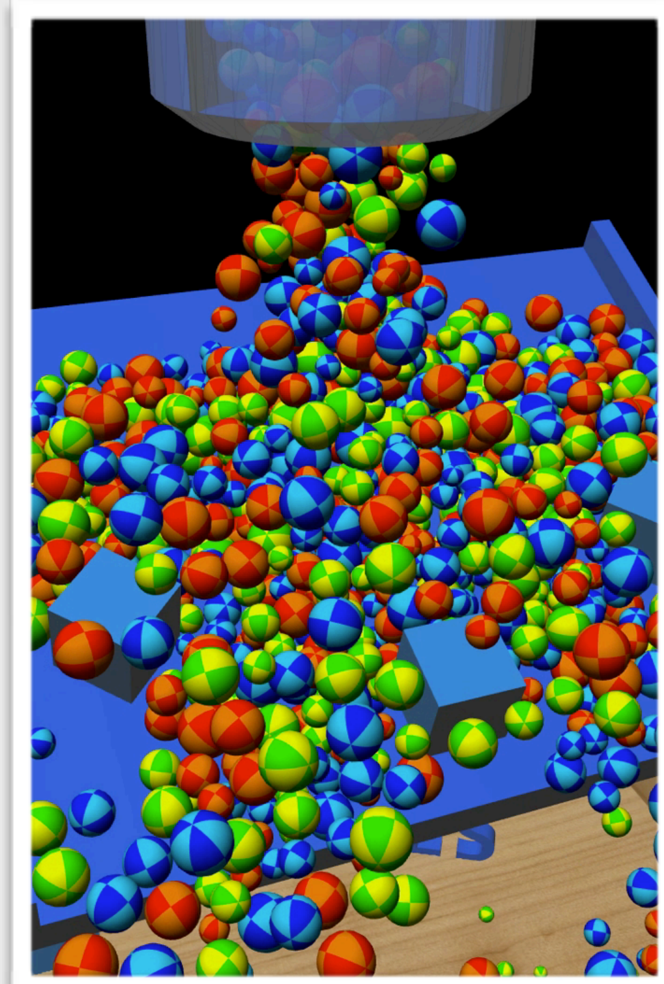


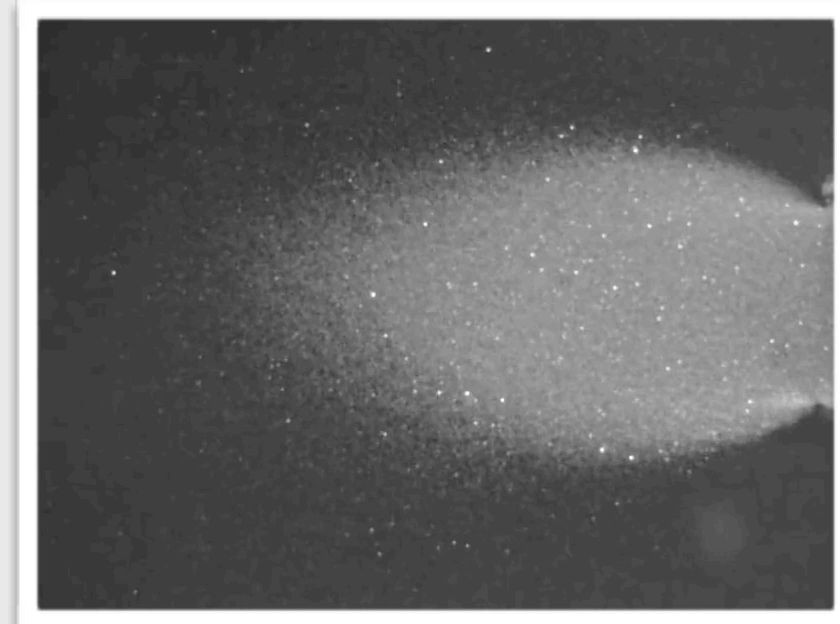

$\mathbf{R}^{-1}\mathbf{df}_\mathbf{x} = \nabla \mathbf{f}_\mathbf{x}$
$\mathbf{R}^{-1}\mathcal{N}_\mathcal{C}(\mathbf{x})$
x_2
$\nabla \mathbf{f}_\mathbf{z}$
$\mathbf{z}$
$\mathbf{t}_\mathbf{z}$
$\mathbf{x}$
$\mathbf{t}_\mathbf{x}$
①
$\mathbf{x} - \nabla \mathbf{f}_\mathbf{x}$
$\nabla \mathbf{f}_{\mathbf{x}^*}$
$\mathbf{x}^*$
$\mathbf{0}$
$\mathcal{C}$
$\mathbf{R}^{-1}\mathcal{N}_\mathcal{C}(\mathbf{x}^*)$
②

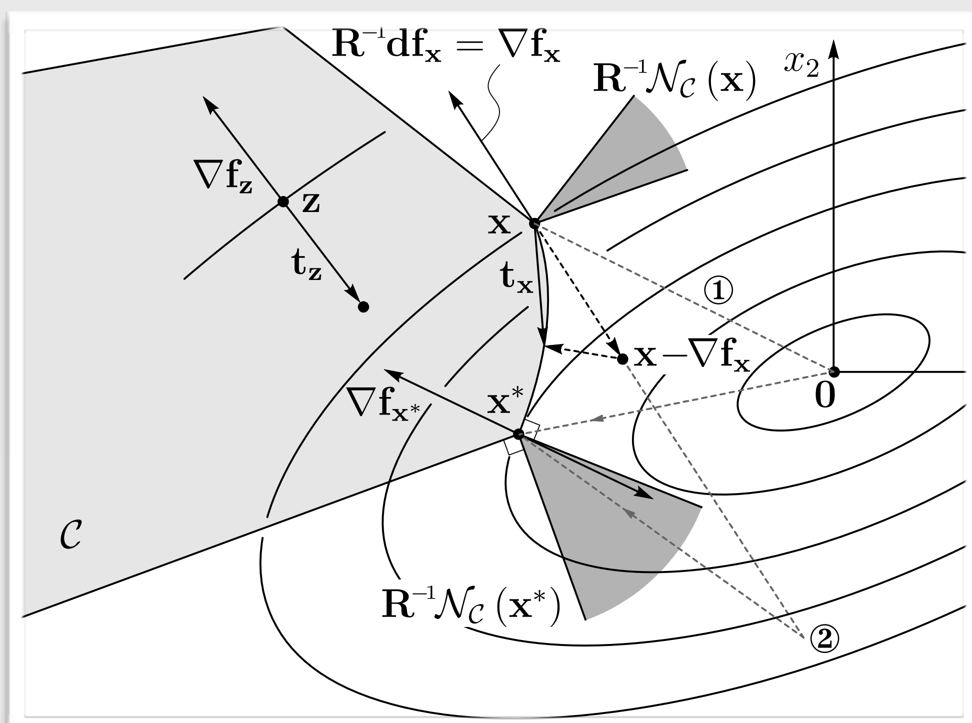





λ_N
μ
1
$\{0\}$
$\mathcal{K}_\mu \in \mathbb{E}^{3*}$
$\boldsymbol{\lambda}$
$\boldsymbol{\lambda}_s$
$\mathcal{C}_T$
$\mathcal{N}_{\mathcal{K}_\mu}(\boldsymbol{\lambda})$
$0 = \langle \boldsymbol{\lambda} \,|\, \mathbf{0} \rangle$
sticking
$\langle \boldsymbol{\lambda} \,|\, \tilde{\gamma} \rangle = 0$
sliding
λ_{T_2}
λ_{T_1}
$\mathbf{0}$
$-\tilde{\gamma}$
μ
1
$\mathcal{K}_\mu^\circ = \mathcal{N}_{\mathcal{K}_\mu}$






$\mathbf{R}^{-1}\mathbf{df_x} = \nabla \mathbf{f_x}$
$\mathbf{R}^{-1}\mathcal{N}_\mathcal{C}(\mathbf{x})$
x_2
$\nabla \mathbf{f_z}$
$\mathbf{z}$
$\mathbf{t_z}$
$\mathbf{x}$
$\mathbf{t_x}$
①
$\mathbf{x} - \nabla \mathbf{f_x}$
$\nabla \mathbf{f_{x^*}}$
$\mathbf{x}^*$
$\mathbf{0}$
$\mathcal{C}$
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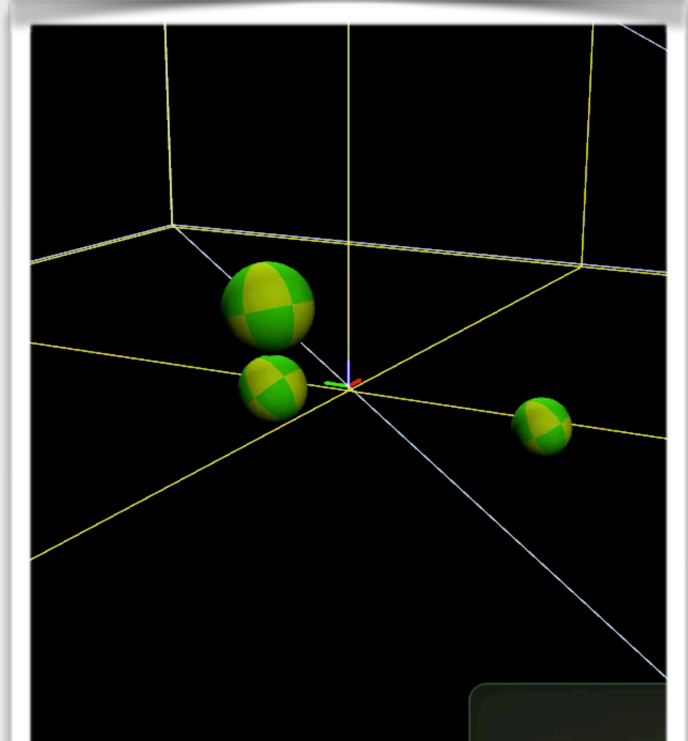
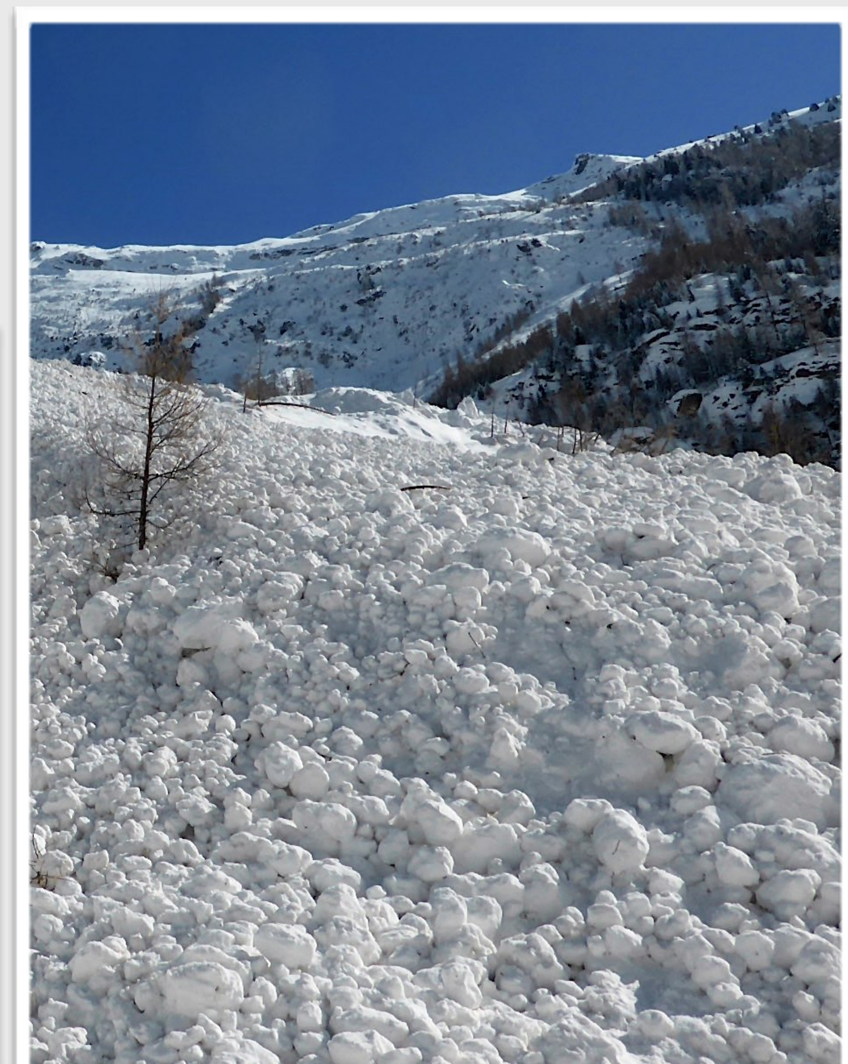
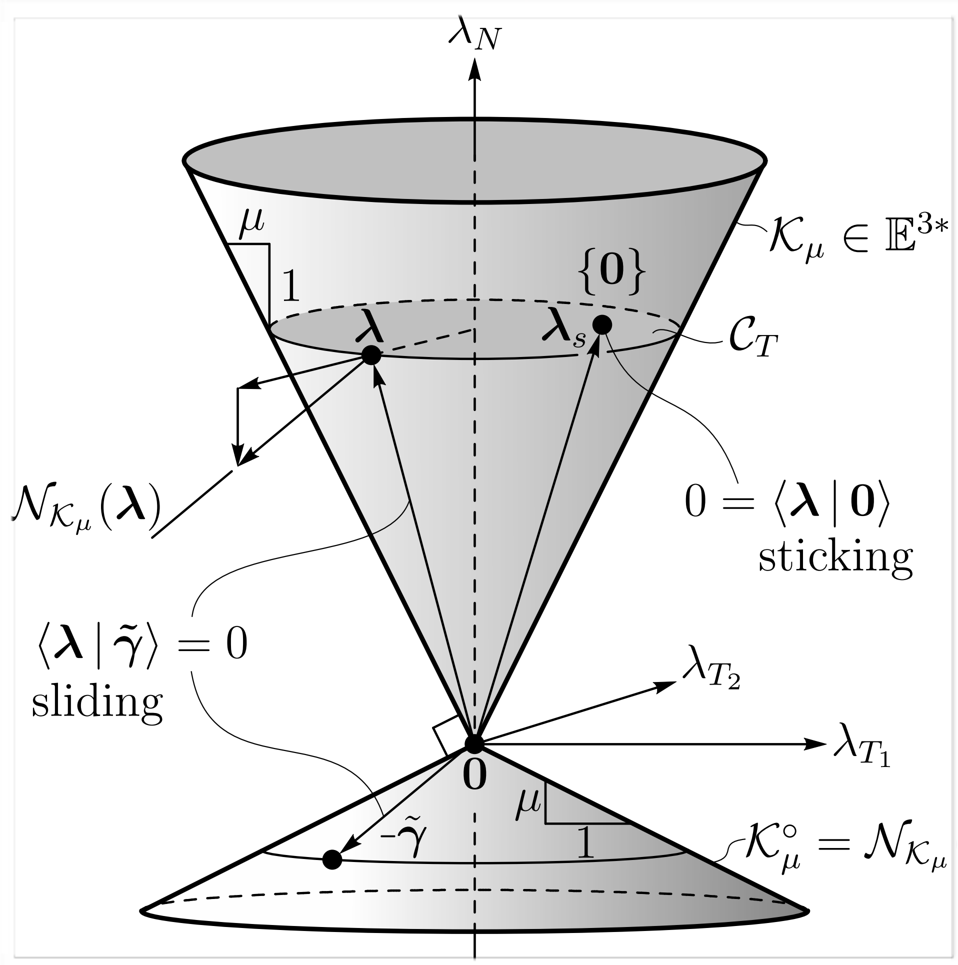
②






λ_N
μ
1
$\{\mathbf{0}\}$
$\mathcal{K}_\mu \in \mathbb{E}^{3*}$
$\boldsymbol{\lambda}$
$\boldsymbol{\lambda}_s$
$\mathcal{C}_T$
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$0 = \langle \boldsymbol{\lambda} \,|\, \mathbf{0} \rangle$
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λ_{T_2}
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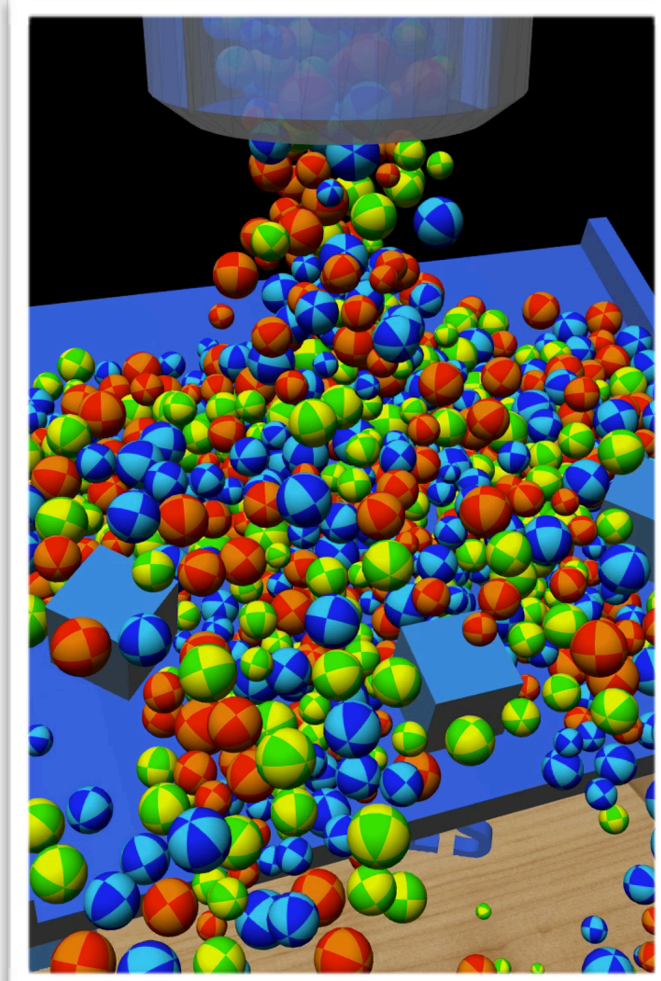

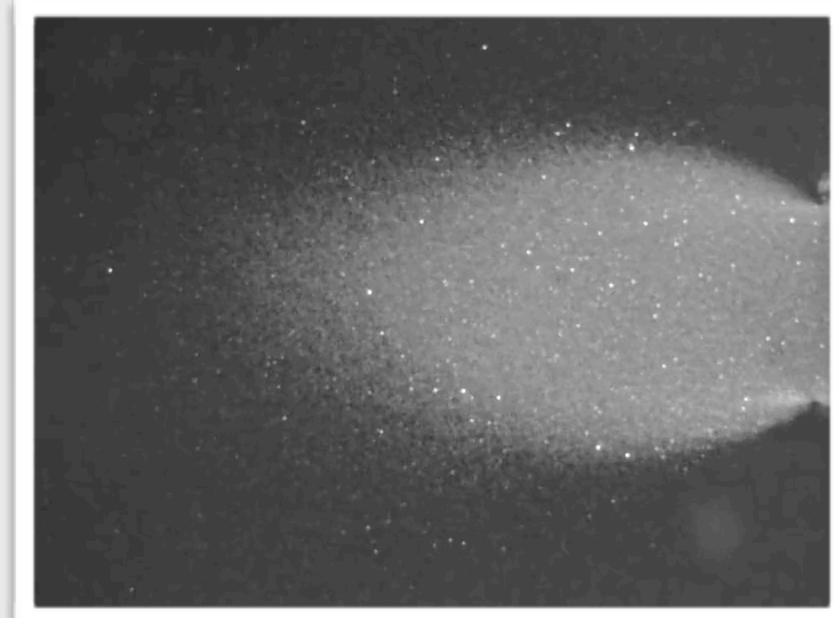

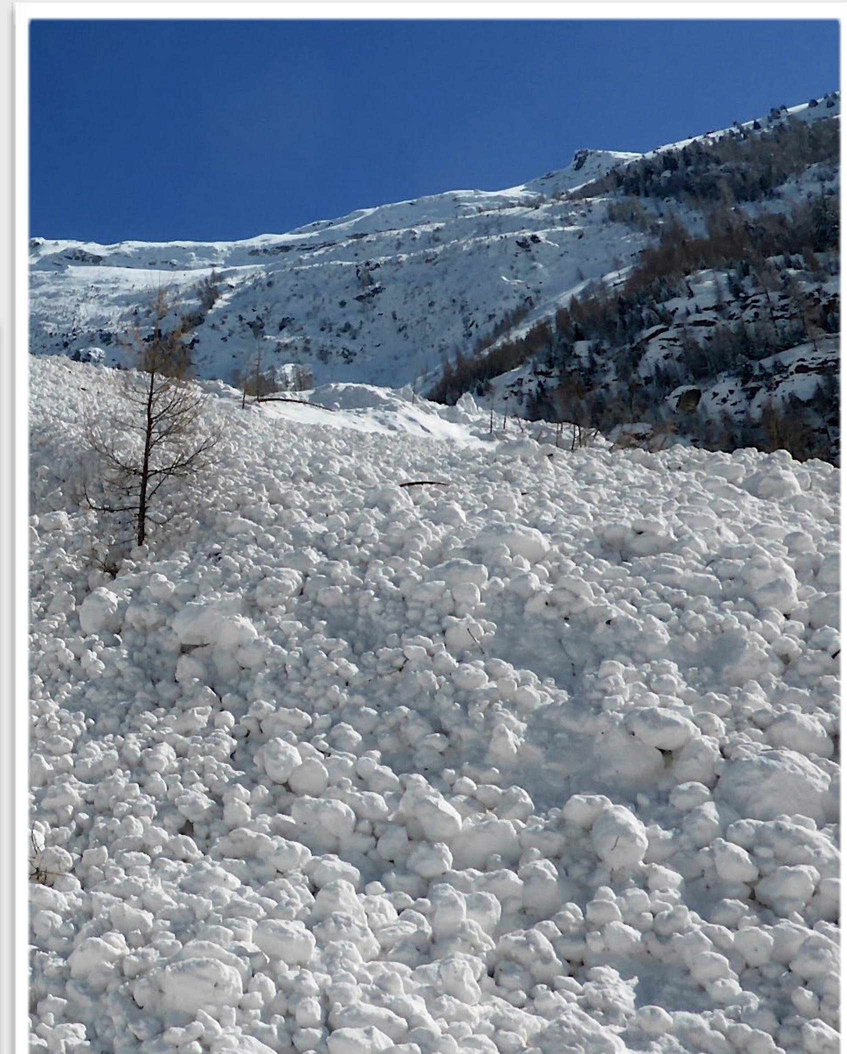

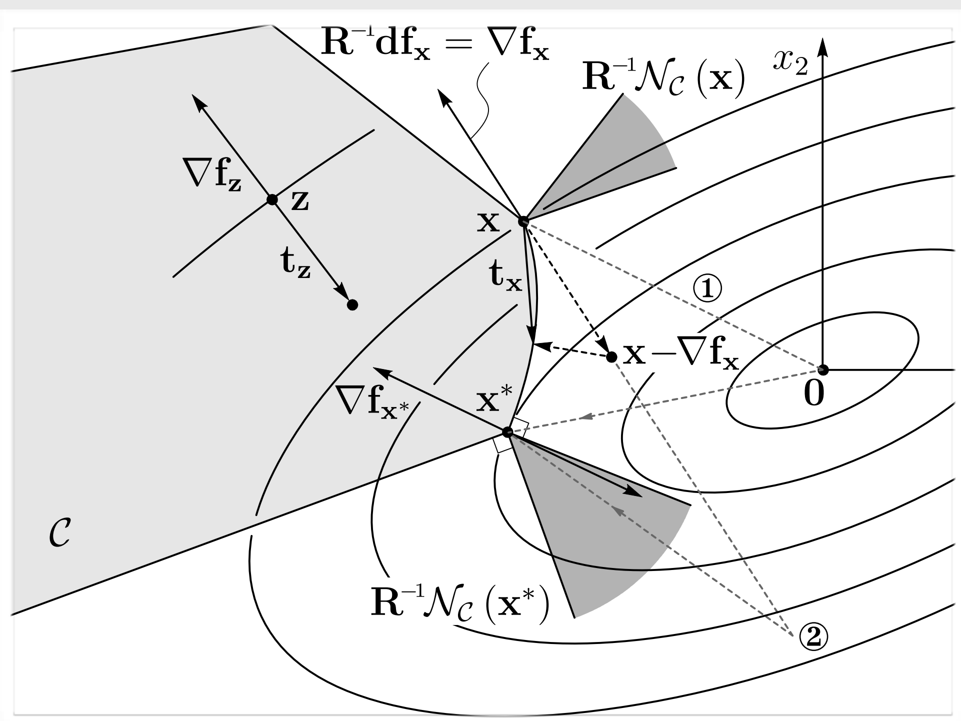
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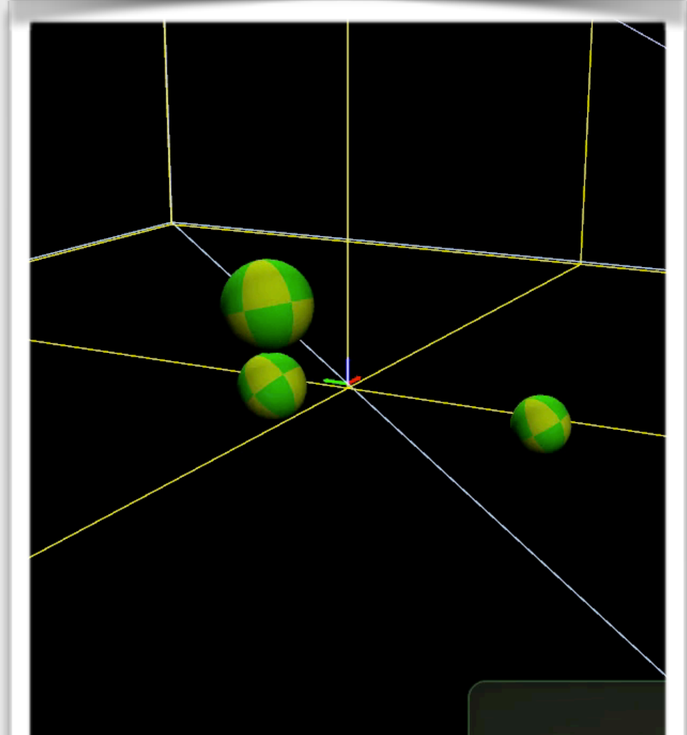
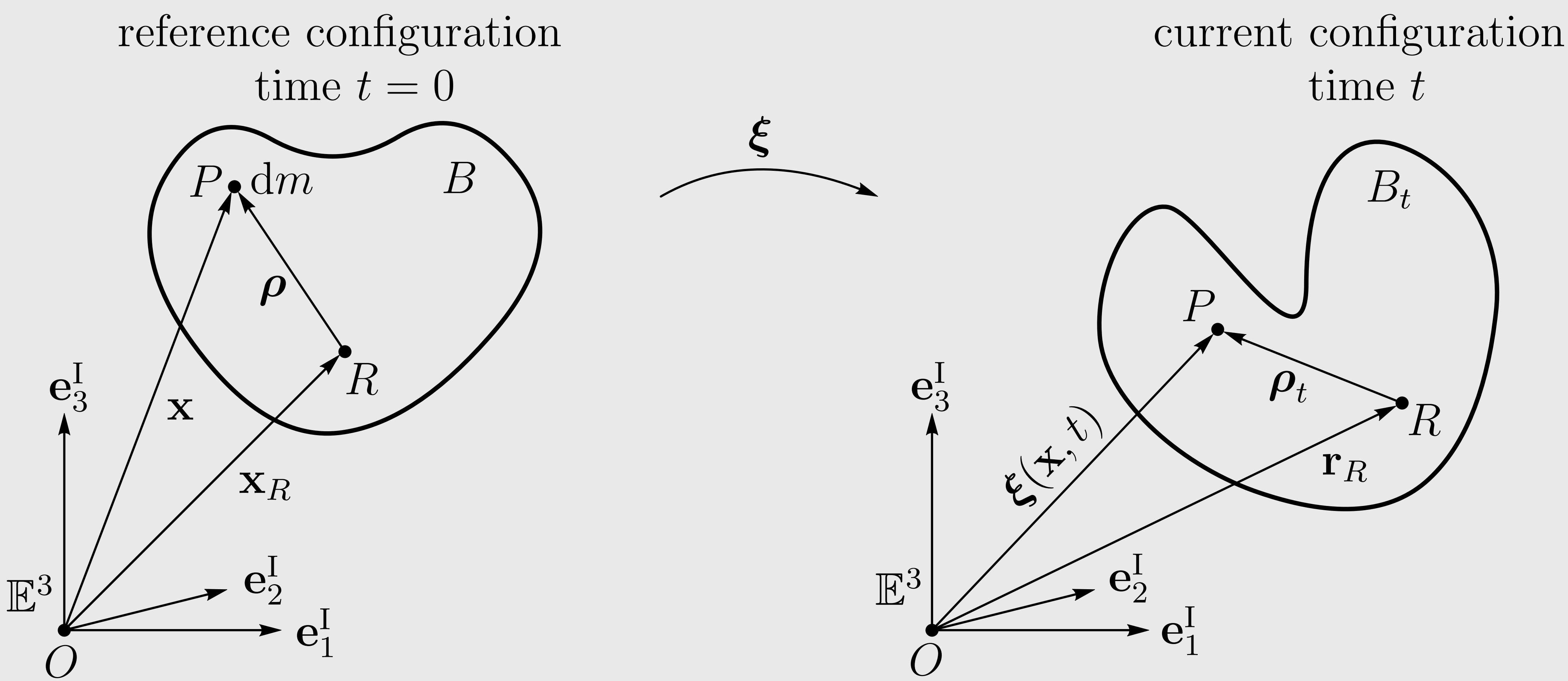
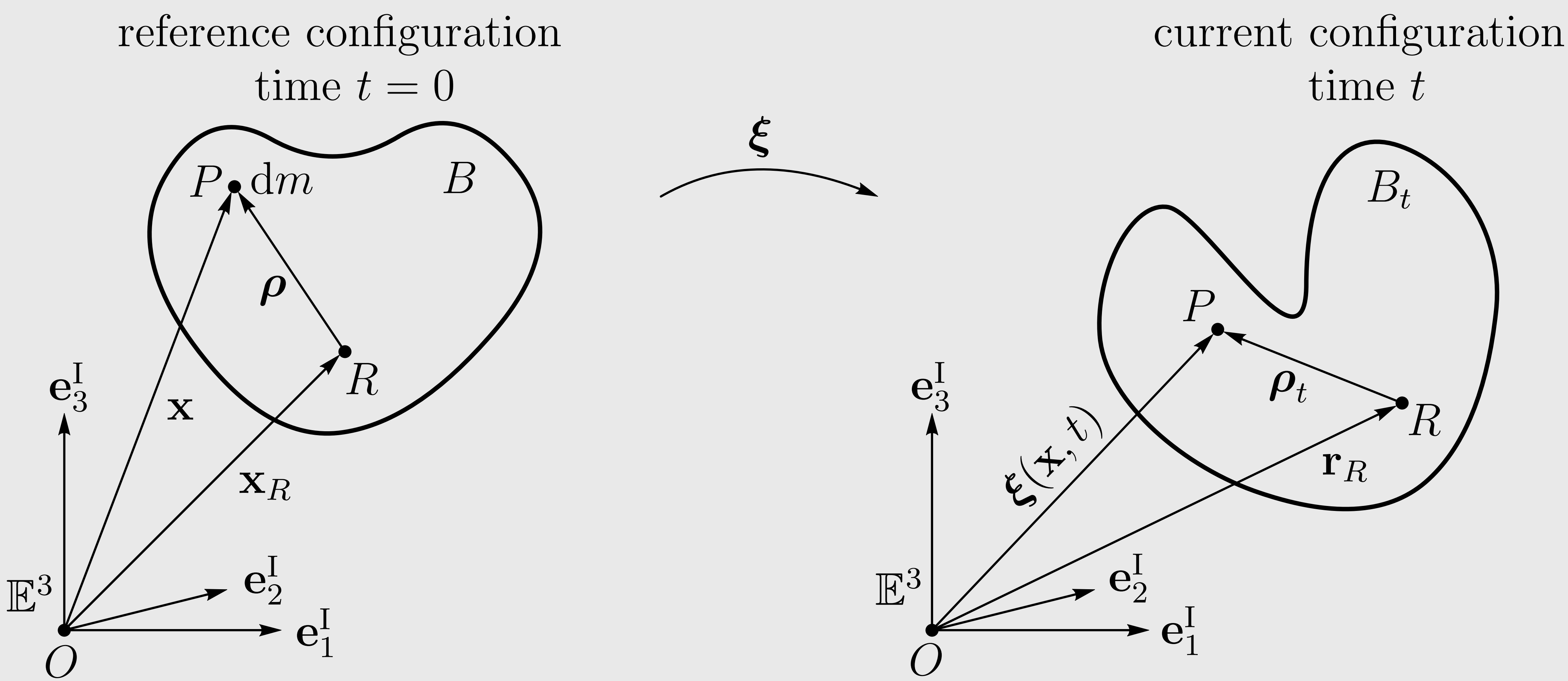


Body Motion


reference configuration
time $t = 0$
current configuration
time t
$\boldsymbol{\xi}$
P
$\mathrm{d}m$
B
$\boldsymbol{\rho}$
R
$\mathbf{x}$
$\mathbf{x}_R$
$\mathbf{e}_3^I$
$\mathbf{e}_2^I$
$\mathbf{e}_1^I$
$\mathbb{E}^3$
O
B_t
$\boldsymbol{\rho}_t$
$\mathbf{r}_R$
$\boldsymbol{\xi}(\mathbf{x}, t)$

Body Motion

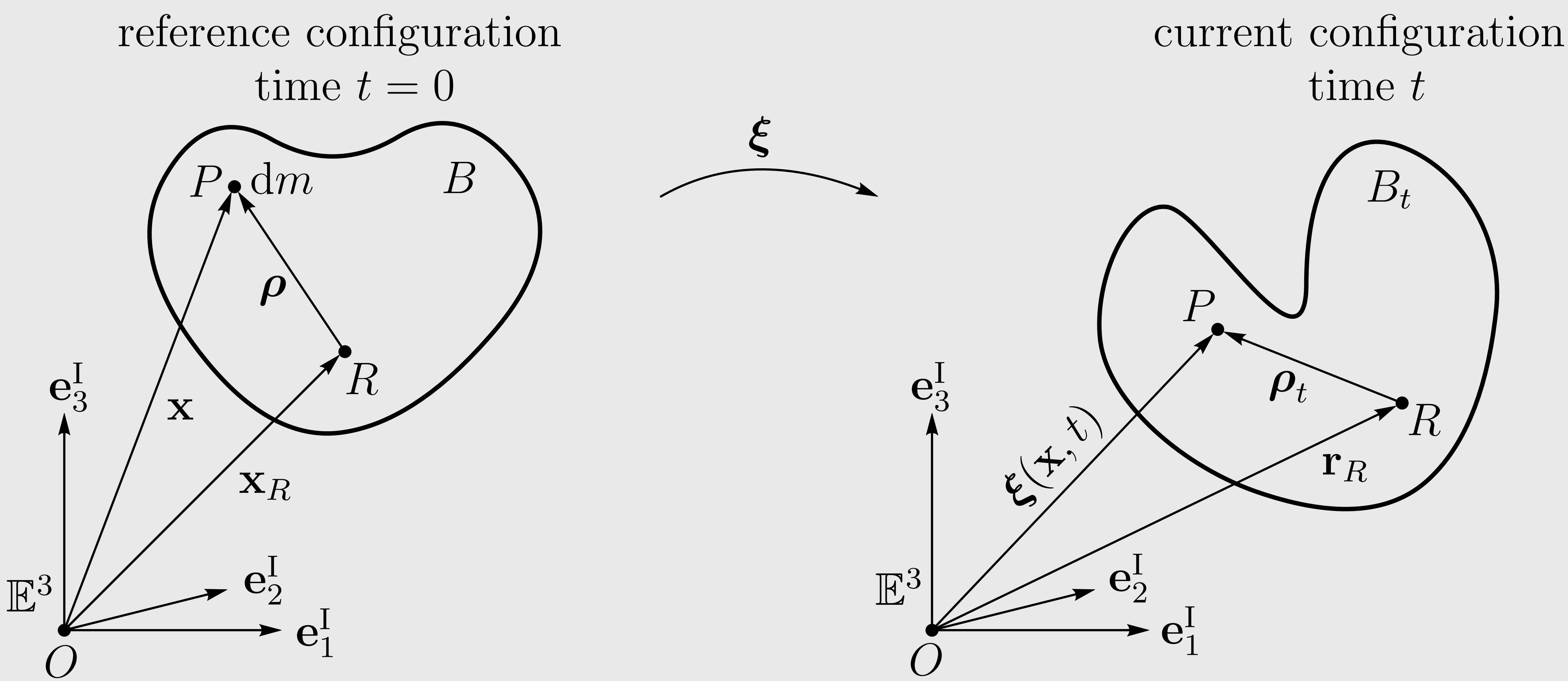


Motion: $\boldsymbol{\xi}(\cdot,t) : B \rightarrow B_t \subset \mathbb{E}^3$

$$\boldsymbol{\rho} \mapsto \boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\rho},t)$$

Inner Product: $(\mathbf{x}\,|\,\mathbf{y}) \Rightarrow \|\mathbf{x}\|_2 := \sqrt{(\mathbf{x}\,|\,\mathbf{x})}$

Body Motion



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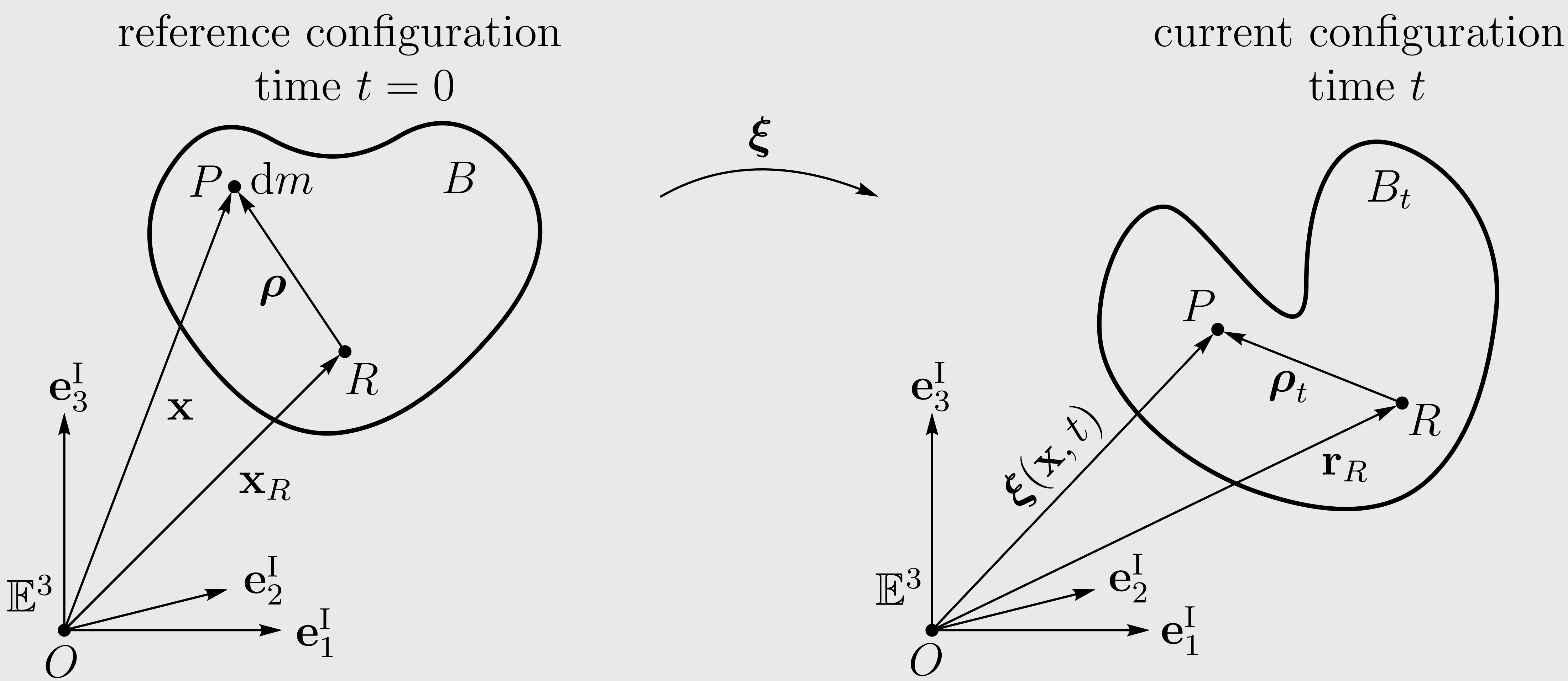
Isometry: $\|\boldsymbol{\xi}(\boldsymbol{\rho}_1, t) - \boldsymbol{\xi}(\boldsymbol{\rho}_2, t)\|_2 = \|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\|_2 \quad \forall \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in B \subset \mathbb{E}^3$

Body Motion



Motion: $\boldsymbol{\xi}(\cdot, t) : B \to B_t \subset \mathbb{E}^3$

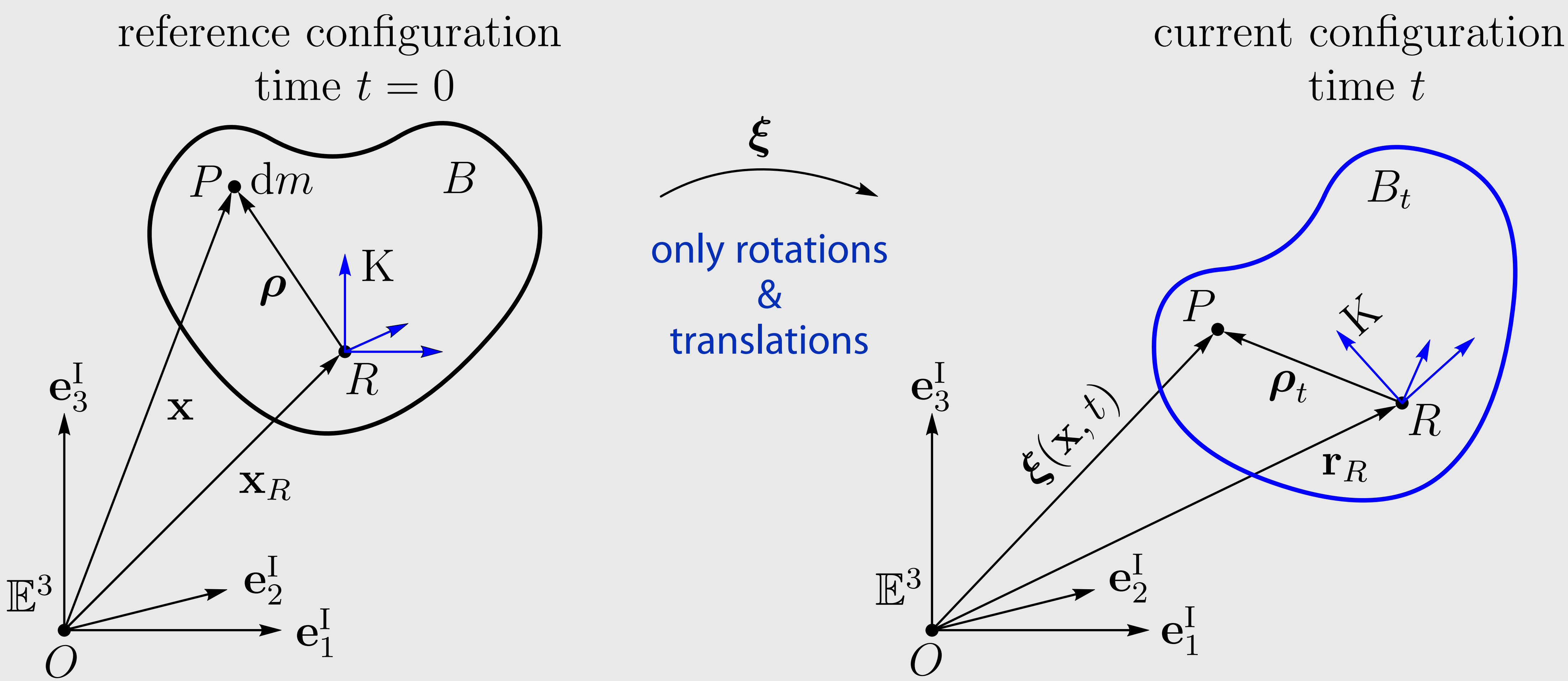
$$\boldsymbol{\rho} \mapsto \boldsymbol{\xi}_{\text{aff}}(\boldsymbol{\rho}, t) := \mathcal{T}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

Mazur-Ulam

Inner Product: $(\mathbf{x} \,|\, \mathbf{y}) \Rightarrow \|\mathbf{x}\|_2 := \sqrt{(\mathbf{x} \,|\, \mathbf{x})}$

Isometry: $\|\boldsymbol{\xi}(\boldsymbol{\rho}_1, t) - \boldsymbol{\xi}(\boldsymbol{\rho}_2, t)\|_2 = \|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\|_2 \quad \forall \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in B \subset \mathbb{E}^3$

Body Motion ⇒ Rigid Body Motion



Motion: $\boldsymbol{\xi}(\cdot, t) : B \;\to\; B_t \subset \mathbb{E}^3$

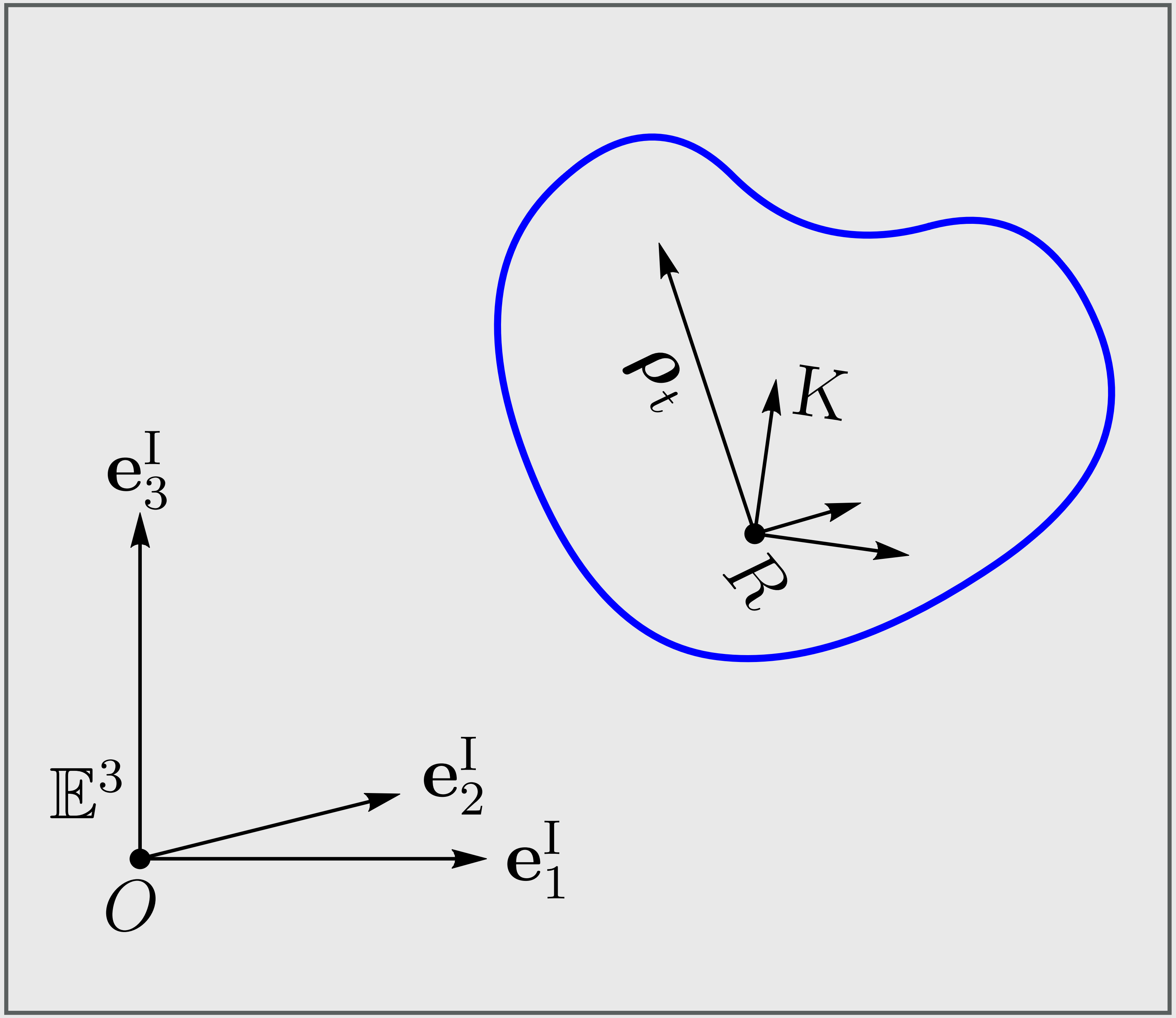
$$\boldsymbol{\rho} \;\mapsto\; \boldsymbol{\xi}_{\mathrm{rig}}(\boldsymbol{\rho}, t) := \mathcal{R}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

Inner Product: $(\mathbf{x}\,|\,\mathbf{y}) \Rightarrow \|\mathbf{x}\|_2 := \sqrt{(\mathbf{x}\,|\,\mathbf{x})}$

Mazur-Ulam

Isometry: $\|\boldsymbol{\xi}(\boldsymbol{\rho}_1, t) - \boldsymbol{\xi}(\boldsymbol{\rho}_2, t)\|_2 = \|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\|_2 \quad \forall \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in B \subset \mathbb{E}^3$

What is a Rotation?



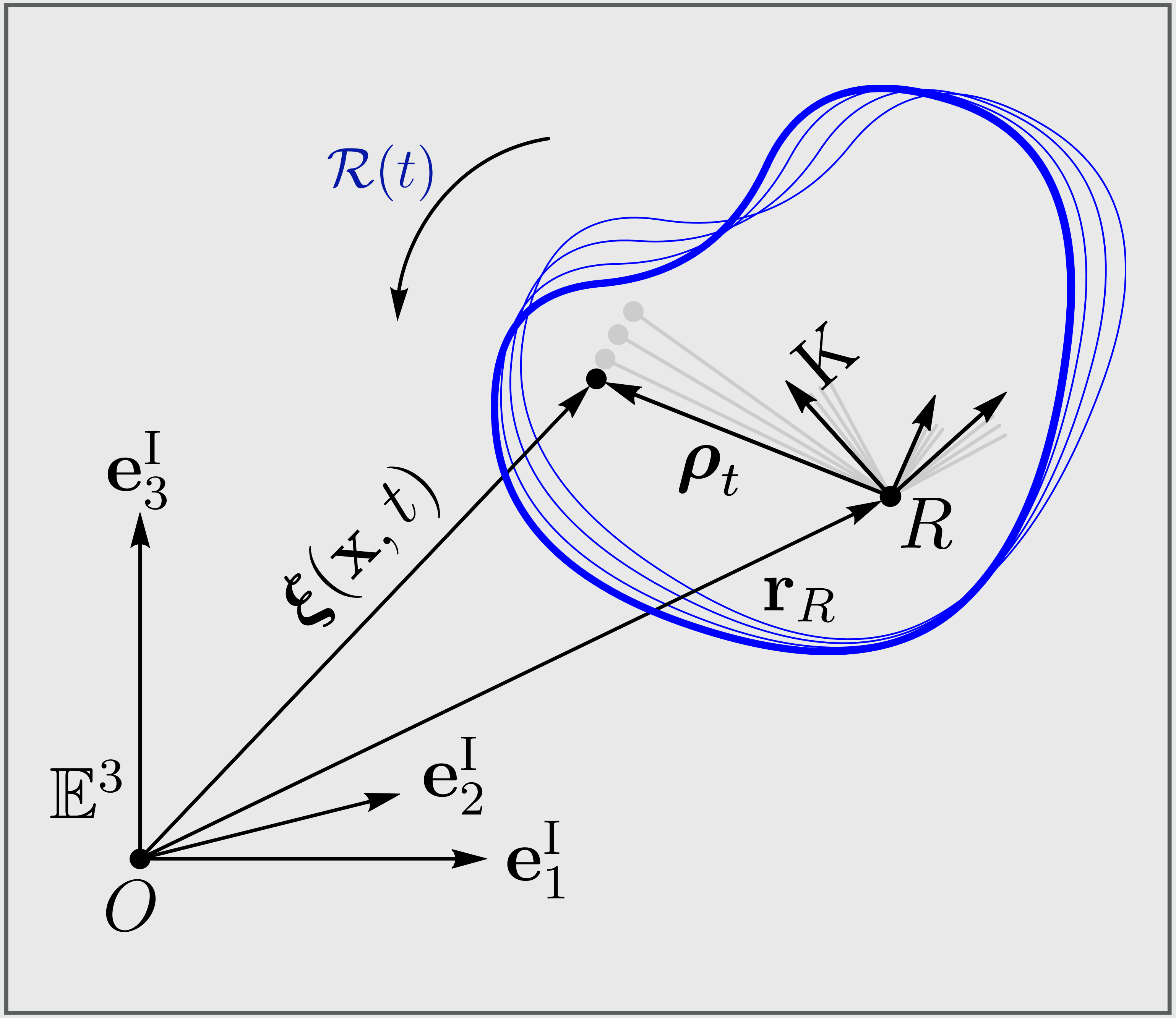
$$\boldsymbol{\xi}_{\mathrm{rig}}(\boldsymbol{\rho}, t) := \mathcal{R}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

A rotation is a linear map which:

- preserves the inner product (preserving angles)
- preserves the induced norm (preserves length)

If we preserve orientation (determinant = 1) too, we look at proper rotations.

What is a Rotation?



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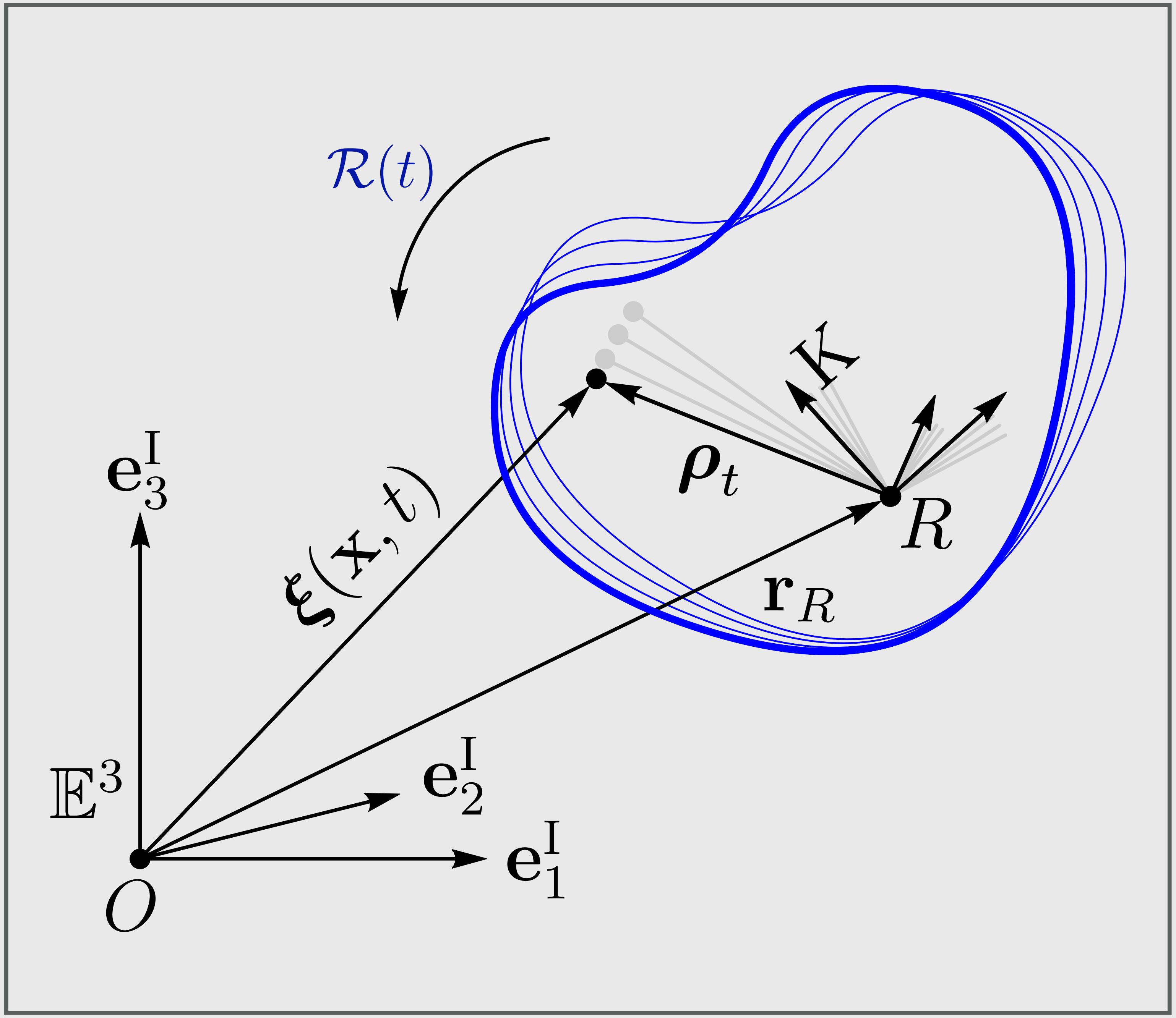

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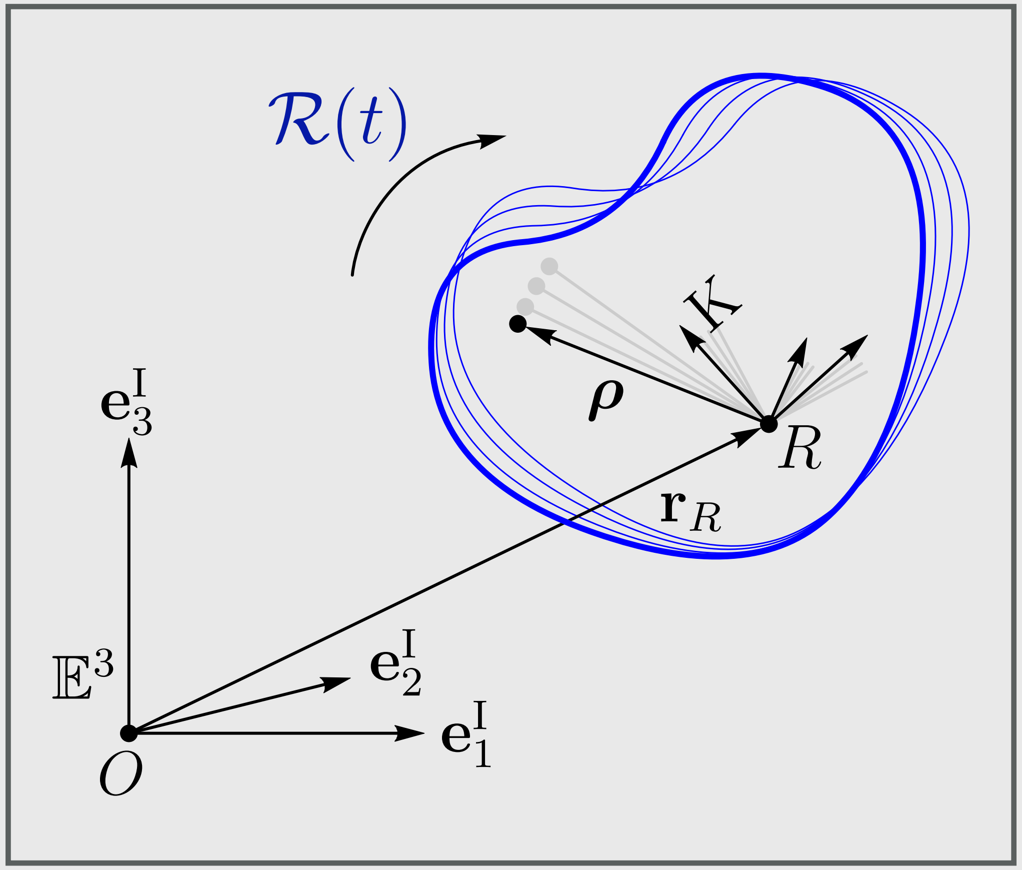
If we preserve orientation (determinant = 1) too, we look at proper rotations.

special orthogonal group: $SO(3) := \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}^{\top}\mathbf{R} = \mathbf{I},\ \det(\mathbf{R}) = 1\}$

Quaternions for Rotations (Rodrigues 1840, Hamilton 1843)



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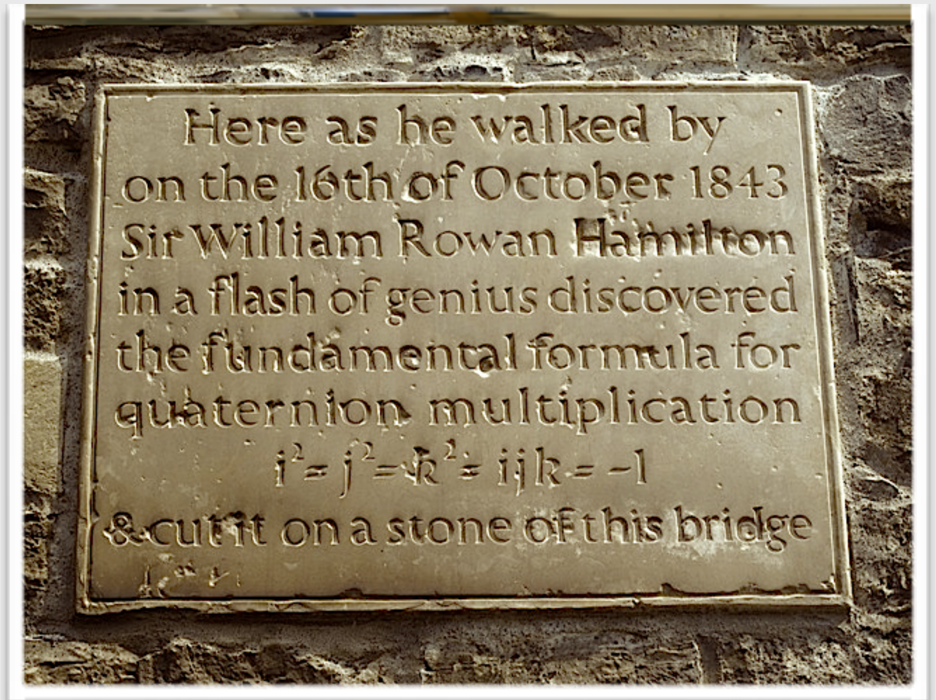

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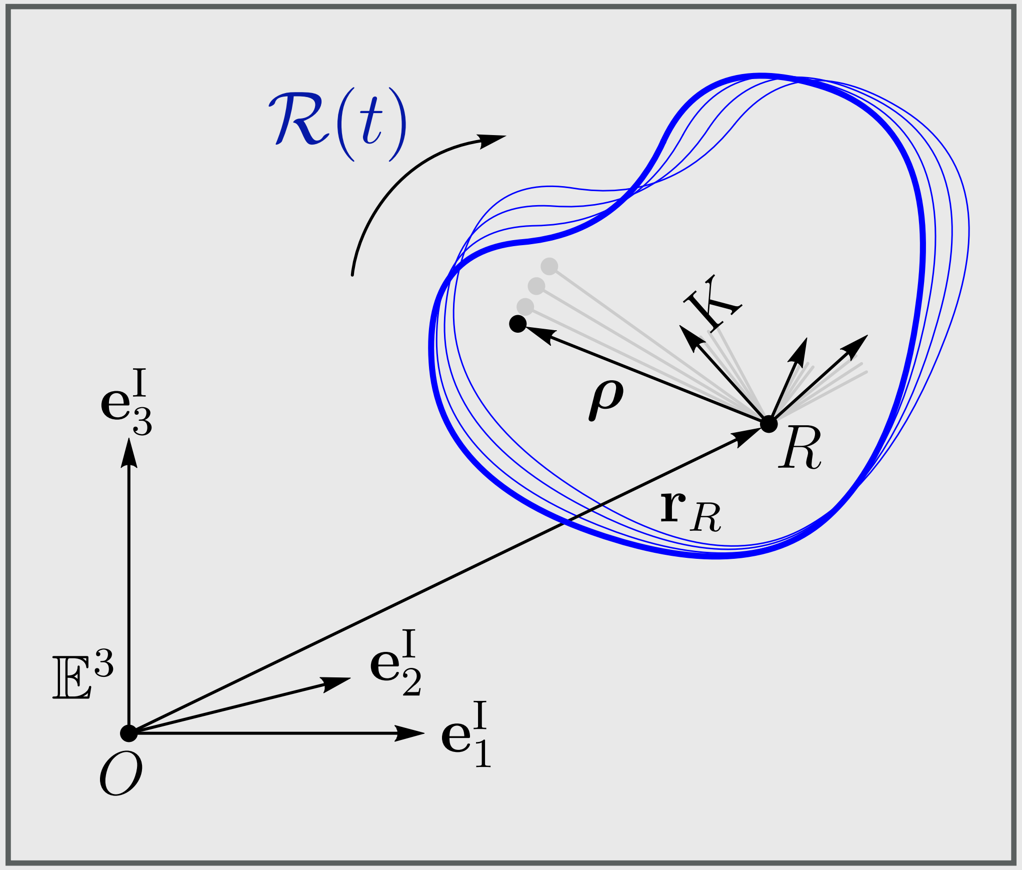


$$\boldsymbol{\xi}_{\mathrm{rig}}(\boldsymbol{\rho}, t) := \mathcal{R}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

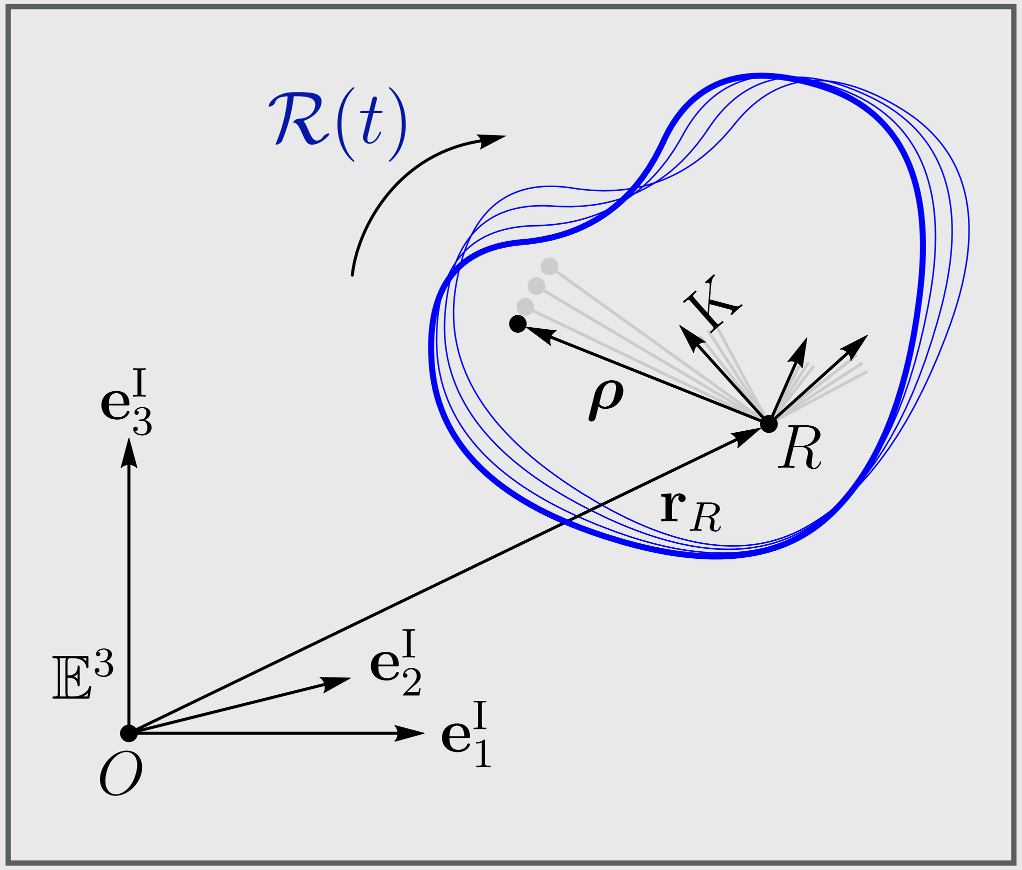
Theory of Couplets (Complex Numbers):

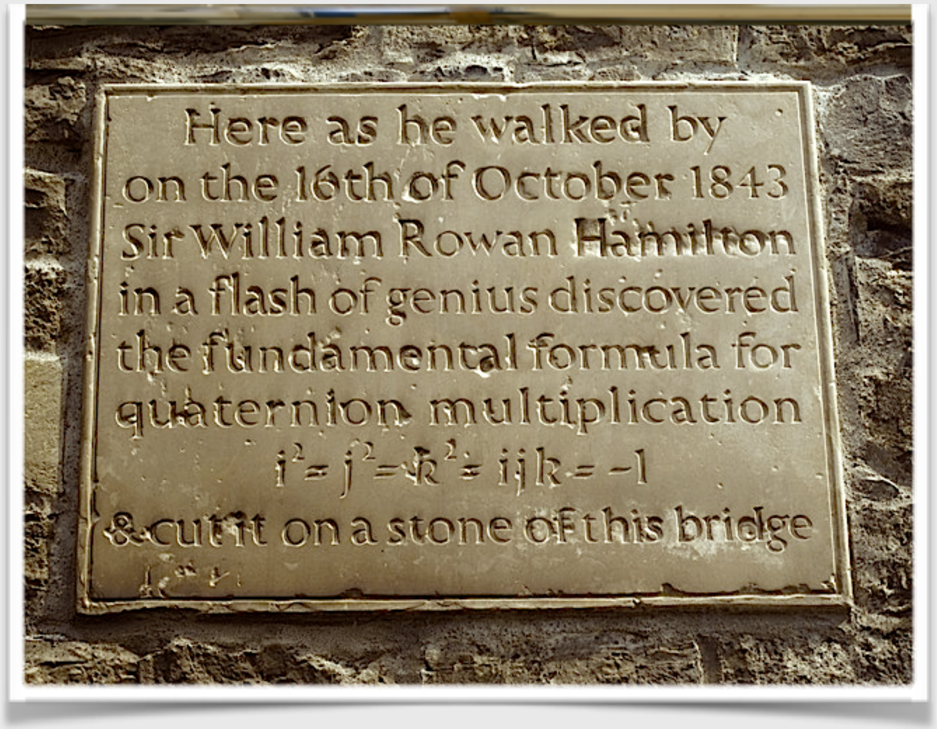
$a + bi \in \mathbb{C} \quad \Leftrightarrow \quad re^{i\varphi} \in \mathbb{C}$ with $i^2 = -1$

Add: $(a, b) + (c, d) = (a + c, b + d)$

Mult: $(a, b) \cdot (c, d) = (ac - bd, ad - bc)$

Quaternions for Rotations (Rodrigues 1840, Hamilton 1843)





$$\boldsymbol{\xi}_{\text{rig}}(\boldsymbol{\rho}, t) := \boxed{\mathcal{R}(t)}(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

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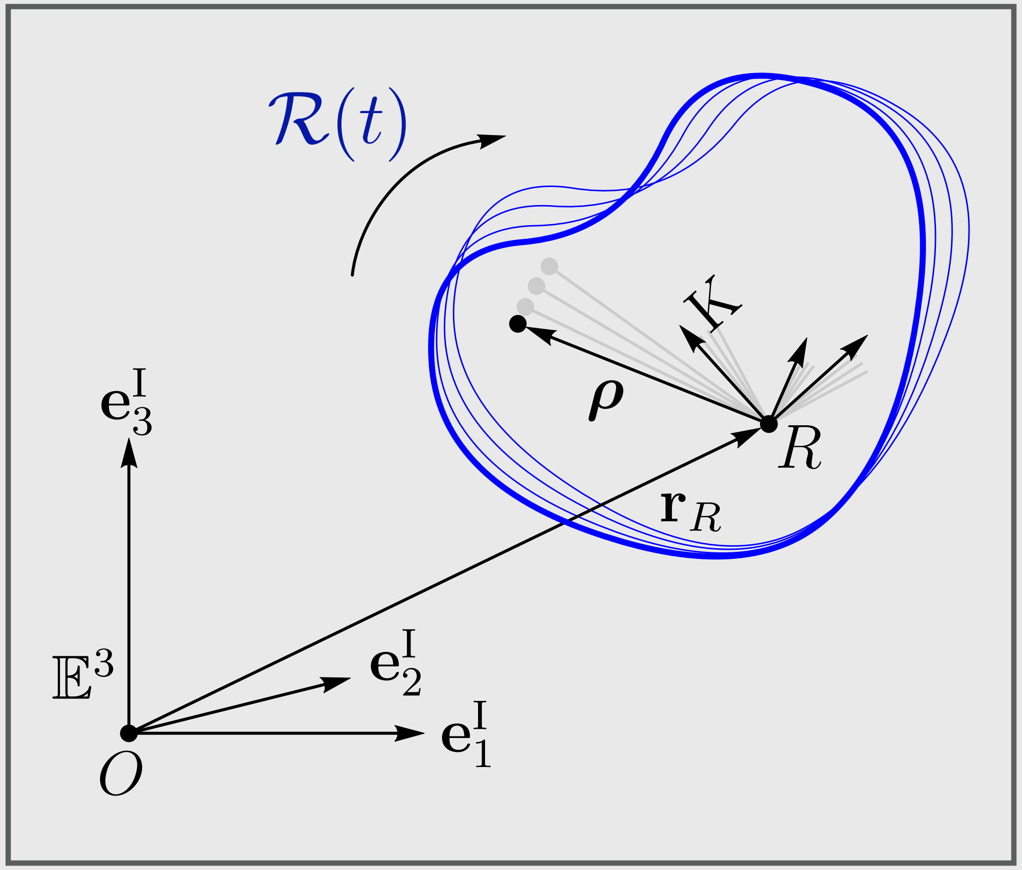
Addition and multiplication leads to rotations in 2D. **How to make it work for triples?**

$a + bi + cj$ with $i^2 = j^2 = -1$ doesn't work!

Quaternions for Rotations (Rodrigues 1840, Hamilton 1843)



$$\boldsymbol{\xi}_{\mathrm{rig}}(\boldsymbol{\rho}, t) := \mathcal{R}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

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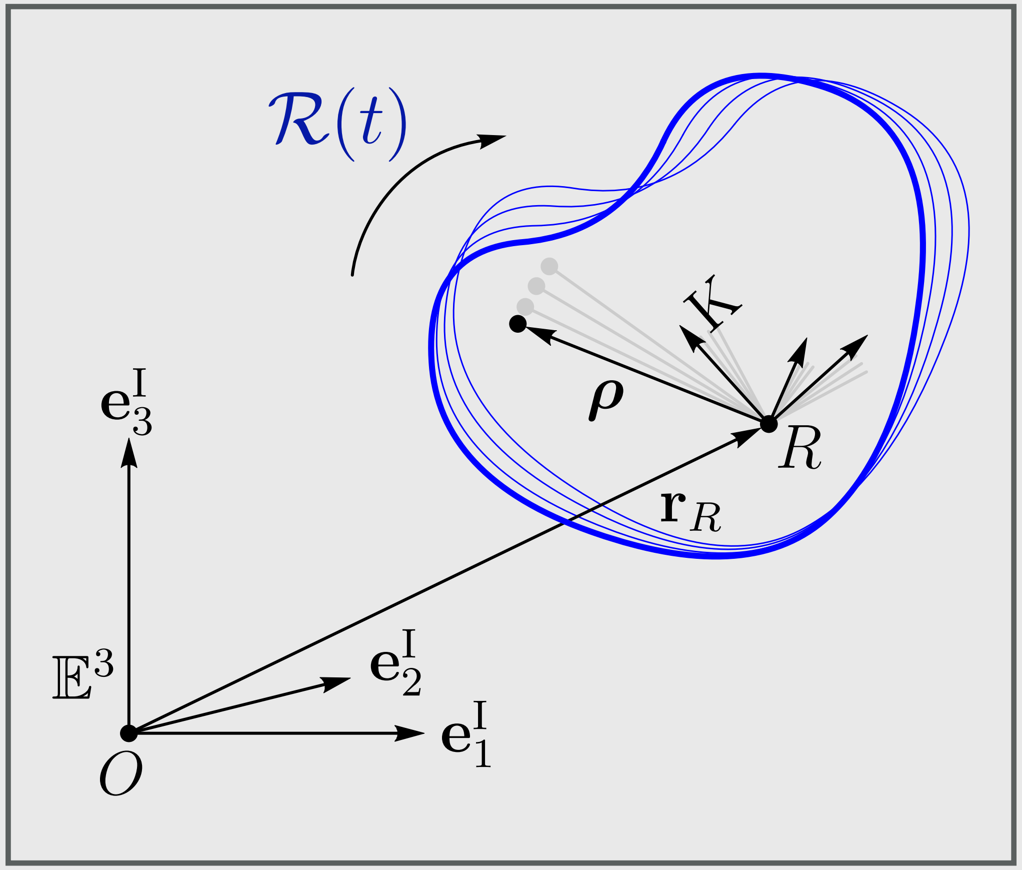
$a + bi + cj$ with $i^2 = j^2 = -1$ doesn't work!

$$\mathbb{H} := \{p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} \ , p_i \in \mathbb{R}\}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ikj} = -1$$

Quaternions for Rotations (Rodrigues 1840, Hamilton 1843)



$$\boldsymbol{\xi}_{\mathrm{rig}}(\boldsymbol{\rho}, t) := \mathcal{R}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

Theory of Couplets (Complex Numbers):

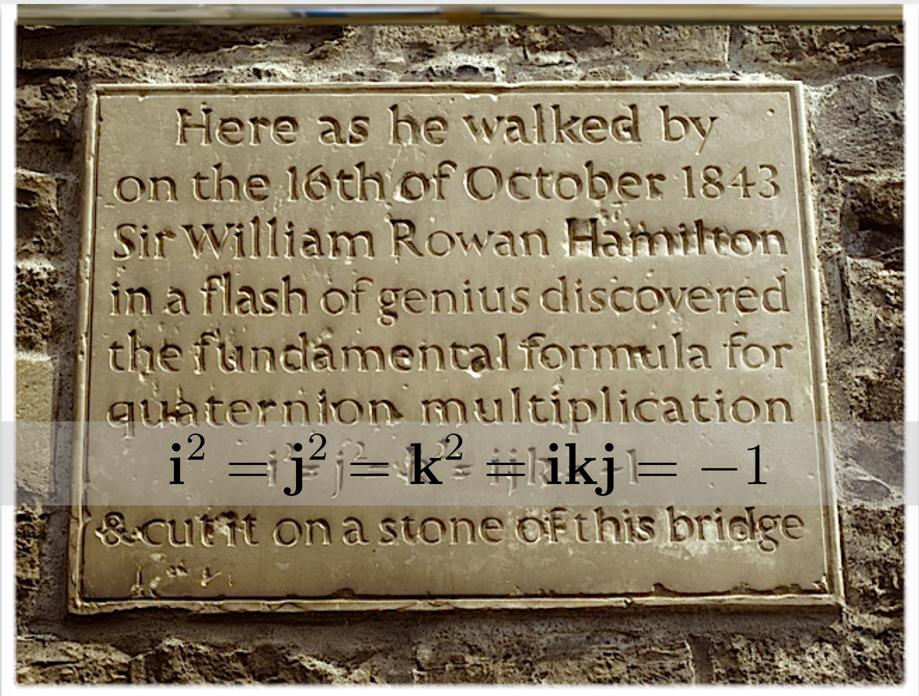
$a + bi \in \mathbb{C} \quad \Leftrightarrow \quad re^{i\varphi} \in \mathbb{C}$ with $i^2 = -1$

Add: $(a, b) + (c, d) = (a + c, b + d)$

Mult: $(a, b) \cdot (c, d) = (ac - bd, ad - bc)$

Addition and multiplication leads to rotations in 2D. **How to make it work for triples?**

$a + bi + cj$ with $i^2 = j^2 = -1$ doesn't work!



$$\mathbb{H} := \{p_0\mathbf{I} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\ , p_i \in \mathbb{R}\} \subset \mathbb{C}^{2\times 2}$$

with:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Quaternions for Rotations

$$\boldsymbol{\xi}_{\mathrm{rig}}(\boldsymbol{\rho}, t) := \mathcal{R}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

$$\mathbf{P} \in \mathbb{H} := \{\underbrace{p_0\mathbf{I}}_{\text{scalar part}} + \underbrace{p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}}_{\text{pure part}}\ , p_i \in \mathbb{R}\} \subset \mathbb{C}^{2\times 2}$$

Basis: $(\mathbf{I}, \mathbf{i}, \mathbf{j}, \mathbf{k})$

Coordinates:

$$\mathcal{K}(\mathbf{P}) = \begin{bmatrix} p_0 \\ \mathbf{p}_r \end{bmatrix} \in \mathbb{R}^4$$

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Rotations in $\mathbb{E}^4$ where P, Q are unit quaternions (= 6 min. parameters):

$$\begin{array}{ccc} \mathbb{H} & \to & \mathbb{H} \\ \mathbf{X} & \mapsto & \mathbf{PXQ} \end{array} \quad \xrightarrow{\mathcal{K}} \quad \begin{array}{ccc} \mathbb{R}^4 & \to & \mathbb{R}^4 \\ \mathbf{x} & \mapsto & \underbrace{\varphi_L(\mathbf{P})\varphi_R(\mathbf{Q})}_{\in\, \mathbb{R}^{4\times 4}}\underbrace{\mathbf{x}}_{\in\, \mathbb{R}^4} \end{array}$$

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make $\mathbf{x}$ pure and $\mathbf{Q} := \mathbf{P}^{-1}$

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scalable body motion:

$${}_{\mathrm{I}}\boldsymbol{\xi}_{\mathrm{scal}} = s\ {}_{\mathrm{I}}\mathbf{R}(\mathbf{p})\ {}_{\mathrm{I}}\boldsymbol{\rho} + {}_{\mathrm{I}}\mathbf{r}_R$$

generalized coordinates:

$$\mathbf{q}(t) := \begin{bmatrix} {}_{\mathrm{I}}\mathbf{r}_R \\ \mathbf{p} \end{bmatrix} \in \mathbb{R}^7$$

scaling factor: $s := \|\mathbf{p}\|$

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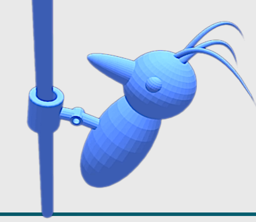
constraint: $\|\mathbf{p}\| = 1 \quad \forall t$



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equation of motions by principles

The Principle of Virtual Work

What the hack is a force?

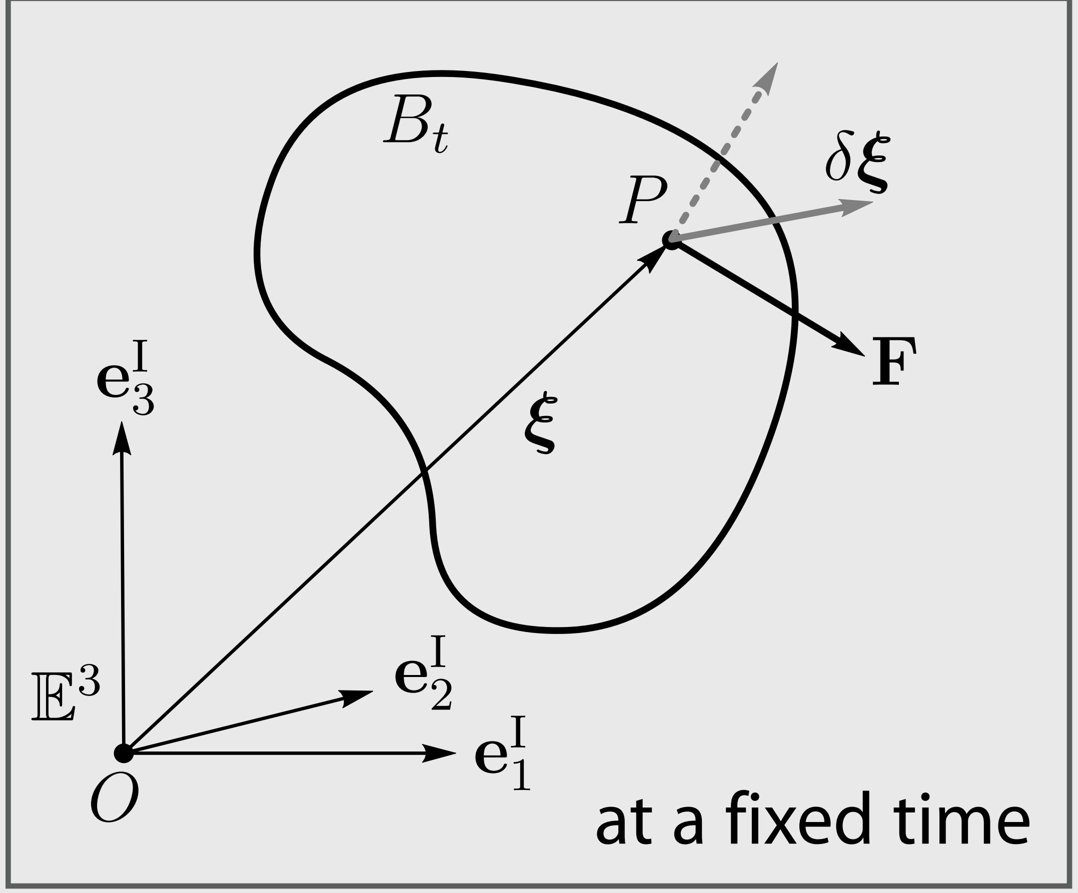
linear functional ⇒ *producing virtual work*

when given a virtual displacement.

$\langle \mathbf{F} \,|\, \delta\boldsymbol{\xi} \rangle =$ $\mathbf{F}(\delta\boldsymbol{\xi}) = \delta W$

duality pairing

force virtual displacement



Principle of Virtual Work:

At any instant of time t, the virtual work δW of a body $B \in \mathbb{E}^3$ vanishes for all virtual displacements $\delta\boldsymbol{\xi}$, that is,

$$\delta W(\delta\boldsymbol{\xi}) = \int_B \langle \mathrm{d}\mathbf{F} \,|\, \delta\boldsymbol{\xi}(\mathbf{x}) \rangle = \delta W^{\mathrm{dyn}}(\delta\boldsymbol{\xi}) + \delta W^{\mathrm{int}}(\delta\boldsymbol{\xi}) + \delta W^{\mathrm{ext}}(\delta\boldsymbol{\xi}) = 0 \quad \forall \delta\boldsymbol{\xi} \;.$$

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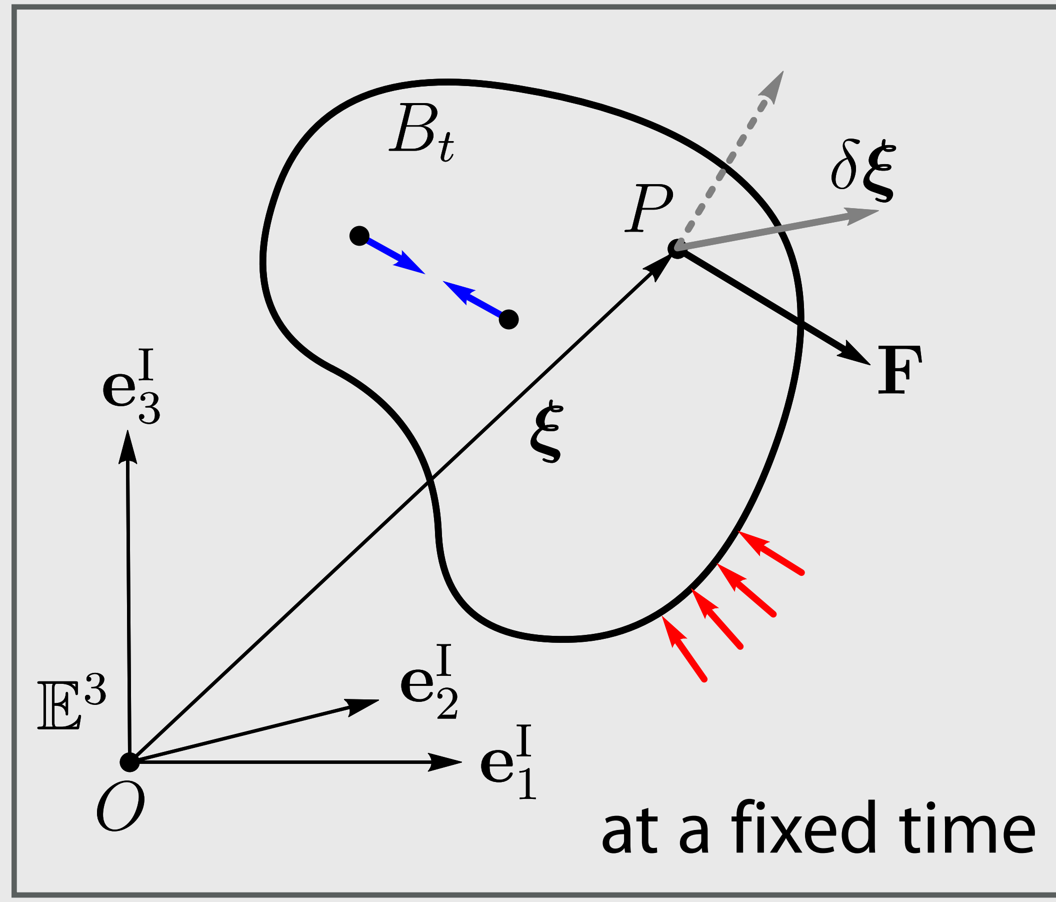

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Applying the Fundamental Axioms

Equations of motion for the rigid body:

rigid body motion: $\boxed{{}_{\mathrm{I}}\boldsymbol{\xi}_{\mathrm{rig}} = {}_{\mathrm{I}}\mathbf{R}(\mathbf{p})\ {}_{\mathrm{I}}\boldsymbol{\rho} + {}_{\mathrm{I}}\mathbf{r}_R}$

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constitutive force law:
Newton's second law

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law of interaction &
perfect constraint
forces ($\infty \rightarrow 6$)

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constitutive force law:
Newton's second law

velocities: $\quad {}_{\mathrm{I}}\dot{\boldsymbol{\xi}}_{\mathrm{rig}} = {}_{\mathrm{K}}\mathbf{R}\ {}_{\mathrm{K}}\tilde{\boldsymbol{\Omega}}\ {}_{\mathrm{K}}\boldsymbol{\rho}_t + {}_{\mathrm{I}}\dot{\mathbf{r}}_R$

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law of interaction & perfect constraint forces ($\infty \to 6$)

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$\dfrac{\mathrm{d}}{\mathrm{d}t}$

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Non-Smooth Dynamics in a Nutshell

Continuous Equations of Motion:

$$\mathbf{M}(\mathbf{q},t)\ \dot{\mathbf{u}} - \mathbf{h}(\mathbf{q},\mathbf{u},t) - \mathbf{W}(\mathbf{q},t)\boldsymbol{\lambda} = \mathbf{0}$$

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q})\ \mathbf{u}$$

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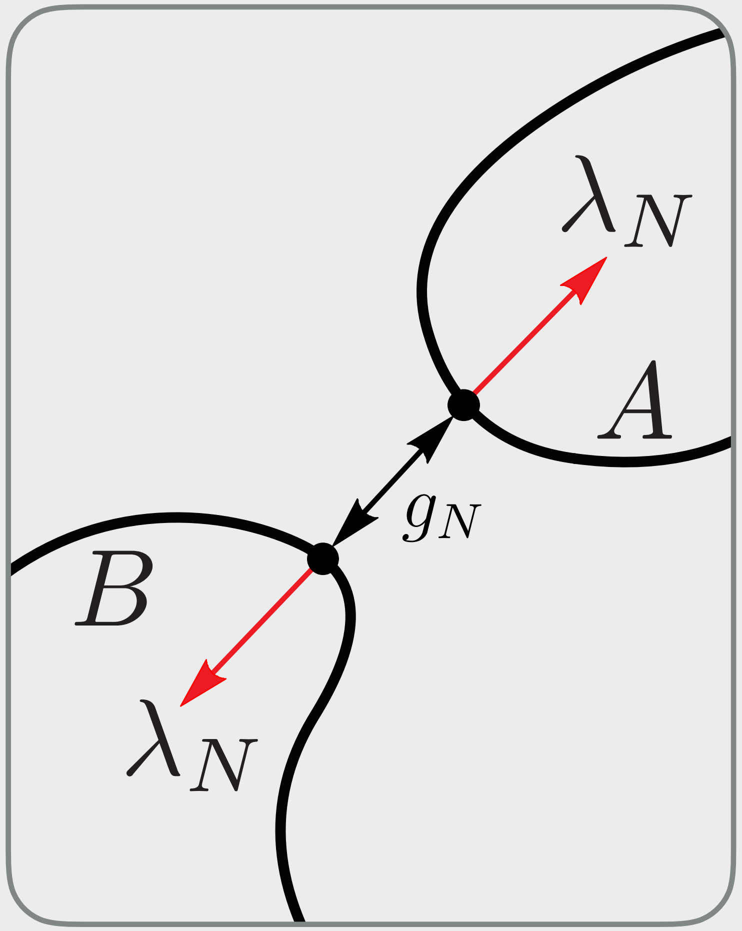
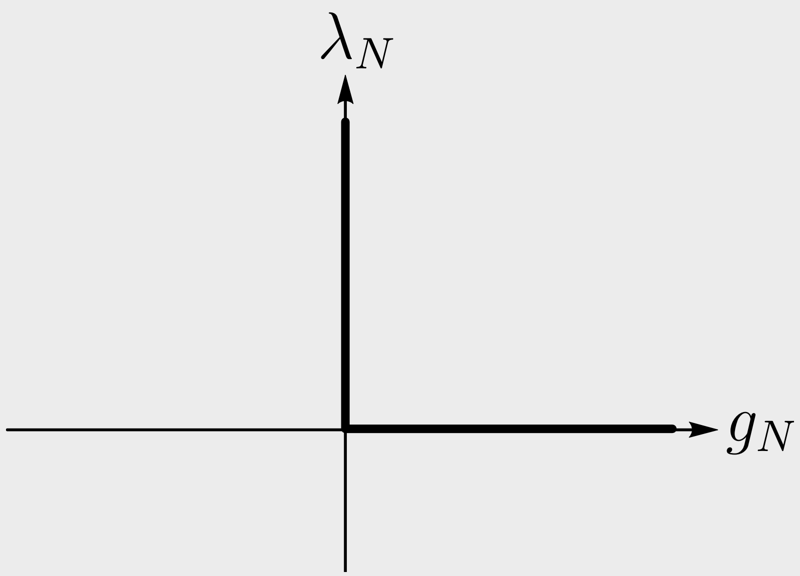


contact configuration

Unilateral Contact:

$$-g_N \in \mathcal{N}_{\mathbb{R}_0^+}(\lambda_N)$$



force laws as normal cone inclusions

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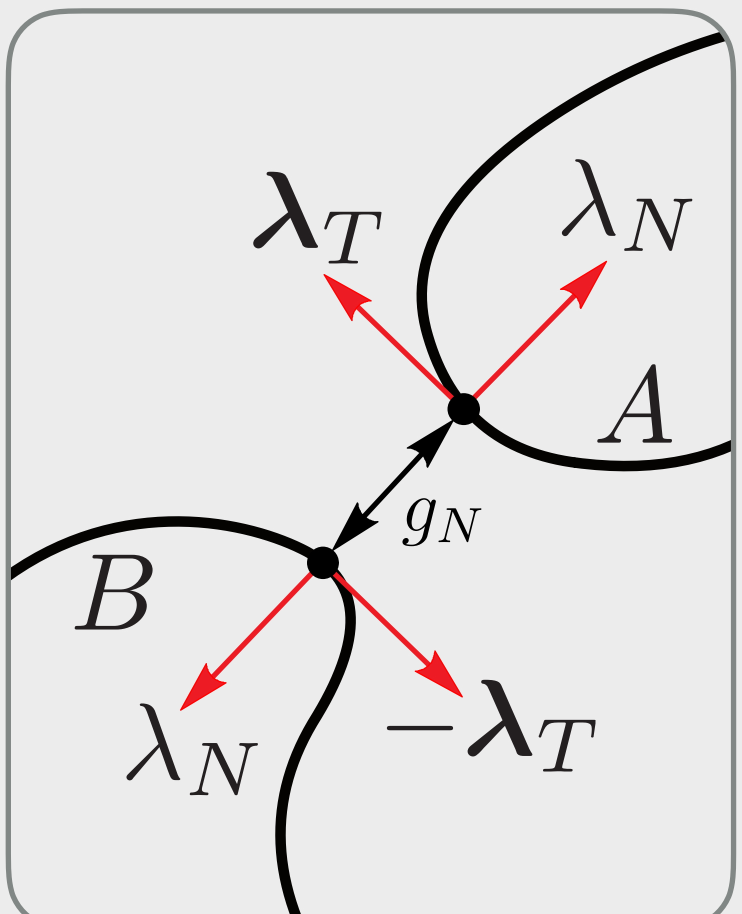

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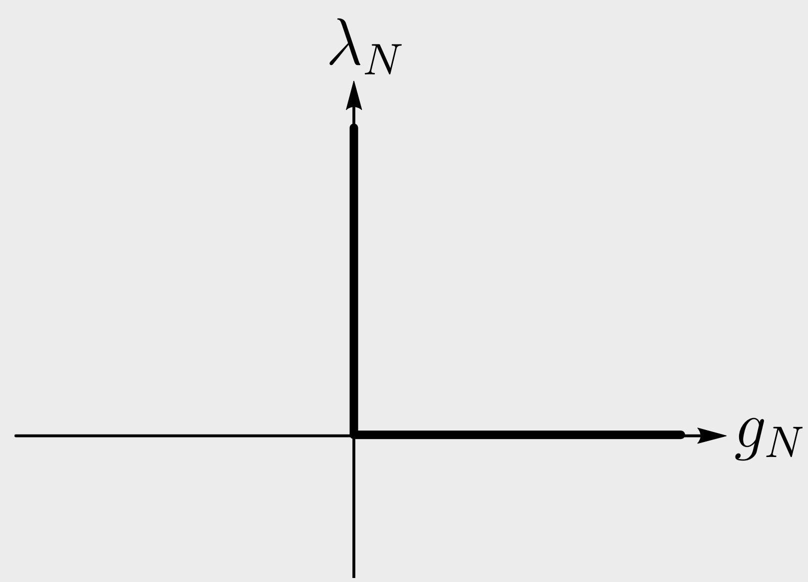

Coulomb Friction:

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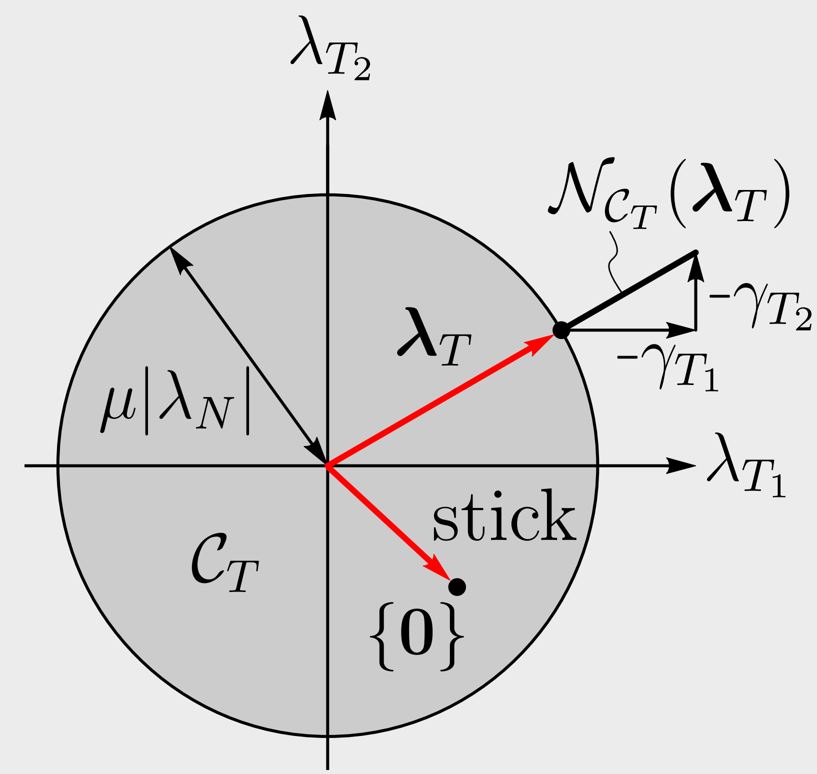

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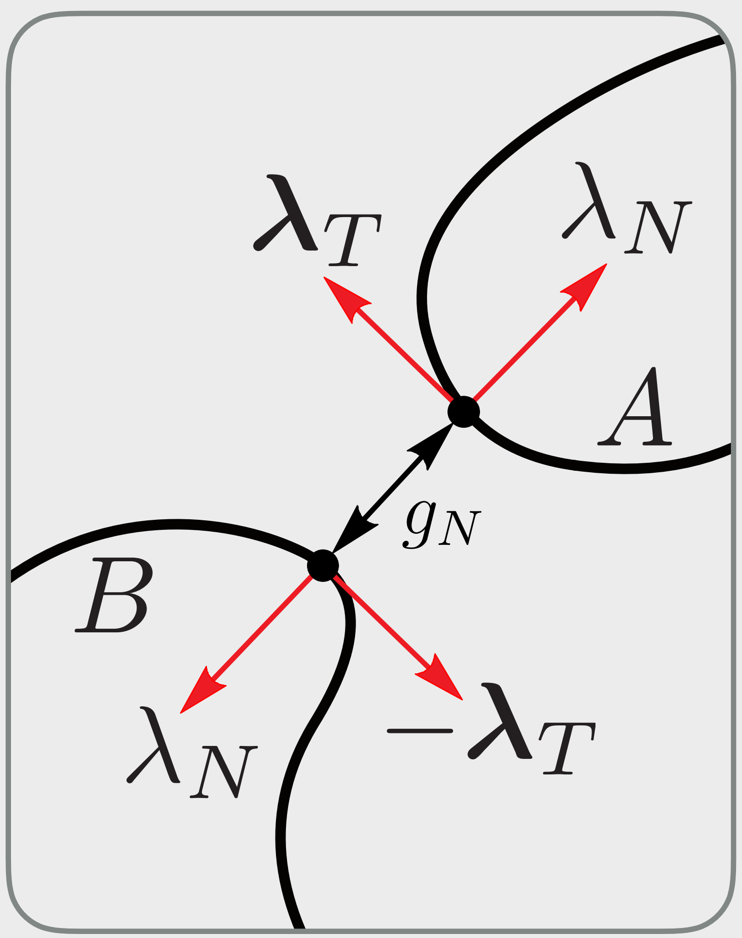

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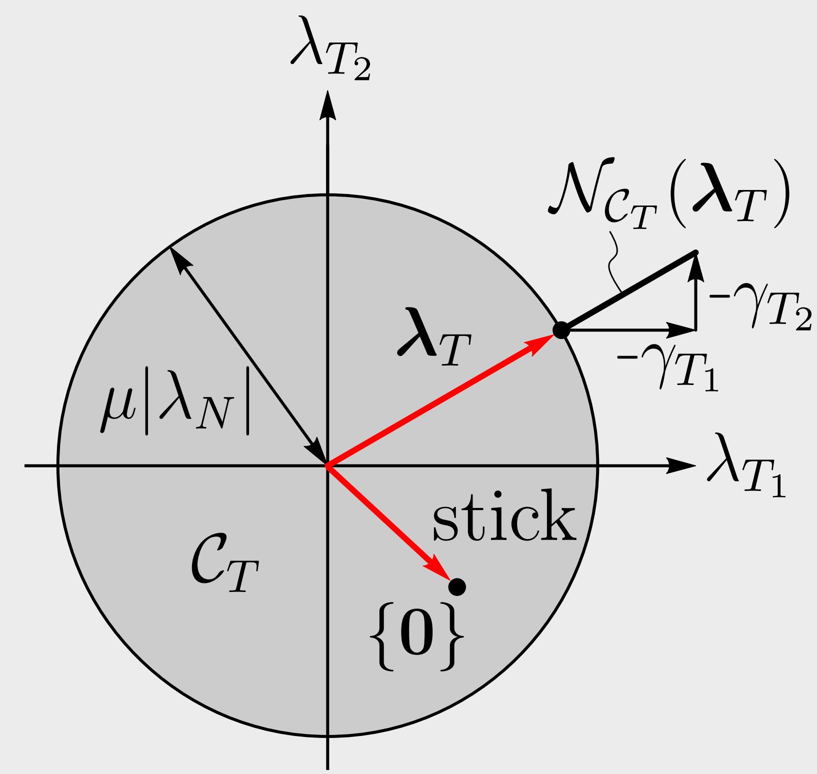

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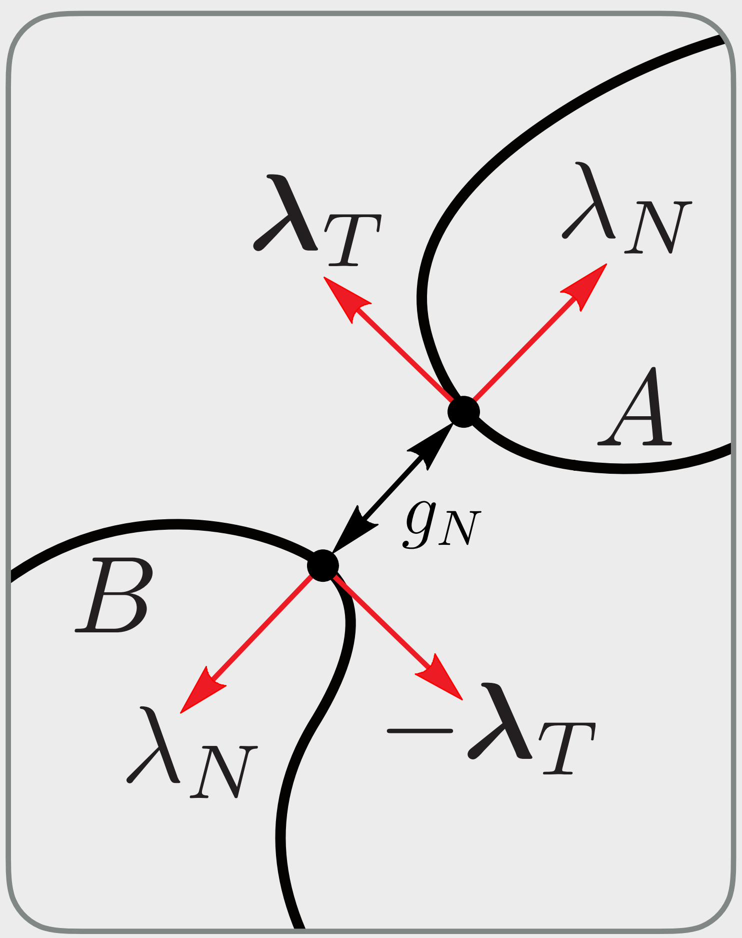

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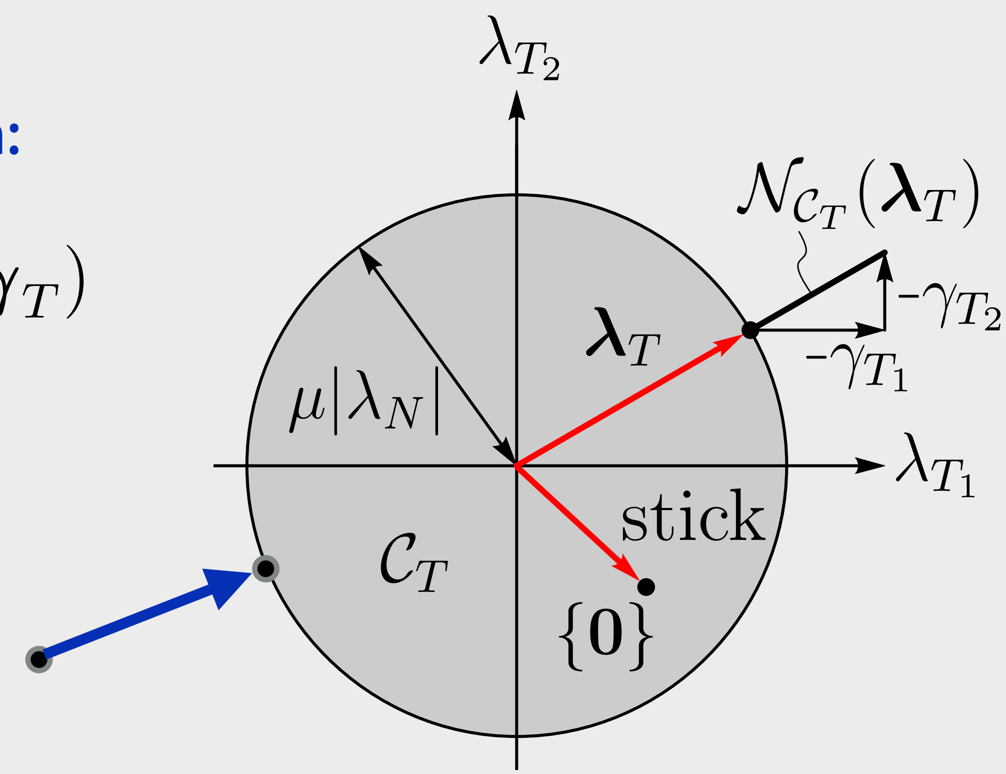


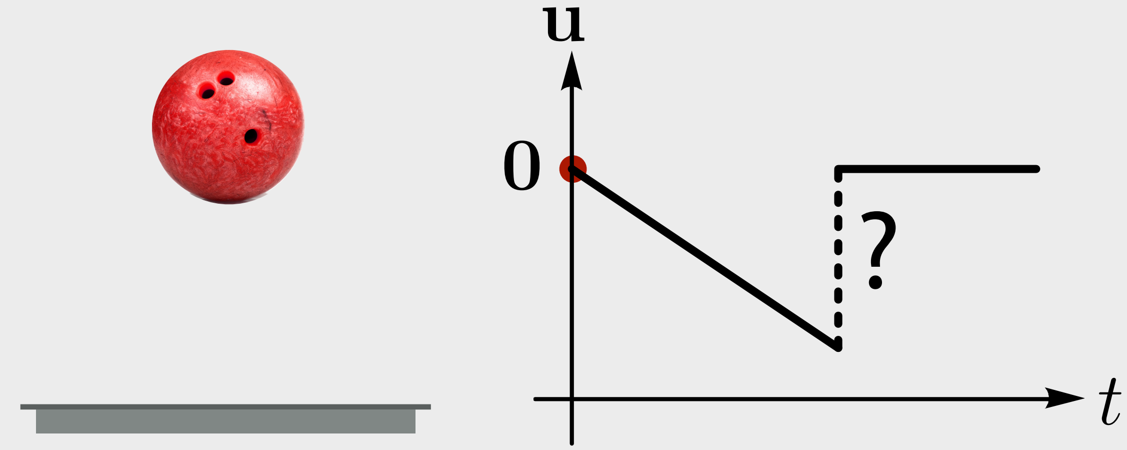
force laws as normal cone inclusions

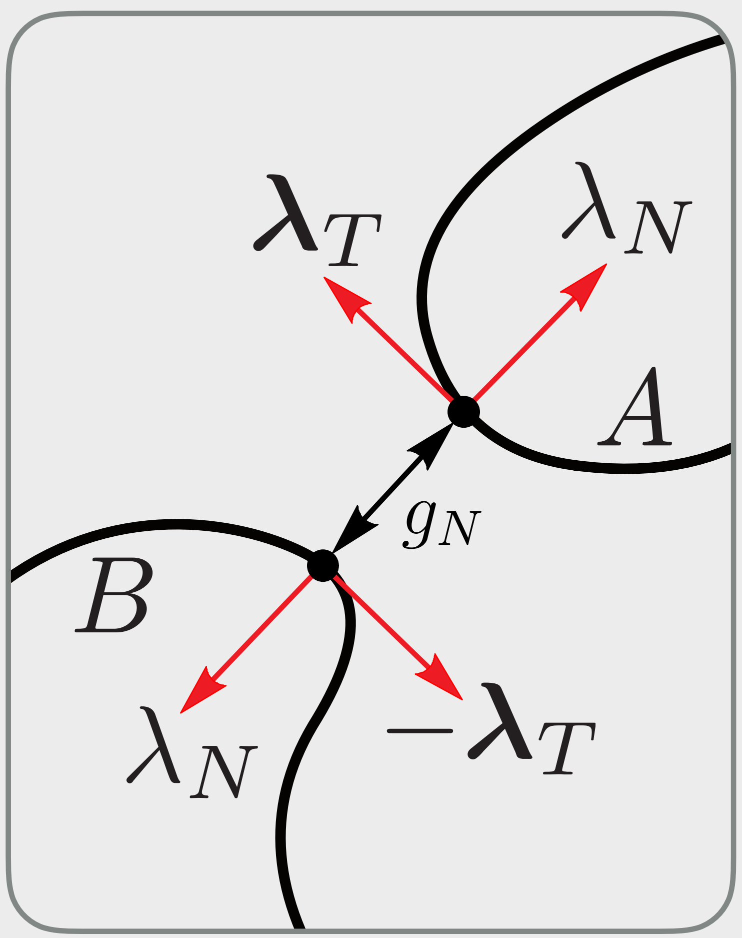
Non-Smooth Dynamics in a Nutshell

Continuous Equations of Motion:

$$\mathbf{M}(\mathbf{q},t)\ \dot{\mathbf{u}} - \mathbf{h}(\mathbf{q},\mathbf{u},t) - \mathbf{W}(\mathbf{q},t)\boldsymbol{\lambda} = \mathbf{0}$$

$$\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q})\ \mathbf{u}$$





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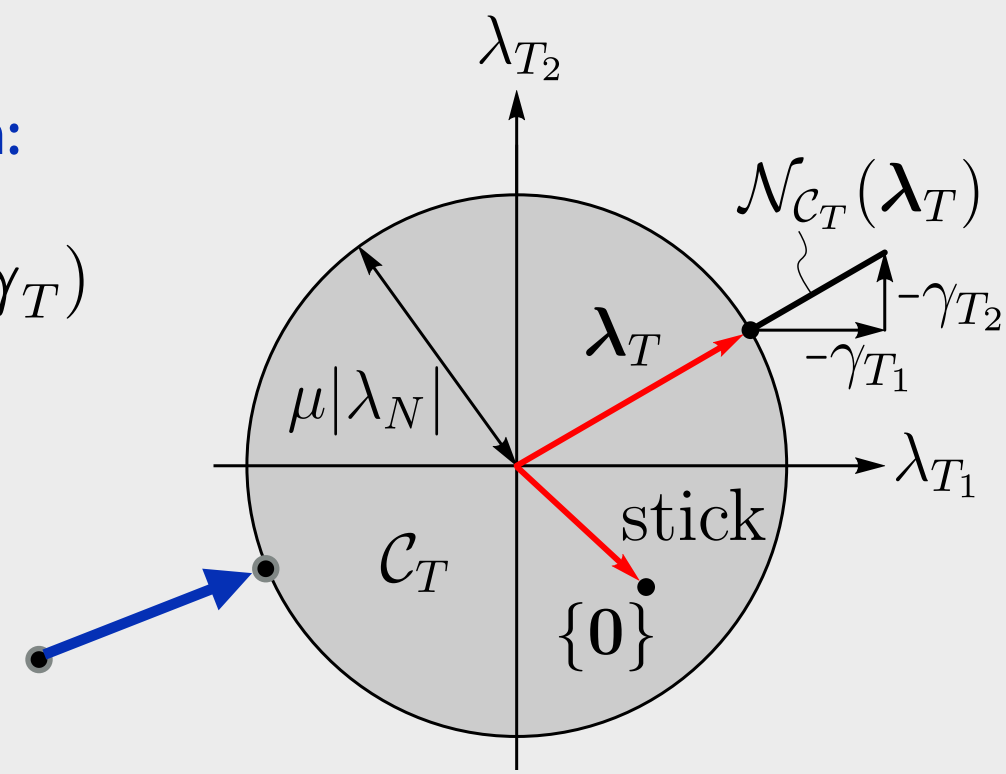

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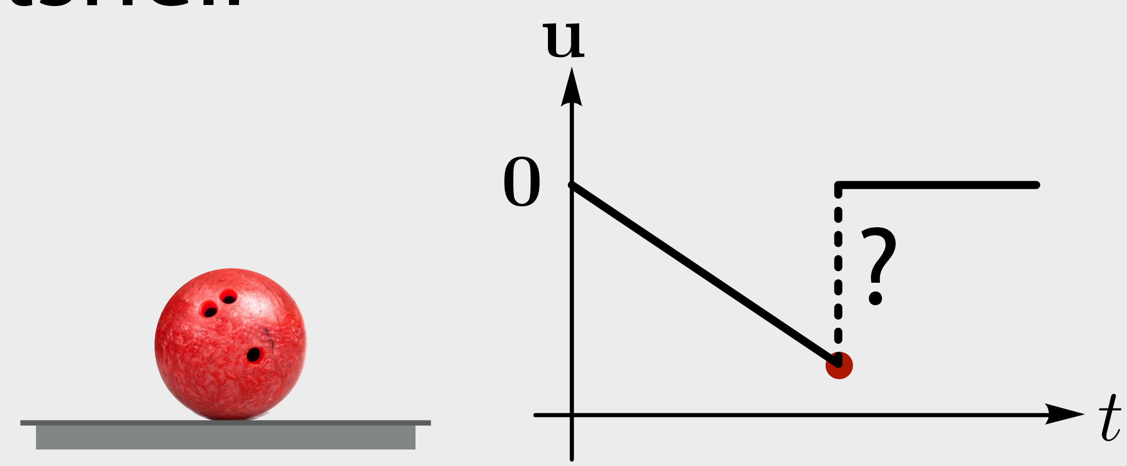

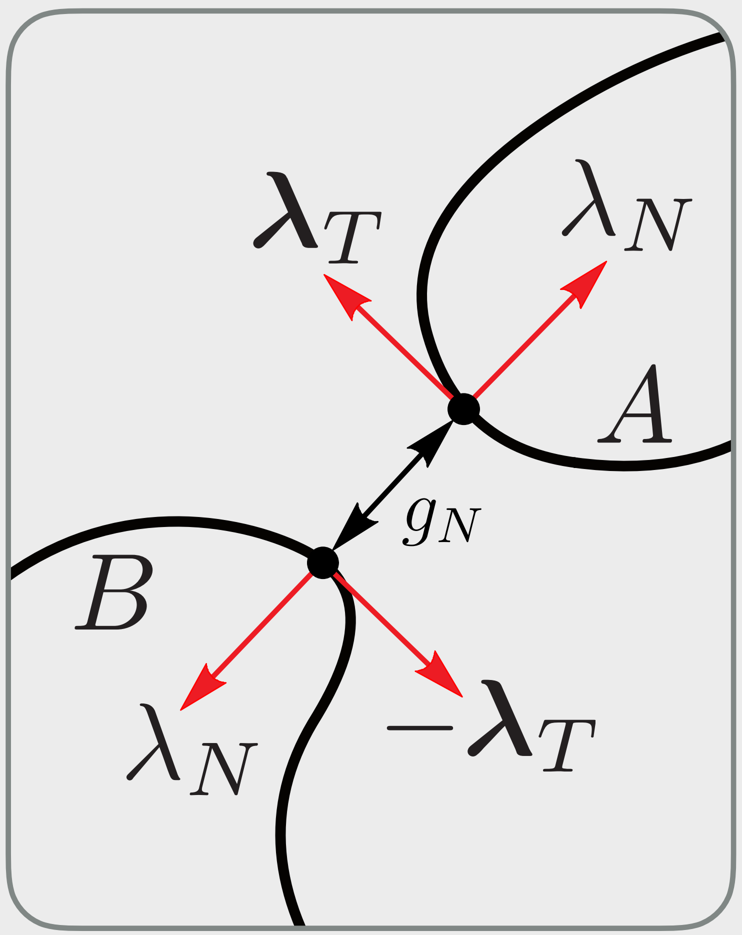

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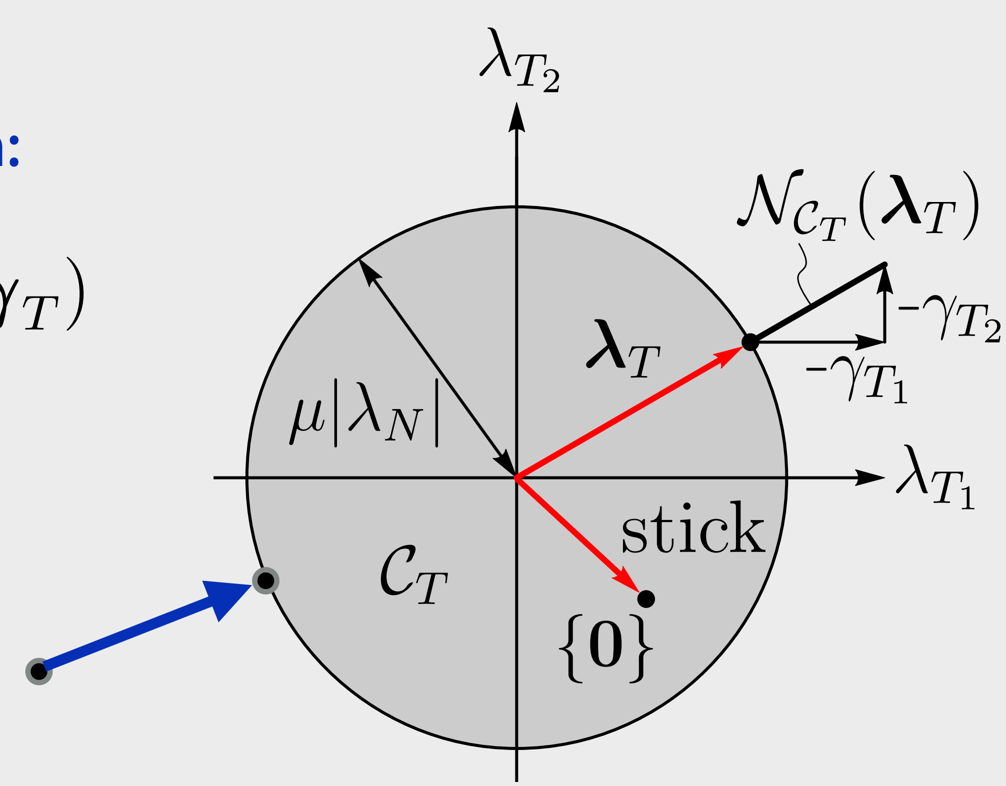

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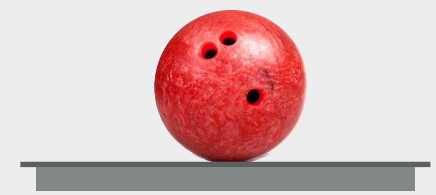

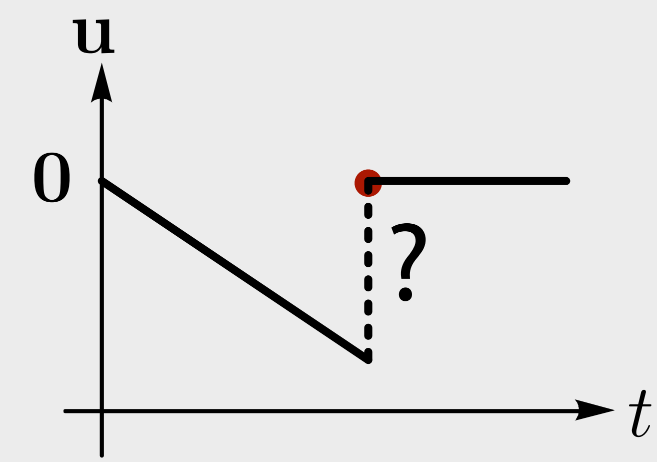

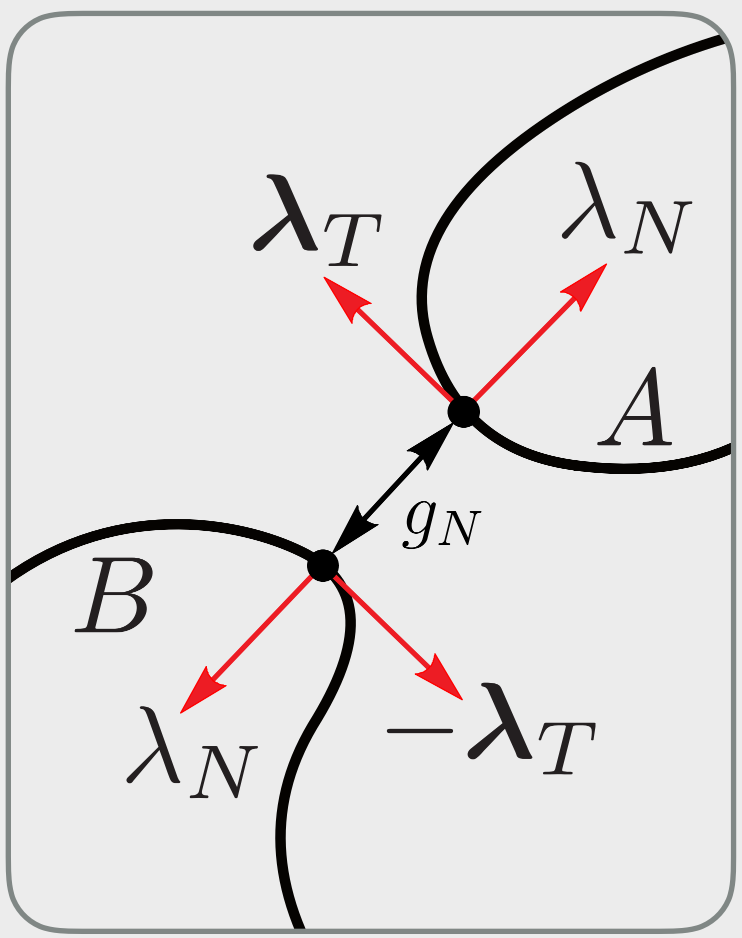

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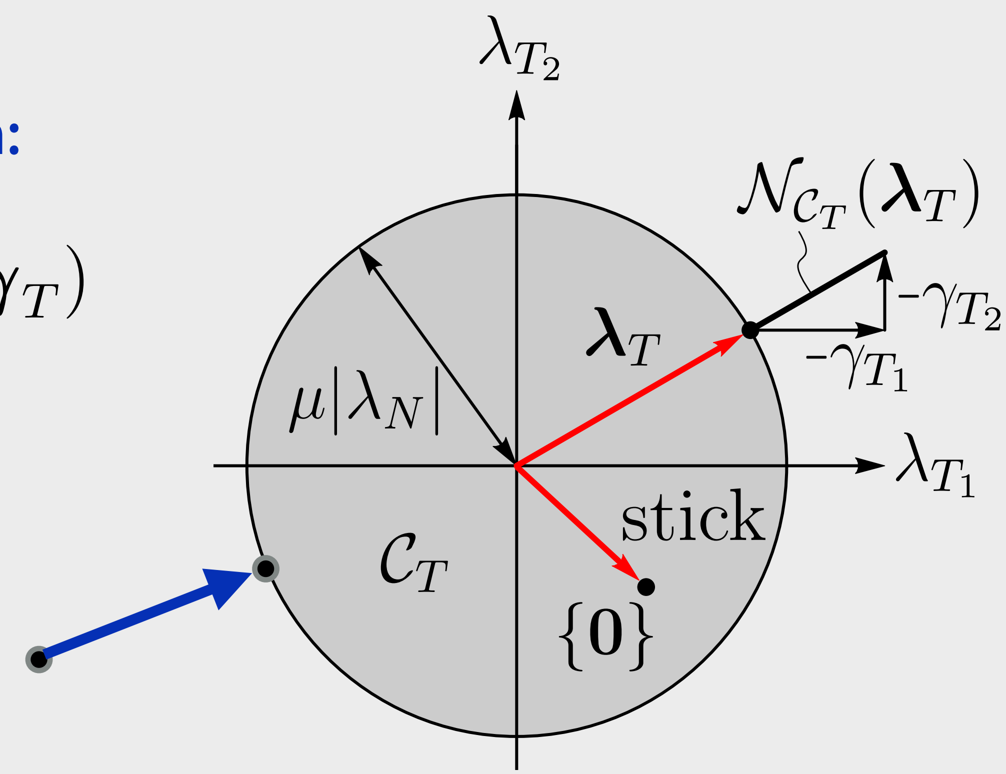

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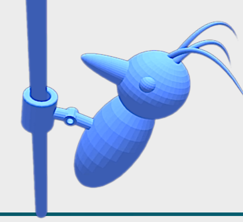

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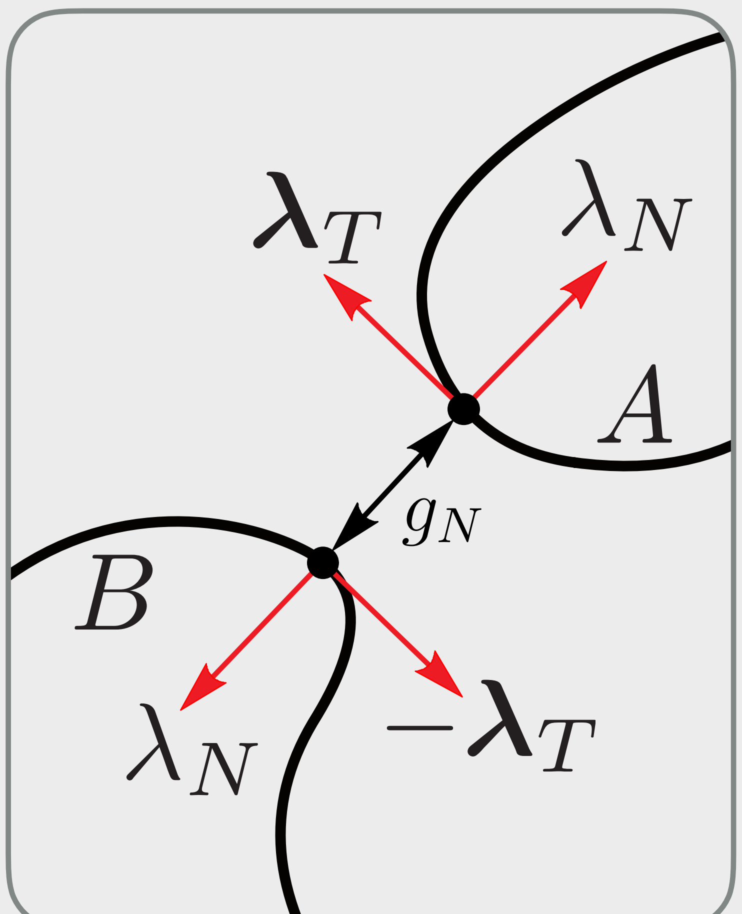
Impact Equation:

$$\mathbf{M}(\mathbf{q},t)\left(\mathbf{u}^{+} - \mathbf{u}^{-}\right) - \mathbf{W}(\mathbf{q},t)\boldsymbol{\Lambda} = \mathbf{0}$$

Newton-type impact law



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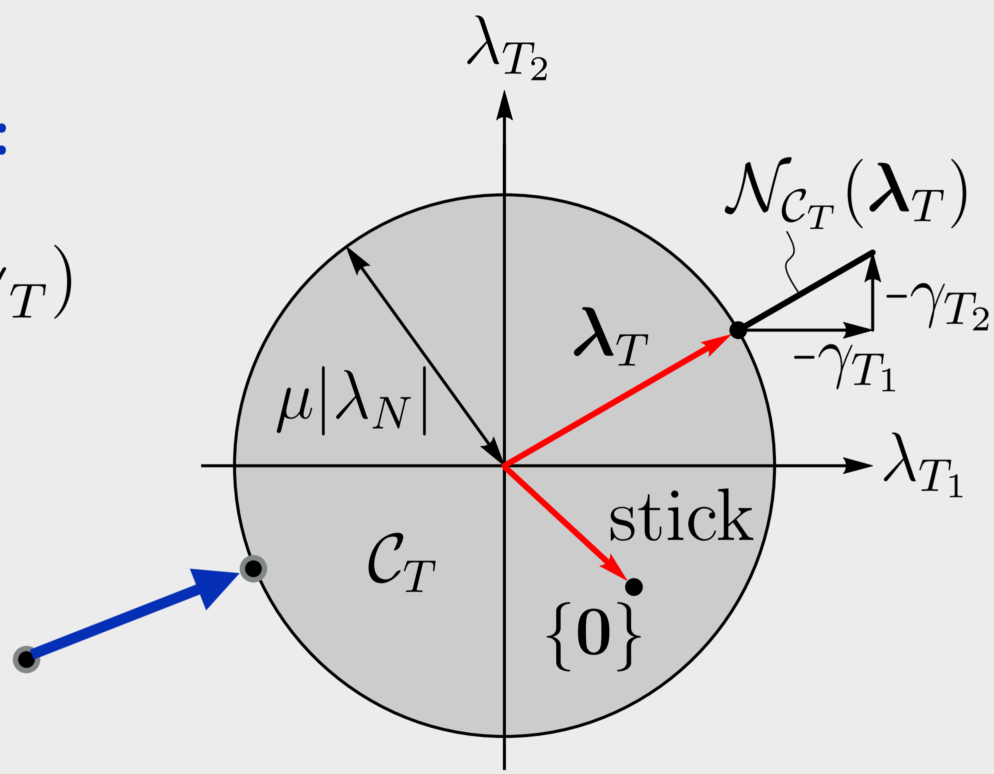
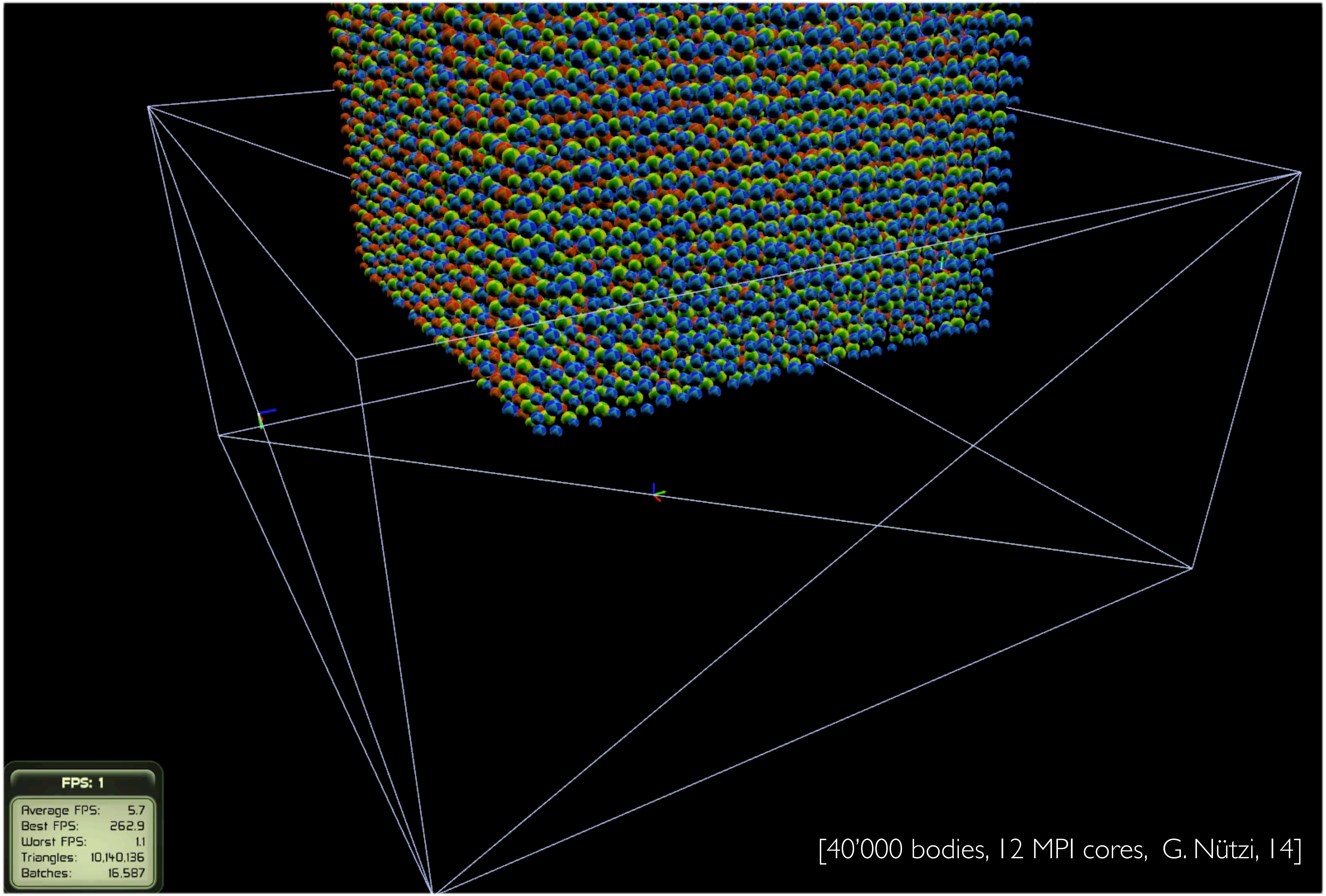


force laws as normal cone inclusions

Non-Smooth Dynamics with Friction and Impacts



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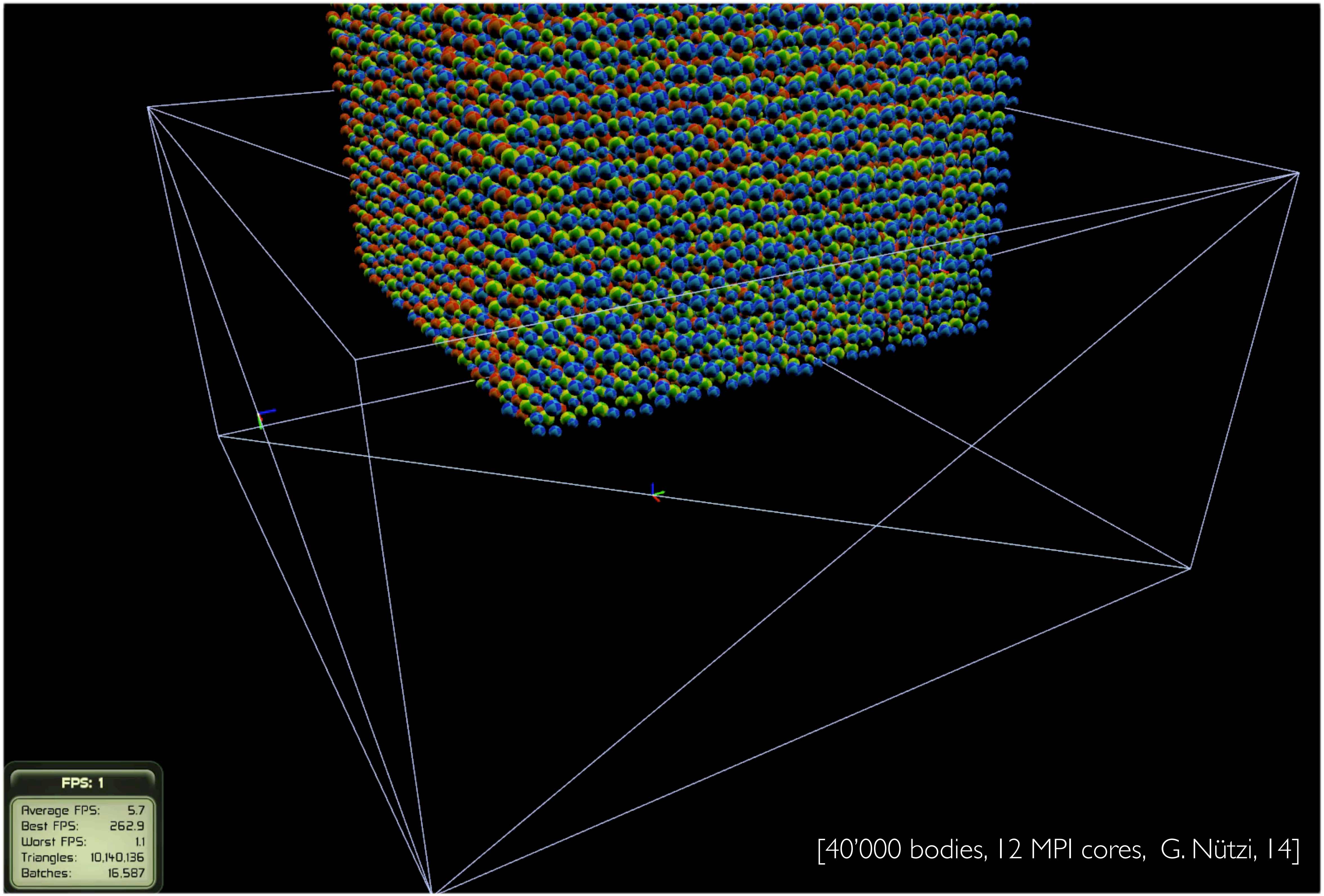

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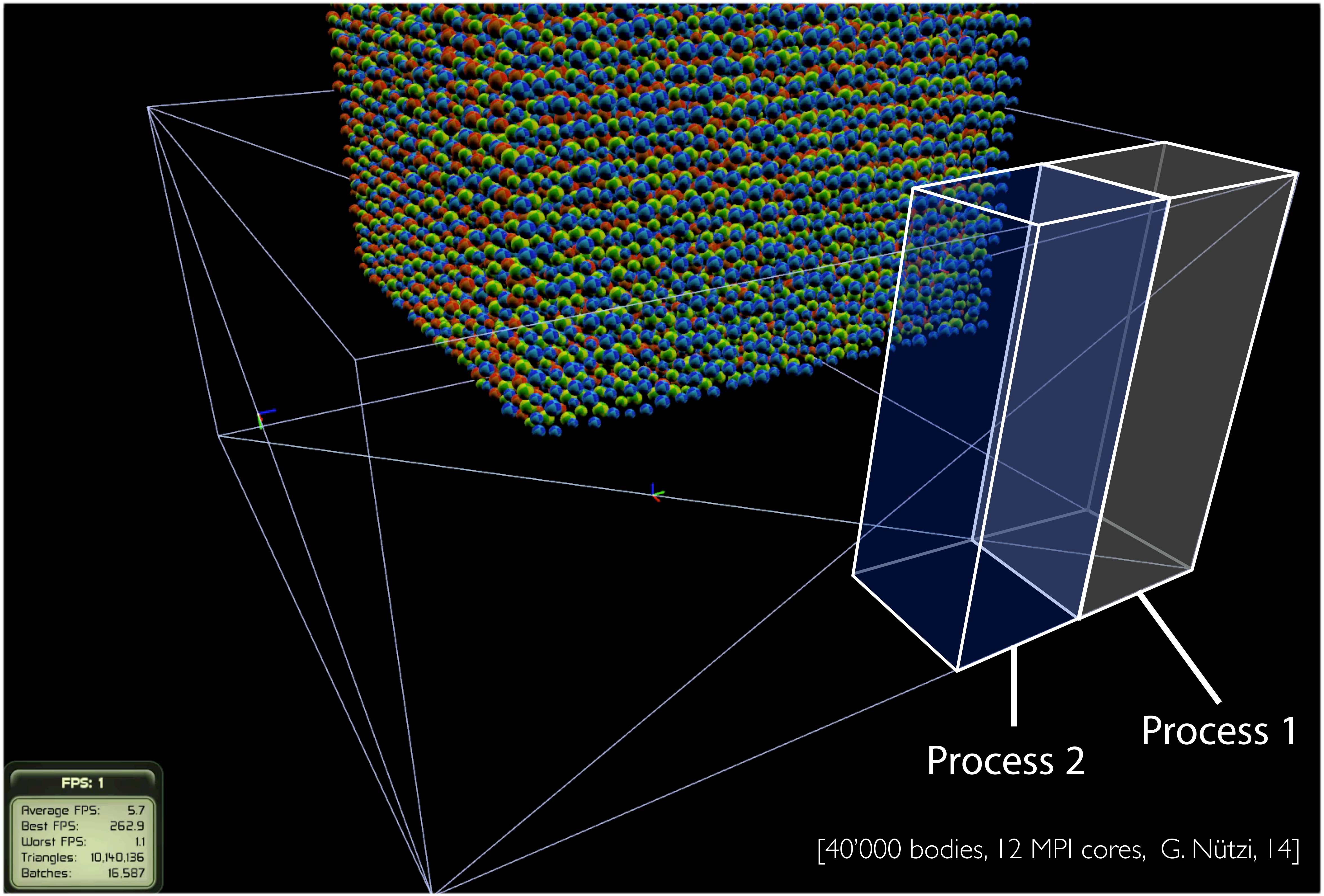


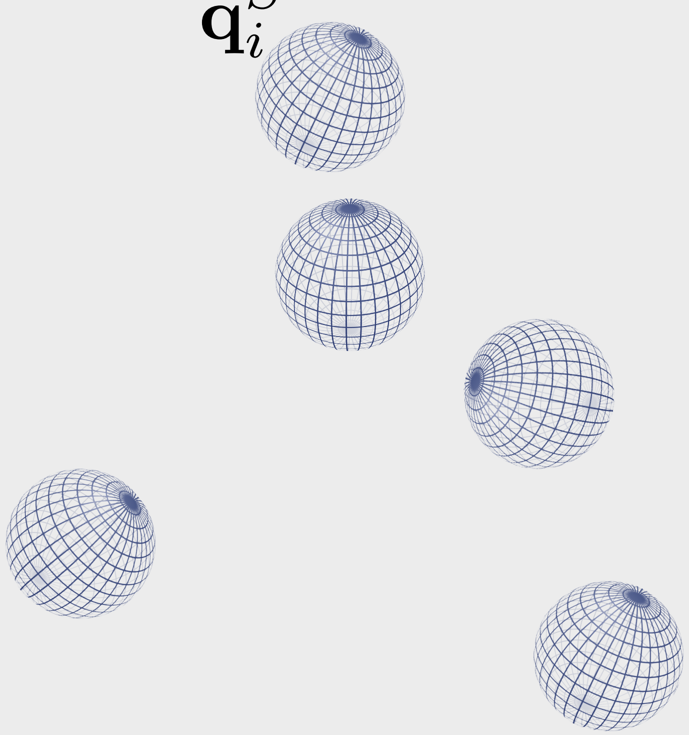
Moreau's Time-Stepping (Jean & Moreau 1986)

$$t_S \xrightarrow{\Delta t/2} t_M \xrightarrow{\Delta t/2} t_E$$

$(\mathbf{q}_i^S, \mathbf{u}_i^S)$ $\quad\quad$ $(\mathbf{q}_i^E, \mathbf{u}_i^E)$

Timestep $\Delta t/2$ → Collision Detection → Solve Contact Problem → Timestep $\Delta t/2$

$\mathbf{q}_i^S$



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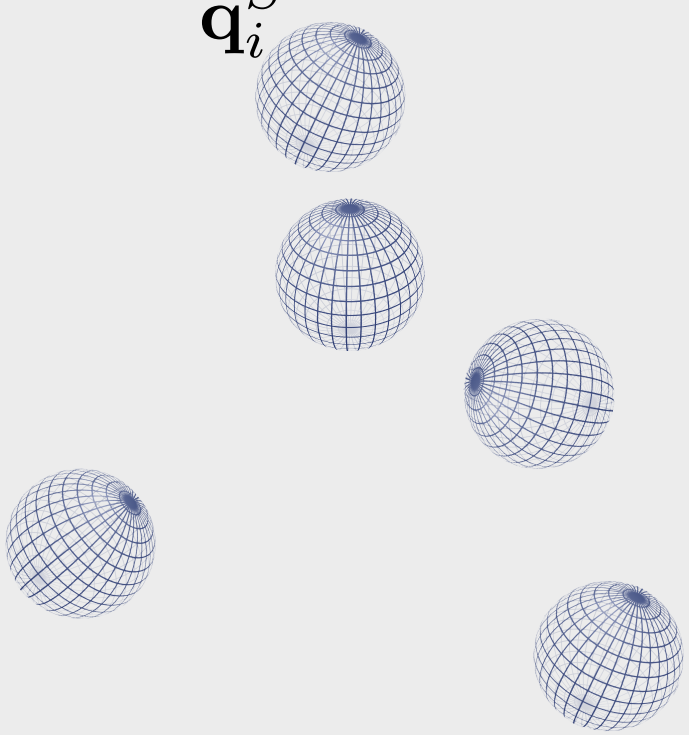
Timestep $\Delta t/2$ → Collision Detection → Solve Contact Problem → Timestep $\Delta t/2$

update to midpoint

$$\mathbf{q}_i^S \rightarrow \mathbf{q}_i^M$$

$\mathbf{q}_i^S$



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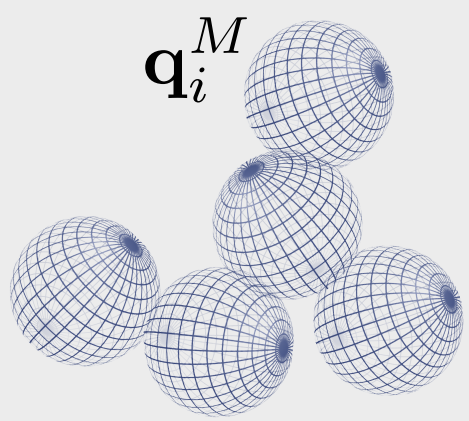

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find all contacts

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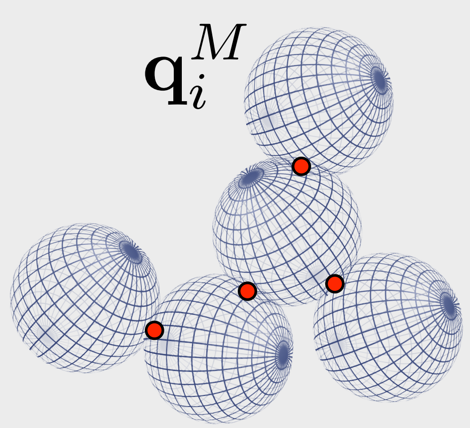

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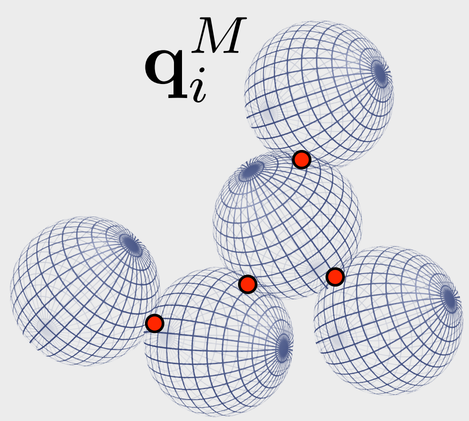

Collision Detection

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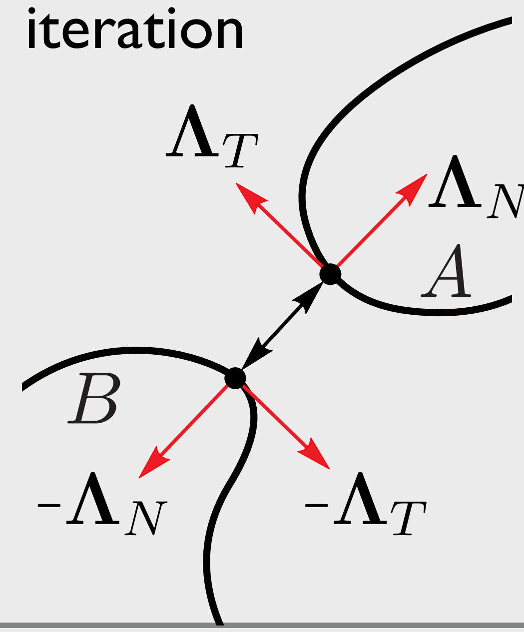
Solve Contact Problem

find percussions and velocities: $\mathbf{u}_i^E, \mathbf{\Lambda}$

contact graph iteration:

- proj. Jacobi iteration
- proj. Gauss-Seidel iteration



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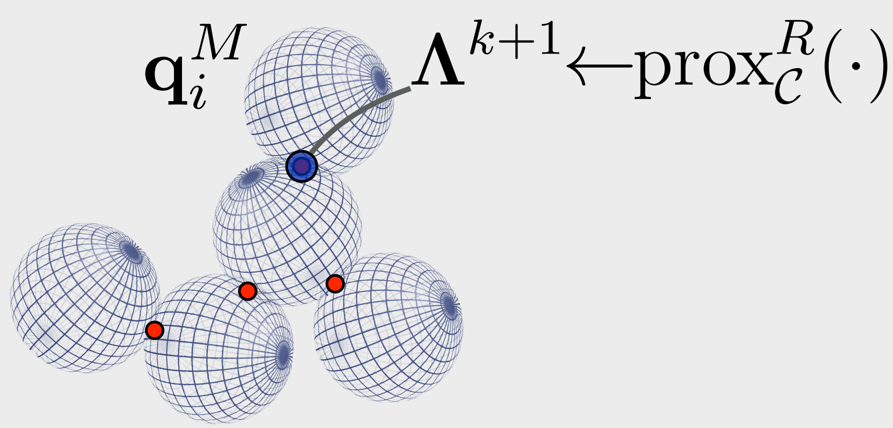

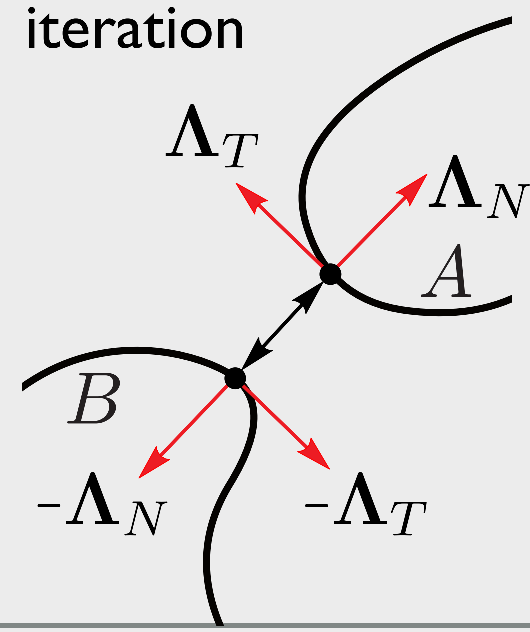

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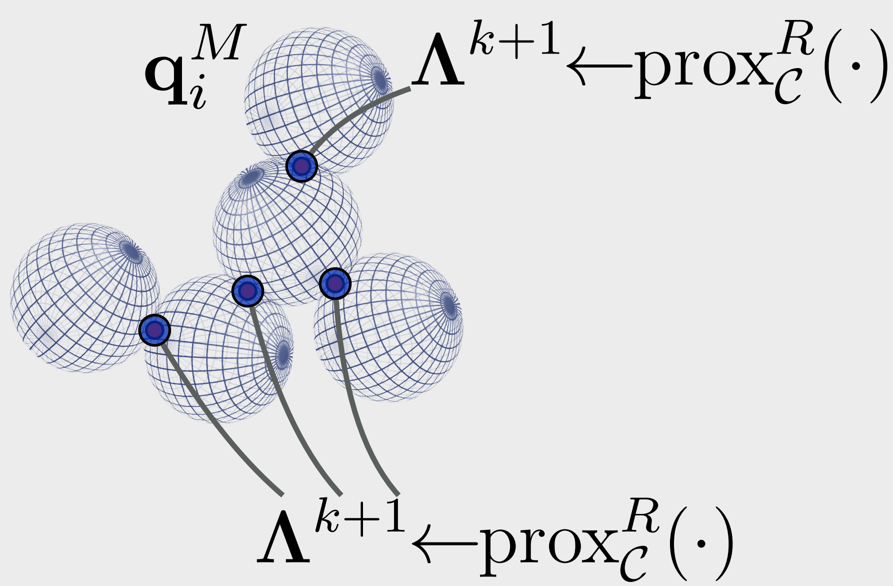

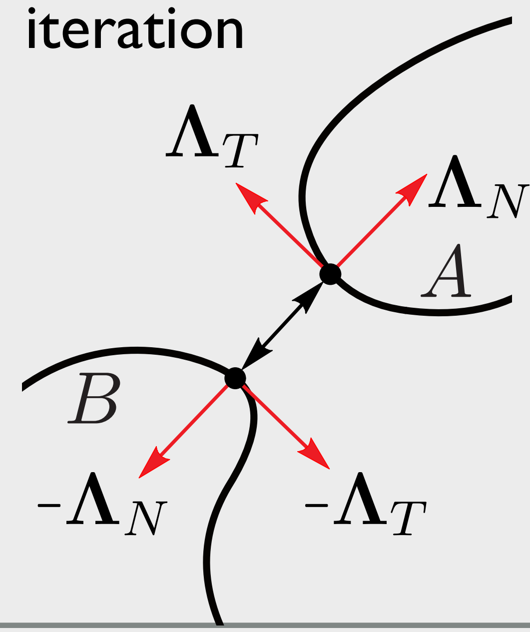

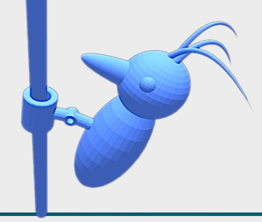

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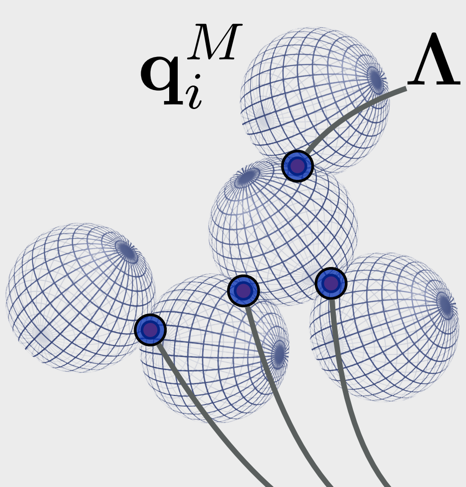

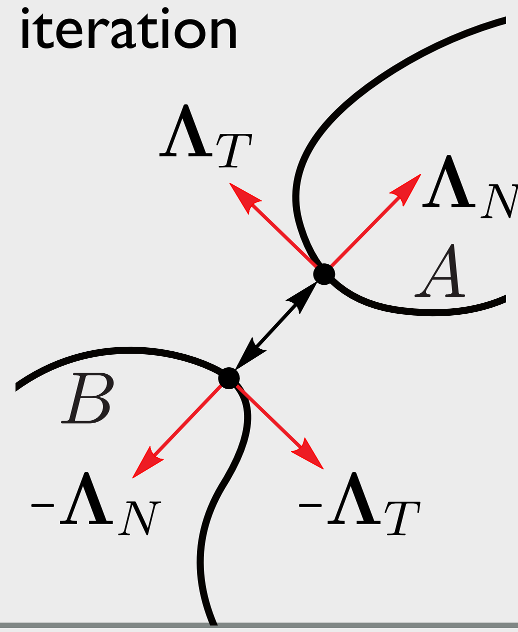

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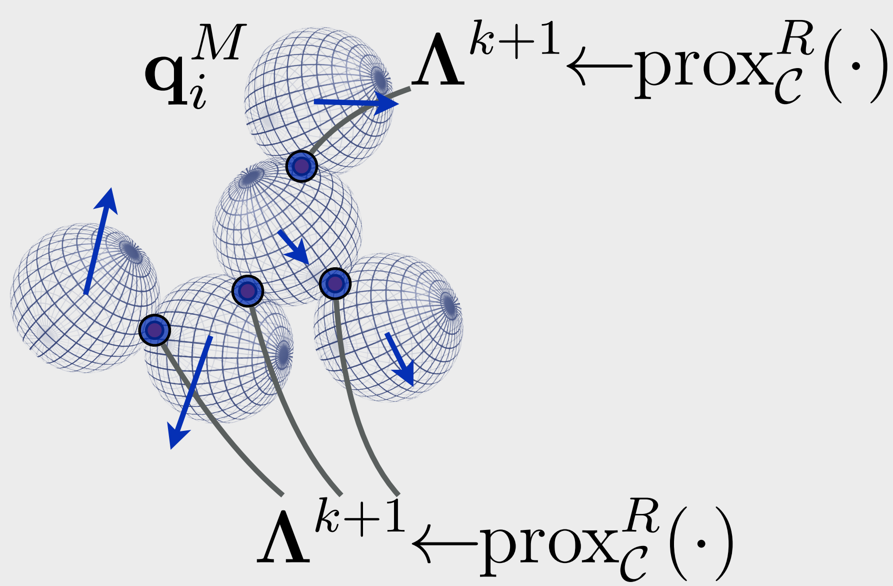

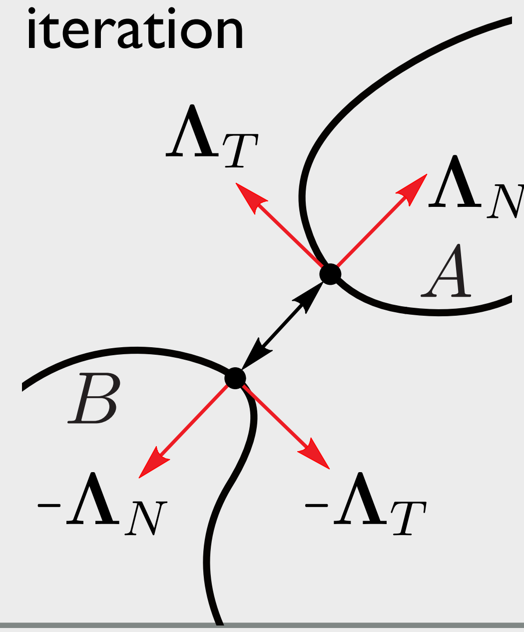

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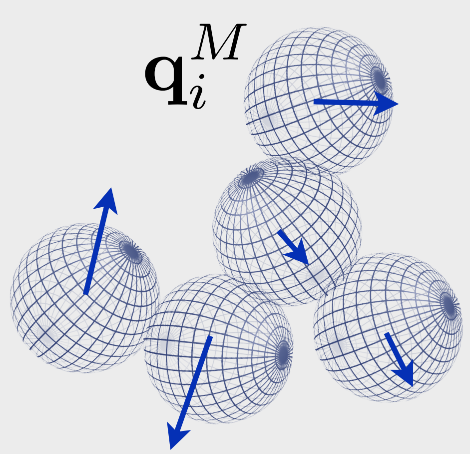

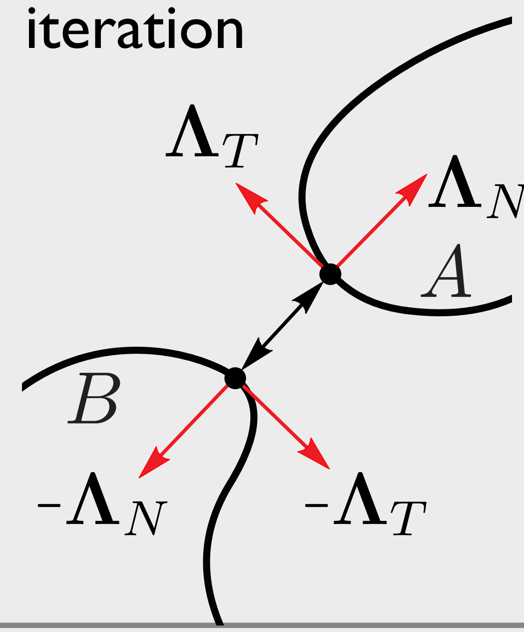

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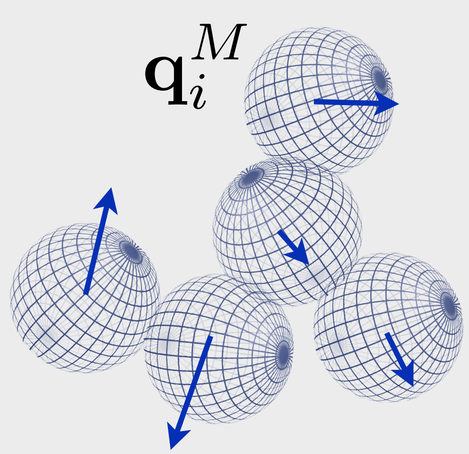

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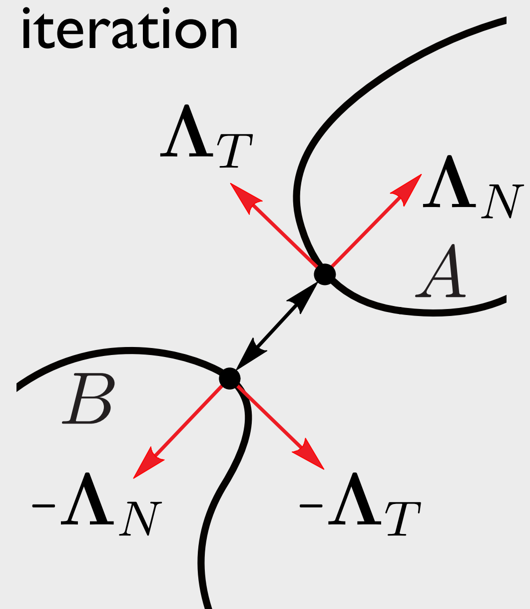

Timestep $\Delta t/2$

update to end point
$(\mathbf{\Lambda}, \mathbf{u}_i^E, \mathbf{q}_i^M) \rightarrow \mathbf{q}_i^E$

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Timestep $\Delta t/2$ → Collision Detection → Solve Contact Problem → Timestep $\Delta t/2$

Timestep $\Delta t/2$

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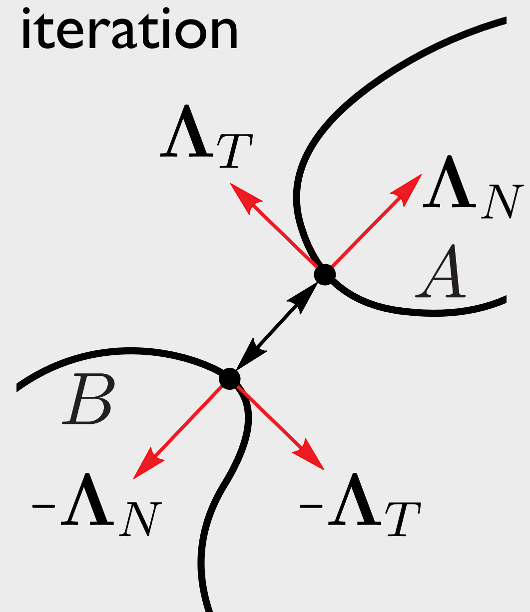
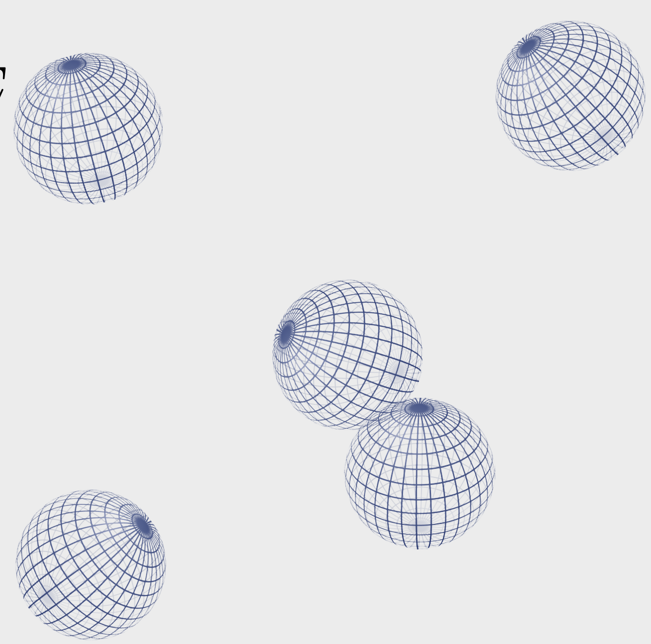


Timestep $\Delta t/2$

update to end point
$(\mathbf{\Lambda}, \mathbf{u}_i^E, \mathbf{q}_i^M) \rightarrow \mathbf{q}_i^E$

$\mathbf{q}_i^E$



Moreau's Time-Stepping (Jean & Moreau 1986)

$$t_S \xrightarrow{\Delta t/2} t_M \xrightarrow{\Delta t/2} t_E$$

$(\mathbf{q}_i^S, \mathbf{u}_i^S)$ — body communication — $(\mathbf{q}_i^E, \mathbf{u}_i^E)$

Timestep $\Delta t/2$ → **Collision Detection** → **Solve Contact Problem** → **Timestep $\Delta t/2$**

update to midpoint
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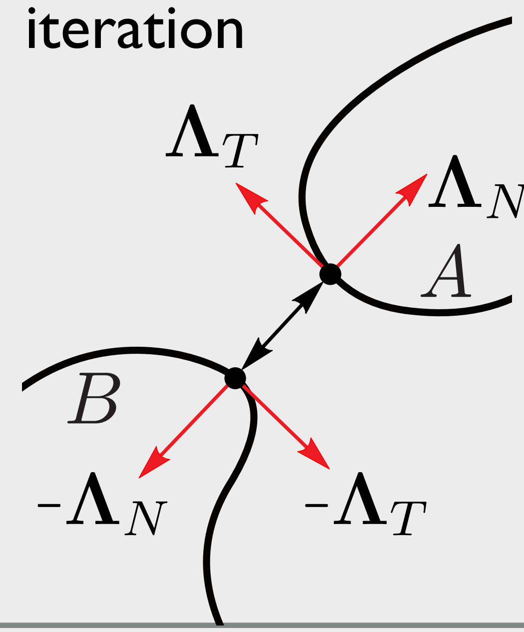
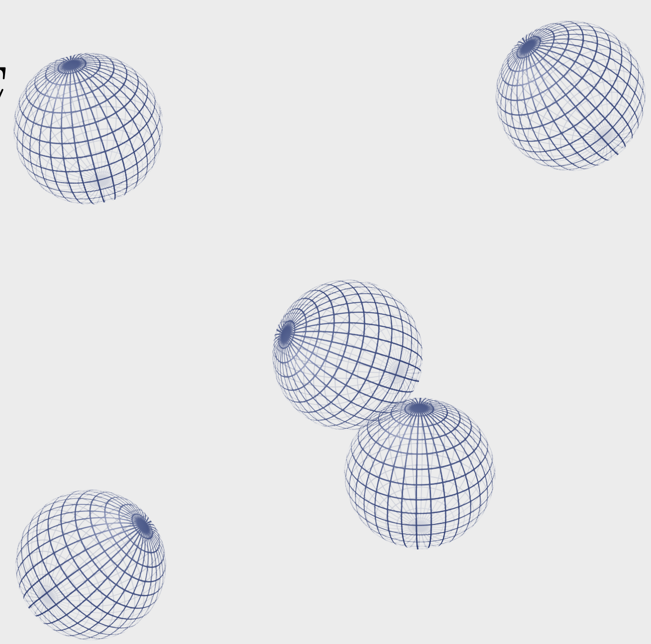


communication

update to end point
$(\mathbf{\Lambda}, \mathbf{u}_i^E, \mathbf{q}_i^M) \rightarrow \mathbf{q}_i^E$

$\mathbf{q}_i^E$



Moreau's Time-Stepping (Jean & Moreau 1986)

t_S —— $\Delta t/2$ ——→ t_M —— $\Delta t/2$ ——→ t_E

$(\mathbf{q}_i^S, \mathbf{u}_i^S)$

body communication

Timestep $\Delta t/2$ → **Collision Detection** → **Solve Contact Problem** → **Timestep $\Delta t/2$**

$(\mathbf{q}_i^E, \mathbf{u}_i^E)$

update to midpoint
$\mathbf{q}_i^S \rightarrow \mathbf{q}_i^M$

find all contacts

collision routines:
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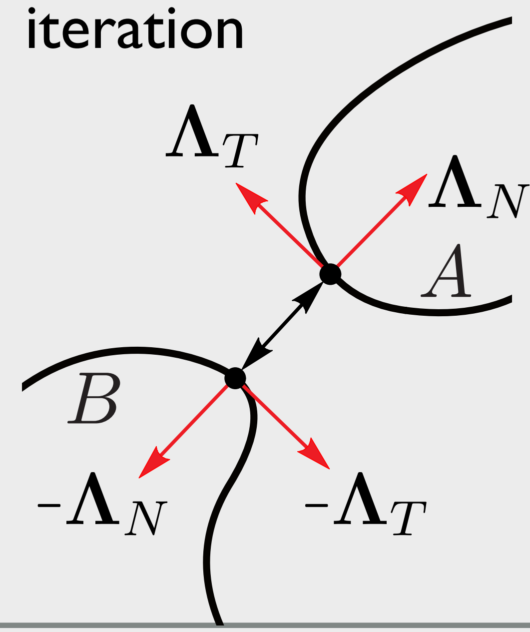
- Mesh-Sphere
- etc...

find percussions and velocities: $\mathbf{u}_i^E, \mathbf{\Lambda}$

contact graph iteration:
- proj. Jacobi iteration
- proj. Gauss-Seidel iteration

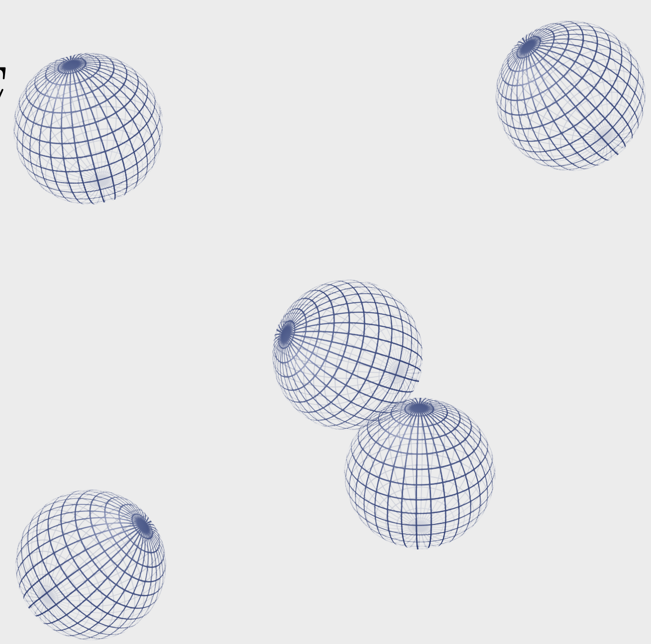


communication

update to end point
$(\mathbf{\Lambda}, \mathbf{u}_i^E, \mathbf{q}_i^M) \rightarrow \mathbf{q}_i^E$

rebuild domain decomposition

$\mathbf{q}_i^E$

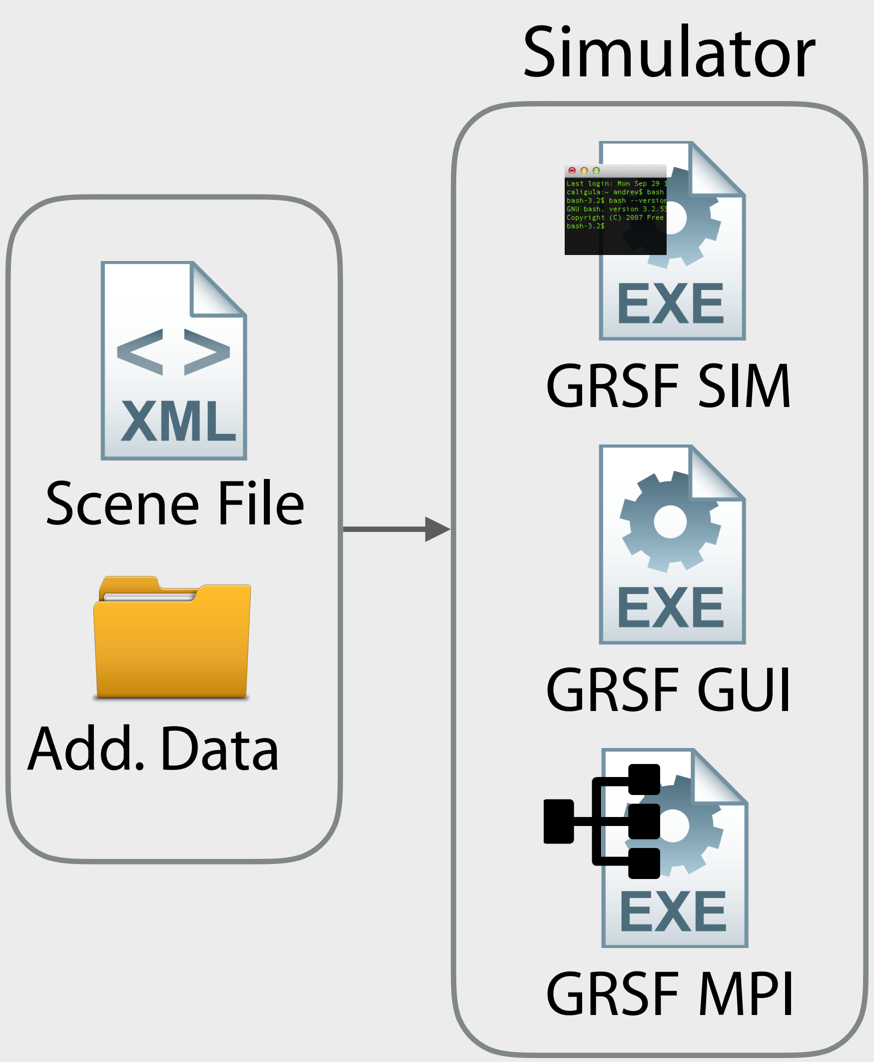


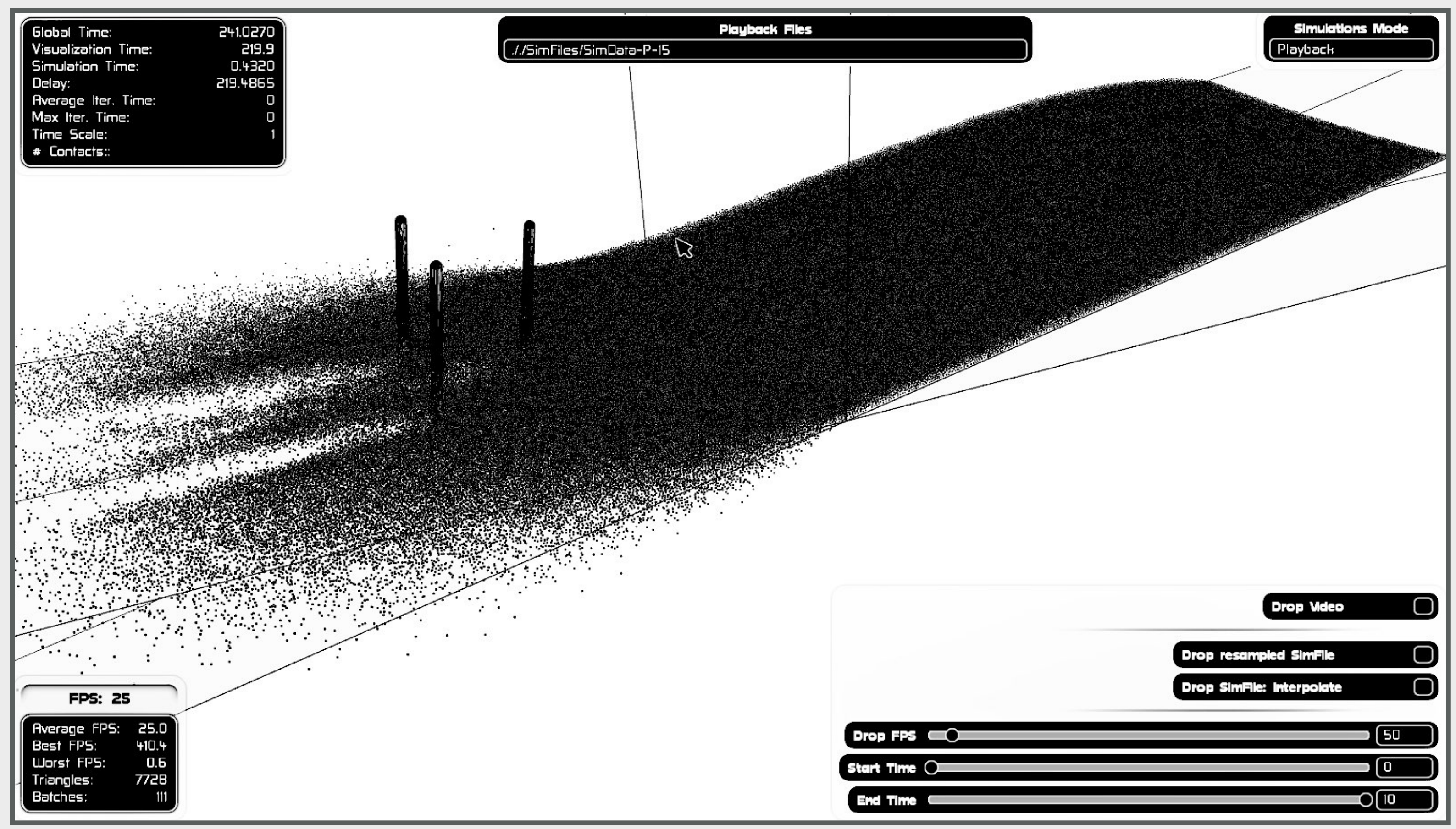
The Granular Rigid Body Simulation Framework

Language: C++11/14 will appear on **github.com**

Features: Large-Scale Non-Smooth Rigid Body Simulations

Dependencies: Eigen, boost, OpenMPI, HDF5, pugixml, Ogre, ApproxMVBB





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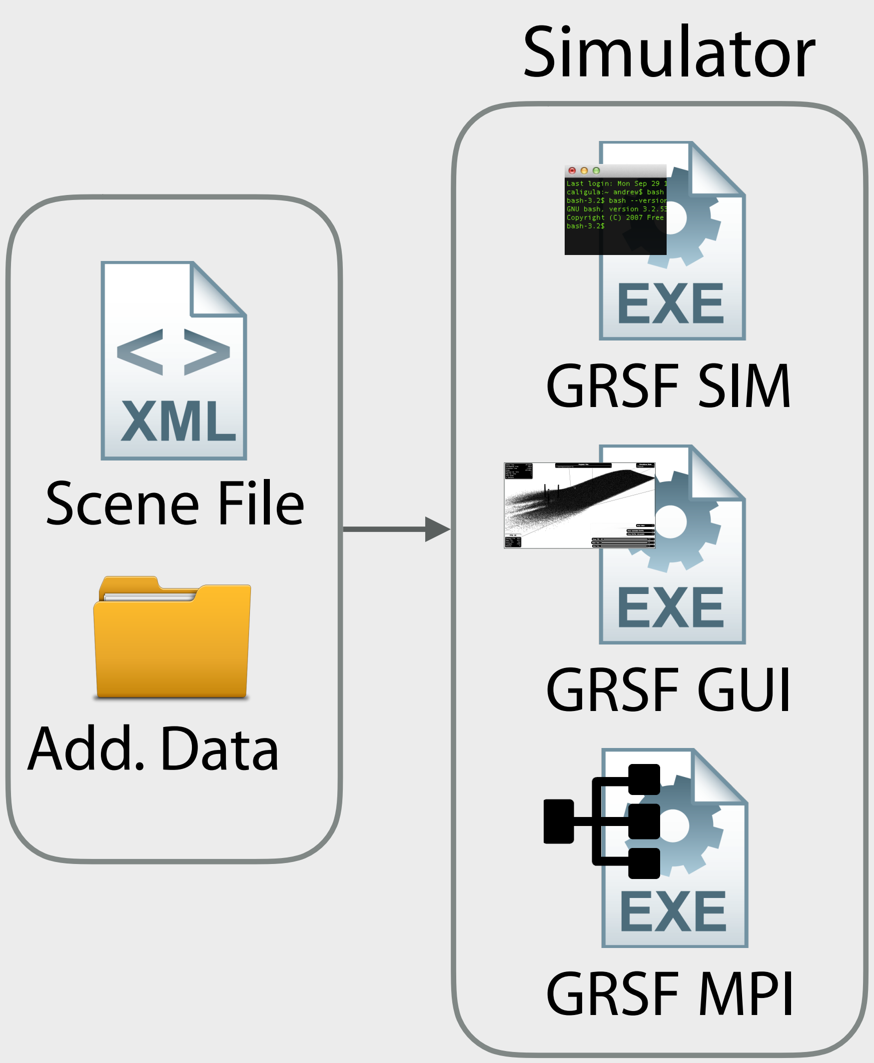

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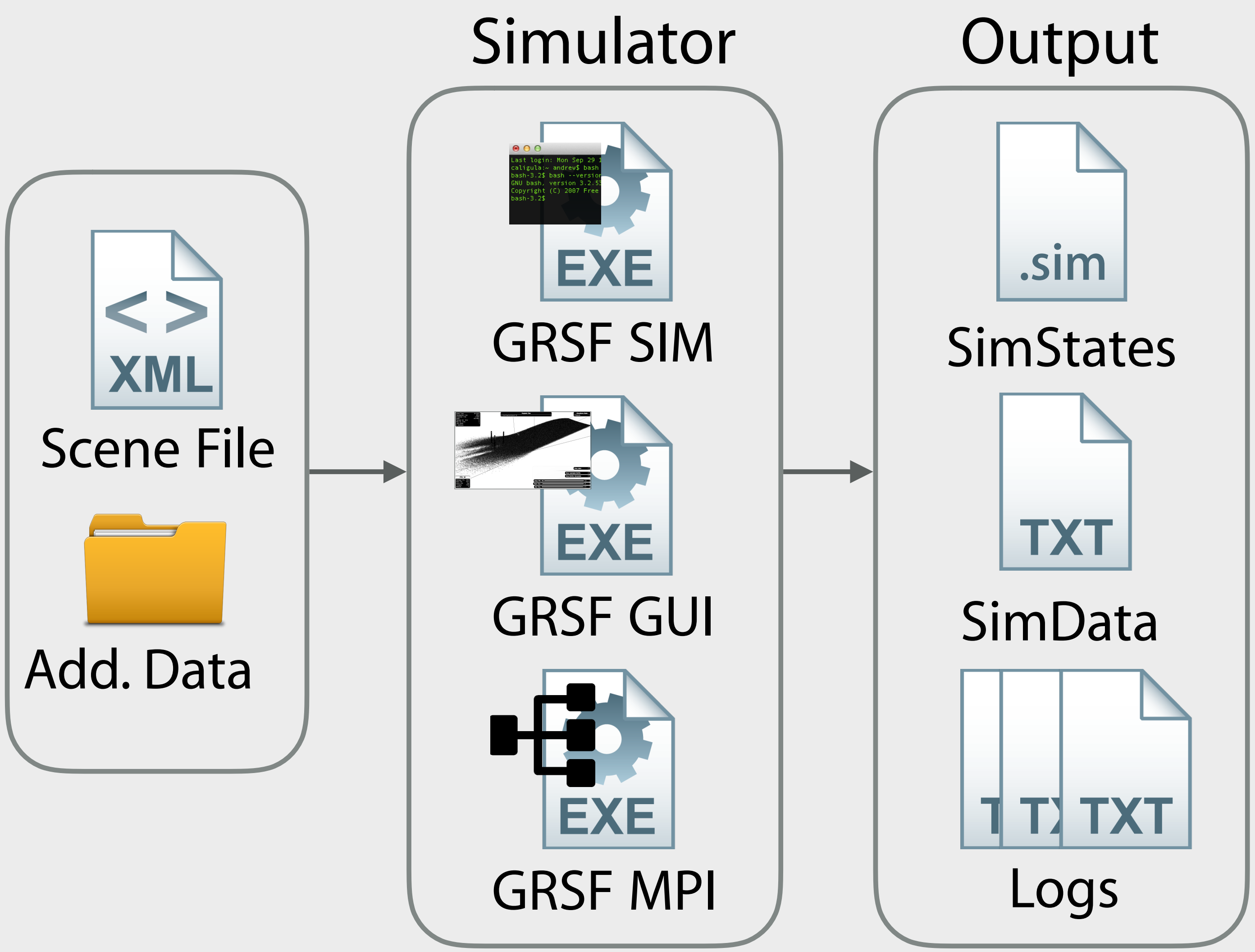

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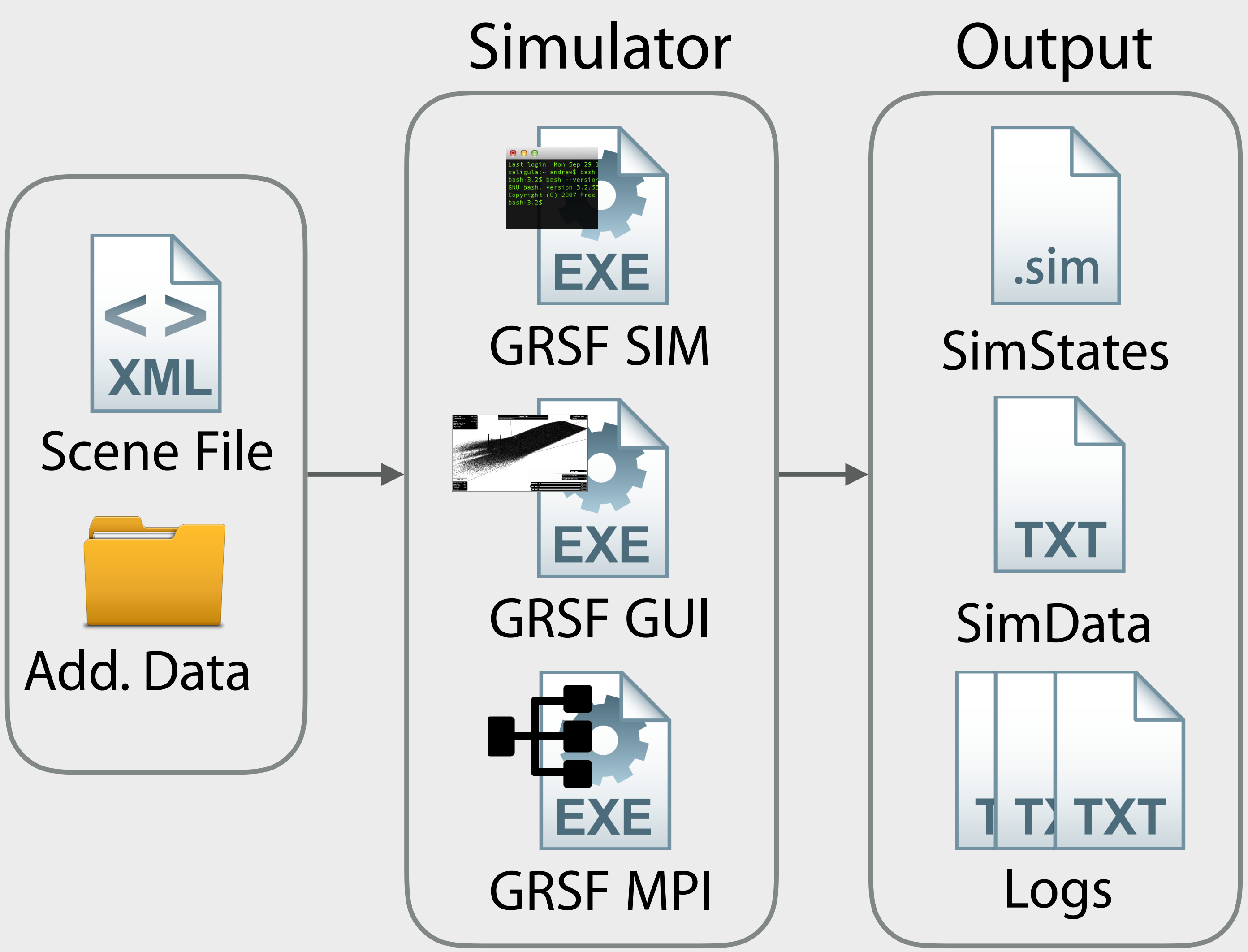

- position/velocity for each body
- add. data per body

$$\mathbf{q}(t) := \begin{bmatrix} {}_{\mathrm{I}}\mathbf{r}_R \\ \mathbf{p} \end{bmatrix} \quad \mathbf{u}(t) := \begin{bmatrix} {}_{\mathrm{I}}\dot{\mathbf{r}}_R \\ {}_{\mathrm{K}}\boldsymbol{\Omega} \end{bmatrix}$$

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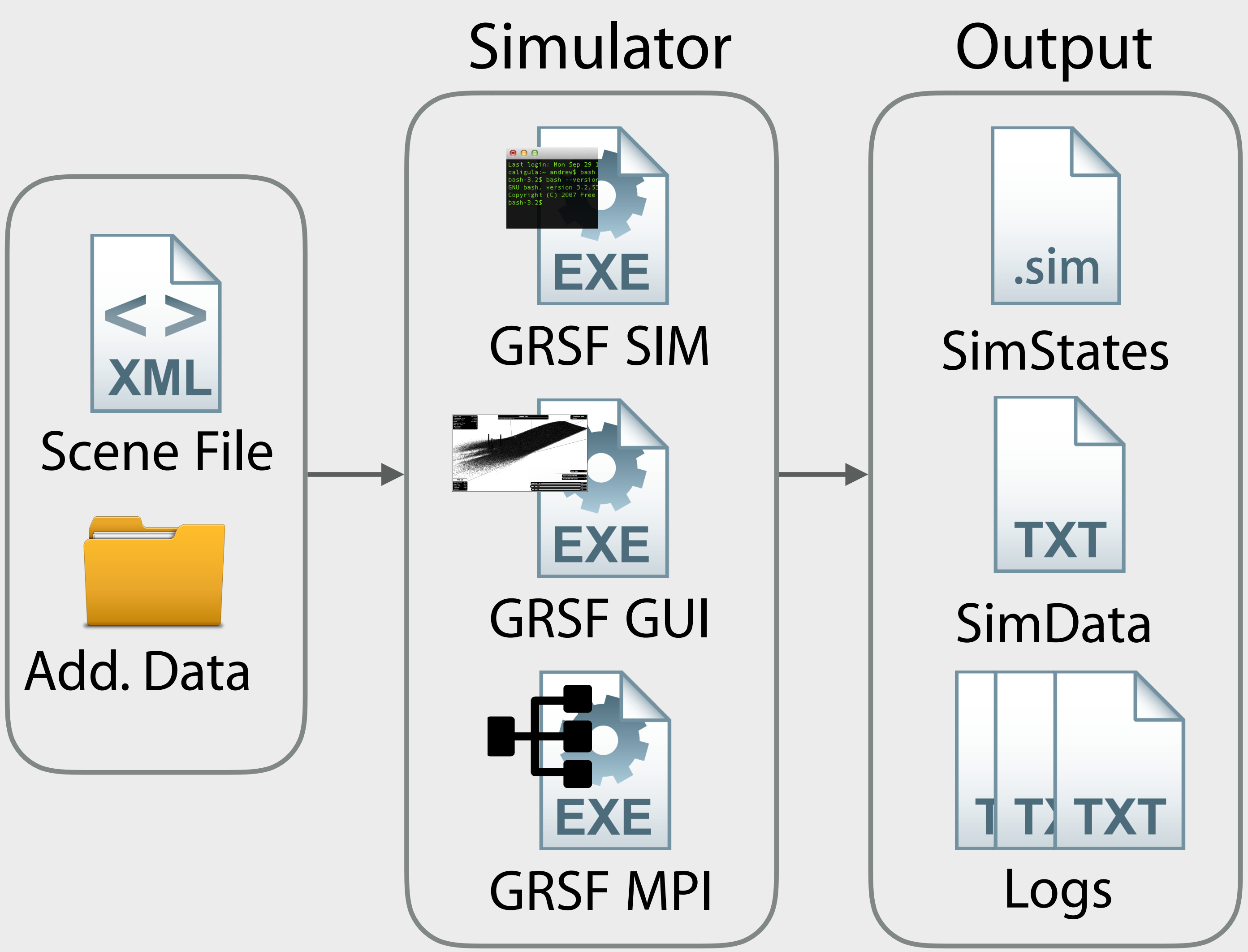

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- profiling information
- solver stats, etc.

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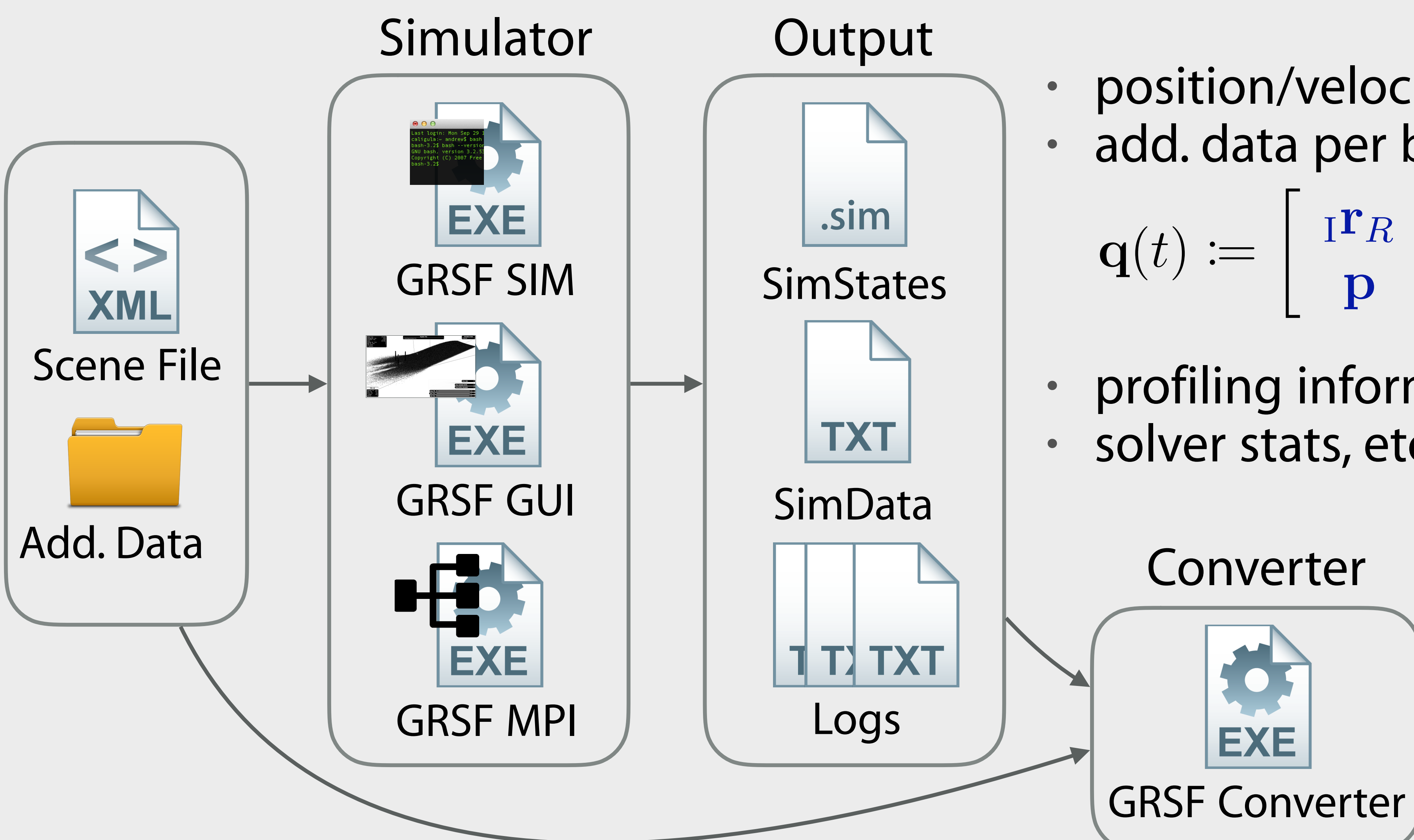

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- profiling information
- solver stats, etc.

- sim file utilities
- exec. graph based analysis
- gridder

Body Communication among Processes

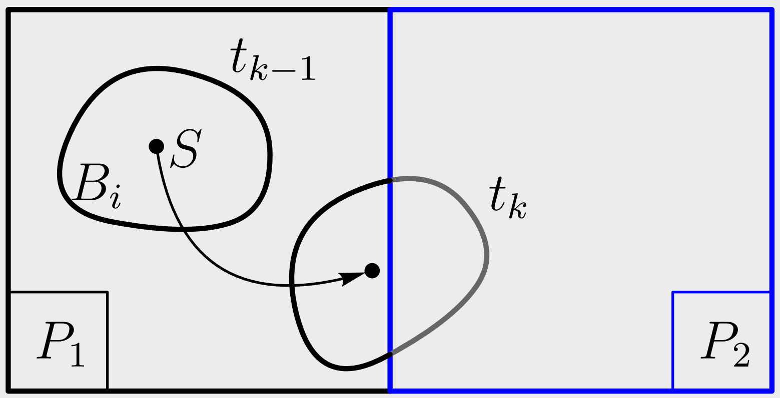
Notification

Update

Removal

Body Communication among Processes

Notification



Update

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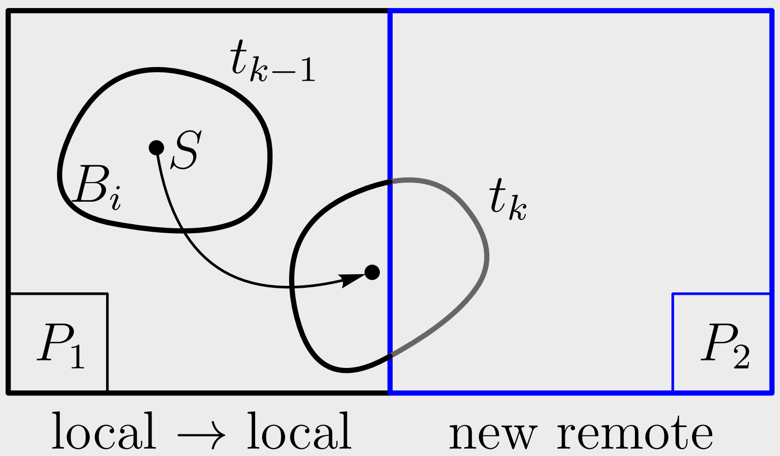


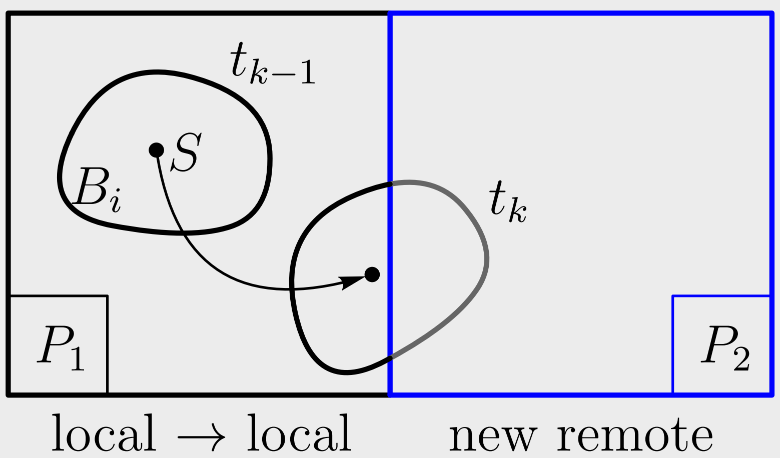
+ 2 more case

Update

Removal

Body Communication among Processes

Notification



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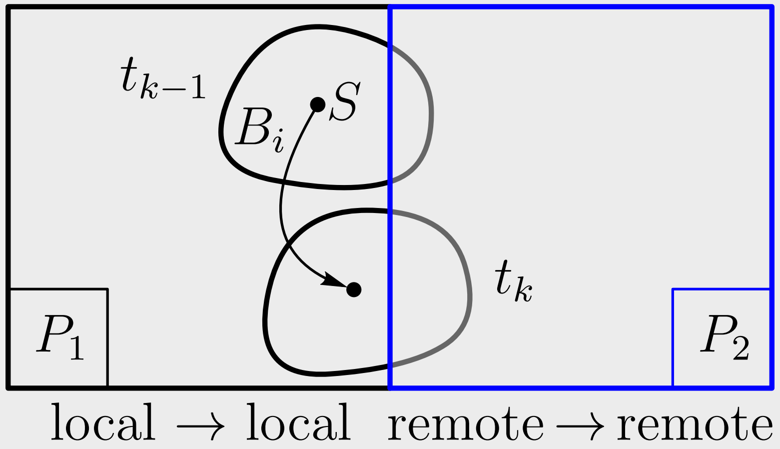
Update



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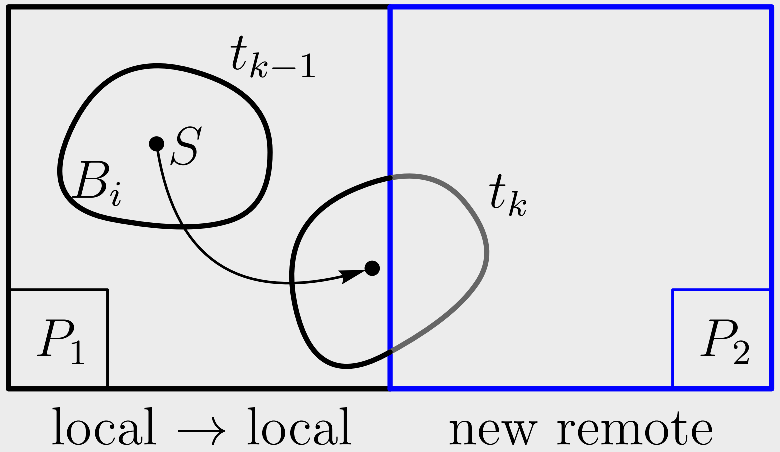

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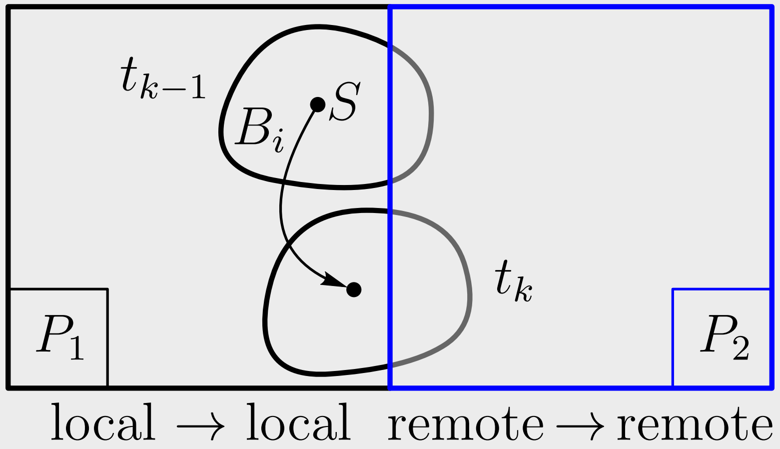

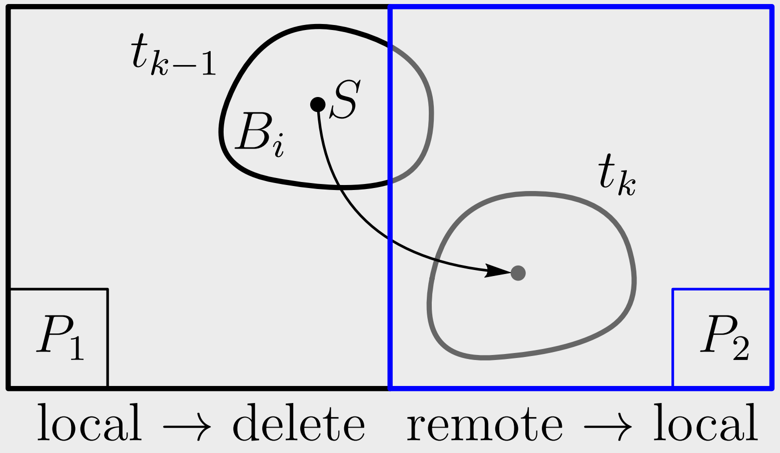

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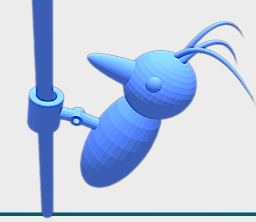

Body Communication among Processes

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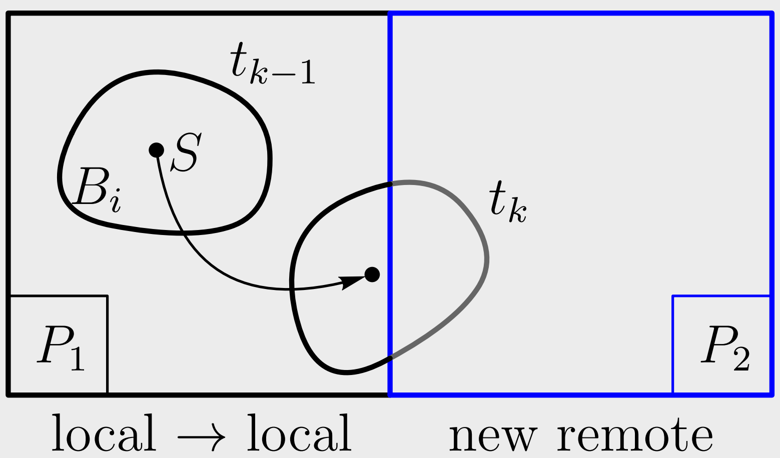
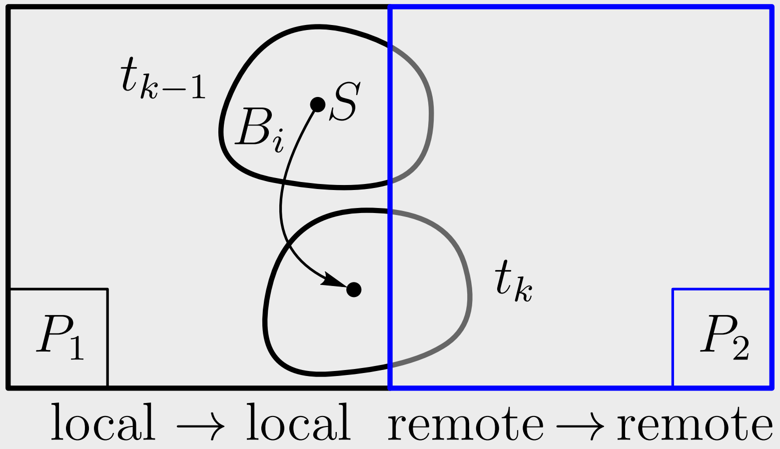
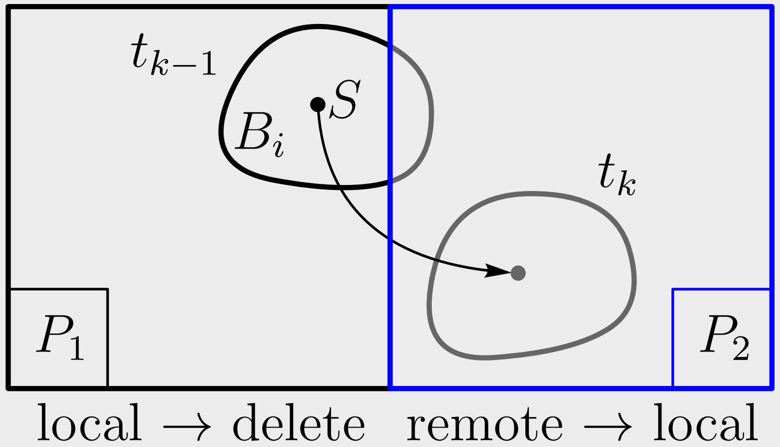


+ 2 more case

Update



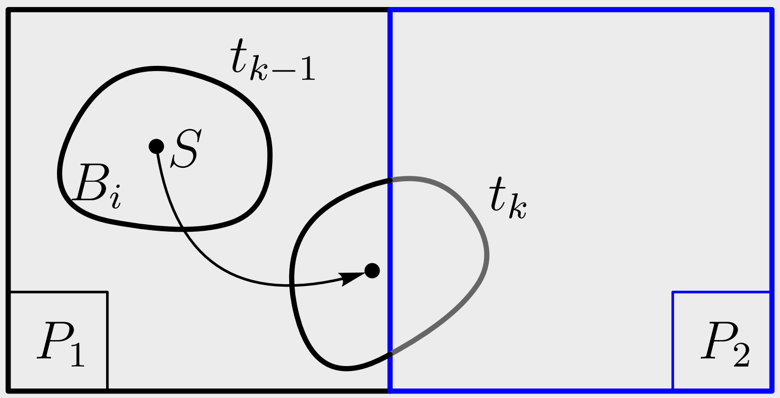


+ 1 more case

Removal

Body Communication among Processes

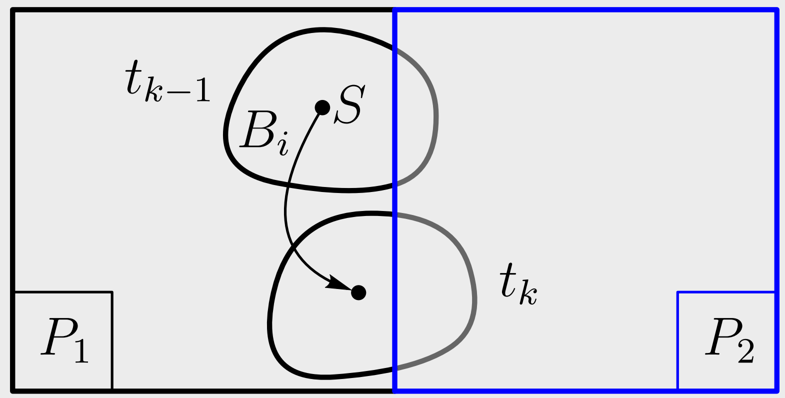
Notification



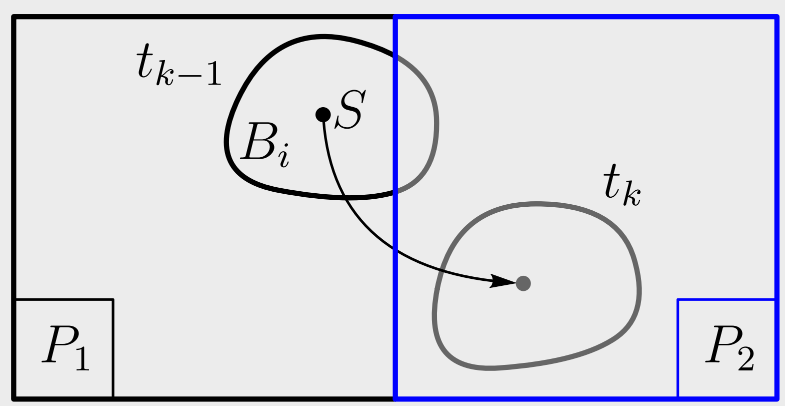
local $\rightarrow$ local new remote

+ 2 more case

Update



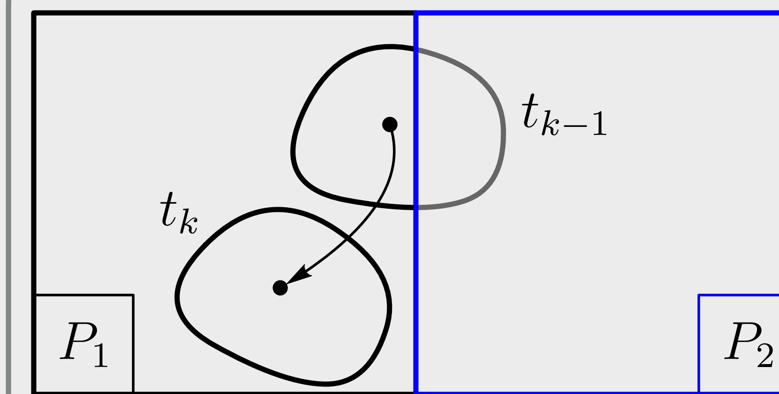
local $\rightarrow$ local remote $\rightarrow$ remote



local $\rightarrow$ delete remote $\rightarrow$ local

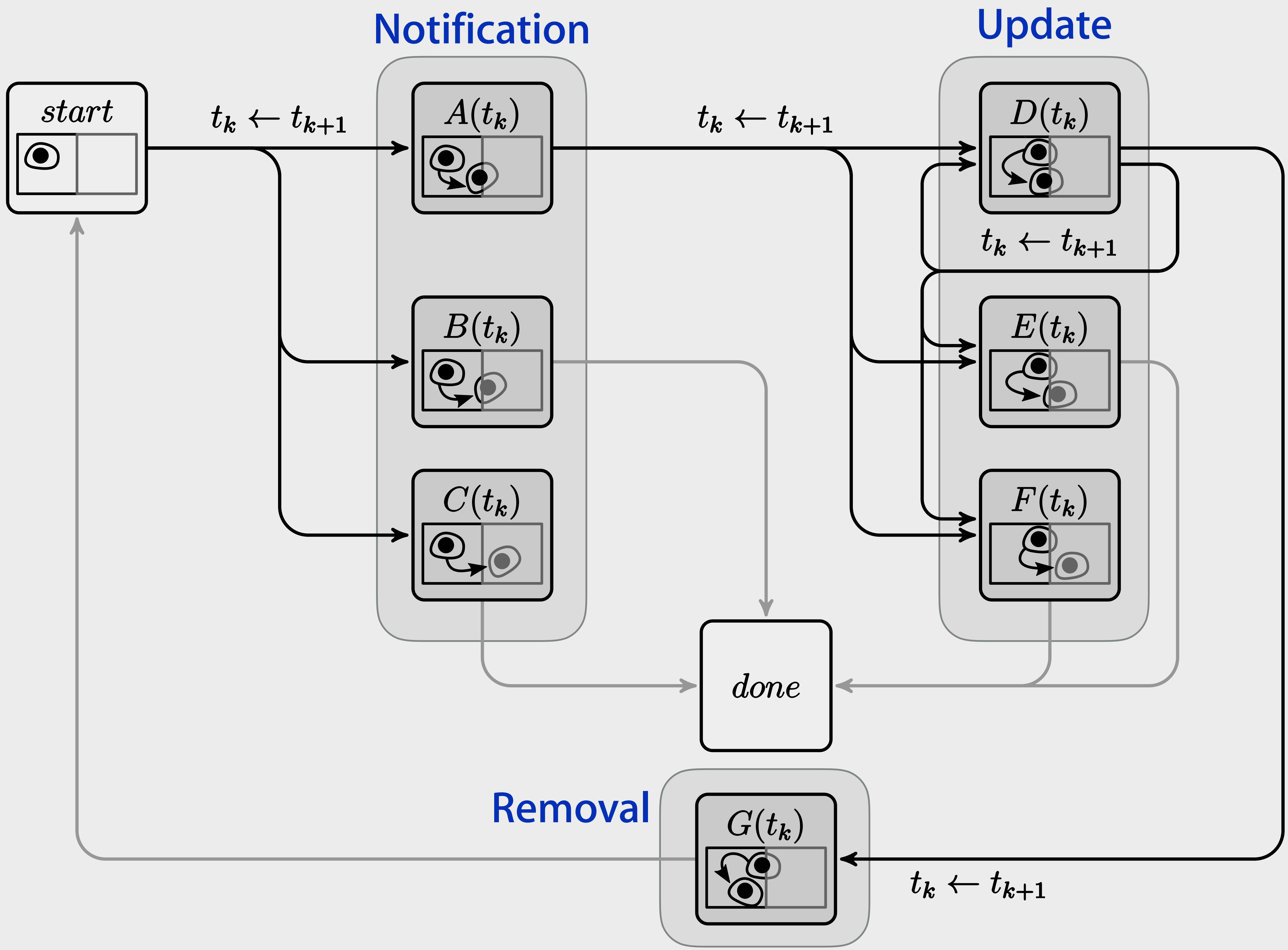
+ 1 more case

Removal



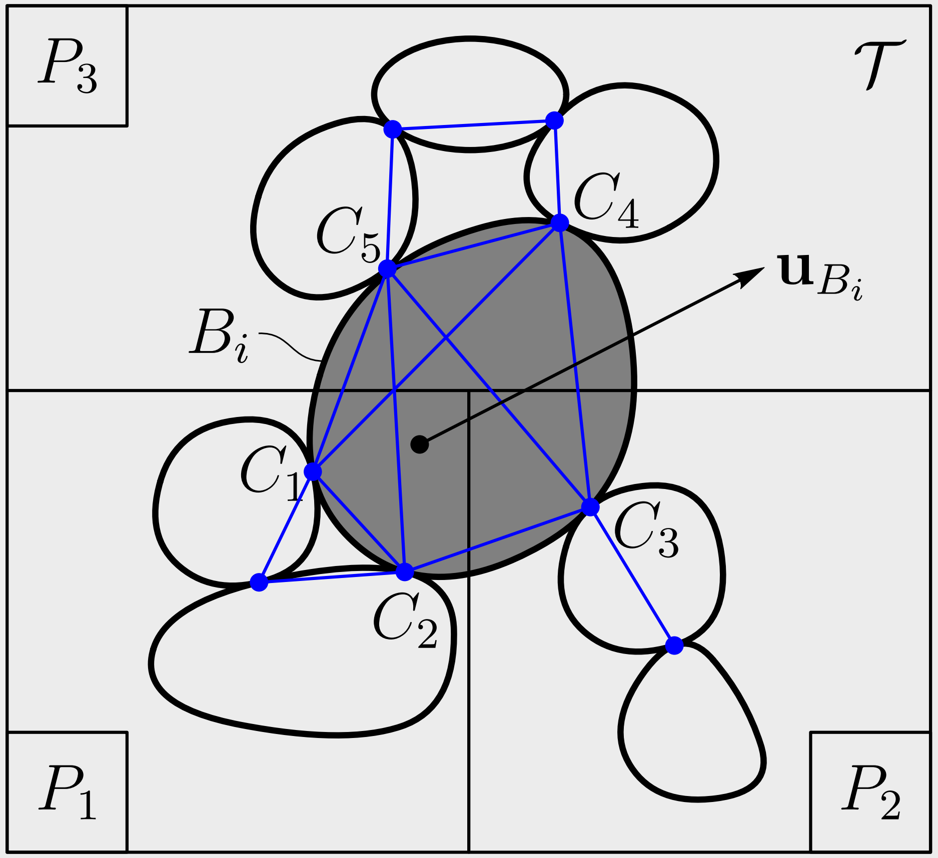
local $\rightarrow$ local remote $\rightarrow$ delete

Body Communication: State Machine



Solving the Discretized Contact Problem

The mass-splitting method



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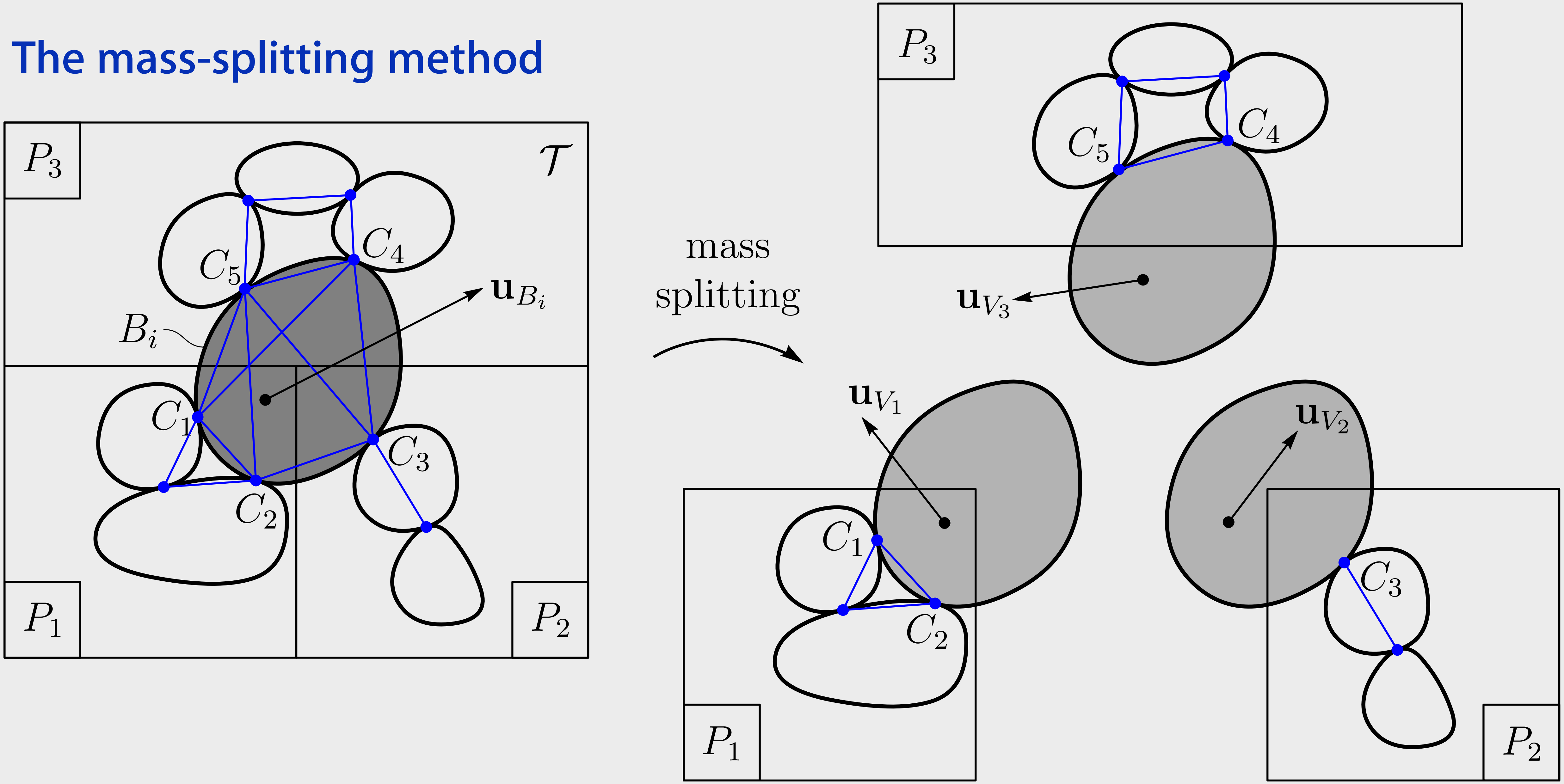

$$\mathbf{M}_{B_i} := \sum_l \alpha_l \mathbf{M}_{V_l}, \quad \mathbf{h}_{B_i} := \sum_l \alpha_l \mathbf{h}_{V_l}$$

with: $\alpha_1 + \cdots + \alpha_p = 1, \quad \alpha_i \geqslant 0$

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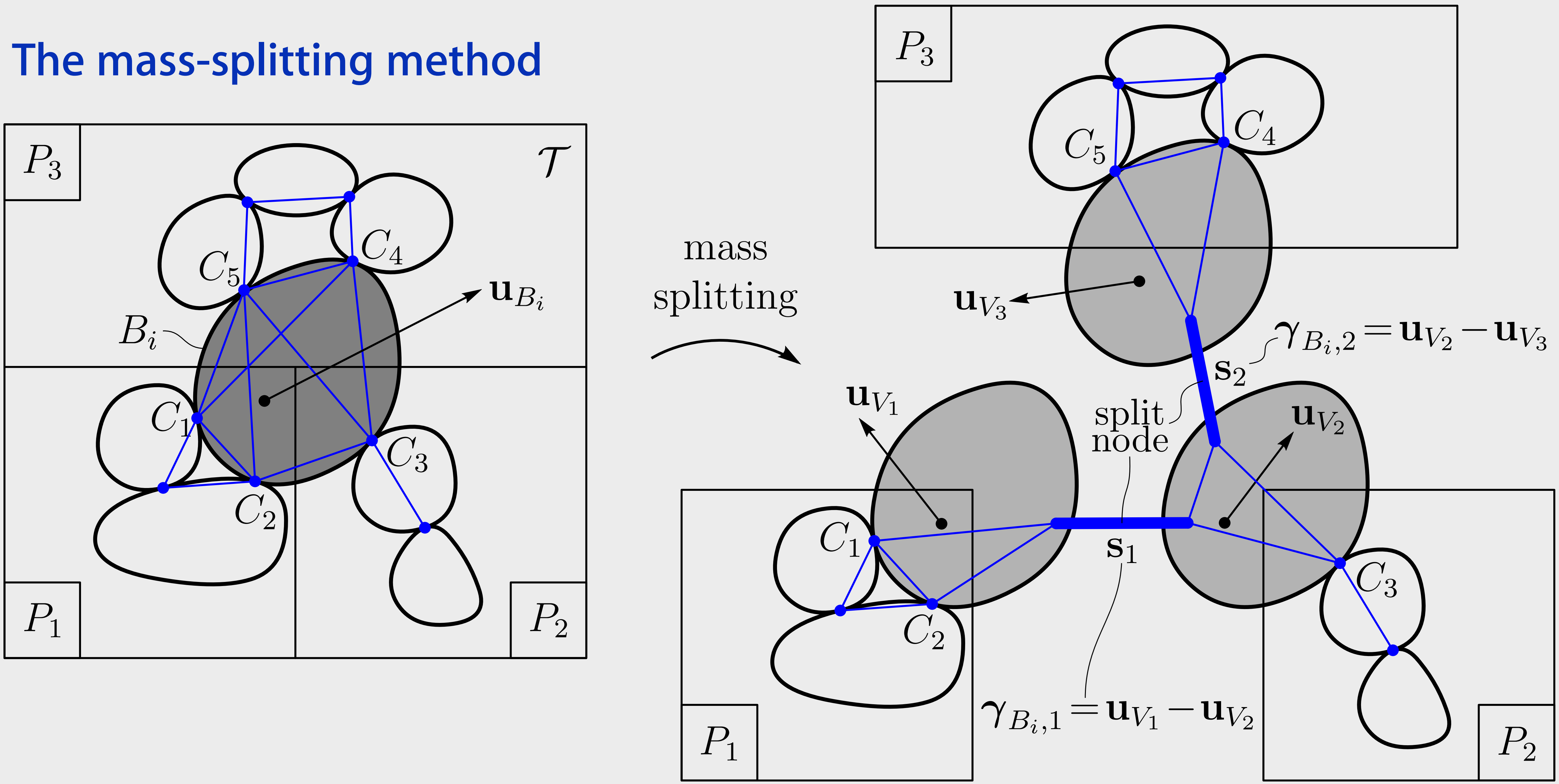

analytical solution of a split-node:
affine combination of velocities

$$\mathbf{M}_{B_i} := \sum_l \alpha_l \mathbf{M}_{V_l}, \quad \mathbf{h}_{B_i} := \sum_l \alpha_l \mathbf{h}_{V_l}$$

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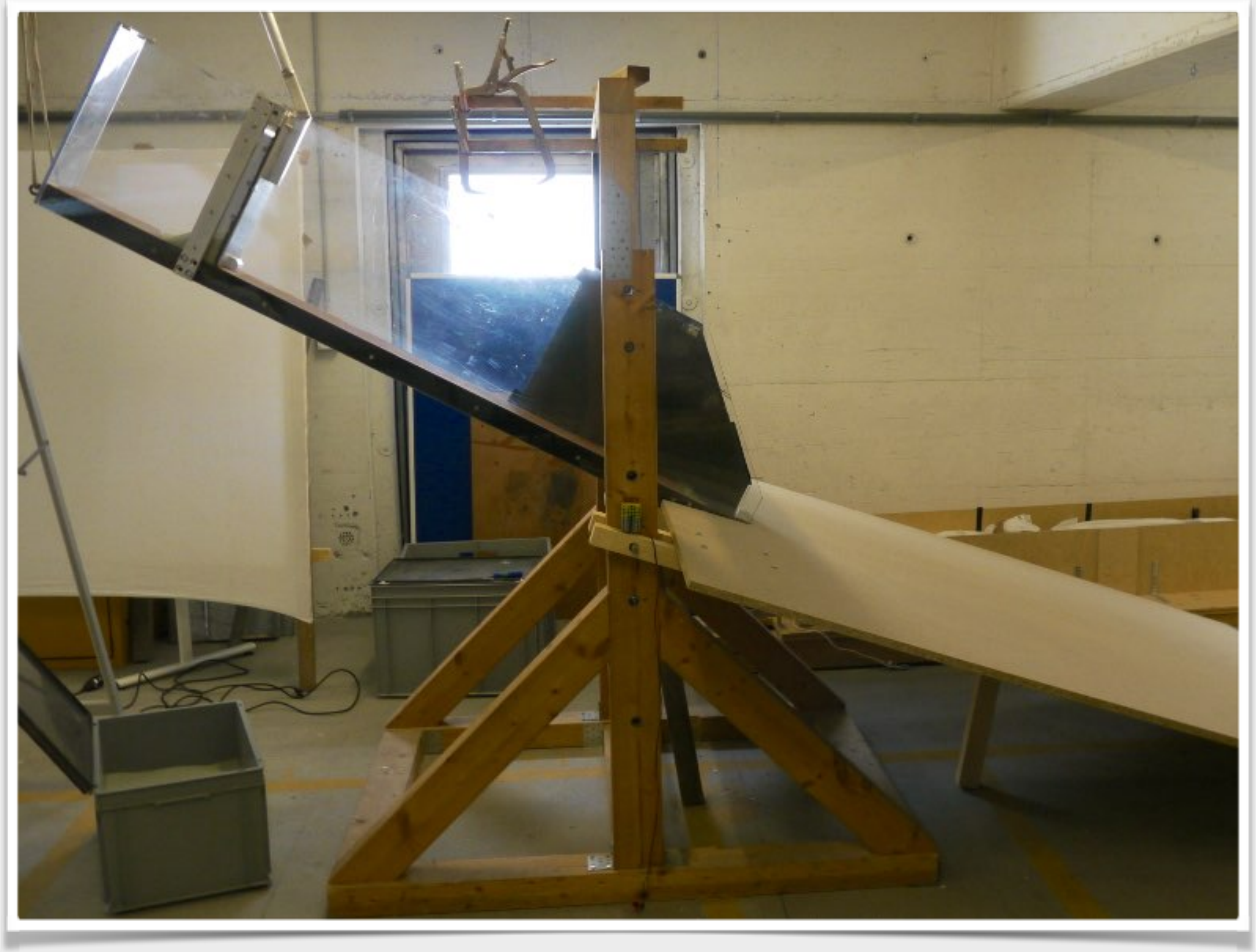
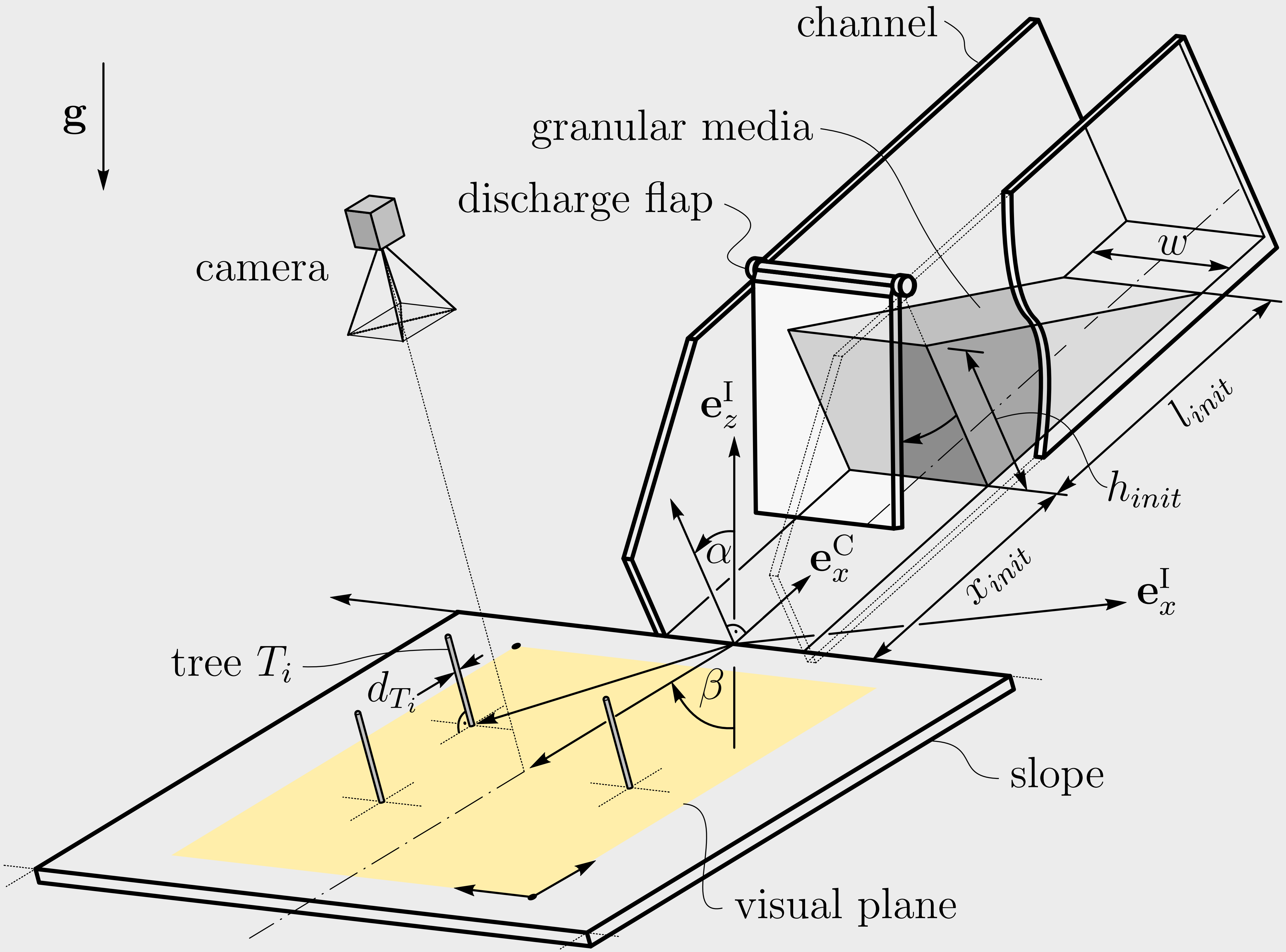
Chute Flow Experiments

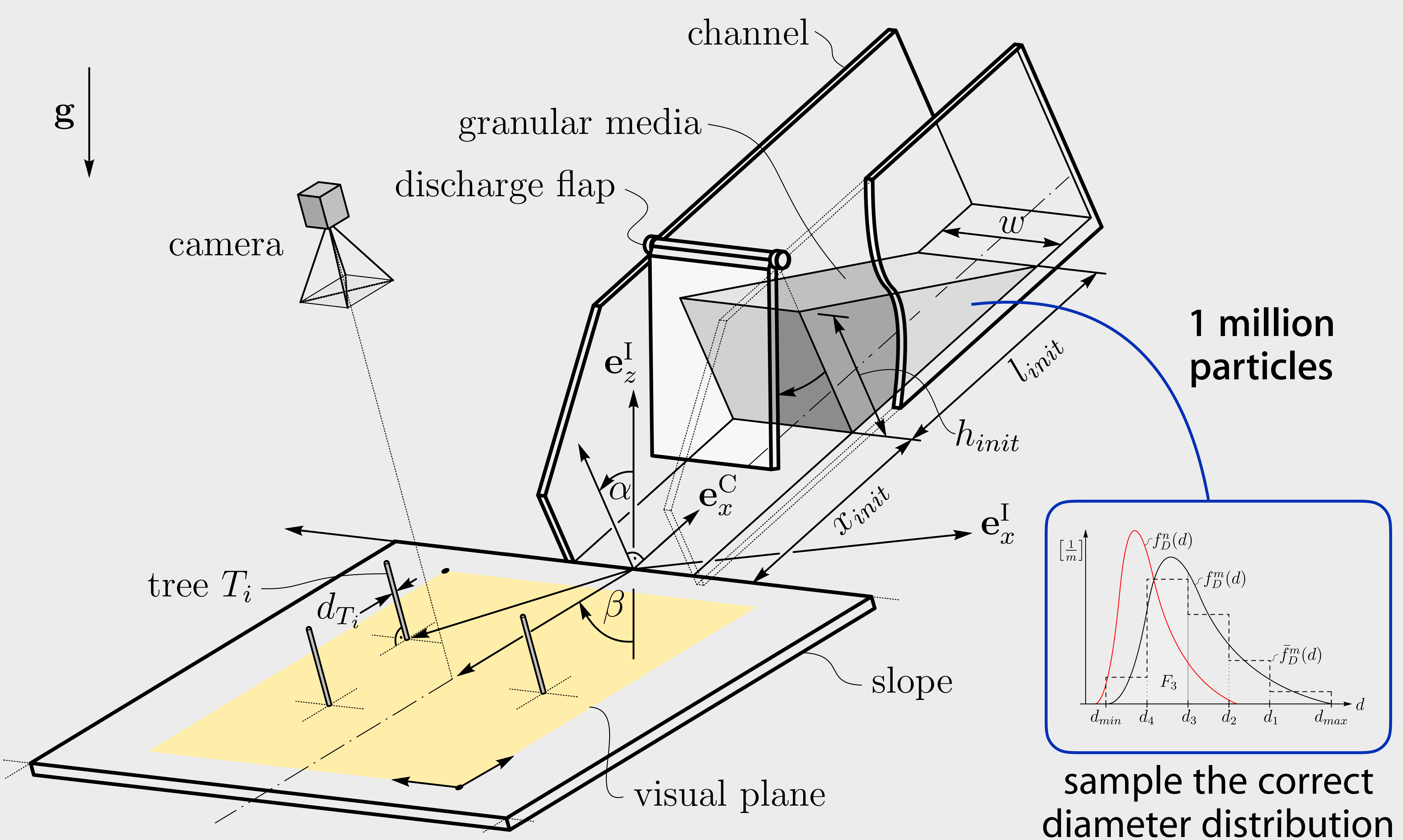
Institute for Snow and Avalanche Research in Davos



Chute Flow Experiments



Chute Flow Experiments



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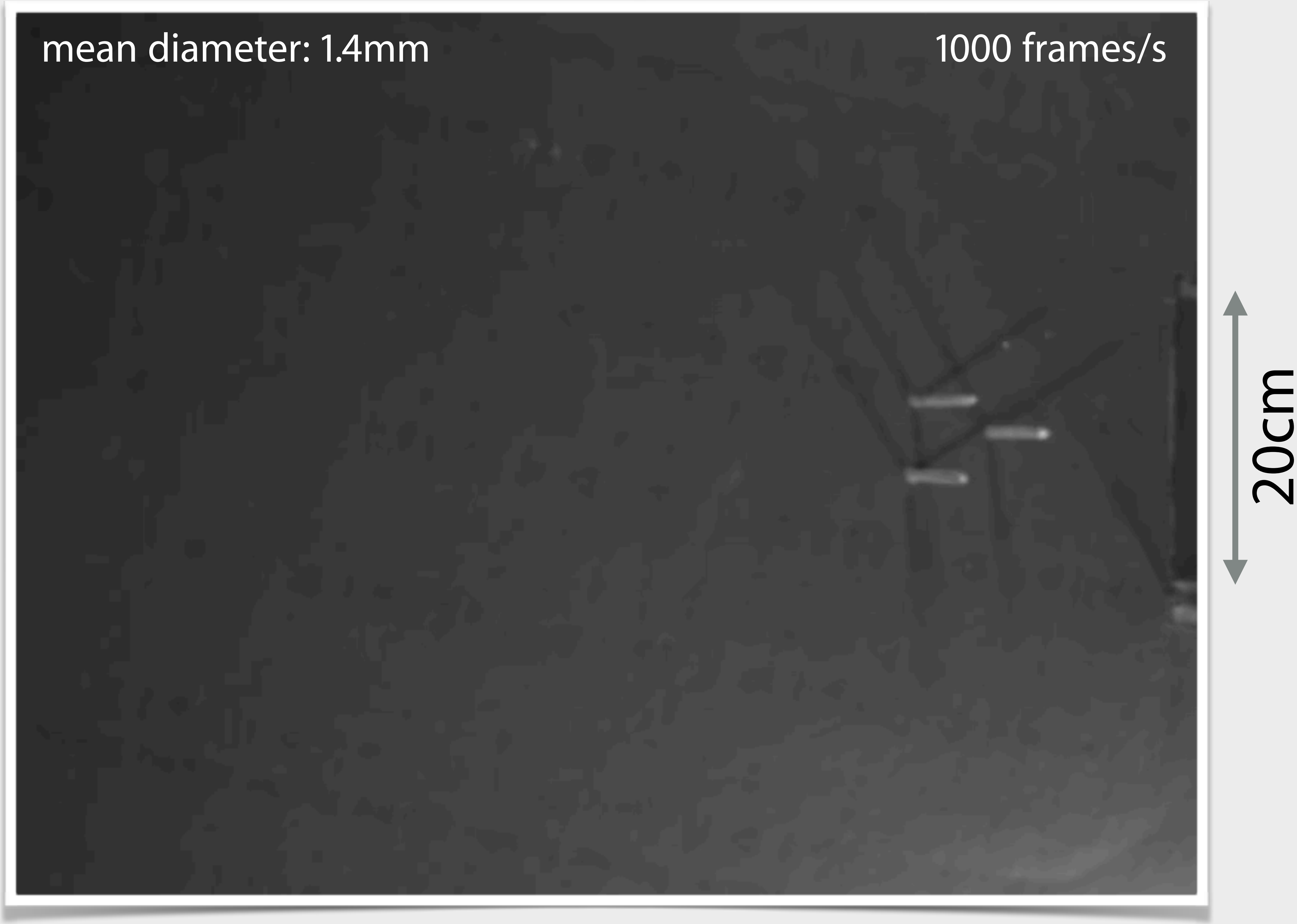

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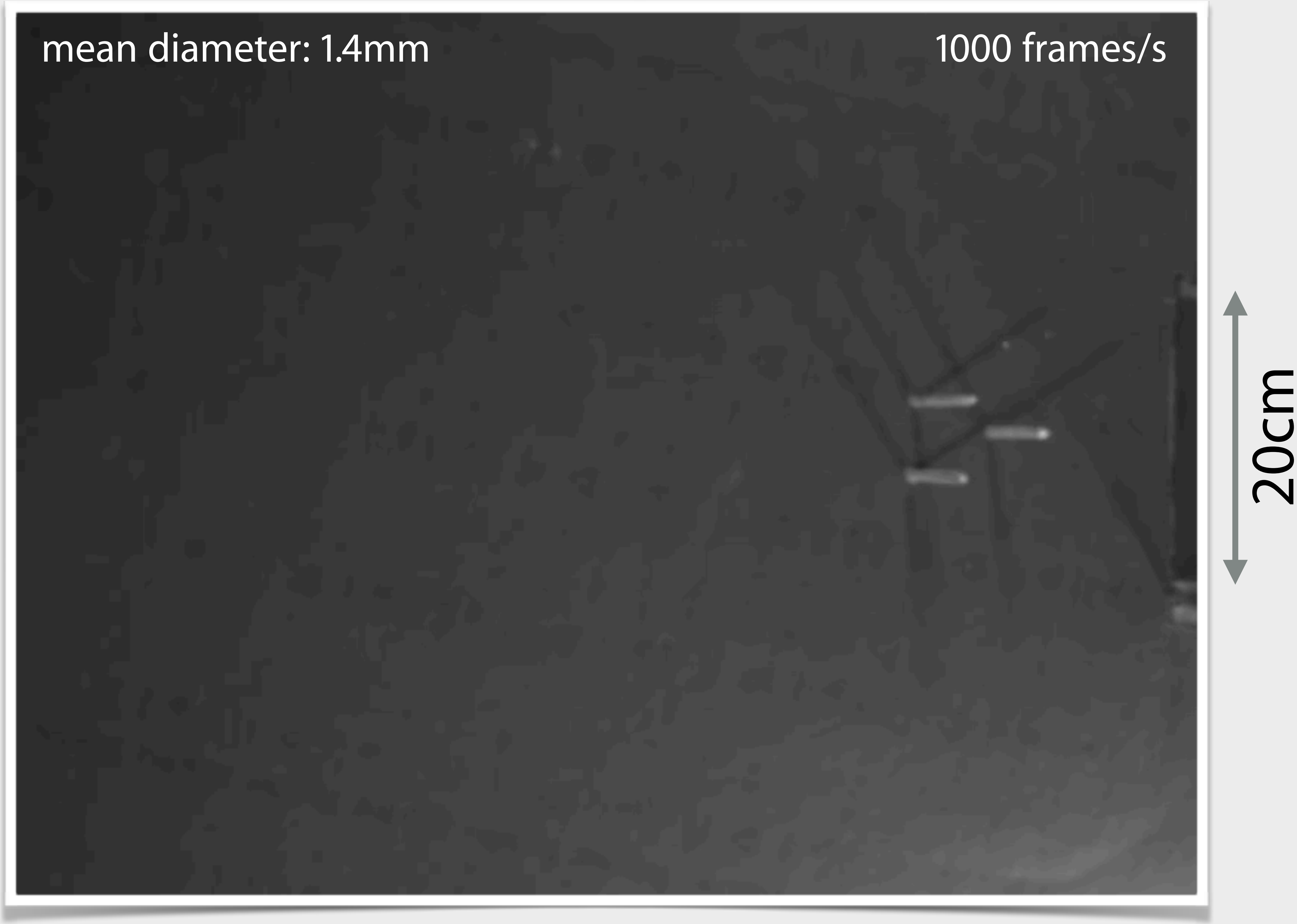
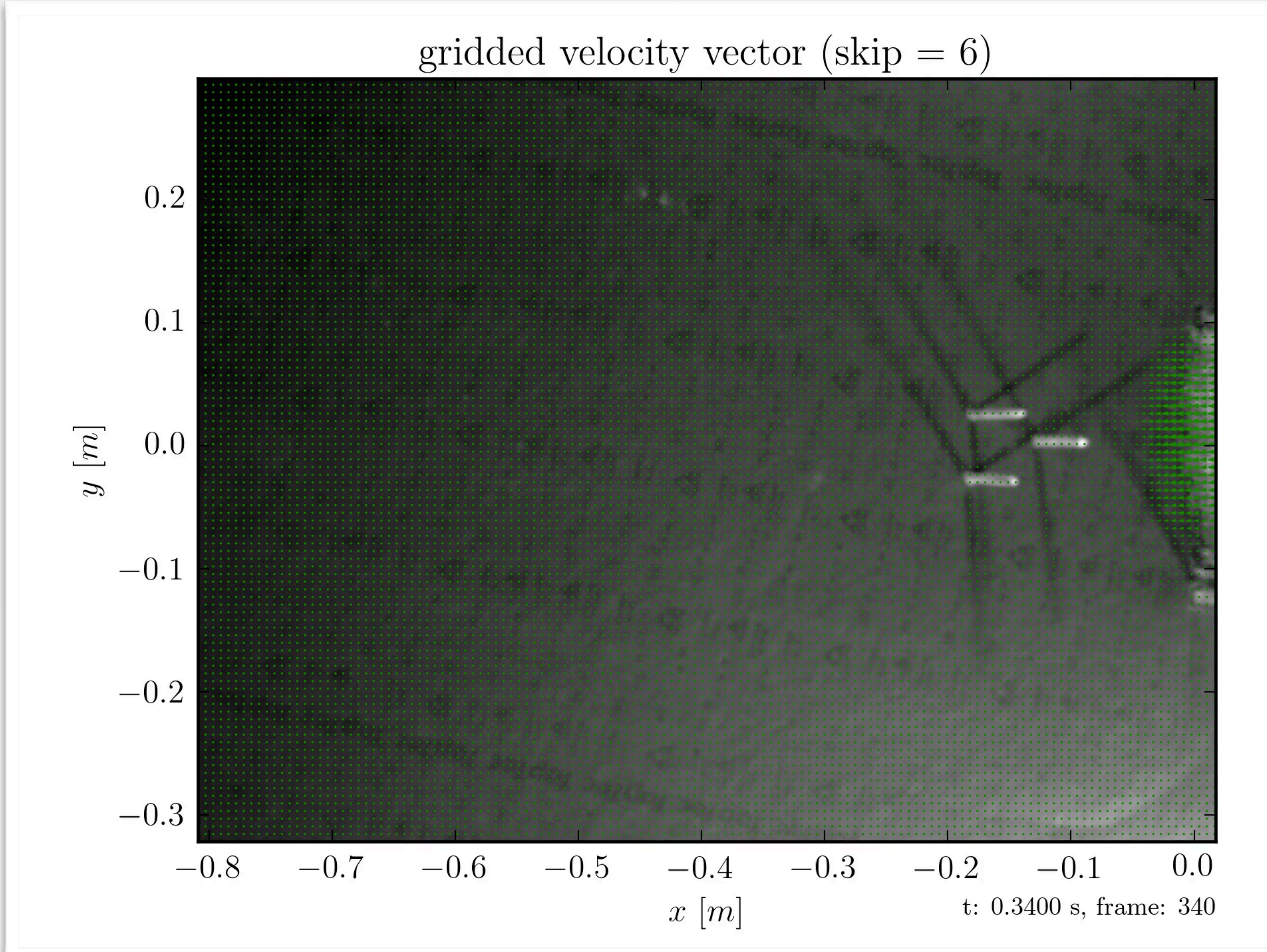


Advection Corrected Velociometry (X. Assay-Davis, 2009)



2000 frames, 4 images/frame, 2 passes = 4h = 75 Gb

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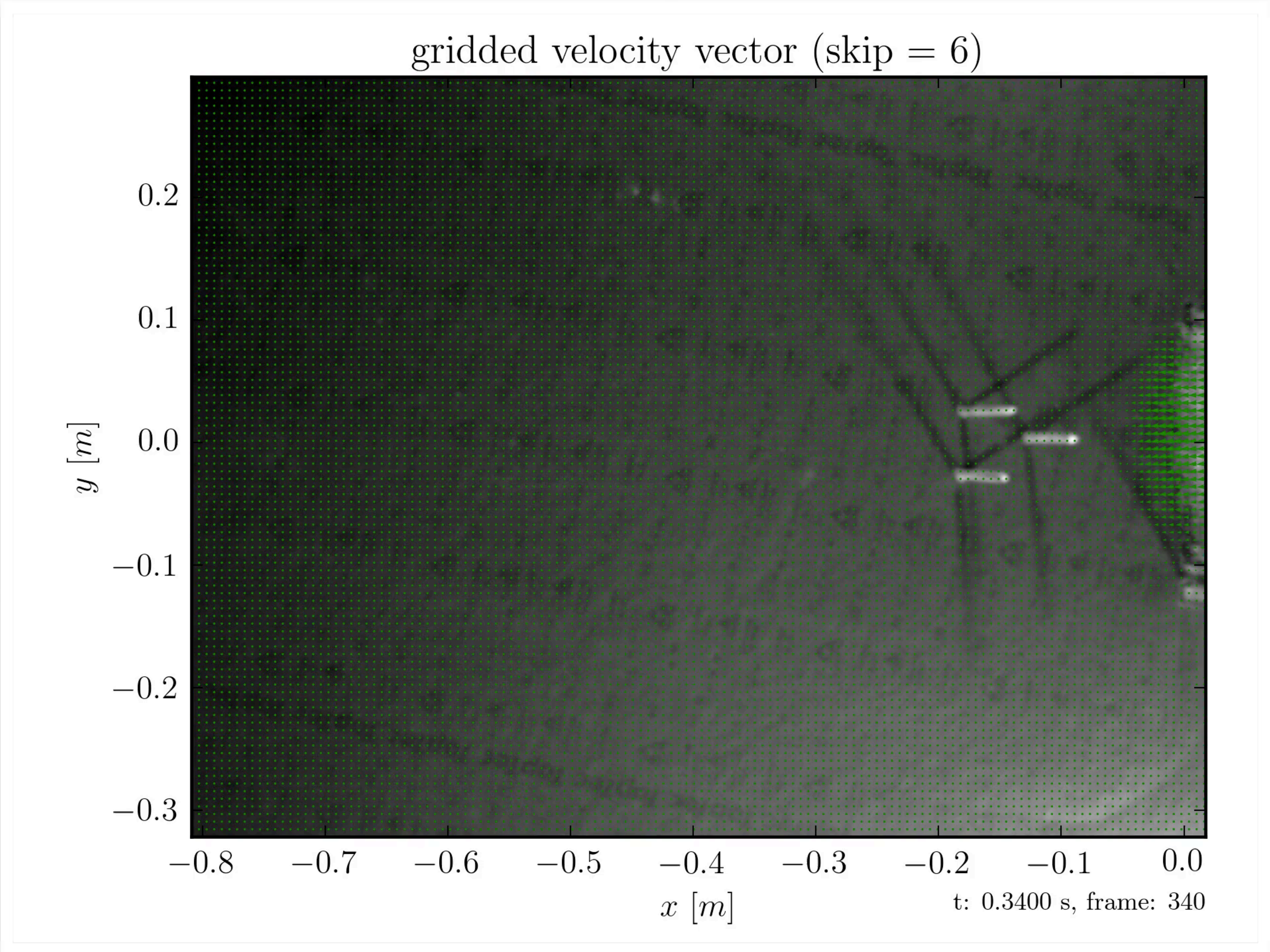

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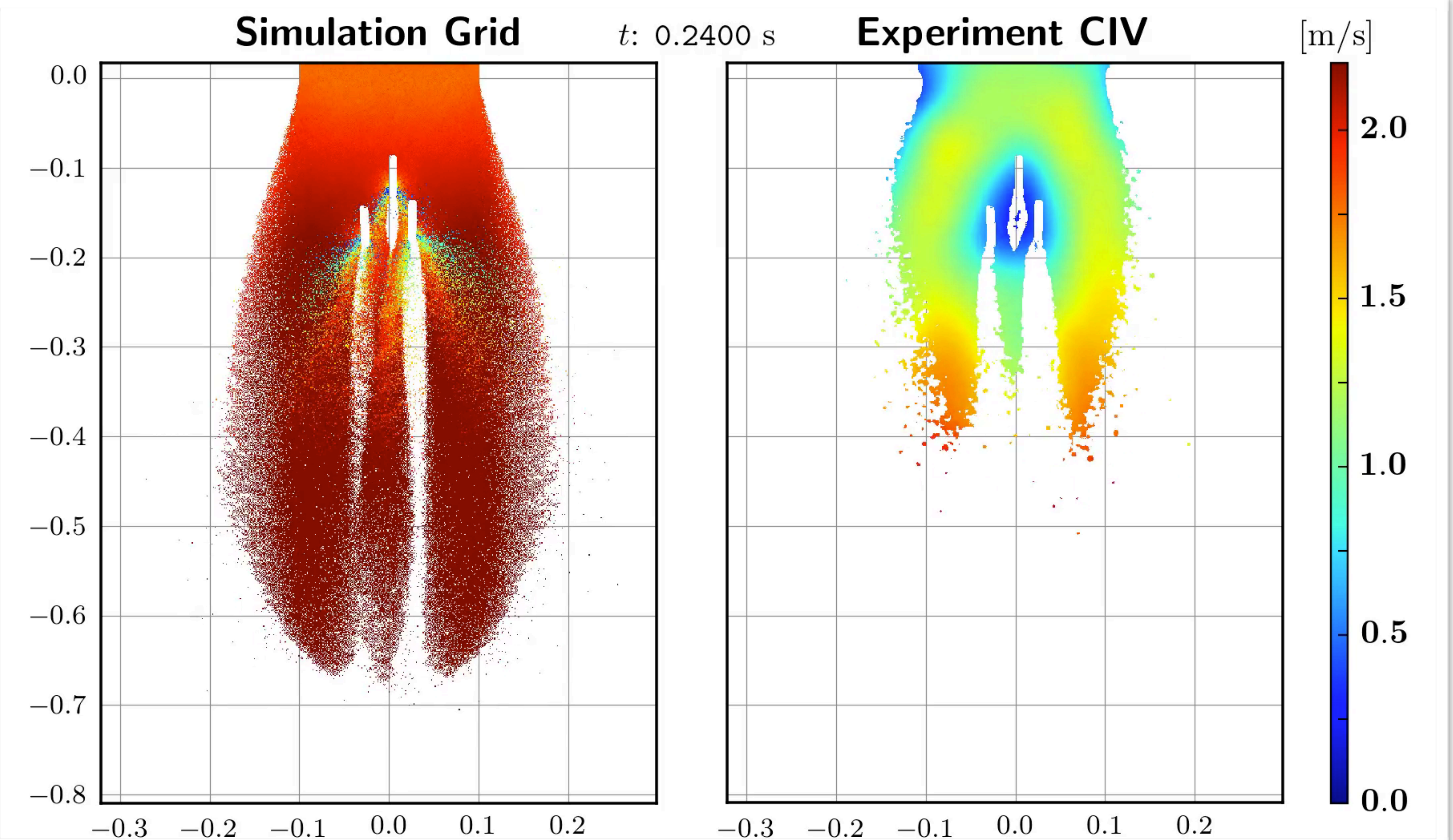
Comparison

$\mu = 0$

$|\Delta t = 0.0002\,\mathrm{s}\,|\,\mathrm{maxIter}{=}1000\,|\,\varepsilon_N, \varepsilon_T = 0$
$|\,n_T = 10^6$



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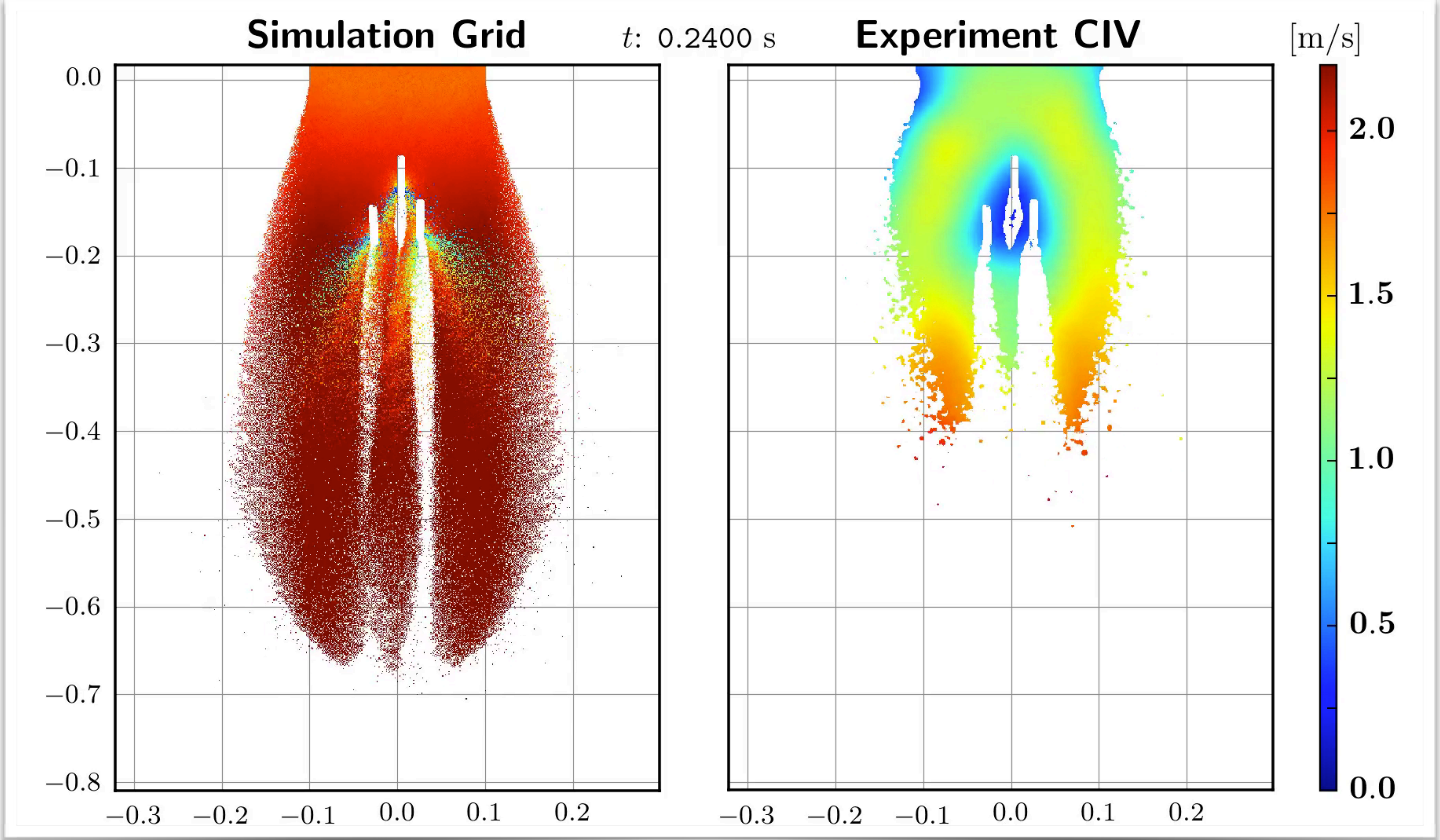

Comparison

$\mu = 0.3$

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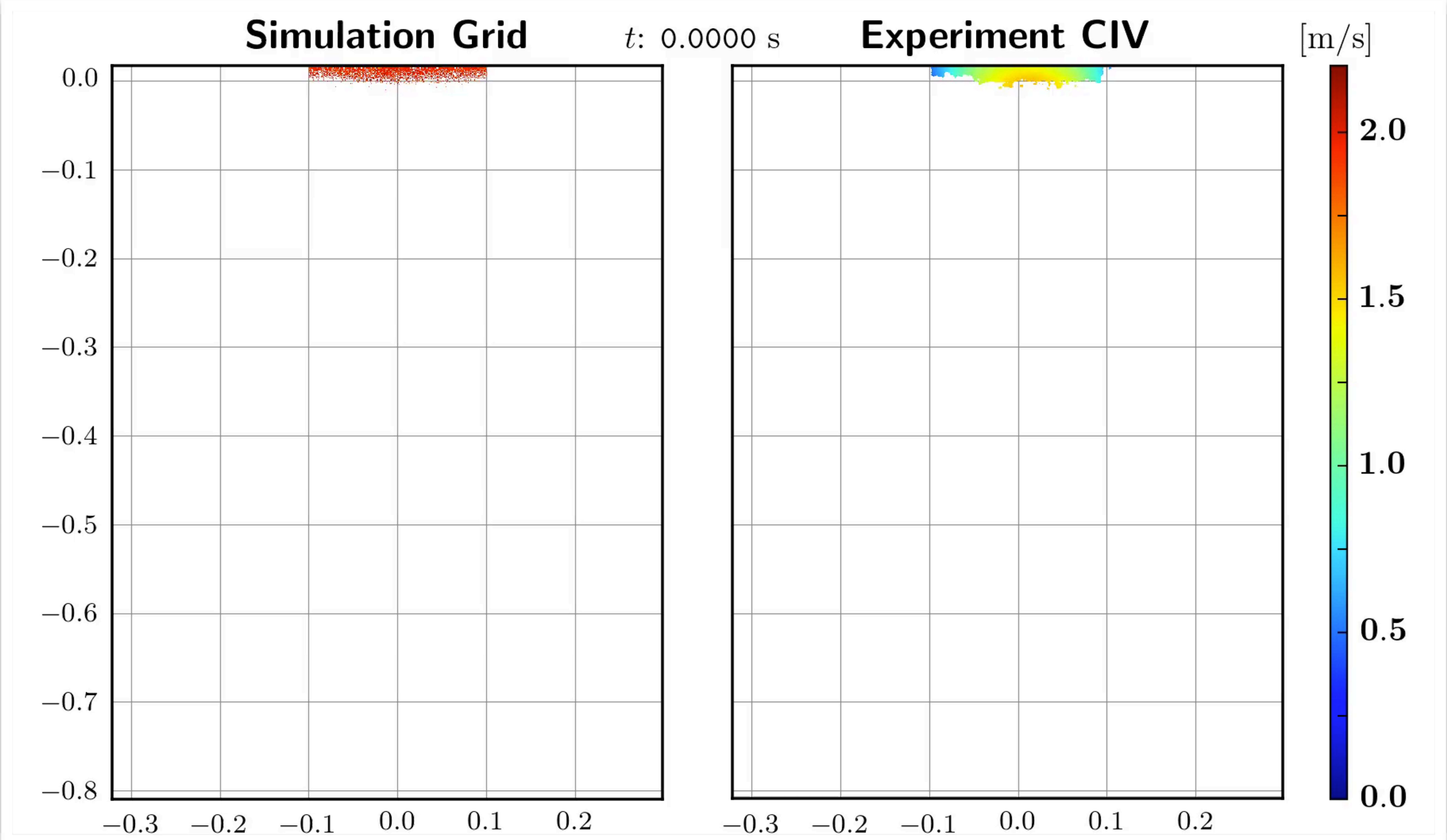

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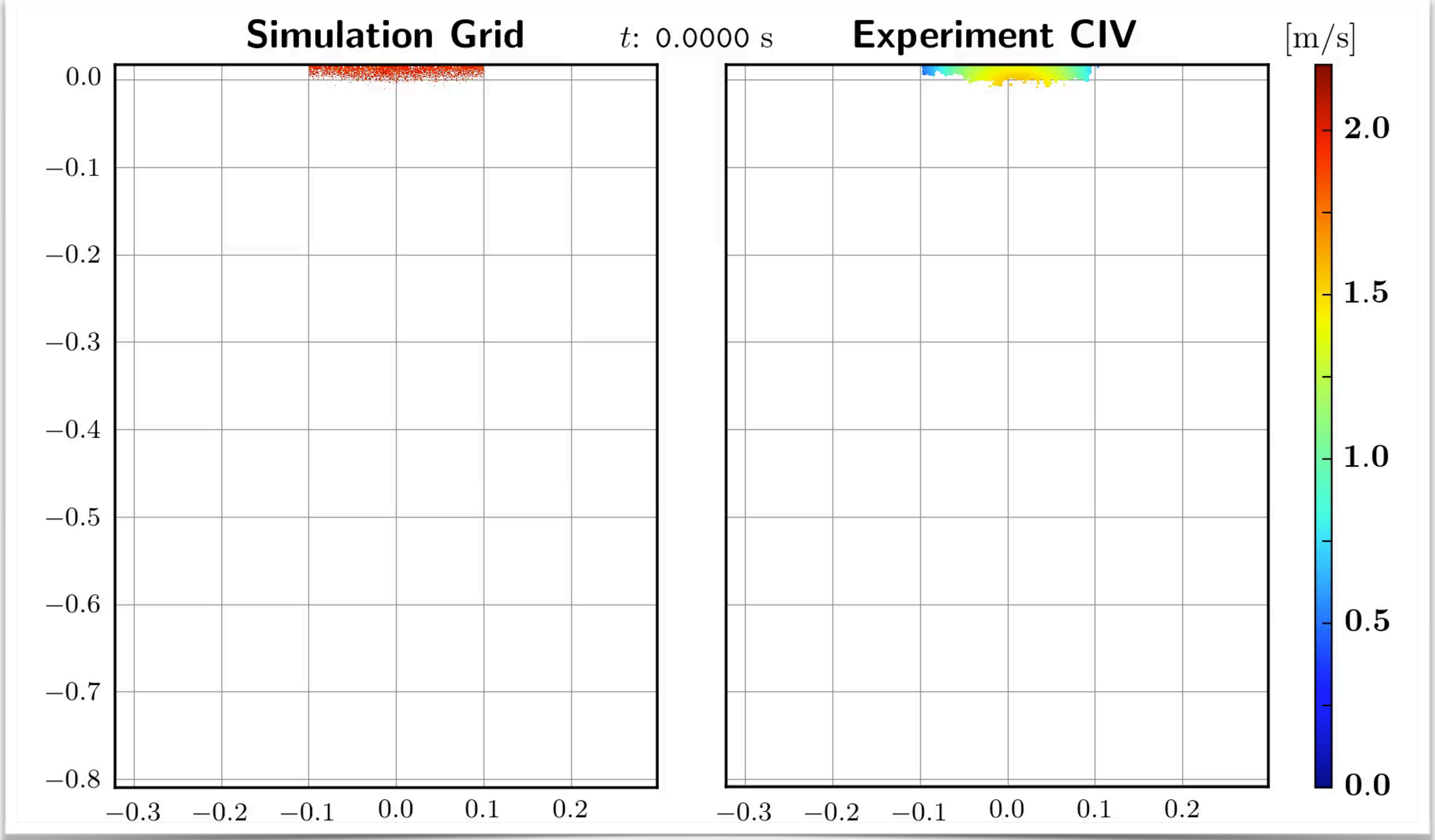
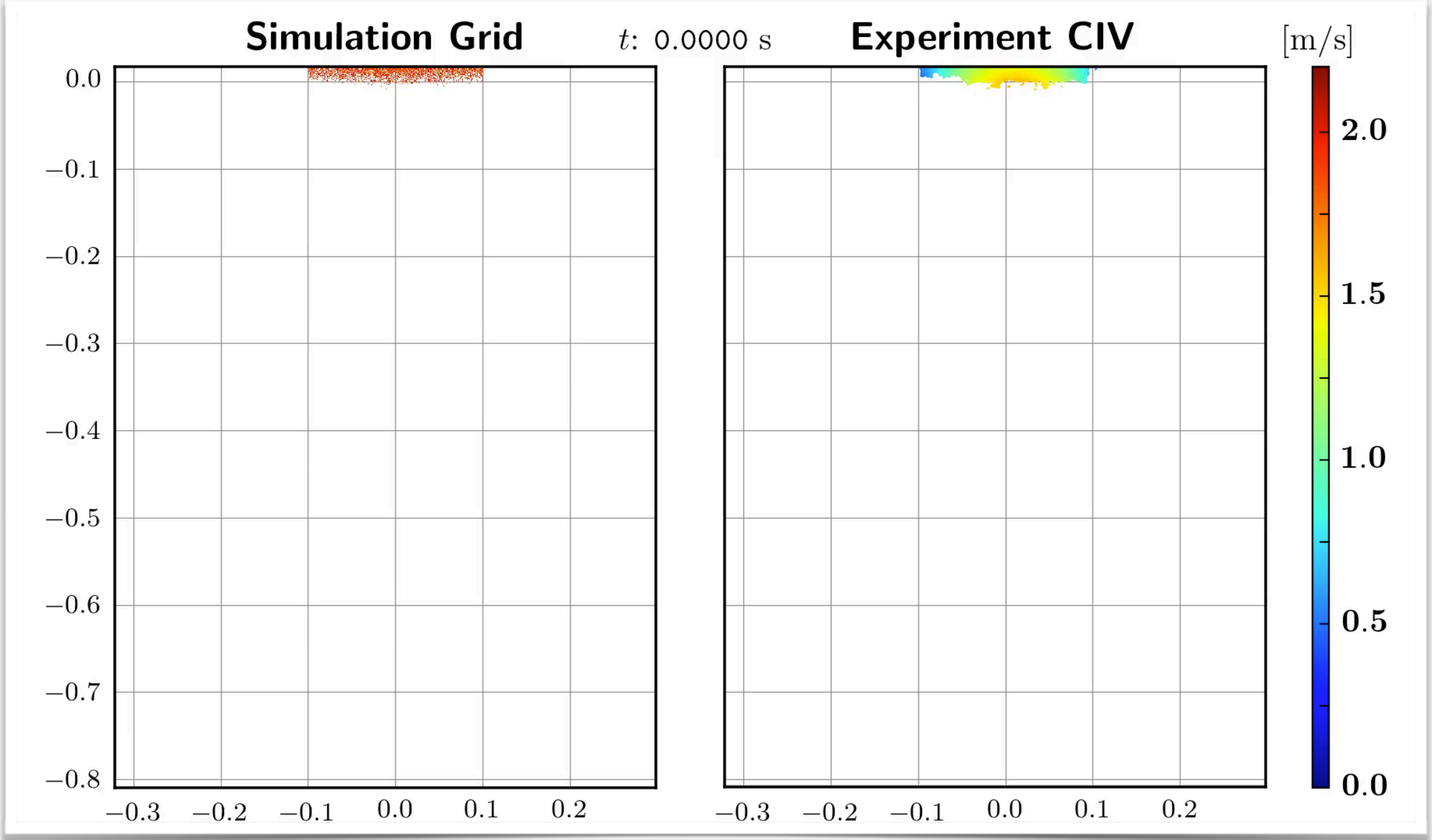


Comparison

$\mu = 0.8$

$|\,\Delta t = 0.0002\,\mathrm{s}\,|\,\mathrm{maxIter{=}1000}\,|\,\varepsilon_N, \varepsilon_T = 0$
$|\,n_T = 10^6$ **best match**



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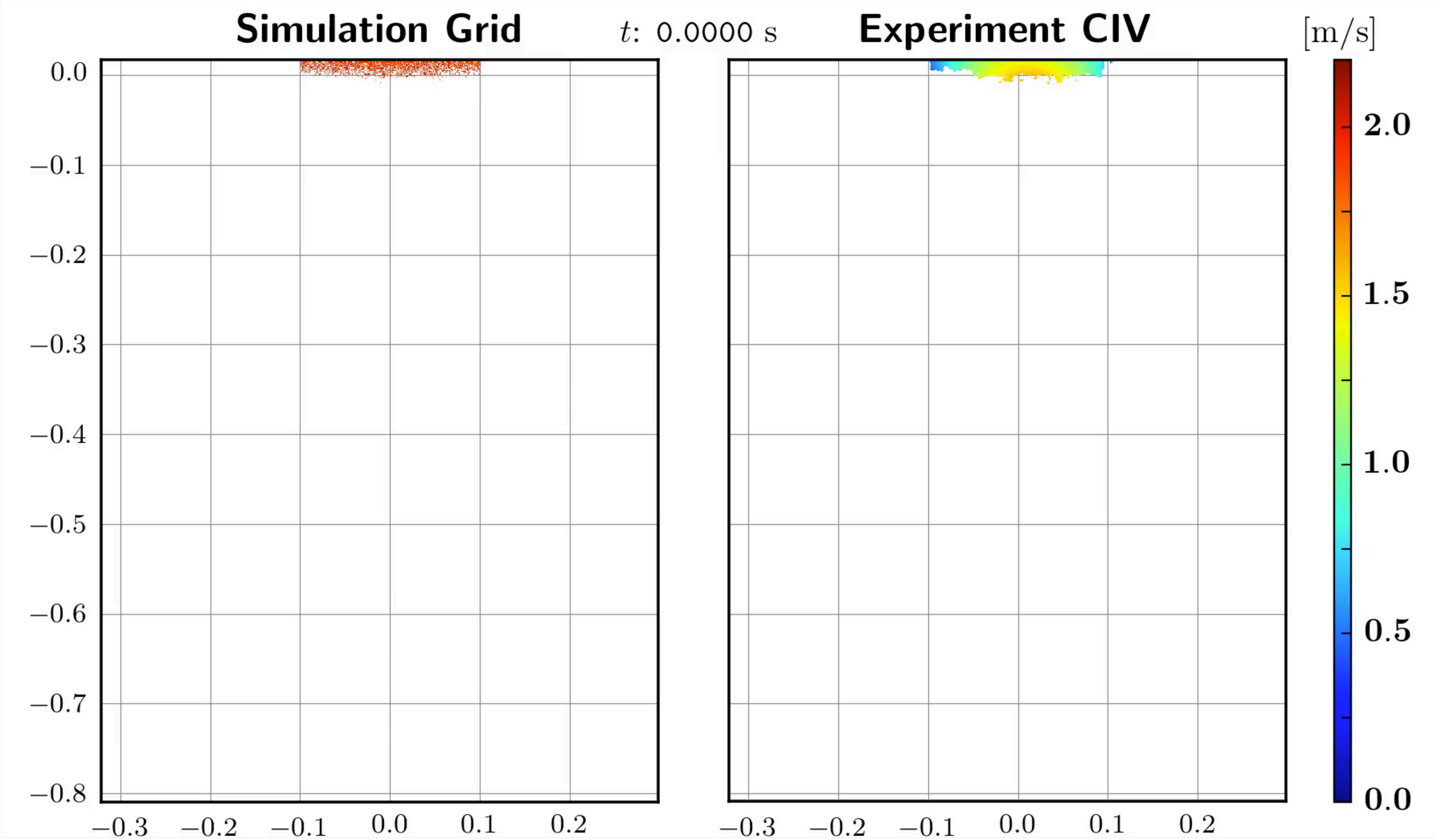

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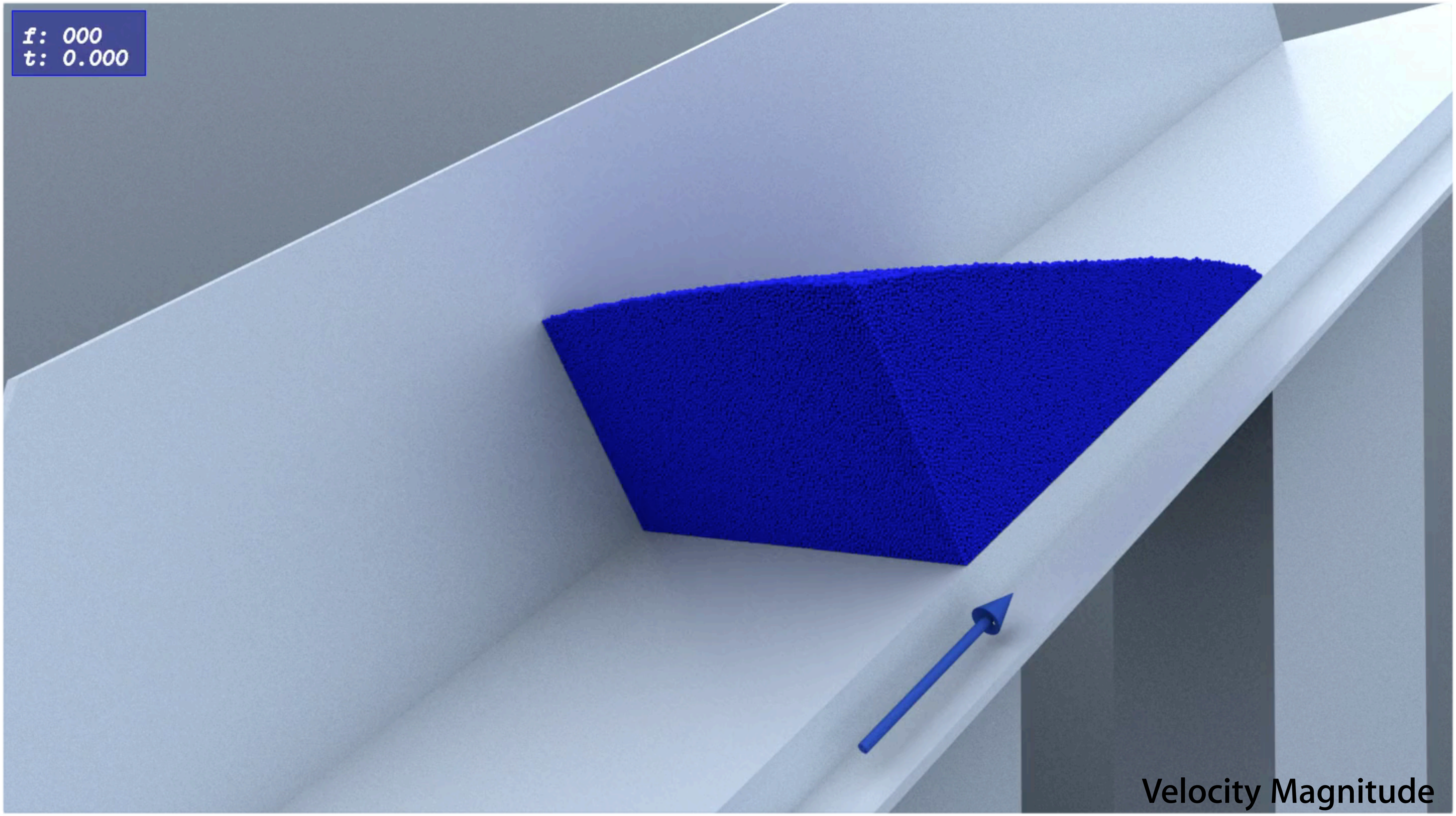
ETH Euler: 384 Cores

GRFS MPI: 12h = 120Gb

Converter/Renderman: 780 Frames in 24h



$\mu = 0.8 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter}{=}1000 \mid \varepsilon_N, \varepsilon_T = 0 \mid n_T = 10^6$

best match

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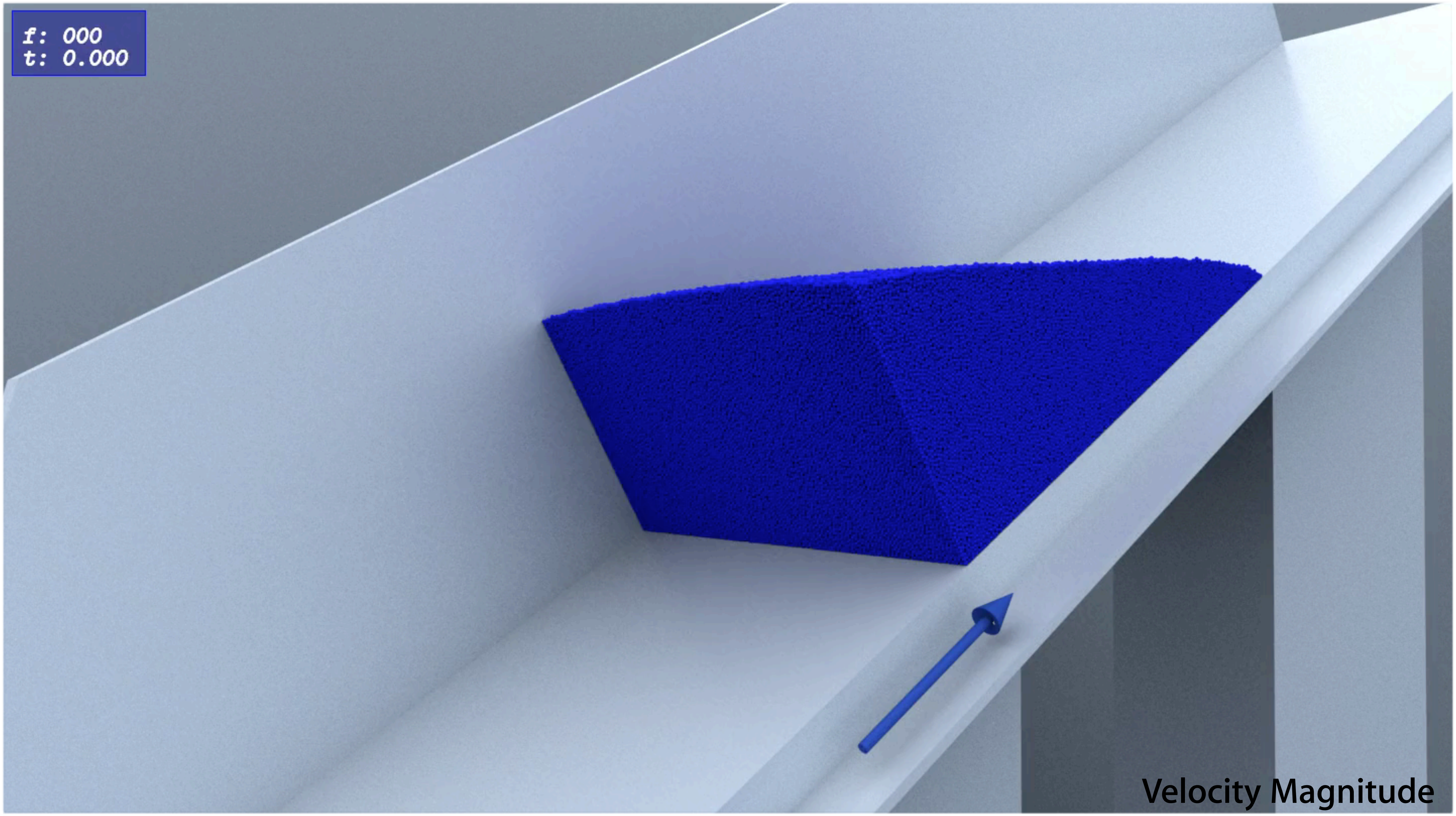

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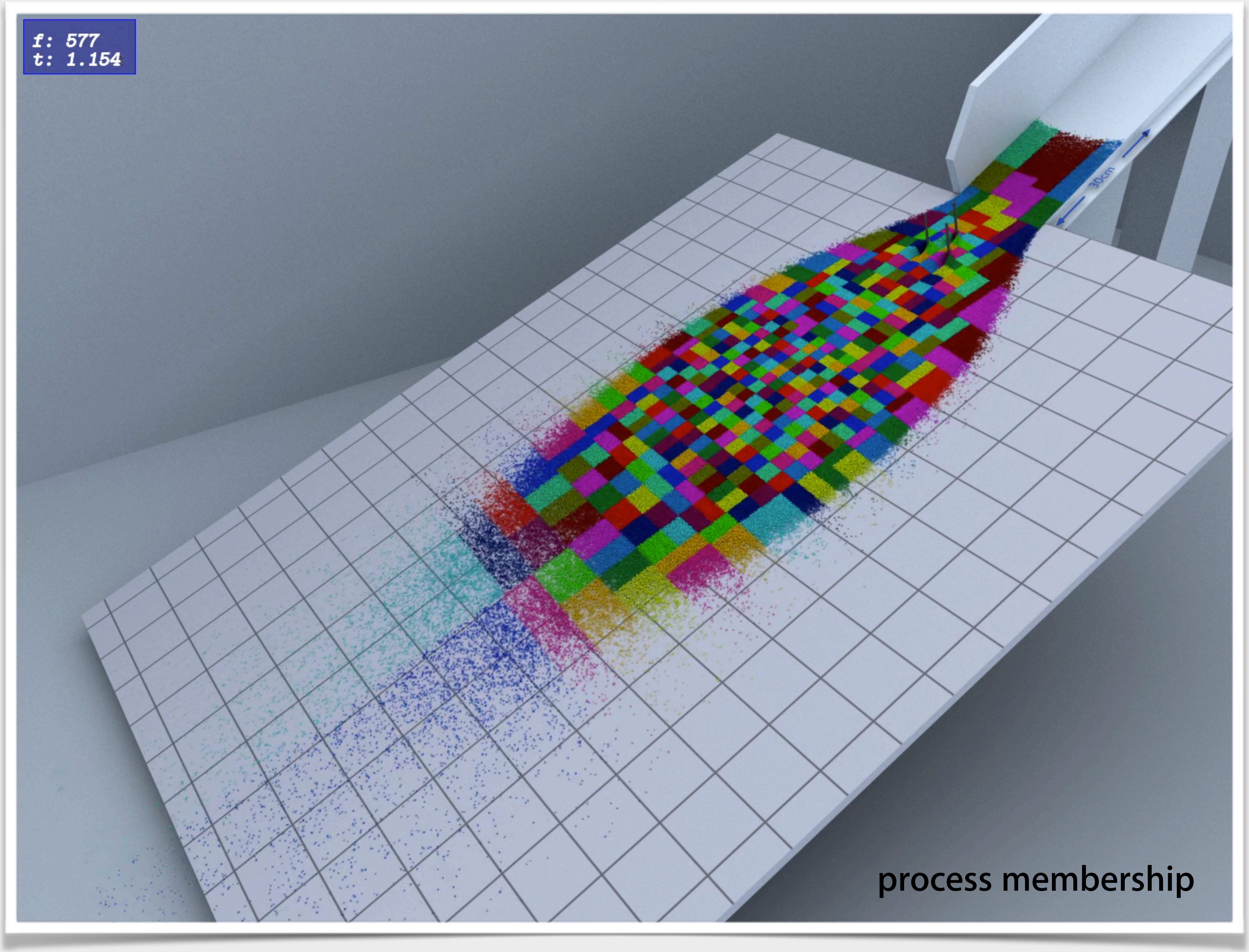
Why Domain Decomposition / Load Balancing?



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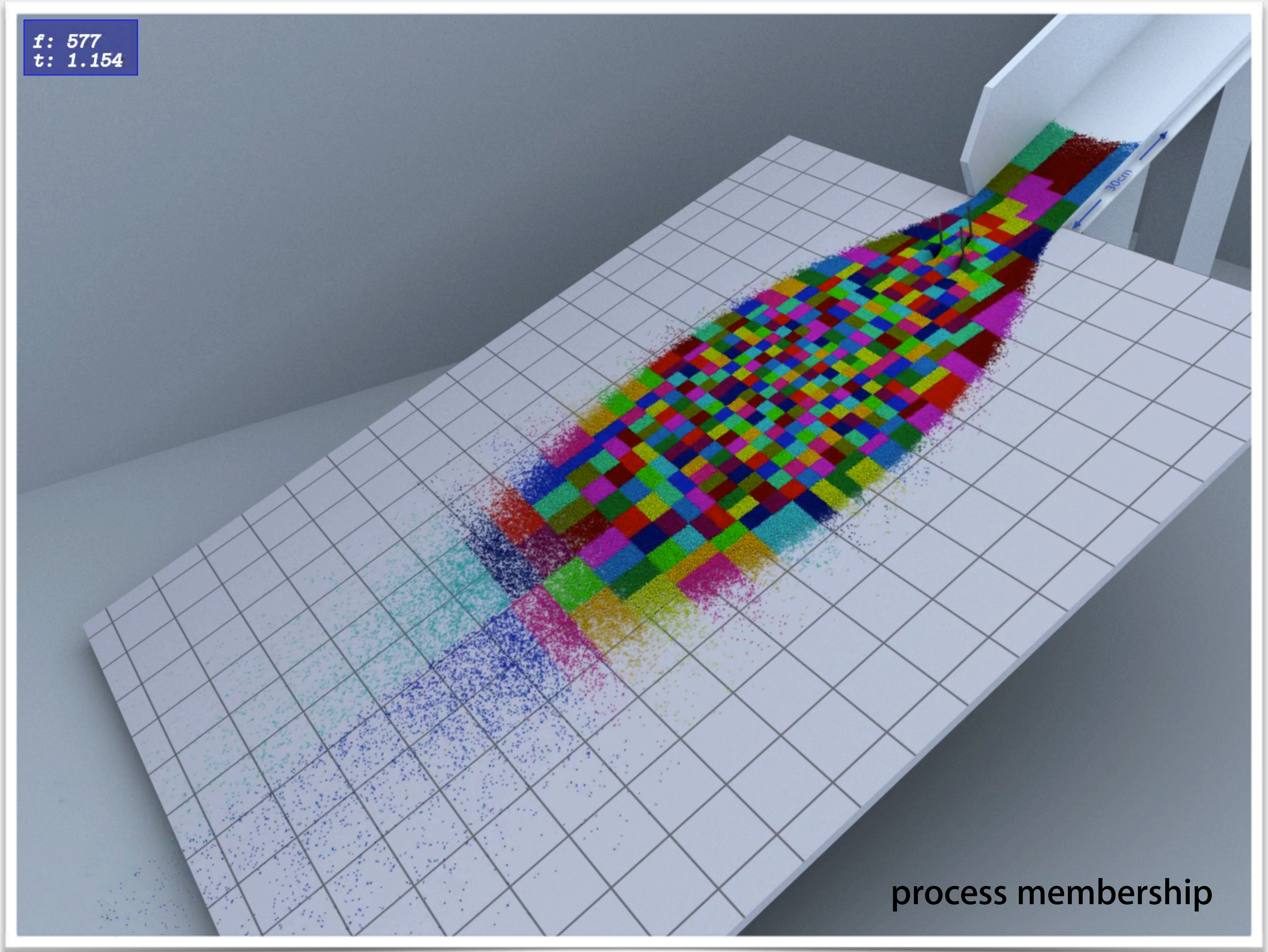
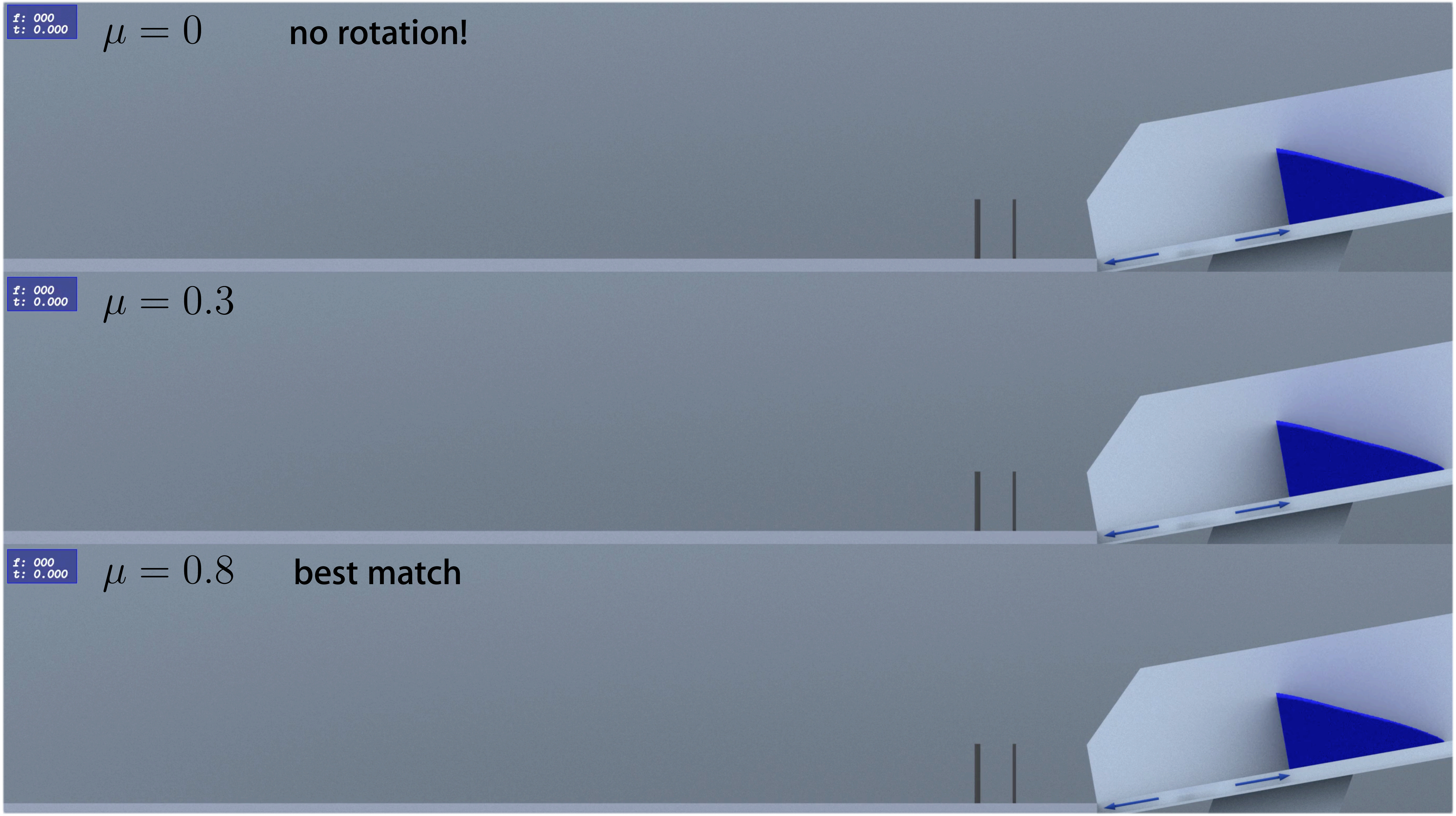
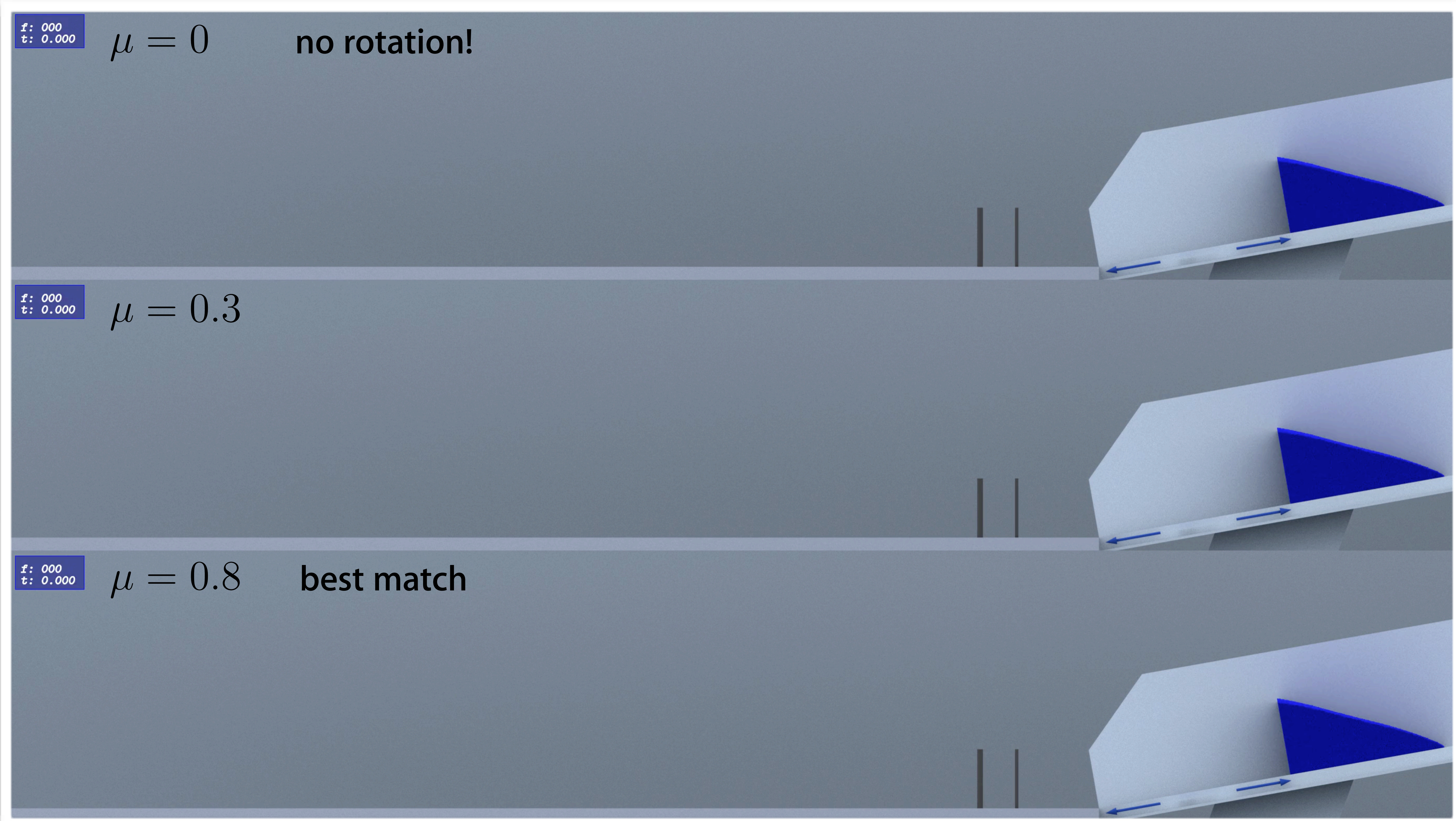




$\Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter{=}1000} \mid \varepsilon_N, \varepsilon_T = 0 \mid n_T = 10^6$

Velocity Magnitude



$\Delta t = 0.0002\,\mathrm{s}$ | maxIter=1000 | $\varepsilon_N, \varepsilon_T = 0$ | $n_T = 10^6$

Velocity Magnitude

Conclusion

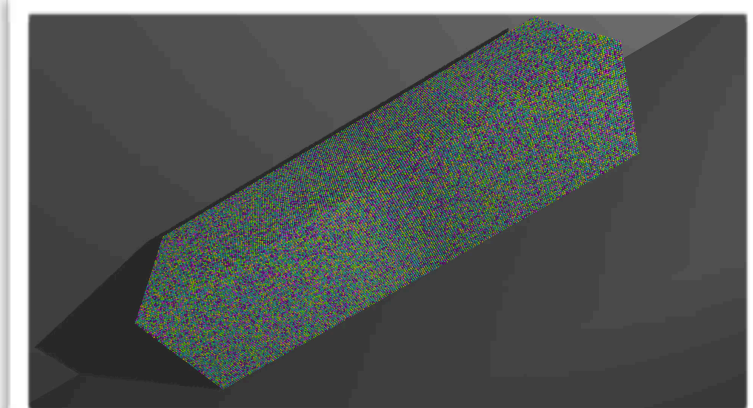
- Modeling granular media within the framework of **non-smooth rigid body dynamics.**
- Quality-conscious **open-source software**:

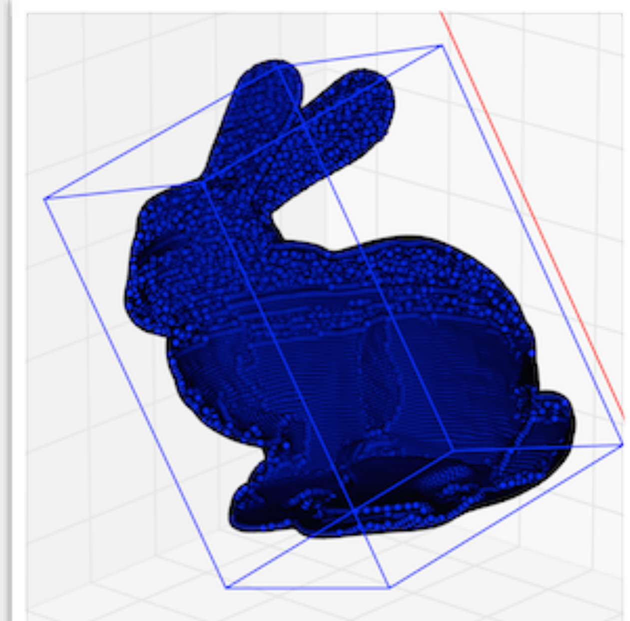
  GRS Framework:
  research tools to simulate granular rigid body dynamics (>10^6 bodies).

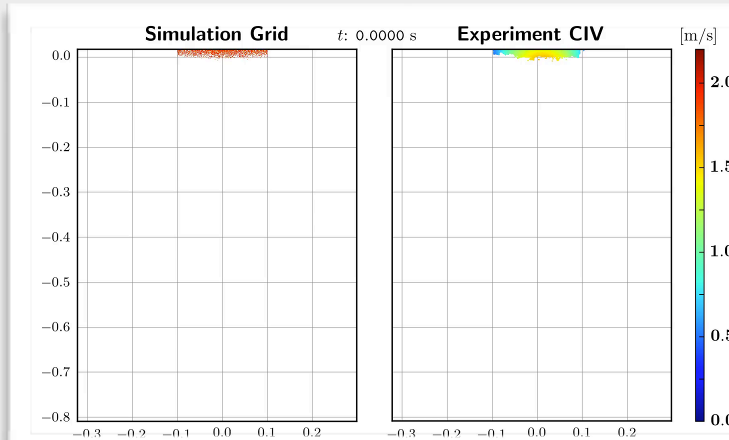
  ApproxMVBB Library:
  approx. minimal volume bounding box, generic kdTree , statistical filtering.

  HPCJobConfigurator:
  job configuration for high-performance computing with **python**

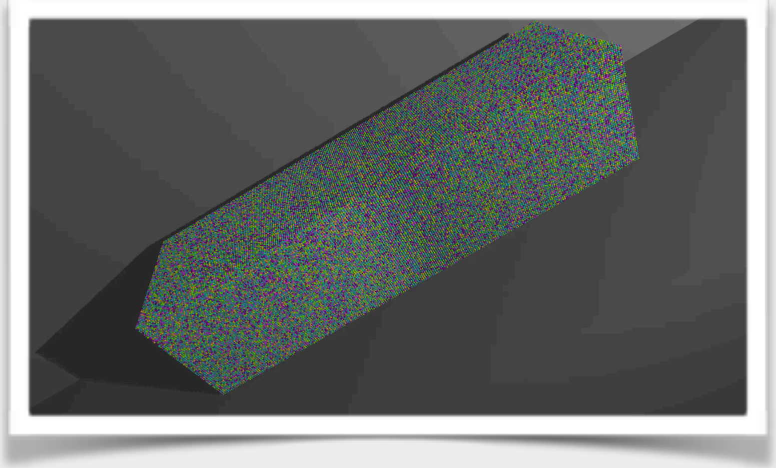
- First step towards visualization and validation of large-scale 3d granular simulations.







Conclusion



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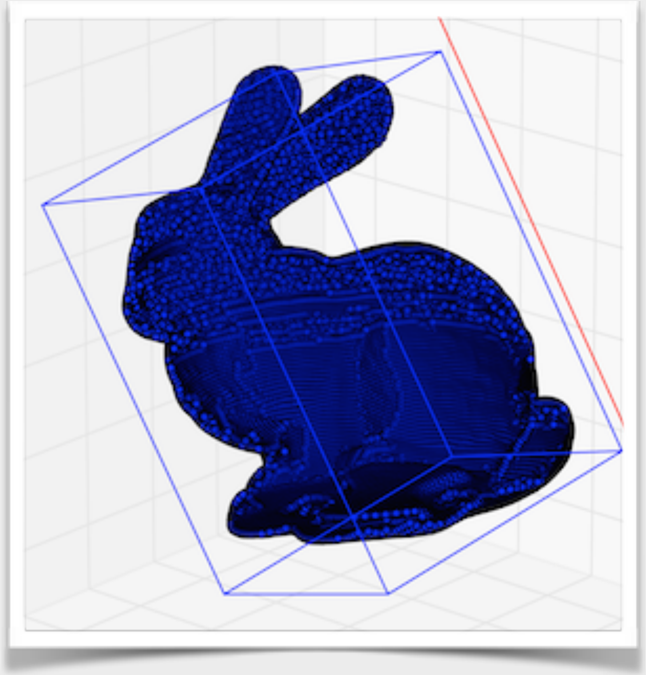

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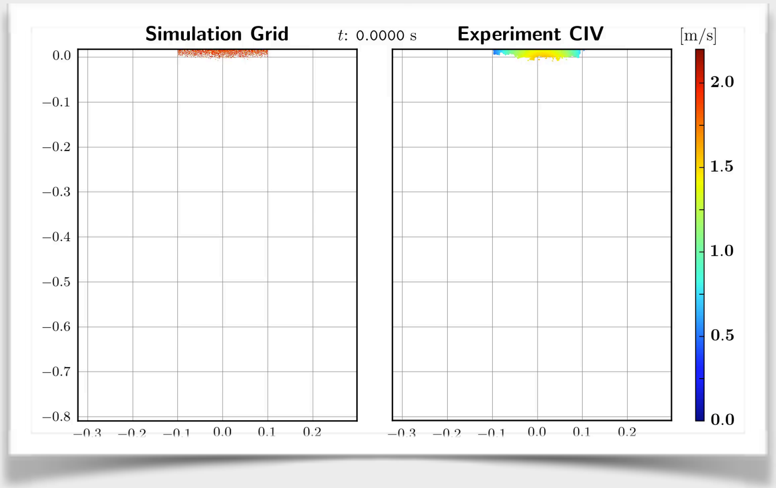
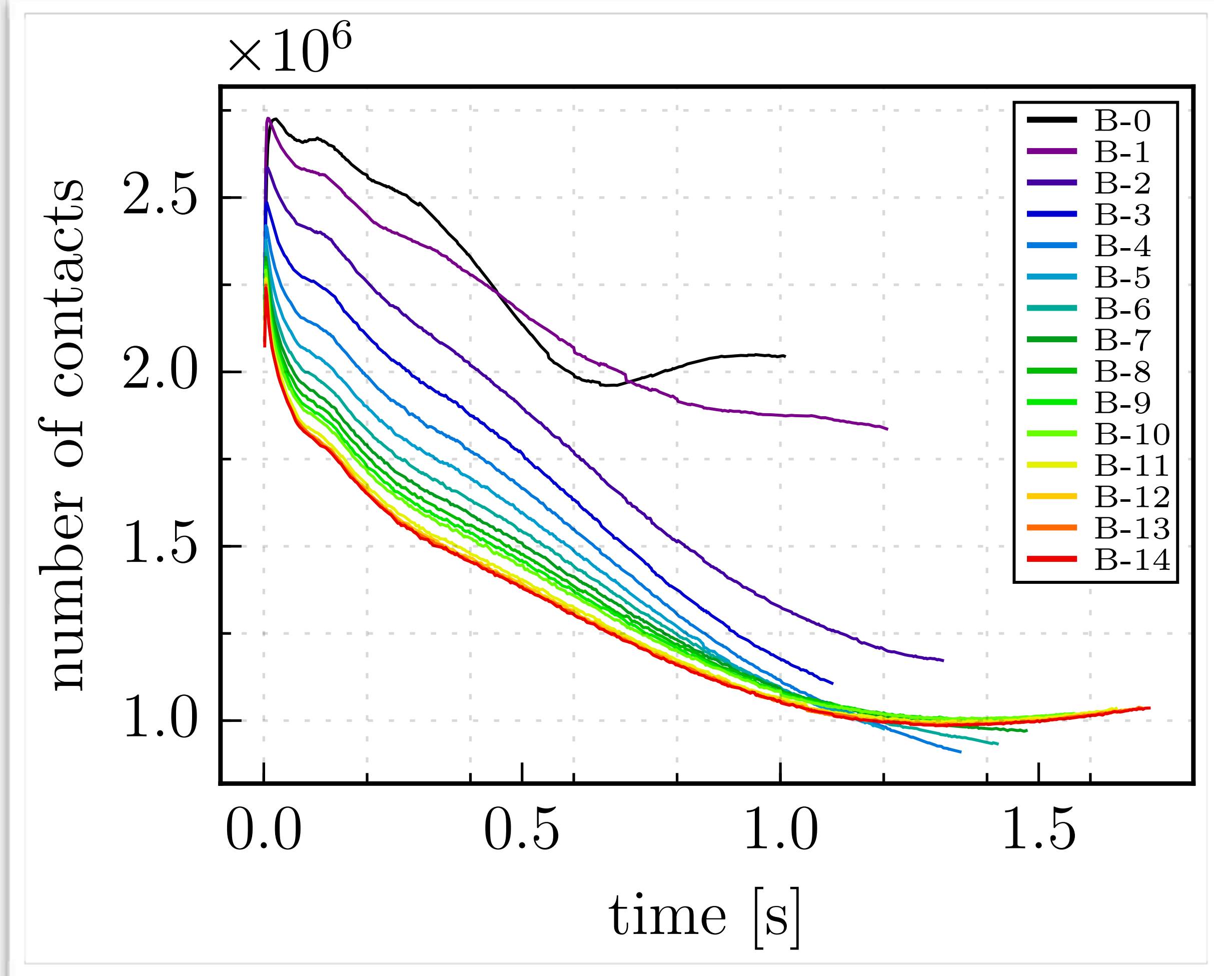
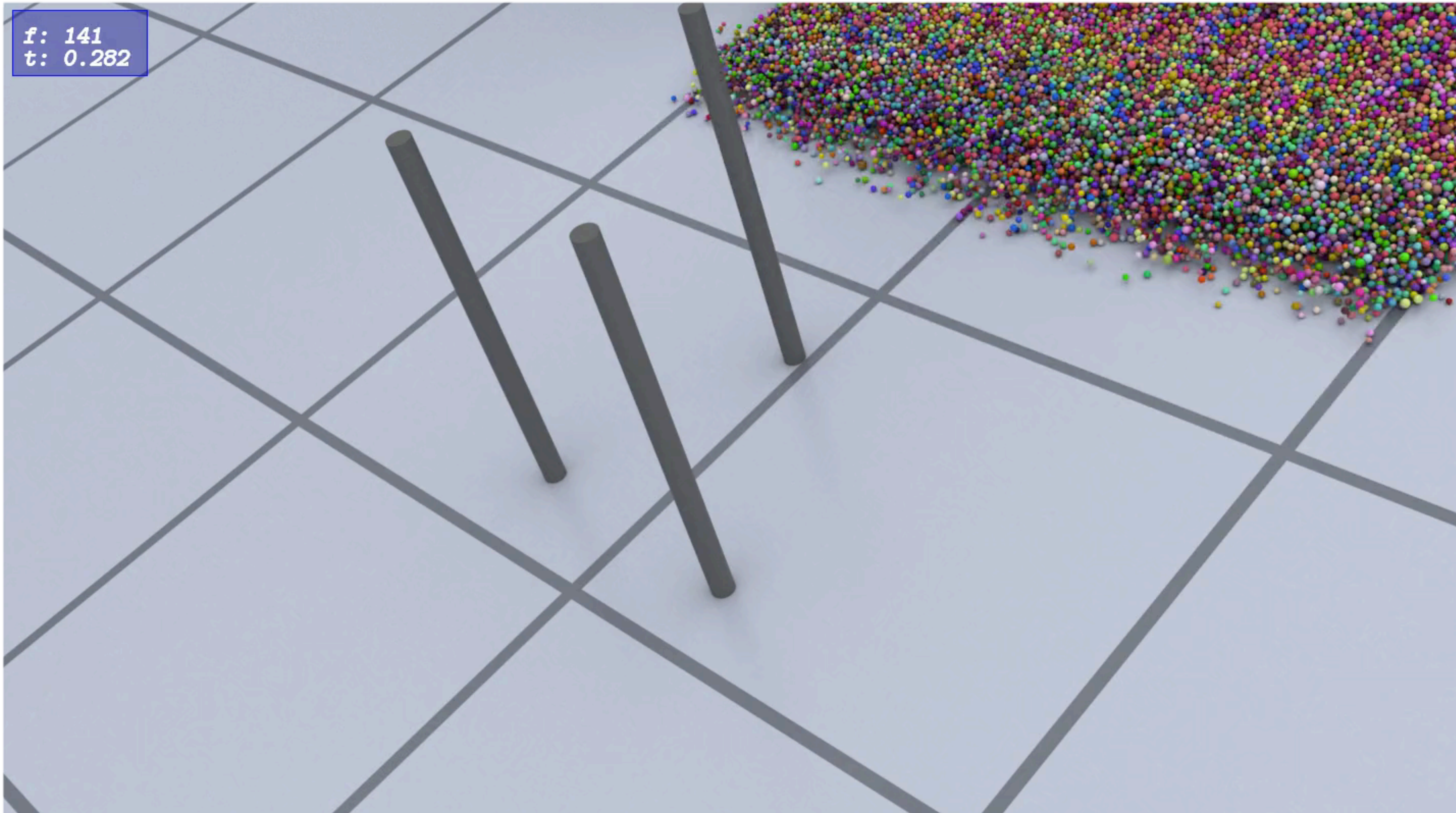


Outlook

- improve **contact problem solvers**:
  - accelerated projected gradient schemes
- more **advanced time-stepping** schemes:
  - variational integration schemes
- improve **experimental evaluations**:
  - velocity fluctuations
  - contact force distribution, stress tensor



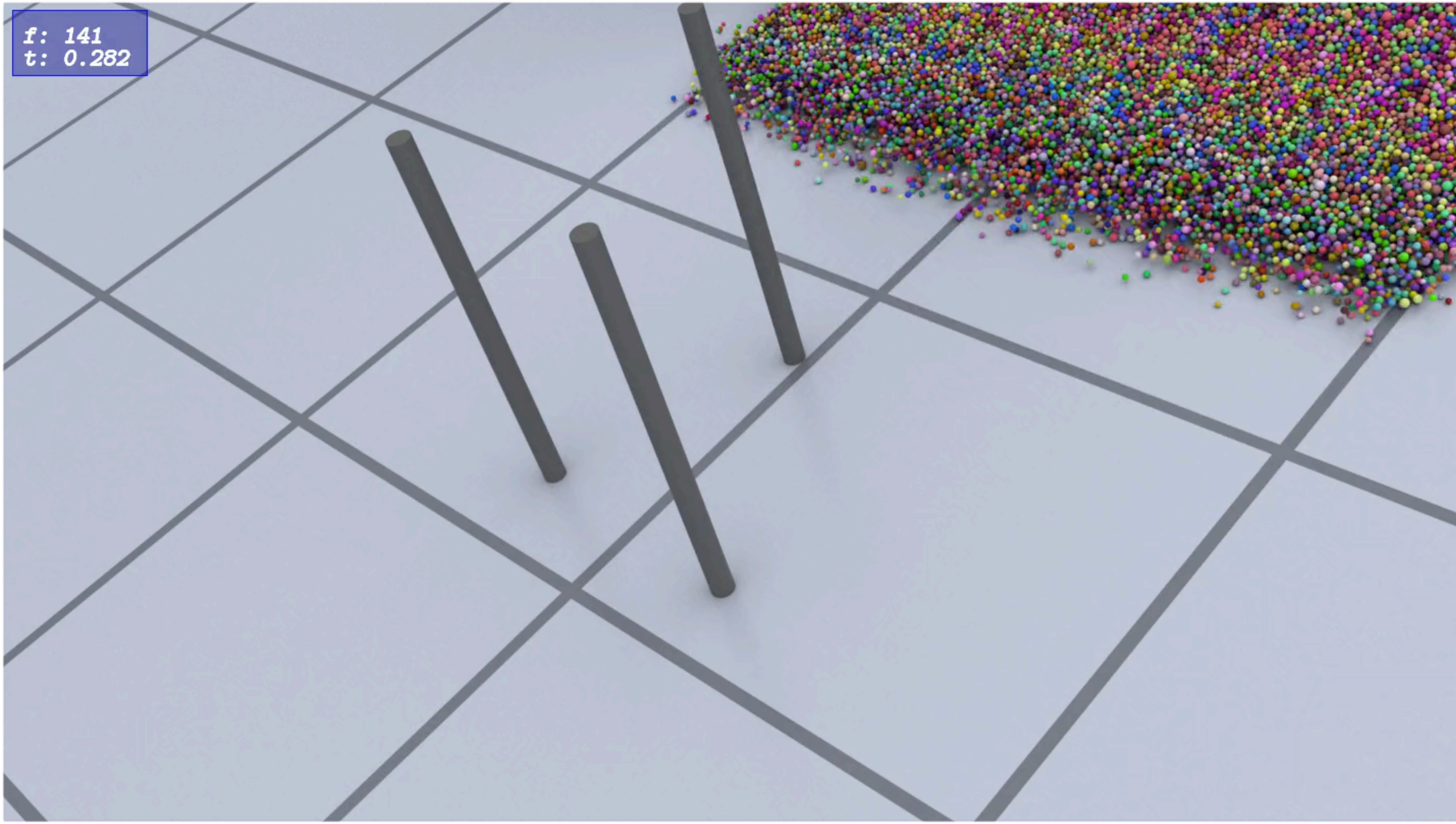
Velocity Magnitude



$\mu = 0.8 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter{=}1000} \mid \varepsilon_N, \varepsilon_T = 0 \mid n_T = 10^6$

best match

Velocity Magnitude



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Thanks for your attention!

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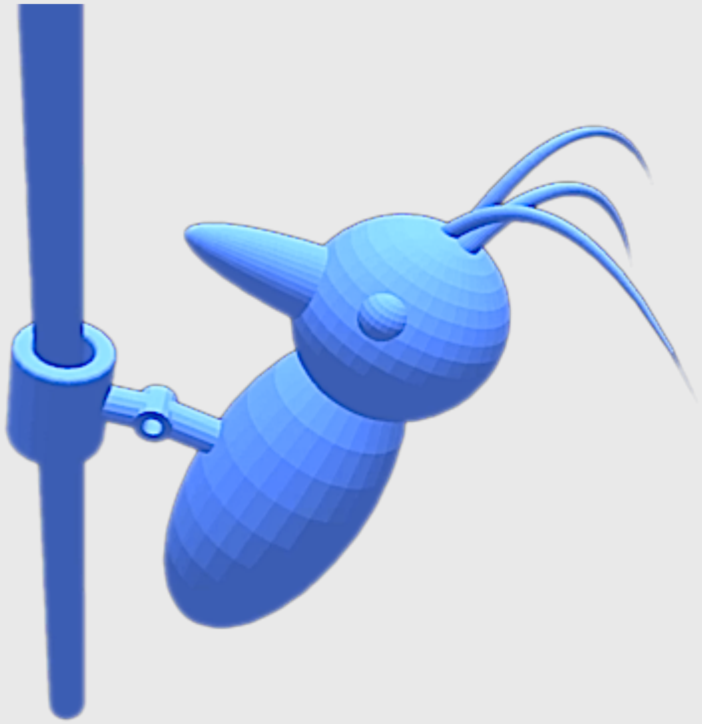

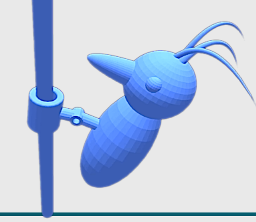

Quaternions for Rotations

$$\boldsymbol{\xi}_{\mathrm{rig}}(\boldsymbol{\rho}, t) := \mathcal{R}(t)(\boldsymbol{\rho}) + \mathbf{r}_R(t)$$

The quaternion space is a non-commutative field.

$$\mathbf{P} \in \mathbb{H} := \{p_0\mathbf{I} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\ , p_i \in \mathbb{R}\} \subset \mathbb{C}^{2\times 2}$$

scalar part

pure part

Basis: $(\mathbf{I}, \mathbf{i}, \mathbf{j}, \mathbf{k})$

Coordinates:

$$\mathcal{K}(\mathbf{P}) = \begin{bmatrix} p_0 \\ \mathbf{p}_r \end{bmatrix} \in \mathbb{R}^4$$

Left multiplication: $\mathbf{PX} \xrightarrow{\mathcal{K}} \varphi_L(\mathbf{P})\mathbf{x}$

Right multiplication: $\mathbf{XP} \xrightarrow[\mathcal{K}]{} \varphi_R(\mathbf{P})\mathbf{x}$

$$\varphi_R(\mathbf{P}) \in \mathbb{R}^{4\times 4}, \quad \mathbf{x} \in \mathbb{R}^4$$

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Rotations in $\mathbb{E}^4$ where $\mathbf{P}, \mathbf{Q}$ are unit quaternions (8 parameters + 2 constraints):

$$\begin{array}{rcl} \mathbb{H} & \to & \mathbb{H} \\ \mathbf{X} & \mapsto & \mathbf{PXQ} \end{array} \xrightarrow{\quad\mathcal{K}\quad} \begin{array}{rcl} \mathbb{R}^4 & \to & \mathbb{R}^4 \\ \mathbf{x} & \mapsto & \varphi_L(\mathbf{P})\varphi_R(\mathbf{Q})\mathbf{x} \end{array}$$

Quaternions for Rotations

The awesome quaternion sandwich: $\mathbf{PXQ} \longleftrightarrow \varphi_L(\mathbf{P})\varphi_R(\mathbf{Q})\mathbf{x}$

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Rotations in $\mathbb{E}^3$ where P is some quaternion and $\mathrm{Q} = \mathrm{P}^{-1}$:

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$$\begin{array}{ccc} \mathbb{R}^4 & \rightarrow & \mathbb{R}^4 \\ \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} & \mapsto & \begin{bmatrix} 0 \\ \mathbf{y} \end{bmatrix} = \frac{1}{s}\varphi_L(\mathbf{P})\varphi_R(\mathbf{P}^*)\begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \end{array} \qquad s := \|\mathbf{P}\|^2$$

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scalable body motion:

$${}_{\mathrm{I}}\boldsymbol{\xi}_{\mathrm{scal}} = \boxed{s\ {}_{\mathrm{I}}\mathbf{R}(\mathbf{p})}\ {}_{\mathrm{I}}\boldsymbol{\rho} + {}_{\mathrm{I}}\mathbf{r}_R$$

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constraint: $\|\mathbf{p}\| = 1 \quad \forall t$

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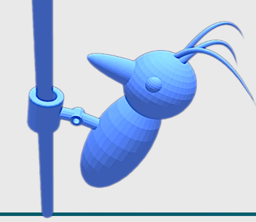

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equation of motions
by principles

Applying the Fundamental Axioms

Equations of motion for the rigid body:

rigid body motion: $\boxed{{}_{\mathrm{I}}\boldsymbol{\xi}_{\mathrm{rig}} = {}_{\mathrm{I}}\mathbf{R}(\mathbf{p})\ {}_{\mathrm{I}}\boldsymbol{\rho} + {}_{\mathrm{I}}\mathbf{r}_R}$

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$$\boxed{0 = \delta W({}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{rig}}) = \delta W^{\mathrm{dyn}} + \delta W_I^{\mathrm{int}} + \delta W_C^{\mathrm{int}} + \delta W^{\mathrm{ext}} \qquad \forall {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{rig}},\ t}$$

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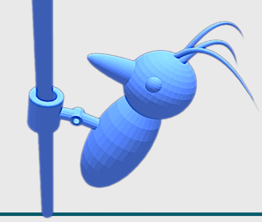
constitutive force law:
Newton's second law



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law of interaction
(S. Eugster 2014)

$$0 = -\int_B {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{rig}}{}^{\top}\, {}_{\mathrm{I}}\ddot{\boldsymbol{\xi}}_{\mathrm{rig}}\,\mathrm{d}m \quad + \quad 0$$

constitutive force law:
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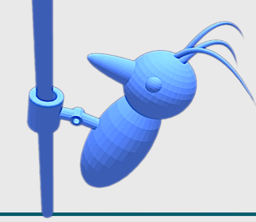

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law of interaction
(S. Eugster 2014)

perfect constraint forces $(\infty \to 6)$ by d'Alembert-Lagrange

$$0 = \boxed{-\int_B {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{rig}}{}^{\top}\; \boxed{{}_{\mathrm{I}}\ddot{\boldsymbol{\xi}}_{\mathrm{rig}}\,\mathrm{d}m}} + 0 + 0$$

constitutive force law:
Newton's second law

Applying the Fundamental Axioms

Equations of motion for the rigid body:

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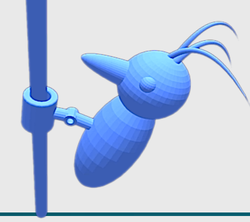

Applying the Fundamental Axioms

Equations of motion for the rigid body:

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law of interaction
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constitutive force law:
Newton's second law

velocities: ${}_\mathrm{I}\dot{\boldsymbol{\xi}}_\mathrm{rig} = {}_\mathrm{K}\mathbf{R}\ {}_\mathrm{K}\tilde{\boldsymbol{\Omega}}\ {}_\mathrm{K}\boldsymbol{\rho}_t + {}_\mathrm{I}\dot{\mathbf{r}}_R$

Applying the Fundamental Axioms

Equations of motion for the rigid body:

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$$\mathbf{q}(t) := \begin{bmatrix} {}_{\mathrm{I}}\mathbf{r}_R \\ \mathbf{p} \end{bmatrix} \in \mathbb{R}^7$$

$$0 = \delta W({}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{rig}}) = \delta W^{\mathrm{dyn}} + \delta W_I^{\mathrm{int}} + \delta W_C^{\mathrm{int}} + \delta W^{\mathrm{ext}} \qquad \forall {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{rig}},\ t$$

law of interaction (S. Eugster 2014)

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Newton's second law

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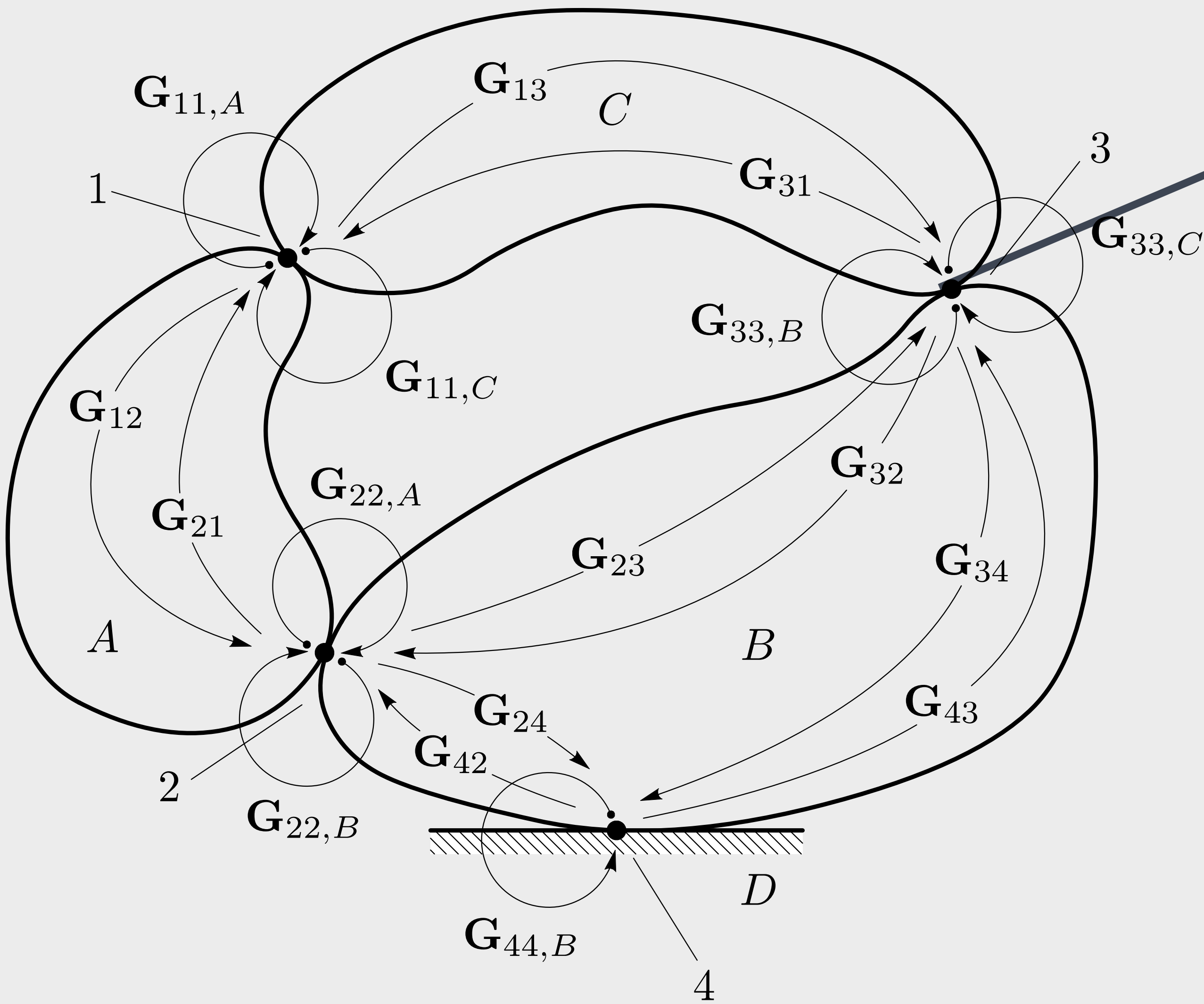
generalized velocities: $$\mathbf{u} := \begin{bmatrix} {}_{\mathrm{I}}\dot{\mathbf{r}}_R \\ {}_{\mathrm{K}}\boldsymbol{\Omega} \end{bmatrix} \in \mathbb{R}^6$$

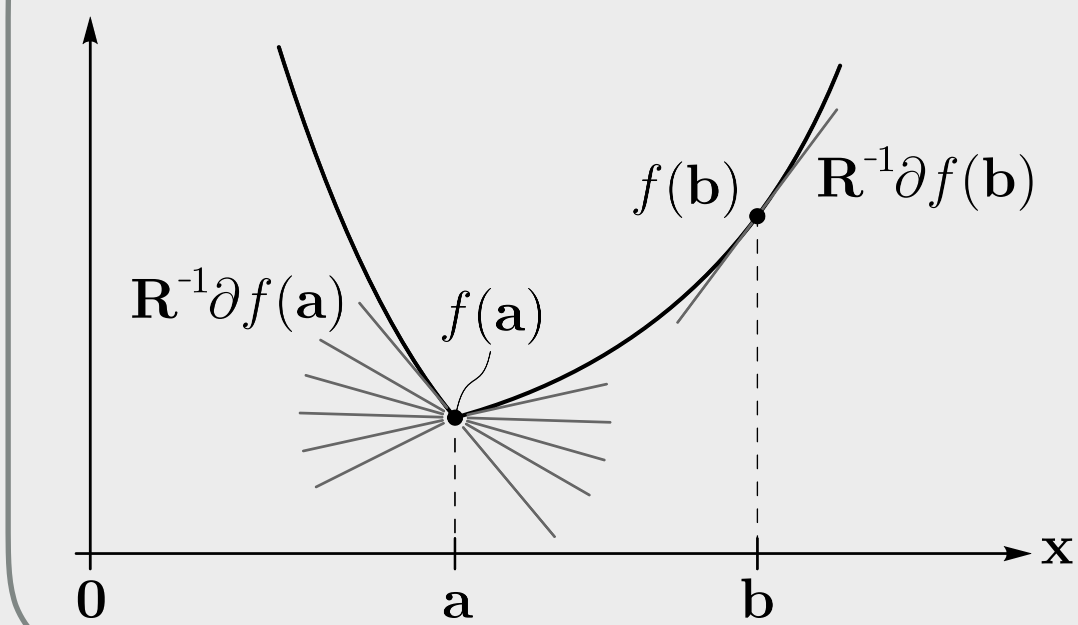
Non-Smooth Dynamics in a Nutshell



contact law

$$-\boldsymbol{\gamma}_i \in \mathcal{N}_{\mathcal{C}_i}(\boldsymbol{\lambda})$$

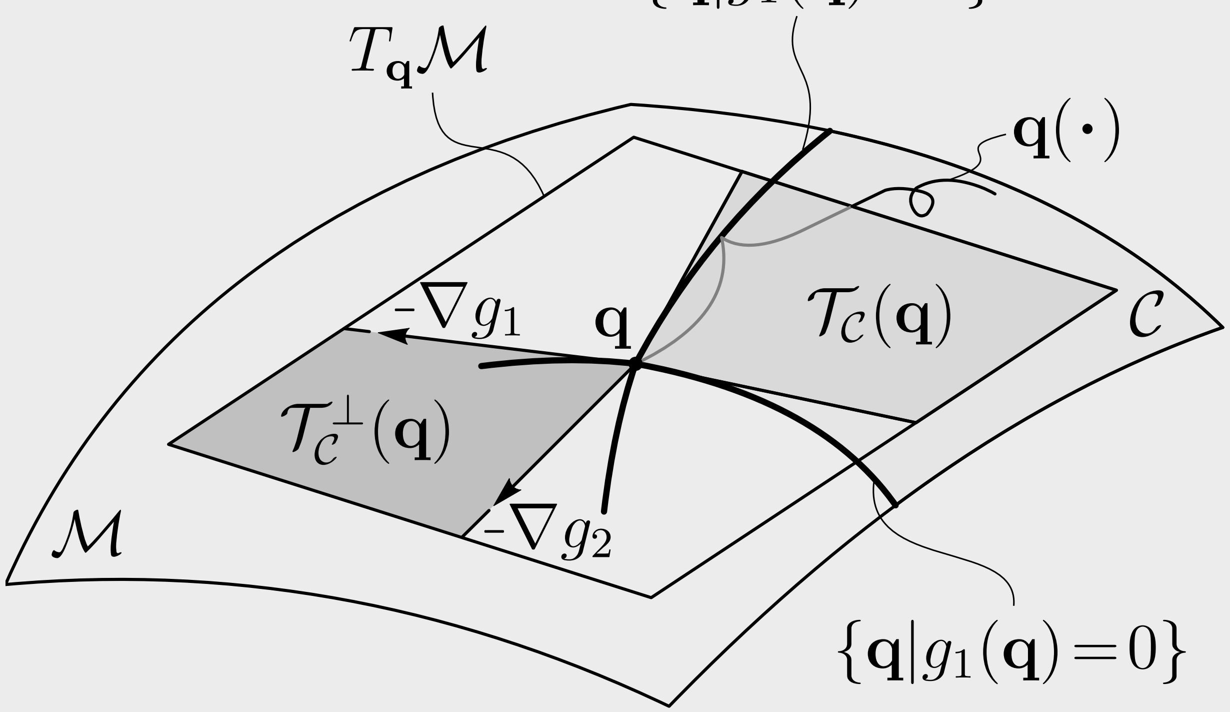
subdifferential

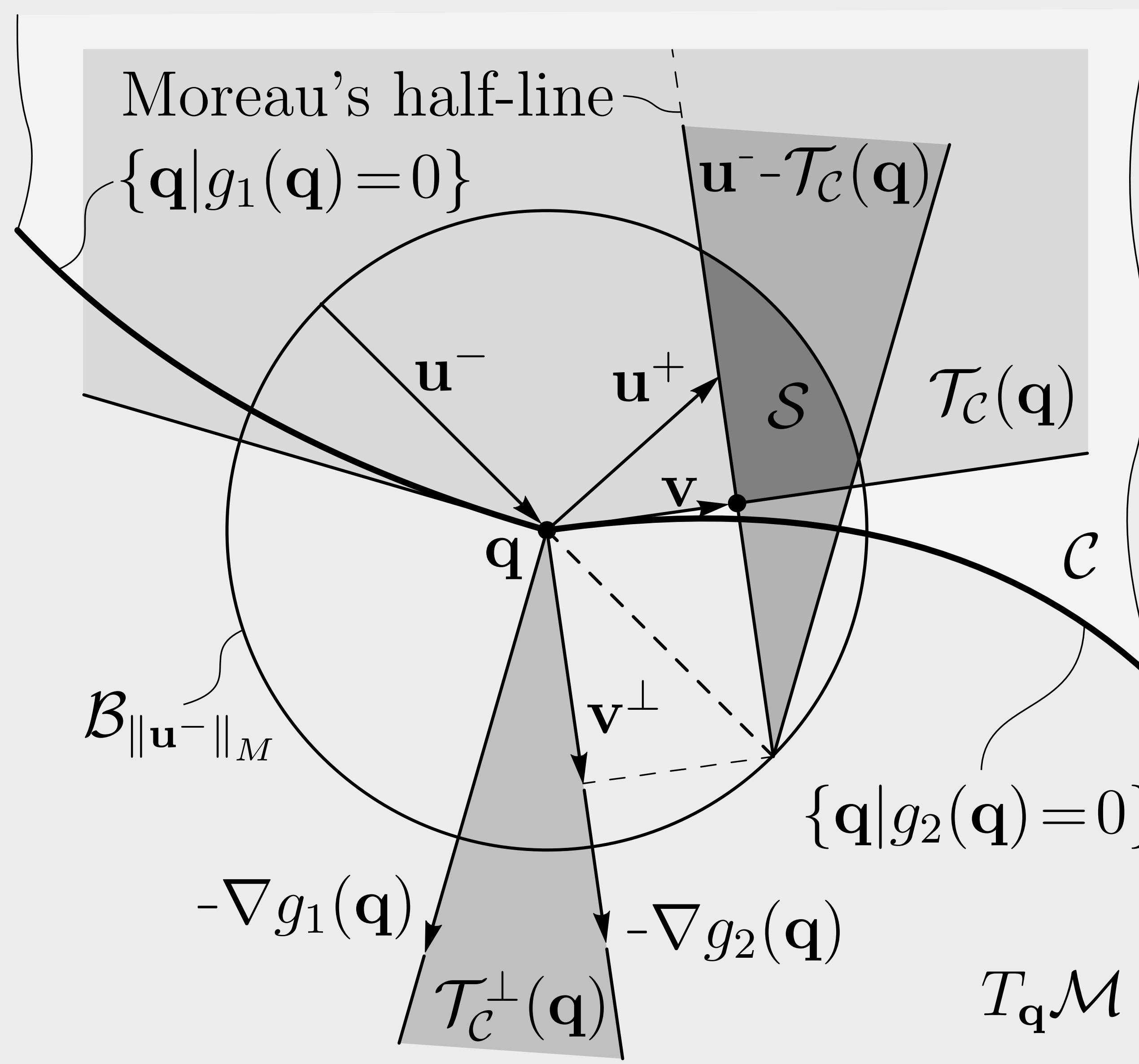


$$\min_{\boldsymbol{\lambda}\in\mathcal{C}} f(\mathbf{x}) = \min_{\boldsymbol{\lambda}\in\mathcal{C}} \frac{1}{2}\boldsymbol{\lambda}^\top \mathbf{G}\boldsymbol{\lambda} + \mathbf{c}^\top\boldsymbol{\lambda} \quad \Leftrightarrow \quad \min_{\boldsymbol{\lambda}} \frac{1}{2}\boldsymbol{\lambda}^\top \mathbf{G}\boldsymbol{\lambda} + \mathbf{c}^\top\boldsymbol{\lambda} + I_{\mathcal{C}}(\boldsymbol{\lambda})$$

necesssary condition for minimizer: $-\mathbf{df}(\mathbf{x}^*) \in \partial I_{\mathcal{C}}(\boldsymbol{\lambda}) \equiv \mathcal{N}_{\mathcal{C}}(\mathbf{x}^*)$

Non-Smooth Dynamics in a Nutshell





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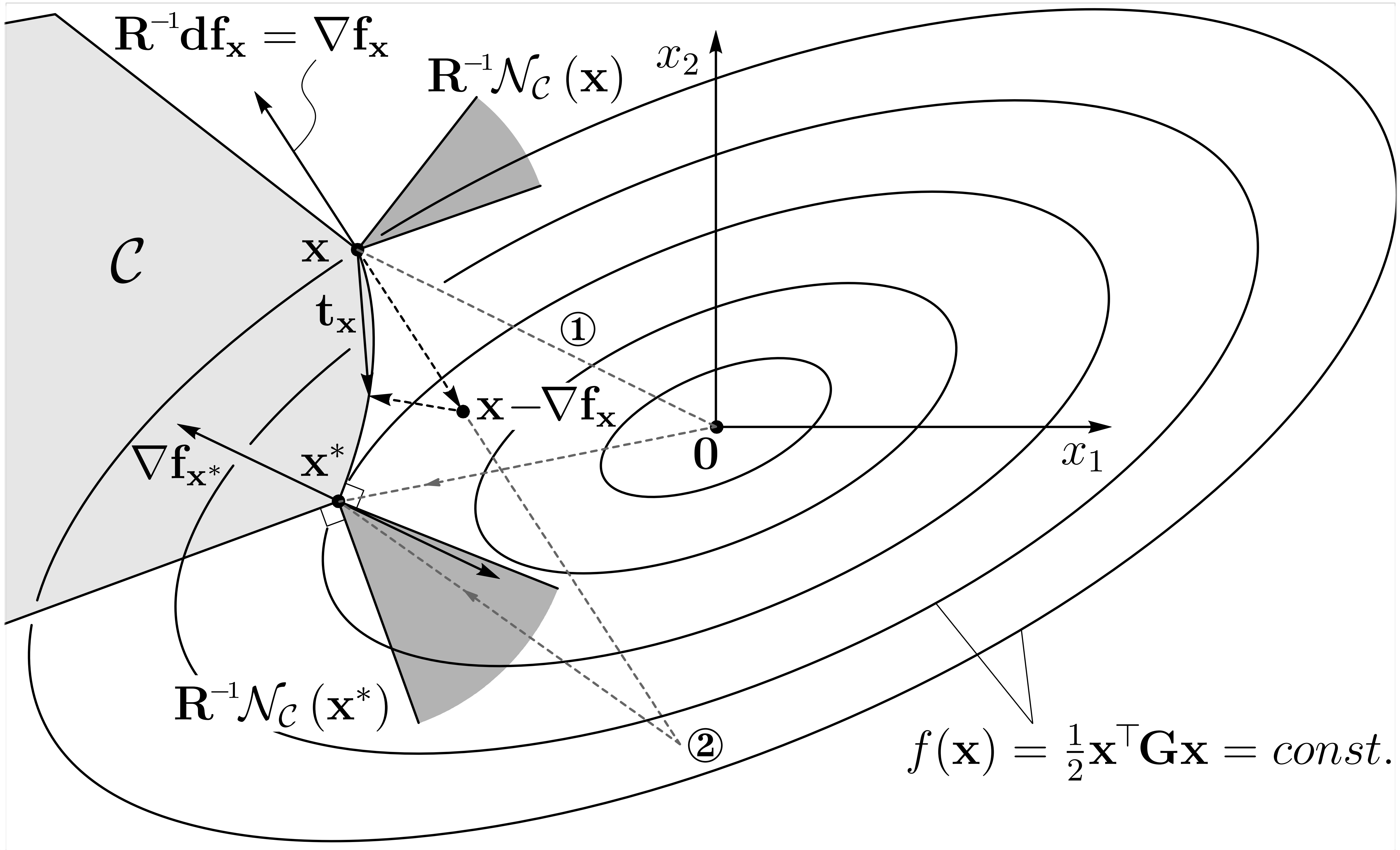
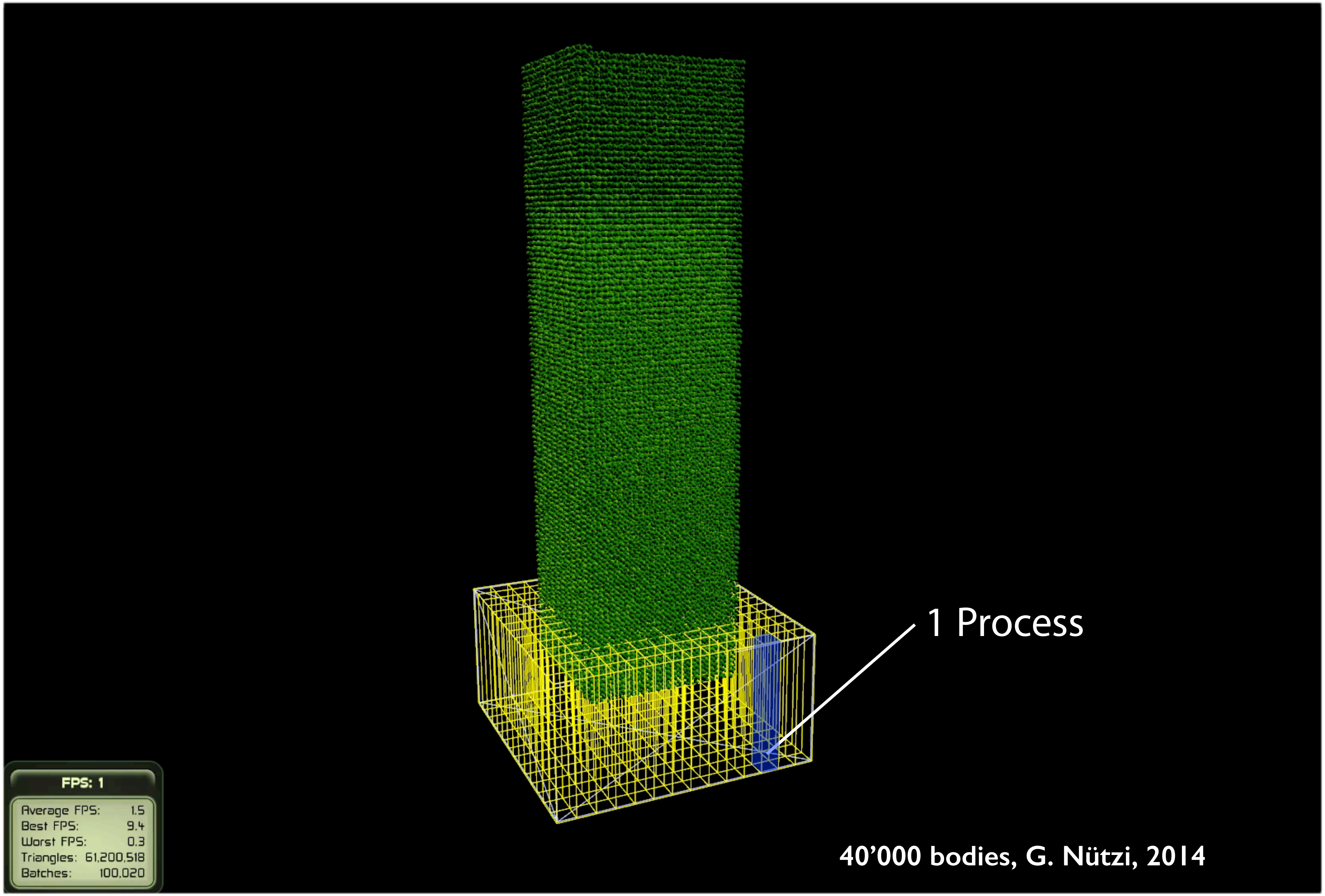


Force Laws for Contacts



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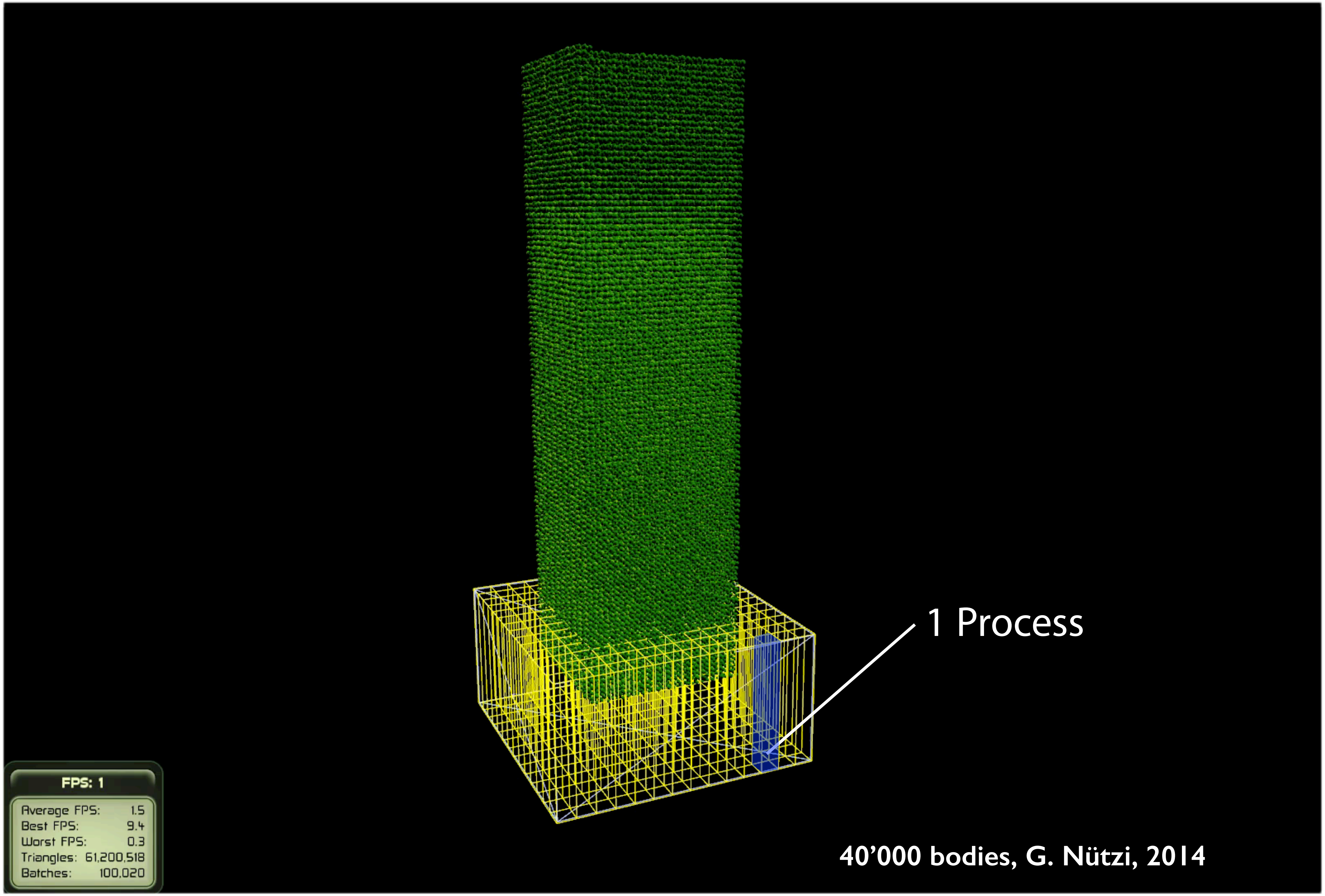

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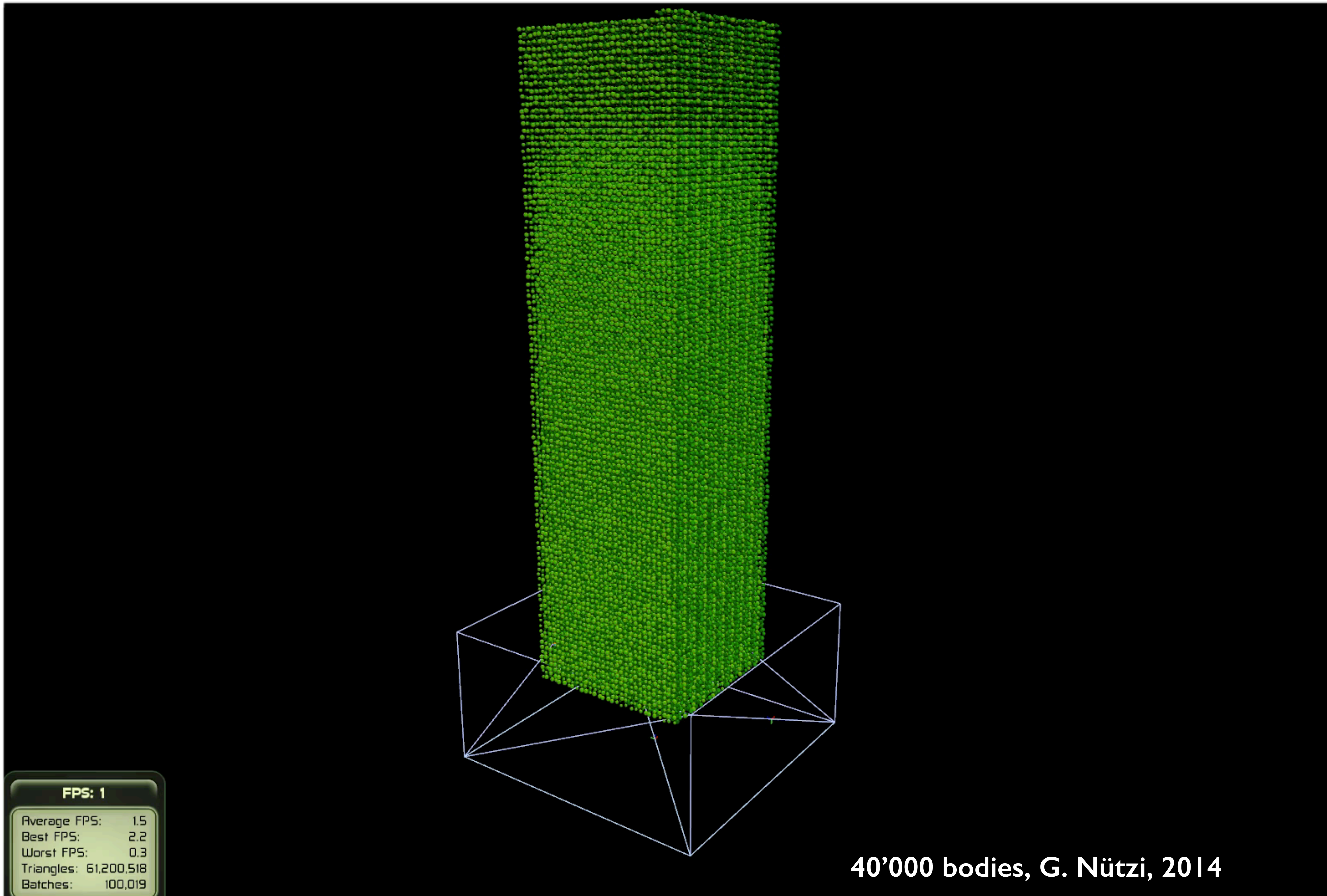

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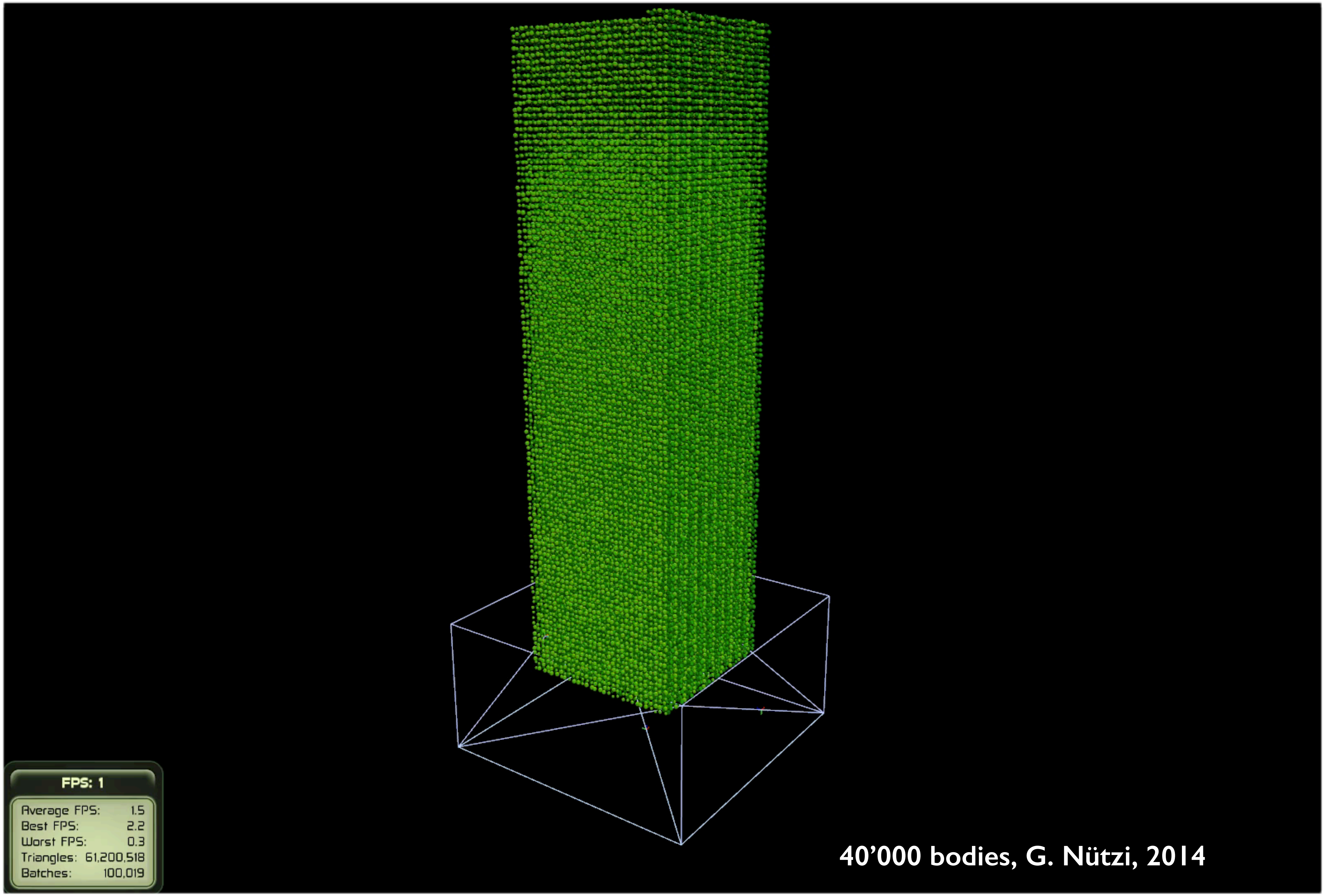
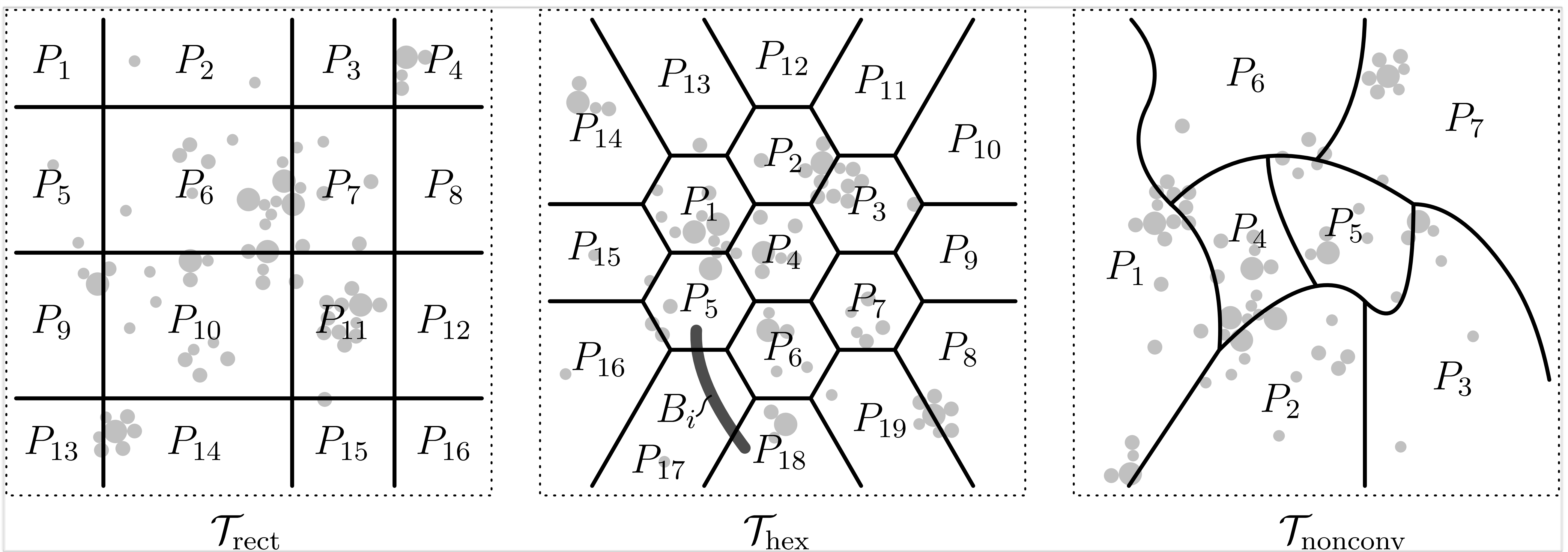


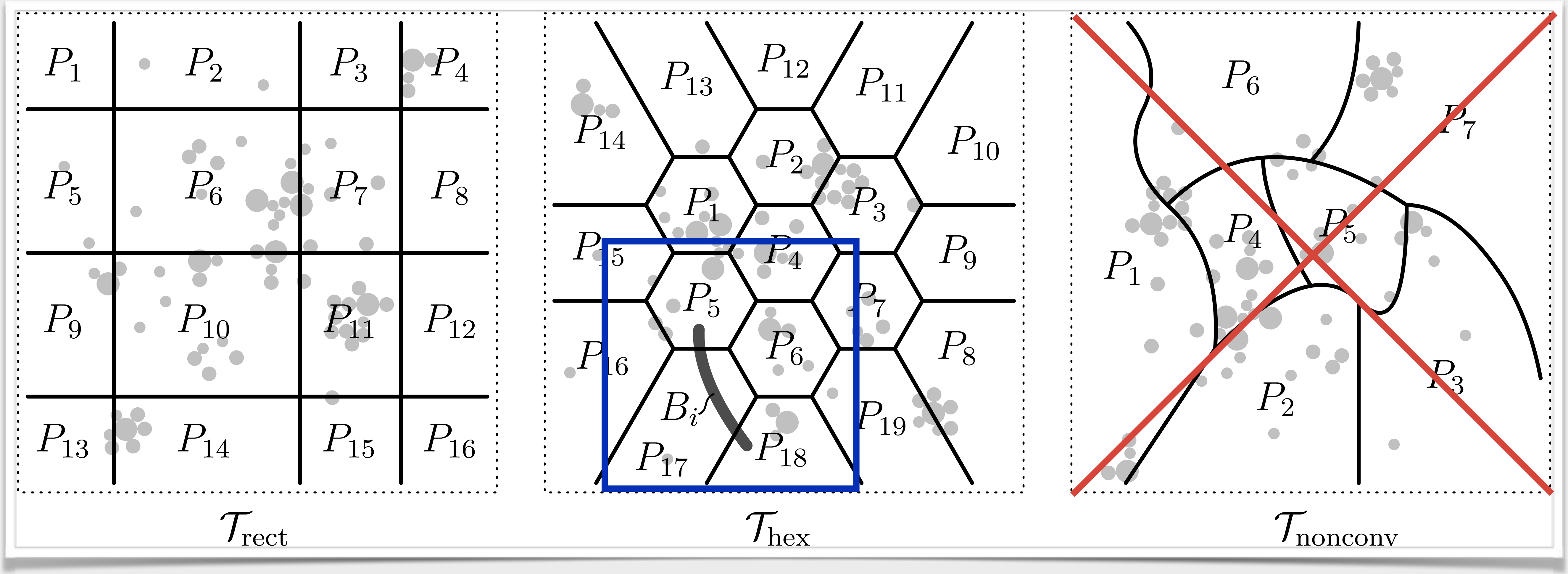
The Granular Rigid Body Simulation Framework

Parallel Implementation: Domain Decomposition



The Granular Rigid Body Simulation Framework

Parallel Implementation: Domain Decomposition



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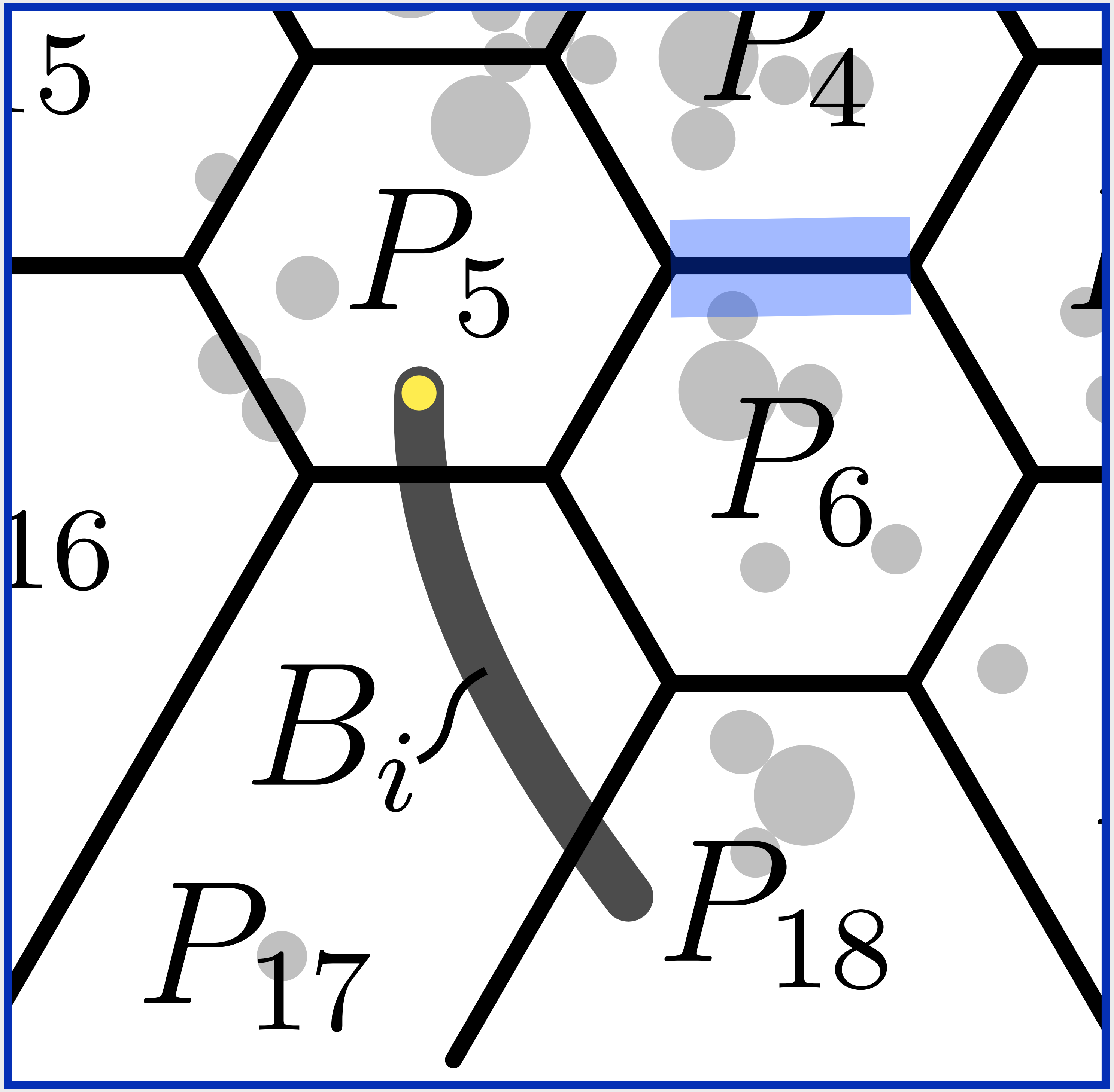

- communicate only with direct neighbourhood
- **restriction** on body size
- **no** epsilon boundary region as in **pe** engine or **chrono**

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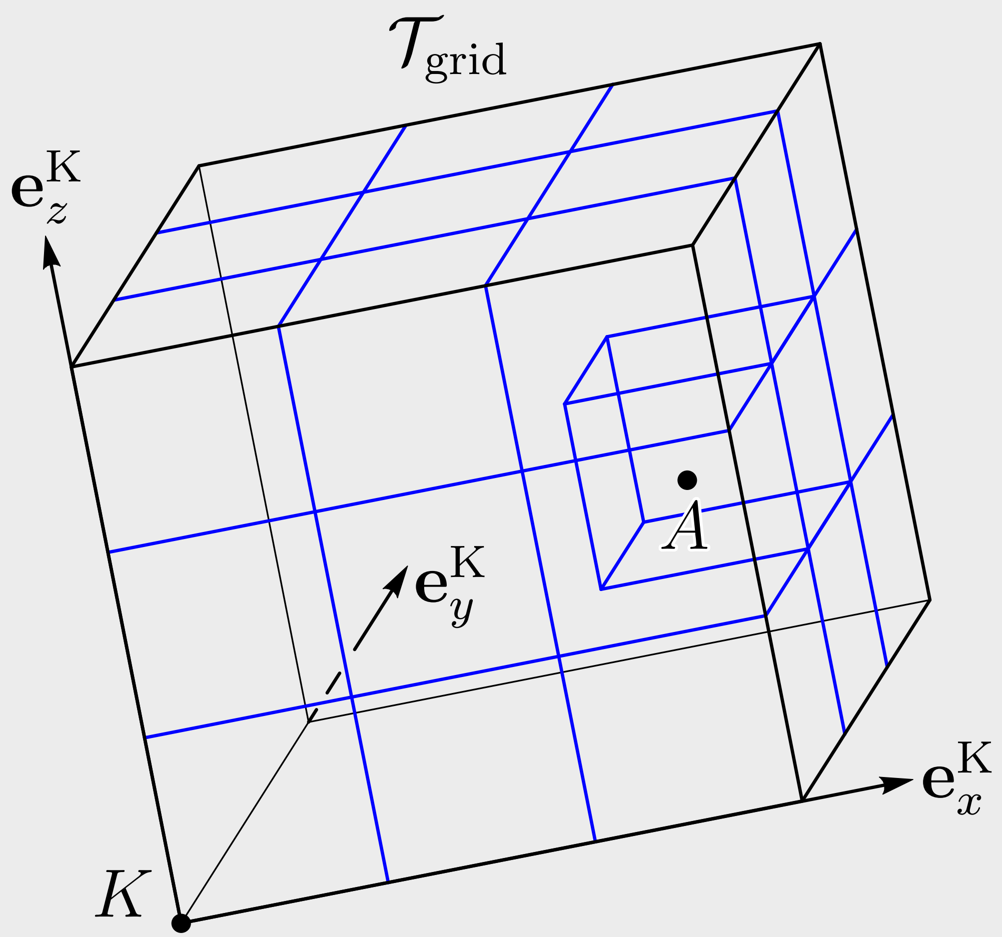

- **unbalanced** workload
- simple data structure
- 26 neighbors

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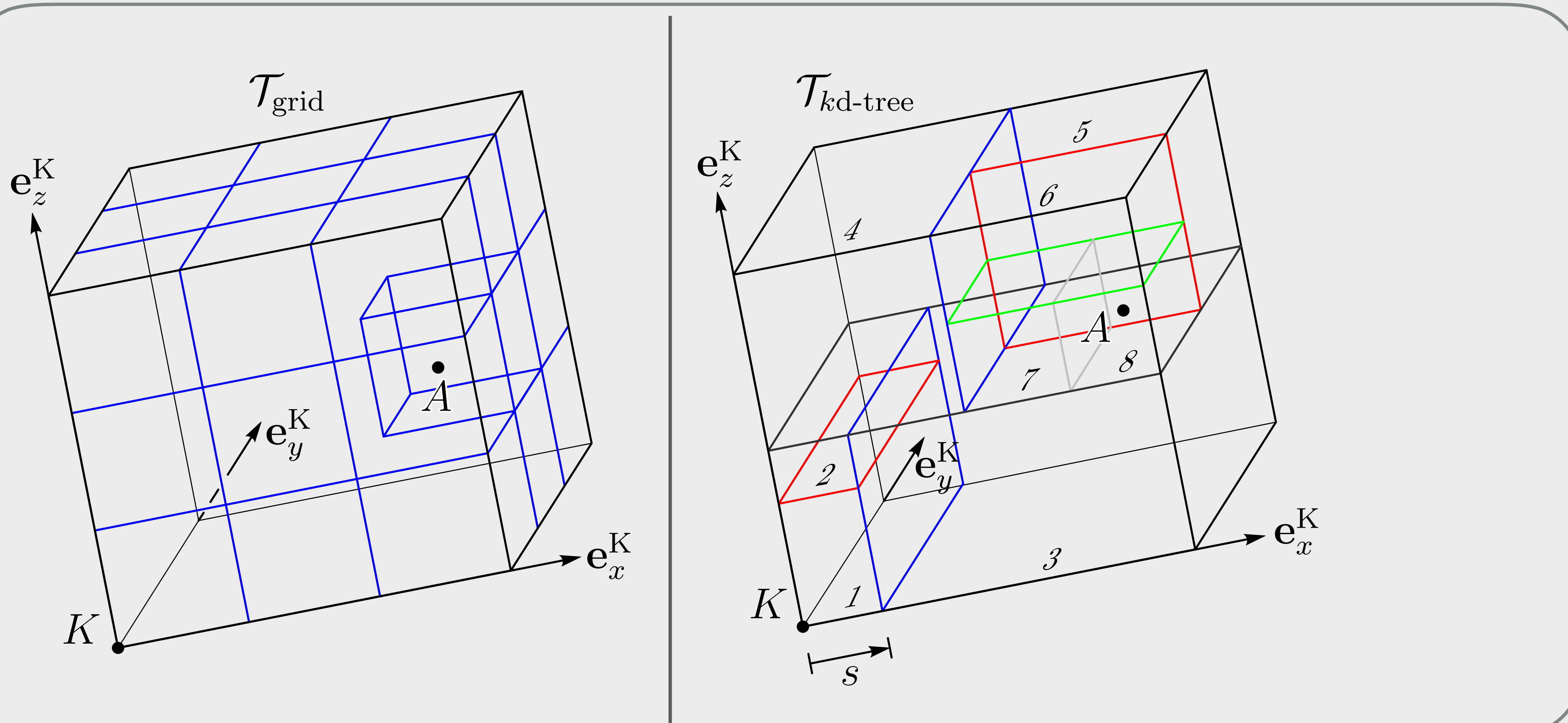

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- **better** balanced workload
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- **<26 neighbors** (mostly)

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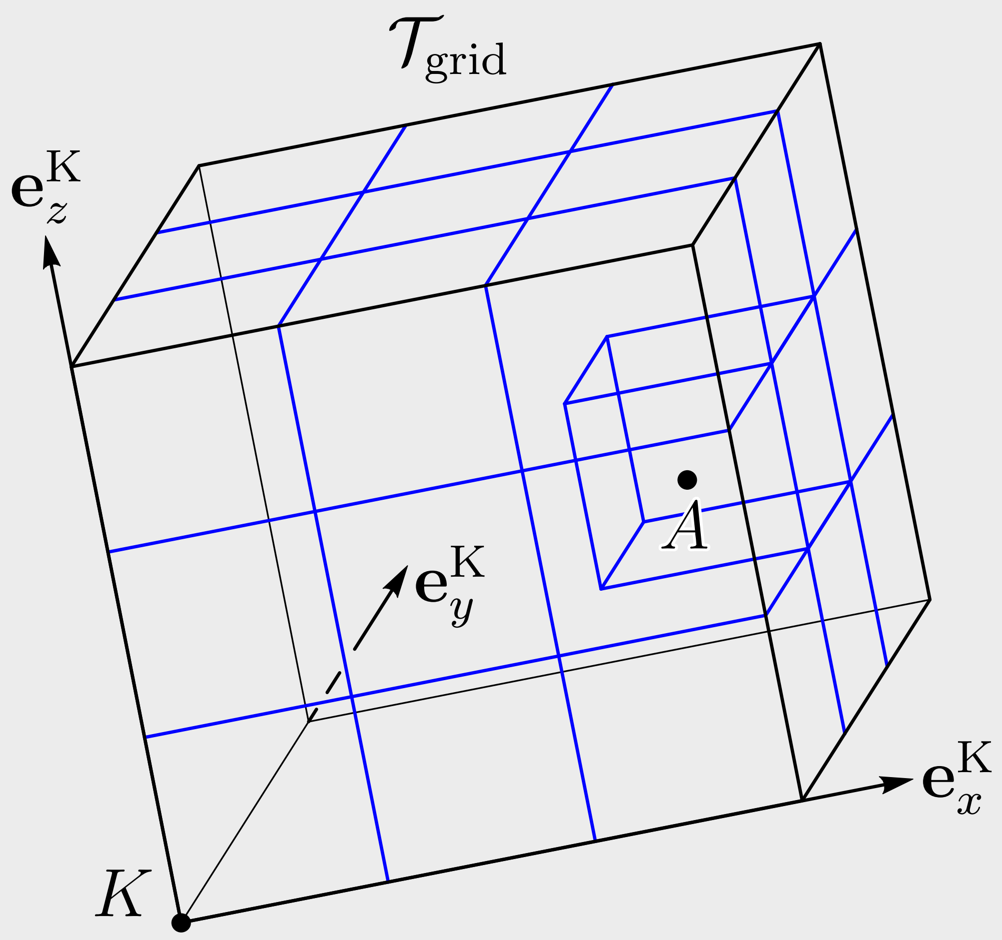

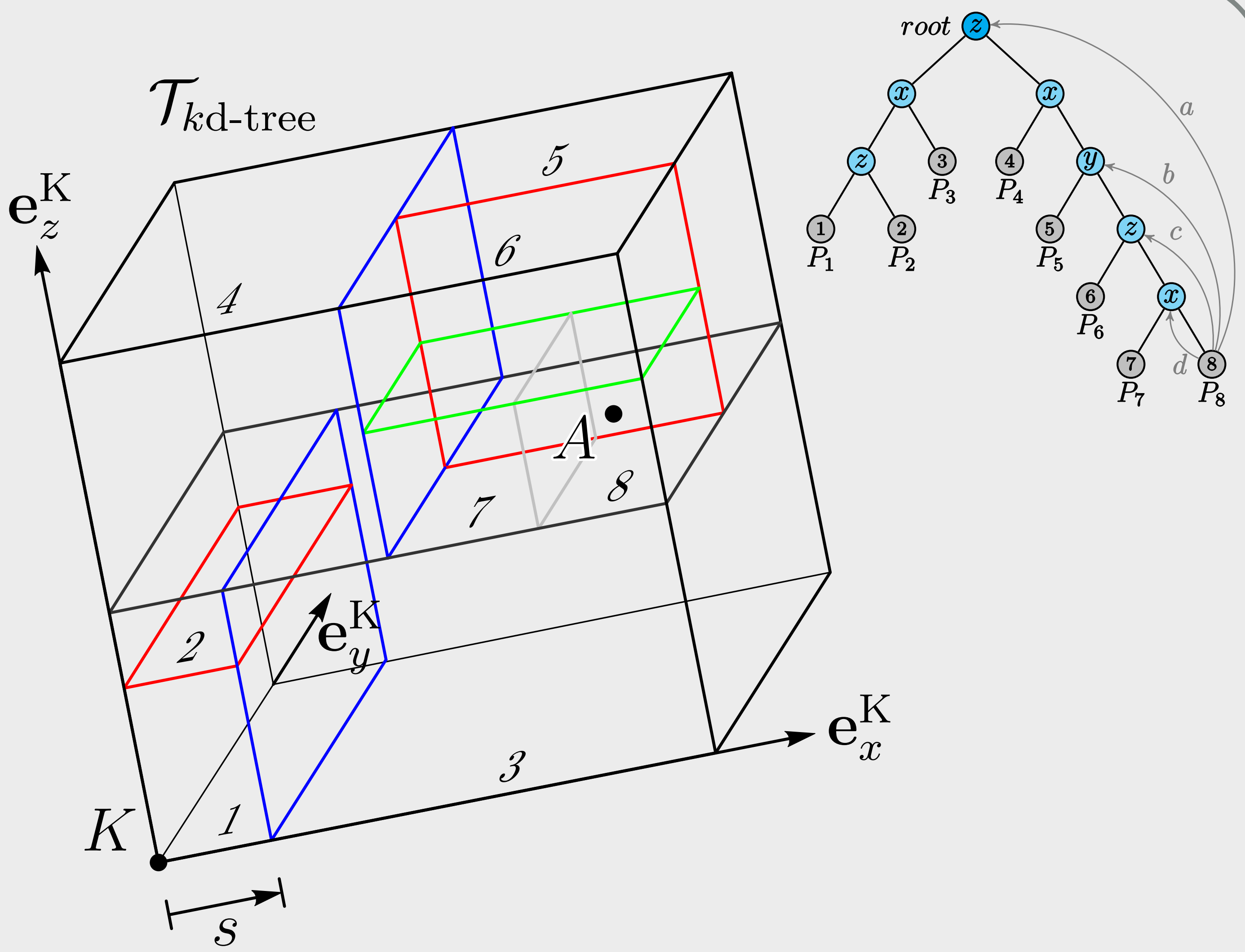

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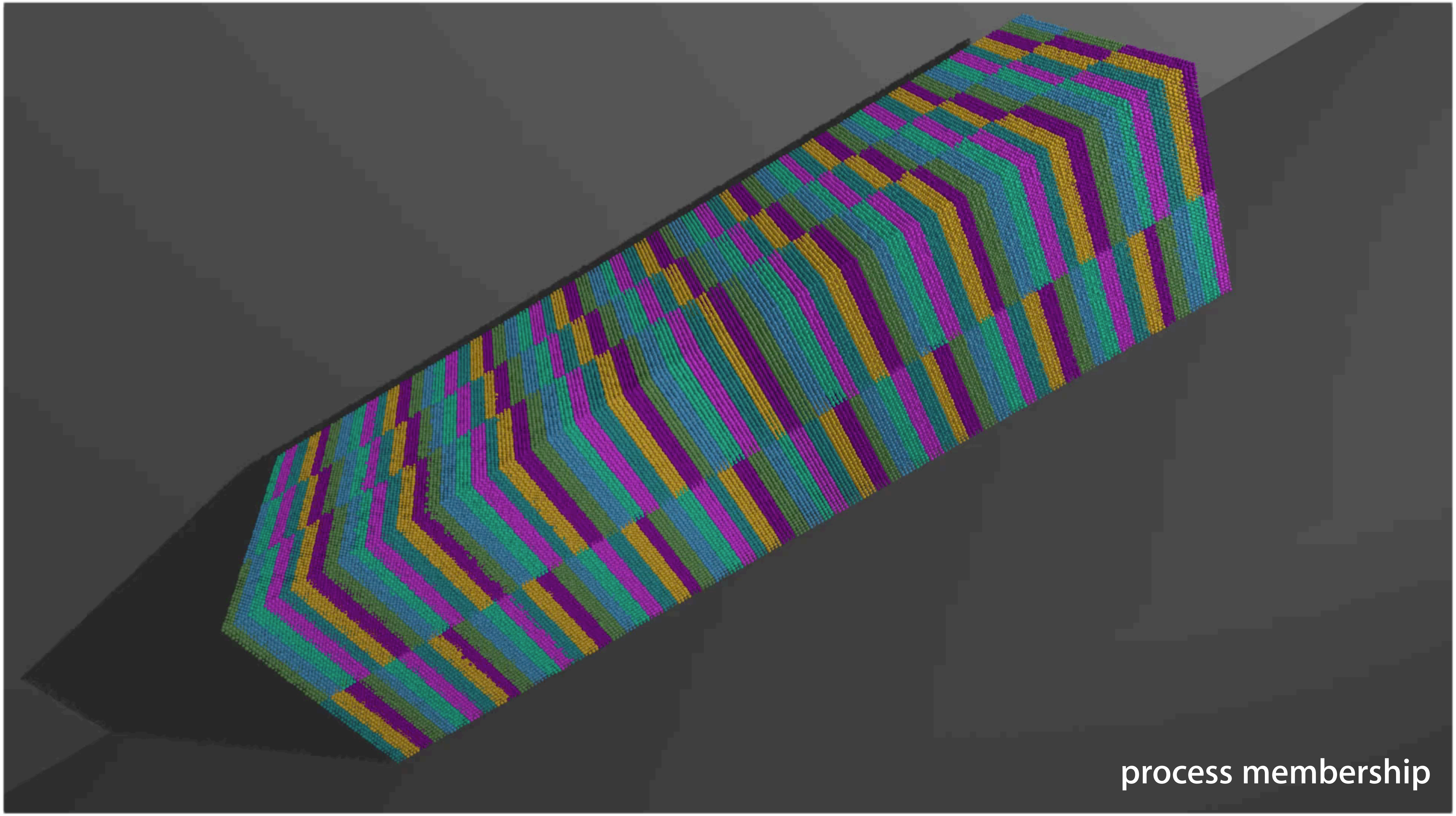
Why Domain Decomposition / Load Balancing?



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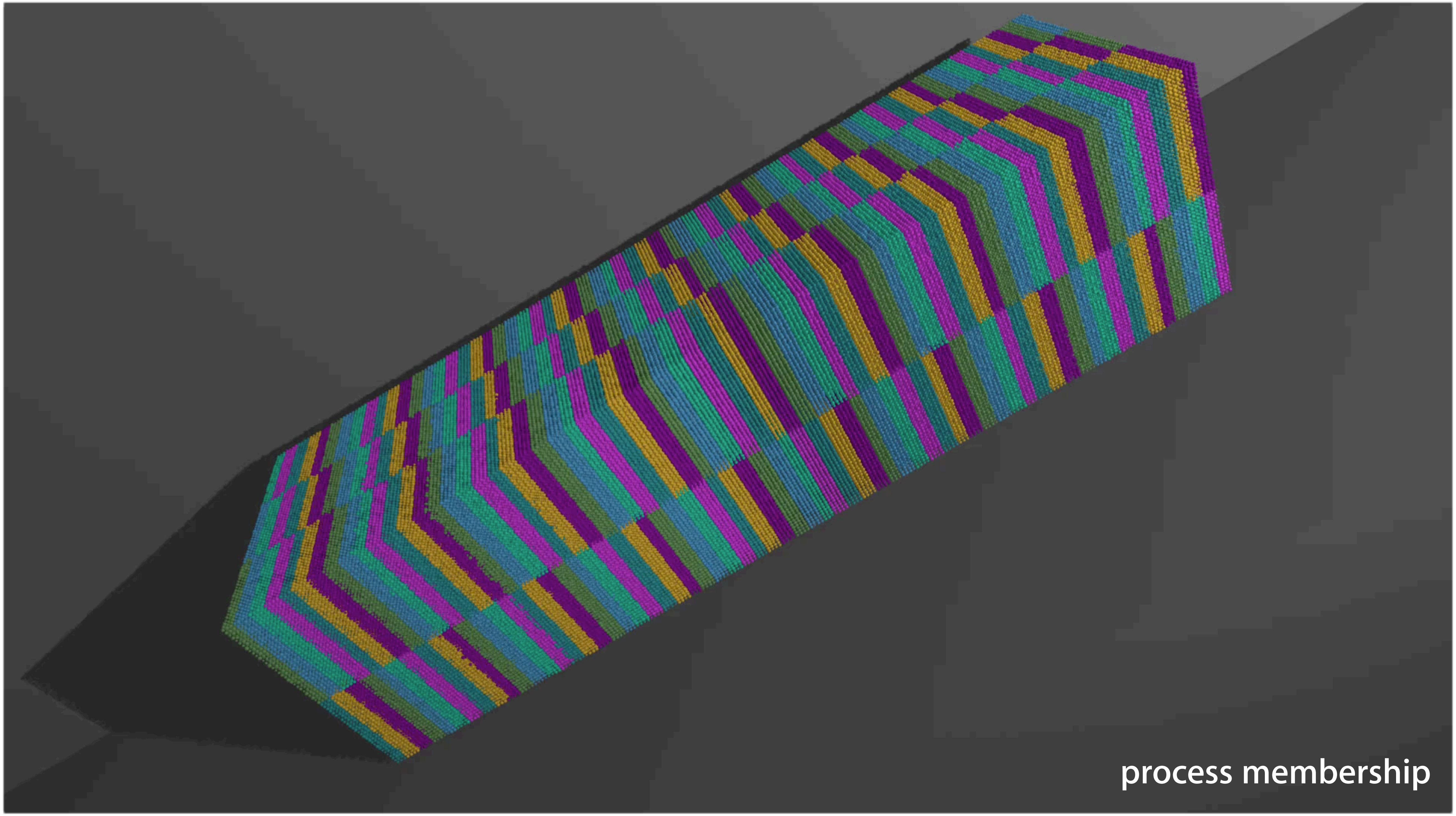

The Granular Rigid Body Simulation Framework

GRSF Converter

Generic Data Conversion Tool

The Granular Rigid Body Simulation Framework

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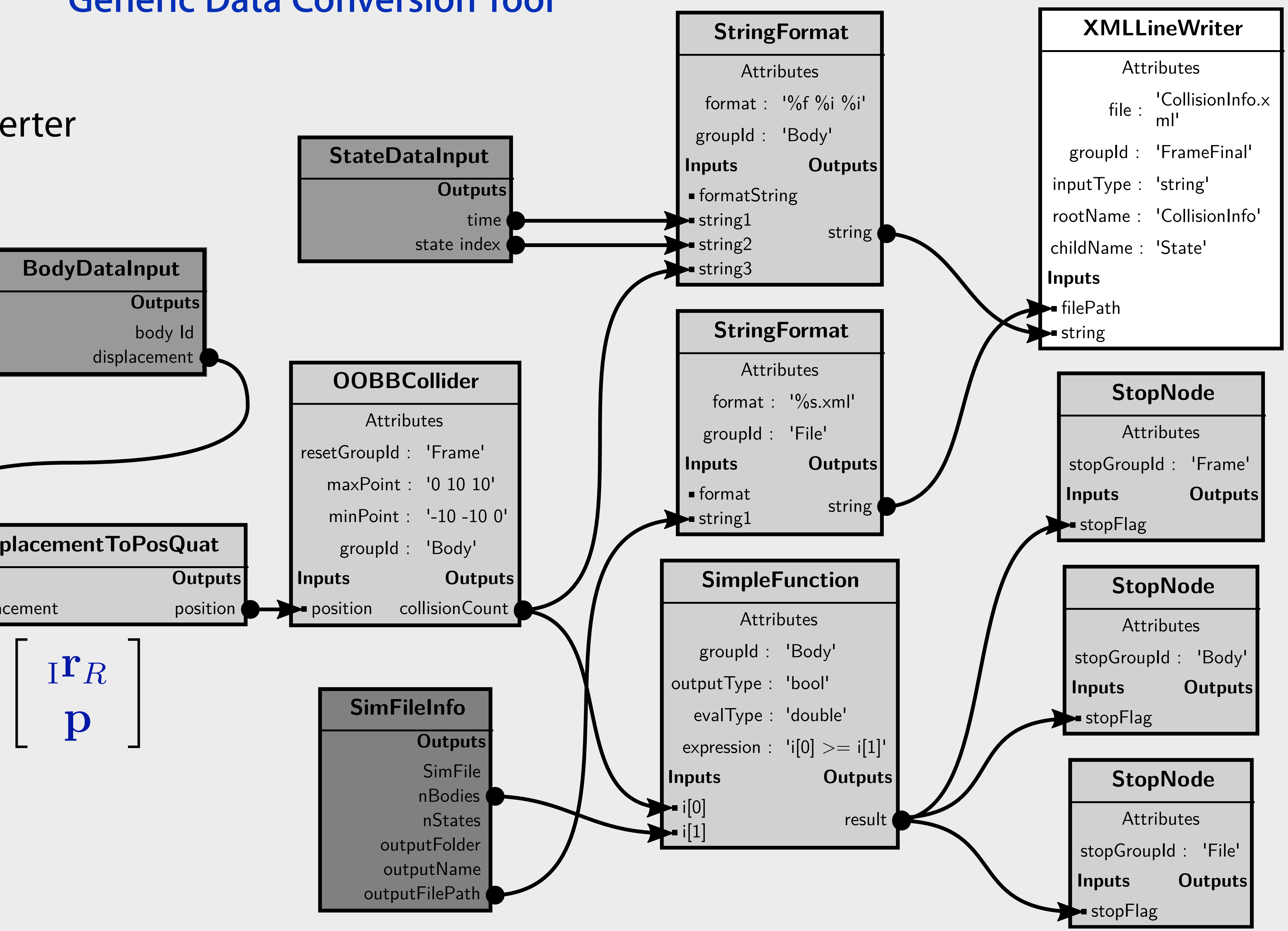

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GRSF Converter

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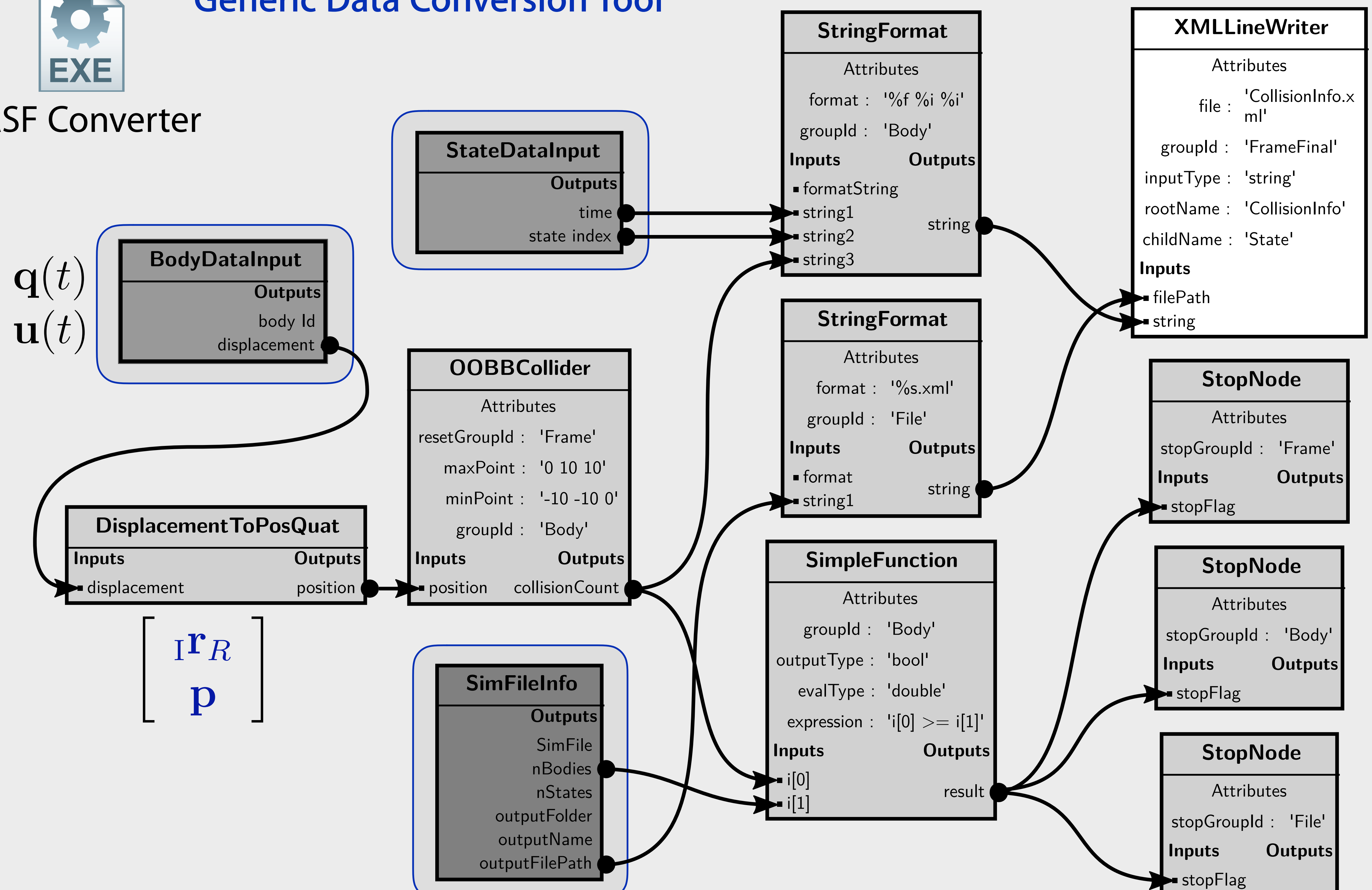

The Granular Rigid Body Simulation Framework

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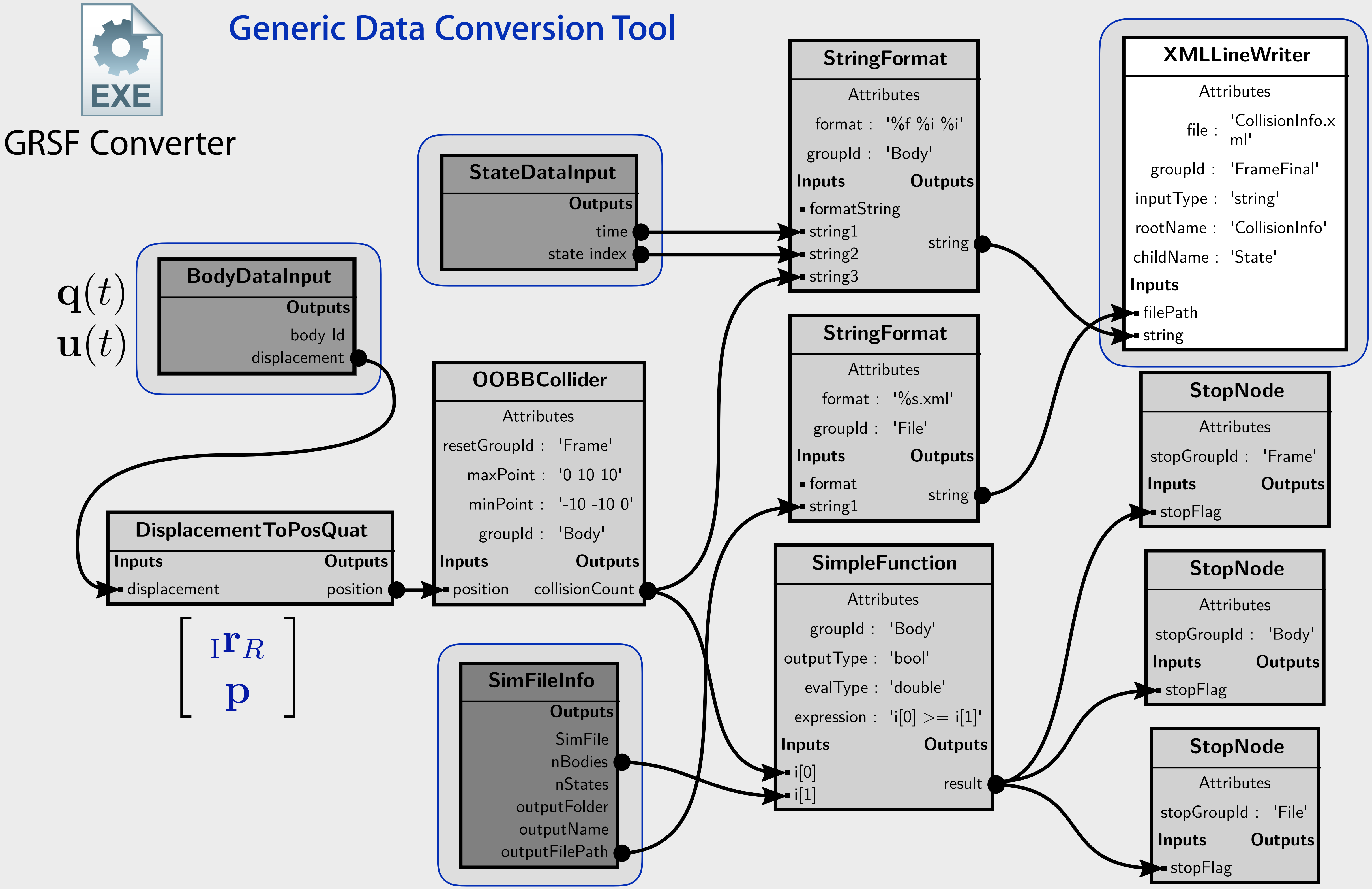


The Granular Rigid Body Simulation Framework

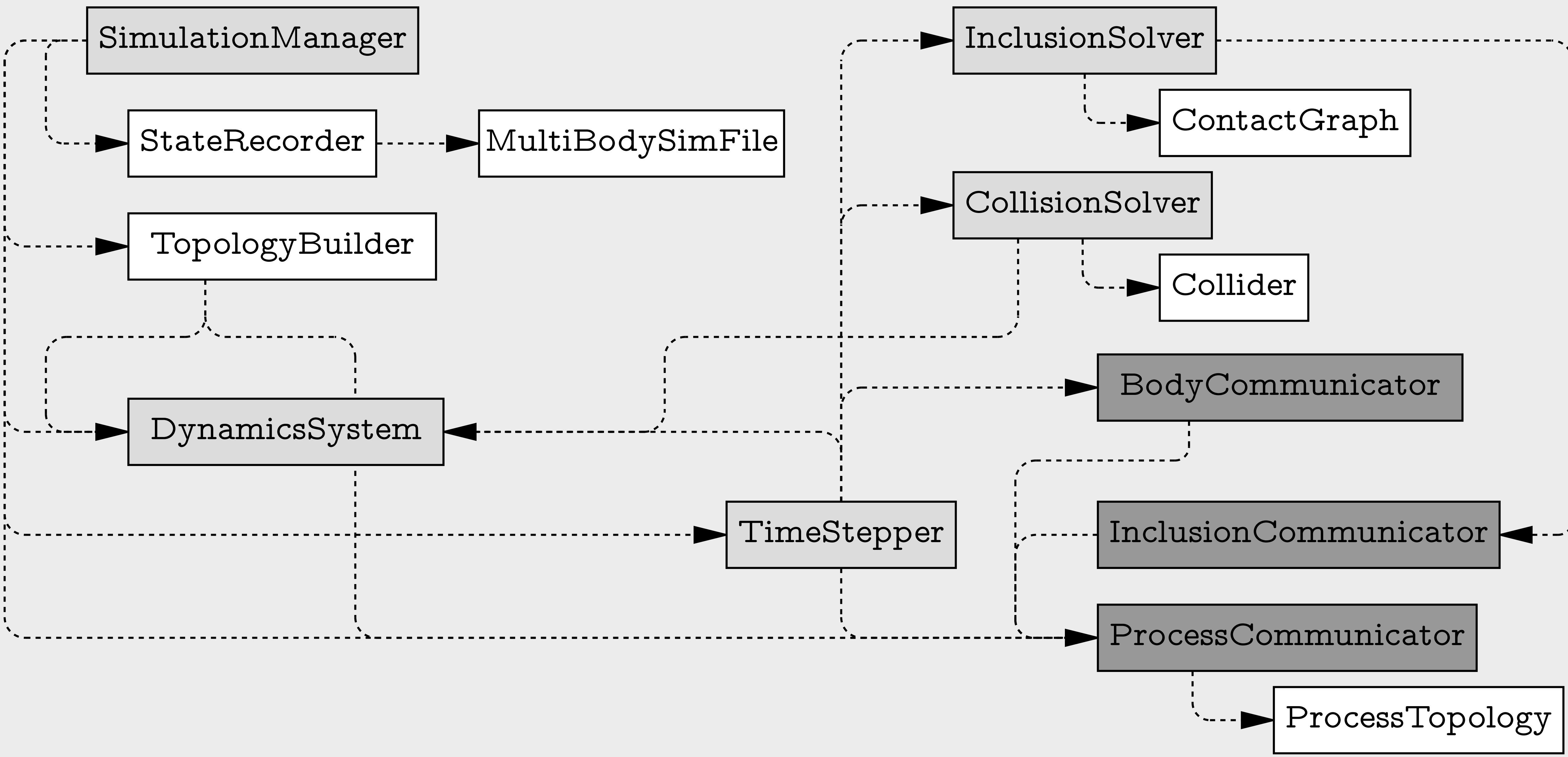
EXE

GRSF Converter

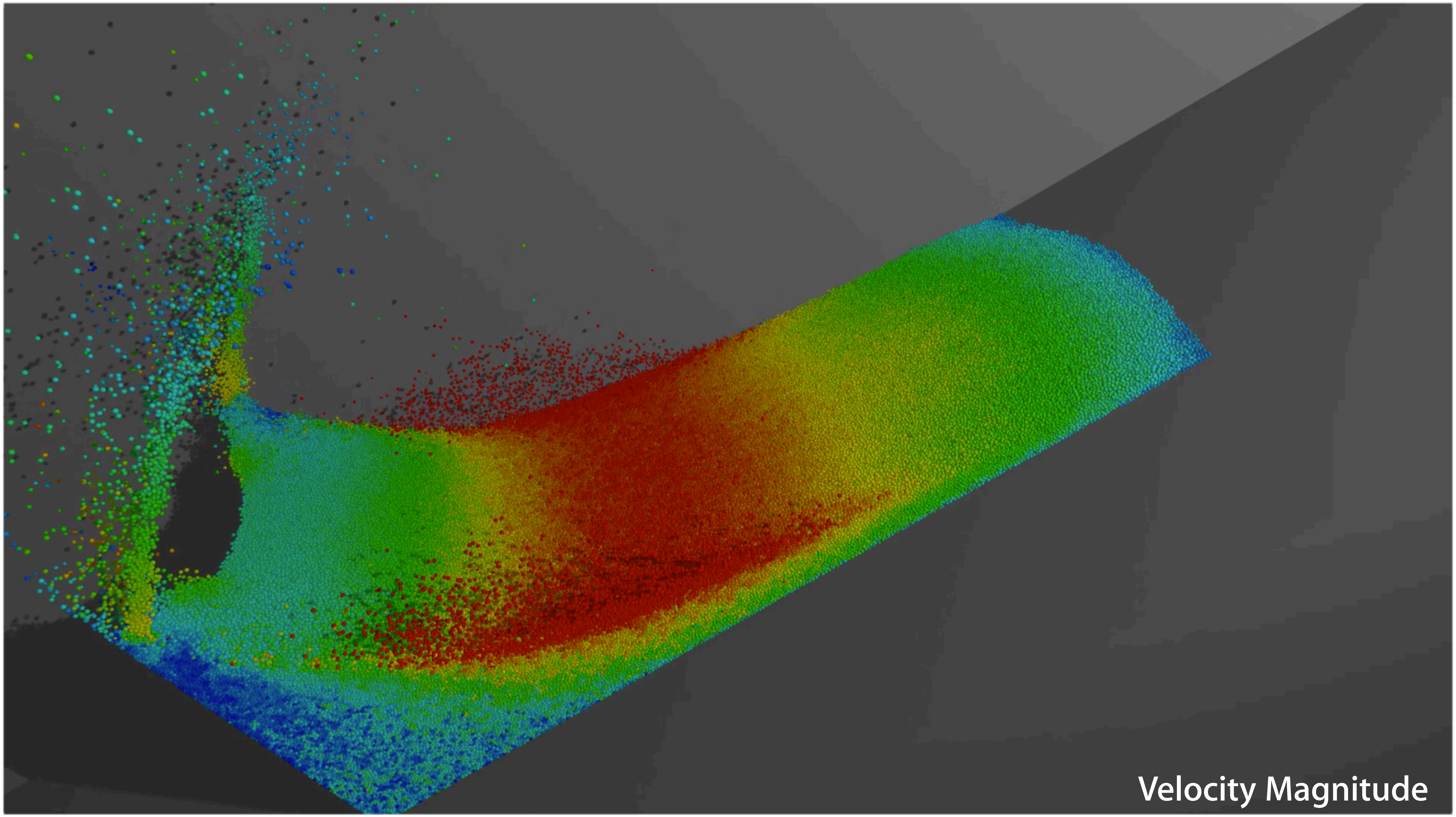
Generic Data Conversion Tool

Google TensorFlow™
2015
tensorflow.org

The Granular Rigid Body Simulation Framework



Drift Correction (cheap on velocity level)



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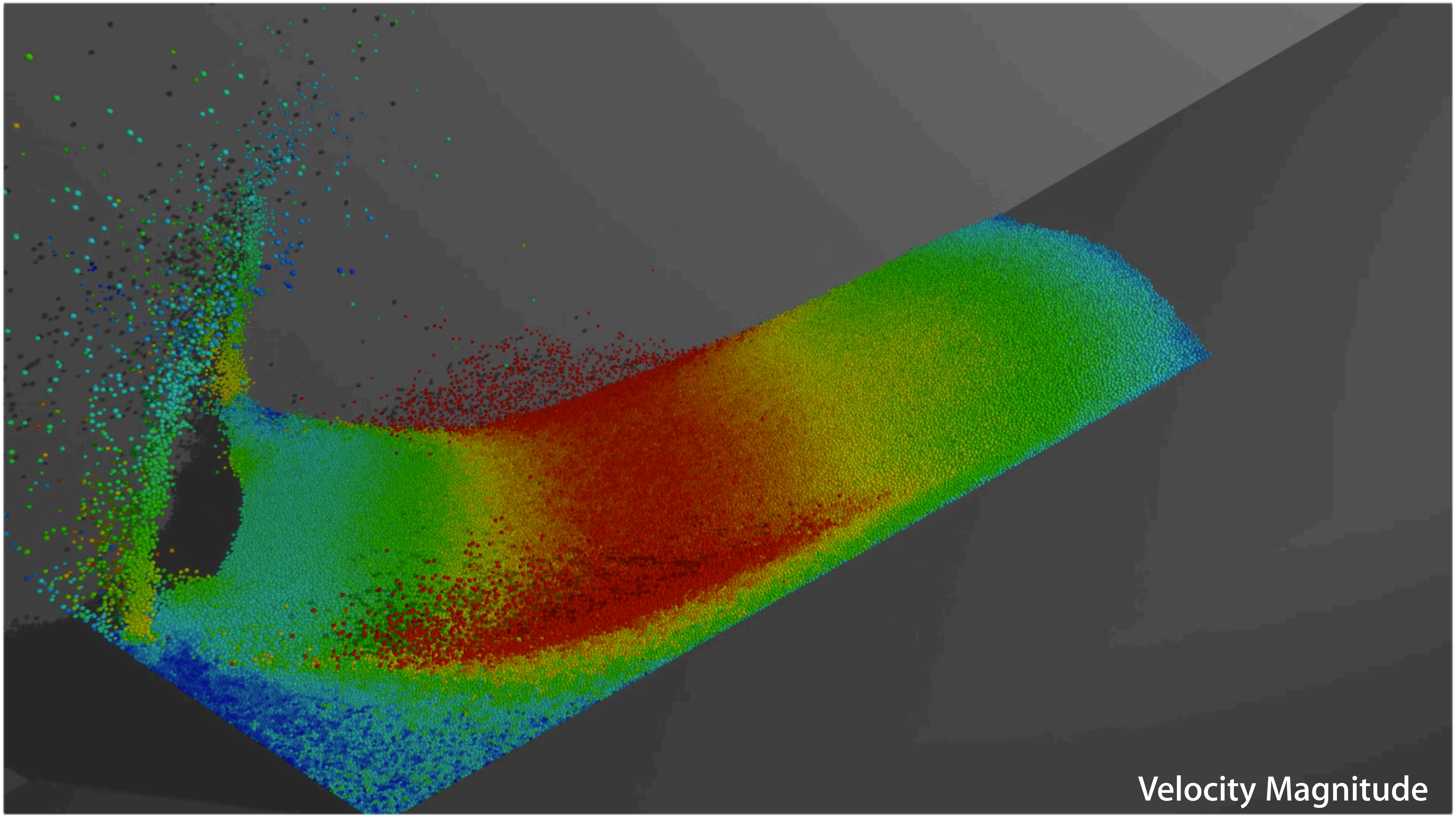

Drift Correction (on displacement level)

$$\mathrm{d}\mathbf{u} = \mathbf{M}(\mathbf{q},t)^{-1}\left(\mathbf{h}(\mathbf{q},\mathbf{u},t)\mathrm{d}t - \mathbf{W}(\mathbf{q},t)\mathrm{d}\mathbf{P}\right)$$
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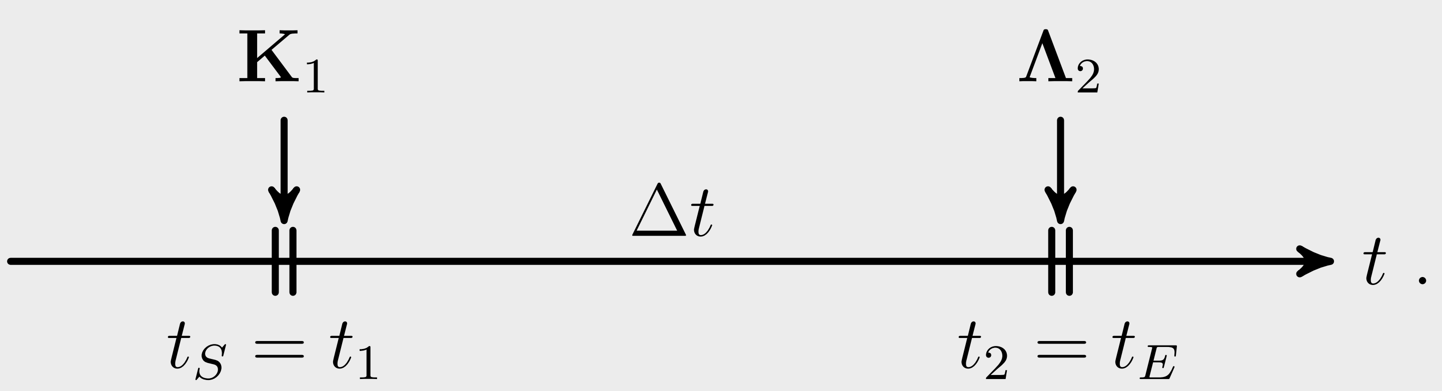
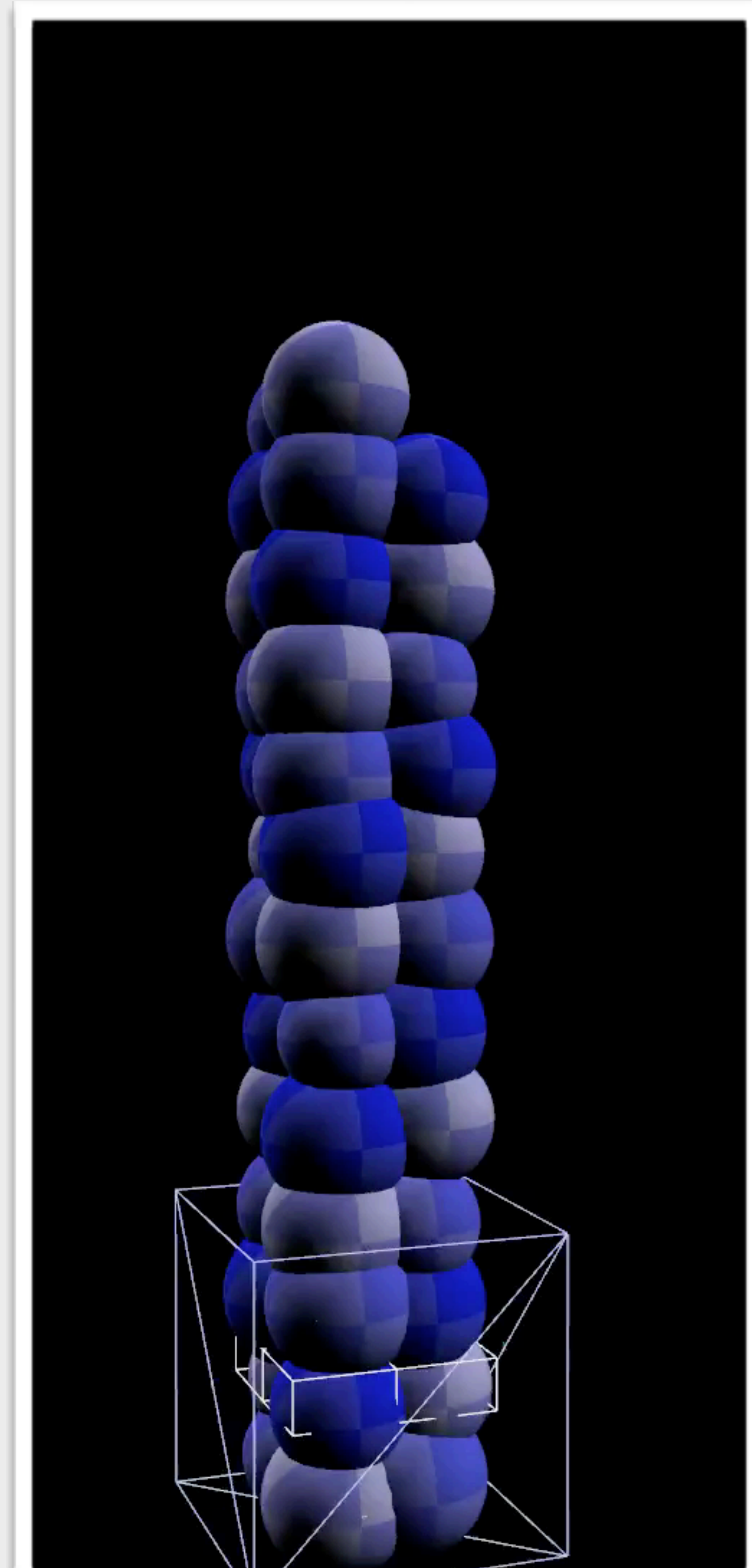
gradually correcting the penetration by using impulse-impulses acting on displacement level





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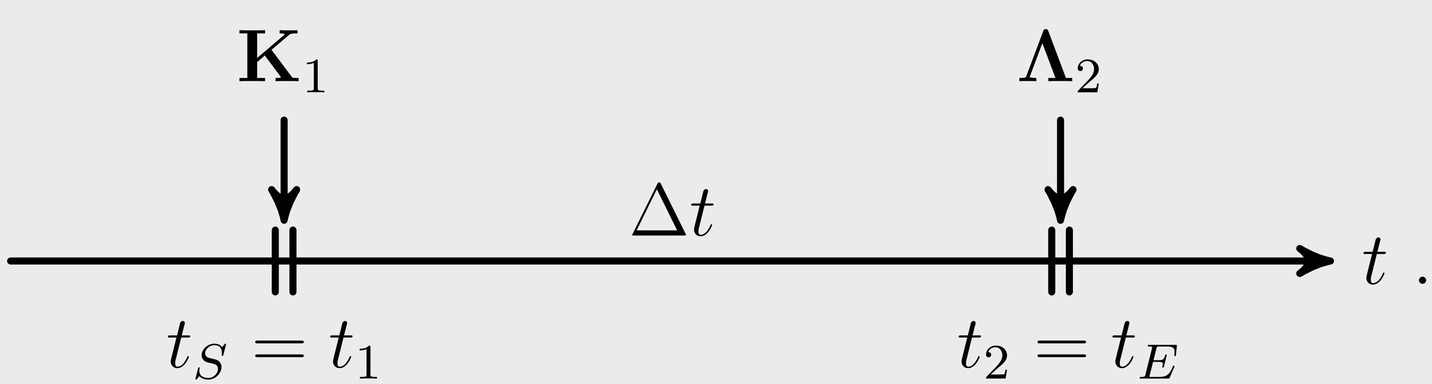

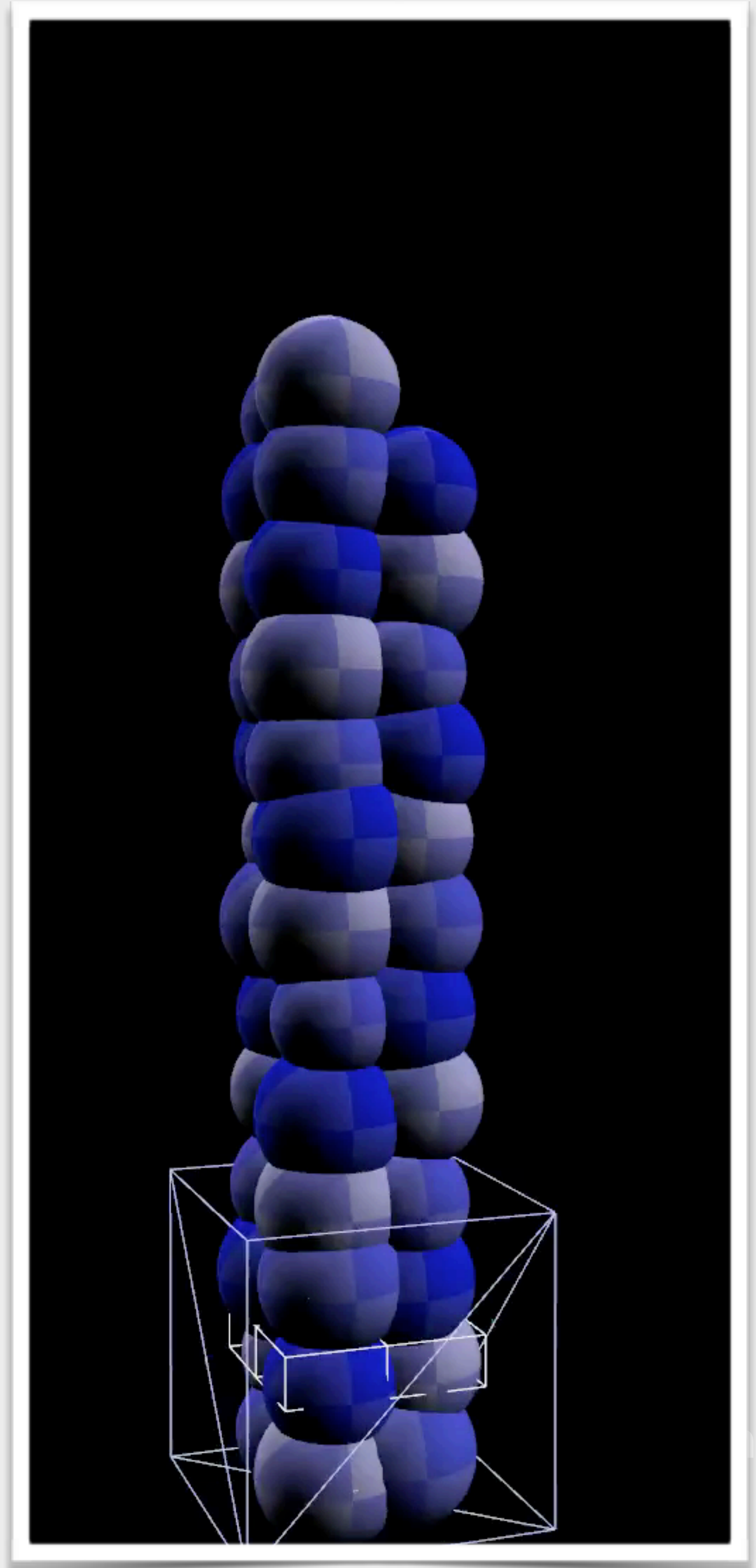

What is Cluster Computing?

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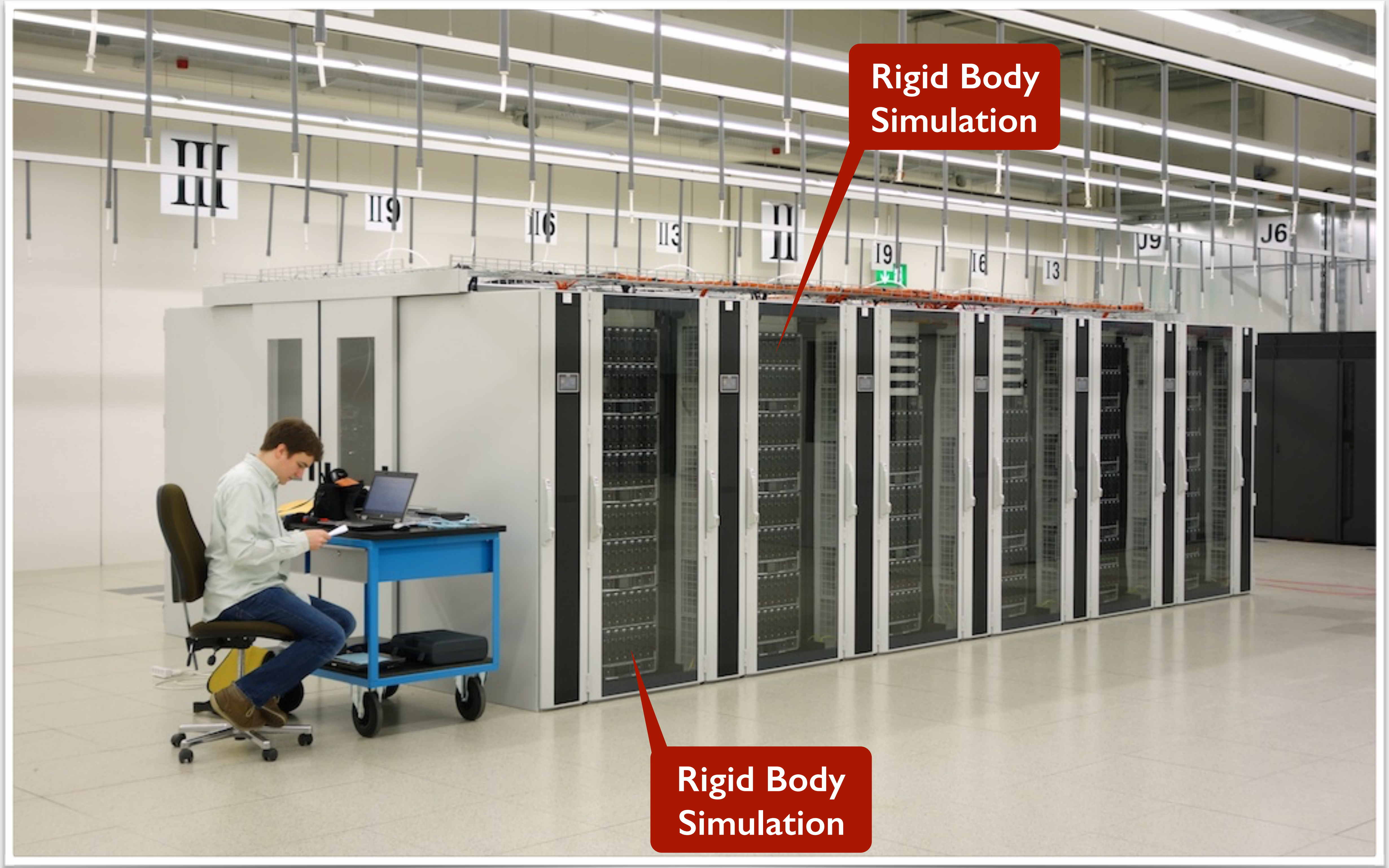

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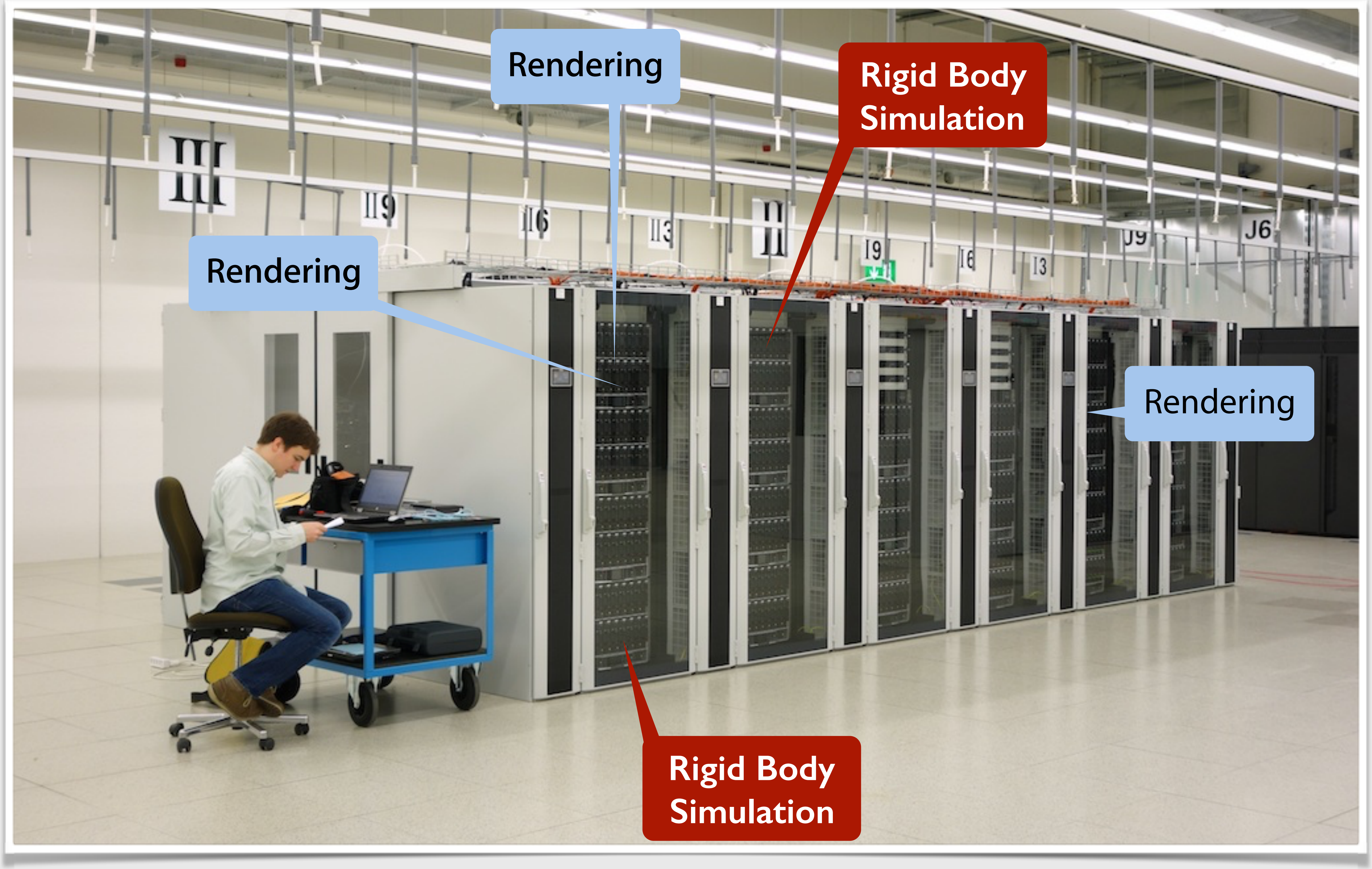

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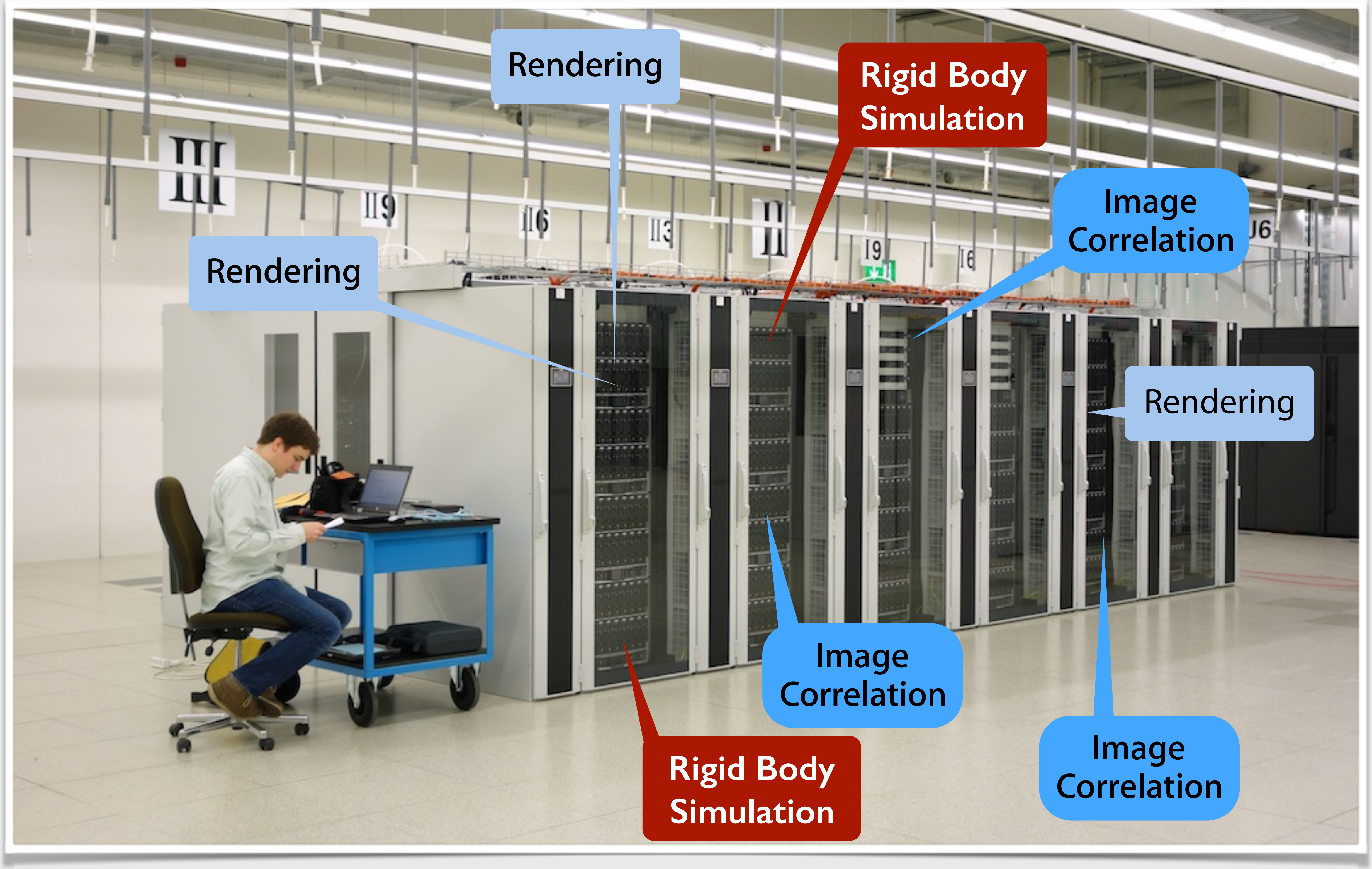

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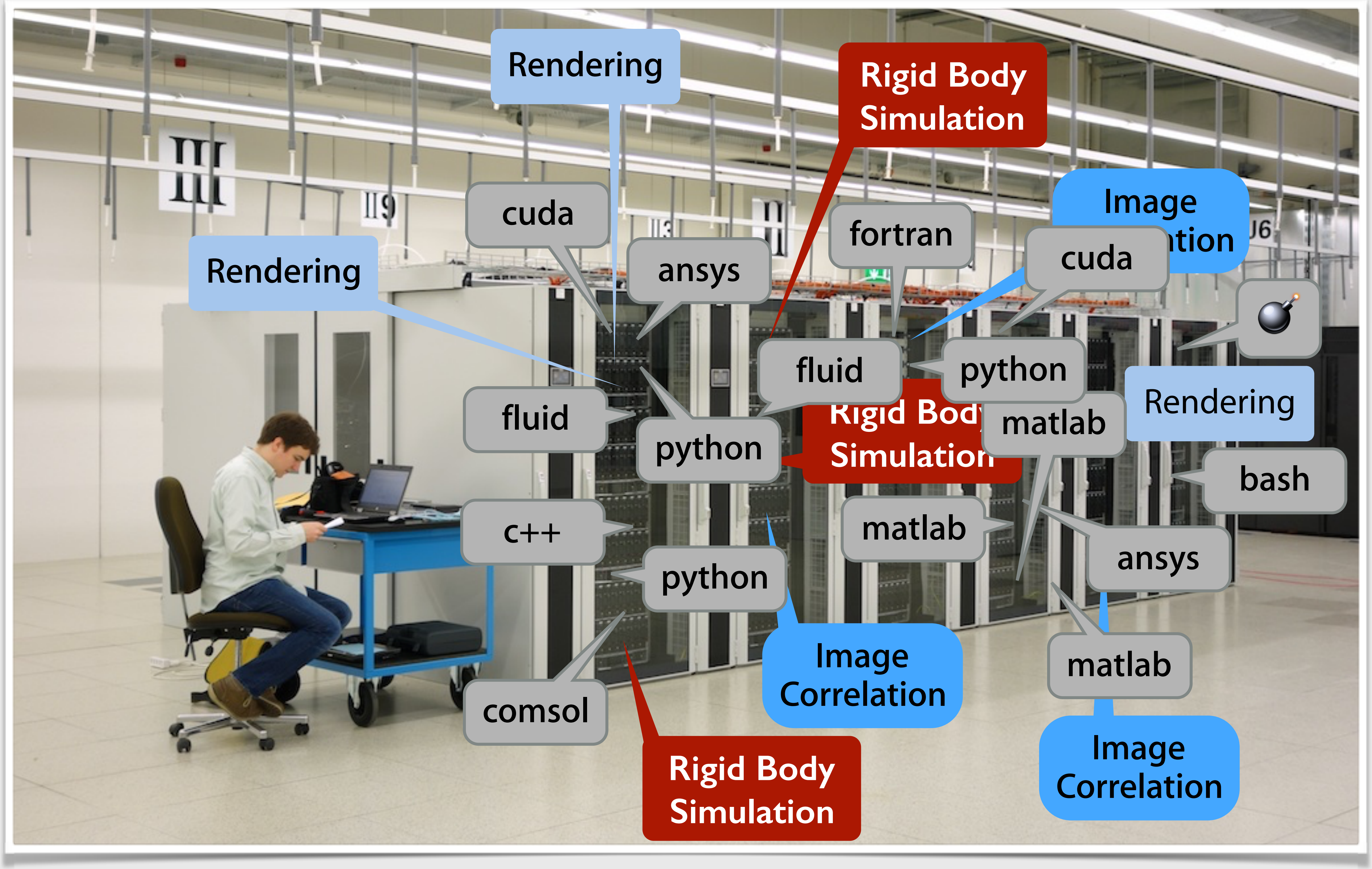
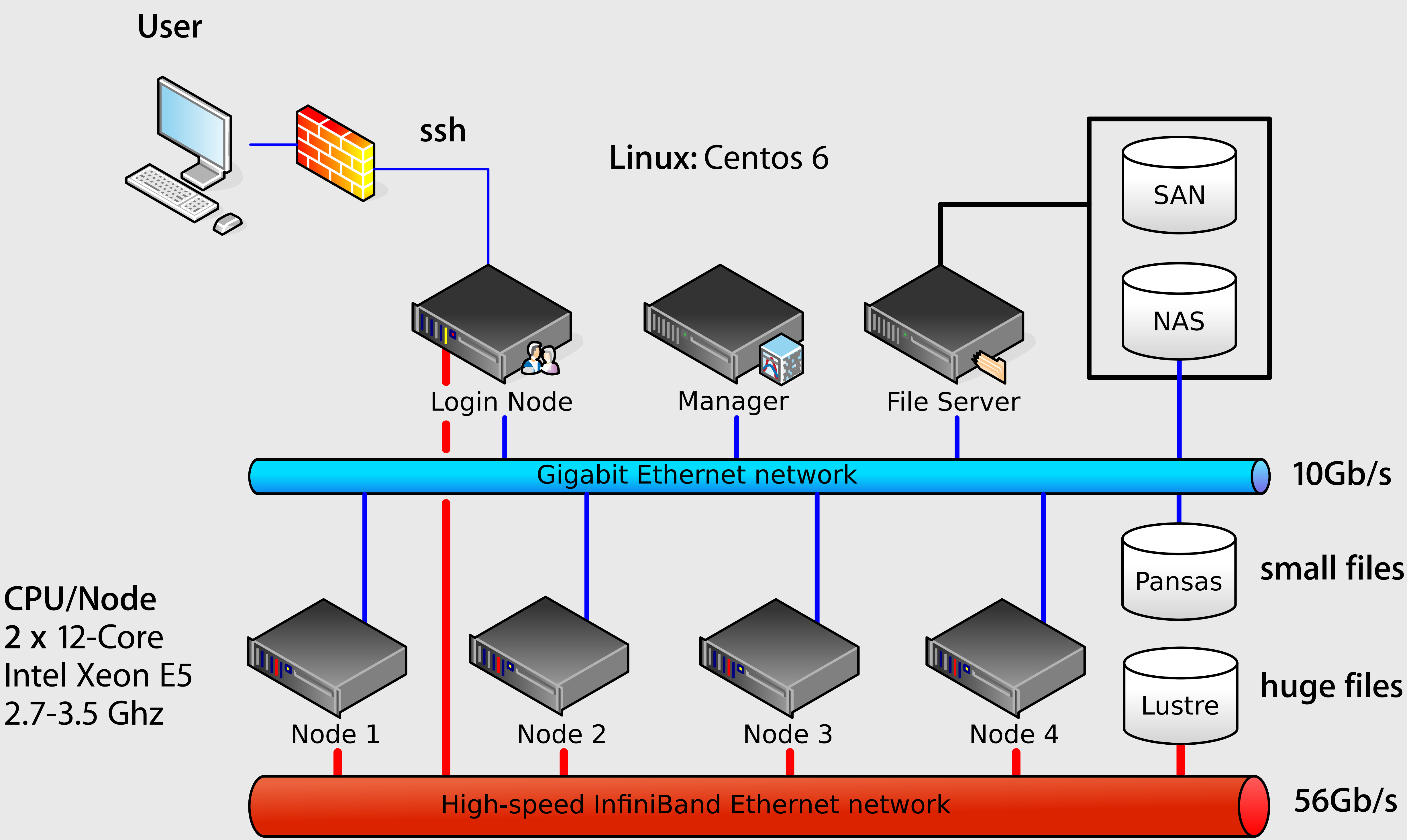
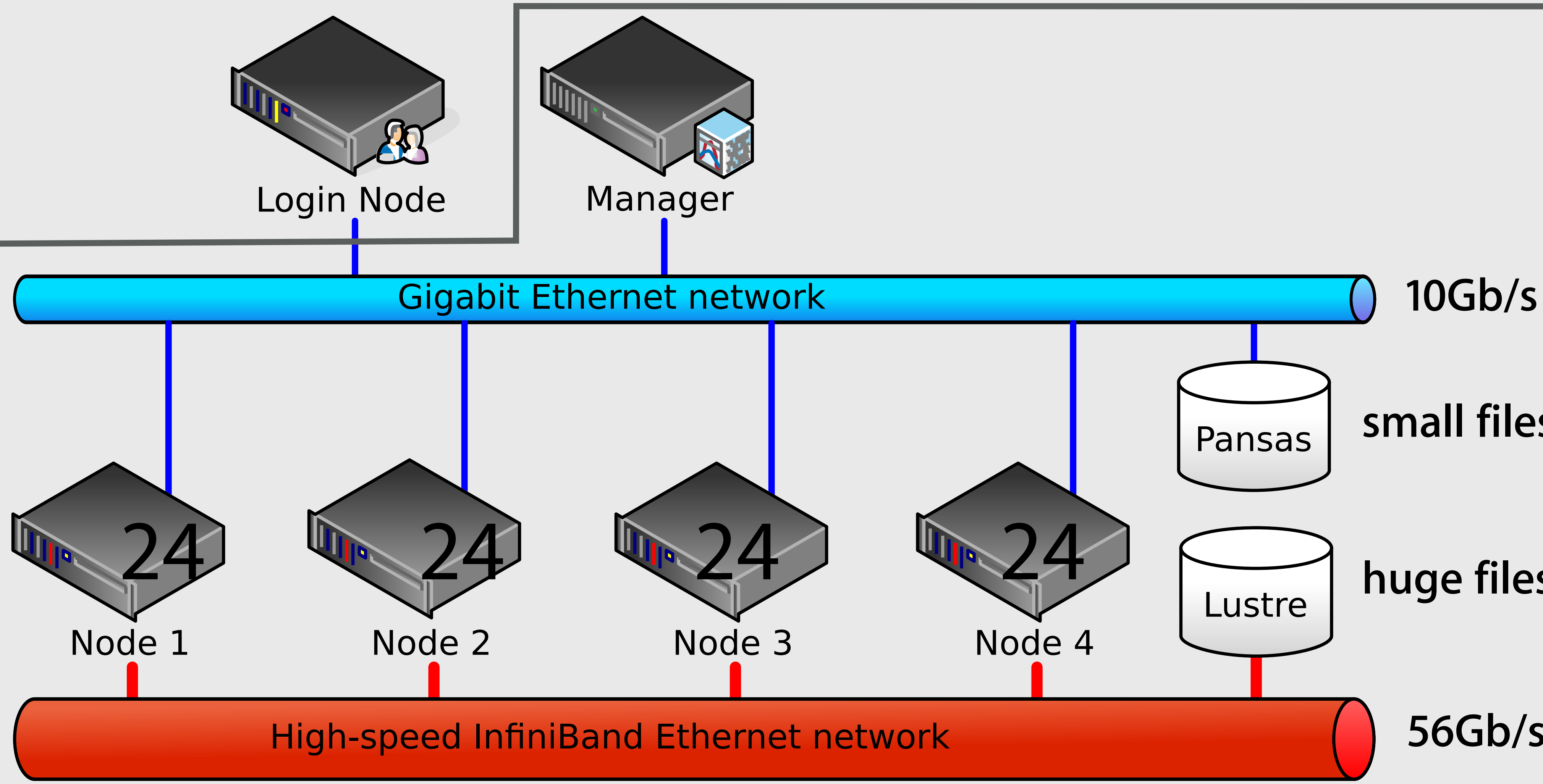


Cluster Architecture



Clean Job Babysitting



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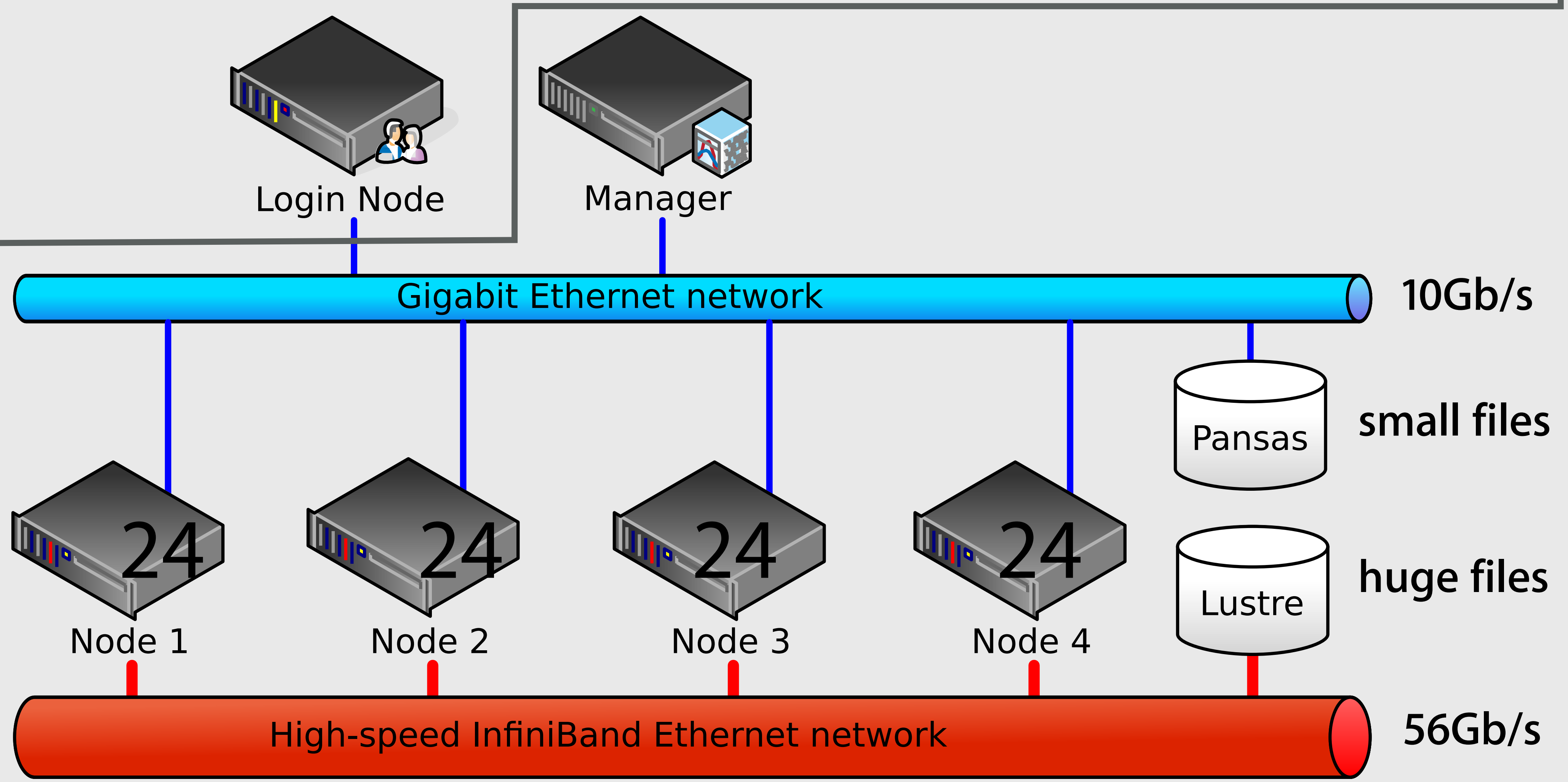

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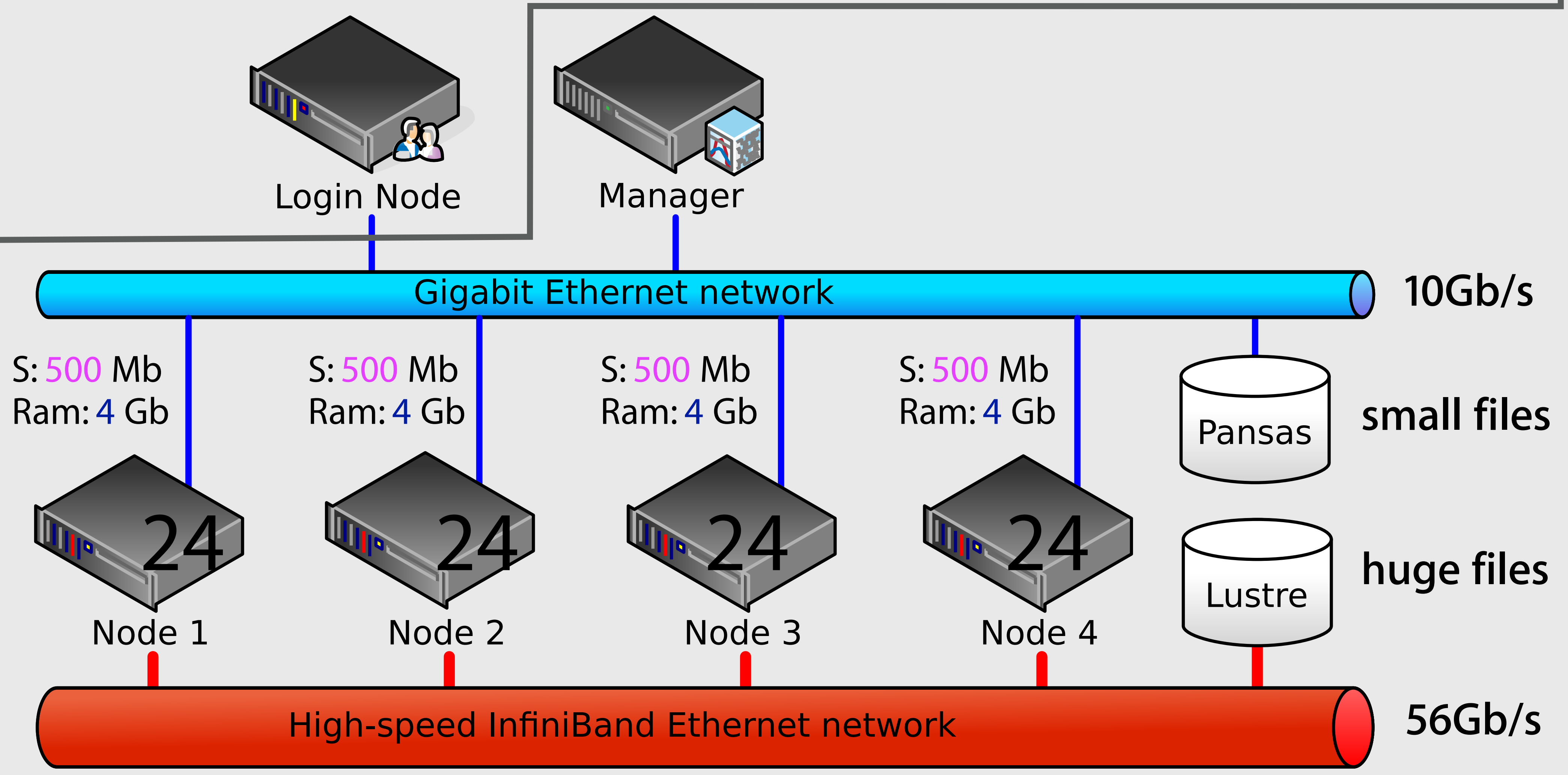

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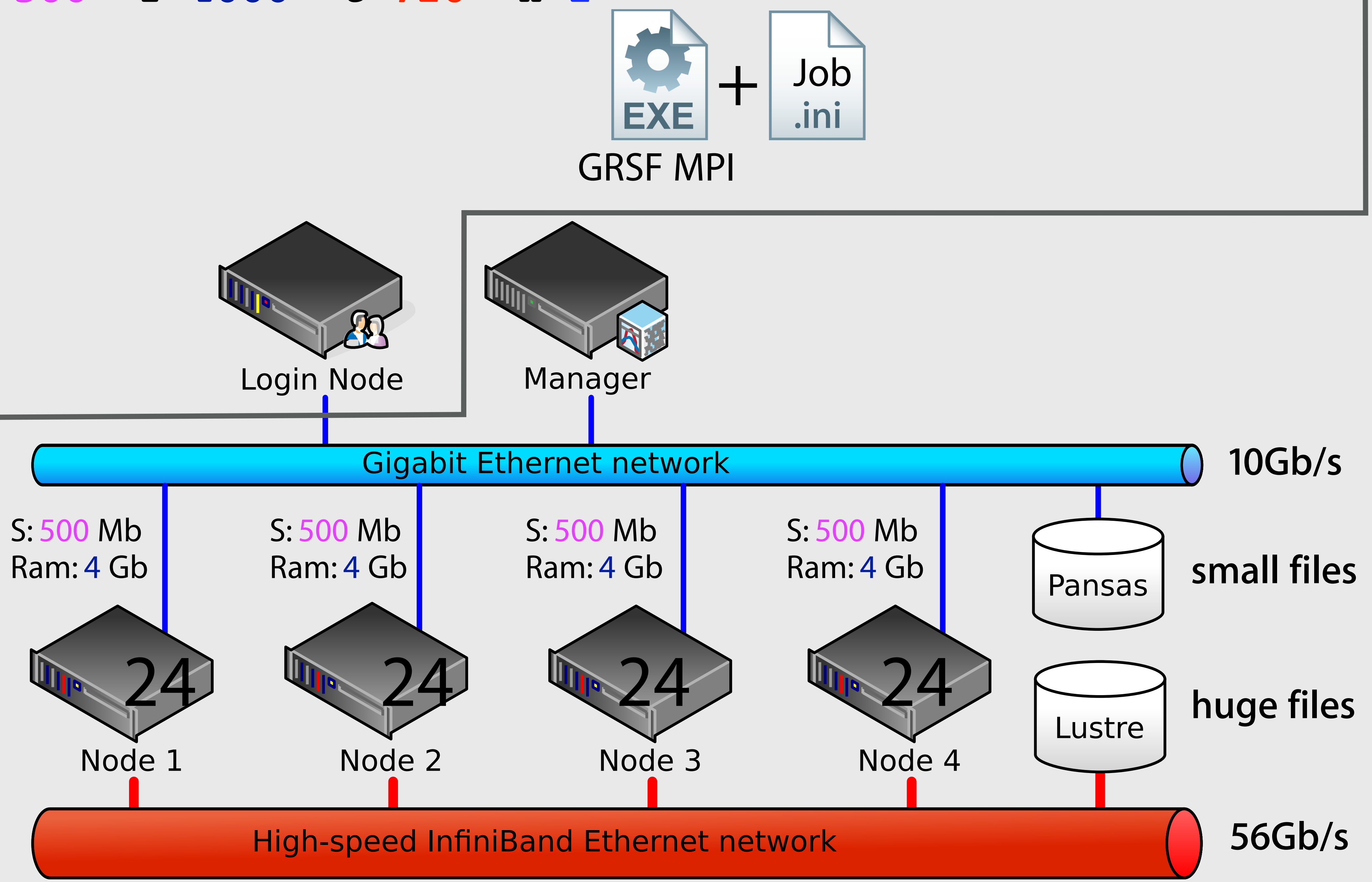

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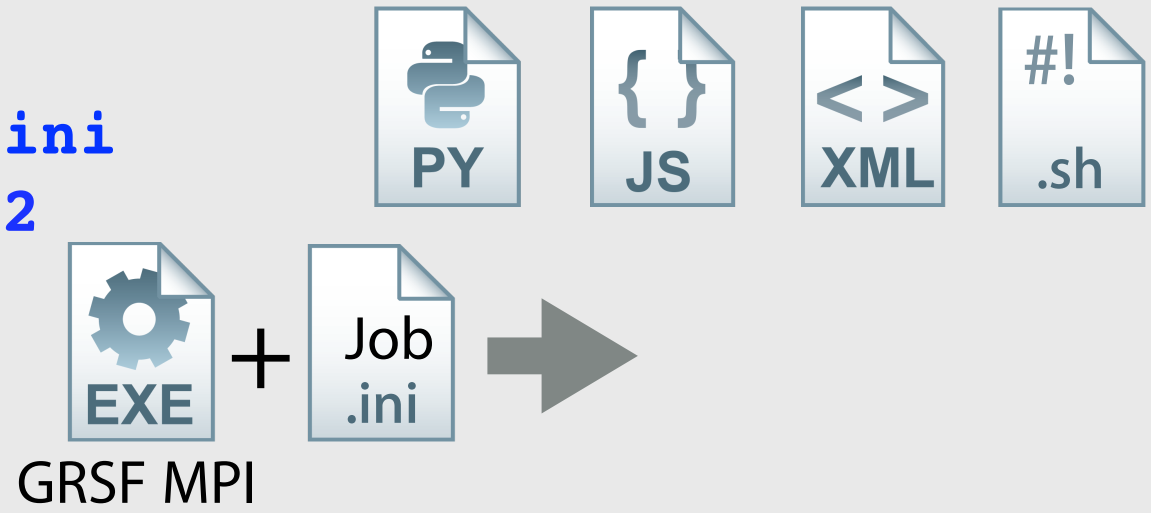

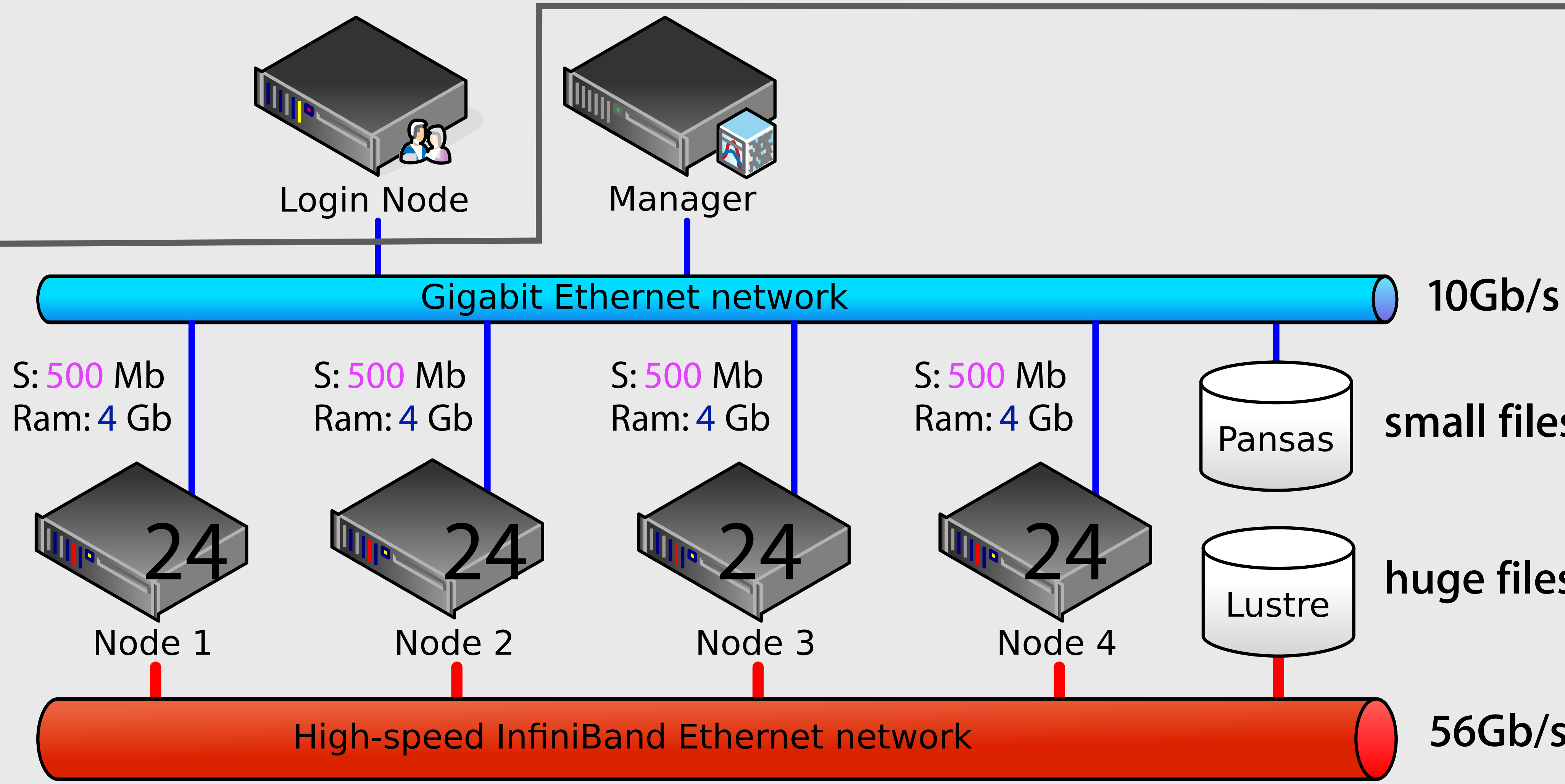

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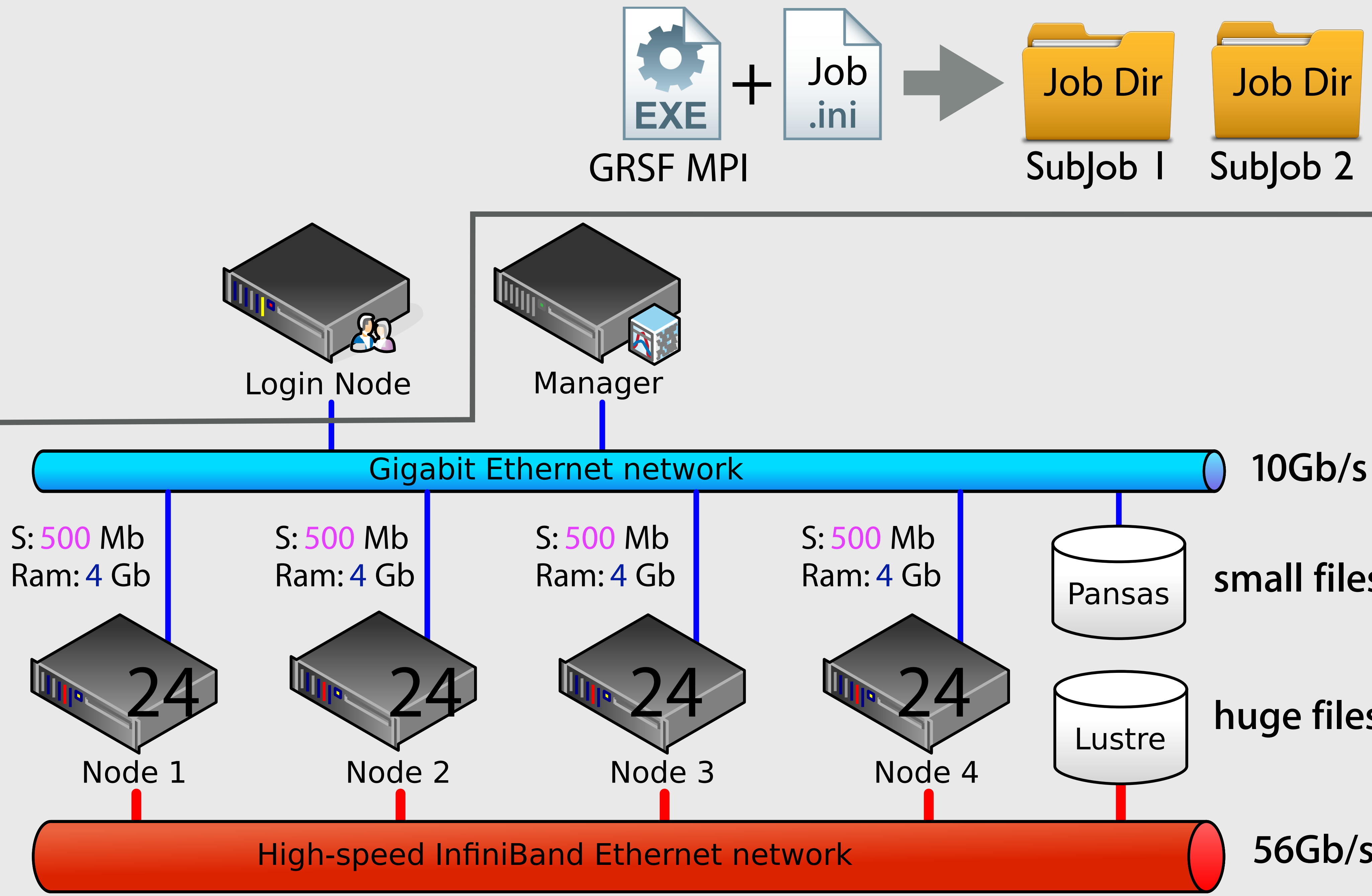

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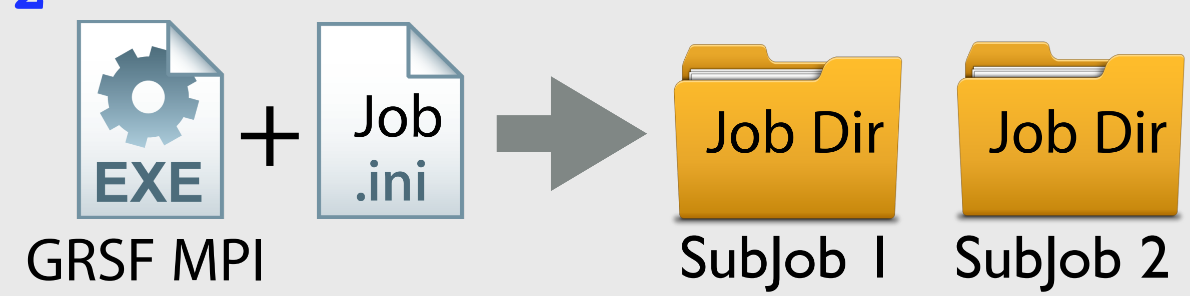

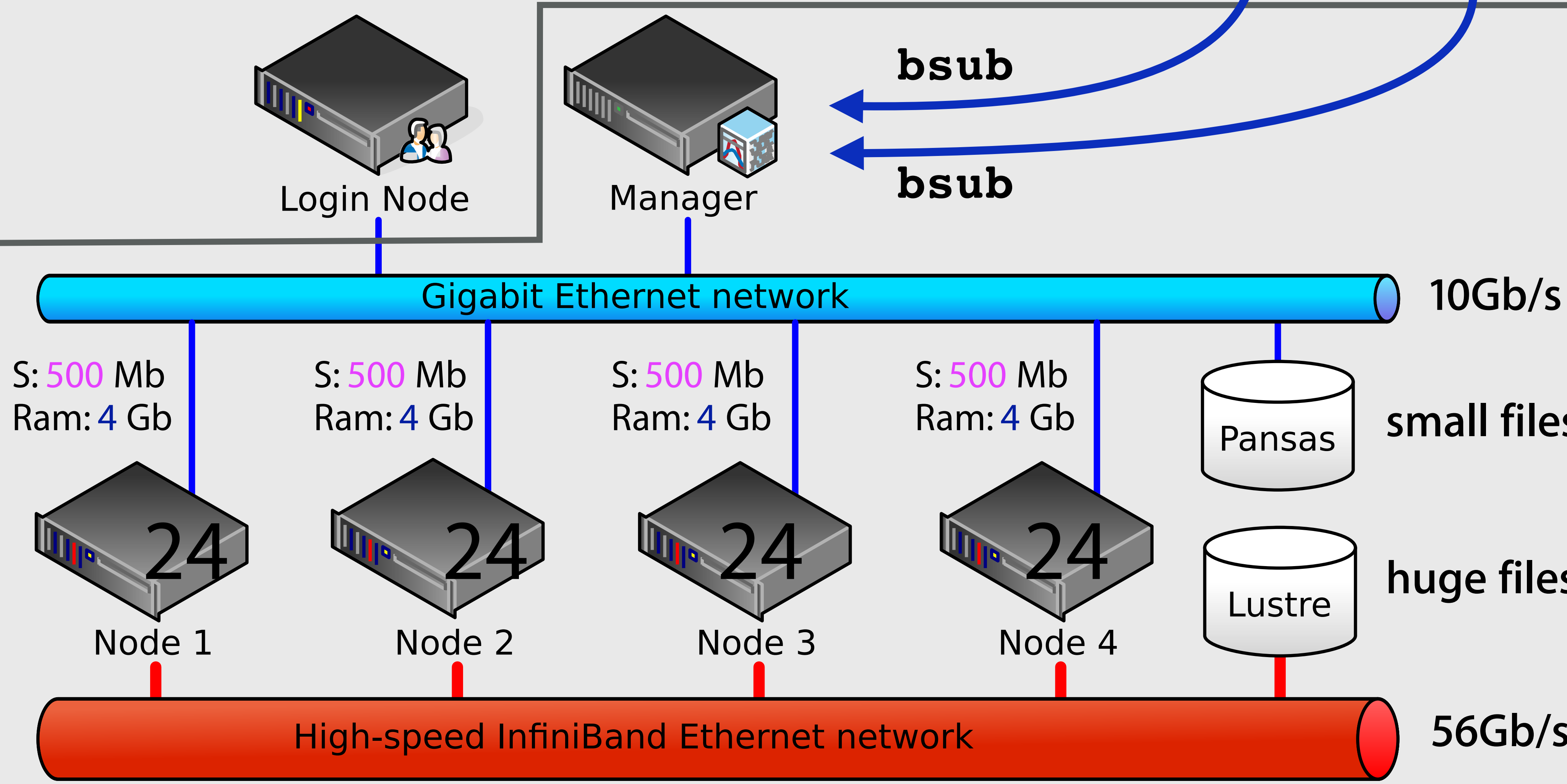

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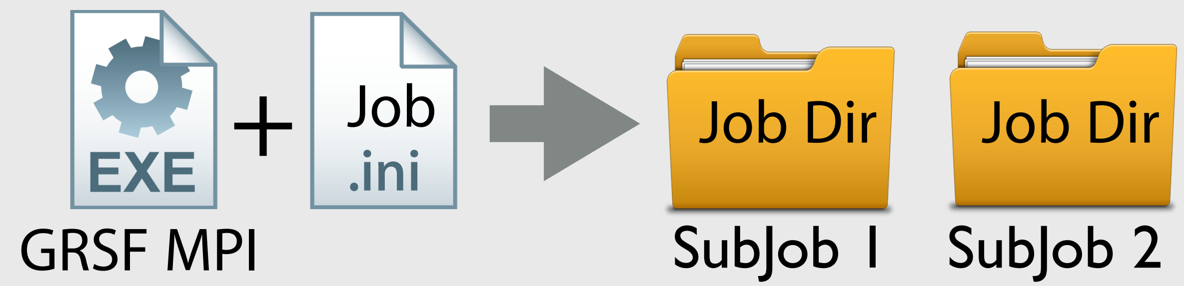

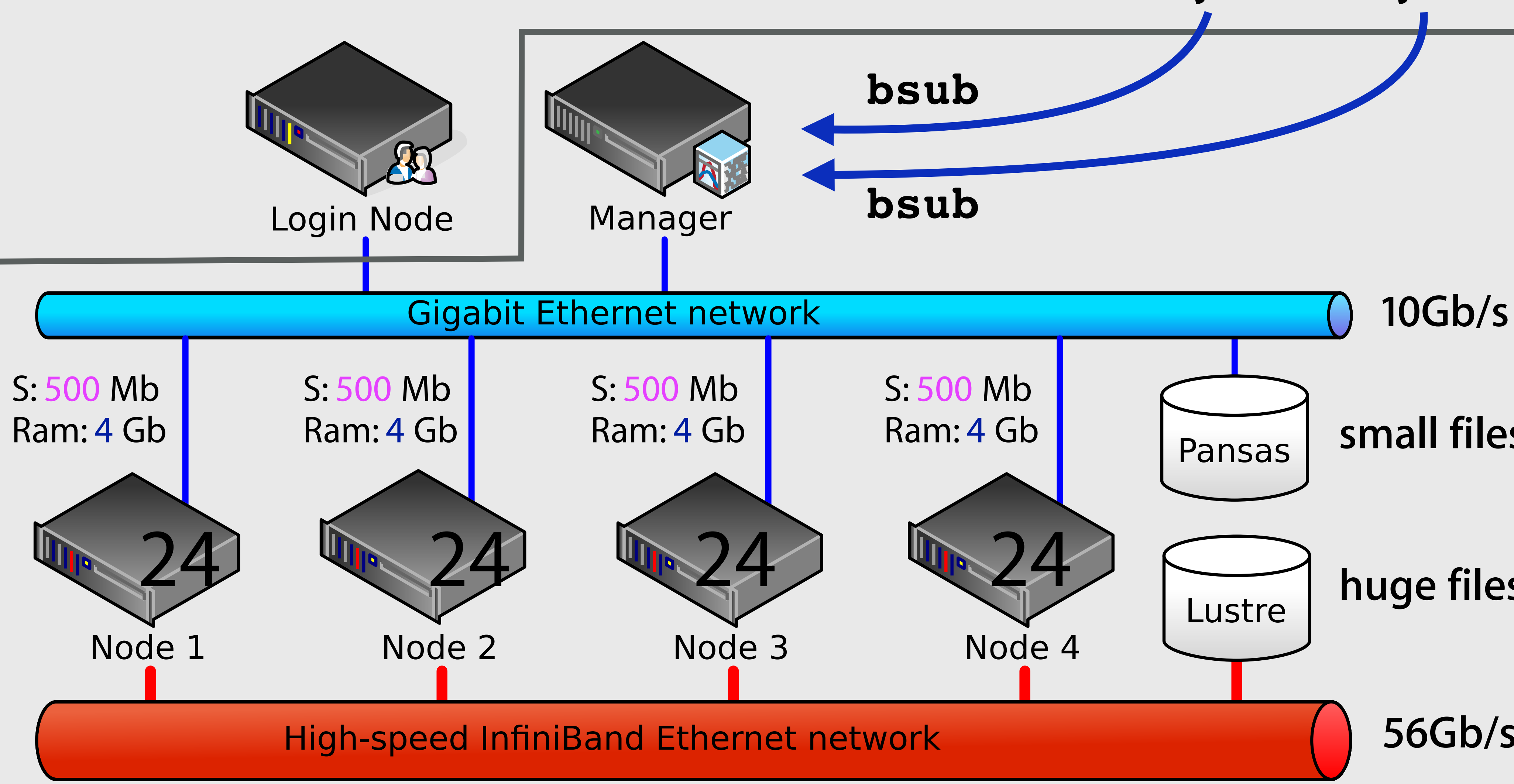

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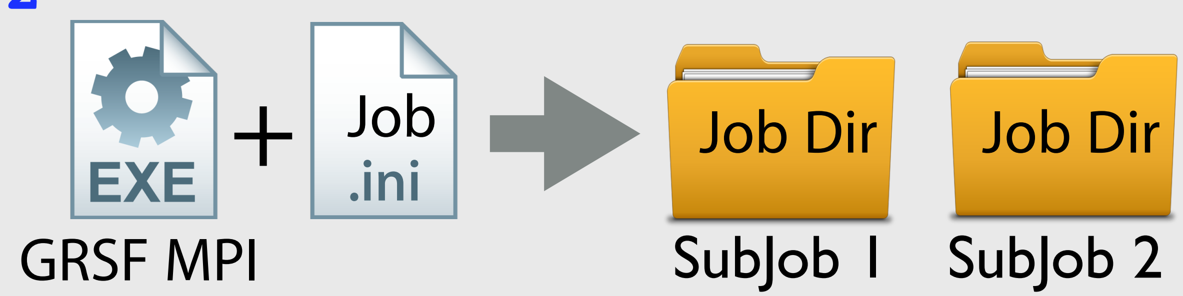

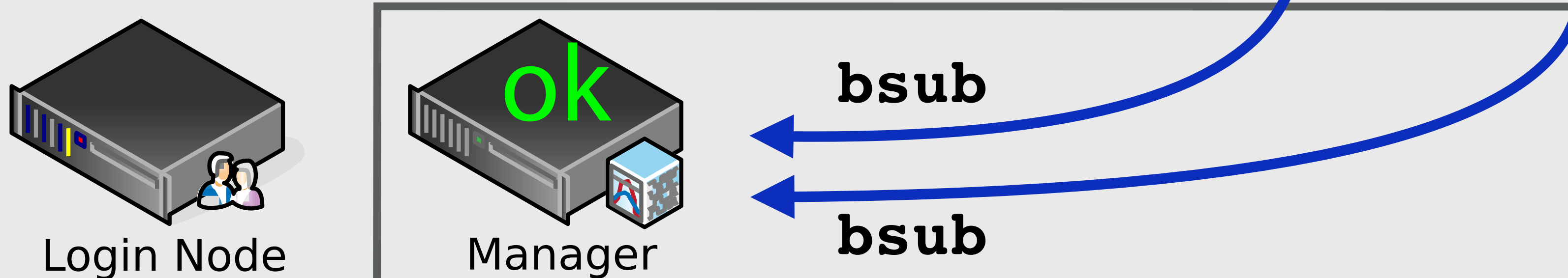

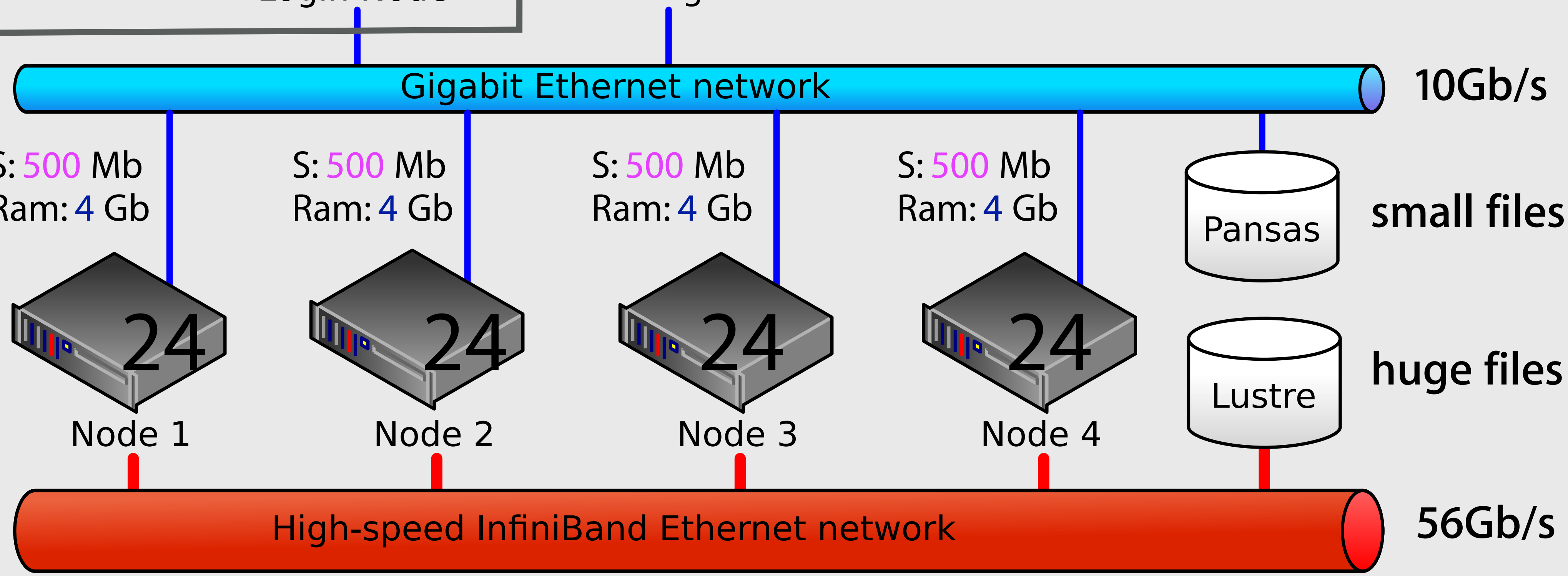

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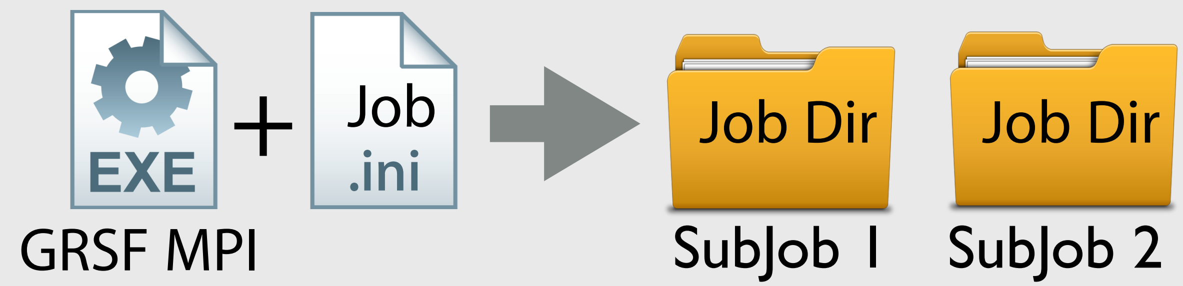

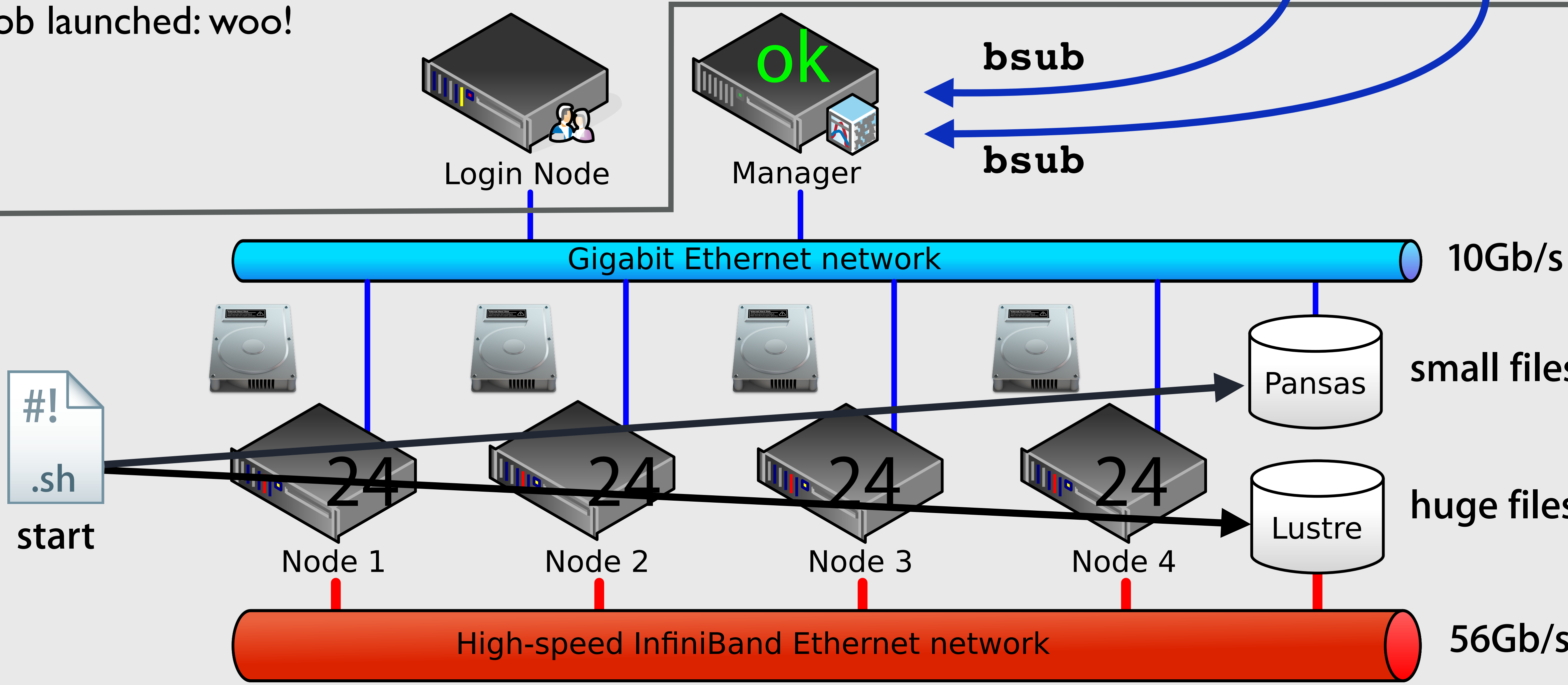

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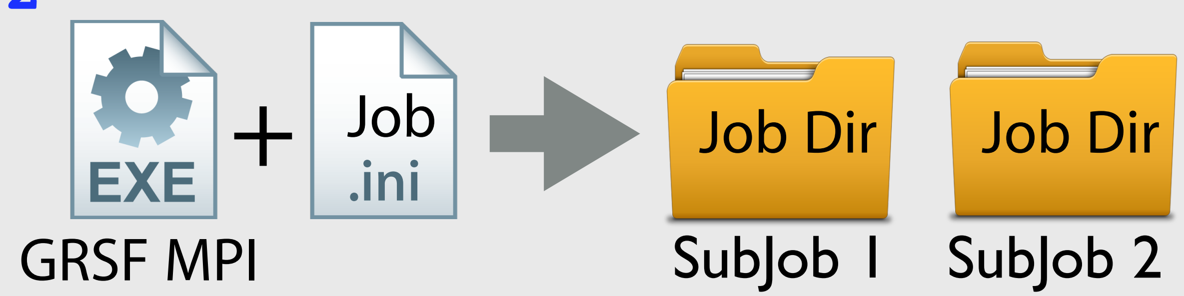

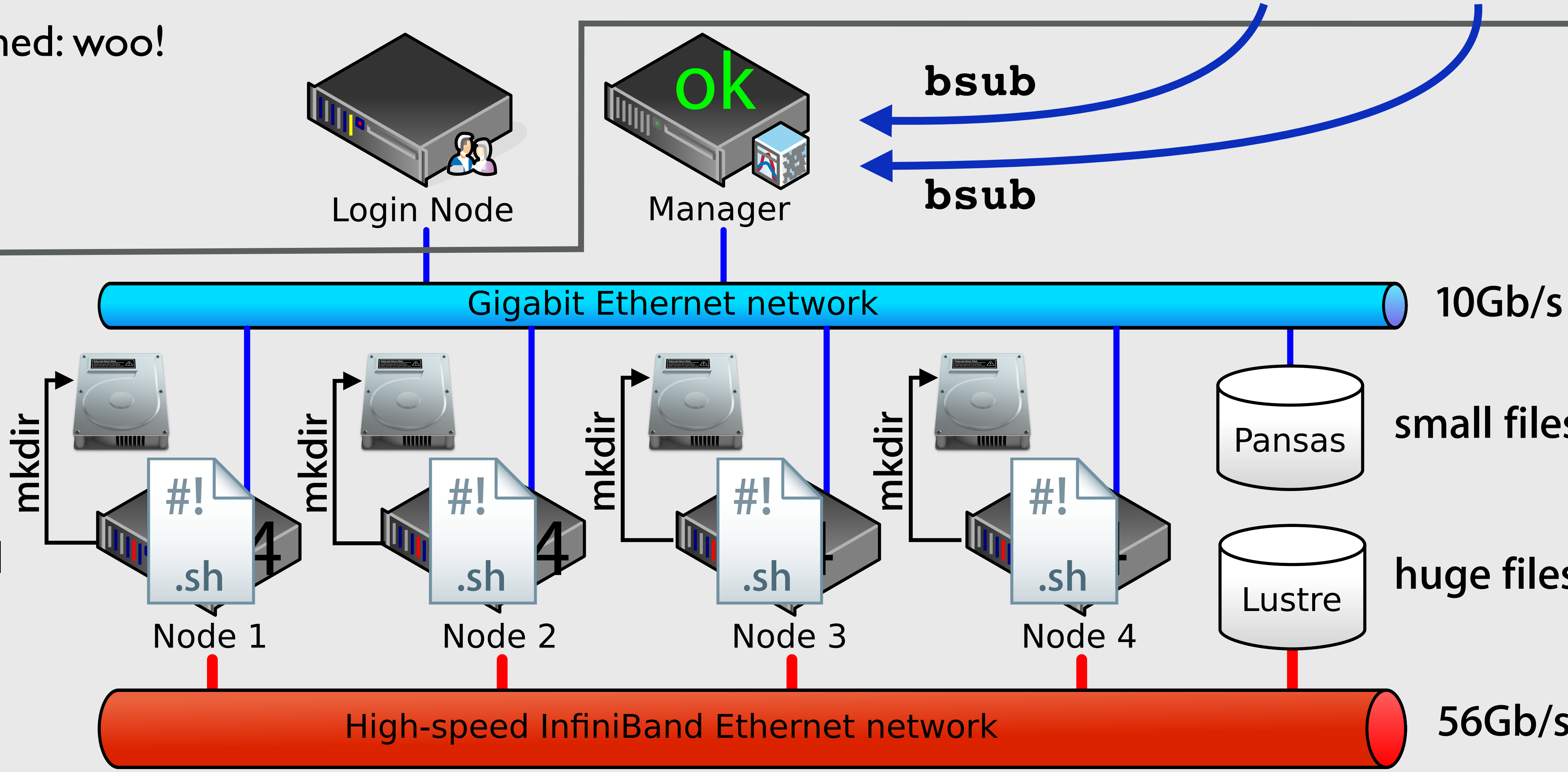

preProcess-PerNode.sh:
I/O from global to scratch

Clean Job Babysitting

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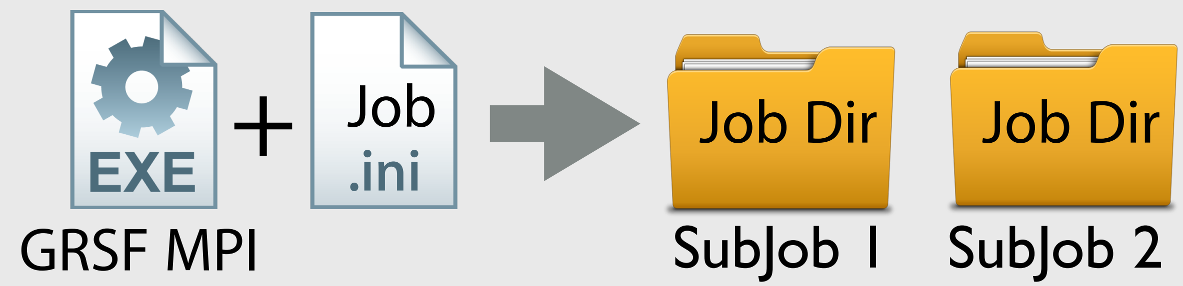

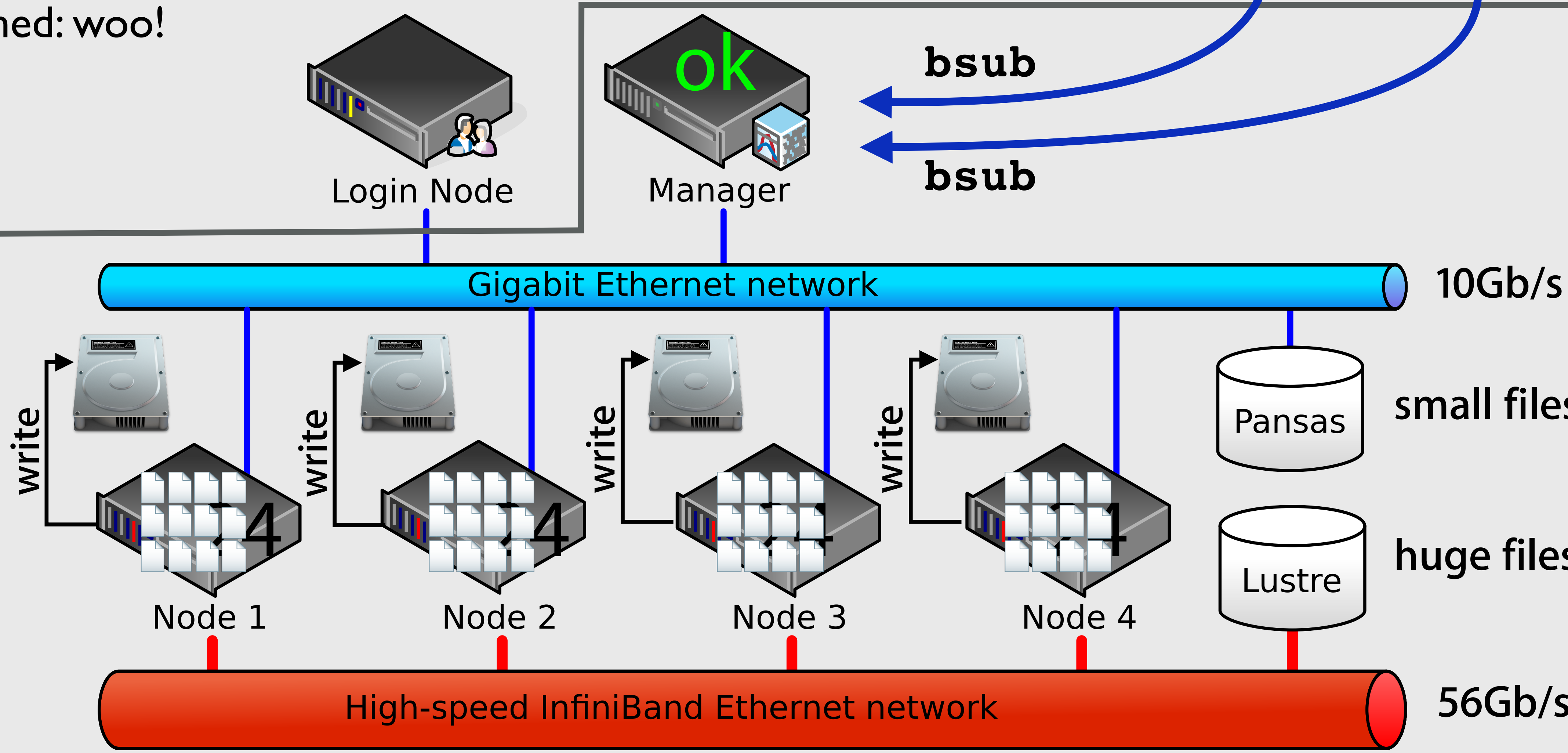

process.sh:
- compute
- do I/O

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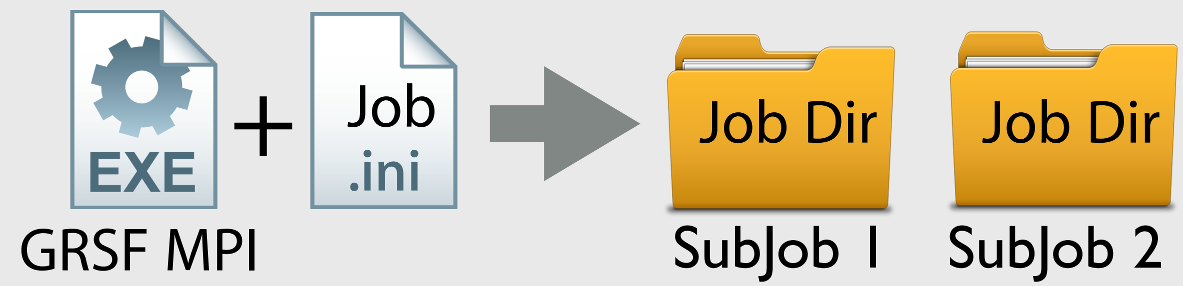

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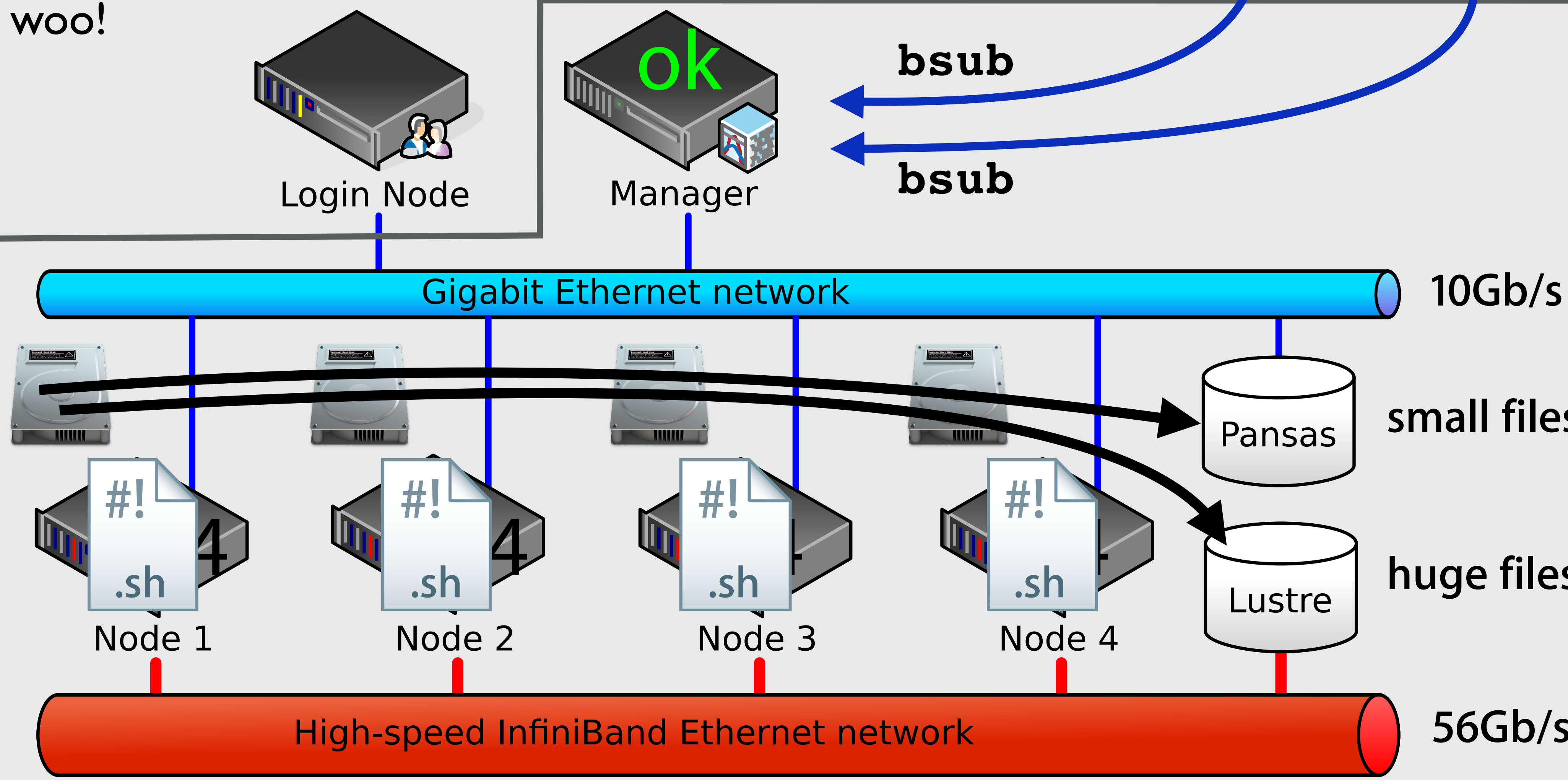

postProcess-PerNode.sh:
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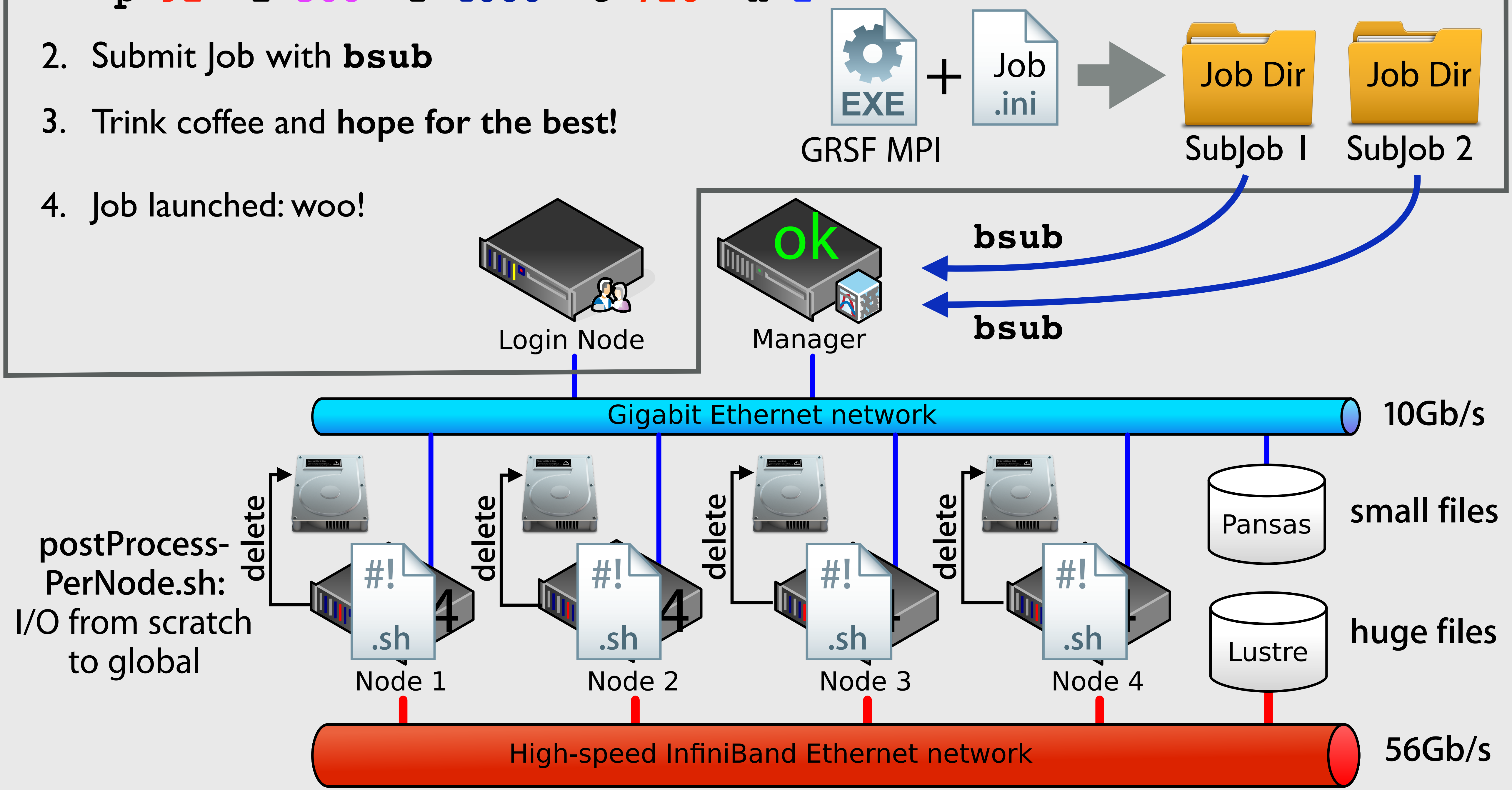

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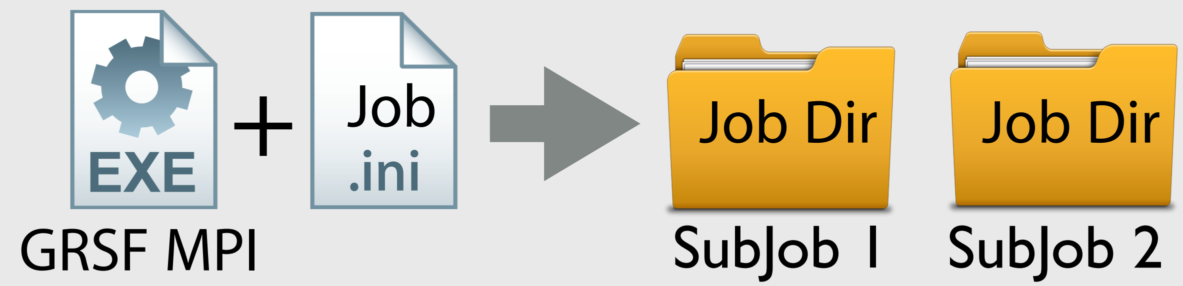

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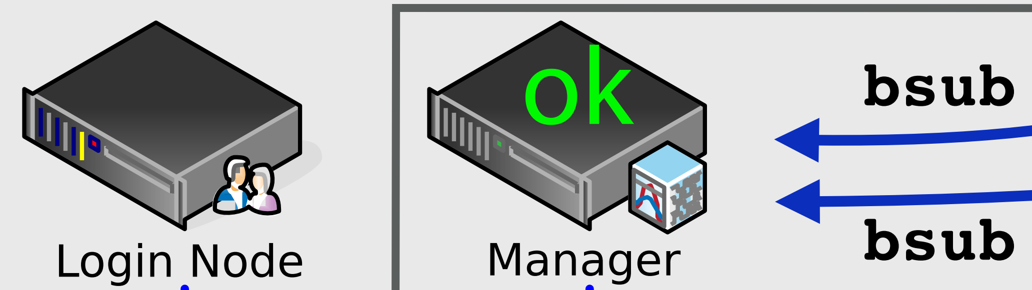

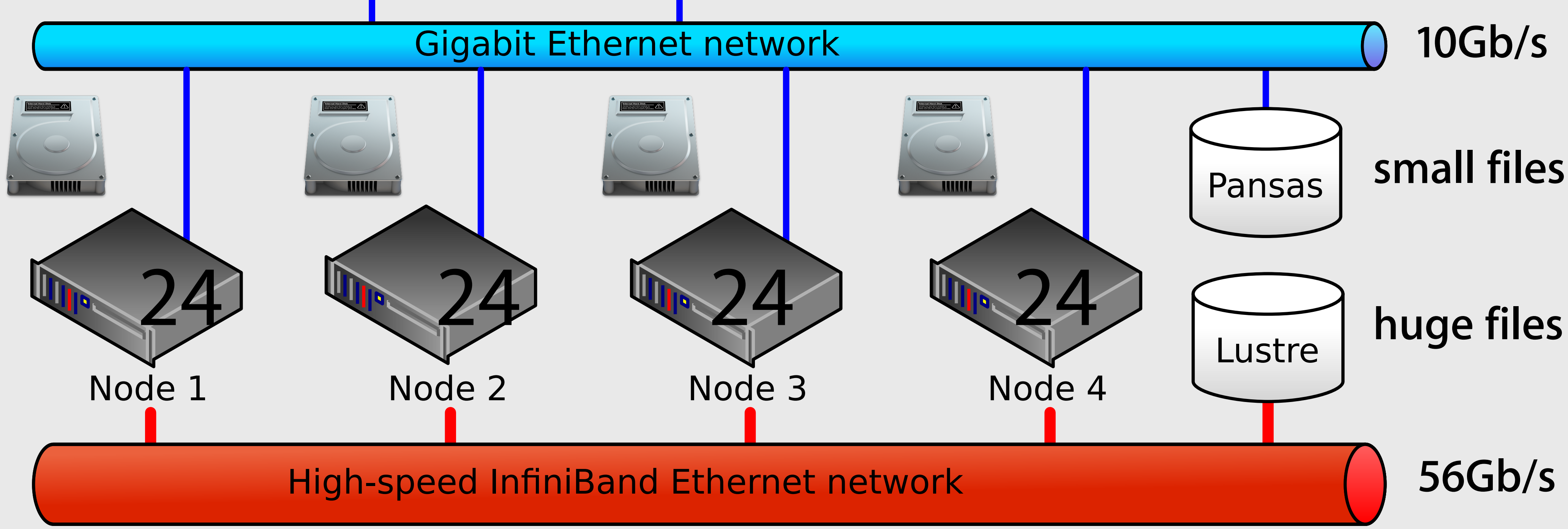

end:
- (file validation)
- notify user

Clean Job Babysitting

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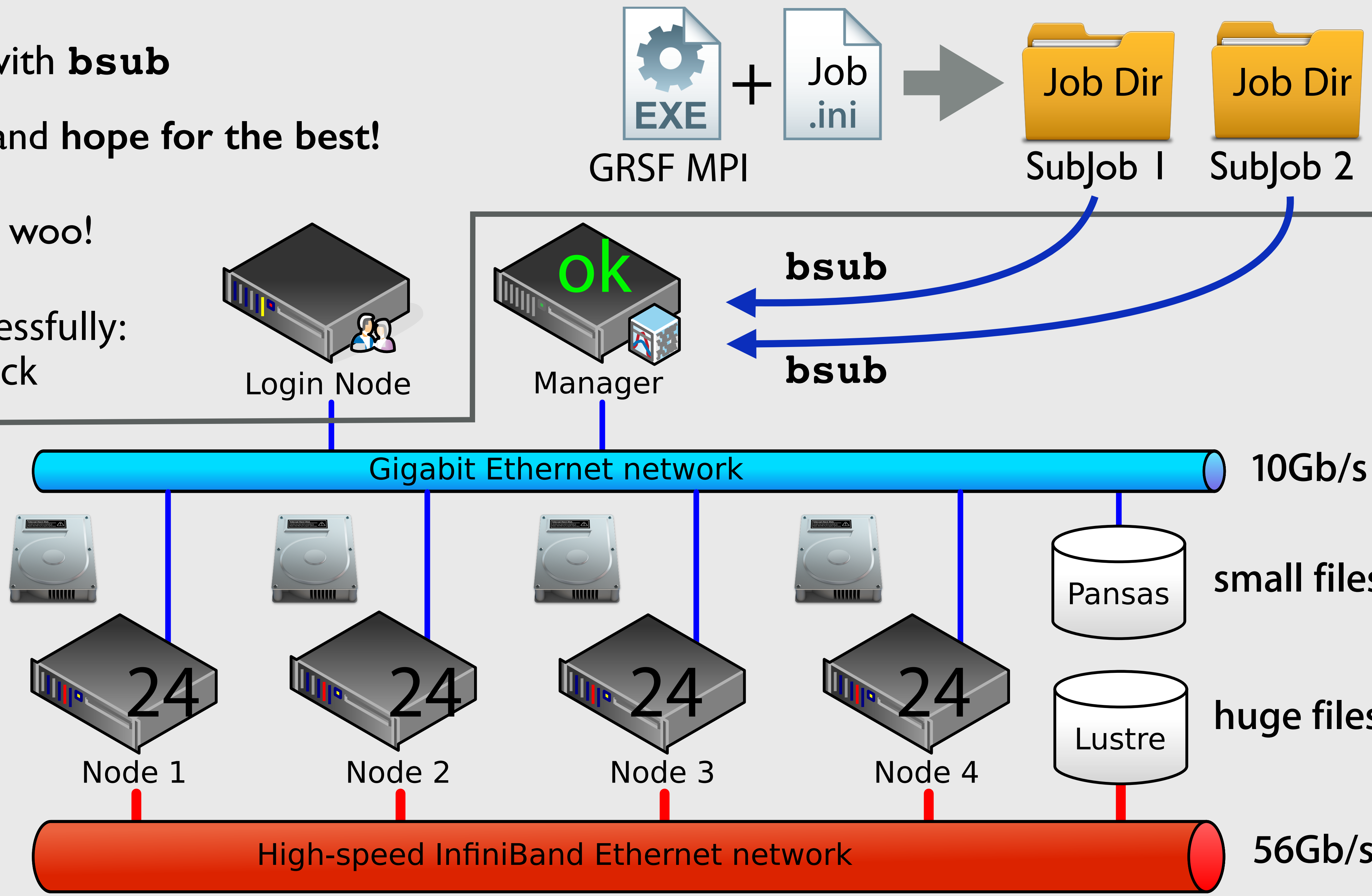
4. Job launched: woo!
5. Job run successfully:
   copy data back



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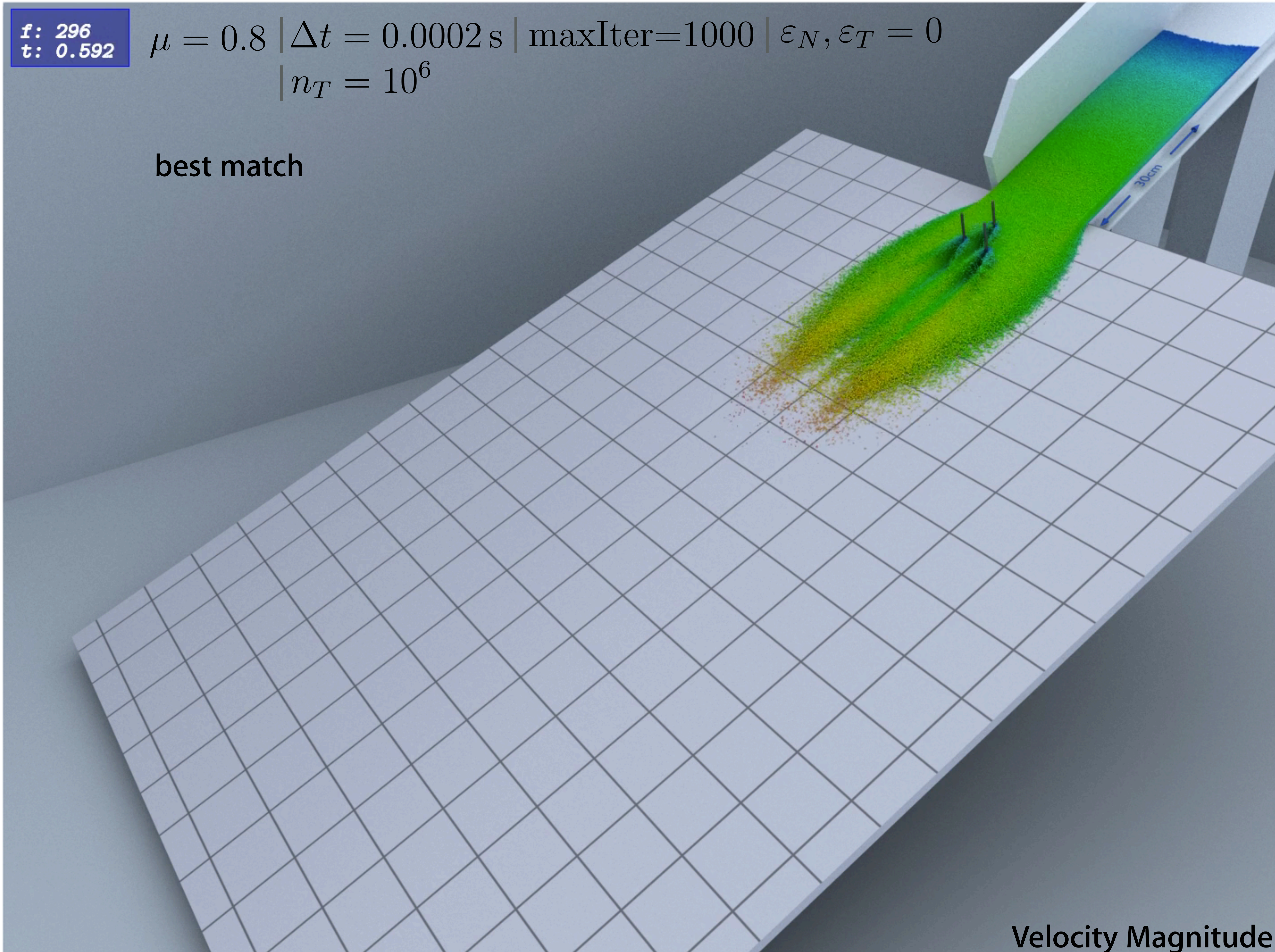
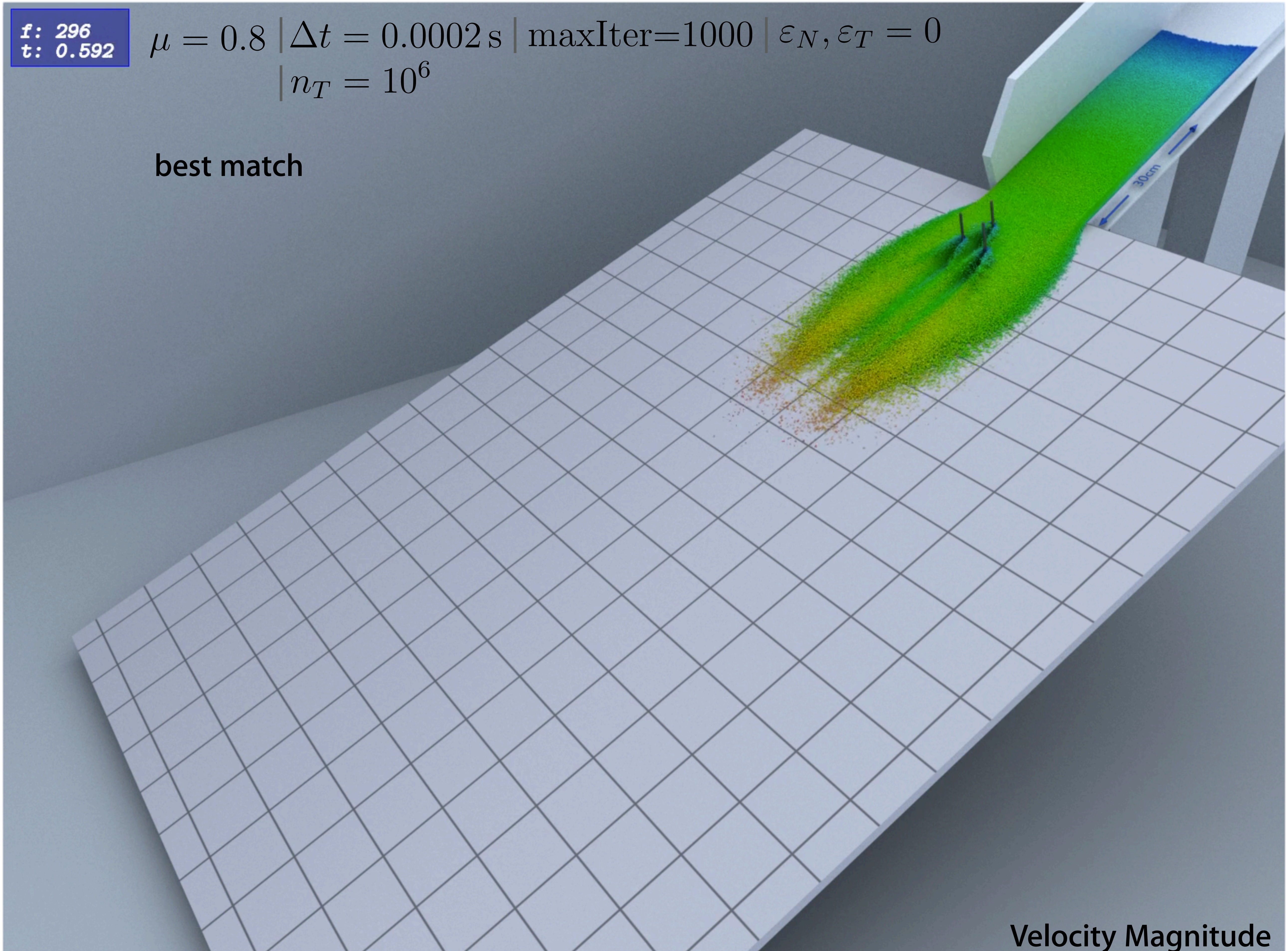

f: 296
t: 0.592
$\mu = 0.8 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter{=}1000} \mid \varepsilon_N, \varepsilon_T = 0 \mid n_T = 10^6$
best match
30cm
Velocity Magnitude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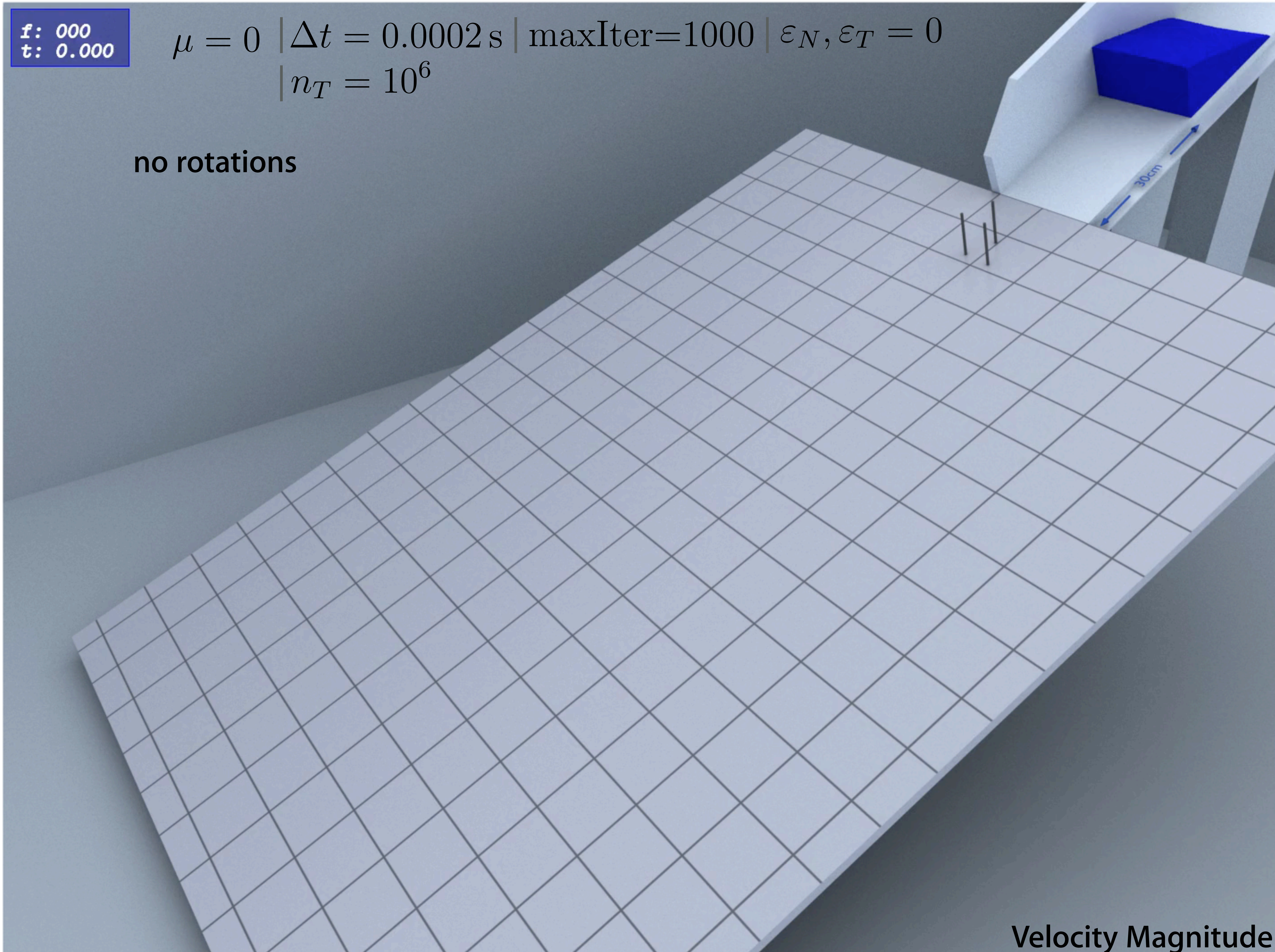
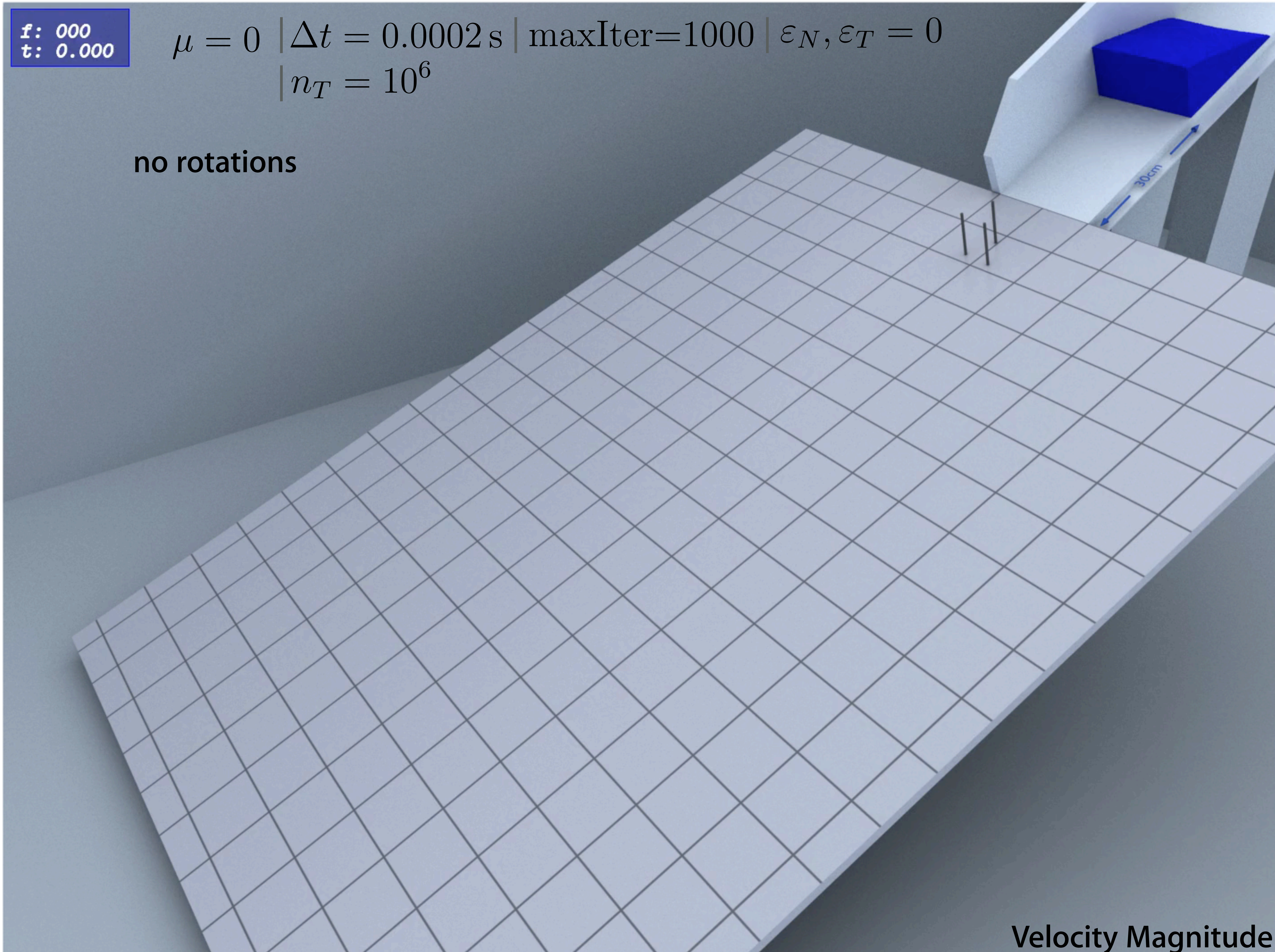

f: 000
t: 0.000
$\mu = 0 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter{=}1000} \mid \varepsilon_N, \varepsilon_T = 0$
$\mid n_T = 10^6$
no rotations
30cm
Velocity Magnitude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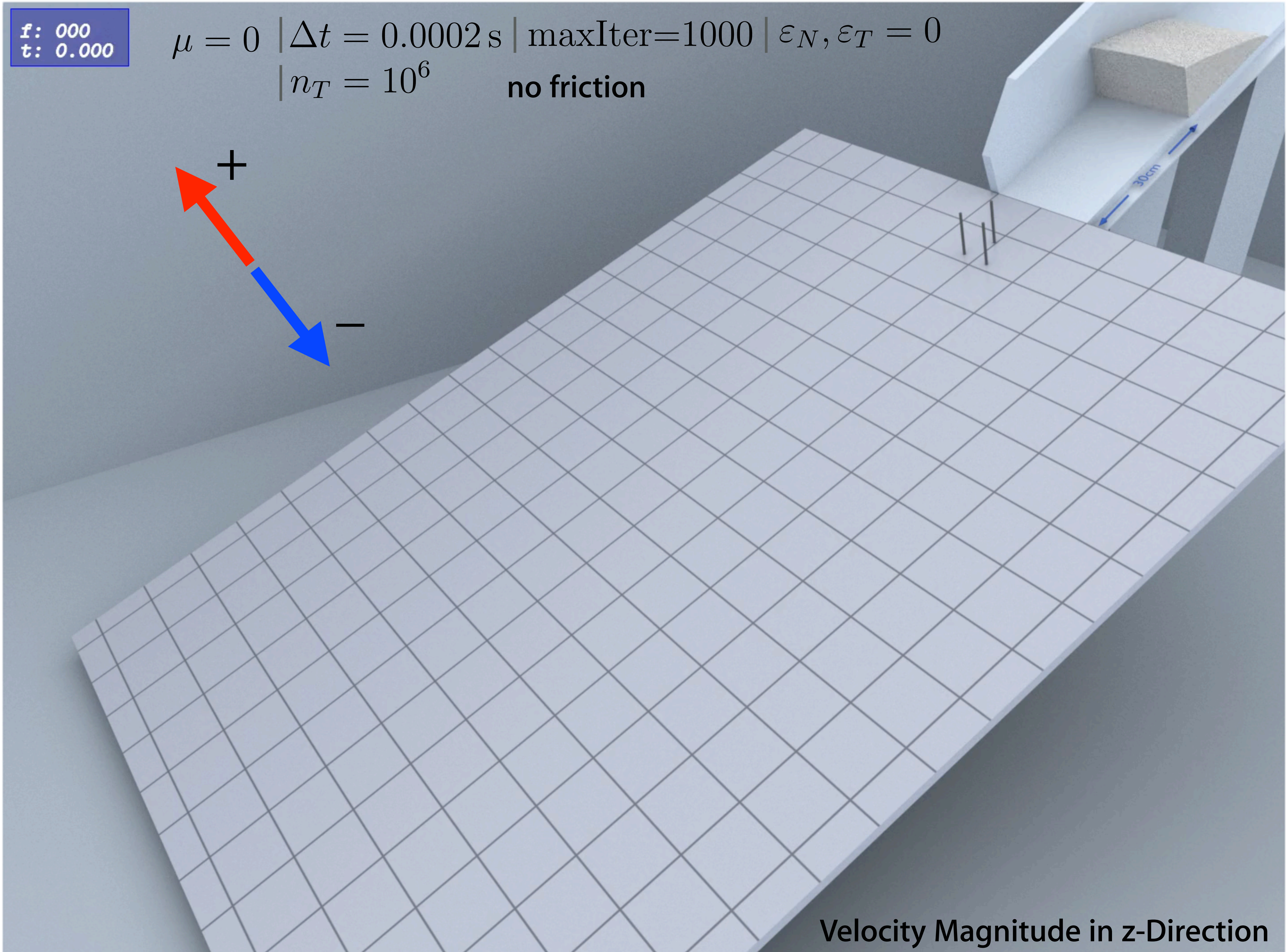
no rotations
30cm
Velocity Magnitude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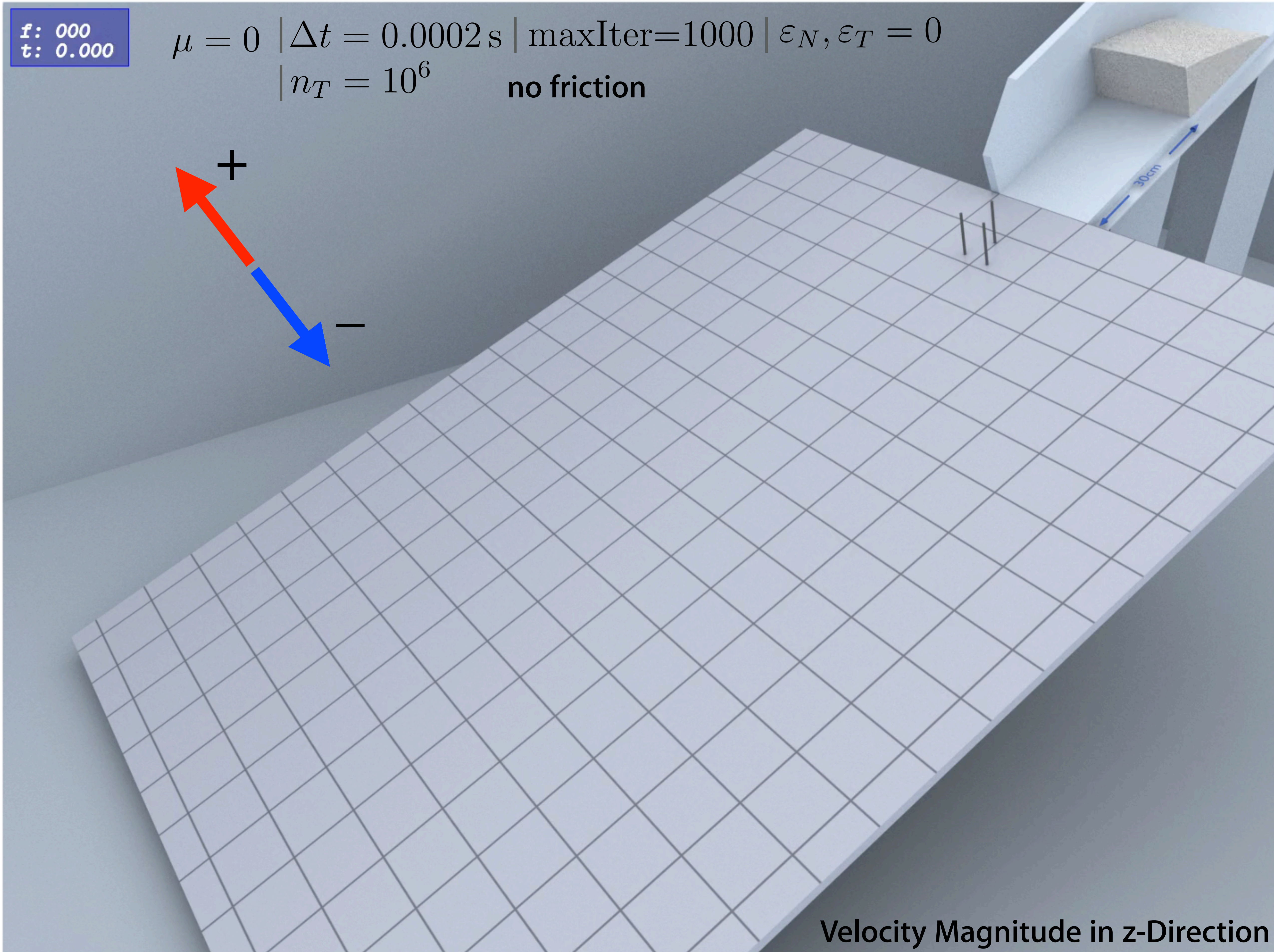
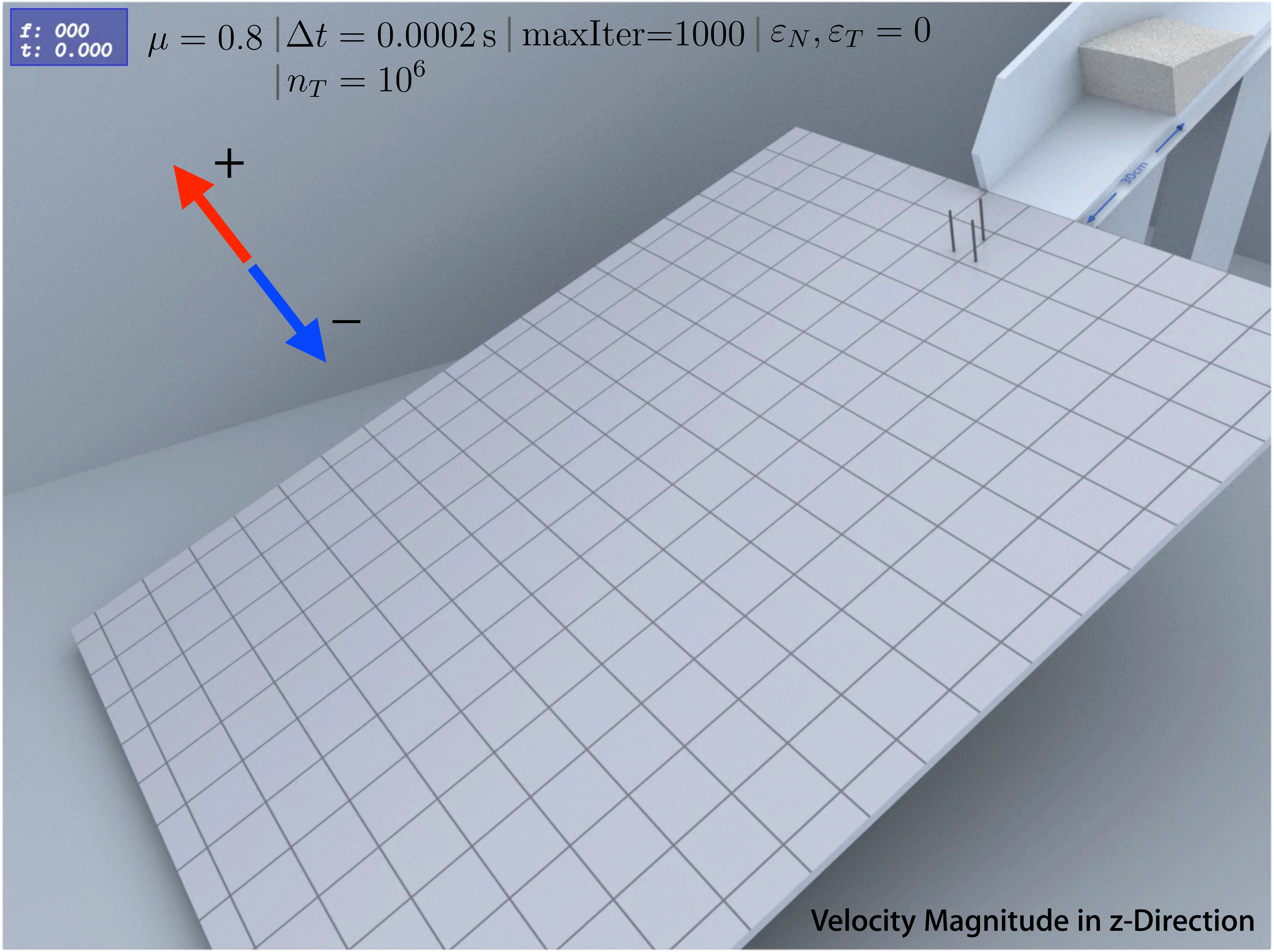
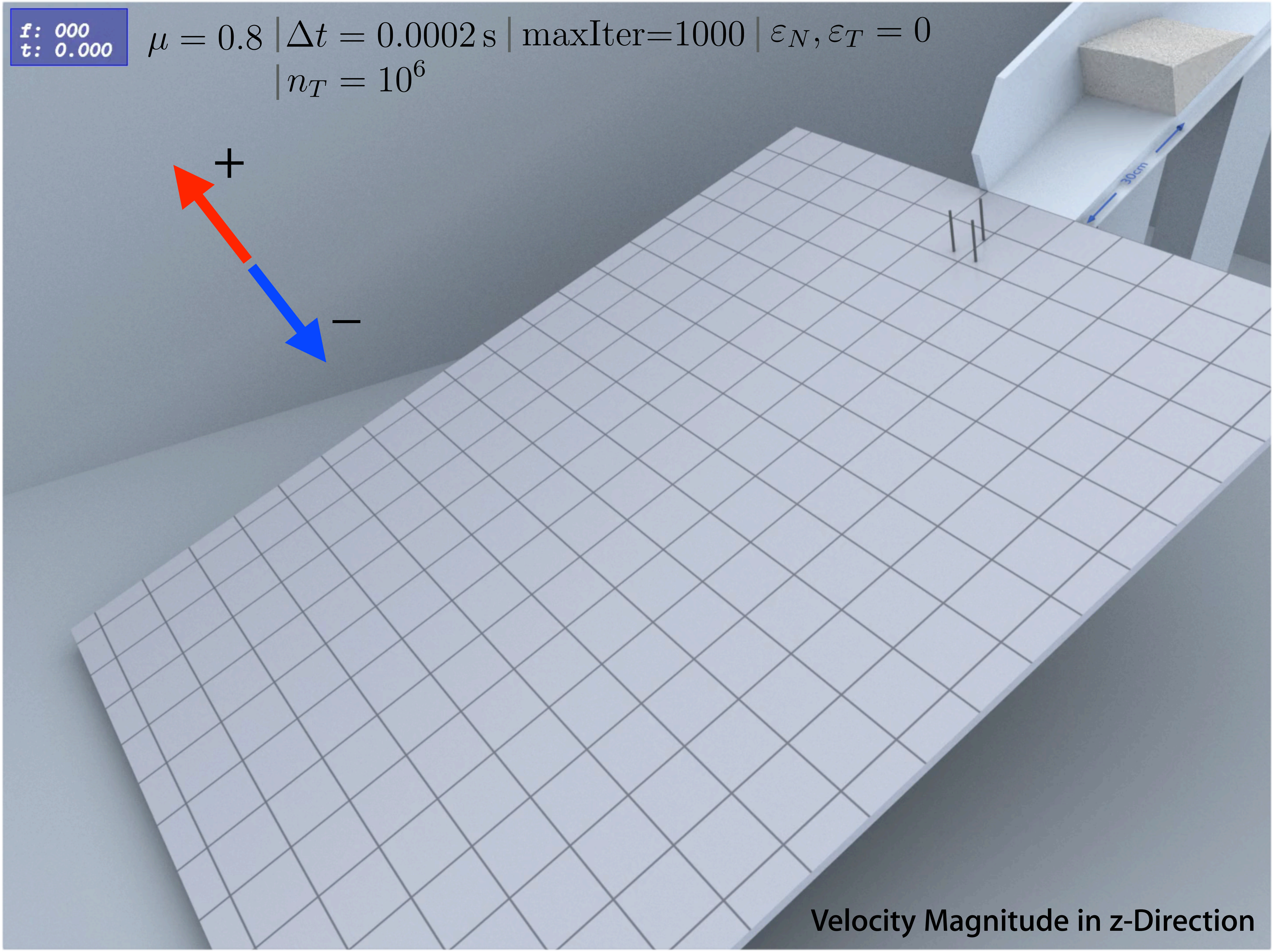
$\mu = 0 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter}{=}1000 \mid \varepsilon_N, \varepsilon_T = 0$
$\mid n_T = 10^6$
no friction
+
−
30cm
Velocity Magnitude in z-Direction

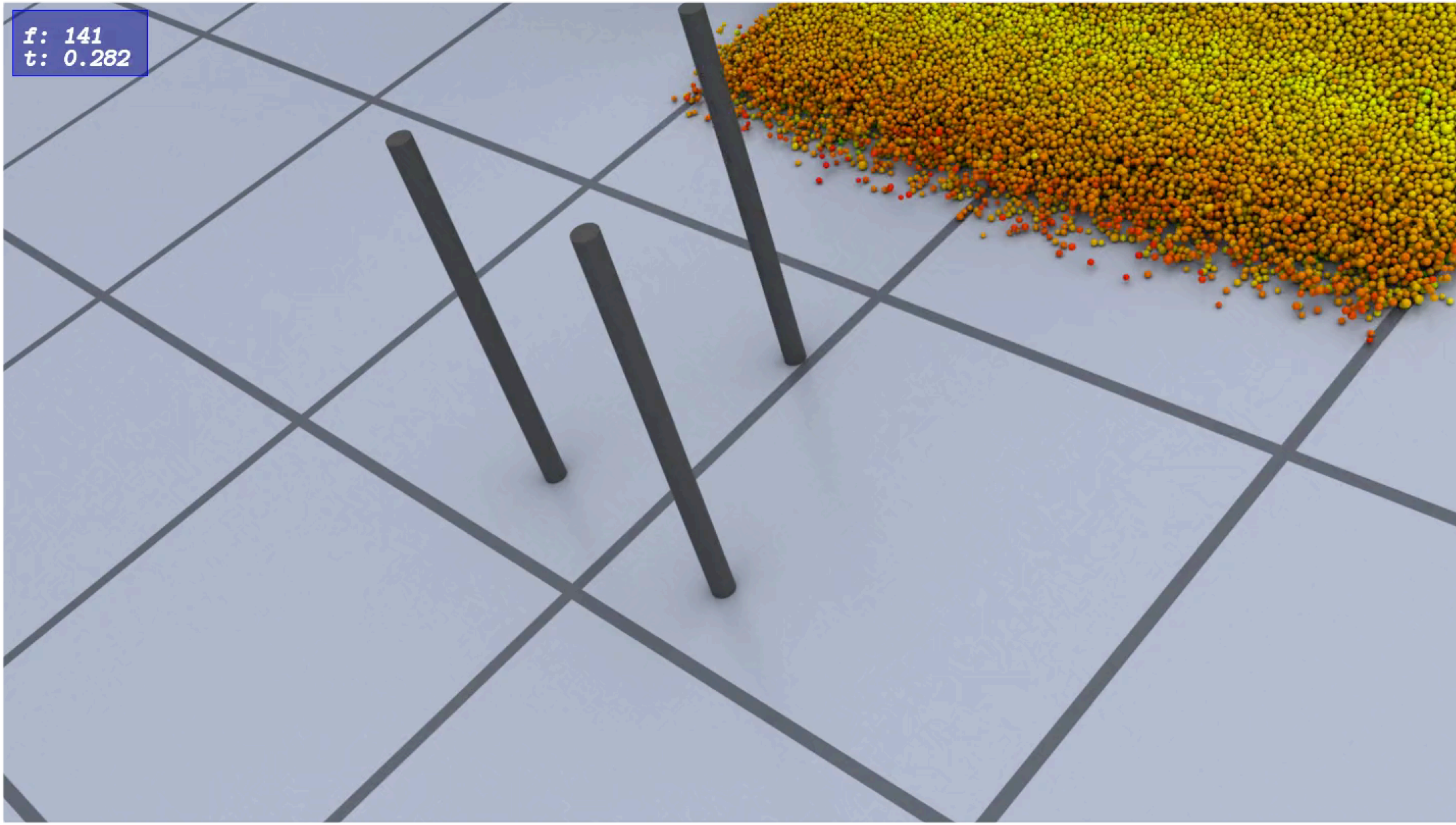


f: 000
t: 0.000
$\mu = 0 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter}{=}1000 \mid \varepsilon_N, \varepsilon_T = 0$
$\mid n_T = 10^6$
no friction
+
−
30cm
Velocity Magnitude in z-Direction


f: 000
t: 0.000
$\mu = 0.8 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter}{=}1000 \mid \varepsilon_N, \varepsilon_T = 0$
$\mid n_T = 10^6$
+
−
30cm
Velocity Magnitude in z-Direction


f: 000
t: 0.000
$\mu = 0.8 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter}{=}1000 \mid \varepsilon_N, \varepsilon_T = 0$
$\mid n_T = 10^6$
+
−
30cm
Velocity Magnitude in z-Direction

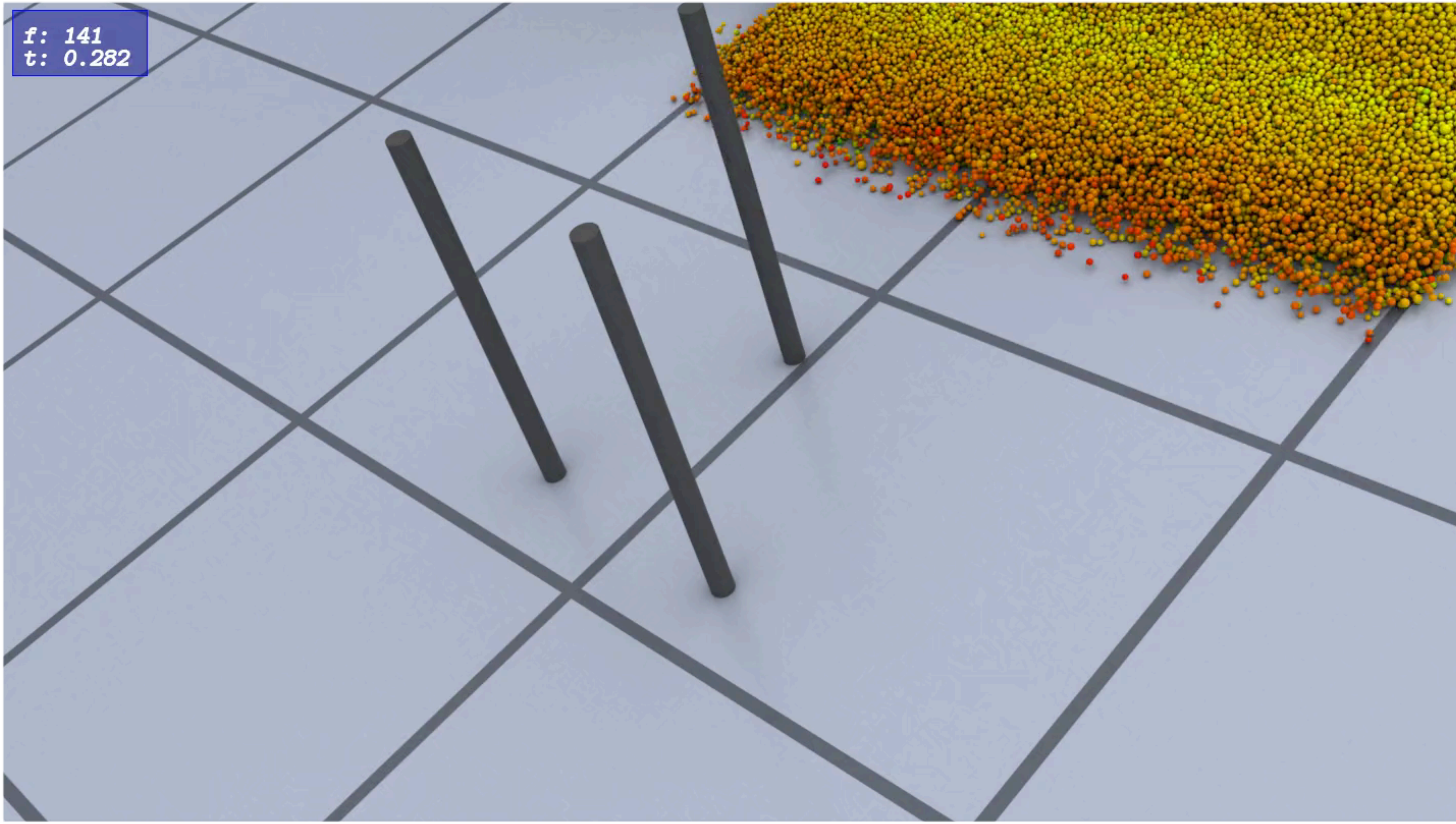
Velocity Magnitude



$\mu = 0.8 \,|\, \Delta t = 0.0002\,\mathrm{s} \,|\, \mathrm{maxIter}{=}1000 \,|\, \varepsilon_N, \varepsilon_T = 0 \,|\, n_T = 10^6$

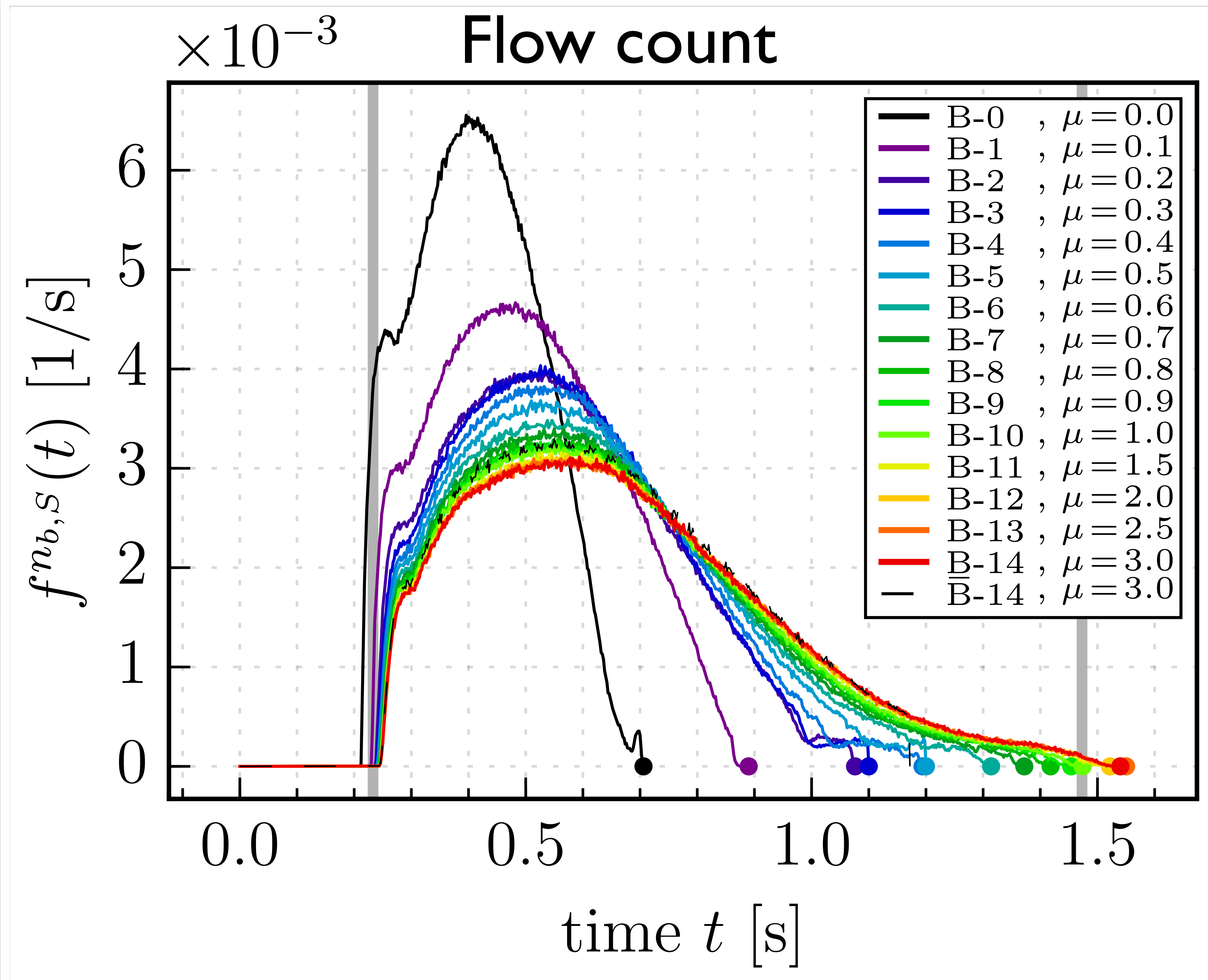
best match

Velocity Magnitude

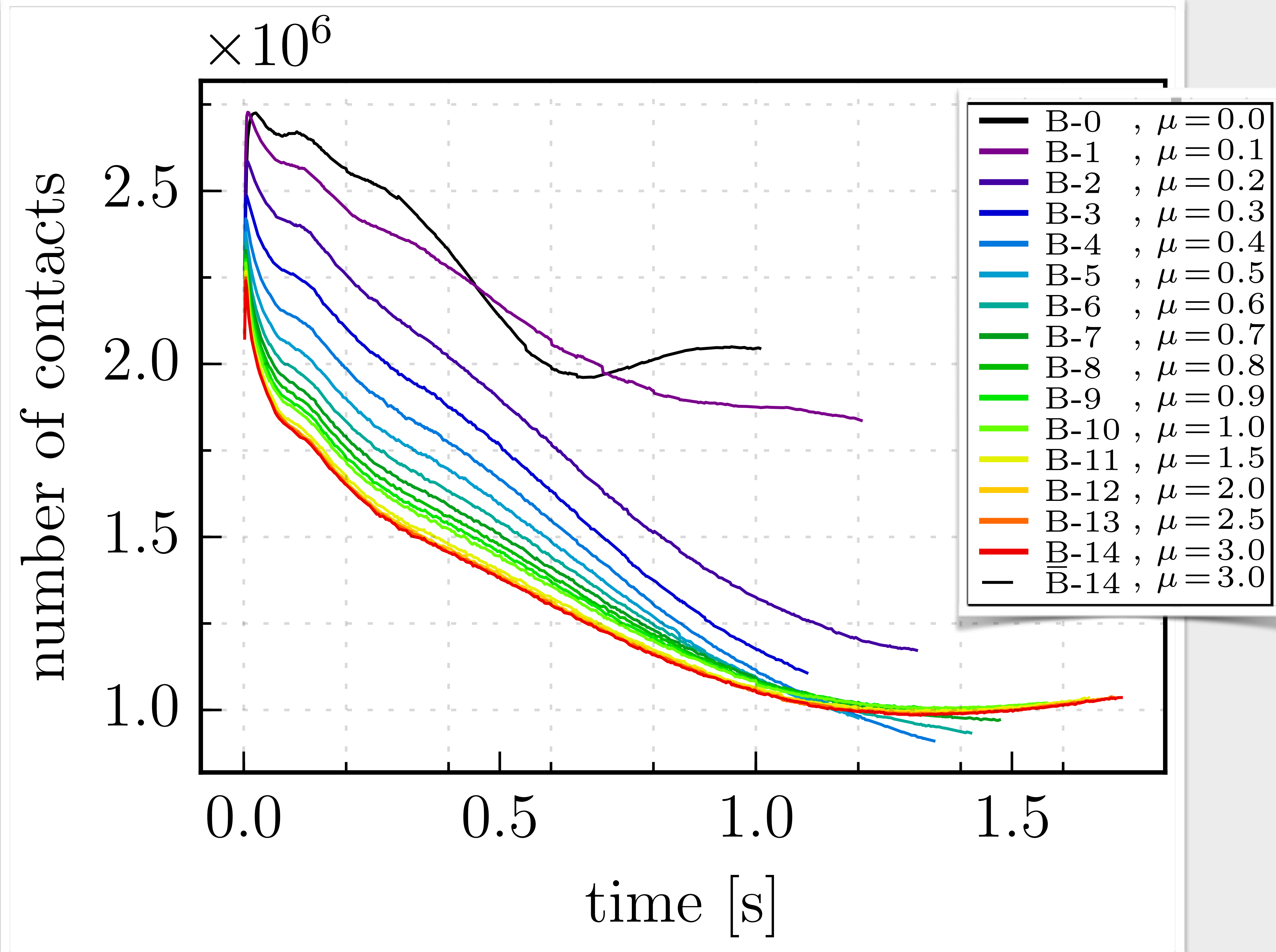


$\mu = 0.8 \mid \Delta t = 0.0002\,\mathrm{s} \mid \mathrm{maxIter{=}1000} \mid \varepsilon_N, \varepsilon_T = 0 \mid n_T = 10^6$ **best match**

Comparison

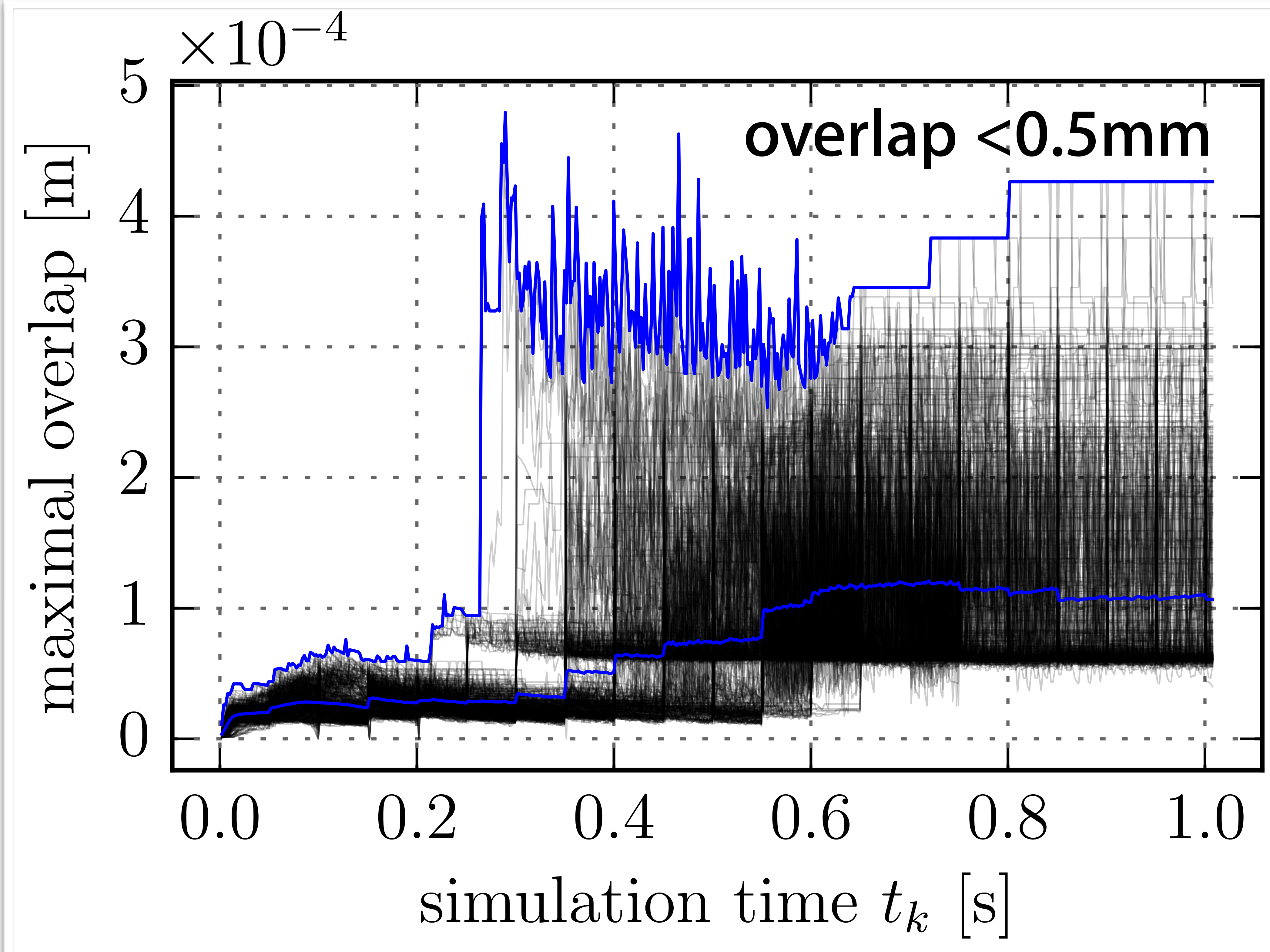

Flow count
×10^−3
$f n_{b,S}(t)$ [1/s]
time t [s]
0
1
2
3
4
5
6
0.0
0.5
1.0
1.5
B-0 , μ = 0.0
B-1 , μ = 0.1
B-2 , μ = 0.2
B-3 , μ = 0.3
B-4 , μ = 0.4
B-5 , μ = 0.5
B-6 , μ = 0.6
B-7 , μ = 0.7
B-8 , μ = 0.8
B-9 , μ = 0.9
B-10 , μ = 1.0
B-11 , μ = 1.5
B-12 , μ = 2.0
B-13 , μ = 2.5
B-14 , μ = 3.0
B-14 , μ = 3.0

Comparison


$\times 10^6$
number of contacts
2.5
2.0
1.5
1.0
0.0
0.5
1.0
1.5
time [s]
B-0 , $\mu = 0.0$
B-1 , $\mu = 0.1$
B-2 , $\mu = 0.2$
B-3 , $\mu = 0.3$
B-4 , $\mu = 0.4$
B-5 , $\mu = 0.5$
B-6 , $\mu = 0.6$
B-7 , $\mu = 0.7$
B-8 , $\mu = 0.8$
B-9 , $\mu = 0.9$
B-10 , $\mu = 1.0$
B-11 , $\mu = 1.5$
B-12 , $\mu = 2.0$
B-13 , $\mu = 2.5$
B-14 , $\mu = 3.0$
$\overline{\text{B}}$-14 , $\mu = 3.0$

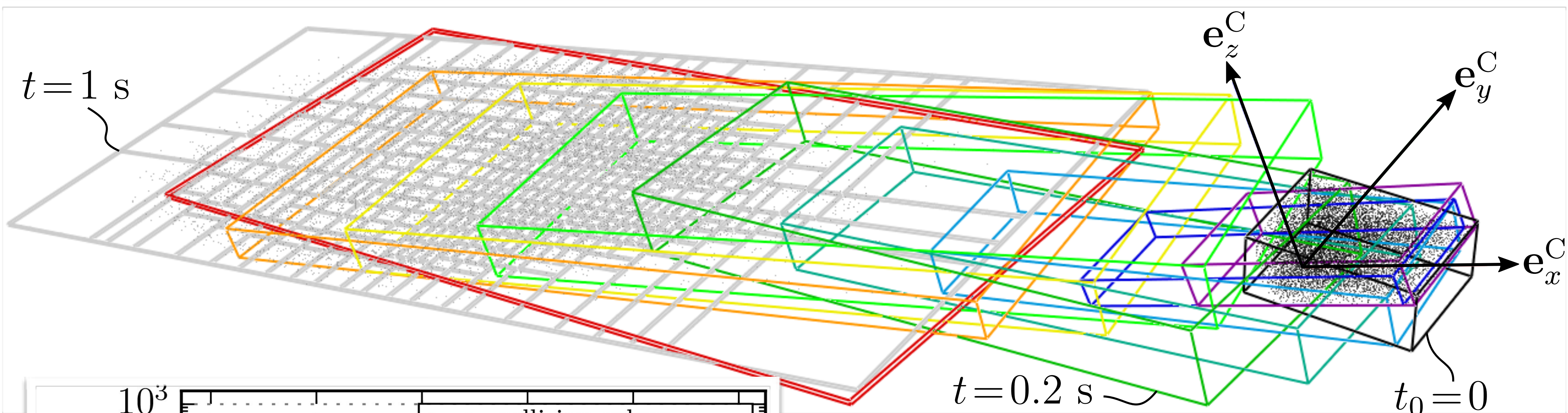
Penetration Depth

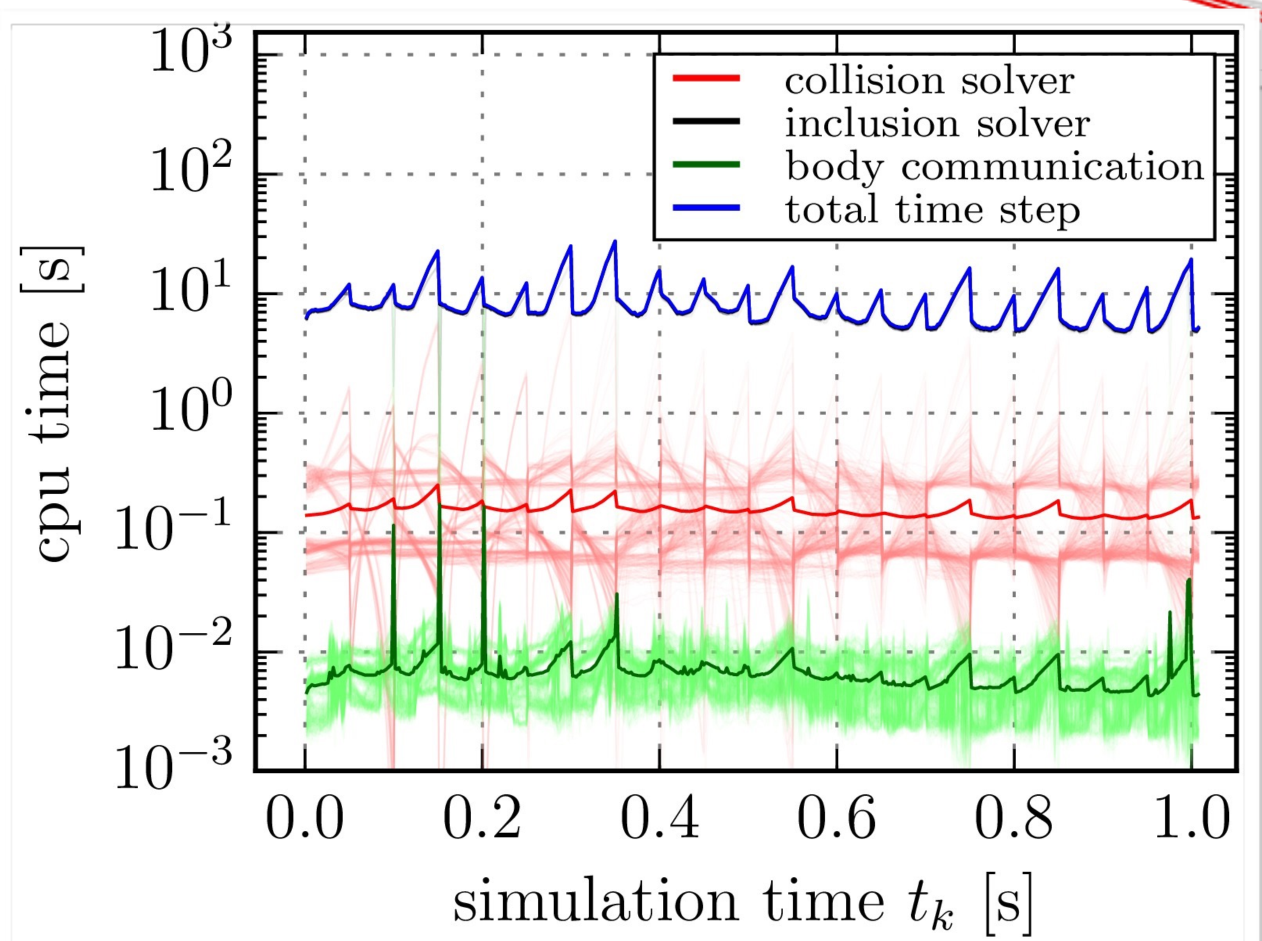
$\mu = 0$ less overlap for $\mu > 0$



Simulation Topologies

$\mu = 0$





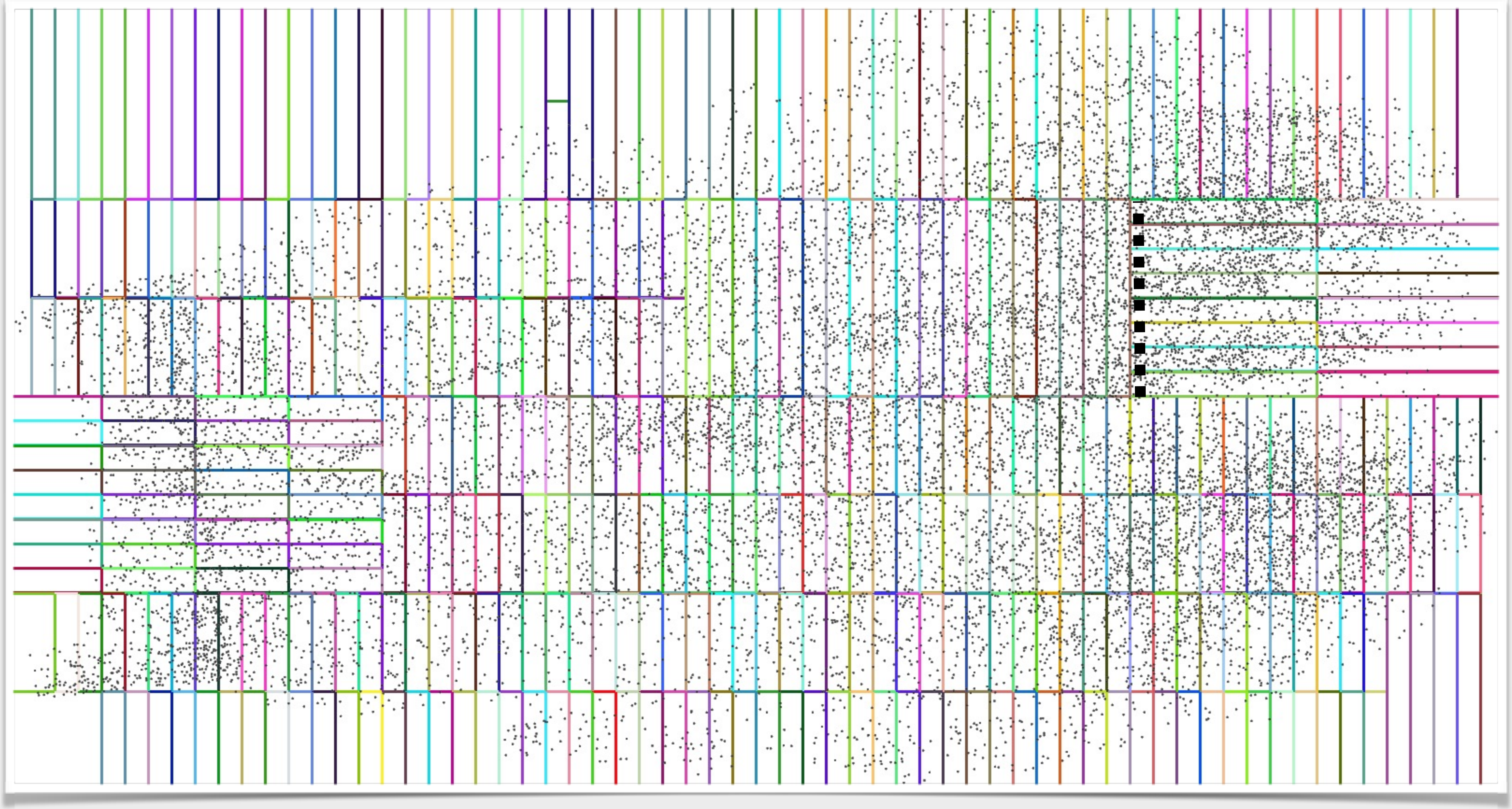
Communication: ~ 0.02 s

Collision: ~ 0.15 s

Time Step: 10 - 30 s

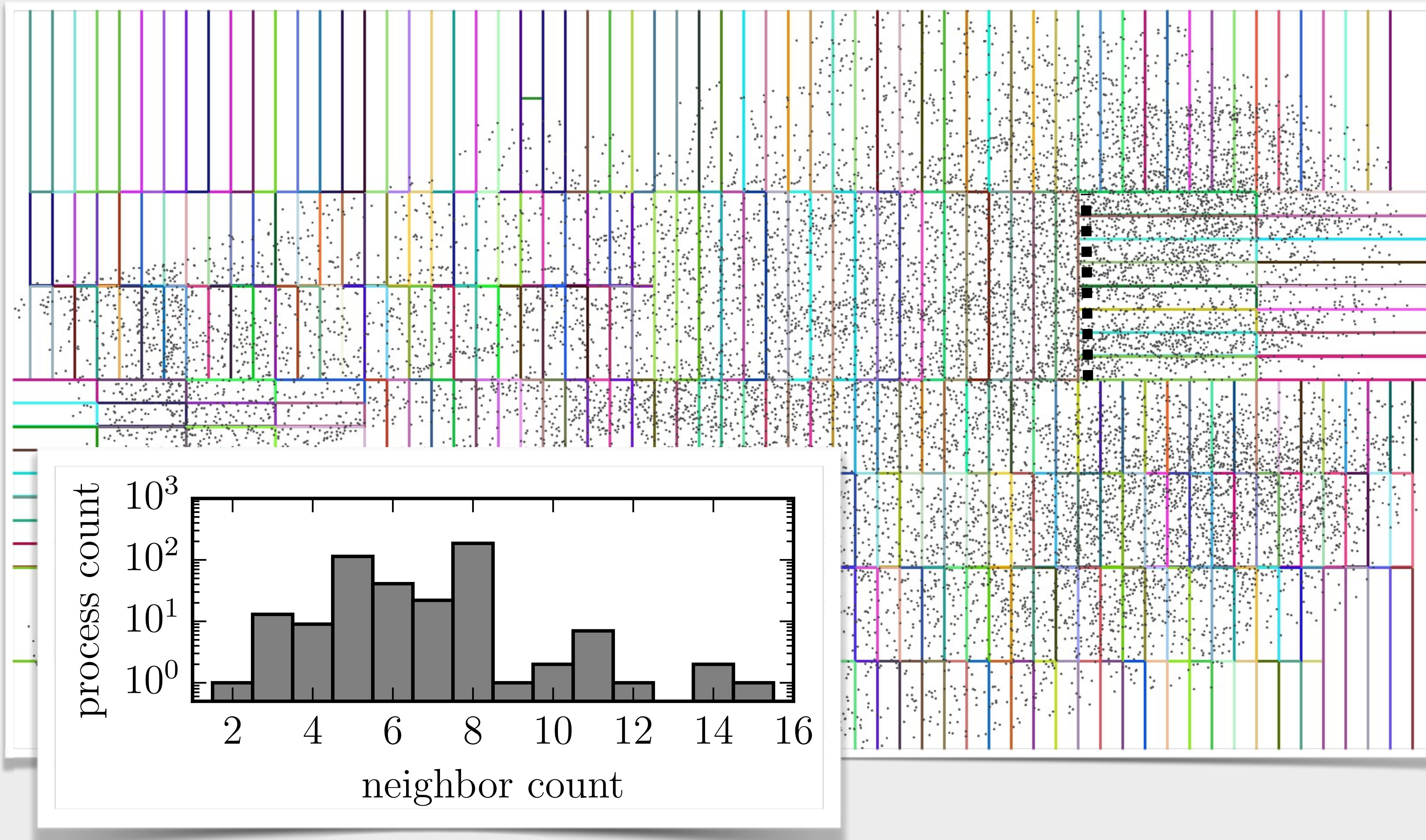
KdTree Topology

worse splitting



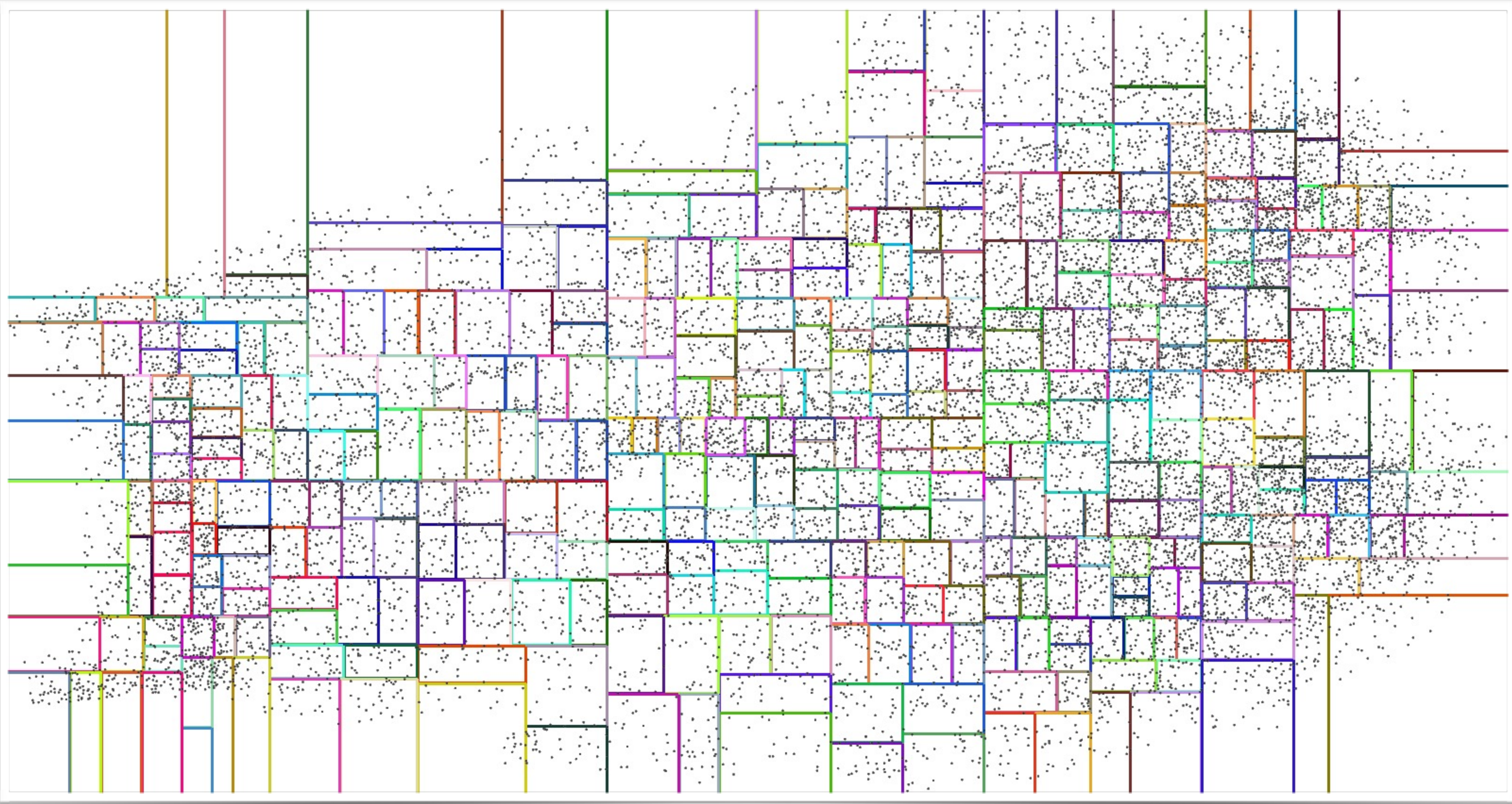
KdTree Topology

worse splitting



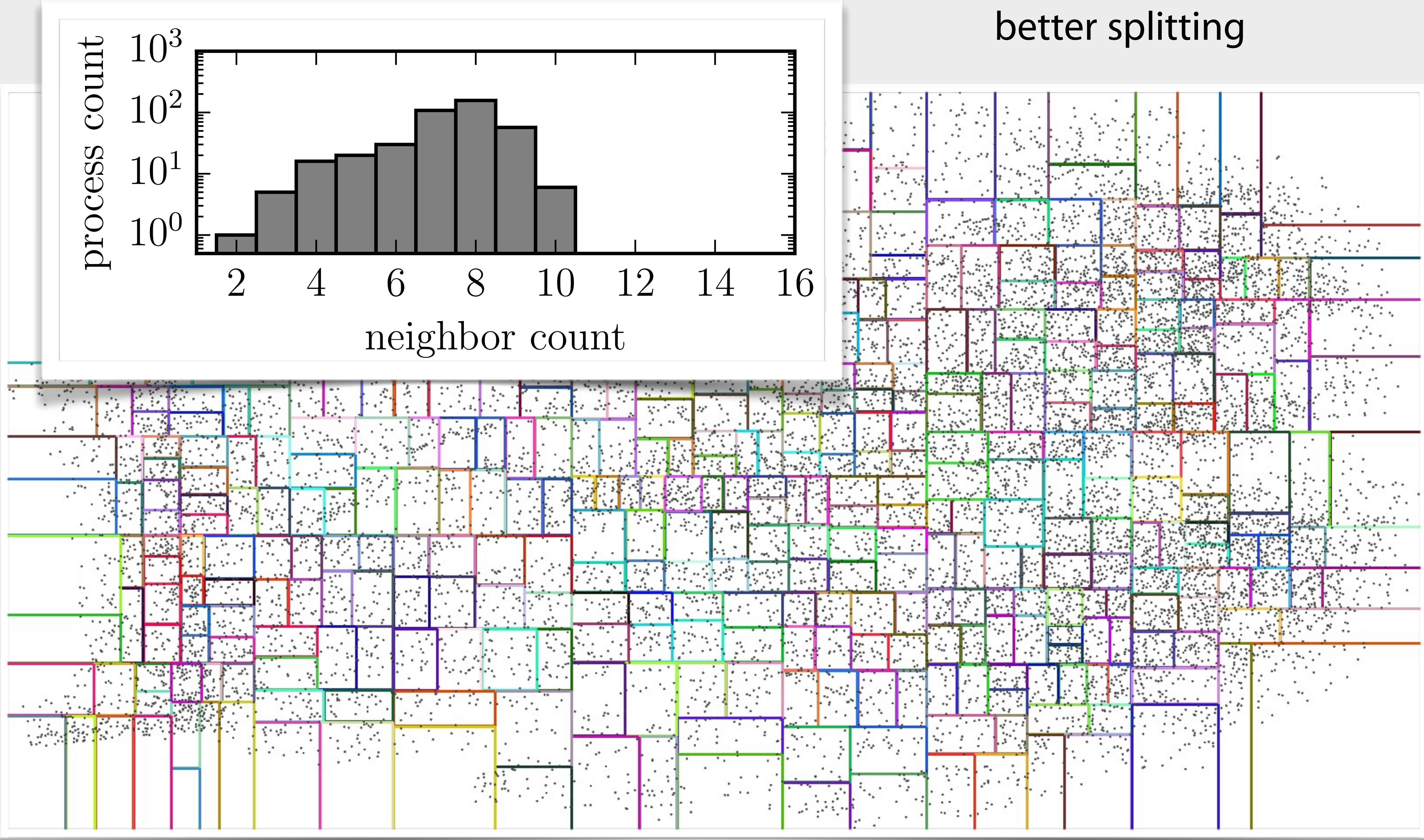
KdTree Topology

better splitting

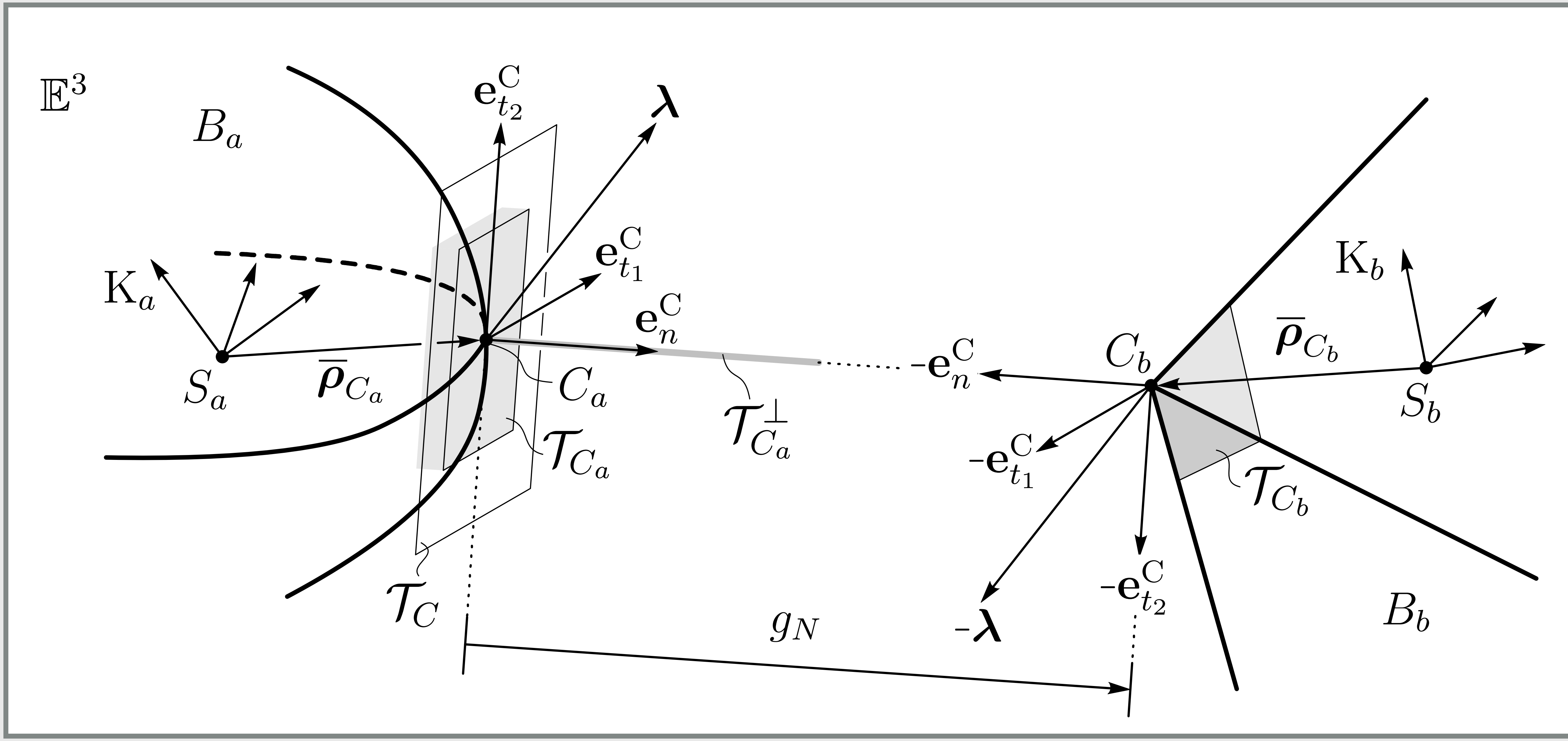


KdTree Topology

better splitting



Contact Configuration



Applying the Fundamental Axioms

Principle of Virtual Work for the Scalable Body:

$$ {}_{\mathrm{I}}\boldsymbol{\xi}_{\mathrm{scal}} = s\ {}_{\mathrm{I}}\mathbf{R}(\mathbf{p})\ {}_{\mathrm{I}}\boldsymbol{\rho} + {}_{\mathrm{I}}\mathbf{r}_R $$

$$ 0 = \delta W({}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}}) = \delta W^{\mathrm{dyn}} + \delta W_I^{\mathrm{int}} + \underbrace{\delta W_C^{\mathrm{int}}}_{=0 \text{ by d'Alembert-Lagrange}} + \delta W^{\mathrm{ext}} \qquad \forall_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}},\ t $$

$$ = \int_B {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}}^{\top}\ {}_{\mathrm{I}}\ddot{\boldsymbol{\xi}}_{\mathrm{scal}}\mathrm{d}m - \delta s\lambda_s - {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}}^{\top}\ {}_{\mathrm{I}}\mathrm{d}\mathbf{F}^{\mathrm{e}} \qquad \forall_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}},\ t\ . $$

$$ \begin{bmatrix} {}_{\mathrm{I}}\delta\mathbf{r}_S \\ \delta s \\ \delta\mathbf{a} \end{bmatrix}^{\top} \left(\begin{bmatrix} m\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathrm{Tr}({}_{\mathrm{K}}\boldsymbol{\Theta}_S) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}_{\mathrm{K}}\boldsymbol{\Theta}_S \end{bmatrix} \dot{\mathbf{u}}_p + \frac{1}{s} \begin{bmatrix} \mathbf{0} \\ -\boldsymbol{\alpha}^{\top}{}_{\mathrm{K}}\boldsymbol{\Theta}_S\boldsymbol{\alpha} \\ (\nu\mathbf{I} + \tilde{\boldsymbol{\alpha}})_{\mathrm{K}}\boldsymbol{\Theta}_S\boldsymbol{\alpha} \end{bmatrix} - \begin{bmatrix} {}_{\mathrm{I}}\mathbf{F}^{\mathrm{e}} \\ E_S^{\mathrm{e}} + \lambda_s \\ {}_{\mathrm{K}}\mathbf{M}_S^{\mathrm{e}} \end{bmatrix} \right) = 0 $$

$$ \forall\ {}_{\mathrm{I}}\delta\mathbf{r}_S, \delta s, \delta\mathbf{a} $$

Applying the Fundamental Axioms

Principle of Virtual Work for the Scalable Body:

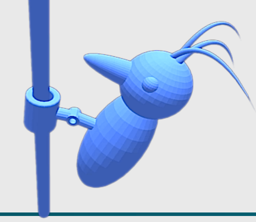
$$\boxed{{}_{\mathrm{I}}\boldsymbol{\xi}_{\mathrm{scal}} = s\;{}_{\mathrm{I}}\mathbf{R}(\mathbf{p})\;{}_{\mathrm{I}}\boldsymbol{\rho} + {}_{\mathrm{I}}\mathbf{r}_R}$$

$$0 = \delta W({}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}}) = \delta W^{\mathrm{dyn}} + \delta W_I^{\mathrm{int}} + \underbrace{\delta W_C^{\mathrm{int}}}_{=0 \text{ by d'Alembert-Lagrange}} + \delta W^{\mathrm{ext}} \qquad \forall_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}},\, t$$

$$= \int_B {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}}{}^{\top}\,{}_{\mathrm{I}}\ddot{\boldsymbol{\xi}}_{\mathrm{scal}}\mathrm{d}m - \delta s\lambda_s - {}_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}}{}^{\top}\,{}_{\mathrm{I}}\mathrm{d}\mathbf{F}^{\mathrm{e}} \qquad \forall_{\mathrm{I}}\delta\boldsymbol{\xi}_{\mathrm{scal}},\, t\;.$$

$$\begin{bmatrix} {}_{\mathrm{I}}\delta\mathbf{r}_S \\ \delta s \\ \delta\mathbf{a} \end{bmatrix}^{\top} \left(\begin{bmatrix} m\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\,\mathrm{Tr}({}_{\mathrm{K}}\boldsymbol{\Theta}_S) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}_{\mathrm{K}}\boldsymbol{\Theta}_S \end{bmatrix} \dot{\mathbf{u}}_p + \frac{1}{s} \begin{bmatrix} \mathbf{0} \\ -\boldsymbol{\alpha}^{\top}{}_{\mathrm{K}}\boldsymbol{\Theta}_S\boldsymbol{\alpha} \\ (\nu\mathbf{I} + \tilde{\boldsymbol{\alpha}})_{\mathrm{K}}\boldsymbol{\Theta}_S\boldsymbol{\alpha} \end{bmatrix} - \begin{bmatrix} {}_{\mathrm{I}}\mathbf{F}^{\mathrm{e}} \\ E_S^{\mathrm{e}} + \lambda_s \\ {}_{\mathrm{K}}\mathbf{M}_S^{\mathrm{e}} \end{bmatrix} \right) = 0$$

$$\forall\; {}_{\mathrm{I}}\delta\mathbf{r}_S, \delta s, \delta\mathbf{a}$$



Applying the Fundamental Axioms

Principle of Virtual Work for the Scalable Body:

scalable body motion: $$\boxed{{}_I\boldsymbol{\xi}_{\mathrm{scal}} = s\ {}_I\mathbf{R}(\mathbf{p})\ {}_I\boldsymbol{\rho} + {}_I\mathbf{r}_R}$$

$$0 = \delta W({}_I\delta\boldsymbol{\xi}_{\mathrm{scal}}) = \delta W^{\mathrm{dyn}} + \delta W_I^{\mathrm{int}} + \underbrace{\delta W_C^{\mathrm{int}}}_{=0 \text{ by d'Alembert-Lagrange}} + \delta W^{\mathrm{ext}} \qquad \forall_I\delta\boldsymbol{\xi}_{\mathrm{scal}},\ t$$

$$= \int_B {}_I\delta\boldsymbol{\xi}_{\mathrm{scal}}^{\top}\ {}_I\ddot{\boldsymbol{\xi}}_{\mathrm{scal}}\mathrm{d}m - \delta s\lambda_s - {}_I\delta\boldsymbol{\xi}_{\mathrm{scal}}^{\top}\ {}_I\mathrm{d}\mathbf{F}^{\mathrm{e}} \qquad \forall_I\delta\boldsymbol{\xi}_{\mathrm{scal}},\ t\ .$$

$$\begin{bmatrix} {}_I\delta\mathbf{r}_S \\ \delta s \\ \delta\mathbf{a} \end{bmatrix}^{\top} \left(\begin{bmatrix} m\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\mathrm{Tr}({}_K\boldsymbol{\Theta}_S) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & {}_K\boldsymbol{\Theta}_S \end{bmatrix} \dot{\mathbf{u}}_p + \frac{1}{s} \begin{bmatrix} \mathbf{0} \\ -\boldsymbol{\alpha}^{\top}\,{}_K\boldsymbol{\Theta}_S\boldsymbol{\alpha} \\ (\nu\mathbf{I} + \tilde{\boldsymbol{\alpha}})\,{}_K\boldsymbol{\Theta}_S\boldsymbol{\alpha} \end{bmatrix} - \begin{bmatrix} {}_I\mathbf{F}^{\mathrm{e}} \\ E_S^{\mathrm{e}} + \lambda_s \\ {}_K\mathbf{M}_S^{\mathrm{e}} \end{bmatrix} \right) = 0$$

$$\forall\ {}_I\delta\mathbf{r}_S, \delta s, \delta\mathbf{a}$$

Fundamental Axioms in Mechanics

Principle of Virtual Work:

At any instant of time t, the virtual work δW of a body $B \in \mathbb{E}^3$ vanishes for all virtual displacements $\delta\boldsymbol{\xi}$, that is,

$$\delta W(\delta\boldsymbol{\xi}) = \int_B \langle \mathrm{d}\mathbf{F}\,, \delta\boldsymbol{\xi}(\mathbf{x})\rangle = \delta W^{\mathrm{dyn}}(\delta\boldsymbol{\xi}) + \delta W^{\mathrm{int}}(\delta\boldsymbol{\xi}) + \delta W^{\mathrm{ext}}(\delta\boldsymbol{\xi}) = 0 \quad \forall \delta\boldsymbol{\xi}\ .$$

Fundamental Axioms in Mechanics

Principle of Virtual Work:

At any instant of time t, the virtual work δW of a body $B \in \mathbb{E}^3$ vanishes for all virtual displacements $\delta\boldsymbol{\xi}$, that is,

$$\delta W(\delta\boldsymbol{\xi}) = \int_B \langle \mathrm{d}\mathbf{F}\,, \delta\boldsymbol{\xi}(\mathbf{x})\rangle = \boxed{\delta W^{\mathrm{dyn}}(\delta\boldsymbol{\xi})} + \boxed{\delta W^{\mathrm{int}}(\delta\boldsymbol{\xi})} + \delta \boxed{W^{\mathrm{ext}}(\delta\boldsymbol{\xi})} = 0 \quad \forall \delta\boldsymbol{\xi}\ .$$

Inertia Contribution:

$$\delta W^{\mathrm{dyn}}(\delta\boldsymbol{\xi}) := -\int_B \delta\boldsymbol{\xi}(\mathbf{x})^\top \boxed{\ddot{\boldsymbol{\xi}}(\mathbf{x})\mathrm{d}m(\mathbf{x})}$$

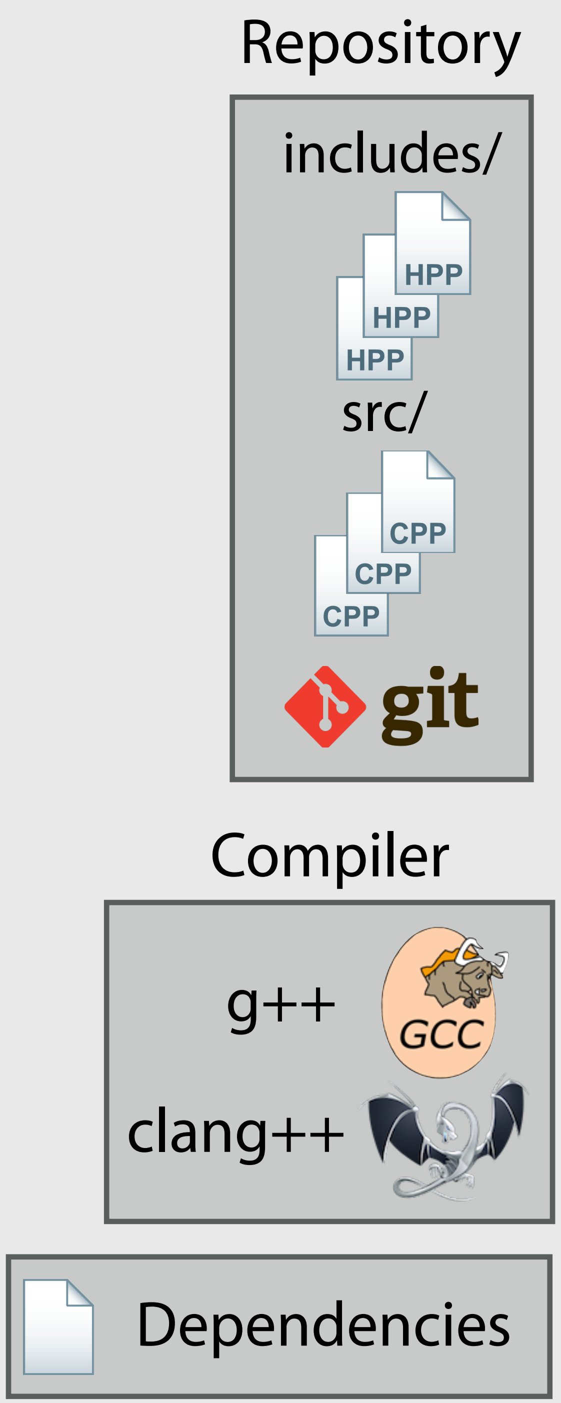
Constitutive force law:
Newton's second law

Law of Interaction as Requirement on the Internal Forces: (S. Eugster 2014)

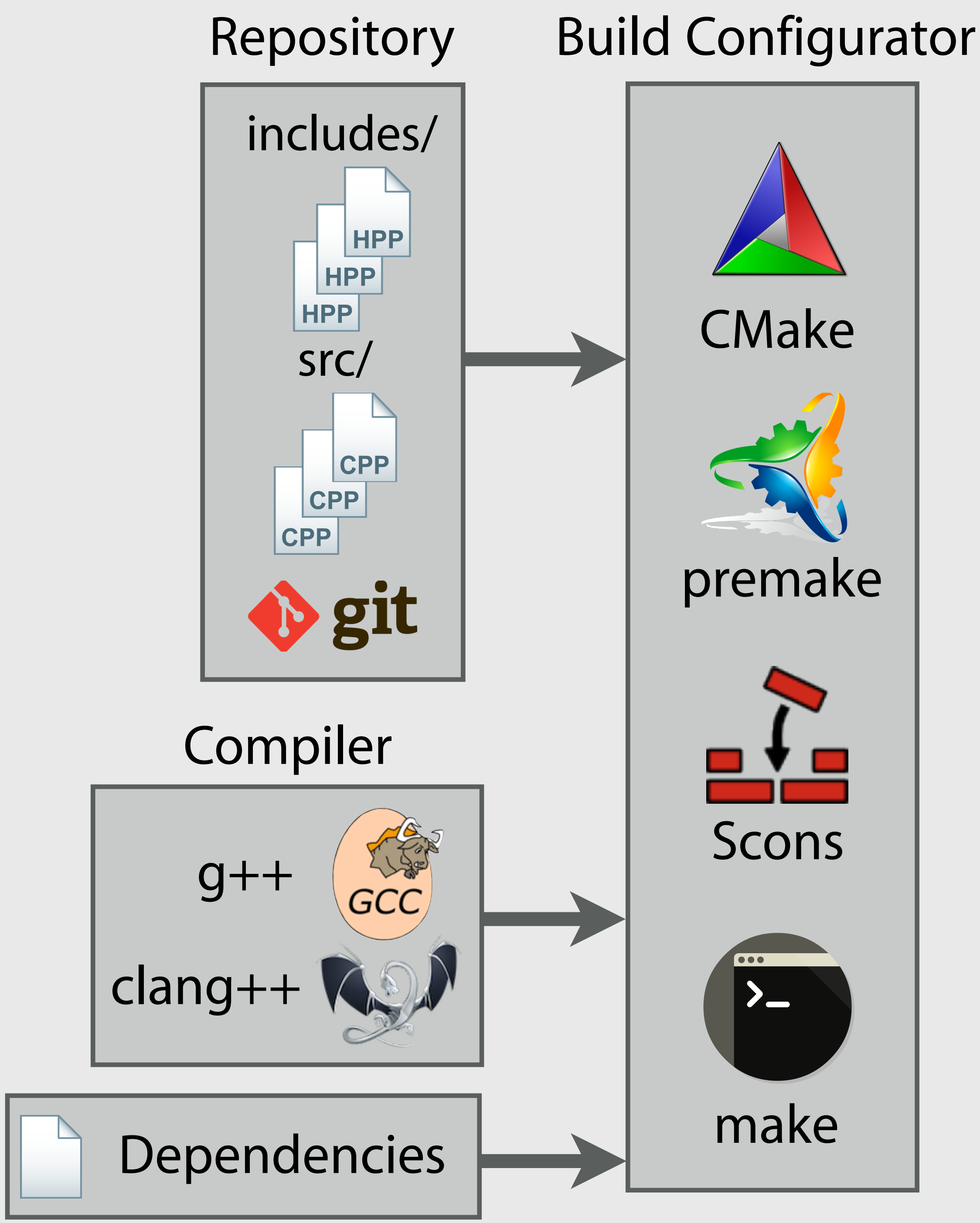
At any instant of time t, the *internal* virtual work $\delta W^{\mathrm{int}}_{B'}$ of any subsystem $B' \subseteq B$ vanishes for all *rigid* virtual displacements $\delta\boldsymbol{\xi}$, that is,

$$\delta W^{\mathrm{int}}_{B'}(\delta\boldsymbol{\xi}) = 0 \quad \forall\ \delta\boldsymbol{\xi} \text{ rigidifying}, \ \ B' \subseteq B\ .$$

Modern Software Development in C++

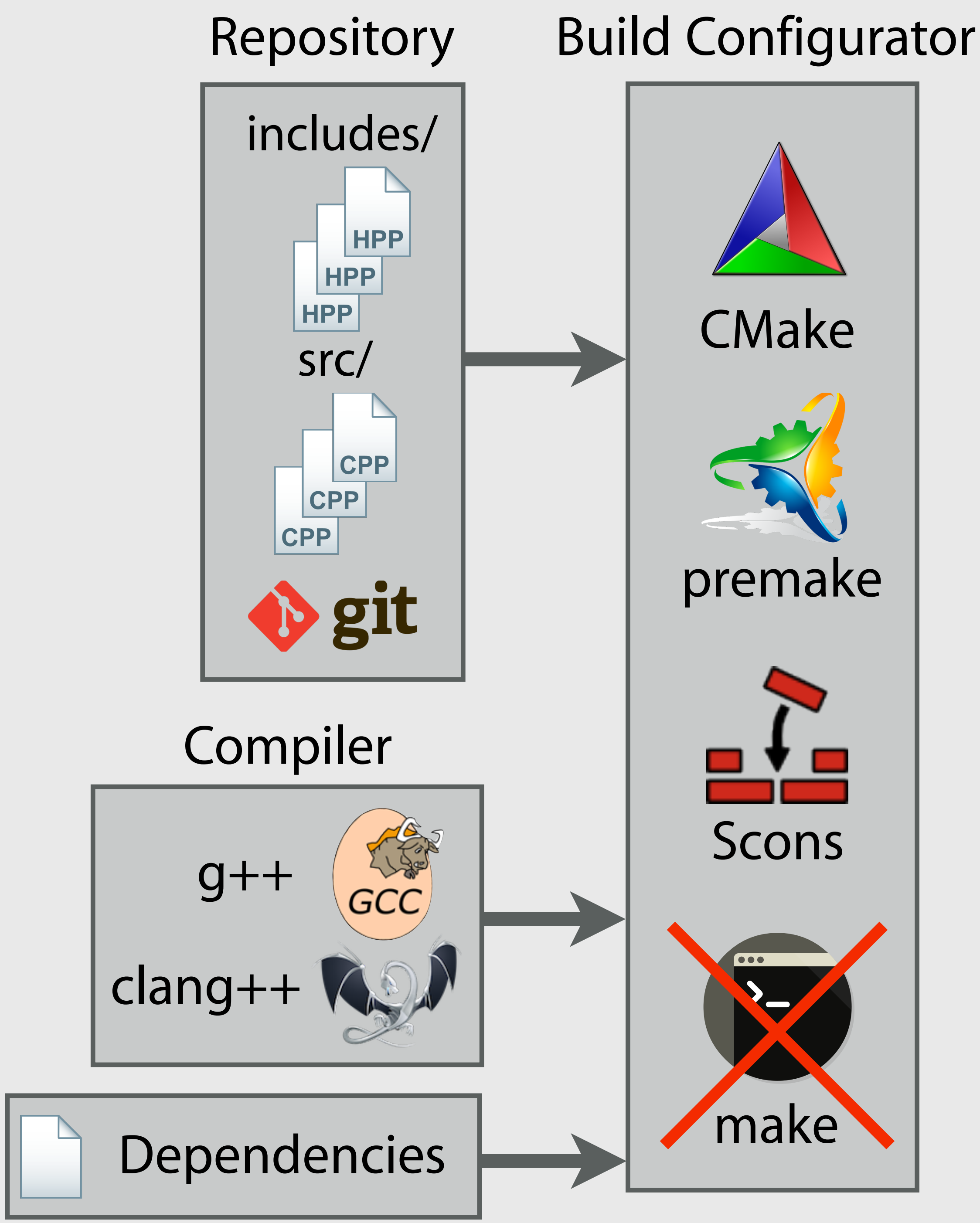


Modern Software Development in C++



- Find Dependencies
- Setup Build Settings

Modern Software Development in C++



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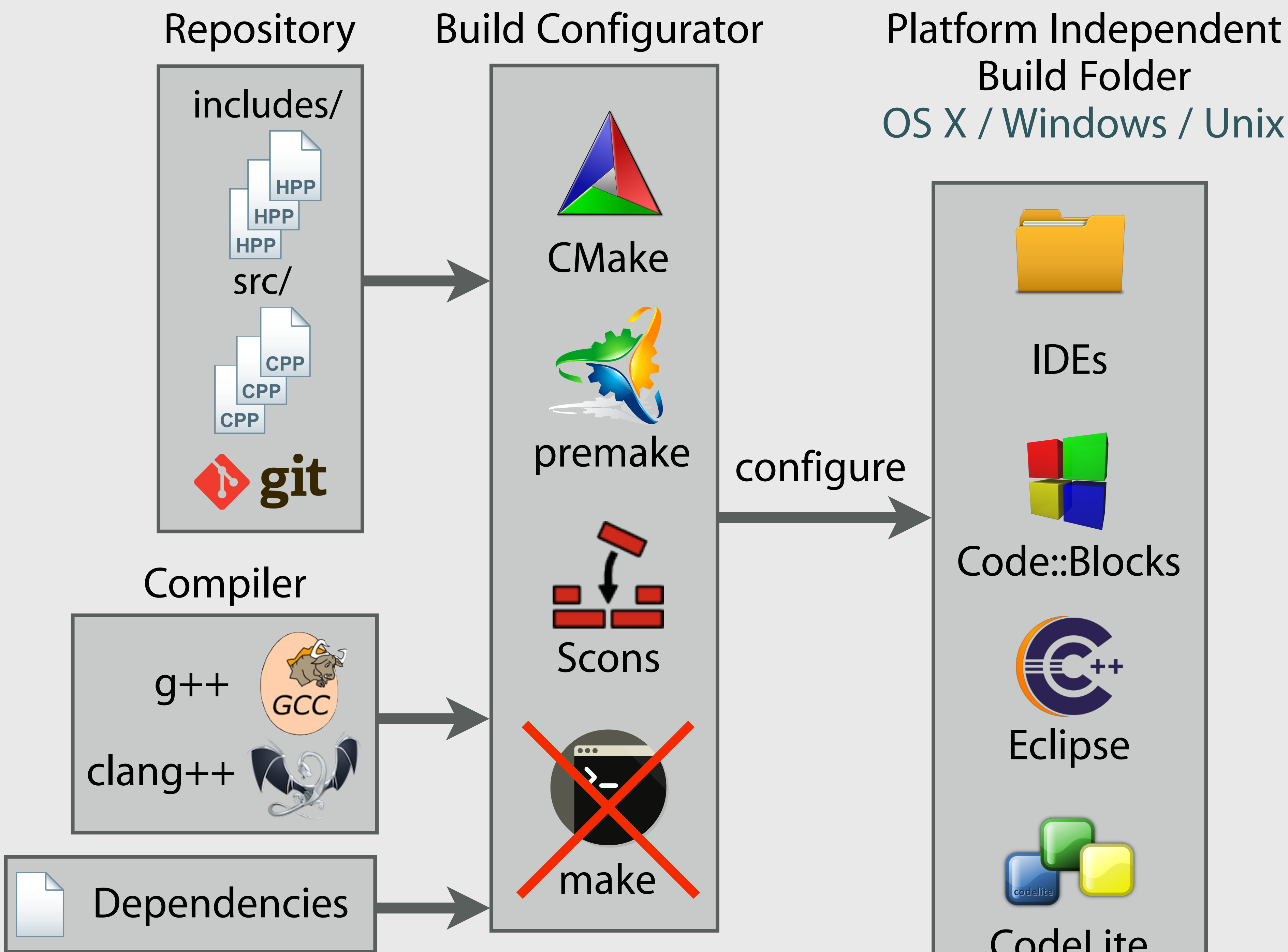
Modern Software Development in C++



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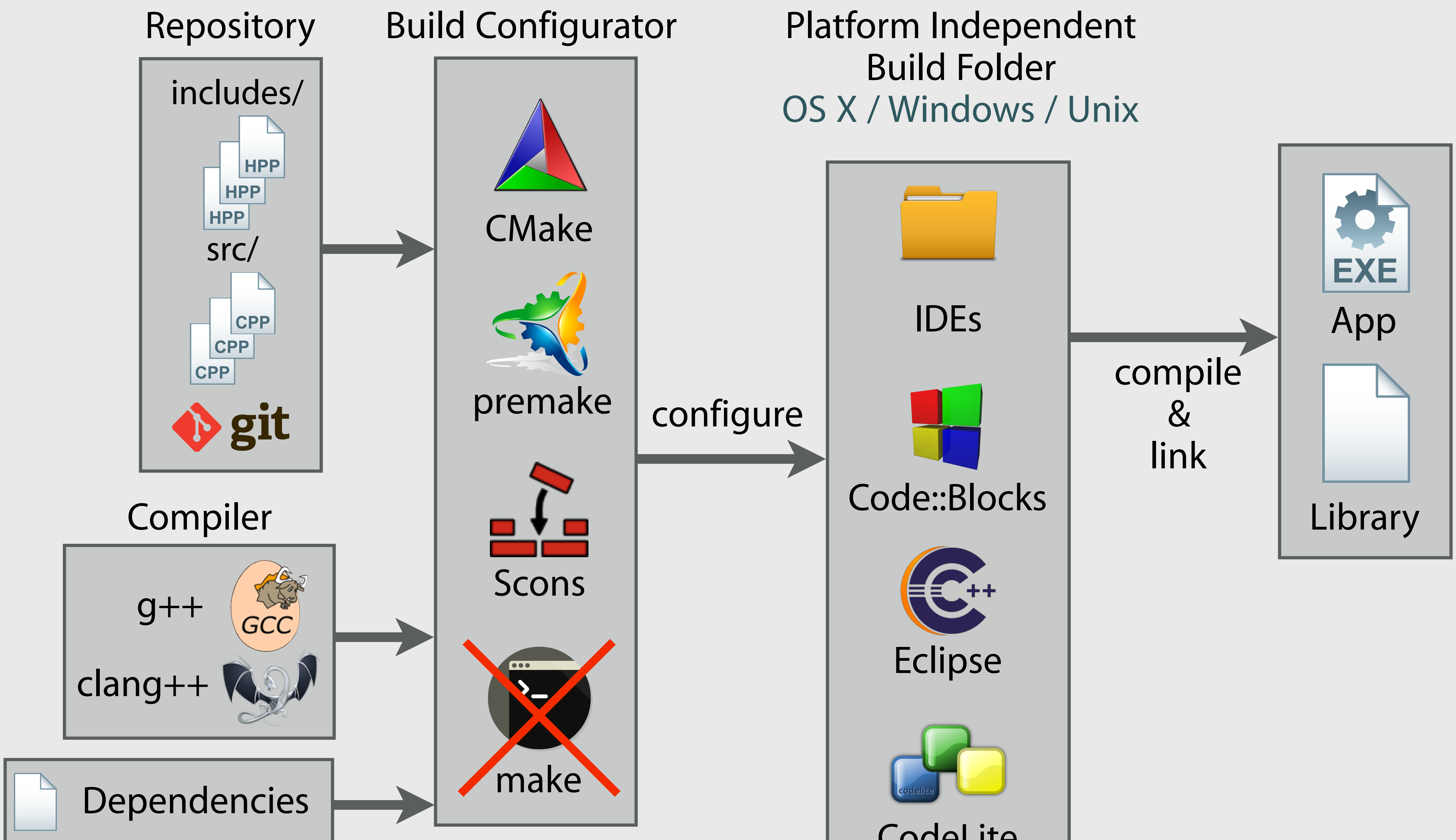
Modern Software Development in C++



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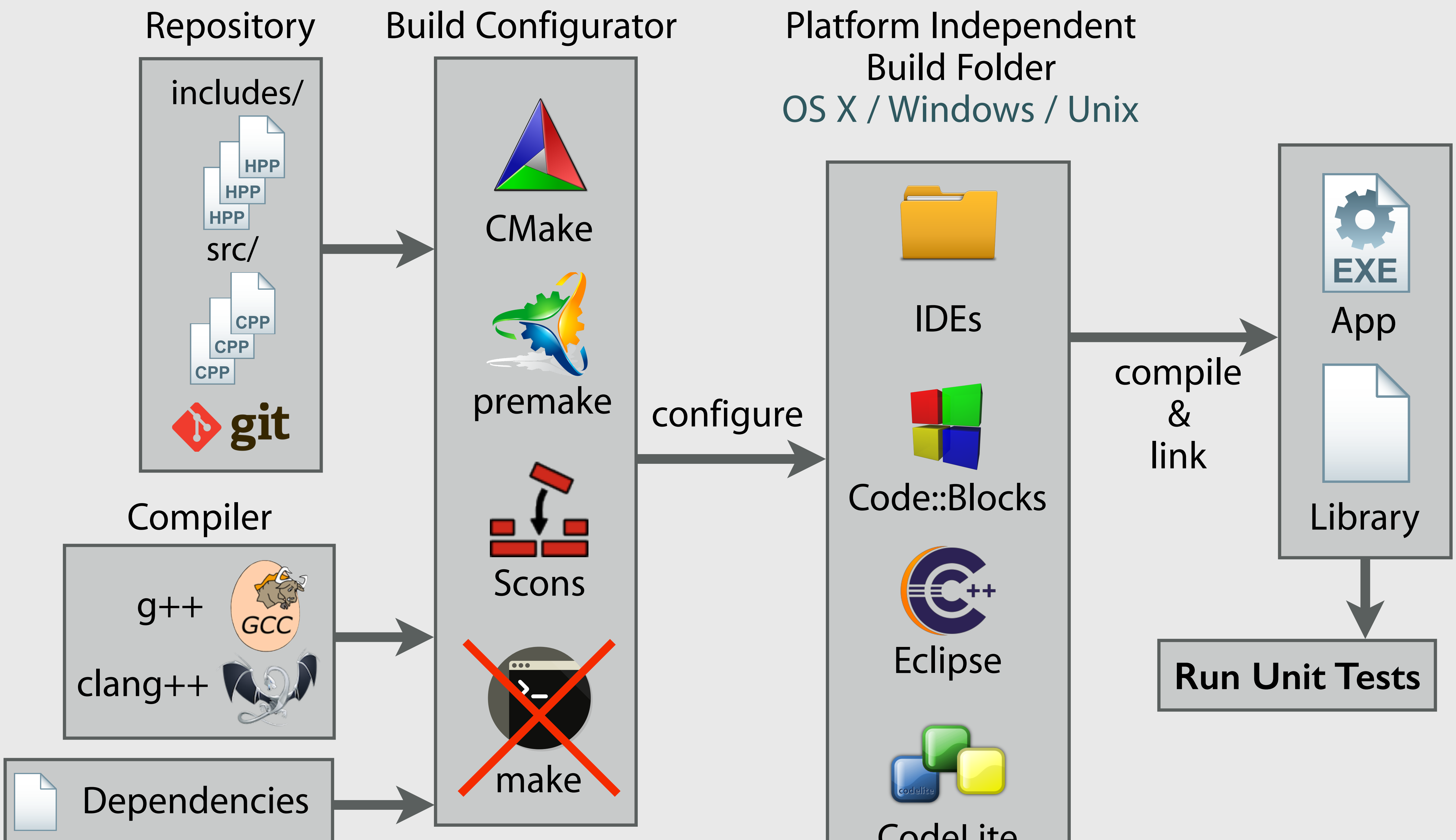
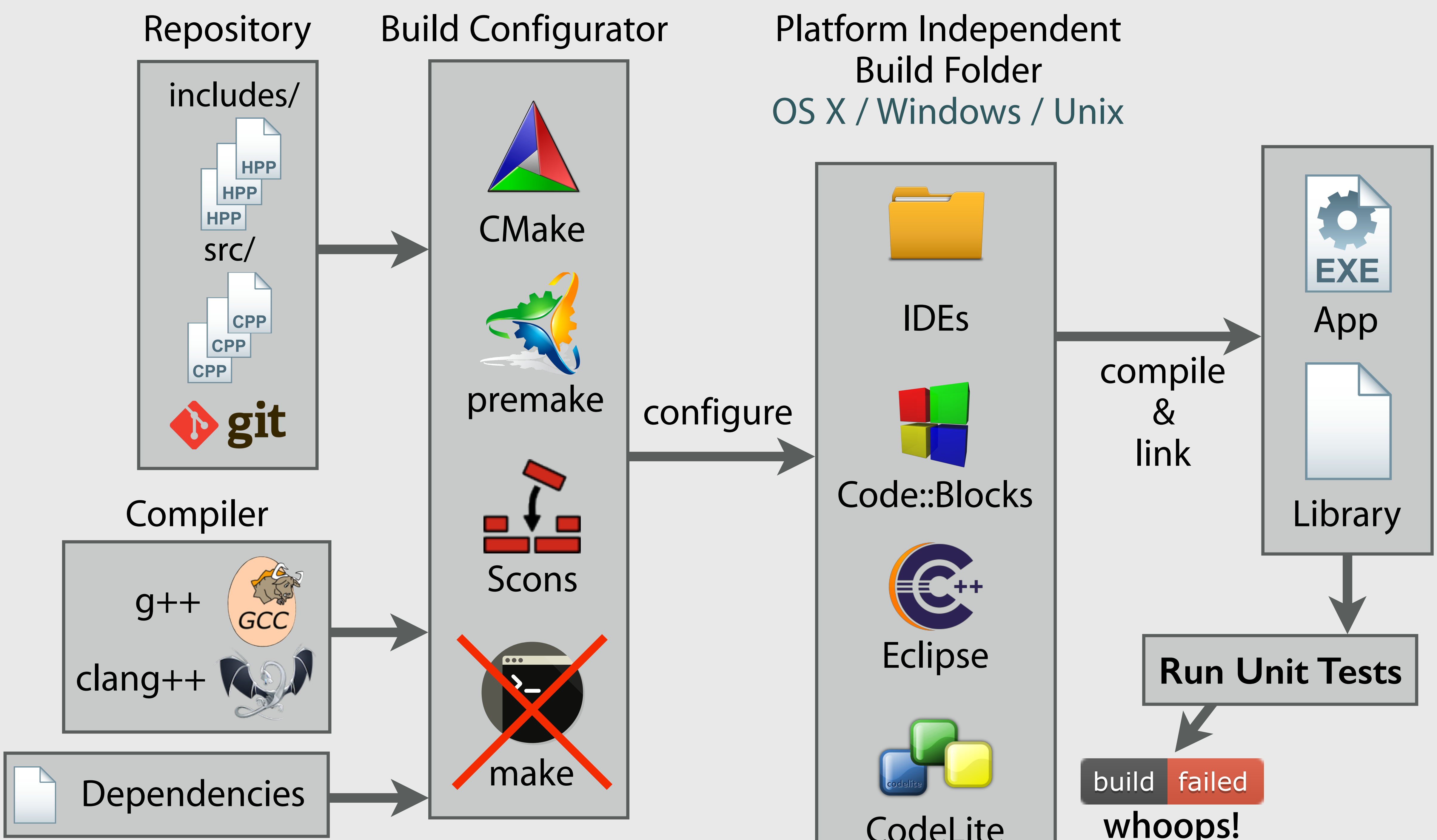
- Setup Build Settings

Modern Software Development in C++



- Find Dependencies
- Setup Build Settings

Modern Software Development in C++



- Find Dependencies
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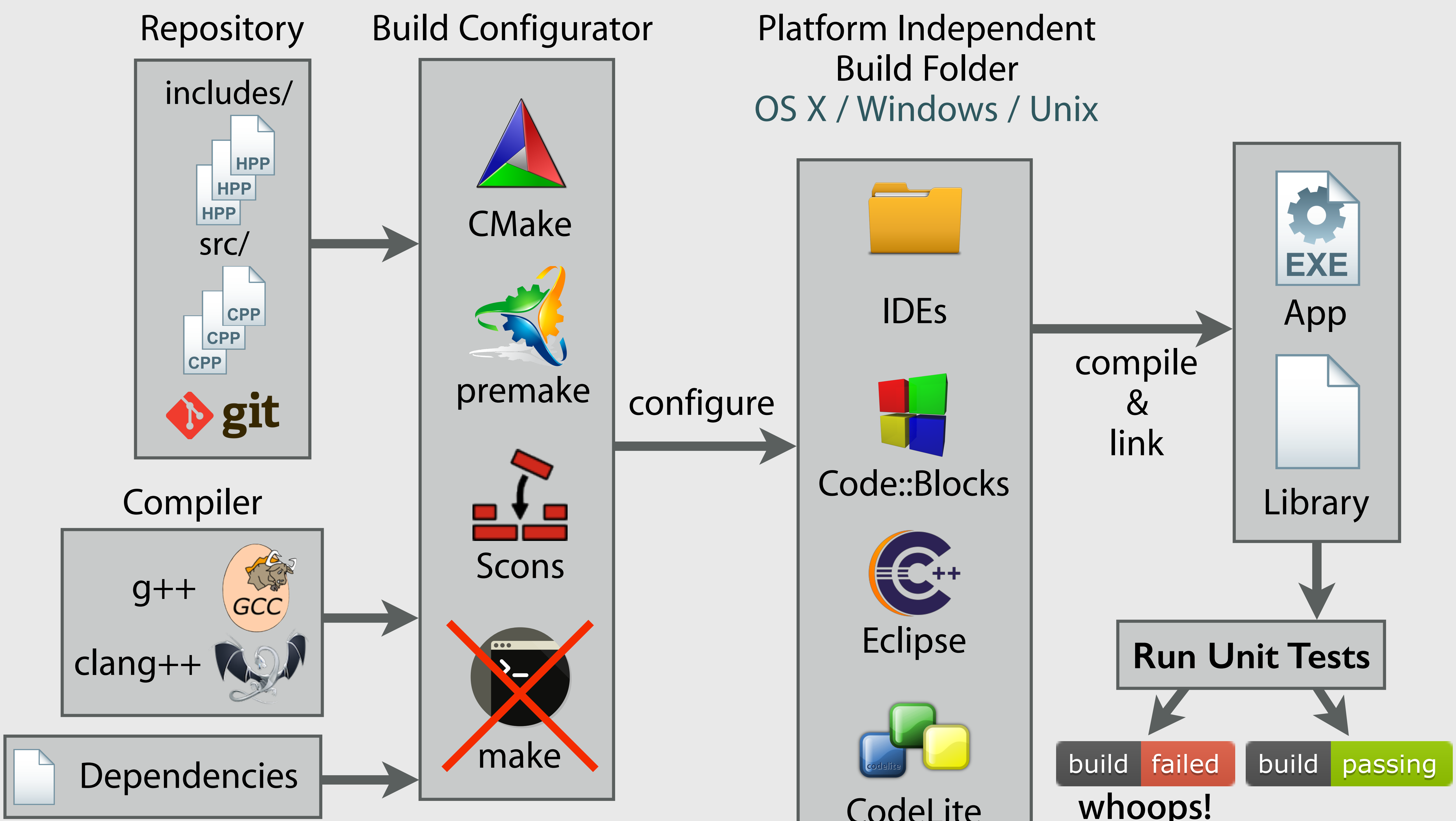
Modern Software Development in C++



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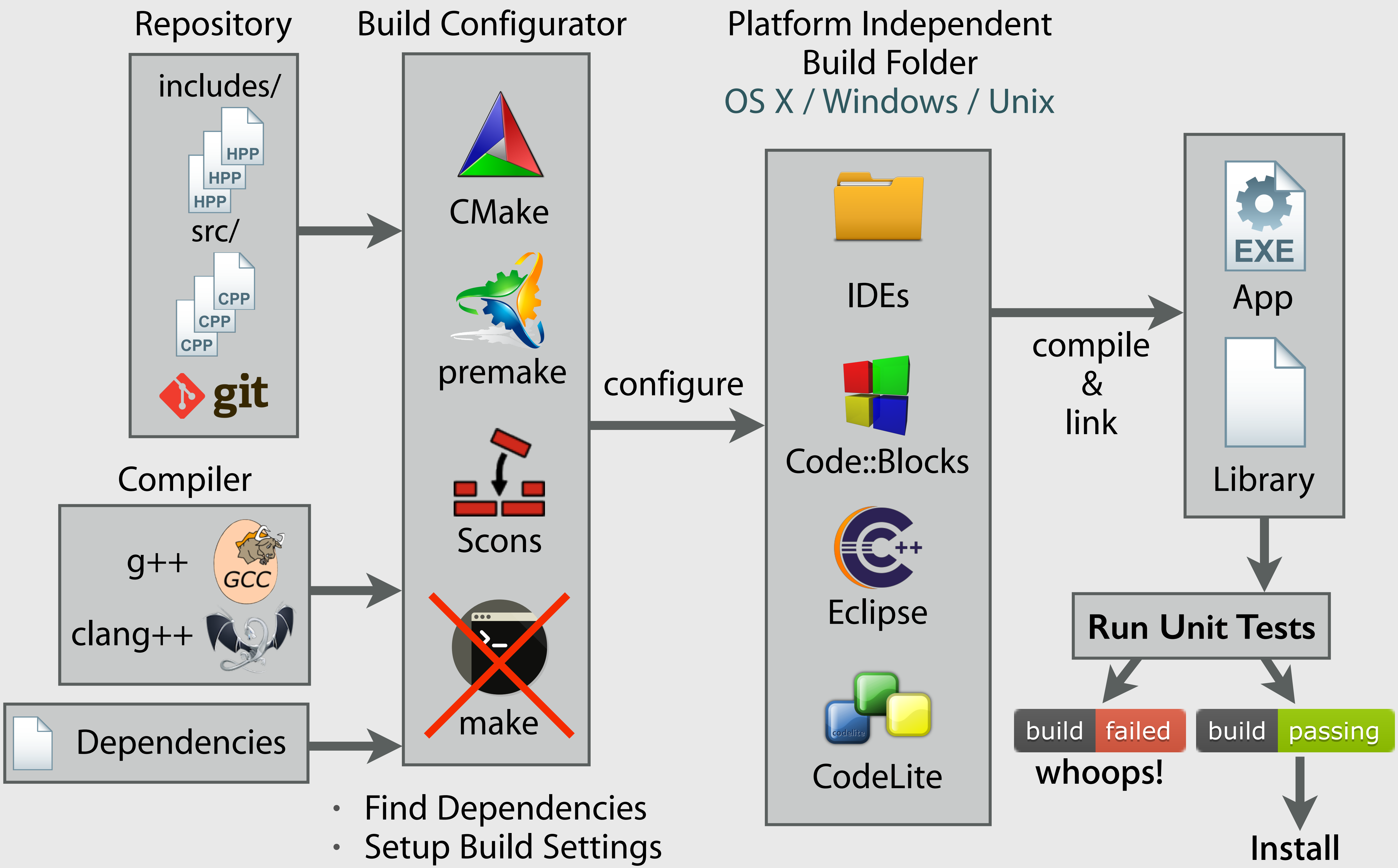
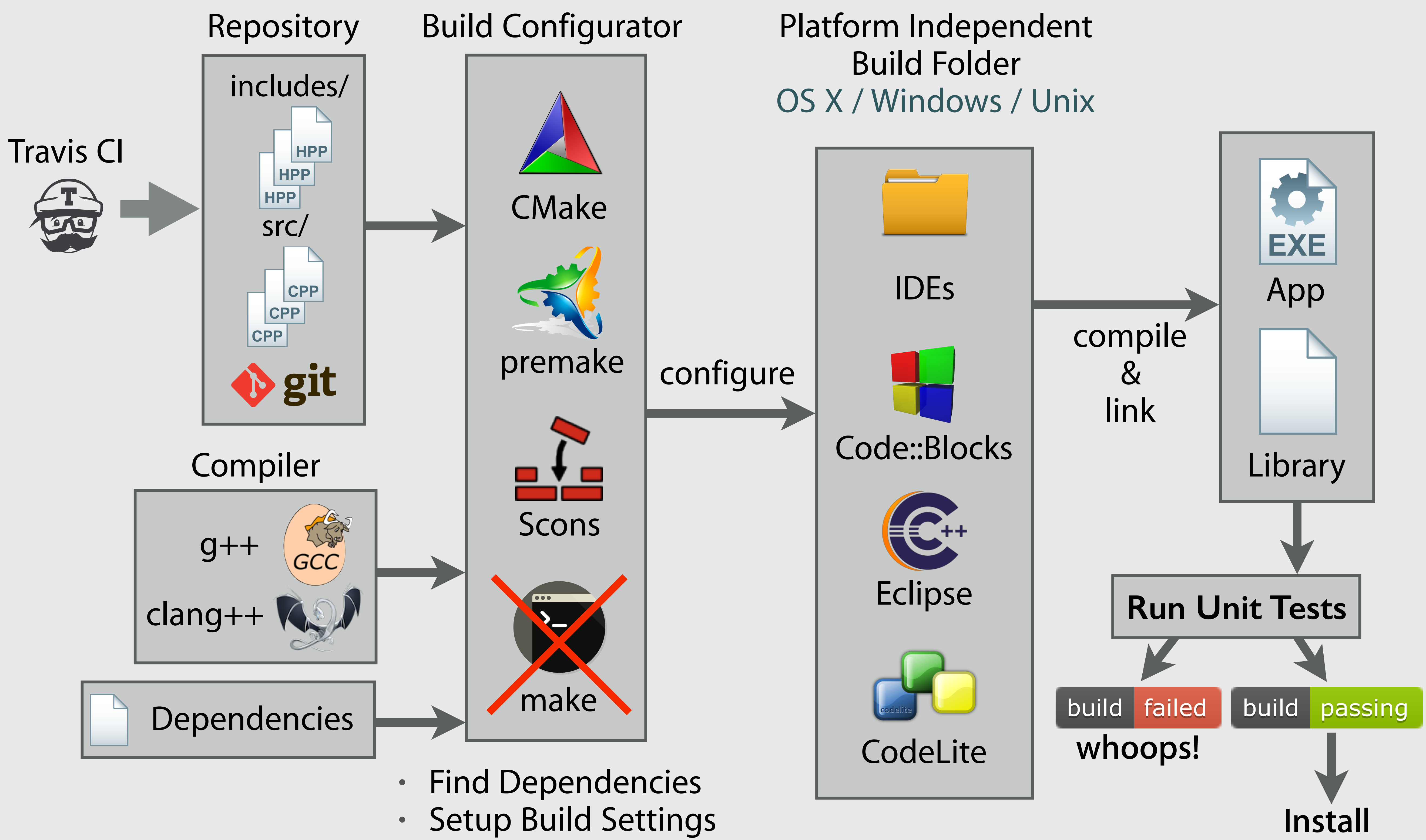
Modern Software Development in C++



Modern Software Development in C++



- Find Dependencies
- Setup Build Settings