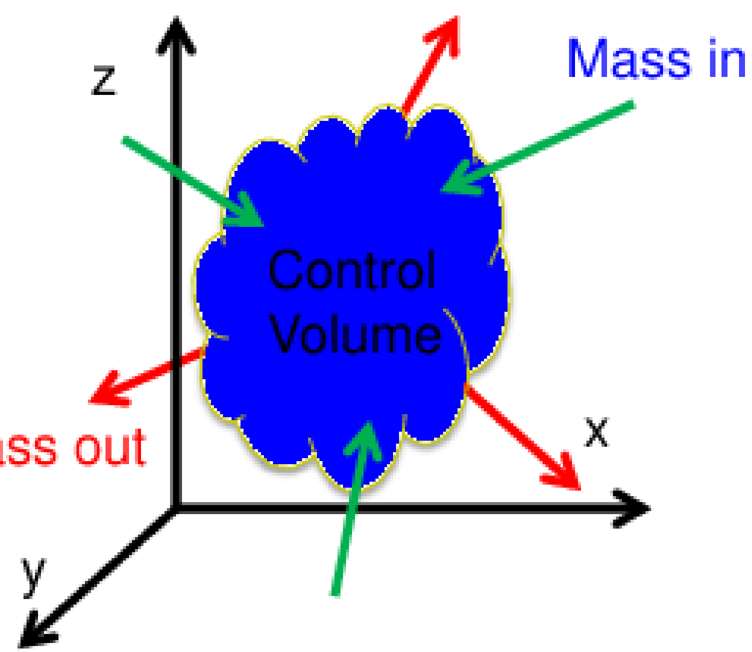
CHAPTER 1.
SINGLE PHASE FLOW EQUATIONS

Single Phase Flow Equations

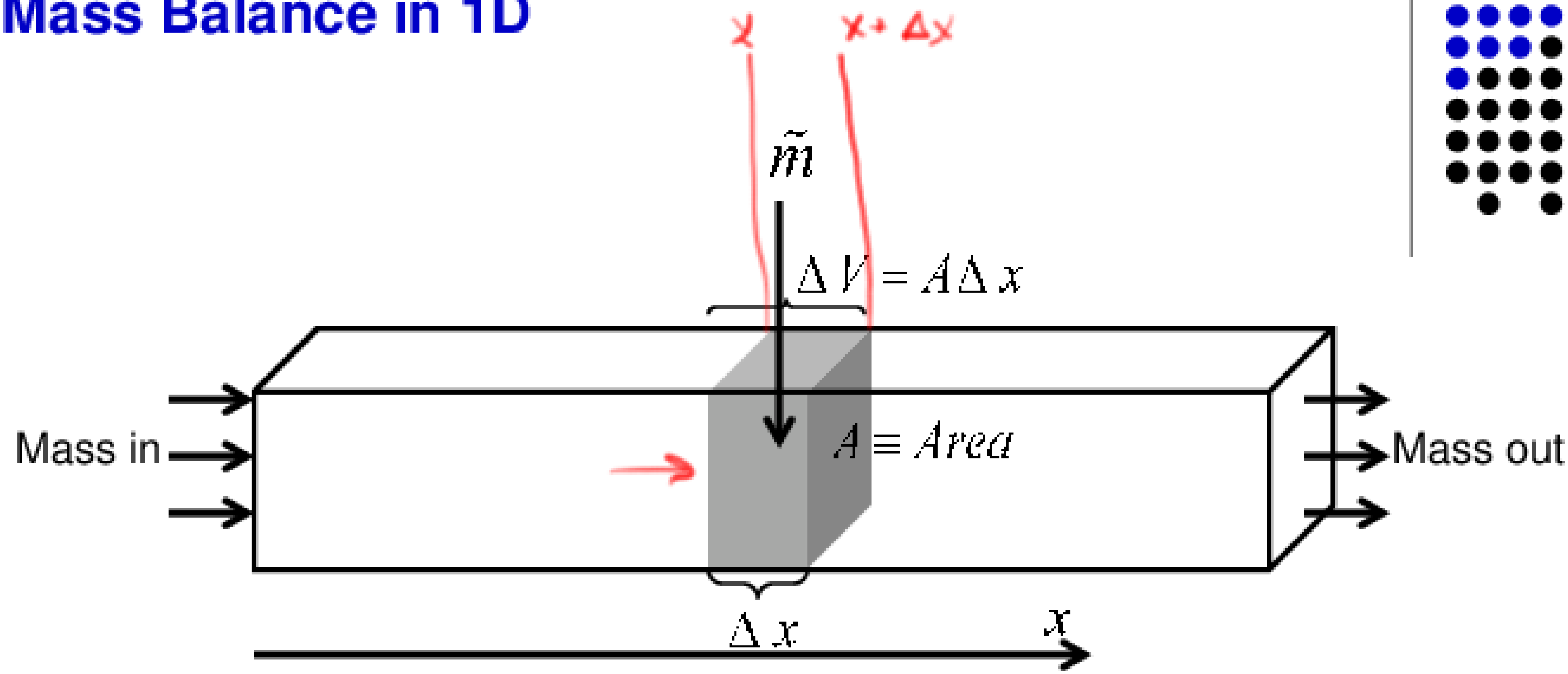
Mass In – Mass Out +/- Generation/Consumption = Accumulation



→ Control volume can be any shape

→ Mass enters and leaves the control volume in any direction

Mass Balance in 1D



$$\underbrace{\dot{m}_x\Big|_x A\Delta t}_{\text{mass in}} - \underbrace{\dot{m}_x\Big|_{x+\Delta x} A\Delta t}_{\text{mass out}} + \underbrace{\tilde{m}\Delta V\,\Delta t}_{\text{generation}} = \underbrace{\rho\phi\Delta V\Big|_{t+\Delta t} - \rho\phi\Delta V\Big|_t}_{\text{accumulation}}$$

$\dot{m} \equiv Mass\ flux\ \left[\frac{M}{L^2T}\right]$	$A = Area\ \left[L^2\right]$	$\tilde{m} \equiv Sink\ or\ source\ rate\ \left[\frac{M}{L^3T}\right]$	$\Delta V = Volume\ \left[L^3\right]$	$\rho \equiv Density\ \left[\frac{M}{L^3}\right]$	$\phi = Porosity\ \left[\frac{L^3}{L^3}\right]$

Continuity Equation in 1D

$$\underbrace{\dot{m}_x\big|_x A\Delta t}_{\text{mass in}} - \underbrace{\dot{m}_x\big|_{x+\Delta x} A\Delta t}_{\text{mass out}} + \underbrace{\tilde{m}\Delta V\Delta t}_{\text{generation}} = \underbrace{\rho\phi\Delta V\big|_{t+\Delta t} - \rho\phi\Delta V\big|_t}_{\text{accumulation}}$$

Divide the mass balance by A∆x∆t

lim ∆x→0
lim ∆t→0

$$\frac{\dot{m}_x\big|_x - \dot{m}_x\big|_{x+\Delta x}}{\Delta x} + \tilde{m} = \frac{\rho\phi\big|_{t+\Delta t} - \rho\phi\big|_t}{\Delta t}$$

Making ∆x and ∆t very small, they become dx and dt

$\dot{m} = \rho u$

$$\lim_{\Delta x \to 0} \frac{\dot{m}_x\big|_x - \dot{m}_x\big|_{x+\Delta x}}{\Delta x} = -\frac{\partial \dot{m}_x}{\partial x} \quad \text{and} \quad \lim_{\Delta t \to 0} \frac{\rho\phi\big|_{t+\Delta t} - \rho\phi\big|_t}{\Delta t} = \frac{\partial(\rho\phi)}{\partial t}$$

Expressing mass flux in terms of velocity ($\dot{m}_x = \rho u_x$), and subtracting $\tilde{m}$ we get the continuity equation to be

$$\boxed{-\frac{\partial(\rho u_x)}{\partial x} = \frac{\partial(\rho\phi)}{\partial t} - \tilde{m}}$$

Continuity Equation

$\dot{m} \equiv$ Mass flux $\left[\frac{M}{L^2T}\right]$	$\tilde{m} \equiv$ Sink or source rate $\left[\frac{M}{L^3T}\right]$	$\rho \equiv$ Density $\left[\frac{M}{L^3}\right]$	ϕ – Porosity $\left|\frac{L^3}{L^3}\right|$	$u \equiv$ velocity $\left[\frac{L}{T}\right]$

Continuity Equation in Multi-Dimensions

In 3D Cartesian coordinates the equation becomes

$$-\left[\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}\right]=\frac{\partial(\rho\phi)}{\partial t}-\tilde{m}$$

More general form of the Continuity Equation

$$\boxed{-\nabla\bullet\rho\vec{u}=\frac{\partial(\rho\phi)}{\partial t}-\tilde{m}}$$

Other names:

- Mass Conservation Equation
- Material Balance Equation

$\rho \equiv Density \left[\frac{M}{L^3}\right]$	$u = velocity \left[\frac{L}{T}\right]$	$\phi = Porosity \left[\frac{L^3}{L^3}\right]$	$\tilde{m} \equiv Sink\ or\ source\ rate \left[\frac{M}{L^3 T}\right]$	$\nabla \equiv Divergence$

Continuity Equation: What is $\tilde{m}$?

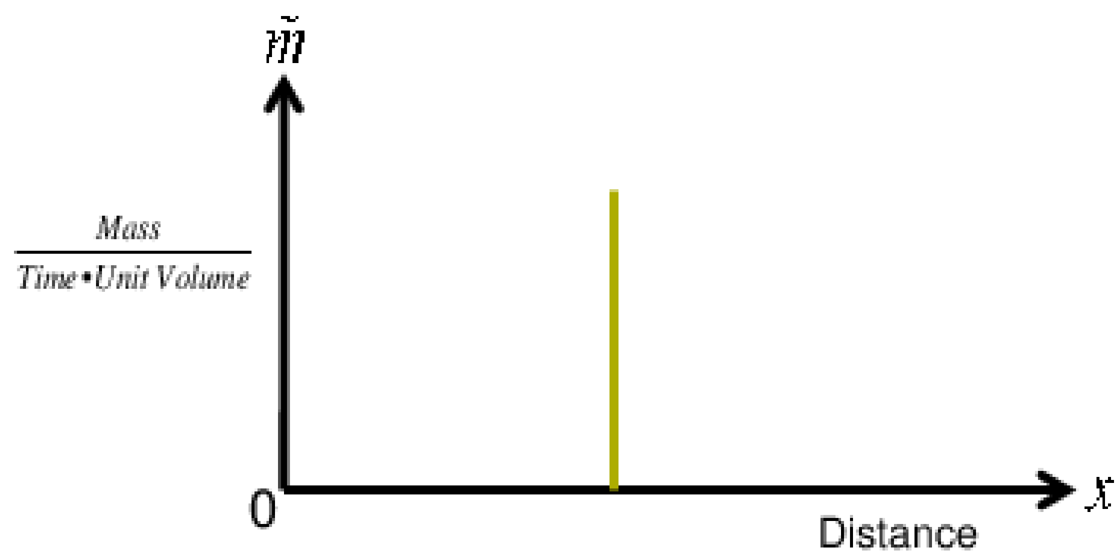
- $\tilde{m}$ accounts for mass entering or leaving a control volume from a source or sink (e.g. well)

- We model injection/production wells as discrete sources/sinks so they occupy no volume

- Taking limits without using this variable creates difficulty

$$\tilde{m} = \frac{\text{mass}}{\text{time-volume}}$$

Units of mass/time per volume of the control volume



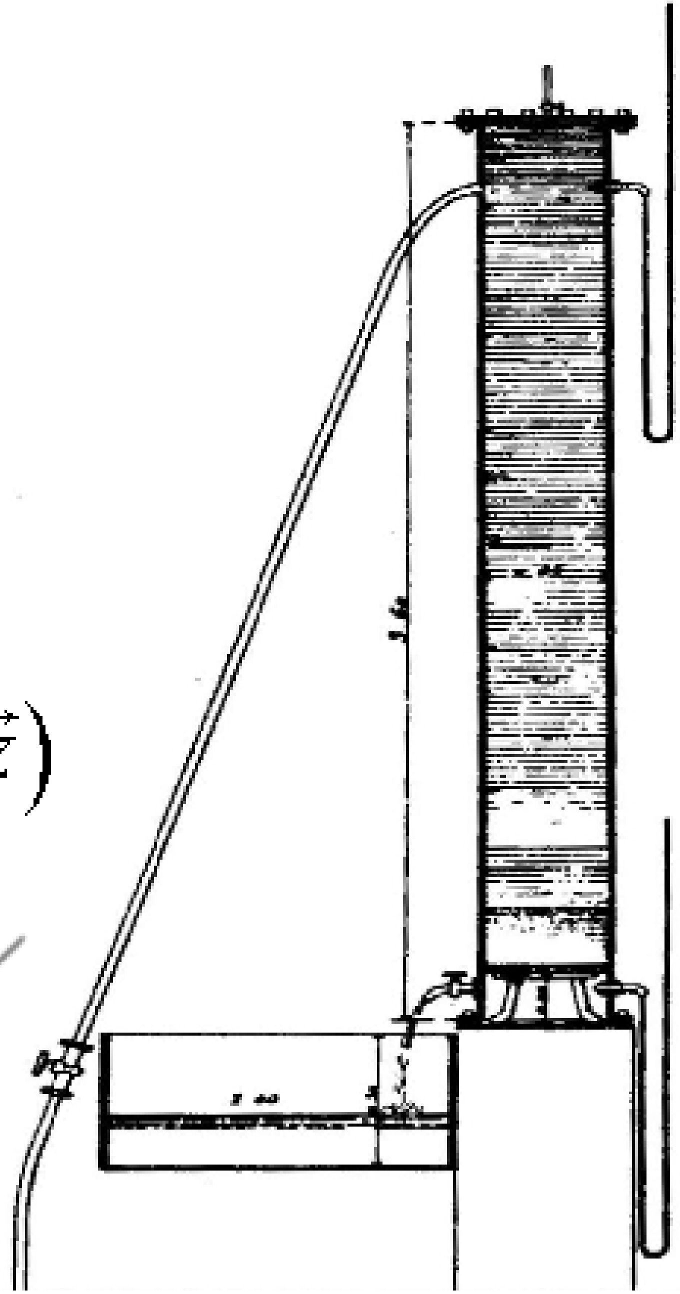
Darcy's Law

$$Q = -\frac{kA\Delta\Phi}{\mu L}$$

$$\vec{u} = -\frac{\vec{k}}{\mu}\left(\nabla p + \rho g \vec{z}\right)$$

1 darcy = 9.869 x 10^{-9} cm^2

1 darcy = 1.062 x 10^{-11} ft^2



1856

THE

PUBLIC FOUNTAINS

OF THE CITY OF DIJON

EXPERIENCE AND APPLICATION

PRINCIPLES TO FOLLOW AND FORMULAS TO BE USED

IN THE QUESTION

OF

THE DISTRIBUTION OF WATER

WORK FINISHES WITH

AN APPENDIX RELATING TO THE WATER SUPPLIES OF SEVERAL CITIES

THE FILTERING OF WATER

AND

THE MANUFACTURE OF STRONG PIPES OF LEAD, SHEET METAL AND BITUMEN

BY

HENRY DARCY

INSPECTOR GENERAL OF BRIDGES AND HIGHWAYS

PARIS

VICTOR DALMONT, EDITOR,

BOOKSELLER OF THE IMPERIAL CORPS OF BRIDGES, HIGHWAYS AND MINES,

1856

Darcy's Experiment

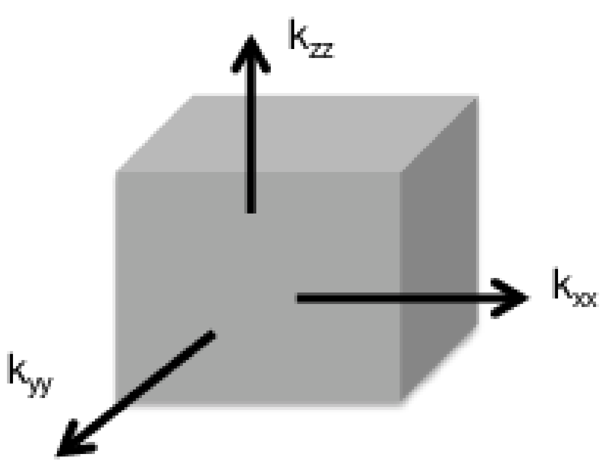
"The Public Fountains of the Town of Dijon", Dalmont, Paris (1856)

$Q \equiv Flow\ Rate \left[\frac{L^3}{T}\right]$	$k \equiv Permeability \left[L^2\right]$	$\Phi \equiv Potential \left[\frac{M}{LT^2}\right]$	$\mu \equiv Viscosity \left[\frac{M}{LT}\right]$	$p \equiv Pressure \left[\frac{M}{LT^2}\right]$	$g \equiv Gravity \left[\frac{L}{T^2}\right]$

Permeability (k)

Medium is **homogeneous** if permeability does not vary with space, but **heterogeneous** if non-uniform, $k=k(x,y,z)$

Medium is **isotropic** if permeability is the same in all directions, but **anisotropic** if $k_x \neq k_y \neq k_z$. Oftentimes $k_x \approx k_y$ but $k_x / k_z \approx 1\text{-}10$



$$\underline{k} = \overbrace{\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix}} \approx \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}$$

Off-diagonal terms are only zero if we choose a coordinate system that makes this true, but generally assume this is the case

Medium could be homogenous and anisotropic or heterogeneous and isotropic

Darcy's Law into Continuity Equation

In 1D and horizontal flow (gravity neglected)

$$u_x = -\frac{k}{\mu}\nabla p = -\frac{k}{\mu}\frac{\partial p}{\partial x}$$

Replacing fluid velocity in the continuity equation

$$-\frac{\partial(\rho u_x)}{\partial x} = \frac{\partial(\rho\phi)}{\partial t} - \tilde{m} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial x}\left(\rho \boxed{\frac{k}{\mu}}\frac{\partial p}{\partial x}\right) = \frac{\partial(\rho\phi)}{\partial t} - \tilde{m}}$$

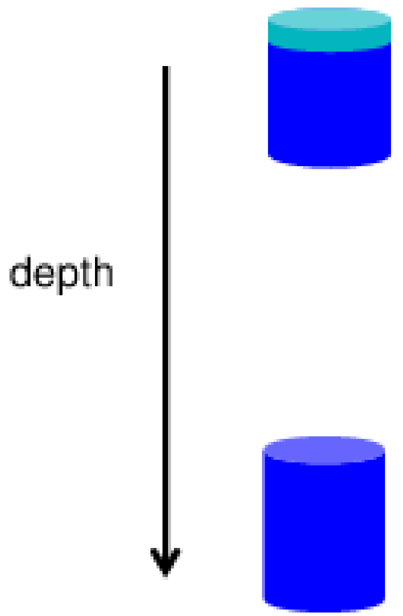
Assuming homogenous and isotropic permeability and viscosity (NOT always the case!)

$$\frac{k}{\mu}\frac{\partial}{\partial x}\left(\rho\frac{\partial p}{\partial x}\right) = \frac{\partial(\rho\phi)}{\partial t} - \tilde{m}$$

$k \equiv$ Permeability $[L^2]$	$\mu \equiv$ Viscosity $\left[\frac{M}{LT}\right]$	$p \equiv$ Pressure $\left[\frac{M}{LT^2}\right]$	$\rho \equiv$ Density $\left[\frac{M}{L^3}\right]$	$\phi -$ Porosity $\frac{L^3}{L^3}$	$\tilde{m} \equiv$ Sink or source rate $\left[\frac{M}{L^3T}\right]$

Formation Volume Factor (B_w)

We measure volume at surface but do the mass balance at the reservoir



Surface/Standard Conditions

$P = 14.7 \ \text{psi}; \ T = 60\ ^\circ\text{F}; \ V = V_{sc}$

Reservoir Conditions

$P = P_R \ ; \ T = T_R; \ V = V_R$

$$\rho_{RC} = \frac{\rho_{SC}}{B_w}$$

$$B_w = \frac{V_{RC}}{V_{SC}} = \frac{\rho_{SC}}{\rho_{RC}}$$

Formation volume factor and ρ depend on both pressure and temperature

- B_o = formation volume factor for oil (> 1.0)
- B_g = formation volume factor for gas (<< 1.0)
- B_w = formation volume factor for water (~1.0)

Formation Volume Factor—Continuity Equation

At reservoir conditions, density is: $\rho = \rho_{RC} = \dfrac{\rho_{SC}}{B_w}$

Replacing density in continuity equation and divide through by ρ_{SC} (a constant)

$$\frac{k}{\mu}\frac{\partial}{\partial x}\left(\frac{\rho_{SC}}{B_w}\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\rho_{SC}}{B_w}\phi\right) - \tilde{m} \quad \Rightarrow \quad \frac{k}{\mu}\frac{\partial}{\partial x}\left(\frac{1}{B_w}\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B_w}\right) - \frac{\tilde{m}}{\rho_{sc}}$$

Using the product rule on the left-hand side of the equation

$$\frac{k}{\mu}\left[\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{1}{B_w}\right)\frac{\partial p}{\partial x}\right] = \frac{\partial}{\partial t}\left(\frac{\phi}{B_w}\right) - \frac{\tilde{m}}{\rho_{sc}}$$

…And chain rule on the left hand side,

$c_E \approx 0$

$$\frac{k}{\mu}\left[\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2} + \cancel{\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)\left(\frac{\partial p}{\partial x}\right)^2}\right] = \frac{\partial}{\partial t}\left(\frac{\phi}{B_w}\right) - \frac{\tilde{m}}{\rho_{sc}}$$

$\rho \equiv Density \left[\frac{M}{L^3}\right]$	$B_w - Volumetric\ Factor \left[\frac{L^3}{L^3}\right]$	$k \equiv Permeability \left[L^2\right]$	$\mu \equiv Viscosity \left[\frac{M}{LT}\right]$	$p \equiv Pressure \left[\frac{M}{LT^2}\right]$	$\tilde{m} \equiv Sink\ or\ source\ rate \left[\frac{M}{L^3T}\right]$