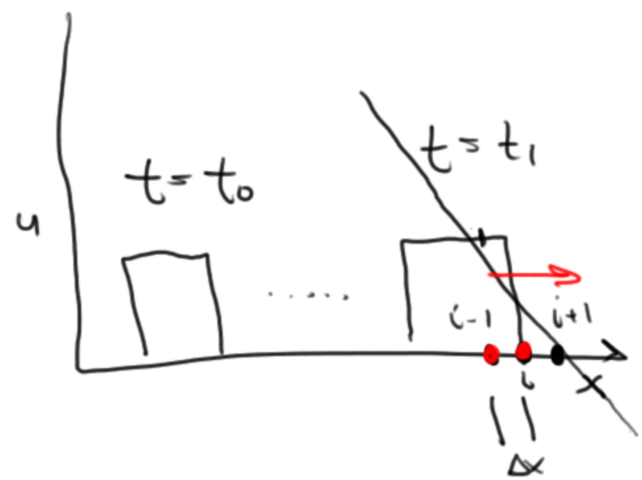
$$\frac{B_o}{B_w}\frac{\partial}{\partial x}\left(\frac{k\,k_{ro}}{\mu_o B_o}\frac{\partial p}{\partial x}\right) + \underbrace{\frac{\partial}{\partial x}\left(\frac{k\,k_{rw}}{\mu_w B_w}\frac{\partial p}{\partial x}\right)}_{=} = \frac{\phi c_t}{B_w}\frac{\partial p}{\partial t} - \left(\frac{B_o}{B_w}\right)\tilde{q}_o - \tilde{q}_w$$

$$\frac{k}{\mu B_w}\left(\frac{\partial k_{rw}}{\partial x}\frac{\partial p}{\partial x} + k_{rw}\frac{\partial^2 p}{\partial x^2}\right)$$

$$A\frac{\partial p}{\partial t} = B\frac{\partial p}{\partial x} + C\frac{\partial^2 p}{\partial x^2}$$

Hyperbolic (nearly)

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$



$$\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

$$\frac{\partial u}{\partial x} = \frac{u_i - u_{i-1}}{\Delta x}$$

$$A \frac{\partial p}{\partial t} = B \frac{\partial p}{\partial x} + C \frac{\partial^2 p}{\partial x^2}$$

$$A \frac{\partial p_i}{\partial t} = B \frac{P_i - P_{i-1}}{\Delta x} + C \frac{P_{i+1} - 2P_i + P_{i-1}}{\Delta x^2}$$

Rearrange

$$A \frac{dP_i}{\partial t} = B \frac{P_{i+1} - P_{i-1}}{2\Delta x} + \left(C + \frac{B \Delta x}{2}\right) \frac{P_{i+1} - 2P_i + P_{i-1}}{\Delta x^2}$$

Review IMPES method (2 phase)

- Discretized mass balances for oil and water
  1. **Implicitly** calculate pressure (overall mass balance) by solving system of equations
  2. **Explicitly** update the saturation equation (water mass balance)

$$\left(\mathbf{T}+\mathbf{J}+\frac{\mathbf{B}}{\Delta t}\right)\mathbf{P}^{n+1}=\frac{\mathbf{B}}{\Delta t}\mathbf{P}^{n}+\mathbf{Q} \qquad \mathbf{S}_w^{n+1}=\mathbf{S}_w^{n}+\mathbf{d}_{12}^{-1}\left[-\mathbf{T}^{\mathbf{w}}\mathbf{P}^{n+1}+\mathbf{Q}_w\right]-\mathbf{C}_{tw}\left(\mathbf{P}^{n+1}-\mathbf{P}^{n}\right)$$

- Explicit methods are only conditionally stable (and IMPES is partially explicit). So time step (Δt) needs to be reasonably small

$$\frac{u\Delta t}{\Delta x}<1$$

- *Upwinding* must be used to determine half relative perms!!!

$$k_{r,i-1/2}=\begin{cases} k_r\left(S_{w,i-1}\right) & \text{if} \quad P_{i-1}>P_i \\ k_r\left(S_{w,i}\right) & \text{if} \quad P_i>P_{i-1} \end{cases}$$

- Boundary conditions and wells are treated similarly to single phase flow

Interblock Transmissibility?

$$T_{w,i-1/2} = \frac{k_{i-1/2} k_{rw,i-1/2} A}{\Delta x_{i-1/2} \mu_w B_w}$$

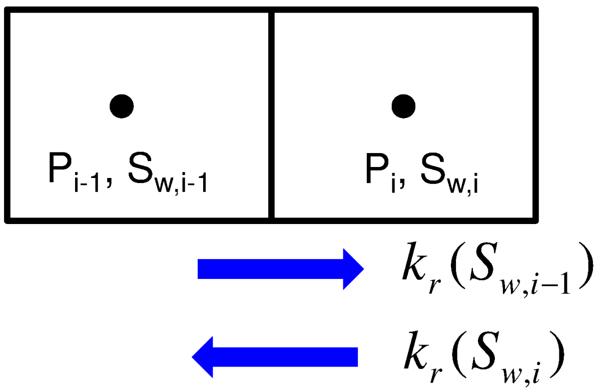
Absolute permeability: use harmonic mean as in single-phase flow:

$$k_{i+\frac{1}{2}} = 2\left(\frac{1}{k_i} + \frac{1}{k_{i+1}}\right)^{-1} \Longrightarrow \left(\frac{kA}{\Delta x}\right)_{i+\frac{1}{2}} = \frac{2k_i A_i k_{i+1} A_{i+1}}{k_i A_i \Delta x_{i+1} + k_{i+1} A_{i+1} \Delta x_i}$$

Relative permeability: use technique called "upwinding"

- Harmonic and arithmetic mean can lead to errors
- Upwinding involves evaluation relative permeability in block flow comes from

$$k_{r,i-1/2} = \begin{cases} k_r\left(S_{w,i-1}\right) & \text{if} \quad P_{i-1} > P_i \\ k_r\left(S_{w,i}\right) & \text{if} \quad P_i > P_{i-1} \end{cases}$$



Wells and well models

As in single phase flow, wells can be constant rate or BHP. Think of each block being composed of rate and BHP wells

$$q_{w,i} = q_{w,i,rate} + q_{w,i,BHP}; \qquad \mathrm{q}_{o,i} = q_{o,i,rate} + q_{o,i,BHP}$$

$$\therefore$$

$$q_{w,i} = q_{w,i,rate} + J_{w,i}\left(P_{wf,i} - P_i\right); \quad \mathrm{q}_{o,i} = q_{o,i,rate} + J_{o,i}\left(P_{wf,i} - P_i\right)$$

In matrix form:

$$\mathbf{Q}_w = \mathbf{Q}_{w,rate} + \mathbf{J}_w\left(\mathbf{P}_{wf} - \mathbf{P}^{n+1}\right); \ \mathbf{Q}_o = \mathbf{Q}_{o,i,rate} + \mathbf{J}_o\left(\mathbf{P}_{wf} - \mathbf{P}^{n+1}\right)$$

$$\mathbf{Q} = \underbrace{\mathbf{Q}_{w,rate} + \left({B_o}/{B_w}\right)\mathbf{Q}_{o,i,rate}}_{\mathbf{Q}_{rate}} + \underbrace{\left(\mathbf{J}_w + {B_o}/{B_w}\,\mathbf{J}_o\right)}_{\mathbf{J}}\mathbf{P}_{wf}$$

Pressure and Saturation Equations::

$$\boxed{\left(\mathbf{T} + \mathbf{J} + \frac{\mathbf{B}}{\Delta t}\right)\mathbf{P}^{n+1} = \frac{\mathbf{B}}{\Delta t}\mathbf{P}^n + \mathbf{Q}}$$

$$\mathbf{J} = \mathbf{J}_w + \left({B_o}/{B_w}\right)\mathbf{J}_o$$

$$\boxed{\mathbf{S}_w^{n+1} = \mathbf{S}_w^n + \mathbf{d}_{12}^{-1}\left[-\mathbf{T}^{\mathbf{w}}\mathbf{P}^{n+1} + \mathbf{Q}_w\right] - \mathbf{C}_{tw}\left(\mathbf{P}^{n+1} - \mathbf{P}^n\right)}$$

B_w=Volumetric Factor [rB/STB] k=perm [mD] Q=rate [ft^3/day] kr = rel perm J'=Productivity [(ft^3/day-psi)] h = thickness [ft] s = skin

Productivity Index for BHP wells

Constant BHP wells have P_{wf} specified:

$$Q_{w,i} = J_w \left(P_{wf} - P_i \right)$$

Productivity indices for each phase:

$$J_w = \frac{2\pi\, k_{rw} k\, h}{\mu_w B_w \left[\ln\left(\frac{r_{eq}}{r_w} \right) + s \right]}; \quad J_o = \frac{2\pi\, k_{ro} k\, h}{\mu_o B_o \left[\ln\left(\frac{r_{eq}}{r_w} \right) + s \right]}$$

Total productivity index is the sum of phase indices weighted by FVF's

$$J = J_w + \left({B_o}/{B_w} \right) J_o = \frac{2\pi\, k\, h}{B_w \ln\left(\frac{r_{eq}}{r_w} + s \right)} \left[\frac{k_{rw}}{\mu_w} + \frac{k_{ro}}{\mu_o} \right]$$

B_w=Volumetric Factor [rB/STB] k=perm [mD] Q=rate [ft^3/day] kr = rel perm J'=Productivity [(ft^3/day-psi)] h = thickness [ft] s = skin

Courant–Friedrichs–Lewy (CFL) condition

(CFL) condition is a necessary condition for stability of IMPES

$$\frac{u\Delta t}{\Delta x} < 1$$