Explicit

t @ n+1

$$p_{i}^{n+1} = p_i^n + \eta\left[p_{i-1}^n - 2p_i^n + p_{i-1}^n \right] \qquad \text{where } \eta = \alpha \frac{\Delta t}{\Delta x^2}$$

i^{th} grid block

$$p_i^{n+1} = \eta\, p_{i-1}^n + (1-2\eta)\, p_i^n + \eta\, p_{i+1}^n$$

$$= [\eta \;,\; 1-2\eta \;,\; \eta] \begin{bmatrix} p_{i-1} \\ p_i \\ p_{i+1} \end{bmatrix}^n$$

$$A\vec{x} = \vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Implicit

$$-\eta\, p_{i-1}^{n+1} + (1+2\eta)\, p_i^{n+1} - \eta\, p_{i+1}^{n+1} = p_i^n$$

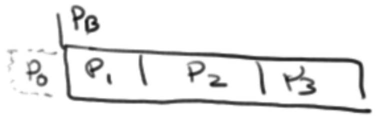
Dirichlet (essential B.C.'s)

$$p = p_B \quad @ \quad x = 0$$

Neumann (natural)

$$q - \frac{\partial p}{\partial x} = 0 \quad @ \quad x = L$$

Block 1



$$\frac{P_0 + P_1}{2} = P_B$$

$$\boxed{P_0 = 2P_B - P_1}$$

$$-\eta P_0^{n+1} + (1+2\eta)P_1^{n+1} - \eta P_2^{n+1} = P_1^n$$

$$-\eta(2P_B - P_1^{n+1}) + (1+2\eta)P_1^{n+1} - \eta P_2^{n+1} = P_1^n$$

$$-\eta\, 2P_B + \eta P_1^{n+1} + P_1^{n+1} + 2\eta P_1^{n+1} - \eta P_2^{n+1} = P_1^n$$

$$\boxed{(1+3\eta)P_1^{n+1} - \eta P_2^{n+1} = P_1^n + 2\eta P_B}$$ Dirichelet @ $x=0$

Neumann $x = L$ N^{th}

$$q_B = \frac{kA}{\mu}\frac{dP}{dx} \Rightarrow \boxed{P_{N+1} = P_N - \frac{q_B \mu \Delta x}{k(hw)}} \qquad \frac{\partial P}{\partial x} = \frac{P_N - P_{N+1}}{\Delta x}$$

P_{N+1}

$$-\eta P_{N-1}^{n+1} + (1+\cancel{2}\eta)P_N^{n+1} - \eta\left[P_N^{n+1} - \frac{q_B\mu\Delta x}{k(hw)}\right] = P_N^n$$

$$-\eta P_{N-1}^{n+1} + (1+\eta)P_N^{n+1} = P_N^n - \left(\frac{\cancel{k}}{\underbrace{\cancel{\mu}\phi c_t}_{\alpha}}\right)\left(\frac{\Delta t}{\Delta x^{\cancel{2}}}\right)\left(\frac{q_B\cancel{\mu}\cancel{\Delta x}}{\cancel{k}\, hw}\right)$$

$$\boxed{-\eta P_{N-1}^{n+1} + (1+\eta)P_N^{n+1} = P_N^n - \frac{q_B \Delta t}{\phi c_t \Delta x\, hw}}$$

We Can Express Equations in Matrix form

*Dirichlet on left and Neumann on right were used in this example

$$\gamma = \begin{cases} 1 & \text{if Dirichlet} \\ 0 & \text{if Neumann} \end{cases}$$

$$1+3\eta = 1 + 1\eta + \gamma_1 2\eta$$

Dirichlet BC

Dirichlet BC

$$\underbrace{\begin{bmatrix} 1+3\eta & -\eta & 0 & \dots & 0 \\ -\eta & 1+2\eta & -\eta & 0 & \dots \\ 0 & \dots & \dots & \dots & 0 \\ \dots & 0 & -\eta & 1+2\eta & -\eta \\ 0 & \dots & 0 & -\eta & 1+\eta \end{bmatrix}}_{\text{tridiagonal matrix}} \underbrace{\begin{bmatrix} P_1^{n+1} \\ \dots \\ P_i^{n+1} \\ \dots \\ P_N^{n+1} \end{bmatrix}}_{\text{unknown pressures}} = \underbrace{\begin{bmatrix} P_1^{n} \\ \dots \\ P_i^{n} \\ \dots \\ P_N^{n} \end{bmatrix}}_{\text{old pressures}} + \gamma \underbrace{\begin{bmatrix} 2\eta P_{B1} \\ \dots \\ 0 \\ \dots \\ 0 \end{bmatrix}}_{\text{boundary conditions}}$$

P_{B_1}

P_{B_N}

Neumann BC

- PDE problem has been transformed into linear system of equations, $AP^{n+1}=b$
- Matrix is tridiagonal, symmetric, and diagonally dominant
- Solve N×N system of equations for each time step. Need good solver for linear system of equations

Implicit

$$-\eta\, p_{i-1}^{n+1} + (1+2\eta)\, p_i^{n+1} - \eta\, p_{i+1}^{n+1} = p_i^n \qquad [\equiv \text{psi}]$$

Multiply through by

$$\frac{A\ \Delta x\ \phi\ c_t}{B_w\ \Delta t} = \underbrace{\frac{V\ \phi\ c_t}{B_w\ \Delta t}} \qquad \text{recall} \quad \eta = \alpha \frac{\Delta t}{\Delta x^2} = \frac{k}{\mu\, \phi\, c_t} \frac{\Delta t}{\Delta x^2}$$

$L^2 \quad L^2 \quad L^3 = V$

$$-\frac{k A}{\mu B_w \Delta x^{L}}\, p_{i+1}^{n+1} + \left(\frac{V_i\, \phi\, c_t}{B_w\ \Delta t} + \frac{2\, k A}{\mu B_w \Delta x} \right) p_i^{n+1} - \frac{k A}{\mu B_w \Delta x}\, p_{i+1}^{n+1} = \frac{V_i\, c_t}{B_w\ \Delta t}\, p_i^n$$

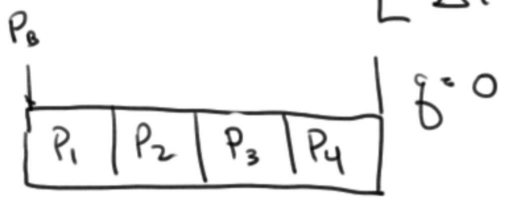
$$\left[\equiv \frac{ft^3}{day}\right]$$

Definitions

$T \Rightarrow$ transmissibility $= \dfrac{kA}{\mu B_w \Delta x} \Rightarrow \dfrac{mD \cdot ft}{cp}$

$B_i =$ accumulation $= \dfrac{V_i\, \phi\, c_t}{B_w} \Rightarrow \dfrac{ft^3}{psi}$

$$-T\,P_{i-1}^{n+1} + \left[\frac{B_i}{\Delta t} + 2T\right] P_i^{n+1} - T\,P_{i+1}^{n+1} = \frac{B_i}{\Delta t} P_i^n$$



$$\frac{1}{\Delta t}\begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & & \vdots \\ \vdots & & B_3 & \vdots \\ 0 & \cdots & & B_4 \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}^{n+1} + \begin{bmatrix} 3T & -T & 0 & 0 \\ -T & 2T & -T & 0 \\ 0 & -T & 2T & -T \\ 0 & 0 & -T & T \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}^{n+1} =$$

$$\frac{1}{\Delta t}\begin{bmatrix} B_1 & & & \\ & B_2 & & \underline{0} \\ & & B_3 & \\ \underline{0} & & & B_4 \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix}^{n} + \underbrace{\begin{bmatrix} 2T P_B \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{Q}}$$

$$\boxed{\underbrace{\left(\vec{\vec{T}} + \frac{1}{\Delta t}\vec{\vec{B}}\right)}_{A} \underbrace{\vec{p}^{\,n+1}}_{\vec{x}} = \underbrace{\frac{1}{\Delta t}\vec{\vec{B}}\, p^n + \vec{Q}}_{\vec{b}}}$$

Explicit

$$\vec{P}^{n+1} = \vec{P}^{n} + \Delta t\, \vec{\vec{B}}^{-1} \left(\vec{Q} - \vec{\vec{T}}\, \vec{P}^{n} \right)$$