1) $\dfrac{\partial p}{\partial t} = \alpha \dfrac{\partial^2 p}{\partial x^2} \Rightarrow$ Finite differences

$$\left(\overset{\Rightarrow}{T} + \frac{\overset{\Rightarrow}{B}}{\Delta t} \right) \vec{p}^{\,n+1} = \frac{\overset{\Rightarrow}{B}}{\Delta t} \vec{p}^{\,n} + \vec{Q} \qquad \text{Implicit}$$

$$\vec{p}^{\,n+1} = \vec{p}^{\,n} + \Delta t \, \overset{\Rightarrow}{B}^{-1} \left[\vec{Q} - \overset{\Rightarrow}{T} \vec{p}^{\,n} \right] \qquad \text{Explicit}$$

$$\left[(1-\theta) \overset{\Rightarrow}{T} + \frac{\overset{\Rightarrow}{B}}{\Delta t} \right] \vec{p}^{\,n+1} = \left(\frac{1}{\Delta t} \overset{\Rightarrow}{B} - \theta T \right) \vec{p}^{\,n} \qquad \text{Mixed}$$

$\theta = 1/2$ Crank - Nicholson

$$\overset{\Rightarrow}{T} = \frac{kA}{\mu B_w \Delta x} \qquad \overset{\Rightarrow}{B}_i = \frac{V_i \phi c_t}{B_w}$$

$\mu = \mu(p^{n+1})$

2) 

$$T_{i-1/2}(P_i - P_{i-1}) + T_{i+1/2}(P_{i+1} - P_i) = B_i \frac{1}{\Delta t} - Q_i^{sc}$$

Fluid Properties Can Vary from Block to Block

$$T_{i+1/2} = \left(\frac{1}{\mu B_w}\right)_{i+\frac{1}{2}} \left(\frac{kA}{\Delta x}\right)_{i+\frac{1}{2}}$$

Option 1. Evaluate at some arithmetic "average" pressure between blocks

$$p_{i+\frac{1}{2}} = \omega p_i + (1-\omega)\, p_{i+1} \quad \Rightarrow \quad \left(\frac{1}{\mu B_w}\right)_{i+\frac{1}{2}} = \left(\frac{1}{\mu B_w}\right) @\, p_{i+\frac{1}{2}}$$

$$\omega = \tfrac{1}{2} \quad \text{or} \quad \frac{(V\phi)_i}{(V\phi)_{i+1}}$$

Option 2. Take arithmetic "average" of block fluid properties

$$\left(\frac{1}{\mu B_w}\right)_{i+\frac{1}{2}} = \omega \left(\frac{1}{\mu B_w}\right)_i + (1-\omega)\left(\frac{1}{\mu B_w}\right)_{i+1}$$

Option 3. Use technique called upwinding (discussed later in multiphase flow)

$T \equiv Transmissibility \left[\frac{L^4 T}{M}\right]$	$k \equiv Permeability \left[L^2\right]$	$\mu \equiv Viscosity \left[\frac{M}{LT}\right]$	$B_w \equiv Volumetric\ Factor \left[\frac{L^3}{L^3}\right]$

Change in Transmissibility Matrix

Mass balance on block "i" now includes interblock transmissibility

$$T(P_{i-1} - P_i) + T(P_{i+1} - P_i) = \frac{V_i \phi c_t}{B_w \Delta t}(P_i^{n+1} - P_i^n) - Q_i^{SC}$$

$$\Downarrow$$

$$T_{i-1/2}(P_{i-1} - P_i) + T_{i+1/2}(P_{i+1} - P_i) = \frac{V_i \phi c_t}{B_w \Delta t}(P_i^{n+1} - P_i^n) - Q_i^{SC}$$

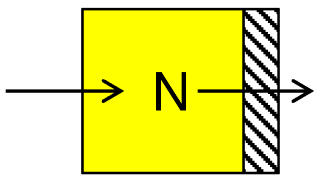
B matrix and Q vector is the same. T (transmissibility) matrix is still symmetric and tri-diagonal, e.g.

$$\mathbf{T} = \begin{bmatrix} T_{1/2} + T_{3/2} & -T_{3/2} & 0 & 0 \\ -T_{3/2} & T_{3/2} + T_{5/2} & -T_{5/2} & 0 \\ 0 & -T_{5/2} & T_{5/2} + T_{7/2} & -T_{7/2} \\ 0 & 0 & -T_{7/2} & T_{7/2} + T_{9/2} \end{bmatrix}$$

$P \equiv Pressure \left[\frac{M}{LT^2}\right]$ | $T \equiv Transmissibility \left[\frac{L^4 T}{M}\right]$ | $Q^{SC} \equiv Source \left[\frac{L^3}{T}\right]$

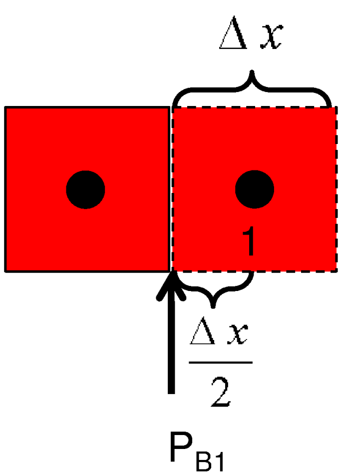
Boundary Conditions

For a Neumann (no flow) boundary, no flow goes through the boundary, which is equivalent to having no interblock transmissibility



$$q_{N+\frac{1}{2}} = T_{N+1/2}\left(P_N - P_{N+1}\right) = 0 \quad \Rightarrow \quad T_{N+1/2} = 0$$

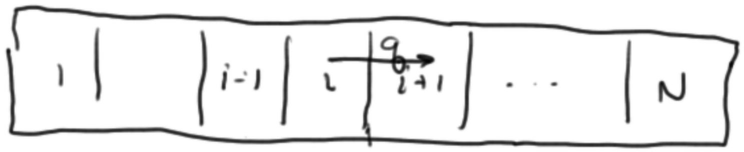
For a Dirichlet (constant pressure) boundary, pressure is defined at the edge, which is Δx/2 distance away from the block center



$$q_{\frac{1}{2}} = T_{1/2}\left(P_0 - P_1\right) = 2T_1\left(P_{B1} - P_1\right)$$

Replace flow equation between blocks with equation describing flow from edge to center of block, all of which has permeability k_1. But the distance traveled is Δx/2 so the effective transmissibility is $2T_1$

$T \equiv Transmissibility \left[\frac{L^4T}{M}\right]$ | $k \equiv Permeability\left[L^2\right]$ | $\mu \equiv Viscosity\left[\frac{M}{LT}\right]$ | $B_w \equiv Volumetric\ Factor\left[\frac{L^3}{L^3}\right]$



$k_{i-1} \quad k_i \quad k_{i+1}$

$$q = \frac{k_{i+1/2} A}{\mu B_w \Delta x} (P_{i+1} - P_i) \Rightarrow P_{i+1} - P_i = \frac{q \mu B_w \Delta x}{k_{i+1/2} A}$$

Write on half-grid block

$$\cancel{P_{i+1/2}} - P_i = \frac{q \mu B_w \Delta x}{k_i A 2}$$

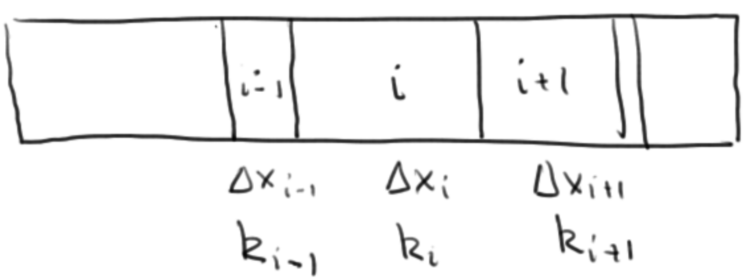
$$P_{i+1} - \cancel{P_{i+1/2}} = \frac{q \mu B_w \Delta x}{k_{i+1} A 2}$$

$$P_{i+1} - P_i = \frac{q \mu B_w \Delta x}{k_i A 2} + \frac{q \mu B_w \Delta x}{k_{i+1} A 2} = \frac{q \mu B_w \Delta x}{2A} \left(\frac{1}{k_i} + \frac{1}{k_{i+1}} \right)$$

$$q = \frac{2A}{\mu B_w \Delta x} \left(\frac{1}{k_i} + \frac{1}{k_{i+1}} \right)^{-1} (P_{i+1} - P_i)$$

$$k_{i+1/2} = 2\left(\frac{1}{k_i} + \frac{1}{k_{i+1}}\right)^{-1} \rightarrow \text{Harmonic mean}$$

$$T_{i+1/2} = \frac{k_{i+1/2}\, A}{\mu B_w \Delta x}$$



$$T_{i+1/2} = \frac{k_{i+1/2}\, A}{\mu B_w \Delta x_{i+1/2}}$$

$$\Delta x_{i+1/2} = \frac{\Delta x_i + \Delta x_{i+1}}{2}$$

$$k_{i+1/2} = \frac{\dfrac{\Delta x_i}{2} + \dfrac{\Delta x_{i+1}}{2}}{\dfrac{\Delta x_i}{2k_i} + \dfrac{\Delta x_{i+1}}{2k_{i+1}}} = \frac{\Delta x_i + \Delta x_{i+1}}{\dfrac{\Delta x_i}{k_i} + \dfrac{\Delta x_{i+1}}{k_{i+1}}}$$

Mass balance

$$T_{i-1/2}(P_{i-1} - P_i) + T_{i+1/2}(P_{i+1} - P_i) = \frac{B_i}{\Delta t}\left(p_i^{n+1} - p_i^n\right) - Q_i^{sc}$$

$$T = \begin{bmatrix} T_{1/2} + T_{3/2} & -T_{3/2} & 0 & 0 \\ -T_{3/2} & T_{3/2} + T_{5/2} & -T_{5/2} & 0 \\ 0 & -T_{5/2} & T_{5/2} + T_{7/2} & -T_{7/2} \\ 0 & 0 & -T_{7/2} & T_{7/2} + \underbrace{T_{9/2}}_{=0} \end{bmatrix}$$

(No Flow) - Neumann

$$q_{N+1/2} = T_{N+1/2}(P_N - P_{N+1}) = 0 \quad \therefore \quad T_{N+1/2}$$

(constant Pressure) - Dirichelet

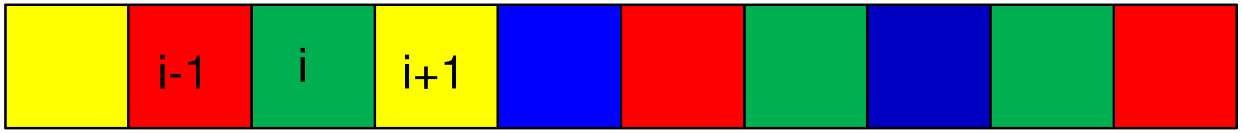
$$q_{1/2} = T_{1/2}(P_0 - P_1) = \overbrace{\boxed{\frac{2\,k_1 A}{\mu B_w \Delta x_1}}}^{T_1}(P_{B_1} - P_1) = 2T_1(P_B - P_1)$$

$$T_{1/2} = 2T_1 \qquad Q_1 = 2T_1 P_{B_1}$$

CHAPTER 5. HETEROGENEITIES

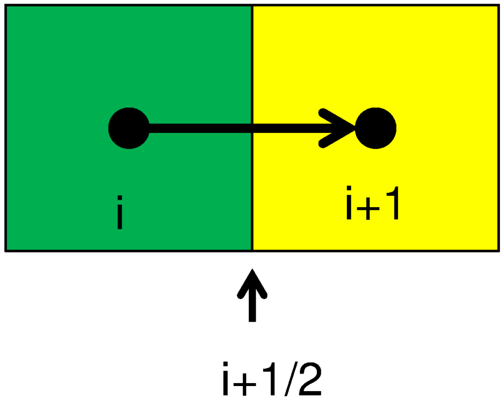
Heterogeneities in Reservoir Simulation

Real reservoirs are often not homogenous in permeability



$$k_{i-1} \neq k_i \neq k_{i+1}$$

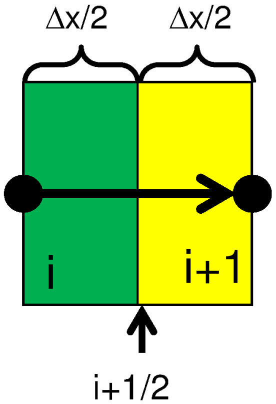
Consider flow from center of block “i” to “i+1” which is governed by Darcy’s Law



$$q = \frac{k_{i+\frac{1}{2}} A}{\mu B_w \Delta x}(P_{i+1} - P_i) \quad \Rightarrow \quad (P_{i+1} - P_i) = \frac{q \mu B_w \Delta x}{k_{i+\frac{1}{2}} A}$$

$T \equiv Transmissibility \left[\frac{L^4 T}{M}\right]$	$k \equiv Permeability \left[L^2\right]$	$\mu \equiv Viscosity \left[\frac{M}{LT}\right]$	$B_w \equiv Volumetric\ Factor \left[\frac{L^3}{L^3}\right]$

Averaging Permeability: Harmonic Average



$$(P_{i+1/2} - P_i) = \frac{q\mu B_w \Delta x}{2k_i A}$$

Pressure drop in green "half block"

$$(P_{i+1} - P_{i+1/2}) = \frac{q\mu B_w \Delta x}{2k_{i+1} A}$$

Pressure drop in yellow "half block"

Add pressure drops across green and yellow "half blocks"

$$(P_{i+1/2} - P_i) + (P_{i+1} - P_{i+1/2}) = \frac{q\mu B_w \Delta x}{2k_i A} + \frac{q\mu B_w \Delta x}{2k_{i+1} A}$$

Simplifying yields:

$$(P_{i+1} - P_i) = \frac{q\mu B_w \Delta x}{2A}\left(\frac{1}{k_i} + \frac{1}{k_{i+1}}\right) \quad \Rightarrow \quad q = \frac{A}{\mu B_w \Delta x} \underbrace{2\left(\frac{1}{k_i} + \frac{1}{k_{i+1}}\right)^{-1}}_{k_{i+\frac{1}{2}}} (P_{i+1} - P_i)$$

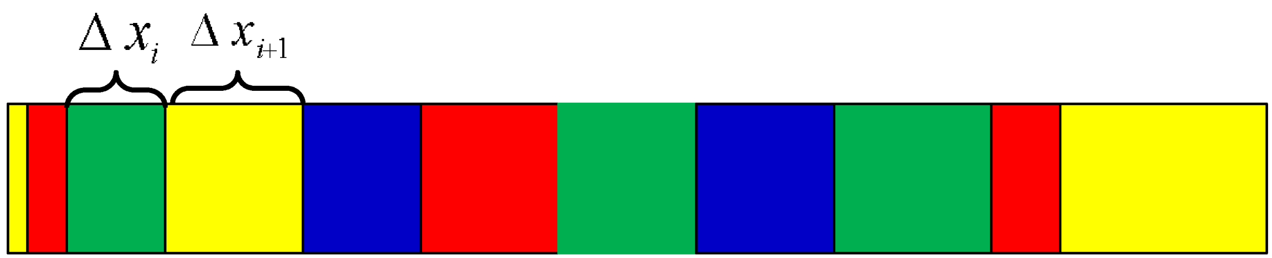
Inter-block permeability and transmissibility can then be defined as:

$$k_{i+\frac{1}{2}} = 2\left(\frac{1}{k_i} + \frac{1}{k_{i+1}}\right)^{-1} \; ; \; T_{i+1/2} = \frac{k_{i+\frac{1}{2}} A}{\mu B_w \Delta x}$$

$q \equiv Flow\ Rate \left[\frac{L^3}{T}\right]$	$k \equiv Permeability\left[L^2\right]$	$\mu \equiv Viscosity\left[\frac{M}{LT}\right]$	$B_w \equiv Volumetric\ Factor\left[\frac{L^3}{L^3}\right]$	$P \equiv Pressure\left[\frac{M}{LT^2}\right]$	$T \equiv Transmissibility\left[\frac{L^4 T}{M}\right]$

Non-Uniform Grid Size?

We may use varying grid sizes when we want more resolution in some areas of the reservoir (e.g near wells)



Interblock transmissibility can now be defined as: $T_{i+1/2} = \dfrac{k_{i+\frac{1}{2}} A}{\mu B_w \Delta x_{i+\frac{1}{2}}}$

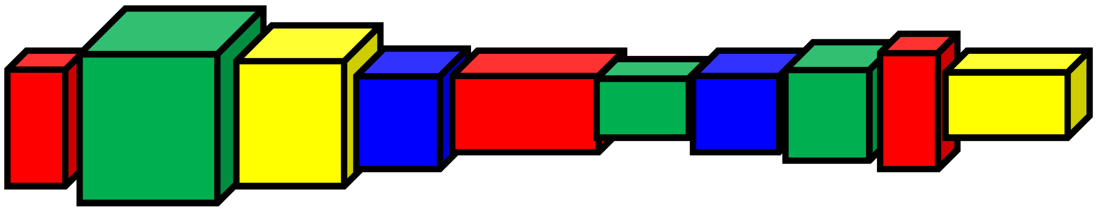
Where the interblock grid size is an arithmetic means and interblock permeability is a harmonic mean:

$$\Delta x_{i+\frac{1}{2}} = \frac{\Delta x_i + \Delta x_{i+1}}{2} \qquad k_{i+\frac{1}{2}} = \frac{\dfrac{\Delta x_i}{2} + \dfrac{\Delta x_{i+1}}{2}}{\dfrac{\Delta x_i}{2k_i} + \dfrac{\Delta x_{i+1}}{2k_{i+1}}} = \frac{\Delta x_i + \Delta x_{i+1}}{\dfrac{\Delta x_i}{k_i} + \dfrac{\Delta x_{i+1}}{k_{i+1}}}$$

$T \equiv Transmissibility \left[\frac{L^4 T}{M}\right]$ | $k \equiv Permeability \left[L^2\right]$ | $\mu \equiv Viscosity \left[\frac{M}{LT}\right]$ | $B_w \equiv Volumetric\ Factor \left[\frac{L^3}{L^3}\right]$

What if Area Changes Too?

Cross sectional area (A_i) can vary from one grid block to the next



Mass balance gives:

$$(P_{i+1}-P_i)=\frac{q\mu B_w}{2}\left(\frac{\Delta x_i}{k_iA_i}+\frac{\Delta x_{i+1}}{k_{i+1}A_{i+1}}\right)\quad\Rightarrow\quad q=\frac{1}{\mu B_w\Delta x_{i+1/2}}\underbrace{\left(\frac{\Delta x_i+\Delta x_{i+1}}{\frac{1}{k_iA_i}+\frac{1}{k_{i+1}A_{i+1}}}\right)}_{kA_{i+1/2}}(P_{i+1}-P_i)$$

Equivalently we can say,

$$T_{i+1/2}=\left(\frac{1}{\mu B_w}\right)_{i+1/2}\left(\frac{kA}{\Delta x}\right)_{i+1/2}\qquad\qquad \left(\frac{kA}{\Delta x}\right)_{i+1/2}=\frac{2k_iA_ik_{i+1}A_{i+1}}{k_iA_i\Delta x_{i+1}+k_{i+1}A_{i+1}\Delta x_i}$$

$T \equiv Transmissibility\left[\frac{L^4T}{M}\right]$	$k \equiv Permeability\left[L^2\right]$	$\mu \equiv Viscosity\left[\frac{M}{LT}\right]$	$B_w \equiv Volumetric\ Factor\left[\frac{L^3}{L^3}\right]$