Darcy's Law into Continuity Equation

In 1D and horizontal flow (gravity neglected)

$$u_x = -\frac{k}{\mu}\nabla p = -\frac{k}{\mu}\frac{\partial p}{\partial x}$$

Replacing fluid velocity in the continuity equation

$$-\frac{\partial(\rho u_x)}{\partial x} = \frac{\partial(\rho\phi)}{\partial t} - \tilde{m} \quad \Longrightarrow \quad \frac{\partial}{\partial x}\left(\rho\frac{k}{\mu}\frac{\partial p}{\partial x}\right) = \frac{\partial(\rho\phi)}{\partial t} - \tilde{m}$$

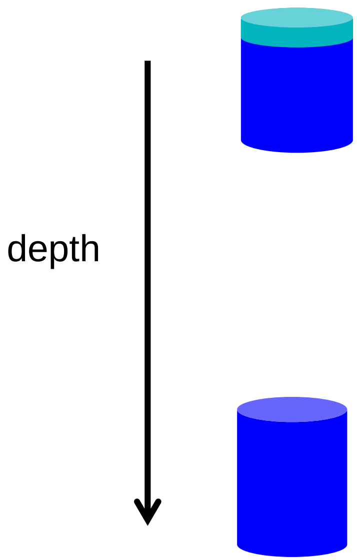
Assuming homogenous and isotropic permeability and viscosity (NOT always the case!)

$$\frac{k}{\mu}\frac{\partial}{\partial x}\left(\rho\frac{\partial p}{\partial x}\right) = \frac{\partial(\rho\phi)}{\partial t} - \tilde{m}$$

$k \equiv Permeability\left[L^2\right]$	$\mu \equiv Viscosity\left[\frac{M}{LT}\right]$	$p \equiv Pressure\left[\frac{M}{LT^2}\right]$	$\rho \equiv Density\left[\frac{M}{L^3}\right]$	$\phi \equiv Porosity\left[\frac{L^3}{L^3}\right]$	$\tilde{m} \equiv Sink\ or\ source\ rate\left[\frac{M}{L^3T}\right]$

Formation Volume Factor (B_w)

We measure volume at surface but do the mass balance at the reservoir



Surface/Standard Conditions

$$P = 14.7 \ \text{psi}; \ T = 60\ °\text{F}; \ V = V_{sc}$$

Reservoir Conditions

$$P = P_R \ ; \ T = T_R; \ V = V_R$$

$$\rho_{RC} = \frac{\rho_{SC}}{B_w}$$

$$\downarrow$$

$$B_w = \frac{V_{RC}}{V_{SC}} = \frac{\rho_{SC}}{\rho_{RC}}$$

Formation volume factor and ρ depend on both pressure and temperature

- B_o = formation volume factor for oil (> 1.0)
- B_g = formation volume factor for gas (<< 1.0)
- B_w = formation volume factor for water (~1.0)

$\Delta V \equiv Volume \left[L^3\right]$	$P \equiv Pressure \left[\frac{M}{LT^2}\right]$	$T \equiv Temperature \ [t]$	$B_w \equiv Volumetric\ Factor\ (Water) \left[\frac{L^3}{L^3}\right]$	$\rho \equiv Density \left[\frac{M}{L^3}\right]$

Formation Volume Factor—Continuity Equation

At reservoir conditions, density is: $\rho = \rho_{RC} = \dfrac{\rho_{SC}}{B_w}$

Replacing density in continuity equation and divide through by ρ_{SC} (a constant)

$$\frac{k}{\mu}\frac{\partial}{\partial x}\left(\frac{\rho_{SC}}{B_w}\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\rho_{SC}}{B_w}\phi\right) - \tilde{m} \quad \Rightarrow \quad \frac{k}{\mu}\frac{\partial}{\partial x}\left(\frac{1}{B_w}\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\phi}{B_w}\right) - \frac{\tilde{m}}{\rho_{sc}}$$

Using the product rule on the left-hand side of the equation

$$\frac{k}{\mu}\left[\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{1}{B_w}\right)\frac{\partial p}{\partial x}\right] = \frac{\partial}{\partial t}\left(\frac{\phi}{B_w}\right) - \frac{\tilde{m}}{\rho_{sc}}$$

…And chain rule on the left hand side,

$$\frac{k}{\mu}\left[\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)\left(\frac{\partial p}{\partial x}\right)^2\right] = \frac{\partial}{\partial t}\left(\frac{\phi}{B_w}\right) - \frac{\tilde{m}}{\rho_{sc}}$$

$\rho \equiv Density \left[\frac{M}{L^3}\right]$	$B_w \equiv Volumetric\ Factor \left[\frac{L^3}{L^3}\right]$	$k \equiv Permeability\left[L^2\right]$	$\mu \equiv Viscosity\left[\frac{M}{LT}\right]$	$p \equiv Pressure \left[\frac{M}{LT^2}\right]$	$\tilde{m} \equiv Sink\ or\ source\ rate \left[\frac{M}{L^3T}\right]$

Expansion of the Time Derivative

Chain and product rule on time derivative (right hand side)

$$\frac{k}{\mu}\left[\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2}+\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)\left(\frac{\partial p}{\partial x}\right)^2\right]=\left[\phi\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)+\frac{1}{B_w}\frac{\partial \phi}{\partial p}\right]\frac{\partial p}{\partial t}-\frac{\tilde{m}}{\rho_{sc}}$$

A few definitions:

$$c_r=\frac{1}{V_P}\frac{\partial V_P}{\partial p}=\frac{1}{\phi}\frac{\partial \phi}{\partial p}\quad \text{(rock compressibility)}$$

$$c_f=-\frac{1}{V}\frac{\partial V}{\partial p}\bigg|_T=\frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_T=B_w\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)=\frac{-1}{B_w}\frac{\partial B_w}{\partial p}\quad \text{(fluid compressibility)}$$

$$c_t=c_r+c_f\quad \text{(total compressibility)}$$

With some manipulation:

$$\frac{k}{\mu}\left[\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2}+\frac{1}{B_w}\underbrace{B_w\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)}_{c_f}\left(\frac{\partial P}{\partial x}\right)^2\right]=\frac{\phi}{B_w}\left[\underbrace{B_w\frac{\partial}{\partial p}\left(\frac{1}{B_w}\right)}_{c_f}+\underbrace{\frac{1}{\phi}\frac{\partial \phi}{\partial p}}_{c_r}\right]\frac{\partial p}{\partial t}-\frac{\tilde{m}}{\rho_{sc}}$$

1D Diffusivity Equation

≈0 slightly compressible fluid

$$\frac{k}{\mu}\left[\frac{1}{B_w}\frac{\partial^2 p}{\partial x^2}+\cancel{\frac{c_f}{B_w}\left(\frac{\partial p}{\partial x}\right)^2}\right]=\frac{\phi c_t}{B_w}\frac{\partial p}{\partial t}-\frac{\tilde{m}}{\rho_{sc}}$$

If the fluid is "slightly compressible" (liquid), the compressibility is small ($< 10^{-5}$) and constant and terms involving can be ignored.

1D diffusivity (with homogenous fluid and reservoir properties) can be written:

$$\boxed{\frac{\partial^2 p}{\partial x^2}=\frac{1}{\alpha}\frac{\partial p}{\partial t}-\frac{\tilde{q}_{SC}}{\lambda}}$$

$$\text{Diffusivity constant} \equiv \alpha = \frac{k}{\mu\phi c_t}$$

$$\text{Mobility} \equiv \lambda = \frac{k}{\mu B_w}$$

$$\text{Source} \equiv \tilde{q}_{SC} = \frac{\tilde{m}}{\rho_{SC}}$$

If no sources or sinks (wells) are present, we get the "heat equation"

$$\frac{\partial^2 p}{\partial x^2}=\frac{1}{\alpha}\frac{\partial p}{\partial t}$$

$k \equiv Permeability\left[L^2\right]$	$\mu \equiv Viscosity\left[\frac{M}{LT}\right]$	$p \equiv Pressure\left[\frac{M}{LT^2}\right]$	$\rho_{SC} \equiv Density\ at\ SC\left[\frac{M}{L^3}\right]$	$\lambda \equiv Mobility = \frac{k}{\mu B}\left[\frac{L^3 T}{M}\right]$	$\tilde{q}_{SC} \equiv Sink/source\ at\ SC\left[\frac{1}{T}\right]$

Generalized Diffusivity (Heat) Equation

$$\boxed{\nabla^2 p = \frac{1}{\alpha}\frac{\partial p}{\partial t} - \frac{\tilde{q}_{SC}}{\lambda}}$$

In 2D (x-y plane)

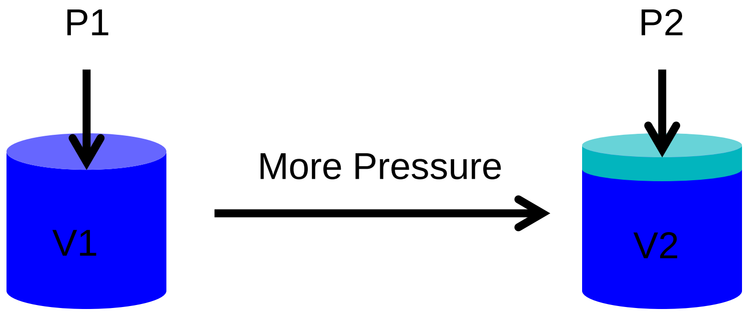
$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\alpha}\frac{\partial p}{\partial t} - \frac{\tilde{q}_{SC}}{\lambda}$$

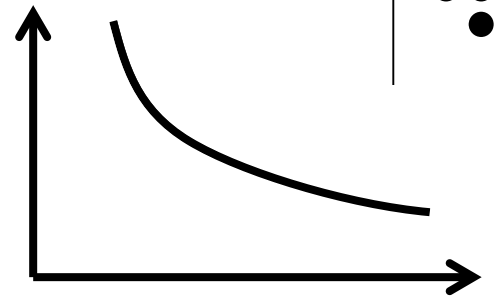
In 3D and potential Φ accounting for gravity

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{\alpha}\frac{\partial p}{\partial t} - \frac{\tilde{q}_{SC}}{\lambda}$$

$p \equiv Pressure\left[\frac{M}{LT^2}\right] \equiv Potential \equiv \Phi$	$\alpha = \frac{\lambda}{\phi c_t} \equiv \left[\frac{L^2}{T}\right]$	$\tilde{q}_{SC} \equiv Sink/source\ at\ SC\left[\frac{1}{T}\right]$	$\lambda = \frac{k}{\mu B} \equiv Mobility\left[\frac{L^3 T}{M}\right]$

Slightly Compressible Fluids: Liquids





Recall fluid compressibility factor (c_f) at constant temperature

$$c_f = -\frac{1}{V}\frac{\partial V}{\partial p}\bigg|_T = \frac{1}{\rho}\frac{\partial \rho}{\partial p}\bigg|_T = B_w \frac{\partial}{\partial p}\left(\frac{1}{B_w}\right) = \frac{-1}{B_w}\frac{\partial B_w}{\partial p}$$

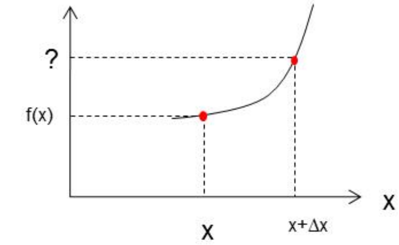
Integrating from a reference point (p^0, ρ^0), to any other point

$$\int_{p^0}^{p} c_f \, dp = \int_{\rho^0}^{\rho} \frac{1}{\rho} d\rho \quad \Rightarrow \quad \rho = \rho^0 e^{c_f (p - p^0)}$$

$c \equiv \text{Compressibility Factor} \left[\frac{LT^2}{M}\right]$	$V \equiv \text{Volume} \left[L^3\right]$	$p \equiv \text{Pressure} \left[\frac{M}{LT^2}\right]$	$\rho \equiv \text{Density} \left[\frac{M}{L^3}\right]$

Taylor Series Expansion

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2!} f''(x)\Delta x^2 + \frac{1}{3!} f'''(x)\Delta x^3 + \ldots$$



Using Taylor series to expand density around a reference density,

$$\rho(p^0 + \Delta p) = \rho(p^0) + \frac{\partial \rho}{\partial p}(p^0)\Delta p + \frac{1}{2!}\rho''(p^0)\Delta p^2 + \frac{1}{3!}\rho''(p^0)\Delta p^3 + \ldots$$

Differentiate exponential equation for density: $\frac{\partial \rho}{\partial p}\left(p^0\right) = \rho^0 c_f e^{c_f(p^0 - p^0)} = \rho^0 c_f$

For slightly compressible ($c_f < 10^{-5}$ psi^{-1}) liquids, higher order terms are small:

Negligible for small c_f

$$\rho(p^0 + \Delta p) = \rho^0 + c_f \rho^0 \Delta p + \cancel{\left[\frac{1}{2!} c_f^2 \rho^0 \Delta p^2 + \frac{1}{3!} c_f^3 \rho^0 \Delta p^3 + \ldots\right]}$$

Therefore,

$$\boxed{\rho \approx \rho^0 \left[1 + c_f (p - p^0)\right]}$$

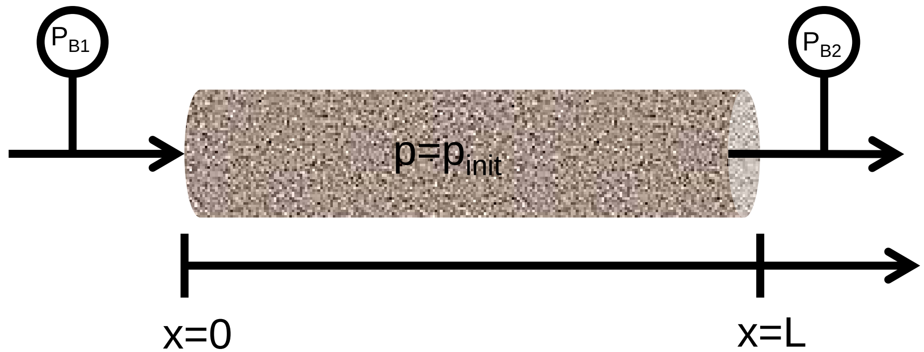
$$\boxed{\frac{1}{B_w} \approx 1 + c_f (p - p^0)}$$

$B_w^0=1$ (assume reference is standard conditions)

$\rho \equiv Density \left[\frac{M}{L^3}\right]$

$p \equiv Pressure \left[\frac{M}{LT^2}\right]$

Simple 1D Problem: Core Flooding



“Heat” Equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{\alpha}\frac{\partial p}{\partial t}$$

$$p(x,0) = p_{init} = 0.0 \ \text{psi}$$

$$p(0,t) = p_{B1} = 1000.0 \ \text{psi}$$

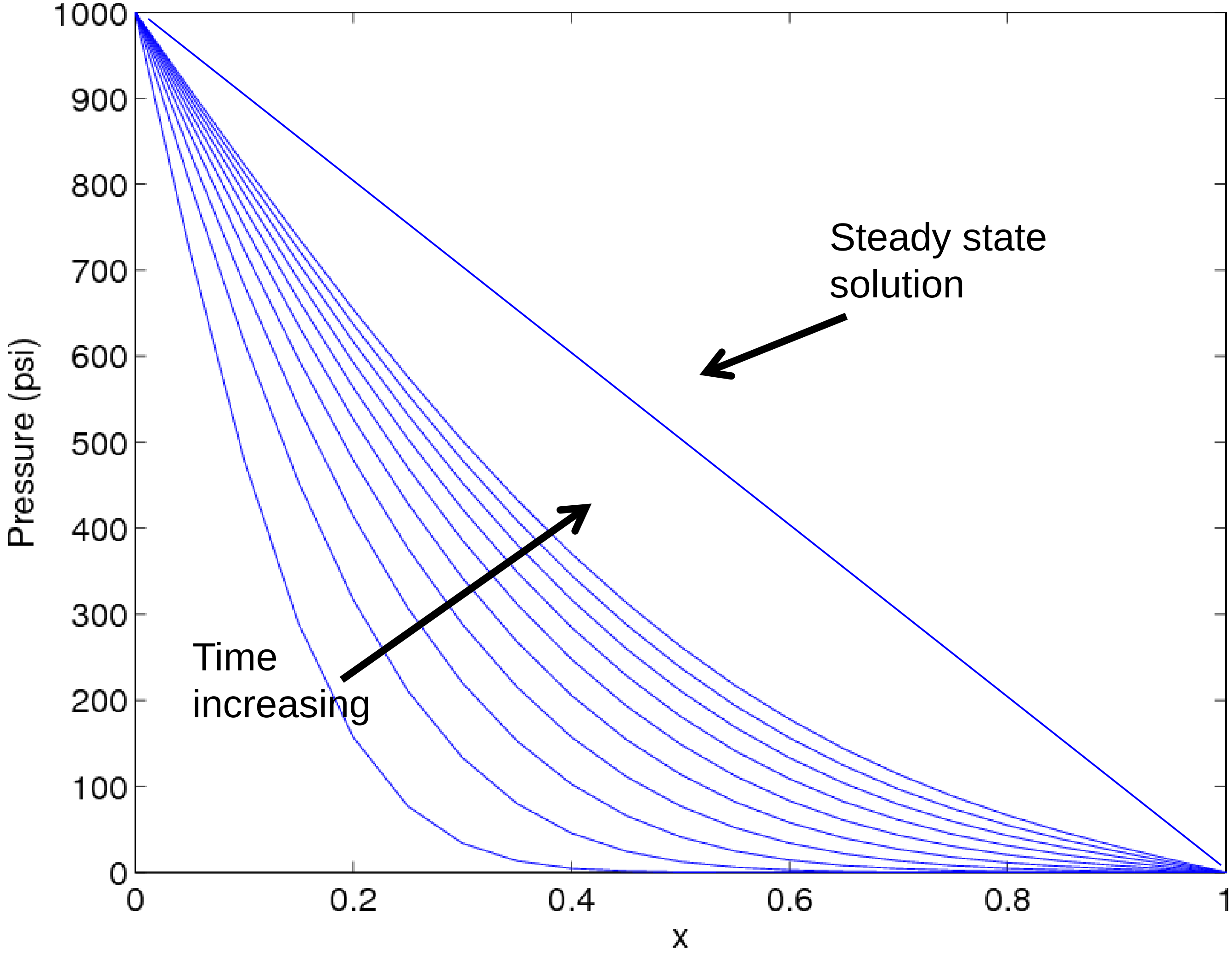
$$p(L,t) = p_{B2} = 0.0 \ \text{psi}$$

Analytical Solution to PDE

$$p(x,t) = p_{B1} - \frac{4p_{init}}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}e^{\frac{\alpha(2n+1)^n\pi^2 t}{4L^2}}\cos\frac{(2n+1)\pi x}{2L}$$

Analytical Solution

$$p\left(x,t\right) = p_{B1} - \frac{4p_{init}}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}e^{\frac{\alpha(2n+1)^n\pi^2 t}{4L^2}}\cos\frac{(2n+1)\pi x}{2L}$$



Real Reservoirs

That was the easy solution…

Real reservoirs have:

- Spatially varying permeability, porosity, etc.
- Time-varying viscosity, formation volume factor
- Geometries that are 2D or 3D and are not on a regular grid
- Sources and sinks (called wells) spaced irregularly throughout the reservoir
- Complex boundary conditions

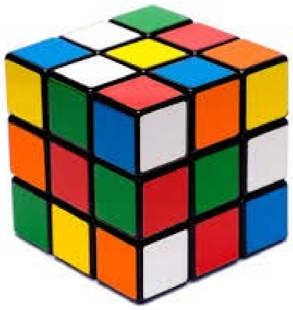
$$\boxed{\nabla\left[\frac{k}{\mu B_w}\left(\nabla p+\rho g\nabla z\right)\right]=\frac{\phi c_t}{B_w}\frac{\partial p}{\partial t}-\tilde{q}_{SC}}$$

(and this is just for single phase flow…)

Solving "ugly" PDE

So how do we solve this complex, nonhomogeneous, 3D PDE?

1. Break the reservoir into manageable blocks that have contain reservoir and fluid properties



2. Write algebraic equations for each block by "discretizing" PDE

$$
\begin{aligned}
&3TP_1 - TP_2 = Q_1 \\
&-TP_1 + 2TP_2 - TP_3 = Q_2 \\
&-TP_2 + 2TP_3 - TP_4 = Q_3 \\
&\vdots \\
&-TP_{N-1} + TP_N = Q_N
\end{aligned}
$$

3. Solve system of linear equations

$$\left(\mathbf{T} + \frac{\mathbf{B}}{\Delta t}\right) P^{n+1} = \frac{\mathbf{B}}{\Delta t} P^{n+1} + \mathbf{Q}$$