

Condition testing

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Contingency table:

		Condition (as determined by "Gold standard")			
		Total population	Condition positive	Condition negative	Prevalence = $\frac{\Sigma \text{ Condition positive}}{\Sigma \text{ Total population}}$
Test outcome	Test outcome positive	True positive	False positive (Type I error)	Positive predictive value (PPV, Precision) = $\frac{\Sigma \text{ True positive}}{\Sigma \text{ Test outcome positive}}$	False discovery rate (FDR) = $\frac{\Sigma \text{ False positive}}{\Sigma \text{ Test outcome positive}}$
	Test outcome negative	False negative (Type II error)	True negative	False omission rate (FOR) = $\frac{\Sigma \text{ False negative}}{\Sigma \text{ Test outcome negative}}$	Negative predictive value (NPV) = $\frac{\Sigma \text{ True negative}}{\Sigma \text{ Test outcome negative}}$
Positive likelihood ratio (LR+) = $\frac{\text{TPR}}{\text{FPR}}$		True positive rate (TPR, Sensitivity, Recall) = $\frac{\Sigma \text{ True positive}}{\Sigma \text{ Condition positive}}$	False positive rate (FPR, Fall-out) = $\frac{\Sigma \text{ False positive}}{\Sigma \text{ Condition negative}}$	Accuracy (ACC) = $\frac{\Sigma \text{ True positive} + \Sigma \text{ True negative}}{\Sigma \text{ Total population}}$	
Negative likelihood ratio (LR-) = $\frac{\text{FNR}}{\text{TNR}}$		False negative rate (FNR) = $\frac{\Sigma \text{ False negative}}{\Sigma \text{ Condition positive}}$	True negative rate (TNR, Specificity, SPC) = $\frac{\Sigma \text{ True negative}}{\Sigma \text{ Condition negative}}$		
Diagnostic odds ratio (DOR) = $\frac{\text{LR+}}{\text{LR-}}$					

In terms of probabilities:

Event	Event		Total
	B ₁	B ₂	
A ₁	P(A ₁ and B ₁)	P(A ₁ and B ₂)	P(A ₁)
A ₂	P(A ₂ and B ₁)	P(A ₂ and B ₂)	P(A ₂)
Total	P(B ₁)	P(B ₂)	1

Joint Probability Marginal (Simple) Probability

Marginal probabilities:

Relative size	Cond B	Cond \bar{B}	Total
Case A	w	x	w+x
Case \bar{A}	y	z	y+z
Total	w+y	x+z	w+x+y+z = 1

$$P(B) = \frac{w+y}{w+x+y+z} = w+y$$

$$P(A) = \frac{w+x}{w+x+y+z} = w+x$$

Condition (determined by gold standard dataset)		
Disease (D)		No Disease (D^c)
Test (+)	TP	FP
Test (-)	FN	TN
	P	N

(being the condition for example a Disease)

Probability of condition positive $\rightarrow p(D) = TP+FN \Rightarrow$ **PREVALENCE**

Probability of condition negative $\rightarrow p(D^c) = FP+TN$

Probability of test outcome positive $\rightarrow p(+) = TP+FP$

Probability of test outcome negative $\rightarrow p(-) = FN+TN$

Where:

$$p(\text{total}) = TP+FP+FN+TN = 1$$

$$p(D^c) = 1 - p(D) \equiv FP+TN = 1 - (TP+FN)$$

$$p(-) = 1 - p(+) \equiv FN+TN = 1 - (TP+FP)$$

Joint probabilities (intersections):

Relative size	Cond B	Cond \bar{B}	Total
Case A	w	x	w+x
Case \bar{A}	y	z	y+z
Total	w+y	x+z	w+x+y+z = 1

$$P(AB) = \frac{w}{w+x+y+z} = w$$

Condition (determined by gold standard dataset)		
Disease (D)		No Disease (D^c)
Test (+)	TP	FP
Test (-)	FN	TN
	P	N

Probability of + and $D \rightarrow p(+ \cap D) = TP$


Probability of - and $D \rightarrow p(- \cap D) = FN$

Probability of + and $D^c \rightarrow p(+ \cap D^c) = FP$


Probability of - and $D^c \rightarrow p(- \cap D^c) = TN$

Conditional probabilities:

Relative size	Cond B	Cond B̄	Total
Case A	w	x	w+x
Case Ā	y	z	y+z
Total	w+y	x+z	w+x+y+z = 1



$$P(A|B) = \frac{w}{w+y} = \frac{P(AB)}{P(B)}$$



$$P(B|A) = \frac{w}{w+x} = \frac{P(AB)}{P(A)}$$

	Condition (determined by gold standard dataset)	
	Disease (D)	No Disease (D ^c)
Test (+)	TP	FP
Test (-)	FN	TN
	P	N

- **Test outcome conditioned on condition**

(the probability of having a positive/negative test result for a patient that has/hasn't the disease)

Probability of + given D → $p(+|D) = \frac{p(+ \cap D)}{p(D)} = \frac{TP}{TP+FN} \Rightarrow$ **SENSITIVITY**

Probability of + given D^c → $p(+|D^c) = \frac{p(+ \cap D^c)}{p(D^c)} = \frac{FP}{FP+TN} \Rightarrow$ **FALSE POSITIVE RATE**

Probability of - given D^c → $p(-|D^c) = \frac{p(- \cap D^c)}{p(D^c)} = \frac{TN}{FP+TN} \Rightarrow$ **SPECIFICITY**

Probability of - given D → $p(-|D) = \frac{p(- \cap D)}{p(D)} = \frac{FN}{TP+FN} \Rightarrow$ **FALSE NEGATIVE RATE**

Where:

$p(+|D^c) = 1 - p(-|D^c) \equiv \frac{FP}{FP+TN} = 1 - \frac{TN}{FP+TN} \rightarrow FPR = 1 - \text{SPECIFICITY}$

$p(-|D) = 1 - p(+|D) \equiv \frac{FN}{TP+FN} = 1 - \frac{TP}{TP+FN} \rightarrow FNR = 1 - \text{SENSITIVITY}$

- **Condition conditioned on test outcome**

(the probability of having/not having the disease if the test result is positive/negative)

Probability of D given + → $p(D|+) = \frac{p(+ \cap D)}{p(+)} = \frac{TP}{TP+FP} \Rightarrow$ **POSITIVE PREDICTIVE VALUE (PPV, PRECISION)**

Probability of D given - → $p(D|-) = \frac{p(- \cap D)}{p(-)} = \frac{FN}{TP+FN} \Rightarrow$ **FALSE OMISSION RATE (FOR)**

Probability of D^c given + → $p(D^c|+) = \frac{p(+ \cap D^c)}{p(+)} = \frac{FP}{TP+FP} \Rightarrow$ **FALSE DISCOVERY RATE (FDR)**

Probability of D^c given - → $p(D^c|-) = \frac{p(- \cap D^c)}{p(-)} = \frac{TN}{FP+TN} \Rightarrow$ **NEGATIVE PREDICTIVE VALUE (NPV)**

Bayes' theorem:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)} \rightarrow p(A|B)p(B) = p(B|A)p(A)$$

Demonstration:

Relative size	Cond B	Cond B̄	Total
Case A	w	x	w+x
Case Ā	y	z	y+z
Total	w+y	x+z	w+x+y+z = 1

$$P(A|B) \times P(B) = \frac{w}{w+y} \times \frac{w+y}{w+x+y+z} = \frac{w}{w+x+y+z} = w = P(AB)$$

$$P(B|A) \times P(A) = \frac{w}{w+x} \times \frac{w+x}{w+x+y+z} = \frac{w}{w+x+y+z} = w = P(AB)$$

SENSITIVITY : $p(+|D) = \frac{p(+ \cap D)}{p(D)}$

PPV : $p(D|+) = \frac{p(+ \cap D)}{p(+)}$

Then:

$$\rightarrow p(+ \cap D) = p(+|D)p(D) = p(D|+)p(+) \Rightarrow TP = \text{SENSITIVITY} * \text{PREVALENCE} = PPV * (TP+FP)$$

Bayes' rule:

$$p(B|A) = \frac{p(A|B)p(B)}{p(A|B)p(B) + p(A|B^c)p(B^c)}$$

Bayes' theorem:

$$p(B|A) = \frac{p(A|B)p(B)}{p(A)}$$

Then:

$$\rightarrow p(A) = p(A|B)p(B) + p(A|B^c)p(B^c)$$

Demonstration:

$$p(+) = p(+|D)p(D) + p(+|D^c)p(D^c) = p(+ \cap D) + p(+ \cap D^c) \Rightarrow TP+FP = TP+FP$$

Application:

$$p(D|+) = \frac{p(+|D)p(D)}{p(+|D)p(D) + p(+|D^c)p(D^c)} \Rightarrow PPV = \frac{\text{SENSITIVITY} * \text{PREVALENCE}}{\text{SENSITIVITY} * \text{PREVALENCE} + (1 - \text{SPECIFICITY}) * (1 - \text{PREVALENCE})}$$

Likelihood ratios:

Bayes' rule:

$$\text{PPV} = p(D|+) = \frac{p(+|D)p(D)}{p(+|D)p(D)+p(+|D^c)p(D^c)}$$

$$\text{FDR} = p(D^c|+) = \frac{p(+|D^c)p(D^c)}{p(+|D)p(D)+p(+|D^c)p(D^c)}$$

Then:

$$\rightarrow \frac{p(D|+)}{p(D^c|+)} = \frac{p(+|D)}{p(+|D^c)} * \frac{P(D)}{P(D^c)}$$

$$\Rightarrow \text{POST TEST ODDS} = \text{DLR}_+ * \text{PRE TEST ODDS}$$

$$\Rightarrow \text{DLR}_+ = \frac{\text{POST TEST ODDS}}{\text{PRE TEST ODDS}}$$

DLR_+ relates the increase in the odds of the disease after a positive test result to the odds of disease prior to the test.

Similarly:

$$\rightarrow \frac{p(D|-)}{p(D^c|-)} = \frac{p(-|D)}{p(-|D^c)} * \frac{P(D)}{P(D^c)}$$

$$\Rightarrow \text{POST TEST ODDS} = \text{DLR}_- * \text{PRE TEST ODDS}$$

$$\Rightarrow \text{DLR}_- = \frac{\text{POST TEST ODDS}}{\text{PRE TEST ODDS}}$$

DLR_- relates the decrease in the odds of the disease after a negative test result to the odds of disease prior to the test.

Type I and II errors

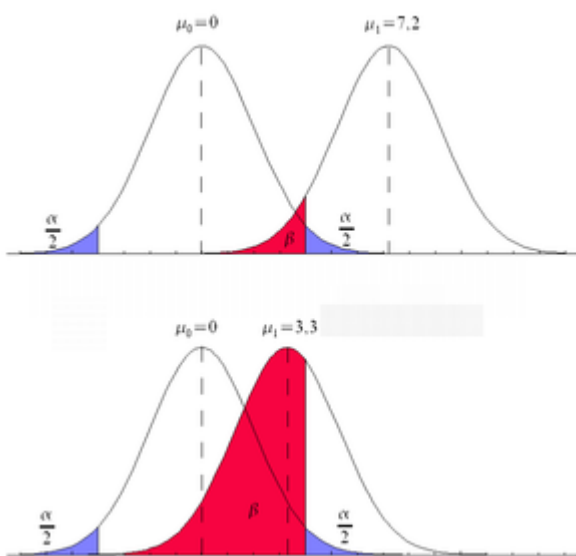
The **FPR** is a **Type I error** (α type error):

$$p(+|D^c) \equiv p(\text{rejecting } H_0 \mid H_0 \text{ is true})$$

We are rejecting the null hypothesis “there are no significant differences with the ‘*normal state*’ population” when we shouldn’t, that is, we are considering that the patient doesn’t belong to the healthy population because he falls into the rejection region determined by the α significance level.

This happens because the two populations (‘*normal state*’ = condition negative = no disease, and ‘*altered state*’ = condition positive = disease) overlap to some extent, so:

- the bigger the significance level, the more we are assuring the ‘integrity’ of the ‘*normal state*’ population: we maximize true positives at the expense of maximizing false positives (bigger $\alpha \rightarrow$ more Type I error)
- the smaller the significance level, the more we are assuring the integrity of the ‘*altered state*’ population: we minimize false positives at the expense of minimizing true positives (smaller $\alpha \rightarrow$ less Type I error)



The **FNR** is a **Type II error** (β type error):

$$p(-|D) \equiv p(\text{not rejecting } H_0 \mid H_0 \text{ is not true})$$

We aren’t rejecting the null hypothesis when we should, that is, we are considering that the patient is no significantly different than the ‘*normal state*’ population because he doesn’t fall into the α region.

Again, this happens because the two populations are overlapping: the more they do (the closer their means are to each other), the less we are able to discern between the two populations \rightarrow the less we reject the null hypothesis (more false negatives). In this case we can’t control the extension of the β region since it is determined by the mean of the alternative population.