Homework 5: Optimization

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import random
import math
from scipy.optimize import minimize
```

Problem 1: Interval Bisection Search and Golden Section Search - 1-D Optimization

Definition of function and its derivative

```
# function of x**2
def x2(x):
    return x**2
```

```
# Derivative of x**2
def x2der(x):
    return 2*x
```

Definition of IBS and GSS method

```
def IBS(derivative, u, l, tol=10**(-4), maxiter=50):
    iteration = 0
    # End condition
    while (u - l > tol) and (iteration < maxiter):
        value = derivative( ( u + l ) /2)
        if value < 0:
            l = value
        else:
            u = value

        iteration += 1

    return ((u + l) /2, iteration)
```

```
def GSS(function, u, l, tol=10**(-4), maxiter=50):
    golden = (1 + 5 ** 0.5) / 2 -1
```

```
    iteration = 0
    # End condition
    while (u - l > tol) and (iteration < maxiter):
        x1 = u - golden*(u - l)
        x2 = l + golden*(u - l)
        value1 = function(x1)
        value2 = function(x2)

        if value1 < value2:
            u = x2
        else:
            l = x1

        iteration += 1

    return ((u + l) /2, iteration)
```

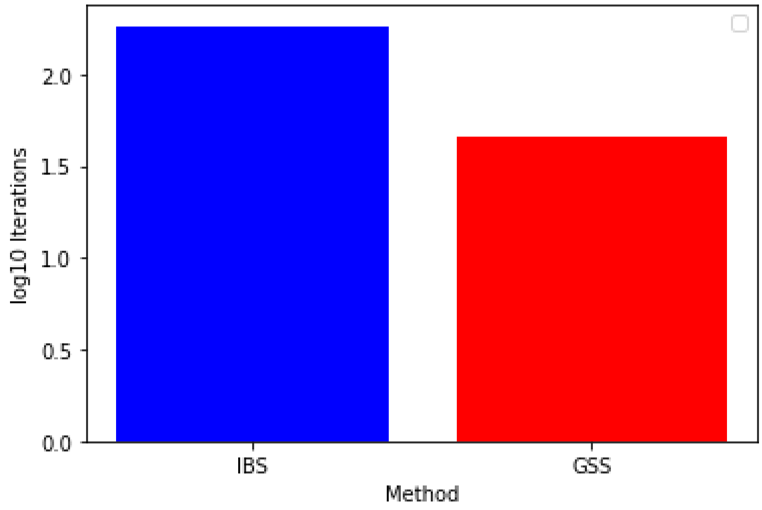
50 IBS and GSS iterations

```
ibs = []
gss = []
maxiterations = 1000
for i in range(50):
    lower = np.random.uniform(-10, 0)
    upper = np.random.uniform(0, 10)
    tolerance = np.random.uniform(10**(-8), 0)
    ibs.append(IBS(x2der, upper, lower, tolerance, maxiter=maxiterations)[1])
    gss.append(GSS(x2, upper, lower, tolerance, maxiter=maxiterations)[1])
```

IBS and GSS comparison

```
plt.bar('IBS', height= np.log10(np.average(ibs)), color='blue')
plt.bar('GSS', height= np.log10(np.average(gss)), color='red')
plt.xlabel('Method')
plt.ylabel('log10 Iterations')
plt.legend()
```

```
No handles with labels found to put in legend.




<matplotlib.legend.Legend at 0x1e2915f6d00>
```



We can clearly see that GSS is in faster than IBS for the experiment conducted

Problem 2: Convexity in 1- and 2-Dimensions

1. Using first and second derivatives confirm or reject that f is convex $f(v)=(1-v)^2$

f' = -2(1-v) we can see that f'(1)=0 which corresponds to the minimum of f.

f'' = 2 We can see that f'' is always positive through its domain. This implies that f is convex.

Let's have a visual approach.

```
def f(x):
    return (1-x)**2
```

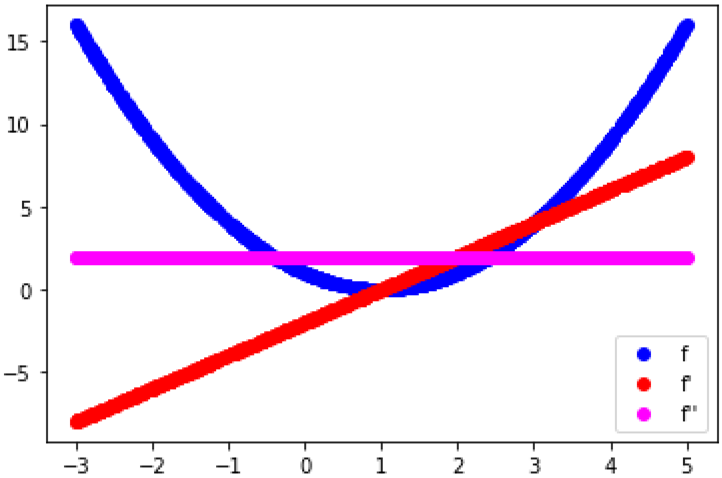
```
def f1(x):
    return -2*(1-x)
```

```
def f2(x):
    return 2
```

```
xs = []
fs = []
f1s = []
f2s = []

for i in range(1000):
    x = np.random.uniform(-3, 5)
    xs.append(x)
```

```
    fs.append(f(x))
    f1s.append(f1(x))
    f2s.append(f2(x))

plt.scatter(xs,fs, color='blue', label='f')
plt.scatter(xs, f1s, color='red', label="f'")
plt.scatter(xs, f2s, color='magenta', label="f''")
plt.legend()
```

```
<matplotlib.legend.Legend at 0x1e291d561f0>
```



2. g(v) and h(v) are convex, is f(v) = g(v) + h(v) convex?

Since g and h are convex, we have that g'' and h'' are > 0. This means that f'' = g'' + h'' is also > 0 and therefore convex.

3. $g(v)=(a-v^2)^2$ where a is a fixed real number.

g' = -4v(a-v^2)

g'' = -4(a-3v^2)

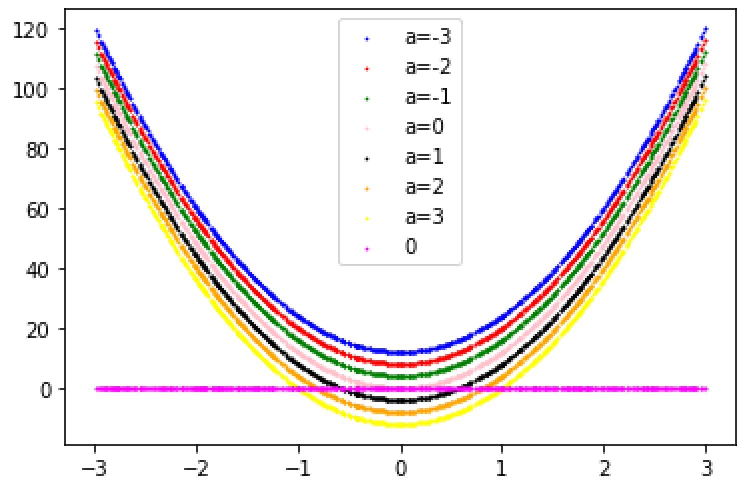
We want g'' = 12v^2 - 4a > 0 → v^2 > $\frac{a}{3}$. This means that if $a \le 0$ g is convex in $\Re$, else g is convex in the ranges $\brack -\infty,-\sqrt{\frac{a}{3}}$ $\bigcup$ $\brack \sqrt{\frac{a}{3}},\infty$.

```
def g2(x,a):
    return -4*(a-3*x**2)
```

```
xs = []
g2s = []

for i in range(1000):
    x = np.random.uniform(-3, 3)
    xs.append(x)
    g2s.append([g2(x,-3),g2(x,-2),g2(x,-1),g2(x,0),g2(x,1),g2(x,2),g2(x,3)])

g2s = np.array(g2s)
zeros = np.zeros(len(g2s))
```

```
plt.scatter(xs, g2s[:,0], s=0.5, color='blue', label='a=-3')
plt.scatter(xs, g2s[:,1], s=0.5, color='red', label='a=-2')
plt.scatter(xs, g2s[:,2], s=0.5, color='green', label='a=-1')
plt.scatter(xs, g2s[:,3], s=0.5, color='pink', label='a=0')
plt.scatter(xs, g2s[:,4], s=0.5, color='black', label='a=1')
plt.scatter(xs, g2s[:,5], s=0.5, color='orange', label='a=2')
plt.scatter(xs, g2s[:,6], s=0.5, color='yellow', label='a=3')
plt.scatter(xs, zeros, s=0.5, color='magenta', label='0')

plt.legend()
```

```
<matplotlib.legend.Legend at 0x1e291e33bb0>
```



4. For a vector v=[v1 v2]^T, let $p(v)=(1−v_1)^2+100∗(v_2−v_1^2)^2$. Calculate the gradient and Hessian of p, and confirm or reject that p is convex everywhere in $\Re^2$

$\nabla p = (2 (200 v_1^3 - 200 v_1 v_2 + v_1 - 1), 200 (y - x^2))$

H=$\begin{bmatrix}(-400 (v_2 - v_1^2) + 800 v_1^2 + 2 & -400 v_1\-400 v_1 & 200\end{bmatrix}$

$\lambda_1 = 600 v_1^2 - \sqrt{360000 v_1^4 - 240000 v_1^2 v_2 + 41200 v_1^2 + 40000 v_2^2 + 39600 v_2 + 9801} - 200 v_2 + 101$ $\lambda_2 = 600 v_1^2 + \sqrt{360000 v_1^4 - 240000 v_1^2 v_2 + 41200 v_1^2 + 40000 v_2^2 + 39600 v_2 + 9801} - 200 v_2 + 101$

It is clear that $\lambda_1 \le \lambda_2 $ therefore, we need to find the conditions that fulfill $\lambda_1>0$ $600 v_1^2 - \sqrt{360000 v_1^4 - 240000 v_1^2 v_2 + 41200 v_1^2 + 40000 v_2^2 + 39600 v_2 + 9801} - 200 v_2 + 101 > 0$

Finally, p is convex when $v_2 < \frac{1}{200} (200 v_1^2 + 1)$

```
def eigen1(x,y):
    return 600*x**2 - math.sqrt(360000*x**4 - 240000*(x**2)*y + 41200*x**2 +
40000*y**2 + 39600*y + 9801) - 200*y + 101
```

```
def eigen2(x,y):
    return 600*x**2 + math.sqrt(360000*x**4 - 240000*(x**2)*y + 41200*x**2 +
40000*y**2 + 39600*y + 9801) - 200*y + 101
```

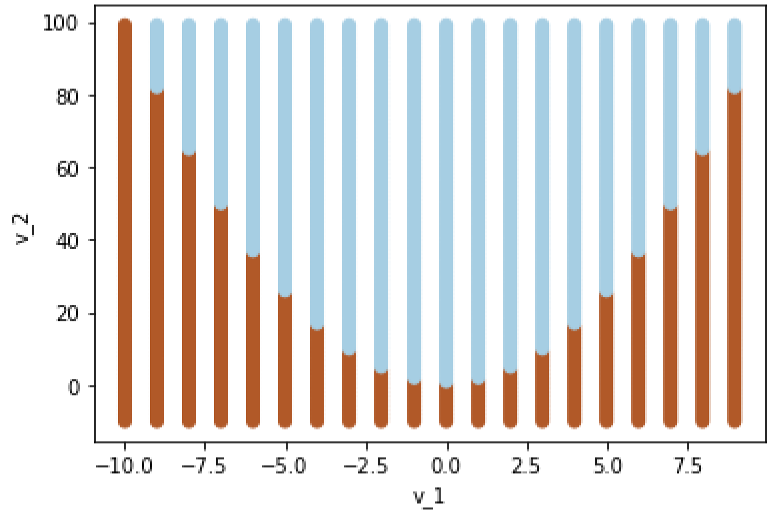
```
x = 1
y = 1
print(eigen1(x,y), eigen2(x,y))
```

```
0.3993607674876216 1001.6006392325123
```

```
x = []
y = []
min_eigen = []

for i in range(-10,10):
    for j in range (-10,100):
        x.append(i)
        y.append(j)
        e1 = eigen1(i,j)
        e2 = eigen2(i,j)
        if e1 < e2:
            min_eigen.append(e1)
        else:
            min_eigen.append(e2)

pos_eigen = [0 if item<0 else 1 if item==0 else 2 for item in min_eigen]
```

The smallest eigenvalue, appears to be negative inside a parabole, therefore we can conclude that p is convex in the dark area corresponding to $v_2 < \frac{1}{200} (200 v_1^2 + 1)$

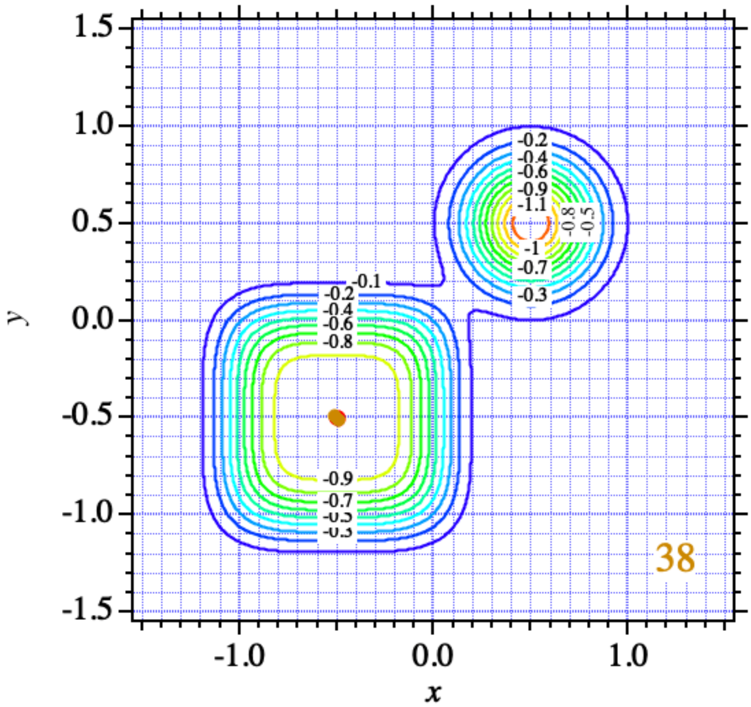
```
plt.scatter(x, y, c=pos_eigen, cmap='Paired')
plt.xlabel('v_1')
plt.ylabel('v_2')

plt.show()
```



Problem 3: Nelder-Mead Search and Gradient Descent

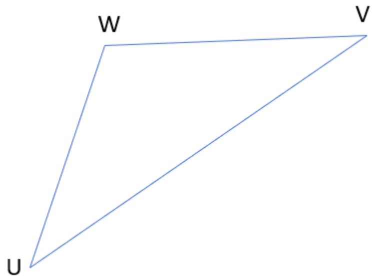
1. How Nelder-Mead Search works

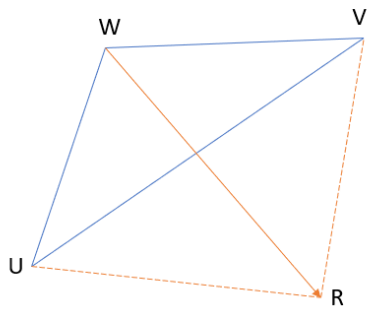
Explanation from https://www.youtube.com/watch?v=vOYlVvT3W80 and GIF from http://takashiida.floppy.jp/en/education-2/gif-nelder-mead/

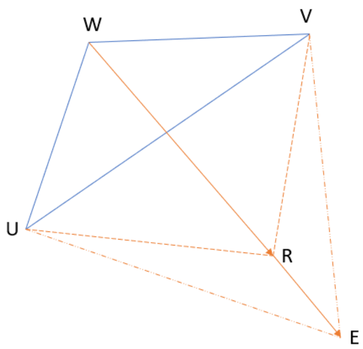


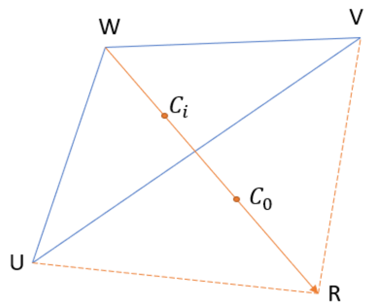
The Nelder-Mead method is a direct search method, meaning that it does not require the computation of the gradient. This makes it very useful for cases where either the gradient is hard to obtain/evaluate or if the function is not smooth. It is based on a n+1 dimension structure called a *simplex*, for example, if we are in $\Re^2$, it will be a triangle (3 points) and in each iteration, this points are updated. Let's take **u**, **v** and **w** as our three points.

In each iteration, there are 6 steps:

1. **Sort**: Sort and label the points according to the function's value $f(u) < f(v) < f(w)$ **u** is the best point 

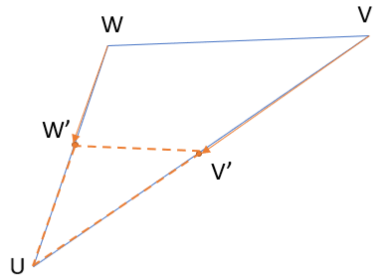
and **w** is the worst.
2. **Reflect**: Reflect the worst point through the centroid of the two remaining points, point **r**. If $f(r) < f(v)$ but $f(u) < f(r)$, then **w = r**. Now **v** is the worst performing point. Go to step 6: Check for Convergence.


3. **Extend**: If $f(r) < f(u)$ then we extend **r** to **e**. If $f(e) < f(r)$ then **w = e** else **w = r**. Go to step 6: Check 

for Convergence.
4. **Contract**: If $f(r) > f(v)$, **w** is contracted to **c_0** (outside) and **c_i** (inside) where $c_i = \frac{1}{4}(w-r)$ and $c_o = \frac{3}{4}(w-r)$ and evaluate $f(c_o)$ and $f(c_i)$. If either performs better than **v**, then **w** = best performing between **c_0** and **c_i**. Go to step 6: Check for Convergence.


5. **Shrink**: If the contracted points do not outperform **v**, then we shrink the simplex towards the **u**, the best performing point with **w = w'** and **v = v'**, where **v'** is the halfway point between **u** and **v** and **w'** is

the halfway point between **u** and **w**. Go to step 6: Check for Convergence.



6. **Check Convergence**: There are many ways to check convergence, but a common one is to check whether the standard deviation between the points of the simplex is below some pre defined tolerance.

2. Nelder-Mead (NMA) and Conjugate Gradient Descent (CG)

```
def objective_p(x):
    return (1 - x[0])**2 + 100*(x[1] - x[0]**2)**2
```

```
maxiterations = 5
optimizer = [1,1]

NMA_distance = []
CG_distance = []

for i in range(10):
    p1 = [np.around([np.random.uniform(-3, 12)], decimals=1)[0],
np.around([np.random.uniform(-6, 60)])[0]]
    p2 = [np.around([np.random.uniform(-3, 12)], decimals=1)[0],
np.around([np.random.uniform(-6, 60)])[0]]
    p3 = [np.around([np.random.uniform(-3, 12)], decimals=1)[0],
np.around([np.random.uniform(-6, 60)])[0]]
    pt = [p1, p2, p3]

    NMA = minimize(objective_p, pt, method='nelder-mead', options=
{'maxiter':maxiterations})
    CG = minimize(objective_p, pt, method='CG', options={'maxiter':maxiterations})

    # evaluate solutions
    solution_NMA = NMA['x']
    evaluation_NMA = objective_p(solution_NMA)

    # euclidean distance
    NMA_distance.append((solution_NMA[0] - optimizer[0])**2 + (solution_NMA[1] -
optimizer[1])**2)

    solution_CG = CG['x']
    evaluation_CG = objective_p(solution_CG)
    CG_distance.append((solution_CG[0] - optimizer[0])**2 + (solution_CG[1] -
optimizer[1])**2)
```