E_{recoil}-Q-Distribution

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1 Derivation

The aim of this paper is to derive the propability density of the quantity

$$\vec{q} = \begin{pmatrix} E_{recoil} \\ Q \end{pmatrix} \tag{1}$$

The quantity $\vec{q}$ depends from

$$\vec{r} = \begin{pmatrix} E_{ion} \\ E_{heat} \end{pmatrix} \tag{2}$$

The dependency is given by

$$T : \vec{r} \to \vec{q}, \begin{pmatrix} E_{ion} \\ E_{heat} \end{pmatrix} \to \begin{pmatrix} E_{recoil} \\ Q \end{pmatrix} = \begin{pmatrix} \left(1 + \frac{V}{\epsilon_\gamma}\right) E_{heat} - \frac{V}{\epsilon_\gamma} E_{ion} \\ \frac{E_{ion}}{\left(1+\frac{V}{\epsilon_\gamma}\right) E_{heat} - \frac{V}{\epsilon_\gamma} E_{ion}} \end{pmatrix} \tag{3}$$

Then the inverse transformation is given by

$$T^{-1} : \vec{q} \to \vec{r}, \begin{pmatrix} E_{recoil} \\ Q \end{pmatrix} \to \begin{pmatrix} E_{ion} \\ E_{heat} \end{pmatrix} = \begin{pmatrix} Q E_{recoil} \\ \frac{1+Q\frac{V}{\epsilon_\gamma}}{1+\frac{V}{\epsilon_\gamma}} E_{recoil} \end{pmatrix} \tag{4}$$

The derivative then is given by

$$T'(\vec{r}) = \begin{pmatrix} \frac{\partial E_{recoil}}{\partial E_{ion}} & \frac{\partial E_{recoil}}{\partial E_{heat}} \\ \frac{\partial Q}{\partial E_{ion}} & \frac{\partial Q}{\partial E_{heat}} \end{pmatrix} = \begin{pmatrix} -\frac{V}{\epsilon_\gamma} & 1 + \frac{V}{\epsilon_\gamma} \\ \frac{1}{E_{recoil}} + \frac{V}{\epsilon_\gamma} \frac{E_{ion}}{E_{recoil}^2} & -\left(1 + \frac{V}{\epsilon_\gamma}\right) \frac{E_{ion}}{E_{recoil}^2} \end{pmatrix} \tag{5}$$

and thus the determinant of it is

$$\det T'(\vec{r}) = \frac{V}{\epsilon_\gamma} \left(1 + \frac{V}{\epsilon_\gamma}\right) \frac{E_{ion}}{E_{recoil}^2} - \left(1 + \frac{V}{\epsilon_\gamma}\right) \left(\frac{1}{E_{recoil}} + \frac{V}{\epsilon_\gamma} \frac{E_{ion}}{E_{recoil}^2}\right) = -\frac{1 + \frac{V}{\epsilon_\gamma}}{E_{recoil}} \tag{6}$$

Obviously for all values for E_{recoil} the determinant doesn't vanish and thus the transformation is invertible in the whole domain according to the inverse function theorem.

If we assume that $\vec{r}$ is a multivariate normal-distributed quantity, that means it follows the propability density

$$f(\vec{r}) = \frac{1}{2\pi\sqrt{\det C}} \exp\left(-\frac{1}{2}(\vec{r} - \vec{r}_0)^T C^{-1} (\vec{r} - \vec{r}_0)\right) \tag{7}$$

with the covariance matrix

$$C = \begin{pmatrix} \sigma_{ion}^2 & \sigma_{ion-heat}^2 \\ \sigma_{ion-heat}^2 & \sigma_{heat}^2 \end{pmatrix} \tag{8}$$

, we get the propability density function $g(\vec{q})$ [1][p. 246]

$$g(\vec{q}) = \frac{f(T^{-1}(\vec{q}))}{|det T'(T^{-1}(\vec{q}))|} \tag{9}$$

$$= \exp\left(-\frac{1}{2}\begin{pmatrix} Q E_{recoil} - \overline{E_{ion}} \\ \frac{1+Q\frac{V}{\epsilon_\gamma}}{1+\frac{V}{\epsilon_\gamma}} E_{recoil} - \overline{E_{heat}} \end{pmatrix}^T \begin{pmatrix} \sigma_{ion}^2 & \sigma_{ion-heat}^2 \\ \sigma_{ion-heat}^2 & \sigma_{heat}^2 \end{pmatrix}^{-1} \begin{pmatrix} Q E_{recoil} - \overline{E_{ion}} \\ \frac{1+Q\frac{V}{\epsilon_\gamma}}{1+\frac{V}{\epsilon_\gamma}} E_{recoil} - \overline{E_{heat}} \end{pmatrix}\right) \tag{10}$$

$$\times \frac{|E_{recoil}|}{2\pi\sqrt{\sigma_{ion}^2 \sigma_{heat}^2 - \sigma_{ion-heat}^4}\left(1 + \frac{V}{\epsilon_\gamma}\right)} \tag{11}$$

2 Statistical moments

In order to determine the means $\langle Q \rangle$ and $\langle E_{recoil} \rangle$ we have to calculate

$$\langle Q \rangle = \int_{-\infty}^{\infty} dQ\, Q \cdot \int_{-\infty}^{\infty} dE_{recoil}\, g(E_{recoil}, Q) \tag{12}$$

$$\langle E_{recoil} \rangle = \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dE_{recoil} E_{recoil} g(E_{recoil}, Q) \tag{13}$$

Obviously the exponent in $g(E_{recoil}, Q)$ is a square polynomial in Q as well as in E_{recoil}. So we can write

$$g(E_{recoil}, Q) = k \cdot |E_{recoil}| \cdot \exp\left(a_{E_{recoil}} E_{recoil}^2 + b_{E_{recoil}} E_{recoil} + c_{E_{recoil}}\right) = k \cdot |E_{recoil}| \cdot \exp\left(a_Q Q^2 + b_Q Q + c_Q\right) \tag{14}$$

Then the means are

$$\overline{E_{recoil}} = \tag{15}$$

$$\overline{Q} = \frac{|E_{recoil}|}{2\pi\sqrt{\sigma_{ion}^2 \sigma_{heat}^2 - \sigma_{ion-heat}^4}\left(1 + \frac{V}{\epsilon_\gamma}\right)}. \tag{16}$$

3 Toy experiments

In order to test the goodness of this density function, one can do monte carlo simulation.This can be done in ROOT by creating many events (E_{Recoil}, Q) from Gaussian distributed quantities E_{Ion} with uncertainty $\sigma_{E_{Ion}}$ and E_{Heat} with uncertainty $\sigma_{E_{Heat}}$ and fixed parameters V, ϵ, where

$$E_{Recoil} = \left(1 + \frac{V}{\epsilon}\right) E_{Heat} - \frac{V}{\epsilon} E_{Ion} \tag{17}$$

$$Q = \frac{E_{Ion}}{E_{Recoil}} \tag{18}$$

Then by filling a TH2D histogram with these events and compare it to another TH2D histogram created from the propability density function (pdf) $g(E_{Recoil}, Q)$ by the TH2D::FillRandom() method, one can do a χ^2 test to test the null hypothesis H_0, that both samples origin from the same distribution. For each bin the quantity

$$z_i = \frac{n_{pdf,i} - n_{mc,i}}{\sqrt{n_{pdf,i} + n_{mc,i}}} \tag{19}$$

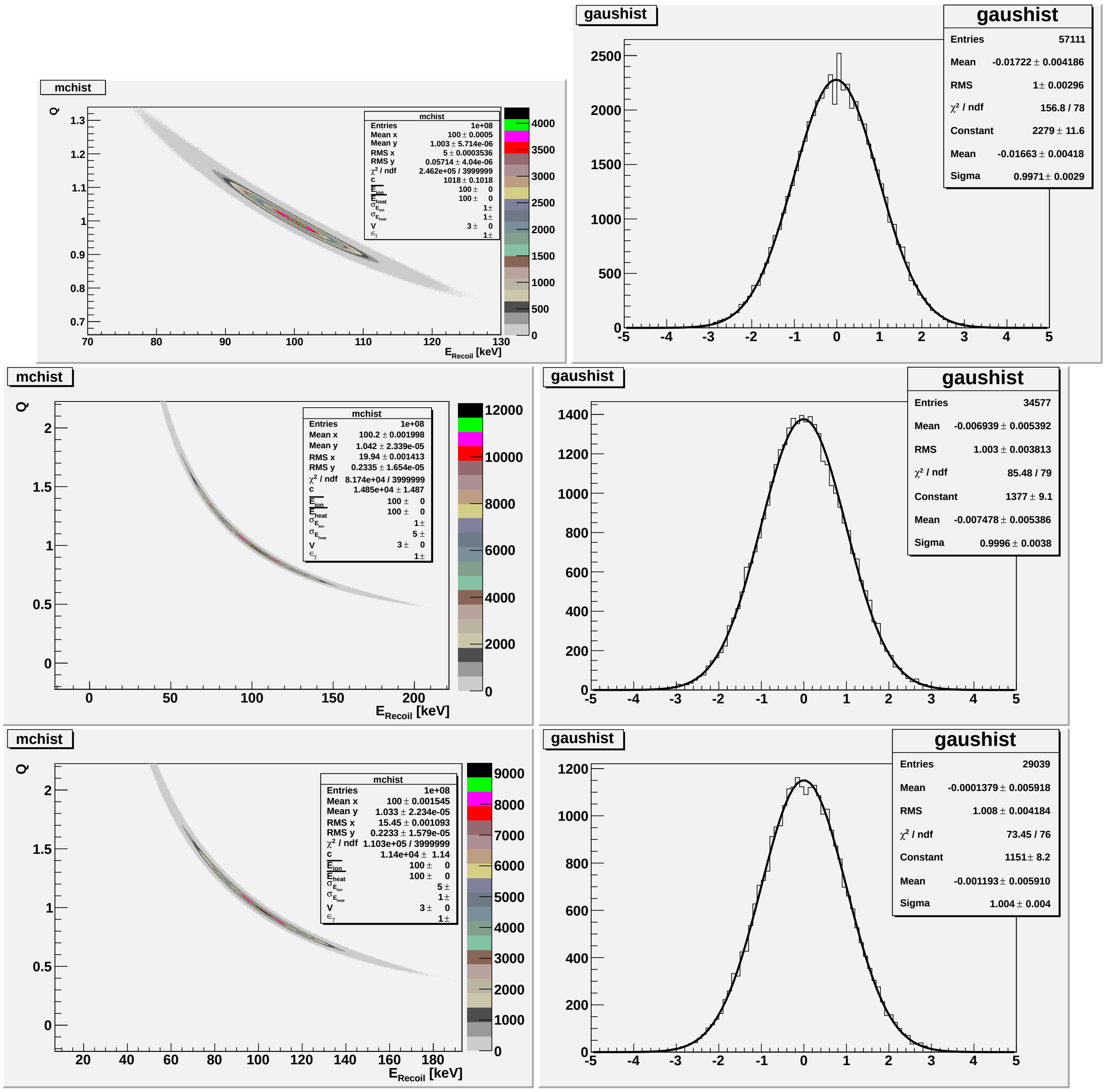
can be determined which should be standard normally distributed for high numbers of events $n_{pdf,i}$ and $n_{mc,i}$. The test then is applied on

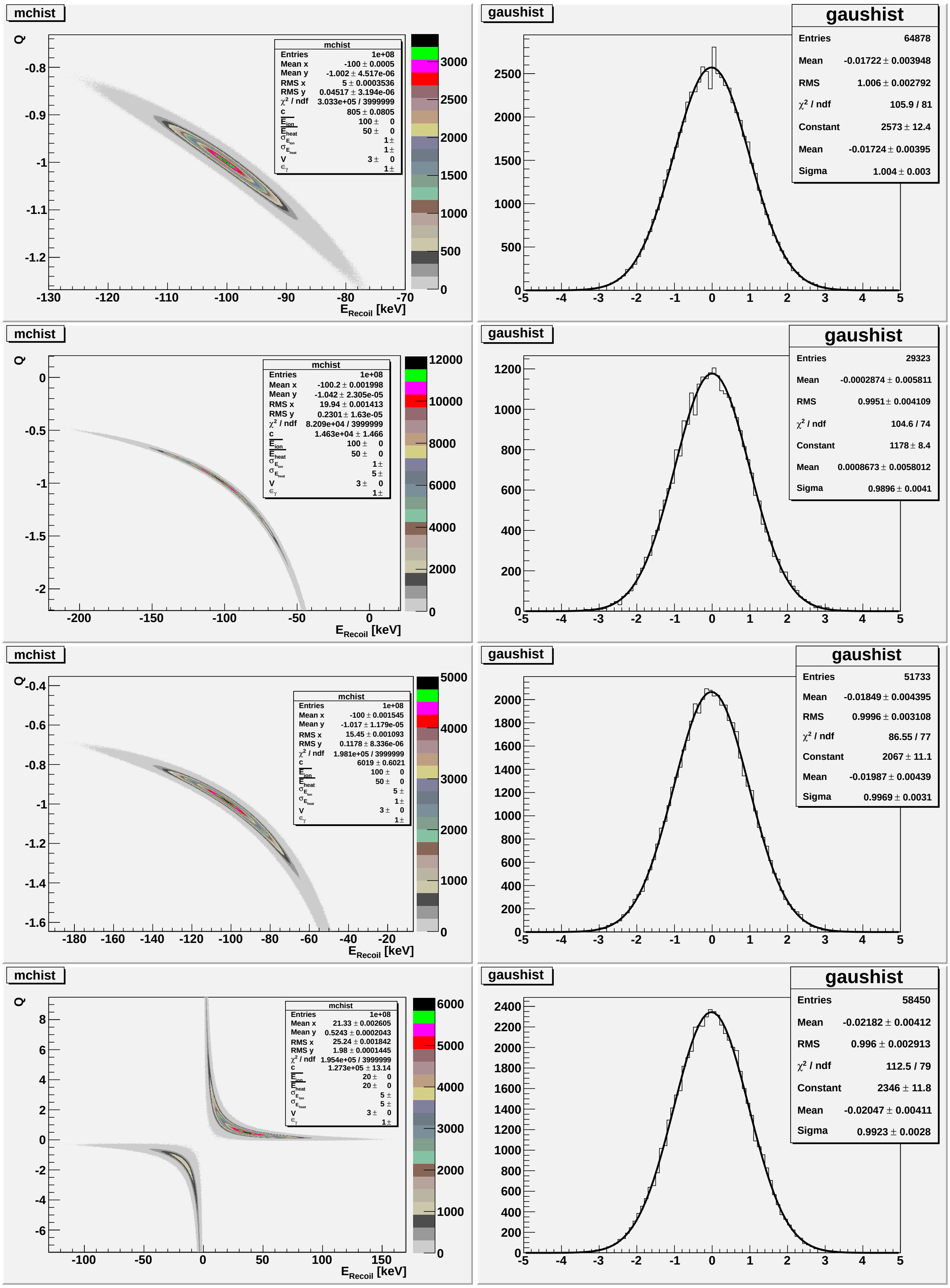
$$\chi^2 = \sum_i z_i^2 \tag{20}$$

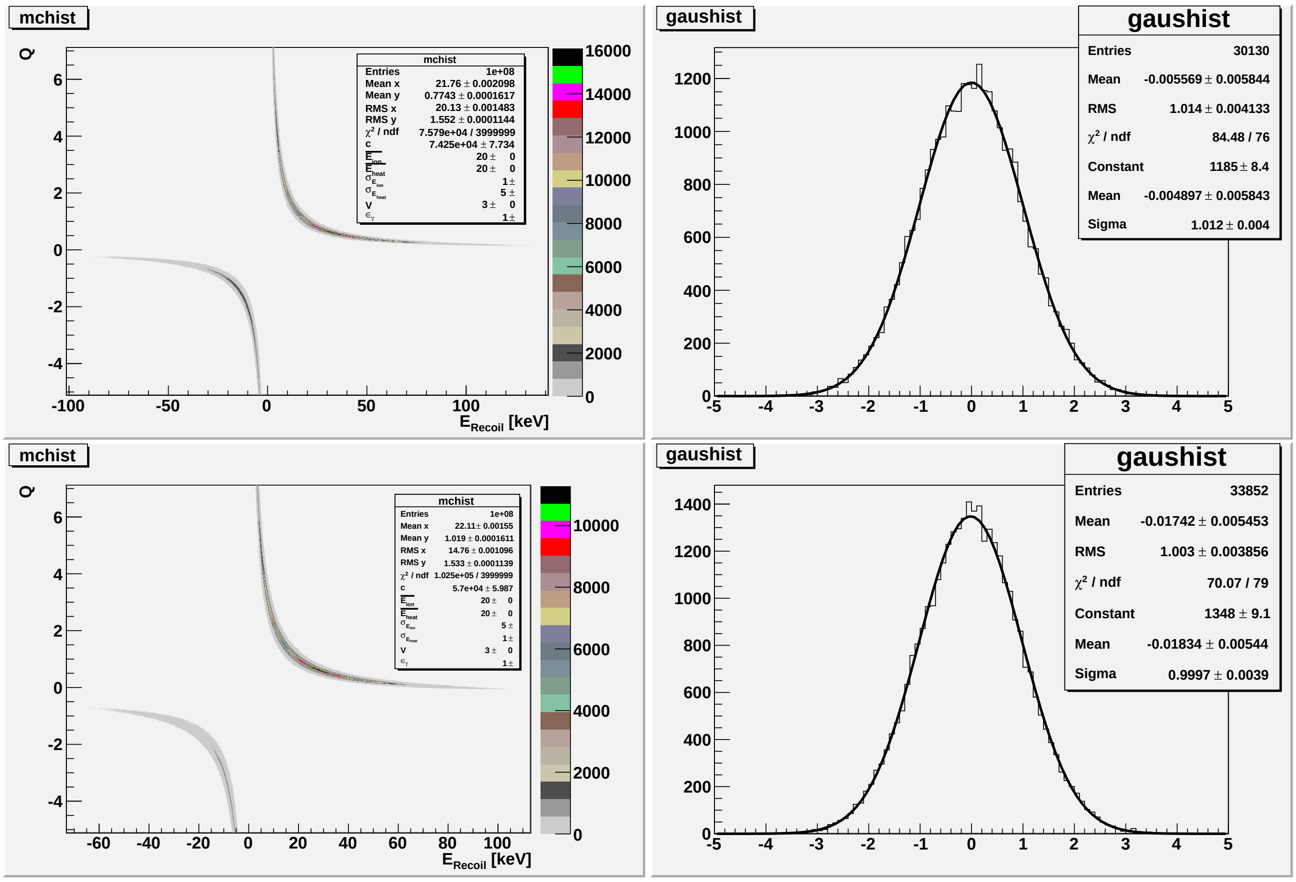
where the sum goes over all bins with more than a certain number of events, which should be high enough to be normally distributed in good approximation.

This procedure is applied for some examples in the following:

Figure 1: Histograms with Monte Carlo events for some parameter combinations of $\bar{E}_{Ion}$,$\bar{E}_{Heat}$, $\sigma_{E_{Ion}}$, and $\sigma_{E_{Heat}}$ fitted with the pdf $f(E_{Recoil}, Q) = c \cdot g(E_{Recoil}, Q)$ and distribution of z_i for minimal number of pdf events $n_{min} > 400$




mchist
Q
E_Recoil [keV]
gaushist
gaushist
Entries 64878
Mean -0.01722 ± 0.003948
RMS 1.006 ± 0.002792
χ² / ndf 105.9 / 81
Constant 2573 ± 12.4
Mean -0.01724 ± 0.00395
Sigma 1.004 ± 0.003
mchist
Q
E_Recoil [keV]
gaushist
gaushist
Entries 29323
Mean -0.0002874 ± 0.005811
RMS 0.9951 ± 0.004109
χ² / ndf 104.6 / 74
Constant 1178 ± 8.4
Mean 0.0008673 ± 0.0058012
Sigma 0.9896 ± 0.0041
mchist
Q
E_Recoil [keV]
gaushist
gaushist
Entries 51733
Mean -0.01849 ± 0.004395
RMS 0.9996 ± 0.003108
χ² / ndf 86.55 / 77
Constant 2067 ± 11.1
Mean -0.01987 ± 0.00439
Sigma 0.9969 ± 0.0031
mchist
Q
E_Recoil [keV]
gaushist
gaushist
Entries 58450
Mean -0.02182 ± 0.00412
RMS 0.996 ± 0.002913
χ² / ndf 112.5 / 79
Constant 2346 ± 11.8
Mean -0.02047 ± 0.00411
Sigma 0.9923 ± 0.0028



The distributions of z_i are well described by normal distribution, the offsets of the means might be a hint for a systematic error due to the random fill process of ROOT. Also the fit parameters c of the likelihood fits have small relative errors. Some z_i-distributions show high fluktuations in the vicinity of 0. This is due to the fact, that z_i is actually discretly distributed, which would be more obvious with smaller binning.

Table 1: χ^2 values for the corresponding parameter combinations and acceptance of the null hypothesis H_0

$\overline{E_{Ion}}$	$\overline{E_{heat}}$	$\sigma_{E_{Ion}}$	$\sigma_{E_{Heat}}$	χ^2 value	ndf	n_{min}	TMath::Prop(χ^2,ndf)	CL of pdf	H_0
100	100	1	1	57179	57110	400	0.418	90.1%	yes
100	100	5	1	34762.5	34576	400	0.239	95.3%	yes
100	100	1	5	29530	29038	400	0.020	96.2%	yes
100	50	1	1	65629	64877	400	0.019	87.4%	yes
100	50	5	1	51710	51732	400	0.526	91.2%	yes
100	50	1	5	29035	29322	400	0.882	96.4%	yes
50	100	1	1	90959	90805	400	0.358	68.7%	yes
50	100	1	5	73109	73203	400	0.597	83.8%	yes
50	100	5	1	61415	61112	400	0.193	88.7%	yes
20	20	5	5	58015	58449	400	0.898	84.7%	yes
20	20	1	5	31008	30129	400	0.0002	89.3%	no
20	20	5	1	34082	33851	400	0.188	86.5%	yes

The confidence level of the propability density function (CL of pdf) gives the percentage of the events in the χ^2 sum to the total sum of all monte carlo events. For the acceptance of the null hypothesis H_0, a significance level of 1% is assumed. That means it is accepted if TMath::Prob(χ^2,ndf)>0.01. So in one case out of 12 the null hypothesis H_0 has to be rejected.

3.1 script

The plots have been created by the script on kalinka in my home directory:

```
/kalinka/home/wegner/ERecoilQDistribution/ERecoilQDist_v20.C
```

This file offers two methods:

a)
```
ERecoilQDist_v20(Double_t anEIonMean = 100,
 Double_t anEHeatMean = 100,
 Double_t anEIonSigma = 1,
 Double_t anEHeatSigma = 1,
 Double_t aNumBinsX = 2000,
 Double_t aNumBinsY = 2000,
 Double_t aNumTimes = 1,
 Long_t aNumEntries = 1E9,
 Double_t aV = 3,
 Double_t anEpsilon = 1,
 Option_t* aFitOption = "`0LI"')
```

This methods creates two histograms "'mchist"' and "'pdfhist"', the first from creating random numbers distributed according the given parameters, the second from the

```
TH1::FillRandom(const char* fname,Int_t ntimes = 500)
```

method according to the theoretical propability density function "'f"'. The histograms have the dimensions (aNumBinsX,aNumBinsY) and the boundaries are chosen, so that it covers $\pm$ aNumSigmas standard deviations calculated from error propagation around the center value

$$\overline{E_{Recoil}} = \left(1 + \frac{V}{\epsilon_\gamma}\right)\overline{E_{Heat}} - \frac{V}{\epsilon_\gamma}\overline{E_{Ion}} \tag{21}$$

$$\overline{Q} = \frac{\overline{E_{Ion}}}{\overline{E_{Recoil}}} \tag{22}$$

Additionally "'mchist"' is fitted with the pdf "'fkt"' alias 'f' and "'aFitOption"' and both histograms are filled with "'aNumEntries"'. "'mchist"' might have less effective entries, as some generated random numbers might be out of range of the histogram. This procedure is applied "'aNumTimes"' and the histrograms filled in a TTree "'tree"' with branchnames "'mc-hists"' and "'pdf-hists"'. Then the tree is stored in a file (TFile "'file"') of the form:

<E_{Ion}>_<E_{Heat}>_<$\sigma_{E_{Ion}}$>_<$\sigma_{E_{Heat}}$>_<aNumEntries>.root

b)
```
void ShowBinGausDistribution(const Char_t* aFileName = "`"',
Int_t anEntry = 0,
Int_t aMinNumEntries = 3000)
```

References

[1] Henze, Stochastik I, Einfuhrung in die Wahrscheinlichkeitstheorie und Statistik, 2004