CFRM 541 Homework 5

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November 18, 2016

Problem #1

Show that for an i.i.d. sample with a normal distribution $N(\mu, \sigma^2)$, the sample mean and the sample variance with divisor n are the joint MLE's of the mean and variance.

We set $\theta = (\mu, \sigma^2)$ and write the log-likelihood:

$l(\theta) = l(\mu, \sigma^2) = \log\left(\frac{1}{(2\pi\sigma^2)^{n/2}}\right) - \Sigma_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma^2}$.

To maximize this we set the partials equal to zero:

$$\frac{\partial}{\partial\mu}\left[\log\left(\frac{1}{(2\pi\sigma^2)^{n/2}}\right) - \Sigma_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0 \tag{1}$$
$$\frac{\partial}{\partial\sigma^2}\left[\log\left(\frac{1}{(2\pi\sigma^2)^{n/2}}\right) - \Sigma_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0. \tag{2}$$

Solving the first, we have:

$$\frac{\partial}{\partial\mu}\left[\log\left(\frac{1}{(2\pi\sigma^2)^{n/2}}\right) - \Sigma_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0 \tag{3}$$
$$0 + \frac{2}{2\sigma^2}\Sigma_{i=1}^n (x_i - \mu) = 0 \tag{4}$$
$$\frac{1}{\sigma^2}\left(\Sigma_{i=1}^n x_i - n\mu\right) = 0 \tag{5}$$
$$\mu = \frac{1}{n}\Sigma_{i=1}^n x_i. \tag{6}$$

Solving the second, we have:

$$\frac{\partial}{\partial\sigma^2}\left[\log\left(\frac{1}{(2\pi\sigma^2)^{n/2}}\right) - \Sigma_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma^2}\right] = 0 \tag{7}$$
$$(2\pi\sigma^2)^{n/2}(-\frac{n}{2})(2\pi\sigma^2)^{-\frac{n}{2}-1}(2\pi) + \frac{\Sigma_{i=1}^n (x_i-\mu)^2}{2\sigma^4} = 0 \tag{8}$$
$$-\frac{n}{2\sigma^2} + \frac{\Sigma_{i=1}^n (x_i-\mu)^2}{2\sigma^4} = 0 \tag{9}$$
$$-\frac{n\sigma^2}{2\sigma^4} + \frac{\Sigma_{i=1}^n (x_i-\mu)^2}{2\sigma^4} = 0 \tag{10}$$
$$\sigma^2 = \frac{1}{n}\Sigma_{i=1}^n (x_i-\mu)^2. \tag{11}$$

These results for the MLE's are indeed the sample mean and variance (except with the divisor of n instead of $n-1$).

Problem #2

Double Exponential Distributions. Slides 10-12 of LS7 show the following for the case of i.i.d. data with a double exponential distribution: For an odd sample size the sample median is the MLE of the mean.

a. Show that for even sample sizes (with no data ties) the MLE is not unique and can have a finite range of values. Describe that range of values in terms of the ordered data.

Our distribution is: $f_{DE}(x_i;\mu,s) = \Pi_{i=1}^n \frac{1}{2s} e^{-\frac{|x_i-\mu|}{s}} = (\frac{1}{2s})^n \Pi_{i=1}^n e^{-\frac{1}{s}\Sigma_{i=1}^n |x_i-\mu|}$.

Slide 10 shows that the log-likelihood is:

$l(\mu) = \log\left(\frac{1}{(2s)^n}\right) - \Sigma_{i=1}^n \frac{|x_i-\mu|}{s}$; slide 11 shows that the derivative of this function (wrt μ) is:

$l'(\mu) = -\frac{1}{s}\Sigma_{i=1}^n \frac{d}{d\mu}|x_i-\mu|$.

Slide 12 writes the solution to this as:

$\Sigma_{i=1}^n SGN(x_i-\mu) = 0$.

For an odd sample size, the sum of the signs of the $x_i - \mu$ will be zero only if we set μ to the middle data point $x_{(n+1)/2}$.

But consider a sample of even size: $x_{(1)}, ..., x_{(\frac{n}{2})}, x_{(\frac{n}{2}+1)}, ..., x_{(n)}$. Here, our sum of the signs of the $x_i - \mu$ will be zero *as long as we choose a* μ *between* $x_{(\frac{n}{2})}$ *and* $x_{(\frac{n}{2}+1)}$. There is thus a continuum of solutions here (since we know that there are no ties among the data points).

Thus we shall have $\mu \in (x_{(\frac{n}{2})}, x_{(\frac{n}{2}+1)})$.

b. Derive the joint MLE for the mean μ and scale parameter s.

We have:

$l(\mu,s) = \log\left(\frac{1}{(2s)^n}\right) - \Sigma_{i=1}^n \frac{|x_i-\mu|}{s}$

For the MLE of μ we have:

$\frac{\partial l}{\partial \mu} = -\frac{1}{s}\Sigma_{i=1}^n SGN(x_i-\mu)$.

Setting this to zero, we find, as per (a):

For odd n, $\mu = x_{(n+1)/2}$; for even n, $\mu \in (x_{(\frac{n}{2})}, x_{(\frac{n}{2}+1)})$.

For the MLE of s we have:

$l(\mu,s) = \log\left(\frac{1}{(2s)^n}\right) - \Sigma_{i=1}^n \frac{|x_i-\mu|}{s} = -n\log(2s) - \frac{1}{s}\Sigma_{i=1}^n |x_i-\mu|$.

Thus:

$$\frac{\partial l}{\partial s} = -\frac{n}{s} + \frac{\Sigma_{i=1}^n |x_i-\mu|}{s^2} = 0 \tag{12}$$

$$-\frac{ns}{s^2} + \frac{\Sigma_{i=1}^n |x_i-\mu|}{s^2} = 0 \tag{13}$$

$$s = \frac{\Sigma_{i=1}^n |x_i-\mu|}{n}. \tag{14}$$

Problem #3

Fisher Information and Asymptotic Variance.

a. Confirm the formula for the Fisher information in Example 1 on slide 23 of LS7.

For this example we have $\theta = \mu$ and $f_1(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}$.

Thus the Fisher Information is:

$$
\begin{aligned}
I(\mu) &= E\left[\left(\frac{\partial}{\partial\mu}\left[\log\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_1-\mu)^2}{2\sigma^2}}\right)\right]\right)^2\right] && (15)\\
&= E\left[\left(0 - \frac{\partial}{\partial\mu}\left[\frac{(x_1-\mu)^2}{2\sigma^2}\right]\right)^2\right] && (16)\\
&= E\left[\left(\frac{\partial}{\partial\mu}\left[\frac{-1}{2\sigma^2}(x_1-\mu)^2\right]\right)^2\right] && (17)\\
&= E\left[\left(\frac{1}{\sigma^2}(x_1-\mu)\right)^2\right] && (18)\\
&= \frac{1}{\sigma^4} E[(x_1-\mu)^2] && (19)\\
&= \frac{1}{\sigma^4}\sigma^2 && (20)\\
&= \frac{1}{\sigma^2}. && (21)
\end{aligned}
$$

b. Derive the formula for the Fisher information in Example 2 on slide 23 of LS7.

For this example we have $\theta = \mu$ and $f_1(x) = \frac{1}{2s} e^{-\frac{|x_1-\mu|}{s}}$.

Thus the Fisher Information is:

$$
\begin{aligned}
I(\mu) &= E\left[\left(\frac{\partial}{\partial\mu}\left[\log\left(\frac{1}{2s} e^{-\frac{|x_i-\mu|}{s}}\right)\right]\right)^2\right] && (22)\\
&= E\left[\left(0 - \frac{1}{s}\frac{\partial}{\partial\mu}|x_1-\mu|\right)^2\right] && (23)\\
&= E\left[\left(\frac{1}{s} SGN(x_1-\mu)\right)^2\right] && (24)\\
&= \frac{1}{s^2} E[SGN(x_1-\mu)^2] && (25)\\
&= \frac{1}{s^2}. && (26)
\end{aligned}
$$

c. By virtue of being an MLE the sample median is an asymptotically optimal estimator of the mean for a double exponential distribution $f_{DE}(x;\mu,s)$. The sample mean is not optimal in this case, and therefore has a larger asymptotic variance than the sample median. What is the formula for the efficiency of the sample mean in this case?

The formula for the efficiency of an estimator is:

$EFF(\theta) = \frac{Var(\theta_{MLE})}{Var(\theta)}$, where $Var(\theta)$ is the multiplicative inverse of θ's Fisher Information.

Thus

$$EFF(\bar{x}) = \frac{Var(MED(x))}{Var(\bar{x})} \tag{27}$$

$$= \frac{I(\bar{x})}{I(MED(x))} \tag{28}$$

$$= s^2 * I(\bar{x}) \tag{29}$$

$$= s^2 \left(\frac{\partial}{\partial \bar{x}} \left[\log \left(\frac{1}{2s} e^{-\frac{|x_1 - \mu|}{s}} \right) \right] \right)^2 \tag{30}$$

$$= \left(s \frac{\partial}{\partial \bar{x}} \left[\log \left(\frac{1}{2s} \right) - \frac{|x_1 - \mu|}{s} \right] \right)^2. \tag{31}$$