CFRM 541 Homework 1

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Problem #1: Martin #1

Do the following:

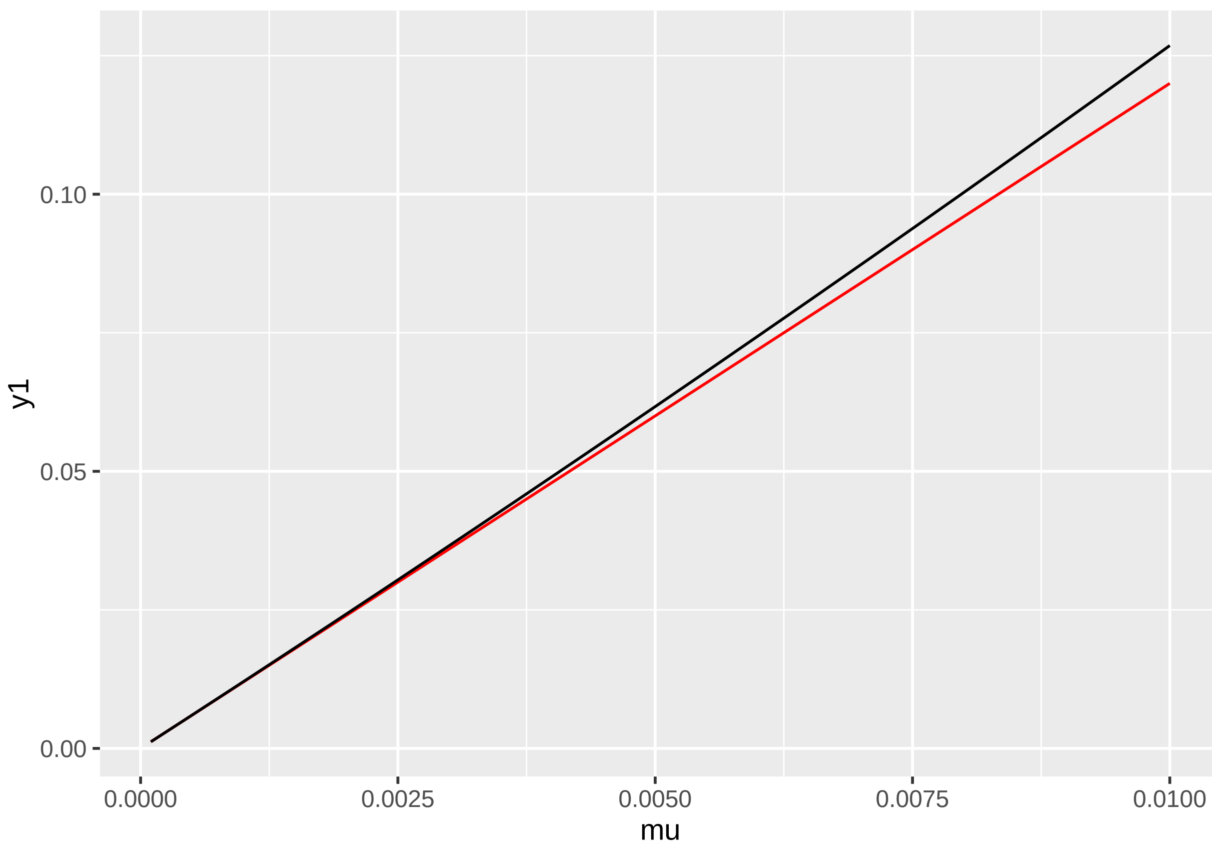
a. Show that for positive values of monthly mean return not greater than $\mu = .01$, the approximation $T * \mu$ given by (1.21) under-estimates the exact result of equation (1.20) by less than 1 percent.

The approximation (1.21) says that $\mu_{0,T} = T * \mu$.

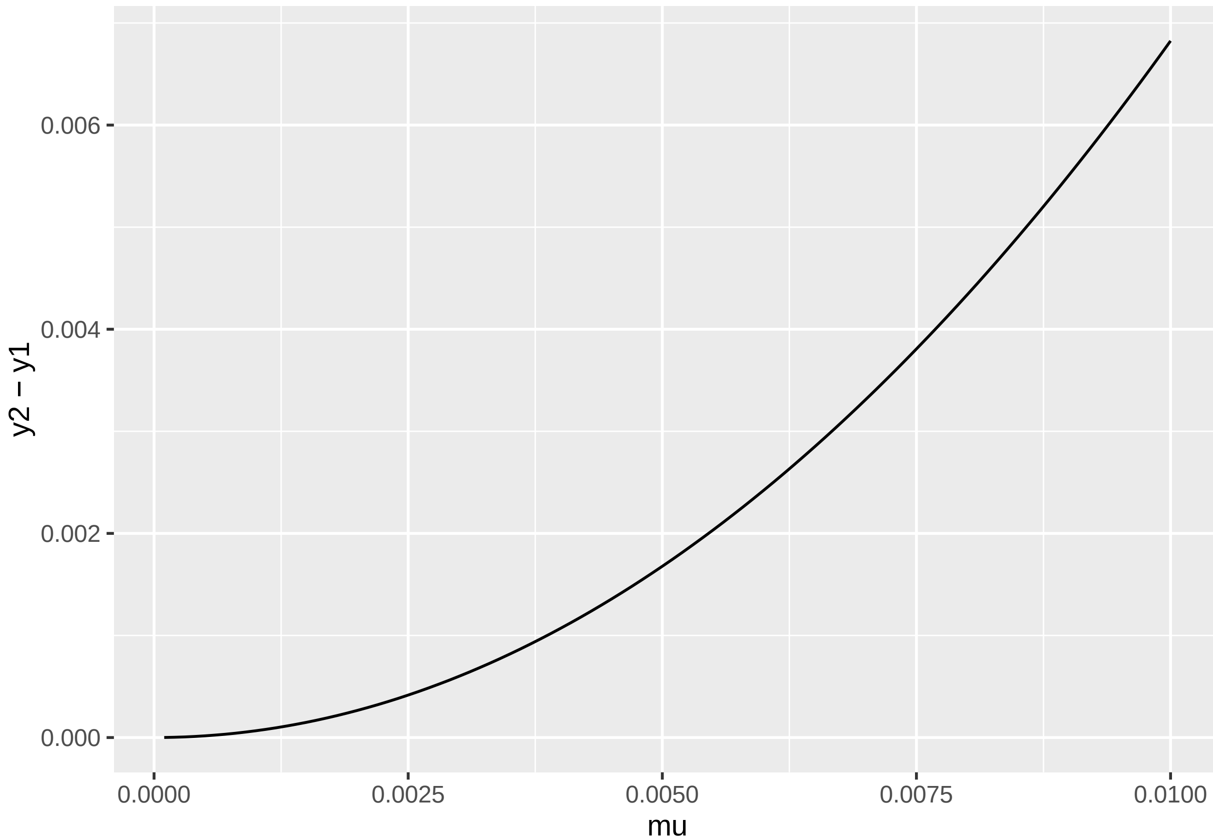
The exact equation (1.20) says that $\mu_{0,T} = (\mu + 1)^T - 1$.

First, the numerical argument using R code:

```
mu <- seq(0.01, 0.0001, by = -0.0001)
T <- 12
y1 <- T * mu
y2 <- (mu + 1)^T - 1
df <- data.frame(mu, y1, y2)
library(ggplot2)
ggplot(df, aes(x = mu)) + geom_line(aes(y = y1), col = "red") +
  geom_line(aes(y = y2), col = "black")
```



```r
ggplot(df, aes(x = mu)) + geom_line(aes(y = y2-y1))
```



For the analytic solution:

If we set $\mu = 0.01$, (1.20) tells us that $\mu_{0,T} = 1.01^T - 1$, whereas (1.21) tells us that $\mu_{0,T} = \frac{T}{100}$. The difference between them, i.e. $(1.01^T - 1) - (\frac{T}{100}) = 0.00683$ for T = 12. Note that the difference = 0 for T = 1.

Moreover, since $\frac{\partial}{\partial \mu}\left((\mu+1)^T - 1 - T\mu\right) = T(\mu+1)^{T-1} - T > 0$ for any positive μ, it is clear that the difference between the exact solution and the approximation will decrease for decreasing μ.

We note here also that, for sufficiently large T, the difference between (1.20) and (1.21) can grow quite large, even for $\mu \leq 0.01$. For example, for T = 1000 we have:

$$T\mu = (1000)(0.01) = 10 \tag{1}$$
$$1.01^T - 1 = 1.01^{1000} - 1 = 20958.16. \tag{2}$$

b. Show that for a windfall constant monthly mean return of 2% the exact formula gives 26.8% for annual mean return, whereas the approximate gives 24%, an underestimate of nearly 3%.

We set $T = 12$, since we are considering the compounding of a monthly mean return into an annual return.

Thus we have:

$1.02^{12} - 1 = 0.268$, and $12 * 0.02 = 0.24$.

Problem #2: Martin #2

Derive the expression (1.22) for the variance of multi-period arithmetic returns under the assumption that the returns are independent with constant mean μ and constant variance σ^2.

(1.22) gives us an expression for the variance of a multi-period (arithmetic) return.

We have:

$$\begin{aligned}
\sigma_T^2 &= E[(r_{0,T} - E[r_{0,T}])^2] && (3)\\
&= E[(r_{0,T} - ((1+\mu)^T - 1))^2] && (4)\\
&= E\left[\left(\prod_{t=1}^{T}(r_{t-1,t}+1) - 1 + 1 - (1+\mu)^T\right)^2\right] && (5)\\
&= E\left[\left(\prod_{t=1}^{T}(r_{t-1,t}+1)\right)^2 - 2(1+\mu)^T\prod_{t=1}^{T}(r_{t-1,t}+1) + (1+\mu)^{2T}\right] && (6)\\
&= E\left[\left(\prod_{t=1}^{T}(r_{t-1,t}+1)\right)^2\right] - 2(1+\mu)^{2T} + (1+\mu)^{2T} && (7)\\
&= E\left[\left(\prod_{t=1}^{T}(r_{t-1,t}+1)\right)^2\right] - (1+\mu)^{2T} && (8)\\
&= E[(r_{0,1}+1)^2] * \ldots * E[(r_{T-1,T}+1)^2] - (1+\mu)^{2T} && (9)\\
&= E[r_{0,1}^2 + 2r_{0,1} + 1] * \ldots * E[r_{T-1,T}^2 + 2r_{T-1,T} + 1] - (1+\mu)^{2T} && (10)\\
&= (E[r_{0,1}^2] - \mu^2 + \mu^2 + 2\mu + 1) * \ldots * (E[r_{T-1,T}^2] - \mu^2 + \mu^2 + 2\mu + 1) - (1+\mu)^{2T} && (11)\\
&= (\sigma^2 + (1+\mu)^2) * \ldots * (\sigma^2 + (1+\mu)^2) - (1+\mu)^{2T} && (12)\\
&= ((1+\mu)^2 + \sigma^2)^T - (1+\mu)^{2T} && (13)
\end{aligned}$$

Problem #3: Martin #3

Use the fact that the log is a concave function to prove the inequality (1.47).

(1.47) says that $\hat{\mu}_{g,T} \leq \hat{\mu}_{a,T}$ for $T > 1$.

We prove it as follows:

Start with (1.46):

$\mu_{g,T} = (\prod_{t=1}^{T}(r_t+1))^{1/T} - 1.$

Take the (natural) log of both sides:

$\log(\mu_{g,T}) = \log((\prod_{t=1}^{T}(r_t+1))^{1/T} - 1)$

Now $\log((\prod_{t=1}^{T}(r_t+1))^{1/T} - 1) < \log((\prod_{t=1}^{T}(r_t+1))^{1/T}) = \frac{1}{T}\log(\prod_{t=1}^{T}(r_t+1)) = \frac{1}{T}\sum_{t=1}^{T}\log(r_t+1).$

The hint tells us that, since $\log(x)$ is concave, $\sum_{t=1}^{T}\log(r_t+1) \leq \log(\sum_{t=1}^{T}(r_t+1))$.

Hence

$$\log(\mu_{g,T}) = \log((\prod_{t=1}^{T}(r_t+1))^{1/T} - 1) \tag{14}$$
$$< \log((\prod_{t=1}^{T}(r_t+1))^{1/T}) \tag{15}$$
$$= \frac{1}{T}\log(\prod_{t=1}^{T}(r_t+1)) \tag{16}$$
$$= \frac{1}{T}\sum_{t=1}^{T}\log(r_t+1) \tag{17}$$
$$\leq \frac{1}{T}\log(\sum_{t=1}^{T}(r_t+1)) \tag{18}$$
$$= \log(\mu_{a,T}). \tag{19}$$

Thus $\mu_{g,T} < \mu_{a,T}$ and $\hat{\mu}_{g,T} < \hat{\mu}_{a,T}$.

Problem #4: Martin #5

Show that if f(r) is any probability density function that is symmetric about a location parameter μ and that has a finite third moment $E[r^3]$, and hence finite third central moment $E[(r-\mu)^3]$, then the coefficient of skewness is zero.

To say that a probability density function f is symmetric about μ is to say that for any r, $f(\mu - r) = f(\mu + r)$. Now $E[(r-\mu)^3] = \int_{-\infty}^{\infty}(r-\mu)^3 f(r)dr$. Consider wlog any r_+ such that $(r_+ - \mu) > 0$. Then, by symmetry, there is some r_- such that $(r_- - \mu) = -(r_+ - \mu)$. And since our choice of r was generic, it is clear that $\int(r-\mu)dr = 0$. Hence $\int_{-\infty}^{\infty}(r-\mu)^3 f(r)dr = 0$ and thus the coefficient of skewness, $\frac{\int(r-\mu)^3 f(r)dr}{\sigma^3}$, is also zero.

Problem #5: Martin #7

Derive the equations (1.57) and (1.58) for the mean and variance of a two-component normal mixture distribution.

(1.57) tells us that $\mu_{nm} = E[r_{nm}] = \pi_1\mu_1 + \pi_2\mu_2$, where π_1 and π_2 are jointly exhaustive probabilities and μ_1 and μ_2 are the means of the two normal components.

Now $\mu_{nm} = E[r_{nm}] = \int_{-\infty}^{\infty} r f_{nm}(r)dr = \int_{-\infty}^{\infty} r(\frac{\pi_1}{\sigma_1\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_1}{\sigma_1})^2}{2}} + \frac{\pi_2}{\sigma_2\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_2}{\sigma_2})^2}{2}})dr$.

But

$$\int_{-\infty}^{\infty} r(\frac{\pi_1}{\sigma_1\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_1}{\sigma_1})^2}{2}} + \frac{\pi_2}{\sigma_2\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_2}{\sigma_2})^2}{2}})dr = \int_{-\infty}^{\infty} r\frac{\pi_1}{\sigma_1\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_1}{\sigma_1})^2}{2}}dr + \int_{-\infty}^{\infty} r\frac{\pi_2}{\sigma_2\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_2}{\sigma_2})^2}{2}}dr \tag{20}$$
$$= \pi_1\int_{-\infty}^{\infty} r\frac{1}{\sigma_1\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_1}{\sigma_1})^2}{2}}dr + \pi_2\int_{-\infty}^{\infty} r\frac{1}{\sigma_2\sqrt{2\pi}}e^{\frac{-(\frac{r-\mu_2}{\sigma_2})^2}{2}}dr \tag{21}$$
$$= \pi_1\mu_1 + \pi_2\mu_2 \tag{22}$$

(1.58) tells us that $\sigma_{mn}^2 = var(r_{nm}) = \sum_{i=1}^{2} \pi_i(\mu_i - \mu_{nm})^2 + \sum_{i=1}^{2} \pi_i \sigma_i^2$.

Now

$$
\begin{aligned}
\sigma_{mn}^2 &= \int_{-\infty}^{\infty} (r-\mu_{nm})^2 \left(\frac{\pi_1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(\frac{r-\mu_1}{\sigma_1})^2}{2}} + \frac{\pi_2}{\sigma_2\sqrt{2\pi}} e^{\frac{-(\frac{r-\mu_2}{\sigma_2})^2}{2}}\right) dr && (23)\\
&= \pi_1 \int_{-\infty}^{\infty} (r-\mu_{nm})^2 \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(\frac{r-\mu_1}{\sigma_1})^2}{2}} dr + \pi_2 \int_{-\infty}^{\infty} (r-\mu_{nm})^2 \frac{1}{\sigma_2\sqrt{2\pi}} e^{\frac{-(\frac{r-\mu_2}{\sigma_2})^2}{2}} dr && (24)\\
&= \pi_1 \left(\int_{-\infty}^{\infty} (r-\mu_1)^2 \frac{1}{\sigma_1\sqrt{2\pi}} e^{\frac{-(\frac{r-\mu_1}{\sigma_1})^2}{2}} dr + (\mu_1-\mu_{nm})^2 \right) + \pi_2 \left(\int_{-\infty}^{\infty} (r-\mu_2)^2 \frac{1}{\sigma_2\sqrt{2\pi}} e^{\frac{-(\frac{r-\mu_2}{\sigma_2})^2}{2}} dr + (\mu_2-\mu_{nm})^2 \right) && (25)\\
&= \pi_1(\sigma_1^2 + (\mu_1-\mu_{nm})^2) + \pi_2(\sigma_2^2 + (\mu_2-\mu_{nm})^2) && (26)\\
&= \sum_{i=1}^{2} \pi_i(\mu_i-\mu_{nm})^2 + \sum_{i=1}^{2} \pi_i \sigma_i^2 && (27)
\end{aligned}
$$