CFRM 541 Homework 2

Greg Damico

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Problem #1

Consider the transformation $Y = X^2$ where the random variable X has distribution function $F_X(x) = P(X \leq x)$ and density function $f_X(x) = F'_X(x)$.

a. Express the distribution function $F_Y(y) = P(Y \leq y)$ in terms of F_X.

Very roughly we have $F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y})$.

But there is a slight difficulty with this reasoning, since X^2 is not, of course, a monotonic function. But it is monotonic (decreasing) on the interval $(-\infty, 0)$ and monotonic (increasing) on the interval $(0, \infty)$.

Now the probability that $X^2 \leq y$ is the probability that BOTH $X \leq \sqrt{y}$ AND $X \geq -\sqrt{y}$.

That is, $P(X^2 \leq y) = P(X \leq \sqrt{y}) - P(X < -\sqrt{y})$.

So we have:

$$F_Y(y) = \begin{cases} 0, & y \leq 0 \\ F_X(\sqrt{y}) - F_X(-\sqrt{y}), & y > 0 \end{cases}$$

b. Derive the density function $f_Y(y)$ from $F_Y(y)$.

$f_Y(y) = F'_Y(y)$. Thus we have:

$$f_Y(y) = \begin{cases} 0, & y \leq 0 \\ \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})), & y > 0 \end{cases}$$

c. Write down the formula for $f_Y(y)$ in the case where $f_X(x)$ is a standard normal density. Confirm that the result is a special case of a chi-squared density.

Suppose $f_X(x)$ is a standard normal density. Then $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$.

Thus:

$$f_Y(y) = \begin{cases} 0, & y \leq 0 \\ \frac{1}{2\sqrt{y}}(\frac{1}{\sqrt{2\pi}} e^{\frac{-y}{2}} + \frac{1}{\sqrt{2\pi}} e^{\frac{-y}{2}}), & y > 0 \end{cases}$$

Simplifying:

$$f_Y(y) = \begin{cases} 0, & y \leq 0 \\ \frac{e^{\frac{-y}{2}}}{\sqrt{2\pi y}}, & y > 0 \end{cases}$$

Now the chi-squared density has the form:

$$f_{\chi^2}(y,k) = \begin{cases} 0, & y \leq 0 \\ \frac{y^{\frac{k}{2}-1}e^{\frac{-y}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}, & y > 0 \end{cases}$$

where k is the number of degrees of freedom. In our case $k = 1$.

Hence:

$$f_{\chi^2}(y) = \begin{cases} 0, & y \leq 0 \\ \frac{y^{\frac{1}{2}-1}e^{\frac{-y}{2}}}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})}, & y > 0 \end{cases}$$

Now $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. So, simplifying:

$$f_{\chi^2}(y) = \begin{cases} 0, & y \leq 0 \\ \frac{e^{\frac{-y}{2}}}{\sqrt{2\pi y}}, & y > 0 \end{cases}$$

This is the same as our expression for $f_Y(y)$ above when $f_X(x)$ is a standard normal density.

Problem #2

Do the following with respect to certain utility functions:

a. Show that log utility is a special case of power utility as $\gamma \to 0$.

A power utility is described by:

$U_p(w) = \frac{w^\gamma - 1}{\gamma}$, for $\gamma < 1$ and $\gamma \neq 0$.

We need to show that $\lim_{\gamma \to 0} U_p(w) = \log(w)$.

If we plug in 0 for γ in the above formula for a power utility, we get:

$U_p(w) = \frac{w^0 - 1}{0} = \frac{0}{0}$.

We can use L'Hôpital's Rule to evaluate our limit. This rule tells us that if we replace numerator and denominator with their derivatives wrt γ and then calculate this new fraction's value in the limit as $\gamma \to 0$, we'll get the same result. Thus we have:

$$\begin{aligned} \lim_{\gamma \to 0} U_p(w) &= \lim_{\gamma \to 0} \frac{w^\gamma - 1}{\gamma} && (1) \\ &= \lim_{\gamma \to 0} \frac{w^\gamma(\log(w))}{1} && (2) \\ &= \log(w). && (3) \end{aligned}$$

b. Derive the expressions for $ara(w)$ and $rra(w)$ for quadratic utility and show that both are increasing functions of wealth.

$ara(w) = \frac{-U''(w)}{U'(w)}$.

Thus for quadratic utility we have:

$U_q(w) = aw - bw^2; a, b > 0$ and

$ara(w) = \frac{2b}{a-2bw}$. We can show that ara(w) is increasing by showing that $\frac{d}{dw}[ara(w)] > 0$ for all w. Now $ara(w) = (2b)(a-2bw)^{-1}$, so

$$\frac{d}{dw}[ara(w)] = (-2b)(a-2bw)^{-2}(-2b) \tag{4}$$
$$= \frac{4b^2}{(a-2bw)^2} \tag{5}$$
$$> 0. \tag{6}$$

$rra(w) = \frac{-wU''(w)}{U'(w)}$.

Thus for quadratic utility we have:

$U_q(w) = aw - bw^2; a, b > 0$ and

$rra(w) = \frac{2bw}{a-2bw}$. Again we consider the derivative:

$$\frac{d}{dw}[rra(w)] = \frac{(a-2bw)(2b) - (2bw)(-2b)}{(a-2bw)^2} \tag{7}$$
$$= \frac{2ab}{(a-2bw)^2} \tag{8}$$
$$> 0. \tag{9}$$

Thus ara(w) and rra(w) are increasing functions of w.

c. Derive the expressions for $ara(w)$ and $rra(w)$ for power utility, thereby making it obvious that the first is decreasing in wealth and the second is constant.

Again, $ara(w) = \frac{-U''(w)}{U'(w)}$.

Thus for power utility we have:

$U_p(w) = \frac{w^{\gamma}-1}{\gamma}$, for $\gamma < 1$ and $\gamma \neq 0$ and

$ara(w) = \frac{-(\gamma-1)w^{\gamma-2}}{w^{\gamma-1}} = \frac{-(\gamma-1)}{w}$.

Now $\frac{d}{dw}[ara(w)] = \frac{\gamma-1}{w^2}$. And since $\gamma < 1$, $\frac{d}{dw}[ara(w)] < 0$, which is to say that ara(w) is a decreasing function of w.

$rra(w) = \frac{-wU''(w)}{U'(w)}$.

Thus:

$rra(w) = \frac{-(\gamma-1)w^{\gamma-1}}{w^{\gamma-1}} = -(\gamma-1)$.

Clearly $\frac{d}{dw}[rra(w)] = 0$. Hence rra(w) is a constant function.

Problem #3

Show that if two utility functions U_1 and U_2 are equivalent by virtue of $U_2 = aU_1 + b, a > 0$, then they both have the same absolute and relative risk aversions.

Suppose $U_2 = aU_1 + b$ for some a > 0.

Then $U_2' = aU_1'$ and $U_2'' = aU_1''$.

Then $ara_{U_2}(w) = \frac{-aU_1''}{aU_1'} = \frac{-U_1''}{U_1'} = ara_{U_1}(w)$.

Similarly, $rra_{U_2}(w) = \frac{-awU_1''}{aU_1'} = \frac{-wU_1''}{U_1'} = rra_{U_1}(w)$.

Problem #4

Carry out the details of the suggested solution method on slide 19 of LS2 to show that $\pi = \frac{\sigma^2}{2} ara(W_0)$.

Slide 19 tells us that $U(W) = E[U(W_0 + H)]$.

We shall expand the LHS to first-order accuracy and the RHS to second-order accuracy.

We have:

$$
\begin{aligned}
U(W) &= E[U(W_0 + H)] && (10)\\
U(W_0) + U'(W_0)(W - W_0) &= E[U(W_0) + U'(W_0)(H) + \frac{1}{2}U''(W_0)H^2] && (11)\\
&= E[U(W_0)] + E[U'(W_0)]E[H] + E[\frac{1}{2}U''(W_0)H^2] && (12)\\
&= U(W_0) + 0 + \frac{1}{2}U''(W_0)\sigma^2 && (13)\\
U'(W_0)(W - W_0) &= \frac{1}{2}U''(W_0)\sigma^2 && (14)\\
-\pi U'(W_0) &= \frac{1}{2}U''(W_0)\sigma^2 && (15)
\end{aligned}
$$

Hence $\pi = -\frac{U''(W_0)\sigma^2}{2U'(W_0)} = \frac{\sigma^2}{2} ara(W_0)$.