Solving Nonlinear Equations (using python)

Assignment 1

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ME7227 : Nonlinear Finite Element Analysis of Solid Continua

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Introduction

Steps involved in solving a nonlinear single-variable equation

Let $F(x) = f$ be a nonlinear equation where $F(x)$ is a nonlinear function in variable 'x'.

1. Start with an initial guess 'x_0'
2. Linearize the function $F(x)$ at 'x_0' using Taylor series expansion.

$$F(x) \approx F_L(x) = \left(F(x_0) - F'(x_0)x_0\right) + F'(x_0)x$$

3. Solve for the solution(say x^*) using the linearized equation $F_L(x^*) = f$.
4. Compute the relative error in 'f' i.e.

$$\text{rel. error} = \frac{f - f^*}{f} = \frac{f - F(x^*)}{f}$$

5. if the relative error is with in the required tolerance, obtained x^* is the solution. Otherwise, replace initial guess with the x^* and repeat steps 2 to 5.

Problem 1

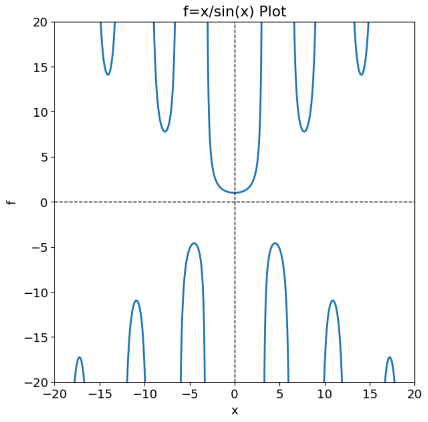
Plotting the function

The nonlinear equation

$$F(x) = \frac{x}{\sin x} = f$$

- Oscillatory in nature
- Multiple solutions are possible for a given 'f'

Plot of the nonlinear function

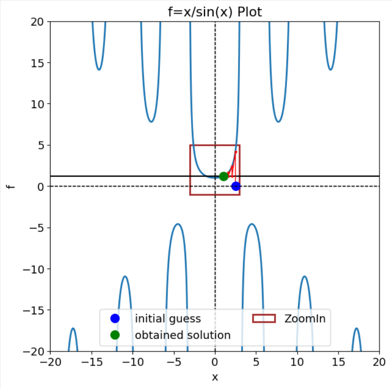


Problem 1

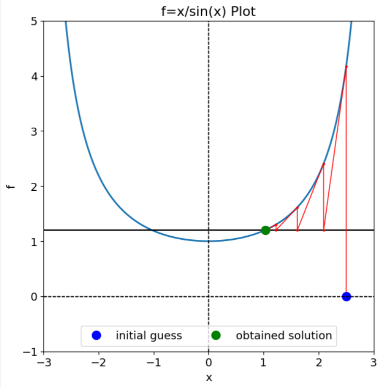
Solution at 'f' = 1.2, Case:1

Initial guess = 2.5, f = 1.2

Solution is : x = 1.026738, Corresponding 'f' = 1.200000, No of iterations = 6



$\Longrightarrow$

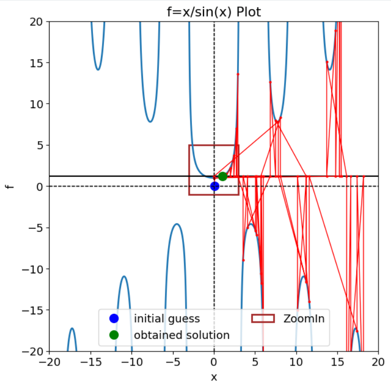


Problem 1

Solution at 'f' = 1.2, Case:2

Initial guess = 0.1001, f = 1.2

Solution is : x = 1.026738, Corresponding 'f' = 1.200000, No of iterations = 33



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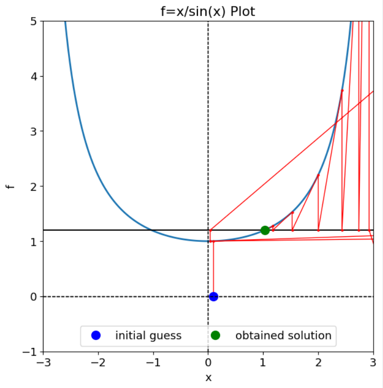
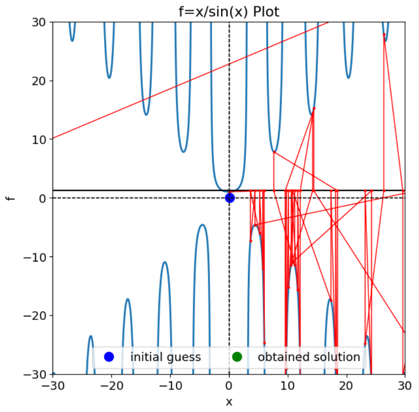


Problem 1

Solution at 'f' = 1.2, Case:3

Initial guess = 0.1000, f = 1.2

Maximum allowed no of iterations(200) reached.



Observations

- From the previous test cases it can be observed that when the initial guess is in the neighborhood of solution, it will most probably converge.
- It is also observed that when the initial guess is far away from solution, it may or may not converge.

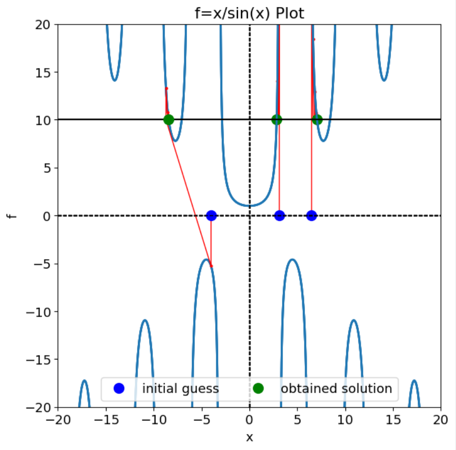
Problem 1

Solution at 'f' = 10

Initial Guess	Solution
6.5	7.068174
3.13	2.852342
-4	-8.423204

Observations

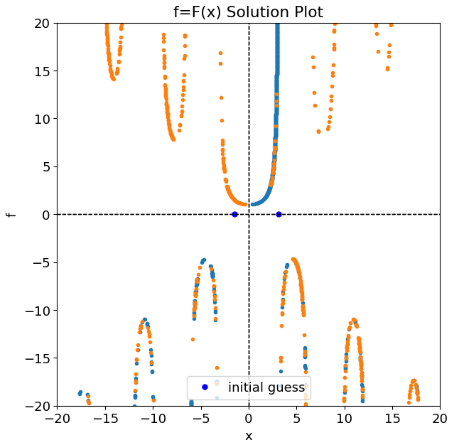
- In case of multiple solutions, the obtained solution really depends on the initial guess.
- It could be either the one in the neighborhood of initial guess or something completely arbitrary.



Problem 1

Plotting the solution curve, $f \in [-20, 20]$

Depends on initial guess (3.12, -1.5)



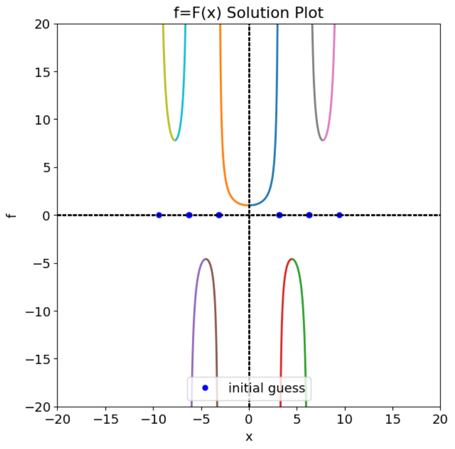
Observations

- For a grid of 1000 points on 'f' axis, and with an arbitrary initial guess the graph plotted gives an incomplete description of the nonlinear function.
- But by carefully choosing the initial guess one could be able to plot any monotonic section of the plot.

Problem 1

Plotting the solution curve, $f \in [-20, 20]$

For multiple initial guesses - combined plot



Observations

- For a grid of 1000 points on 'f' axis, and with an arbitrary initial guess the graph plotted gives an incomplete description of the nonlinear function.
- But by carefully choosing the initial guess one could be able to plot any monotonic section of the graph.
- And with many such initial guesses it is possible to get complete description of the nonlinear function.

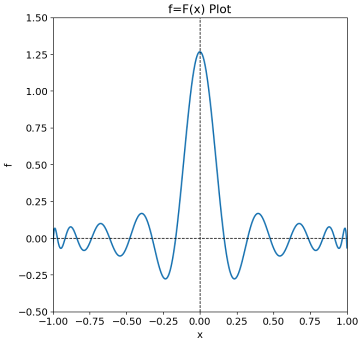
Problem 2

Plotting the function

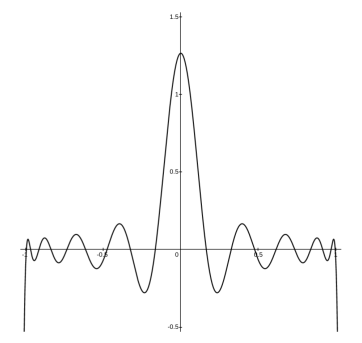
The function

$$F(x) = \frac{1}{15}\sum_{k=0}^{n}(-1)^k\left(\frac{(2n+1)!}{(2n-2k)!\,(2k+1)!}\right)\left(1-x^2\right)^{(n-k)}\cdot x^{2k} = f, \quad n=9$$

Plotted using python



Verified using Desmos



Problem 2

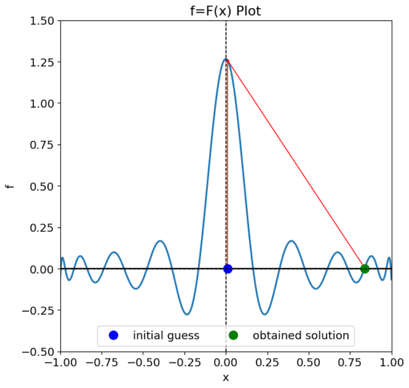
Solution at 'f' = 0.0001

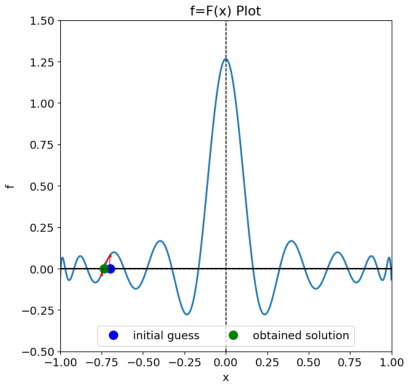
Initial guess = 0.01, f = 0.0001

Solution is : x = 0.837203

Initial guess = -0.7, f = 0.0001

Solution is : x = -0.735685



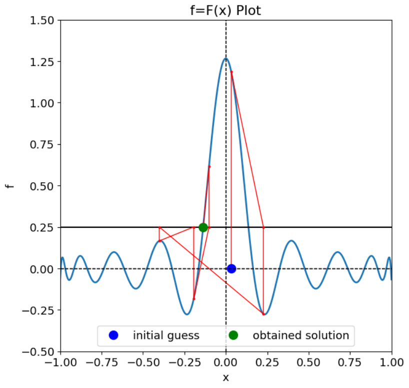


Problem 2

Solution at 'f' = 0.25

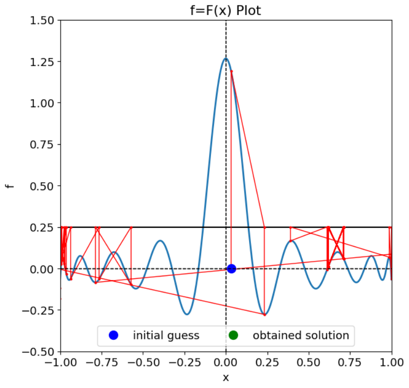
Initial guess = 0.033, f = 0.25

Solution is : x = -0.136611



Initial guess = 0.032, f = 0.25

Maximum iterations reached - 200



Problem 2
Observations

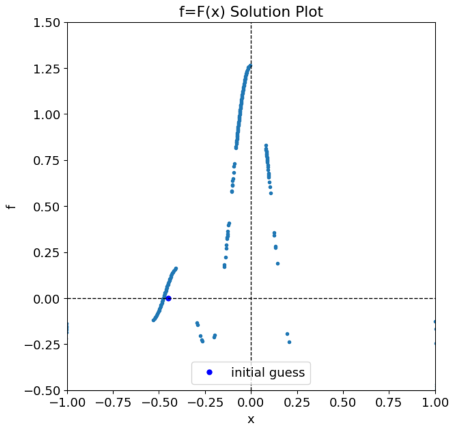
Observations made from previous test cases

1. At f=0.0001 for any value of initial guess x_0, the solution converges except for the 'x' values corresponding to peaks.
2. At f=0.25, similar to the problem 1, the convergence of the solution most probably will occur when the initial guess is in the neighborhood of the solution. Otherwise, it may or may not converge.

Problem 2

Plotting the solution curve, $f \in [-0.5, 1.5]$

For a single initial guess



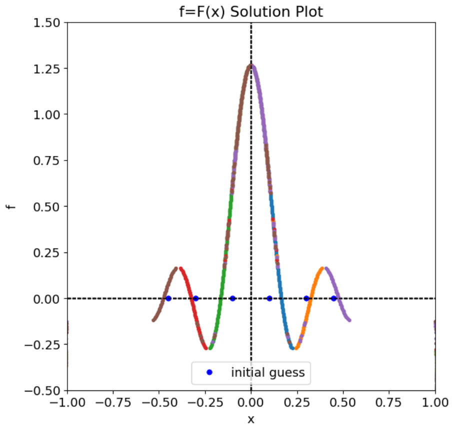
Observations

- For a grid of 1000 points on 'f' axis, and with an arbitrary initial guess the graph plotted gives an incomplete description of the nonlinear function.

Problem 2

Plotting the solution curve, $f \in [-0.5, 1.5]$

For multiple initial guesses - combined plot



Observations

- For a grid of 1000 points on 'f' axis, and with an arbitrary initial guess the graph plotted gives an incomplete description of the nonlinear function.
- But, in contrast to problem 1, instead of choosing initial guess in such a way that it gives a monotonic section of the graph, one can also take multiple random initial guesses and combine all the obtained data points to get the complete description of the nonlinear function.

System of equations

2 equations - decoupled

Procedure

Let $\mathcal{F}(\mathbf{x}) = 0$ be a system of nonlinear equations

$$F_1 = \frac{x_1}{\sin x_1} - f_1 = 0, \quad F_2 = \frac{x_2}{\sin x_2} - f_2 = 0$$

The solution is obtained using the directional derivative.

$$\implies D\,\mathcal{F}(\mathbf{x}^k)[\mathbf{u}] = -\mathcal{F}(\mathbf{x}^k)$$

$$\implies \frac{\partial \mathcal{F}(\mathbf{x}^k)}{\partial \mathbf{x}}\mathbf{u} = -\mathcal{F}(\mathbf{x}^k)$$

$$\begin{bmatrix} \frac{\partial F_1(x_1,x_2)}{\partial x_1} & \frac{\partial F_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1,x_2)}{\partial x_1} & \frac{\partial F_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -F_1(x_1, x_2) \\ -F_2(x_1, x_2) \end{bmatrix}$$

$$\implies \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & 0 \\ 0 & \frac{\partial F_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -F_1(x_1) \\ -F_2(x_2) \end{bmatrix}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{u}$$

Example

$$F_1 = \frac{x_1}{\sin x_1} - 1.2 = 0$$

$$F_2 = \frac{x_2}{\sin x_2} - 1.5 = 0$$

For initial guess

$$\begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

The solution at iteration 200

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.02673829 \\ 1.49578157 \end{bmatrix}$$

The solution is matching with the solutions obtained form the problem 1.

System of equations

2 equations - coupled

Example - 1

Let $\mathcal{F}(\mathbf{x}) = 0$ be a system of nonlinear equations

$$F_1 = x_1^2 + 2x_1x_2 + x_2 + 4 = 0$$
$$F_2 = x_2^2 + 6x_1x_2 + x_1^2 + 6 = 0$$

$$\begin{bmatrix} \frac{\partial F_1(x_1,x_2)}{\partial x_1} & \frac{\partial F_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1,x_2)}{\partial x_1} & \frac{\partial F_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -F_1(x_1,x_2) \\ -F_2(x_1,x_2) \end{bmatrix}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{u}$$

For initial guess

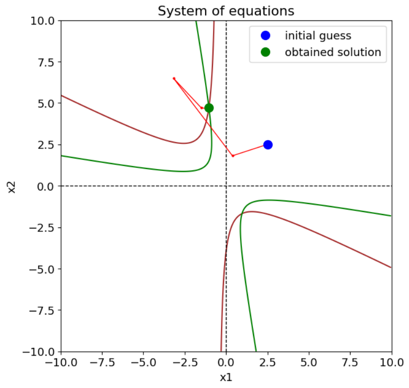
$$\begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

The solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1.0370868 \\ 4.72507334 \end{bmatrix}$$

Number of iterations $= 8$

Example Plot - 1



System of equations

2 equations - coupled

Example - 2

Let $\mathcal{F}(\mathbf{x}) = 0$ be a system of nonlinear equations

$$F_1 = x_1^2 + 2x_1x_2 + x_2 + 4 = 0$$
$$F_2 = x_2^2 + 6x_1x_2 + x_1^2 + 6 = 0$$

$$\begin{bmatrix} \frac{\partial F_1(x_1,x_2)}{\partial x_1} & \frac{\partial F_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1,x_2)}{\partial x_1} & \frac{\partial F_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -F_1(x_1, x_2) \\ -F_2(x_1, x_2) \end{bmatrix}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{u}$$

For initial guess

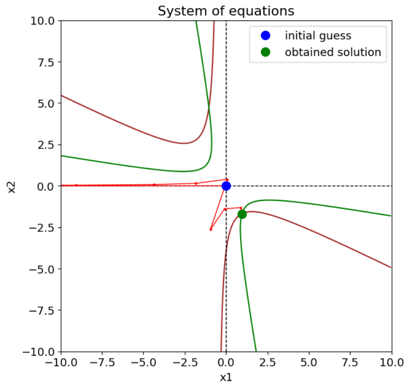
$$\begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix} = \begin{bmatrix} 0.00001 \\ 0.00001 \end{bmatrix}$$

The solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.96774125 \\ -1.6816735 \end{bmatrix}$$

Number of iterations $= 25$

Example Plot - 2



Example - 3

Let $\mathcal{F}(\mathbf{x}) = 0$ be a system of nonlinear equations

$$F_1 = x_1^2 + 2x_1x_2 + x_2 + 4 = 0$$
$$F_2 = x_2^2 + 6x_1x_2 + x_1^2 + 6 = 0$$

$$\begin{bmatrix} \frac{\partial F_1(x_1,x_2)}{\partial x_1} & \frac{\partial F_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial F_2(x_1,x_2)}{\partial x_1} & \frac{\partial F_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -F_1(x_1,x_2) \\ -F_2(x_1,x_2) \end{bmatrix}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{u}$$

For initial guess

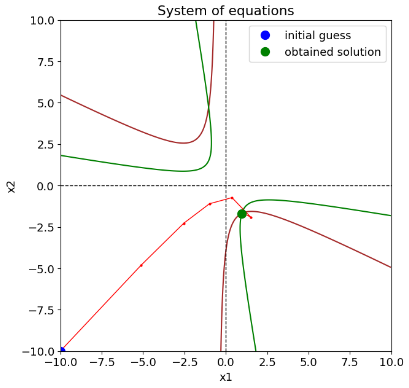
$$\begin{bmatrix} x_1^o \\ x_2^o \end{bmatrix} = \begin{bmatrix} -10 \\ -10 \end{bmatrix}$$

The solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.96774125 \\ -1.6816735 \end{bmatrix}$$

Number of iterations $= 10$

Example Plot - 3



Observations

The algorithm is able to predict one of the existing solutions based on the initial guess.

Conclusions

Observations made from the used solution algorithms

Single-variable nonlinear equations

1. In case of single-variable nonlinear equations, when the initial guess is in the neighborhood of the solution, convergence will most probably occur. Otherwise, convergence may or may not occur depending on the nonlinear behavior of the function.
2. The complete description of the behavior of the function is not possible with the data points obtained from one initial guess (since it is a nonlinear function). Instead, the combined plot obtained using multiple initial guesses can describe the function completely.

System of nonlinear equations

1. The algorithm is able to predict any one of the solutions (if there exists multiple) which depends on the initial guess.