AN INTRODUCTION TO (QUANTUM) SYMMETRY

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ABSTRACT. The study of symmetry is fundamental in mathematics and has wide-ranging applications. For example, a famous theorem of Emmy Noether asserts a correspondence between symmetries of physical systems and conservation laws in physics. Students in abstract algebra learn early on about symmetries of simple geometric objects (i.e., dihedral groups). In this talk I will discuss some of the basics of invariant theory, or the study of symmetries on polynomial rings. We will see how parts of this theory generalize (and don't generalize) to the noncommutative setting. This will lead into a brief introduction of Hopf algebras, otherwise sometimes known as quantum groups.

INTRODUCTION

The study of symmetry is fundamental in mathematics. A famous theorem of Emily Noether asserts a correspondence between differential symmetries of physical systems and conservation laws [Noe18]. In introductory abstract algebra, students learn early on about symmetries of simple geometric objects.

The study of symmetry expands far beyond simple geometric shapes and physical systems. Given any algebraic object, one can ask for its symmetries. Formally, one is asking what groups *act* on the object. That is, given a group and an object, one can ask what actions preserve both the defining properties of the object as well as the properties of the group.

Consider polynomials in two variables over $\mathbb{C}$, denoted by $\mathbb{C}[x, y]$. This is a *ring* because we can add and multiply polynomials (and get another polynomial), and these two operations distribute over one another. It is also a *vector space* over $\mathbb{C}$, because you can add polynomials or multiply polynomials by a complex number (and get another polynomial). Moreover, these three operations (addition, multiplication, and scalar multiplication) all play nicely together. That makes $\mathbb{C}[x, y]$ into an *algebra*.

An *algebra automorphism* of $\mathbb{C}[x, y]$ is a bijective map that is both a ring homomorphism *and* a linear transformation.The algebra automorphisms of $\mathbb{C}[x, y]$ are all pretty easy to describe[1] and are all defined by their action on x and y. However, we will only be concerned with *linear automorphisms*, which have the form[2]

$$x \mapsto ax + by \quad y \mapsto cx + dy \qquad a, b, c, d \in \mathbb{C} \quad ad - bc \neq 0.$$

Let τ be the automorphism that switches x and y. What polynomials $p \in \mathbb{C}[x, y]$ are *fixed* by the action of τ, i.e., satisfy $\tau(p) = p$? We observe that the automorphism τ fixes the polynomial $x + y$. Similarly, $\tau(xy) = yx = xy$ (it sure helps that x and y commute), so xy is fixed. We call $x + y$ and xy *invariants* of the action of τ.

If p and q are invariants of τ, then $\tau(p+q) = p+q$ and $\tau(pq) = pq$, so $p+q$ and pq are also invariants. Any scalar is invariant, so the invariants of τ form a subring of $\mathbb{C}[x, y]$ (called the *invariant ring* or *subring of invariants*), which we denote by $\mathbb{C}[x, y]^\tau$. Another invariant

[1] See the work of Jung [Jun42] and Van der Kulk [vdK53]

[2] For those in the know, this is just the group $\mathrm{GL}_n(\mathbb{C})$.

is $(x-y)^2$ but

$$(x-y)^2 = x^2 - 2xy + y^2 = (x+y)^2 - 4xy,$$

so $(x-y)^2$ is *generated by* the invariants $x+y$ and xy. In this case, it turns out that the invariant ring is another polynomial ring, $\mathbb{C}[x+y, xy]$.

But this does not always happen! Consider the automorphism σ such that $\sigma(x) = -x$ and $\sigma(y) = -y$. Now the polynomials x^2, y^2, and xy are all fixed. None of these generate the other, but they do satisfy the relation $x^2y^2 = (xy)^2$. So the invariant ring $\mathbb{C}[x,y]^\sigma$ is $\mathbb{C}[x^2, y^2, xy]/(x^2y^2 - (xy)^2)$. This is not a polynomial ring[3].

We can in fact characterize exactly when the invariant ring is a polynomial ring. A finite linear automorphism of $\mathbb{C}[x,y]$ is a *reflection* if one of its eigenvalues[4] is 1. By the Shephard-Todd-Chevalley Theorem [ST54, Che55], $\mathbb{C}[x,y]^G$ is a polynomial ring exactly when G is a finite group of linear automorphisms generated by reflections[5].

Symmetries of noncommutative polynomial rings

Now we consider the algebra $\mathbb{C}_{-1}[x,y]$, or polynomials *skewed* by -1. As a set, this is just polynomials in two variables, but x and y satisfy the relation $xy = -yx$ (or $xy + yx = 0$). Something funny happens when we look at invariants.

Consider τ again which switches x and y. Then $\tau(x+y) = y + x = x + y$ so $x+y$ is still invariant. But $\tau(xy) = yx = -xy$, so xy is not invariant! It turns out that the invariant ring is quite exotic[6]

Here is another example. Consider the map σ given by $x \mapsto -x$ and $y \mapsto y$. The invariant ring in this case is generated by the invariants x^2 and y. These satisfy

$$x^2y = x(xy) = x(-yx) = -(xy)x = -(-yx)x = yx^2.$$

Hence, x^2 and y commute, so $\mathbb{C}_{-1}[x,y]^\sigma = \mathbb{C}[x^2, y]$, a *commutative* polynomial ring. This is a common "quantum" phenomenon[7]. Even though the fixed ring is "nice", it is not isomorphic to the original algebraic object.

The automorphisms of $\mathbb{C}_{-1}[x,y]$ are much more limited because they must not only be bijective linear transformations, but they need to respect the defining relation (so the automorphism applied to $xy + yx$ should be 0). It turns out that they are all linear:

$$x \mapsto ax, y \mapsto by \quad \text{or} \quad x \mapsto ay, y \mapsto ax \qquad a, b \in \mathbb{C}\backslash\{0\}.$$

At play here is the principle of *quantum rigidity*. There are several more or less equivalent statements of this principle but the essence of all of them is that the noncommutativity of the relations prevents the existence of (many) classical symmetries (e.g. group actions). To find new symmetries, we need to dig a big deeper.

[3]In algebraic geometry terms, there is an affine variety associated to a commutative ring. The variety associated to a polynomial ring is *smooth* or *nonsingular*. But the affine variety associated to this second example has singularities. In particular, it is an example of a *Kleinian singularity.*

[4]This actually implies that the other eigenvalue is a root of unity ξ. The more technical definition, however, is that the automorphism fixes a codimension one subspace.

[5]This theorem is true for higher dimension polynomial rings as well.

[6]A pair of generators are $X = x+y$ and $Y = (x-y)(xy)$. These satisfy the relation $YX^2 = X^2Y$, $Y^2X = XY^2$, and $Y^2 - \frac{1}{4}X^2(XY + YX)$. See [KKZ15, Remark 2.6].

[7]Ant-man: Do you guys just put quantum in front of everything?

Hopf algebras

Warning: Throughout this section I'm going to use, frequently, the symbol[8] $\otimes$. In some sense, this is just a formality and you should not think too hard about it. The *tensor product* of two algebras A and B, denoted $A \otimes B$, is *like* the cartesian product $A \times B$, but behaves a bit different. It is *bilinear* (instead of linear), so[9]

$$a \otimes (b + b') = a \otimes b + a \otimes b' \quad \text{and} \quad (a + a') \otimes b = a \otimes b + a' \otimes b.$$

But $a \otimes b + a' \otimes b' \neq (a + a') \otimes (b + b')$ in general.

For the purpose of this talk, you should think of $\otimes$ as formally representing multiplication in an algebra. So, we might write the relation in $A = \mathbb{C}_{-1}[x, y]$ by $x \otimes y + y \otimes x$. When we have a automorphism σ acting on A, then $\sigma(x \otimes y) = \sigma(x) \otimes \sigma(y)$. So there is some sort of "splitting" going on $\sigma \mapsto \sigma \otimes \sigma$. This is the opposite of multiplication, which we call *comultiplication*. We also have a rule that $\sigma(1_A) = 1_A$ because σ acts trivially on all scalars. This is known as the *counit*.

Formally, an (associative) algebra A is a $\mathbb{C}$-vector space along with a multiplication map $\mu : A \otimes A \to A$ and a unit map $u : \mathbb{C} \to A$. We can express associativity and the interaction between μ and u through commutative diagrams:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[dr, "\mu"] & \mathbb{C} \otimes A \arrow[l, "u \otimes \mathrm{id}"'] \arrow[d, "\mu"] \\
A \otimes \mathbb{C} \arrow[u, "\mathrm{id} \otimes u"] \arrow[r, "\mu"] & A
\end{tikzcd}$$

A *coalgebra* is a $\mathbb{C}$-vector space C along with maps $\Delta : C \to C \otimes C$, called *comultiplication* and $\epsilon : C \to \mathbb{C}$ called the *counit*. The comultiplication should be *coassociative* and the maps Δ and ϵ should be compatible:

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes \mathrm{id}"'] \\
C \otimes C \arrow[u, "\mathrm{id} \otimes \Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \epsilon"'] & \mathbb{C} \otimes C \\
C \otimes \mathbb{C} & C \arrow[l, "- \otimes 1"] \arrow[u, "1 \otimes -"'] \arrow[ul, "\Delta"]
\end{tikzcd}$$

A *bialgebra* B is an algebra (B, μ, u) and a *coalgebra* (B, Δ, ϵ) satisfying a whole bunch of compatibility conditions such as:

$$\begin{tikzcd}
H \otimes H \arrow[r, "\mu"] \arrow[d, "\Delta \otimes \Delta"'] & H \arrow[r, "\Delta"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "\mathrm{id} \otimes \tau \otimes \mathrm{id}"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\mu \otimes \mu"']
\end{tikzcd}$$

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[8] X-men!

[9] I'm obscuring another important point as well. We think of these algebras as $\mathbb{C}$-algebras (so they are $\mathbb{C}$-vector spaces. That means that there is a scalar multiplication operation and $c(a \otimes b) = (ca) \otimes b = a \otimes (cb)$.

But a compact way of stating the condtions is just to say that Δ and ϵ are required to be homomorphisms of $\mathbb{C}$-algebras.

Finally, a *Hopf algebra*[10] is a bialgebra $H = (H, \mu, u, \Delta, \epsilon)$ along with a map $S : H \to H$, called the *antipode*, which satisfies the following diagram:

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{C} \arrow[rr, "u"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id} \otimes S"'] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

The antipode is also called the *coinverse* by some. One should think of it as somehow reflecting inverses in the coalgebra[11].

The polynomial ring $\mathbb{C}[x]$ is an example of a Hopf algebra. The algebra structure is the usual one. Comultiplication, counit, and antipode[12] are given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \epsilon(x) = 0 \qquad S(x) = -x.$$

Another simple example is a group algebra. Let G be a group. Then the group algebra $\mathbb{C}G$ is, as a set, "polynomials" in the variables $g \in G$. Multiplication is defined by the group operation. For example, if $G = C_3 = \langle g \rangle$ (the cyclic group of order 3), then elements are of the form $a + bg + cg^2$ with $a, b, c \in \mathbb{C}$, and multiplication is

$$(a + bg + cg^2)(a' + b'g + c'g^2) = (aa' + bc' + cb') + (ab' + ba' + cc')g + (ac' + bb' + ca')g^2.$$

The comultiplication, counit, and antipode are given by

$$\Delta(g) = g \otimes g \qquad \epsilon(g) = 1 \qquad S(g) = g^{-1}.$$

Our last example is a little unusual. As a vector space over $\mathbb{C}$, T has basis the elements $\{1, g, x, gx\}$. It is an algebra where the relations are

$$g^2 = 1 \qquad x^2 = 0 \qquad xg = -gx.$$

Comultiplication, counit, and antipode are given by

$$\begin{aligned} \Delta(g) &= g \otimes g & \epsilon(g) &= 1 & S(g) &= g \\ \Delta(x) &= g \otimes x + x \otimes 1 & \epsilon(x) &= 0 & S(x) &= -gx. \end{aligned}$$

This Hopf algebra T is called the *Sweedler algebra* and it is an example of a *Taft algebra*.

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[10] Quantum symmetries expert Chelsea Walton calls Hopf algebras "groups on crack".

[11] Formality footnote. Let H be Hopf algebra. For $f, g \in \mathrm{Hom}_{\mathbb{C}}(H, H)$ define a multiplication $*$, called *convolution*, by $(f * g)(a) = m(f \otimes g)\Delta(a) = \sum f(a_1)g(a_2)$. Convolution makes $\mathrm{Hom}(H, H)$ into a monoid and the antipode S is a (left and right) inverse of id_H under $*$.

[12] Since we haven't said that S is a homomorphism (it is) we really should say that $S(x^i) = (-x)^i$.

Hopf actions

Suppose that $A = \mathbb{C}_{-1}[u, v]$. The variable change is necessary because we are now using x for something else. We can define an *action* of the Sweedler algebra T on A by

$$g(u) = -u \qquad g(v) = v \qquad x(u) = 0 \qquad x(v) = u.$$

We want the action to respect the relations on T. For example, $g^2(u) = u = 1_T(u)$ and $g^2(v) = v = 1_T(v)$. Similarly, $x^2(u) = x(0) = 0$ and $x^2(v) = x(u) = 0$. For the last relation $(xg + gx = 0)$,

$$(xg + gx)(u) = xg(u) + gx(u) = x(-u) + g(0) = 0 + 0 = 0$$
$$(xg + gx)(v) = xg(v) + gx(v) = x(v) + g(u) = u + (-u) = 0.$$

We also want the action to respect the relation on A $(uv + vu = 0)$. But this requires applying T to products of elements in A. In order to do this, we use the coproduct. So we think of the relation of A as $u \otimes v + v \otimes u$ and recall that $\Delta(g) = g \otimes g$, suppressing the tensor product we have:

$$g(uv + vu) = g(u)g(v) + g(v)g(u) = (-u)v + v(-u) = -(uv + vu) = 0.$$

The action of x is trickier to work out, so we do it in a couple steps:

$$x(uv) = g(u)x(v) + x(u)v = (-u)u + 0v = -u^2$$
$$x(vu) = g(v)x(u) + x(v)u = v0 + (u)u = u^2.$$

So, $x(uv + vu) = x(uv) + x(vu) = -u^2 + u = 0$. We say that A is a T-module algebra.

A few more words just to formalize the above. Suppose H is a Hopf algebra and A and algebra which is also an H-module[13]. Then A is a *H-module algebra* if for all $h \in H$, $h(1_A) = \epsilon(h)1_A$ and $h(ab) = \sum h_1(a)h_2(b)$ where $\Delta(h) = \sum h_1 \otimes h_2$. Invariants of a Hopf action are slightly different. In the context of the definition above,

$$A^H = \{a \in A : h(a) = \epsilon(h)a \text{ for all } h \in H\}.$$

Consider again the Sweedler algebra T and $A = \mathbb{C}_{-1}[u, v]$ with action given above. The invariants are elements *fixed* by g and *killed* by x. We already computed the elements fixed by g, $\mathbb{C}[u^2, v]$. Since u^2 is killed by x (and so any polynomial in u^2 is killed), we only need to compute what powers of v are killed by x. Note

$$x(v^2) = g(v)x(v) + x(v)v = vu + uv = 0.$$

Inductively we can show that any even power of v is killed, so $A^T = \mathbb{C}[u^2, v^2]$.

References

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[13] A module is basically just a vector space over a ring.