Ozone groups and centers of skew polynomial rings

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Setup

Let $\mathbb{k}$ be a field of characteristic zero.

Let $\mathbf{p} = (p_{ij}) \in M_n(\mathbb{k}^\times)$ be multiplicatively antisymmetric ($p_{ii} = 1$ for all i and $p_{ij} = p_{ji}^{-1}$ for all $i \neq j$).

The skew polynomial ring $S_\mathbf{p}$ is the $\mathbb{k}$-algebra

$$S_\mathbf{p} = \mathbb{k}_\mathbf{p}[x_1, \dots, x_n] = \frac{\mathbb{k}\langle x_1, \dots, x_n \rangle}{(x_j x_i = p_{ij} x_i x_j)}$$

- $S_\mathbf{p}$ is AS regular
- $S_\mathbf{p}$ has global and GK dimension n
- $S_\mathbf{p}$ is PI if and only if each p_{ij} is a root of unity

Motivation

In case $S_\mathbf{p}$ is PI, we want to understand the properties of $S_\mathbf{p}$ and its center $Z = ZS_\mathbf{p}$. For example, when is Z Gorenstein or regular (a polynomial ring)?

Example ($n = 2$)

Let $S = \Bbbk_{\mathbf{p}}[x_1, x_2]$. Then

$$\mathbf{p} = \begin{pmatrix} 1 & p_{12} \\ p_{12}^{-1} & 1 \end{pmatrix}$$

where p_{12} is an ℓth root of unity.

Which monomials are central?

$$(x_1^i x_2^j)x_1 = p_{12}^j x_1(x_1^i x_2^j)$$
$$(x_1^i x_2^j)x_2 = p_{12}^{-i} x_2(x_1^i x_2^j)$$

So $x_1^i x_2^j$ central if and only if $i \equiv j \equiv 0 \mod \ell$. Thus,

$$Z(S) = \Bbbk[x_1^\ell, x_2^\ell]$$

The case $n = 3$ is already significantly harder. Here we will give one way of attacking this problem.

The ozone group

Let $\phi_i \in \mathrm{Aut}_{\mathrm{gr}}(S_{\mathbf{p}})$ denote conjugation by x_i:

$$\phi_i(f) = x_i^{-1} f x_i \quad \text{for all } f \in S_{\mathbf{p}}$$

Let $O = \langle \phi_1, \dots, \phi_n \rangle$, which is a subgroup of $\mathrm{Aut}_{\mathrm{gr}}(S_{\mathbf{p}})$.

It is clear that

$$Z = ZS_{\mathbf{p}} = S_{\mathbf{p}}^{O},$$

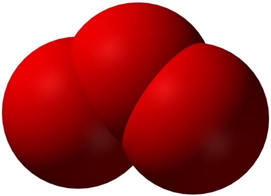
so we can employ tools from (noncommutative) invariant theory.

One can show that $O = \mathrm{Aut}_{Z-\mathrm{alg}}(S)$.

The ozone group

Some other famous ozones:

The ozone molecule



The G.I. Joe character

The Motown funk band

The ozone group

Definition

Let A be a noetherian PI AS regular algebra with center Z. The ozone group of A is

$$\mathsf{Oz}(A) = \mathsf{Aut}_{Z-\mathrm{alg}}(A).$$

In general we have

$$1 \leq |\,\mathsf{Oz}(A)| \leq \mathsf{rank}(A_Z).$$

The ozone group can be used to characterize skew polynomial rings.

Theorem (CGWZ)

Suppose $\Bbbk = \overline{\Bbbk}$ and A is generated in degree 1. Then A is a skew polynomial ring if and only if $\mathsf{Oz}(A)$ is abelian and $|\mathsf{Oz}(A)| = \mathsf{rank}(A_Z)$.

The ozone group

Example

Let A be the quantum Heisenberg algebra

$$\Bbbk\langle x, y, z\rangle/(zx = pxz, zy = p^{-1}yz, yx = pxy - z^2).$$

where p is a primitive ℓth root of unity.

Set $\Omega = (yx - p^2xy)$. The center of A is generated by x^ℓ, y^ℓ, z^ℓ, and $\Omega z^{\ell-1}$.

Let $\phi \in \mathrm{Oz}(A)$. A computation shows

$$\phi(x) = \epsilon_1 x, \quad \phi(y) = \epsilon_2 y, \quad \phi(z) = \epsilon_3 z$$

where each ϵ_i is an ℓth root of unity.

In order to fix $\Omega z^{\ell-1}$ and satisfy $0 = \phi(yx - pxy + z^2)$, we must have

$$\epsilon_3 = 1 \quad \text{and} \quad \epsilon_2 = \epsilon_1^{-1}.$$

This implies that $\mathrm{Oz}(A) \cong C_\ell$.

The ozone group

Lemma

If A and B are noetherian PI AS regular algebras, then

$$\mathrm{Oz}(A \otimes B) = \mathrm{Oz}(A) \times \mathrm{Oz}(B).$$

Hence, every finite abelian group is realizable as the ozone group of a noetherian PI AS regular algebra.

Example

Let A be the 3-dimensional Sklyanin algebra $S(1, 1, -1)$

$$\Bbbk\langle x, y, z\rangle/(xy + yx = z^2, yz + zy = x^2, zx + xz = y^2).$$

A similar computation to the previous one shows that the ozone group of A is trivial.

We conjecture that the ozone group is abelian for every PI AS regular algebra.

For non-connected algebras the ozone group may be non-abelian.

The mozone

One can ask if there is a “Galois-like” correspondence for the ozone group.

Definition

Let A be a noetherian PI AS regular algebra with center Z.

(1) A subring R of A is called ozone if R is AS regular and $Z \subseteq R \subseteq A$.

(2) The set of all ozone subrings of A is denoted by $\Phi_Z(A)$.

(3) If R is a minimal element in $\Phi_Z(A)$ via inclusion, then R is called a mozone subring of A.

Proposition (CGWZ)

Let $S = S_{\mathbf{p}}$ be PI and let O be the ozone group of S. Let H denote the subgroup of O generated by reflections. Then S^H is a mozone subring of S.

Reflections

Let $S = S_{\mathbf{p}}$ be PI, let $Z = ZS_{\mathbf{p}}$, and let O be the ozone group of S.

Since the automorphisms of O are diagonal, a reflection of O is a classical reflection.

Let H denote the subgroup of O generated by reflections.

Theorem (Kirkman, Kuzmanovich, Zhang (2010))

Let G be a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(S)$. Then S^G has finite global dimension if and only if G is generated by reflections of S. In this case, S^G is again a skew polynomial ring.

By the above theorem, Z is regular if and only if $O = H$.

Theorem (Kirkman, Kuzmanovich, Zhang (2009))

Let G be a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(S)$. Then S^G is Gorenstein if and only if G/H acts on S^H with trivial homological determinant.

Reflections

Proposition (CGWZ)

Set

$$\mathfrak{f}_i = \gcd\{d_i \mid x_1^{d_1} \cdots x_i^{d_i} \cdots x_n^{d_n} \in Z\}.$$

Then

$$H = \prod_{i=1}^{n} \langle r_i \rangle \quad \textit{where} \quad r_i : x_j \mapsto \begin{cases} x_j & j \neq i \\ c_i x_i & j = i \end{cases}$$

for some root of unity c_i. Moreover, the order of c_i is $\mathfrak{f}_i$, so

$$S^H = \mathbb{k}_{\mathbf{q}}[x_1^{\mathfrak{f}_1}, \ldots, x_n^{\mathfrak{f}_n}]$$

and

$$\mathfrak{f}_i = \min\{d_i > 0 \mid x_1^{d_1} \cdots x_i^{d_i} \cdots x_n^{d_n} \in Z\}.$$

An immediate consequence is that O contains no reflections if and only if each $f_i = 1$.

Auslander's Theorem

Let A be an algebra and let G a subgroup of $\mathrm{Aut}(A)$. The Auslander map $A\#G \to \mathrm{End}(A_{A^G})$ is given by

$$a\#g \mapsto \begin{pmatrix} A & \to & A \\ b & \mapsto & ag(b) \end{pmatrix}$$

Auslander's original theorem says that for A a polynomial ring, the Auslander map is an isomorphism if and only if G is small (contains no reflections).

Theorem (CGWZ)

The following are equivalent:

(1) The Auslander map is an isomorphism for (S, O).

(2) O is small (in the classical sense).

(3) $f_i = 1$ for all i. (There is an element of the form $x_1^{a_1} \cdots x_i \cdots x_n^{a_n} \in Z$.)

We can work this out explicitly (in terms of the parameters) for small n.

Auslander's Theorem

First, note that for the parameters $\mathbf{p} = (p_{ij})$ we can find some ℓth root of unity ξ (where ℓ is minimal) such that $p_{ij} = \xi^{b_{ij}}$ for some integers b_{ij}.

$$\mathbf{p} = \begin{pmatrix} 1 & p_{12} & p_{13} \\ p_{12}^{-1} & 1 & p_{23} \\ p_{13}^{-1} & p_{23}^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi^{b_{12}} & \xi^{b_{13}} \\ \xi^{-b_{12}} & 1 & \xi^{b_{23}} \\ \xi^{-b_{13}} & \xi^{-b_{23}} & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix} \in \mathrm{M}_3(\mathbb{Z}/\ell\mathbb{Z})$$

The matrix $B = (b_{ij})$ is (honestly) anti-symmetric.

Recall that the Pfaffian of (a skew-symmetric matrix) B is

$$\mathrm{pf}(B) = \sqrt{\det(B)}$$

Auslander's Theorem

Theorem (CGWZ)

($n = 2$) The Auslander map is not *an isomorphism for* (S, O).

($n = 3$) The Auslander map is an isomorphism if and only if $\gcd(b_{ij}, \ell) = 1$ *for each* $i \neq j$.

($n = 4$) The Auslander map is an isomorphism if and only if $\mathrm{pf}(B) = 0 \mod \ell$ *and there does not index* j *and integer* k *such that* $kb_{ij} = 0 \mod \ell$ *for all but one* i.

In case $n = 3$, $\mathrm{pf}(B)$ is automatically zero. This demonstrates that the Pfaffian plays an important role in analyzing these algebras.

Regular center

Recall that, in the $n = 2$ case, Z is always regular.

There is an algorithm for working out this problem in general which is explained in our paper. Here is the key lemma:

Lemma

Let $\overline{B}$ be the matrix obtained from B by reduction mod ℓ. Let $\overline{K} = \ker(\overline{B})$ and let $K \subset \mathbb{Z}^n$ be its inverse image.

Then $Z = \Bbbk[x_1^{\mathfrak{f}_1}, \ldots, x_n^{\mathfrak{f}_n}]$ if and only if $\mathfrak{f}_i \mathbf{e}_i \in K$ for each i.

Equivalently, $f_i \mathbf{e}_i \otimes 1 \in K \otimes \mathbb{Z}_{(p)}$ for every prime $p \mid \ell$ and each i.

For each $p \mid \ell$, we work out an explicit generating set of $K \otimes \mathbb{Z}_{(p)}$ in the cases above. These can then be glued together to get a generating set for K.

Regular center

Let $n = 3$. The Smith normal form $D = LBR$ of B over the ring $\mathbb{Z}_{(p)}$ is

$$D = \begin{bmatrix} b_{12} & 0 & 0 \\ 0 & -b_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_{23}/b_{12} & -b_{13}/b_{12} & 1 \end{bmatrix}, R = \begin{bmatrix} 0 & 1 & b_{23}/b_{12} \\ 1 & 0 & -b_{13}/b_{12} \\ 0 & 0 & 1 \end{bmatrix}.$$

The kernel of $D_{(p)}$ is generated, as a $\mathbb{Z}_{(p)}$-module, by $p^N\mathbf{e}_1$, $p^N\mathbf{e}_2$, $\mathbf{e}_3$.

Applying R to these gives that $K_{(p)}$ is generated as a $\mathbb{Z}_{(p)}$-module by $p^N\mathbf{e}_i$ and

$$\frac{1}{b_{12}}\begin{bmatrix} b_{23} \\ -b_{13} \\ b_{12} \end{bmatrix}.$$

Theorem (CGWZ)

($n = 3$) Z is regular if and only if the orders of p_{12}, p_{13}, and p_{23} are pairwise coprime.

($n = 4$) Let $\rho = \gcd(\ell, \mathrm{pf}(B))$, $c_{ij} = \gcd(b_{ij}, \rho)$, ω be a primitive ρth root of unity, and set $q_{ij} = \omega^{c_{ij}}$. Then Z is regular if and only if the orders of the $\{q_{ij}\}_{i<j}$ are pairwise coprime.

We introduce here some "new" invariants. Several of these are "ozone versions" of invariants defined by Kirkman and Zhang (2021).

Ozone Invariants

- *The ozone Jacobian of S is* $\mathfrak{oj}_S := \prod_{i=1}^n x_i^{\mathfrak{f}_i - 1}$.
- *The ozone arrangement of S is* $\mathfrak{oa}_S := \prod_{\mathfrak{f}_i > 1}^n x_i$.
- *The ozone Jacobian of S is* $\mathfrak{od}_S := \prod_{\mathfrak{f}_i > 1}^n x_i^{\mathfrak{f}_i} = \mathfrak{oj}_S \mathfrak{oa}_S$.
- *The product of generators of S is* $\mathfrak{pg}_S := \prod_{i=1}^n x_i$.

The first three are algebra invariants (up to a nonzero scalar) but the last one is not (depends on the presentation).

When Z is Gorenstein, then $\mathfrak{od}_S$ is the same as $\mathfrak{j}_{S,O}$ as defined by Kirkman and Zhang.

Gorenstein center

Theorem (CGWZ)

The following are equivalent.

(1) Z is Gorenstein.

(2) $\mathfrak{o}\mathfrak{j}_S \mathfrak{p}\mathfrak{g}_S = \prod_{i=1}^n x_i^{\mathfrak{f}_i}$.

(3) For all i, we have $\prod_{j=1}^n p_{ij}^{\mathfrak{f}_j} = 1$.

Again, when $n = 2$, Z is regular so Gorenstein.

Theorem (CGWZ)

($n = 3$) Z is Gorenstein if and only if

$$\overline{B}(b'_{23}, b'_{13}, b'_{12})^T = 0 \quad \textit{where} \quad b'_{ij} = \gcd(b_{ij}, \ell)$$

($n = 4$) Z is Gorenstein if and only if

$$\frac{\ell}{\gcd(\mathrm{pf}(B), \ell)} \overline{B}(v_1, v_2, v_3, v_4)^T = 0 \quad \textit{where} \quad v_i = \gcd(\ell, \{b_{jk} \mid j, k \neq i\})$$

But wait! There's more!

Corollary (CGWZ)

(1) S is Calabi-Yau if and only $\mathfrak{pg}_S \in Z$ if and only if Z is Gorenstein and Auslander's Theorem holds for (S, O).

(2) If S is Calabi-Yau, then Z is not regular.

Questions

- *Characterize $S_{\mathbf{p}}$ when $ZS_{\mathbf{p}}$ is a hypersurface ring, or a complete intersection*

$$\textit{regular} \Rightarrow \textit{hypersurface} \Rightarrow \textit{complete intersection} \Rightarrow \textit{Gorenstein}$$

- *For A a PI AS regular algebra, is there a semisimple Hopf algebra H such that*

$$Z(A) = A^H?$$

- *Is there a version of previous corollary for A?*

- *Can we define the ozone invariants for A so that they control properties of the center?*

Thank You!

Thanks James!