Algebras similar to the 2×2 Jordanian matrix algebra

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Quantum groups

Quantum groups are an important source of examples in noncommutative algebra. Their constructions, representation theory, automorphism groups, and (noncommutative algebraic) geometry are all actively studied. In some cases we have a full picture and in others only a partial understanding on one or more of these points.

One well-studied class of quantum groups are quantum matrix algebras. These can be understood as quantum coordinate rings of functions on $n \times n$ matrices.

We wish to understand algebras which are close to quantum groups but perhaps do not fall under that classification.

Constructing $M_q(2)$

Throughout, let $\mathbb{k}$ be an algebraically closed field of characteristic zero.

For $q \in \mathbb{k}^\times$, the quantum plane is the $\mathbb{k}$-algebra

$$\mathcal{O}_q(\mathbb{k}^2) = \mathbb{k}\langle x_1, x_2 \mid x_1 x_2 - q x_2 x_1 \rangle.$$

Classically, the 2×2 *quantum matrix algebra* relative to q is the unique $\mathbb{k}$-algebra on generators x_{11}, x_{12}, x_{21}, x_{22} such that there exist homomorphisms

$$\begin{aligned} \mathcal{O}_q(\mathbb{k}^2) &\to M_q(2) \otimes \mathcal{O}_q(\mathbb{k}^2) \\ x_i &\mapsto x_{i1} \otimes x_1 + x_{i2} \otimes x_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{O}_q(\mathbb{k}^2) &\to \mathcal{O}_q(\mathbb{k}^2) \otimes M_q(2) \\ x_j &\mapsto x_1 \otimes x_{1j} + x_2 \otimes x_{2j}. \end{aligned}$$

Constructing $M_q(2)$

Making the identifications $b = x_{11}$, $c = x_{22}$, $a = x_{12}$, $d = x_{21}$, we get the relations

$$\begin{array}{lll} ac = qca & dc = qcd & da = ad \\ ba = qab & bd = qdb & bc = cb + (q - q^{-1})ad. \end{array}$$

The element $z = bc - qad$ is central in $M_q(2)$ and known as the *quantum determinant*.

Constructing $M_J(2)$

We can repeat the above process starting with the Jordan plane,

$$\mathcal{J} = k\langle x_1, x_2 \mid x_2x_1 - x_1x_2 + x_2^2\rangle.$$

The Jordanian matrix algebra $M_J(2)$ is the $\mathbb{k}$-algebra on generators a, b, c, d subject to the following relations.

$$\begin{aligned} ac &= ca + c^2 & bc &= cb + ca + cd + c^2 \\ dc &= cd + c^2 & bd &= db + cb + cd - ad + d^2 \\ da &= ad - cd + ca & ba &= ab + cb + cd - ad + a^2. \end{aligned}$$

The *Jordanian determinant* is $z = ad - cb - cd$.

Properties of $M_J(2)$

- $M_q(2)$ and $M_J(2)$ are noetherian domains of GK and global dimension 4.
- Both $M_q(2)$ and $M_J(2)$ are Artin-Schelter regular.
- The prime spectrum and automorphism group of $M_q(2)$ and $M_J(2)$ are computable.
- $M_q(2)$ is birationally equivalent to a quantum affine space.
- $M_J(2)$ is birationally equivalent to $A_1(\mathbb{k}[s,t])$.

I will discuss algebras similar to quantum matrices (specifically $M_J(2)$), sharing *most* of these properties.

Skew polynomial rings

Let R be a ring. For $\sigma \in \mathrm{Aut}(R)$, a σ-derivation δ is a $\mathbb{k}$-linear R-map such that

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s \text{ for all } r, s \in R.$$

The *skew polynomial ring* (or Ore extension) $R[x; \sigma, \delta]$ is defined via the commutation rule $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

Both $\mathcal{O}_q(\mathbb{k}^2)$ and $\mathcal{J}$ are skew polynomial rings over $\mathbb{k}[c]$.

$M_q(2)$ and $M_J(2)$ are iterated skew polynomial rings over $\mathcal{O}_q(\mathbb{k}^2)$ and $\mathcal{J}$, respectively.

Involutive skew polynomial extensions

Definition

Let R be a ring. We say $S = R[a;\sigma_1,\delta_1][d;\sigma_2;\delta_2]$ is an ***involutive skew polynomial extension*** *(ISPE) of R if there exists an involution $\tau \in \mathrm{Aut}(S)$ such that $\tau(a) = d$, $\tau(d) = a$, and $\tau(r) = r$ for all $r \in R$.*

If $R = \Bbbk$, there are only three ISPEs,

- $S = \mathcal{O}_q(\Bbbk^2)$ with $q = 1$ or -1;
- $S = \Bbbk\langle x_1, x_2 \mid x_1x_2 + x_2x_1 - 1\rangle$.

Involutive skew polynomial extensions

Proposition

Let S be an ISPE of R and suppose the following conditions hold:

- *R is affine and commutative with generators $r_1, \ldots, r_n$ and $\sigma_1 = \mathrm{id}_R$.*
- *$u = d - a$ is normal in S.*
- *$r_i^{-1}\delta_1(r_i) = r_j^{-1}\delta_1(r_j)$ for all $i, j \in \{1, \ldots, n\}$.*
- *$a\left(r_i^{-1}u\right) = \left(r_i^{-1}u\right)a$.*

Then S is birationally equivalent to $A_1(R)$.

Pre-matrix algebras

Definition

*A **pre-matrix algebra** is an ISPE of the polynomial ring* $\Bbbk[c]$.

The base rings in the skew polynomial construction of $M_q(2)$ and $M_J(2)$ are pre-matrix algebras. In particular, the involution τ may be thought of as the transposition operator on 2×2 matrices.

Pre-matrix algebras - differential operator case

For any $f \in \mathbb{k}[c]$, define the differential operator ring

$$R_f = \mathbb{k}\langle a, c \mid ca - ac + f \rangle.$$

This is a skew polynomial ring $\mathbb{k}[c][a; \delta_1]$ with $\delta_1(c) = f$.

Proposition

Suppose $f \in k[c]$, $f \notin \mathbb{k}$. If $P = R_f[d; \sigma_2; \delta_2]$ is a pre-matrix algebra, then there exists $g \in k[c]$ such that $\sigma_2(c) = c$, $\delta_2(c) = f$, $\sigma_2(a) = a - g$, and $\delta_2(a) = ga$.

We denote the above pre-matrix algebra by $P(f, g)$. The base ring of the Jordanian matrix algebra $M_J(2)$ is $P(c^2, c)$.

Pre-matrix algebras - differential operator case

Proposition

Choose $f, g \in \Bbbk[c]$ such that $f \notin \Bbbk$ and set $P = P(f, g)$.

- *The algebra P is a noetherian domain with* $\operatorname{gldim} P = \operatorname{GKdim} P = 3$.
- *The following elements of P are normal: f, $d - a$, and $c - \alpha$ for any $\alpha \in \Bbbk$ such that $f(\alpha) = 0$.*
- *The ideals $(d - a)$ and $(c - \alpha)$ are prime in P. Moreover, these are all of the height one prime ideals of P if and only if there do not exist integers $n, m_1, \ldots, m_k$ $(n \neq 0)$ such that*

$$g = -\frac{1}{n} \sum_{\substack{\alpha \\ f(\alpha)=0}} m_i((c-\alpha)^{-1} f).$$

Pre-matrix algebras - differential operator case

Suppose henceforth that $f = cg$ and set $u = d - a$, $P = P(f, g)$.

Proposition

For any $\alpha, \lambda, \mu \in \mathbb{k}^{\times}$, $\eta \in \mathbb{k}$, and $h \in \mathbb{k}[u, c]$ there is an automorphism π of P such that $\pi(g) = \lambda g$ determined by the assignments

$$a \mapsto \lambda a + h, \quad c \mapsto \varepsilon c, \quad \text{and } u \mapsto \mu u + \eta c.$$

Moreover, any automorphism of P has the above form.

Proposition

The pre-matrix algebra P is birationally equivalent to $A_1(\mathbb{k}[t])$.

Generalized Jordanian matrix algebras

Definition

We say $M = P(f, g)[b; \sigma_3, \delta_3]$ *is a* ***Generalized Jordanian Matrix Algebra*** *(GJMA) if it is birationally equivalent to* $M_J(2)$ *and the involution* τ *extends to* M *with* $\tau(b) = b$.

Generalized Jordanian matrix algebras

Any irreducible factor p of f determines an inner automorphism σ. For $\theta \in P$, denote by δ the σ-derivation of P determined by θ,

$$\delta(x) = \theta x - \sigma(x)\theta, \quad x \in P.$$

Denote by $G(f, p, \theta)$ the algebra $P[b; \sigma, \delta]$.

Proposition

$G(f, p, \theta)$ is a GJMA if

$$\begin{aligned} \delta(u) &= \tau(\delta(a)) - \delta(a), \\ \delta(c) &= cu^{-1}\delta(u) = cu^{-1}\left(\tau(\delta(a)) - \delta(a)\right), \end{aligned}$$

and $\delta(x) \in P$ for $x = a, u, c$.

Generalized Jordanian matrix algebras

Proposition

Suppose $\deg g \geq 1$. *The ring* $G\left(f, g, g^{-1}\left(a^2 + (u-g)a\right)\right)$ *is a GJMA which we denote by* $\mathcal{G}_f$. *Moreover,* $\mathcal{G}_{c^2} = M_J(2)$.

Set $h = g'c$ and $\gamma = h + u + 2a$. The relations for $\mathcal{G}_f$ are

$$ac = ca + f, au = ua + ug, cu = uc,$$
$$bc = cb + c\gamma, bu = ub + u\gamma, ba = (a+h)b + (h-u)a.$$

Generalized Jordanian matrix algebras

Proposition

- *The algebra* $\mathcal{G}_f$ *is a noetherian domain with* $\operatorname{gldim} \mathcal{G}_f = \operatorname{GKdim} \mathcal{G}_f = 4$.
- *The center of* $\mathcal{G}_f$ *is* $\mathbb{k}[z]$ *where* $z = gb + (g-u)a - a^2$.
- *The height one prime ideals of* $\mathcal{G}_f$ *are* (c), (u), *and* $(z-\xi)$ *for* $\xi \in \mathbb{k}^\times$.

Future work

- Can we classify all GJMAs?
- Is the automorphism group of $\mathcal{G}_f$ computable? (Have partial results.)
- Hopf actions of $\mathcal{G}_f$?
- Can the definition of a pre-matrix algebra be generalized so as to construct the base ring for two-parameter quantum matrix algebras and the mixed quantum matrix algebra (with $\mathcal{O}_q(\mathbb{k}^2)$ and $\mathcal{J}$ relations)?

Thank You!