Enveloping Algebras of Poisson Superalgebras

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The Big Idea

We construct and study the universal enveloping algebra of Poisson superalgebras, generalizing the idea and several theorems from Poisson algebras.

Background

Poisson algebras arise in various fields of study including Hamiltonian mechanics and algebraic geometry. We wish to study Poisson superalgebras, a generalization which has enjoyed increased attention due to the development of supersymmetry theories.

Definition

A *superalgebra* is a $\mathbb{Z}_2$-graded algebra $R = R_0 \oplus R_1$. An element x is even if $x \in R_0$ and odd if $x \in R_1$, and we write $|x|$ for the parity of x; such even or odd elements are called *homogeneous*. A superalgebra is *supercommutative* if $xy = (-1)^{|x||y|}yx$ for any homogeneous x, y. A *Poisson superalgebra* is a supercommutative superalgebra R with bracket $\{\cdot,\cdot\}$ satsifying the following for all $x, y, z \in R$:

$$\begin{aligned}
0 &= \{x,y\} + (-1)^{|x||y|}\{y,x\} \\
0 &= (-1)^{|x||z|}\{x,\{y,z\}\} + (-1)^{|x||y|}\{y,\{z,x\}\} + (-1)^{|y||z|}\{z,\{x,y\}\} \\
0 &= \{x,yz\} - (-1)^{|x||y|}y\{x,z\} - \{x,y\}z.
\end{aligned}$$

The first two relations make $(R,\{\cdot,\cdot\})$ into a Lie superalgebra.

We extend the idea of the universal enveloping algebra, first defined and constructed by Oh in [Oh99], to Poisson superalgebras, and extend several well-known theorems to the super case.

Definition

Let R be a Poisson superalgebra. A triple (U,α,β) satisfies property $\mathbf{P}$ (with respect to R) if U is an algebra, $\alpha : R \to U$ is an algebra homomorphism, and $\beta : R \to U$ is a linear map such that

$$\begin{aligned}
\beta(\{x,y\}) &= \beta(x)\beta(y) - (-1)^{|x||y|}\beta(y)\beta(x) \\
\alpha(\{x,y\}) &= \beta(x)\alpha(y) - (-1)^{|x||y|}\alpha(y)\beta(x) \\
\beta(xy) &= \alpha(x)\beta(y) + (-1)^{|x||y|}\alpha(y)\beta(x).
\end{aligned}$$

The *universal enveloping algebra* of R is a triple $(U(R),\alpha,\beta)$ that is universal with respect to property $\mathbf{P}$.

Lie-Rinehart superalgebras and the PBW theorem

In order to prove the PBW theorem for Poisson superalgebras, we first prove a new version of the PBW theorem for Lie-Rinehart superalgebras. In this section alone we work over a commutative ring S rather than a field k.

Definition

A *Lie-Rinehart superalgebra* is a pair (A, L), where A is a supercommutative superalgebra, L is a Lie superalgebra as well as an A-supermodule, together with a Lie superalgebra and A-supermodule morphism $\rho : L \to \mathrm{Der}(A)$, where $\mathrm{Der}(A)$ is the Lie superalgebra of superderivations of A, such that for $x, y \in L, a \in A$:

$$[x, ay] = (-1)^{|a||x|}a[x,y] + \rho(x)(a)y.$$

As with Poisson superalgebras, one can define the universal enveloping algebra of a Lie-Rinehart superalgebra.

Definition

Let (A, L) be a Lie-Rinehart superalgebra (A, L). A triple (U, f, g) satisfies property $\mathbf{R}$ (with respect to (A, L)) if U is an algebra, $f : R \to U$ is an algebra homomorphism, and $g : L \to U$ is a linear map such that

$$\begin{aligned}
g([x,y]) &= g(x)g(y) - (-1)^{|x||y|}g(y)g(x) \\
f(x(a)) &= g(x)f(a) - (-1)^{|a||x|}f(a)g(x) \\
g(ax) &= f(a)g(x).
\end{aligned}$$

The *universal enveloping algebra* of a (A, L) is a triple $(V(A, L), \alpha, \beta)$ that is universal with respect to property $\mathbf{R}$.

There is a filtration on $V(A, L)$ as follows: for $p \geq 0$, let V_p denote the left A-subsupermodule of $V(A, L)$ generated by products of at most p elements of L, and let $V_{-1} = 0$. We denote the associated graded superalgebra with respect to this filtration by $\mathrm{gr}(V(A, L))$. We remark the following PBW theorem is independent of the result in [Rin63].

PBW Theorem

If A and L are free S-supermodules, then the canonical epimorphism $S_A(L) \to \mathrm{gr}(V(A, L))$ is an isomorphism.

PBW theorem for Poisson superalgebras

Let Ω_A^{ev} denote the even Kähler superdifferentials over A. For any Poisson superalgebra A, the pair $(A, \Omega_A^{\mathrm{ev}})$ is a Lie-Rinehart superalgebra if Ω_A^{ev} is given the bracket

$$[a\mathrm{d}f, b\mathrm{d}g] = (-1)^{|b||f|}ab\mathrm{d}\{f,g\} + a\{f,b\}\mathrm{d}g - (-1)^{|af||bg|}b\{g,a\}\mathrm{d}f$$

and the anchor map is $\rho(\mathrm{d}f) = \{f,\cdot\}$. There is a unique isomorphism $\Lambda : U(A) \to V(A, \Omega_A^{\mathrm{ev}})$ between the two enveloping algebras, so the following PBW theorem is obtained.

PBW Theorem

Let A be a Poisson superalgebra, and consider the filtration defined on $U(A)$ by Λ. Then there is an A-superalgebra isomorphism

$$S_A(\Omega_A^{\mathrm{ev}}) \cong \mathrm{gr}\, U(A).$$

Further Results

- If R is a Poisson Hopf superalgebra, then the enveloping algebra $U(R)$ is also a Hopf superalgebra
- If $A = R[x]$ is a Poisson-Ore extension of a Poisson superalgebra R, then the enveloping algebra $U(A)$ can be expressed as an iterated Ore extension

  $$U(A) = U(R)[m_x;\sigma_1,\eta_1][h_x;\sigma_2,\eta_2]$$

  for appropriate σ_i, η_i.

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