Techniques of Integration

7.1 Integration by parts

The integral $\int xe^{x^2}\ dx$ can be evaluated easily using u-substitution. Set $u = x^2$ so $du = 2x\ dx$. Then

$$\int xe^{x^2}\ dx = \frac{1}{2}\int (2x)e^{x^2}\ dx = \frac{1}{2}\int e^u\ du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

But what about an integral like $\int xe^x\ dx$. Suddenly, u-substitution does not apply because we no longer have a composition of functions but a product. So, we would like some analog of the product rule from differentiation.

Let f and g be differentiable functions.

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives

$$f(x)g(x) = \int f'(x)g(x)\ dx + \int f(x)g'(x)\ dx.$$

After rearranging we have the integration by parts formula,

$$\int f(x)g'(x)\ dx = f(x)g(x) - \int g(x)f'(x)\ dx.$$

Integration-by-parts

Let f and g be differentiable functions. Set $u = f(x), v = g(x)$, then the differentials are du $= f'(x)\ dx$ and dv $= g'(x)\ dx$. We then have the following formula,

$$\int u\ dv = uv - \int v\ du.$$

Example. We evaluate $\int xe^x\ dx$ using integration-by-parts. Let $u = x$ and $dv = e^x\ dx$. Then $du = dx$ and $v = e^x$. Using the IBP formula gives

$$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x + C.$$

If we had chosen $u = e^x$ and $dv = x\ dx$, then we would have $v = \frac{1}{2}x^2$ which is more complicated, instead of simpler than the original.

These notes are derived primarily from *Calculus, Early Transcendentals* by James Stewart (8ed). Most of this material is drawn from Chapter 7. Last Updated: February 16, 2023

Example. We evaluate $\int x \sin x \; dx$ using integration-by-parts. Let $u = x$ and $dv = \sin x \; dx$. The rationale behind this choice is that we will differentiate u, because we want u to be *simpler*, and and integrate dv. This choice gives $du = dx$ and $v = -\cos x$. Thus, the IBP formula gives,

$$\int u \; dv = -x \cos x - \int -\cos x \; dx = -x \cos x + \sin x + C.$$

The pnuemonic LIPET is useful for helping to choose what to set u. This stands for

L(ogarithms)I(nverse trig)P(olynomials)E(xponentials)T(trig).

Example. We evaluate $\int \ln x \; dx$ using IBP. Set $u = \ln x$ and $dv = \; dx$, so $du = \frac{1}{x} \; dx$ and $v = x$. By the IBP formula,

$$\int \ln x \; dx = x \ln x - \int x \frac{1}{x} \; dx = x \ln x - \int \; dx = x \ln x - x + C.$$

Example. We evaluate $\int x^2 \sin 4x \; dx$ using integration-by-parts. For reasons similar to the previous example, we make the choice $u = x^2$ and $dv = \sin 4x \; dx$. Then $du = 2x \; dx$ and $v = -\frac{1}{4} \cos 4x$. The IBP formula gives,

$$\int x^2 \sin 4x \; dx = x^2 \left(-\frac{1}{4} \cos 4x \right) - \int -\frac{1}{4} \cos 4x \cdot 2x \; dx = -\frac{1}{4} x^2 \cos 4x + \frac{1}{2} \int x \cos 4x \; dx.$$

To evaluate the remaining integral we must use IBP again. Set $u = x$ and $dv = \cos 4x \; dx$, so $du = \; dx$ and $v = \frac{1}{4} \sin 4x$. So we have

$$\begin{aligned}
\int x^2 \sin 4x \; dx &= -\frac{1}{4} x^2 \cos 4x + \frac{1}{2} \int x \cos 4x \; dx \\
&= -\frac{1}{4} x^2 \cos 4x + \frac{1}{2} \left(\frac{1}{4} x \sin 4x - \int \frac{1}{4} \sin 4x \; dx \right) \\
&= -\frac{1}{4} x^2 \cos 4x + \frac{1}{8} x \sin 4x - \frac{1}{8} \int \sin 4x \; dx \\
&= -\frac{1}{4} x^2 \cos 4x + \frac{1}{8} x \sin 4x - \frac{1}{8} \left(-\frac{1}{4} \cos 4x \right) + C \\
&= -\frac{1}{4} x^2 \cos 4x + \frac{1}{8} x \sin 4x + \frac{1}{32} \cos 4x + C.
\end{aligned}$$

In the last example we had to use IBP multiple times. In these situations we use *tabular integration* to organize our work. Our table consists of a column to keep track of signs (alternating +,-,+,-,...), a column for the derivatives of u, and a column for integrals of v. The sign goes with u, but each term in the u column is matched with the corresponding term *one down* in the dv column.

sign	u	dv
$+$	x^2	$\sin 4x$
$-$	$2x$	$-\frac{1}{4}\cos 4x$
$+$	2	$-\frac{1}{16}\sin 4x$
$-$	0	$\frac{1}{64}\cos 4x$

Multiplying with the arrows (watch the sign column!) and adding gives

$$\int x^2 \sin 4x \; dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + C.$$

Evaluating definite integrals with IBP is similar. Note that, unlike with u-substitution, there is no need to change the bounds. However, we must be careful to evaluate *both parts* of the decomposition.

Integration by Parts for definite integrals

Let f and g be differentiable functions. Then

$$\int_a^b f(x)g'(x) \; dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x) \; dx.$$

Example. We evaluate $\int_1^4 \sqrt{t}\ln t \; dt$ using IBP. Set $u = \ln t$ and $dv = \sqrt{t} \; dt$. Then $du = \frac{1}{t} \; dt$ and $v = \frac{2}{3}t^{3/2}$. By the IBP formula,

$$\begin{aligned}\int_1^4 \sqrt{t}\ln t \; dt &= \frac{2}{3}t^{3/2}\ln t\bigg|_1^4 - \int_1^4 \frac{2}{3}t^{3/2}\cdot\frac{1}{t} \; dt = \frac{2}{3}(8\ln 4 - 1\cdot 0) - \frac{2}{3}\int_1^4 t^{1/2} \; dt \\ &= \frac{16}{3}\ln 4 - \frac{2}{3}\left[\frac{2}{3}t^{3/2}\right]_1^4 = \frac{16}{3}\ln 4 - \frac{4}{9}[8-1] = \frac{16}{3}\ln 4 - \frac{28}{9}.\end{aligned}$$

At times we may need to combine methods, such as u-substitution, to make our calculations easier.

Example. We evaluate $\int x^5 e^{x^2} \; dx$ using u-substitution and IBP. First we make a substitution. Let $y = x^2$ so $dy = 2x \; dx$. Then

$$\int x^5 e^{x^2} \; dx = \int x^4 e^{x^2}(x \; dx) = \frac{1}{2}\int y^2 e^y \; dy.$$

Set $u = y^2$ and $dv = e^y$. We will then need to apply IBP twice in order to reduce u to a constant. We do this using tabular integration as before:

sign	u	dv
$+$	y^2	e^y
$-$	$2y$	e^y
$+$	2	e^y
$-$	0	e^y

This gives

$$\frac{1}{2}\int y^2e^y\ dy = \frac{1}{2}\left(y^2e^y - 2ye^y + 2e^y\right) + C = \frac{1}{2}\left(x^4e^{x^2} - 2x^2e^{x^2} + 2e^{x^2}\right) + C.$$

In the next example, one could try using tabular integration, but one would end up just going around in circles, so this method is not effective. However, we can use a similar idea to compute the integral. This trick is called *Roundin' the Corner*.

Example. We evaluate $\int e^{2x}\cos x\ dx$. Using IBP twice gives

$$\begin{aligned}\int e^{2x}\cos x\ dx &= e^{2x}\sin x - \int 2e^{2x}\sin x\ dx\\ &= e^{2x}\sin x - 2\left(-e^{2x}\cos x + \int 2e^{2x}\cos x\ dx\right)\\ &= e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\ dx.\end{aligned}$$

Now we observe that the remaining integral is the same as the initial integral but with a coefficient of -4. Adding $4\int e^{2x}\cos x\ dx$ to both sides gives

$$5\int e^{2x}\cos x\ dx = e^{2x}\sin x + 2e^{2x}\cos x + C.$$

Thus,

$$\int e^{2x}\cos x\ dx = \frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x + C.$$

7.2 Trigonometric Integrals

We will use several trigonometric identities to simplify integration problems into something we can solve using u-substitution or IBP. Two of the most important will be the Pythagorean identity:

$$\sin^2 x + \cos^2 x = 1$$

and the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x)) \text{ or } \cos^2 x = \frac{1}{2}(1 + \cos(2x)).$$

Example. We evaluate $\int \sin^2 x \; dx$. Recall that $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$. Then

$$\begin{aligned}\int \sin^2 x \; dx &= \int \frac{1}{2}(1 - \cos(2x)) \; dx = \frac{1}{2}\int 1 - \cos(2x) \; dx \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + C = \frac{x}{2} - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

Integrals of the form $\int \sin^m x \cos^n x \; dx$

(1) If n is odd, save one factor of $\cos x$ and rewrite the rest using $\cos^2 x = 1 - \sin^2 x$.
(2) If m is odd, save one factor of $\sin x$ and rewrite the rest using $\sin^2 x = 1 - \cos^2 x$.
(3) If both are even, use the half-angle identities.
(4) It is sometimes helpful to use,

$$\sin x \cos x = \frac{1}{2}\sin 2x \text{ or } \cos^2(2x) = \frac{1}{2}(1 + \cos 4x).$$

Example. We evaluate $\int \sin^3 x \; dx$. Save one factor of $\sin x$ and rewrite the rest using the Pythagorean identity:

$$\int \sin^3 x \; dx = \int \sin x(\sin^2 x) \; dx = \int \sin x(1 - \cos^2 x) \; dx.$$

Now let $u = \cos x$ so $du = -\cos x \; dx$. Then

$$\begin{aligned}\int \sin^3 x \; dx &= \int \sin x(1 - \cos^2 x) \; dx = -\int 1 - u^2 \; du \\ &= -\left(u - \frac{1}{3}u^3\right) + C = \frac{1}{3}\cos^3 x - \cos x + C.\end{aligned}$$

Example. We evaluate $\int_0^\pi \cos^4\theta\ d\theta$ using the half-angle identity:

$$\begin{aligned}
\int_0^\pi \cos^4\theta\ d\theta = \int_0^\pi \left[\frac{1}{2}(1+\cos 2\theta)\right]^2\ d\theta &= \int_0^\pi \frac{1}{4}\left[1+2\cos 2\theta+\cos^4(2\theta)\right]\ d\theta \\
&= \frac{1}{4}\int_0^\pi 1+2\cos(2\theta)+\frac{1}{2}(1+\cos(4\theta))\ d\theta \\
&= \frac{1}{4}\int_0^\pi \frac{3}{2}+2\cos(2\theta)+\frac{1}{2}\cos(4\theta)\ d\theta \\
&= \frac{1}{4}\left[\frac{3}{2}\theta+\sin(2\theta)+\frac{1}{8}\sin(4\theta)\right]_0^\pi = \frac{3}{8}\pi.
\end{aligned}$$

Integrals of the form $\int \tan^m x \sec^n x\ dx$

(1) If n is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1+\tan^2 x$;

(2) If m is odd, save a factor of $\sec x\tan x$ and use $\tan^2 x = \sec^2 x - 1$.

Example. We evaluate $\int \sec^4 x \tan^4 x\ dx$ using the guidelines above,

$$\int \sec^4 x \tan^4 x\ dx = \int \sec^2 x(1+\tan^2 x)\tan^4 x\ dx.$$

Now let $u=\tan x$ so $du = \sec^2 x$. Then

$$\int \sec^2 x(1+\tan^2 x)\tan^4 x\ dx = \int (1+u^2)u^4\ du.$$

Example. We evaluate $\int_0^{\pi/3} \sec^3 x\tan x\ dx$ by rewriting $u=\sec x$ so $du=\sec x\tan x\ dx$. Then

$$\begin{aligned}
\int_0^{\pi/3} \sec^3 x\tan x\ dx &= \int_0^{\pi/3} \sec^2 x(\sec x\tan x)\ dx \\
&= \int_1^2 u^2\ du = \left[\frac{u^3}{3}\right]_1^2 = \frac{8}{3}-\frac{1}{3} = \frac{7}{3}.
\end{aligned}$$

At times we will also need the following integrals,

$$\int \tan x\ dx = \ln|\sec x| + C \text{ and } \int \sec x\ dx = \ln|\sec x+\tan x| + C.$$

The key examples to think about are

$$\int \frac{x}{1+x^2}\,dx \text{ and } \int \frac{x^2}{1+x^2}\,dx.$$

The first one we can evaluate using the substitution method. On the other hand, our current methods don't seem adequate to handle the second. The trick is to use a sort of 'reverse substitution'. Instead of choosing a function to be $u(x)$ and then determining du, we instead assign x to be a function. A convenient choice will be certain trig functions which allows us to simplify the expression using the trigonometric identities.

Example. We want to evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}$. We can simplify the expression by setting $x = \sec\theta$, so $dx = \sec\theta\tan\theta\,d\theta$. So that we can take the inverse later, we require $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$. We have

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \int \cos\theta\,d\theta = \sin\theta + C.$$

Now $\theta = \operatorname{arcsec}(x)$ and so,

$$\sin\theta + C = \sin(\operatorname{arcsec}(x)) + C = \frac{\sqrt{x^2-1}}{x} + C.$$

Trig substitution

Let a be a nonzero constant. We make the following substitution for each expression:

(1) $\sqrt{a^2-x^2}$, let $x = a\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$;
(2) $\sqrt{a^2+x^2}$, let $x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$;
(3) $\sqrt{x^2-a^2}$, let $x = a\sec\theta, 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$.

The bounds above are necessary to ensure that we can take the inverse, e.g., $\theta = \arcsin(x/a)$.

Example. We evaluate $\displaystyle\int x^3\sqrt{9-x^2}\,dx$. Let $x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, so $dx = 3\cos\theta\,d\theta$. Then

$$\begin{aligned}\int x^3\sqrt{9-x^2}\,dx &= \int 3^3\sin^3\theta\sqrt{9-9\sin^2\theta}(3\cos\theta\,d\theta) = 3^5\int \sin^3\theta\cos\theta\sqrt{1-\sin^2\theta}\,d\theta \\ &= 3^5\int \sin^3\theta\cos^2\theta\,d\theta = 3^5\int \sin\theta(1-\cos^2\theta)\cos^2\theta\,d\theta.\end{aligned}$$

Now let $u = \cos\theta$ so $du = -\sin\theta\,d\theta$. Then

$$\int x^3\sqrt{9-x^2}\,dx = 3^5\int \sin\theta(\cos^4\theta-\cos^2\theta)\,d\theta = 3^5\int u^4-u^2\,du = 3^5\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]+C.$$

Since $x = 3\sin\theta$, then $\theta = \arcsin(x/3)$. Then

$$u = \cos\theta = \cos\left(\arcsin\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9-x^2}}{3}.$$

Hence,

$$\begin{aligned}
\int x^3\sqrt{9-x^2}\,dx &= 3^5\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right] + C \\
&= 3^5\left[\frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3\right] + C \\
&= \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C.
\end{aligned}$$

Example. We evaluate $\int_0^2 x^3\sqrt{x^2+4}\,dx$. Let $x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ so $dx = 2\sec^2\theta\,d\theta$. We must adjust the bounds as well. When $x=0$, $\theta=0$ and when $x=2$, $\theta=\frac{\pi}{4}$. Then

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^2+4}\,dx &= \int_0^{\pi/4}(2\tan\theta)^3\sqrt{4\tan^2\theta+4}(2\sec^2\theta\,d\theta) \\
&= 2^5\int_0^{\pi/4}\tan^3\theta(\sec^3\theta)\,d\theta \\
&= 2^5\int_0^{\pi/4}(\sec^2\theta-1)\sec^2\theta(\tan\theta\sec\theta)\,d\theta.
\end{aligned}$$

Let $u=\sec\theta$ so $du = \tan\theta\sec\theta\,d\theta$. Now the bounds become $\sec(0)=1$ and $\sec(\frac{\pi}{4}) = \sqrt{2}$. Hence,

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^2+4}\,dx &= 2^5\int_0^{\pi/4}(\sec^4\theta-\sec^2\theta)(\tan\theta\sec\theta)\,d\theta \\
&= 2^5\int_1^{\sqrt{2}} u^4-u^2\,du = 2^5\left[\frac{1}{5}u^5-\frac{1}{3}u^3\right]_1^{\sqrt{2}} \\
&= 2^5\left[\frac{4\sqrt{2}}{5}-\frac{2\sqrt{2}}{3}-\frac{1}{5}+\frac{1}{3}\right] = 2^5\left[\frac{2}{15}\sqrt{2}+\frac{2}{15}\right] = \frac{64}{15}(\sqrt{2}+1).
\end{aligned}$$

7.4 Partial Fractions

We begin with an integral we could have done in Calculus I using long division of fractions.

Example. We will evaluate $\displaystyle\int \frac{x^3-1}{x+3}\,dx$. Polynomial long division gives

$$\frac{x^3-1}{x+3} = x^2-3x+3-\frac{10}{x+3}.$$

Hence,

$$\int \frac{x^3-1}{x+3}\,dx = \int (x^2-3x+3)-\frac{10}{x+3}\,dx = \frac{1}{3}x^3-\frac{3}{2}x^2+3x-10\ln|x+3|+C.$$

Another way to evaluate integrals involving quotients of rational functions is to use *partial fraction decomposition*. To do this, we factor the denominator and then try to find numerators that, if we were to find a common denominator, would give us the original numerator.

Partial Fraction Decomposition (linear factors)

Assume the a_i are all distinct. Then we decompose the rational function as

$$\frac{P}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\cdots+\frac{A_n}{x-a_n}.$$

Example. We evaluate $\displaystyle\frac{x+4}{2x^2-5x-3}\,dx$ using PFD. Since $2x^2-5x-3=(2x+1)(x-3)$, then

$$\frac{x+4}{2x^2-5x-3} = \frac{A}{2x+1}+\frac{B}{x-3}.$$

To find A and B, we find a common denominator on the right-hand side, so $A(x-3)+B(2x+1) = x+4$. That is, $A+2B=1$ and $-3A+B=4$. Solving gives $A=-1$ and $B=1$. Thus,

$$\int \frac{x+4}{2x^2-5x-3}\,dx = \int \frac{-1}{2x+1}-\frac{1}{x-3}\,dx = -\frac{1}{2}\ln|2x-1|-\ln|x+3|+C.$$

There is a shortcut for finding A and B. The reader is cautioned, however, that this method is only effective when all factors are distinct.

The equation $A(x-3)+B(2x+1)=x+4$ holds for *all* x. In particular, it holds for $x=3$. Making this substitution gives

$$A(3-3)+B(2\cdot(3)+1) = 3+4 \Rightarrow 7B=7 \Rightarrow B=1.$$

Similarly, choosing $x=-1/2$ gives

$$A(-1/2-3)+B(2(-1/2)-1) = (-1/2)+4 \Rightarrow (-7/2)A = 7/2 \Rightarrow A=-1.$$

The next case is when the denominator is a product of linear factors but some are repeated. We treat each power of that factor as an individual factor.

Partial Fraction Decomposition (one repeated factor)

Assume the a_i are all distinct and $r > 1$. Then we decompose the rational function as

$$\frac{P}{(x-a_1)^r(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_r}{(x-a_1)^r} + \frac{B_2}{x-a_2} + \cdots + \frac{B_n}{x-a_n}.$$

Example. We evaluate $\displaystyle\int \frac{x^2-2x-2}{x^3+2x^2+x}\,dx$ using PFD. We have

$$\frac{x^2-2x-2}{x^3+2x^2+x} = \frac{x^2-2x-2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

This leads to the equation

$$A(x+1)^2 + Bx(x+1) + Cx = x^2 - 2x - 2.$$

If we set $x = 0$, then $A(1)^2 = -2$ so $A = -2$. If we set $x = -1$, then $C(-1) = (-1)^2 - 2(-1) - 2$, so $C = -1$. We cannot use this trick to find B, but we can substitute for A and C,

$$-2(x^2+2x+1) + B(x^2+x) - 1(x) = x^2 - 2x - 2.$$

Comparing the coefficient of x^2 gives $-2 + B = 1$, so $B = 3$. Now

$$\int \frac{x^2-2x-2}{x^3+2x^2+x}dx = \int \frac{-2}{x} + \frac{3}{x+1} - \frac{1}{(x+1)^2}\,dx = -2\ln|x| + 3\ln|x+1| + \frac{1}{x+1} + C.$$

If the denominator contains an irreducible quadratic factor, then we include the term

$$\frac{Cx+D}{ax^2+bx+c}$$

and use the integration,

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Example. We evaluate $\displaystyle\int \frac{3x^3+3x^2-3x-1}{x^4+x^2}\,dx$ using PFD. Write

$$\frac{3x^3+3x^2-3x-1}{x^4+x^2}\,dx = \frac{3x^3+3x^2-3x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}.$$

This implies the equation

$$Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = 3x^3 + 3x^2 - 3x - 1.$$

Let $x = 0$, which implies $B = -1$. Now we have

$$\begin{aligned} Ax(x^2+1) - (x^2+1) + (Cx+D)x^2 &= 3x^3 + 3x^2 - 3x - 1 \\ Ax(x^2+1) + (Cx+D)x^2 &= 3x^3 + 4x^2 - 3x. \end{aligned}$$

The only coefficient of x^2 on the left is D, so $D = 4$. The only coefficient of x on the left is A, so $A = -3$. Comparing coefficients of x^3 gives $A + C = 3$, so $C = 6$. Now we have

$$\begin{aligned}
\int \frac{3x^3 + 3x^2 - 3x - 1}{x^2(x^2+1)}\ dx &= \int -\frac{3}{x} - \frac{1}{x^2} + \frac{6x+4}{x^2+1}\ dx \\
&= -3\ln|x| + \frac{1}{x} + \int \frac{6x}{x^2+1} + \frac{4}{x^2+1}\ dx \\
&= -3\ln|x| + \frac{1}{x} + 3\ln|x^2+1| + 4\arctan(x) + C.
\end{aligned}$$

Some integrals are still beyond algebraic methods for integration. At times we still need to resort to the definition of integration to approximate a definite integral.

Recall that we can divide the domain of integration $[a, b]$ into a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of regular subintervals, so $\Delta x = (b - a)/n$. Then we can approximate the integral using left or right-hand endpoints,

$$\int_a^b f(x)\ dx \approx L_n = \sum_{i=1}^n f(x_{i-1})\Delta x, \qquad \int_a^b f(x)\ dx \approx R_n = \sum_{i=1}^n f(x_i)\Delta x.$$

There are other methods which may at times give us a better result.

Midpoint Rule

Let P be a regular partition of $[a, b]$. Let $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$ be the midpoint of the interval $[x_{i-1}, x_i]$. Then

$$\int_a^b f(x)/dx \approx M_n = \Delta x \sum_{i=1}^n f(\overline{x_i}).$$

The actual value of an integral lies between L_n and R_n, so we can get a reasonable approximation of the actual value by averaging the two. This is the Trapezoidal Rule. Some integrals are better approximated with trapezoids instead of rectangles.

At points $y_i = f(x_i)$ and $y_j = f(x_j)$, we can find the area of the trapezoid formed by connecting y_i and y_j. The area of this trapezoid is $A = \frac{1}{2}(x_j - x_i)(y_i + y_j)$. Thus, if we have N regular intervals of length Δx, then we can approximate the area under f by

$$\begin{aligned} T_N &= \frac{1}{2}\Delta x(f(x_0) + f(x_1)) + \frac{1}{2}\Delta x(f(x_1) + f(x_2)) + \cdots + \frac{1}{2}\Delta x(f(x_{N-1}) + f(x_N)) \\ &= \frac{1}{2}\Delta x\left((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{N-1}) + f(x_N))\right) \\ &= \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{N-1}) + f(x_N)\right). \end{aligned}$$

Trapezoidal Rule

Let P be a regular partition of $[a, b]$ and $x_i = a + i\Delta x$. Then

$$\int_a^b f(x)\ dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Example. We will approximate the integral $\displaystyle\int_1^4 \frac{dx}{x}$ using the midpoint rule and trapezoidal rule with $N = 6$.

The partition is $\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$ and the midpoints of the partition are $\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}$.

We have $\Delta x = \frac{4-1}{6} = \frac{1}{2}$ so the midpoint rule gives,

$$\begin{aligned}
M_6 &= \frac{1}{2}\left(f(5/4) + f(7/4) + f(9/4) + f(11/4) + f(13/4) + f(15/4)\right) \\
&= \frac{1}{2}\left(\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \frac{4}{13} + \frac{4}{15}\right) = \frac{62024}{45045} \approx 1.37693.
\end{aligned}$$

The trapezoidal rule gives

$$\begin{aligned}
T_6 &= \frac{1/2}{2}\left(f(1) + 2f(3/2) + 2f(2) + 2f(5/2) + 2f(3) + 2f(7/2) + f(4)\right) \\
&= \frac{1}{4}\left(1 + 2\left(\frac{2}{3}\right) + 2\left(\frac{1}{2}\right) + 2\left(\frac{2}{5}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{7}\right) + \frac{1}{4}\right) \\
&= \frac{1}{4}\left(1 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{4}\right) = \frac{787}{560} \approx 1.40536
\end{aligned}$$

The actual value is $\ln(4) \approx 1.38629$.

The trapezoidal rule and the midpoint rule are comparable in certain circumstances.

- If $f(x)$ is concave up then $M_n \leq \int_a^b f(x)\, dx \leq T_n$.
- If $f(x)$ is concave down then $T_n \leq \int_a^b f(x)\, dx \leq M_n$.

If we take the average of the two, then we might hope to reduce our error.

Simpson's Rule

Let P be a regular partition of $[a,b]$ and suppose n is even and $x_i = a + i\Delta x$. Then

$$\int_a^b f(x)\, dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \right.$$
$$\left. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Example. We will approxmiate the integral $\int_1^4 \frac{dx}{x}$ using Simpson's rule with $N = 6$ ($\Delta x = \frac{1}{2}$),

$$\begin{aligned}
S_6 &= \frac{1/2}{3}\left(f(1) + 4f(3/2) + 2f(2) + 4f(5/2) + 2f(3) + 4f(7/2) + f(4)\right) \\
&= \frac{1}{6}\left(1 + 4\left(\frac{2}{3}\right) + 2\left(\frac{1}{2}\right) + 4\left(\frac{2}{5}\right) + 2\left(\frac{1}{3}\right) + 4\left(\frac{2}{7}\right) + \frac{1}{4}\right) = \frac{3497}{2520} \approx 1.38770.
\end{aligned}$$

There is a shortcut for computing Simpson's Rule in terms of the trapezoidal and midpoint rule:

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n.$$

In our previous example, we could get

$$S_{12} = \frac{1}{3}T_6 + \frac{2}{3}M_6 = \frac{2997637}{2162160} = 1.38641.$$

Consider the two integrals, $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$. We can evaluate these integrals by rewriting the integral as a limit. Though they look similar, the two have very different answers,

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}(\ln t - \ln 1) = \infty$$
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}(1 - \frac{1}{t}) = 1.$$

Improper Integrals, Type I

(1) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx.$

(2) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx.$

An improper integral $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ are called *convergent* if the corresponding limit exists and is finite, and *divergent* if the limit does not exist or is infinite.

Example. We evaluate $\displaystyle\int_{-\infty}^{-1} e^{2x}\,dx$ by rewriting as a limit,

$$\int_{-\infty}^{-1} e^{2x}\,dx = \lim_{t\to-\infty}\int_t^{-1} e^{2x}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}e^{2x}\right]_t^{-1} = \frac{1}{2}\lim_{t\to-\infty}\left[e^{-2} - e^{2t}\right] = \frac{1}{2}e^{-2}.$$

Example (p-integral Type I)**.** Let p be a real number and consider the integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$. We have already consider the case of $p = 1$, so suppose $p \neq 1$. Then,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^t = \frac{1}{1-p}\lim_{t\to\infty}\left[t^{1-p} - 1\right] = \frac{1}{p-1} + \lim_{t\to\infty}\frac{1}{t^{p-1}}.$$

If $p - 1 > 0$ (so $p > 1$), then the limit goes to 0. Otherwise, the limit goes to ∞. This gives

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Example. Consider $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\,dx$. First (always first!) we rewrite as a limit,

$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx.$$

We will employ IBP. Let $u = \ln x$, $dv = x^{-3}\,dx$ so $du = x^{-1}\,dx$ and $v = -\frac{1}{2}x^{-2}$. Then,

$$\begin{aligned}\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx &= \lim_{t\to\infty}\left[(\ln x)\left(-\frac{1}{2}x^{-2}\right)\right]_1^t + \frac{1}{2}\int_1^t x^{-3}\,dx \\ &= \lim_{t\to\infty} -\frac{1}{2}\frac{\ln t}{t^2} + \frac{1}{2}\left[-\frac{1}{2}x^{-2}\right]_1^t = \lim_{t\to\infty} -\frac{1}{2}\frac{\ln t}{t^2} - \frac{1}{4}\left[t^{-2} - 1\right] = \frac{1}{4}.\end{aligned}$$

If both $\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^b f(x)\ dx$ are convergent, then we define

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_a^\infty f(x)\ dx + \int_{-\infty}^b f(x)\ dx.$$

Example. Consider $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\ dx$. We break up the integral and evaluate,

$$\begin{aligned}\int_{-\infty}^{\infty} xe^{-x^2}\ dx &= \int_{-\infty}^{0} xe^{-x^2}\ dx + \int_0^\infty xe^{-x^2}\ dx = \lim_{s\to-\infty}\int_s^0 xe^{-x^2}\ dx + \lim_{t\to\infty}\int_0^t xe^{-x^2}\ dx \\ &= \lim_{s\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_s^0 + \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^t = -\frac{1}{2}\left[\lim_{s\to-\infty}\left(1-e^{-s^2}\right) + \lim_{t\to\infty}\left(e^{-t^2}-1\right)\right] \\ &= -\frac{1}{2}\left[(1-0)+(0-1)\right] = 0.\end{aligned}$$

Suppose a function $f(x)$ is continuous except at a point $x = b$. We can evaluate $\int_a^b f(x)\ dx$ by turning it into a limit approaching b.

Improper Integrals, Type II

(1) If f is continuous on $[a, b)$ and is discontinuous at b, then $\displaystyle\int_a^b f(x)\ dx = \lim_{t\to b^-}\int_a^t f(x)\ dx$.

(2) If f is continuous on $(a, b]$ and is discontinuous at a, then $\displaystyle\int_a^b f(x)\ dx = \lim_{t\to a^+}\int_t^b f(x)\ dx$.

An improper integral $\int_a^b f(x)\ dx$ is called *convergent* if the corresponding limit exists and is finite, and *divergent* if the limit does not exist or is infinite.

Example. Consider $\displaystyle\int_0^1 \frac{dx}{x}$. Since the function does not exist at $x = 0$, we set

$$\int_0^1 \frac{dx}{x} = \lim_{t\to0^+}\int_t^1 \frac{dx}{x} = \lim_{t\to0^+}\left[\ln x\right]_t^1 = \lim_{t\to0^+}(\ln 1 - \ln t) = \lim_{t\to0^+}\ln t = \infty,$$

so the integral diverges.

Example. Consider $\displaystyle\int_2^3 \frac{dx}{\sqrt{3-x}}$. Since the function does not exist at $x = 3$, we set

$$\int_2^3 \frac{dx}{\sqrt{3-x}} = \lim_{t\to3^-}\int_2^t \frac{dx}{\sqrt{3-x}} = \lim_{t\to3^-}\left[-2\sqrt{3-x}\right]_2^t = \lim_{t\to3^-}\left(-2\sqrt{3-t}+2\right) = 2.$$

Example. Consider $\int_0^1 \frac{\ln x}{\sqrt{x}}\ dx$. Since the function does not exist at $x = 0$, we set

$$\begin{aligned}\int_0^1 \frac{\ln x}{\sqrt{x}}\ dx &= \lim_{t\to0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\ dx = \lim_{t\to0^+}\left[\left[(\ln x)(2\sqrt{x})\right]_t^1 - \int_t^1 2\frac{\sqrt{x}}{x}\ dx\right] \\ &= \lim_{t\to0^+}\left[-2\sqrt{t}\ln t - 2\int_t^1 \frac{1}{\sqrt{x}}\ dx\right] = \lim_{t\to0^+}\left[-2\sqrt{t}\ln t - 4\left[\sqrt{x}\right]_t^1\right] \\ &= \lim_{t\to0^+}\left[-2\sqrt{t}\ln t - 4(1-\sqrt{t})\right] = -4 - 2\lim_{t\to0^+}\sqrt{t}\ln t.\end{aligned}$$

To evaluate the remaining integral, we use L'Hospital's rule.

$$\lim_{t\to 0^+} \sqrt{t}\ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-1/2}} = \lim_{t\to 0^+} \frac{1/t}{-\frac{1}{2}t^{-3/2}} = -2\lim_{t\to 0^+} \sqrt{t} = 0.$$

Hence, $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = -4$.

Example (p-integral Type II)**.** Let p be a real number and consider the integral $\displaystyle\int_0^1 \frac{dx}{x^p}$. We have already consider the case of $p = 1$, so suppose $p \neq 1$. Then,

$$\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \left[\frac{1}{-p+1}x^{-p+1}\right]_t^1 = \frac{1}{1-p}\lim_{t\to 0^+} \left(1 - t^{1-p}\right).$$

The limit is 0 if $p < 1$ and otherwise it is infinite. This gives

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1. \end{cases}.$$

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Another way to evaluate improper integrals is to compare them to other functions in which we know the convergence.

Comparison Theorem (for Type I integrals)

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.
(1) If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$.
(2) If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.
(3) If $\int_a^\infty g(x)\,dx$ converges or $\int_a^\infty f(x)\,dx$ diverges, then the theorem gives us no information.

A similar result holds for Type II integrals.

Example. Consider $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$. Since $2 + e^{-x} \geq 1$ for all $x \geq 1$, then

$$\frac{2+e^{-x}}{x} \geq \frac{1}{x} \text{ for all } x \geq 1.$$

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, then by the Comparison Theorem so does the given integral.

Example. Consider $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$. For $0 \leq x \leq 1$, $0 \leq e^{-x} \leq 1$. Thus, on the given interval,

$$0 \leq \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, then by the Comparison Theorem so does the given integral.

Applications of Integration

6.2 Volume

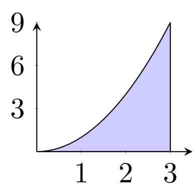
Let S be a solid. We take a vertical slice at some point x_i^* in our interval. At that point, the object will have base area $A(x_i^*)$ and width Δx (assuming regular subintervals). This gives us that the volume on that interval can be approximated by $V_i = A(x_i^*)\Delta x$. We can approximate by summing these areas at sample points in each interval.

Volume of a solid S

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional areas of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the *volume* of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\ dx.$$

Example. Consider the region bounded by $y = x^2$ and $x = 3$.



When this region is rotated about the x-axis, the area of each cross-section $A(x)$ is $\pi(f(x))^2 = \pi x^4$. Hence,

$$V = \int_0^3 \pi x^4 = \pi \left[\frac{1}{5}x^5\right]_0^3 = \frac{243}{5}\pi.$$

The previous example demonstrates a general procedure for finding volumes of objects. We call these objects *solids of revolution.*

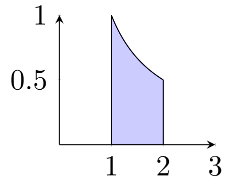
Disk method

If $f(x) \geq 0$ is continuous on $[a, b]$, then the solid obtained by rotating the region under the graph about the x-axis has volume

$$V = \int_a^b \pi f(x)^2\ dx.$$

These notes are derived primarily from *Calculus, Early Transcendentals* by James Stewart (8ed). Most of this material is drawn from Chapters 6 and 8. Last Updated: March 3, 2023

Example. Consider the region bounded by $y = 1/x$, $x = 1$, $x = 2$, and $y = 0$ (the x-axis).



When the region is rotated about the x-axis, the radius of the disk at x is given by $1/x$. Thus,

$$V = \int_1^2 \pi \left(\frac{1}{x}\right)^2 \, dx = \pi \left[-\frac{1}{x}\right]_1^2 = \frac{1}{2}\pi.$$

When we have a region bounded by two curves, we can think about one of them cutting out a shape from the other.

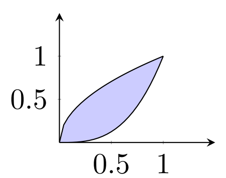
Washer method

Suppose a region is bounded between the curves $y = f(x)$ and $y = g(x)$, both continuous, with $f(x) \geq g(x)$ on $[a, b]$. The volume obtained by rotating the region about the x-axis is

$$V = \int_a^b \pi \left((R_{\text{outer}})^2 - (R_{\text{inner}})^2\right) = \int_a^b \pi \left(f(x)^2 - g(x)^2\right).$$

Be careful! Many students make the mistake of integrating the square of the difference, and not the difference of the squares.

Example. Consider the region enclosed by the curves $y = x^3$ and $y = \sqrt{x}$. The curves intersect at $x = 0$ and $x = 1$.



We rotate this region about the x-axis. The outer radius is given by $\sqrt{x}$ and the inner radius is given by x^3. Thus, the volume is

$$V = \int_0^1 \pi \left((\sqrt{x})^2 - (x^3)^2\right) \, dx = \pi \int_0^1 x - x^6 \, dx = \pi \left[\frac{1}{2}x^2 - \frac{1}{7}x^7\right]_0^1 = \frac{5\pi}{14}.$$

Thus far, we have confined ourselves to rotating about the x-axis (the line $y = 0$). But now we are in a position to generalize this. To rotate about the line $y = c$, the inner and outer radii need to be measured relative to the line $y = c$:

- If $f(x) \geq g(x) \geq c$, then $R_{outer} = f(x) - c$ and $R_{inner} = g(x) - c$.
- If $c \geq f(x) \geq g(x)$, then $R_{outer} = c - g(x)$ and $R_{inner} = c - f(x)$.

We then proceed as before.

Example. Consider the region from the previous example. The only difference here is that we rotate about the line $y = 2$. Now, the graph $y = \sqrt{x}$ is 'closer' to our axis. The distance from $\sqrt{x}$ to $y = 2$ at any point x is $R_{inner} = 2 - \sqrt{x}$. Similarly, the difference from x^3 to $y = 2$ at any point x is $R_{outer} = 2 - x^3$. Thus, the volume is

$$\begin{aligned}
V &= \int_0^1 \pi \left((2 - x^3)^2 - (2 - \sqrt{x})^2\right)\, dx \\
&= \pi \int_0^1 (4 - 4x^3 + x^6) - (4 - 4\sqrt{x} + x)\, dx \\
&= \pi \int_0^1 -4x^3 + x^6 + 4\sqrt{x} - x\, dx \\
&= \pi \left[-x^4 + \frac{1}{7}x^7 + \frac{8}{3}x^{3/2} - \frac{1}{2}x^2\right]_0^1 = \frac{55\pi}{42}.
\end{aligned}$$

The last case to cover is rotation about the y-axis. This requires integrating with respect to y as opposed to x. Otherwise the method is identical.

Example. Consider the region bounded by $y = \ln x$, $x = e$, and the x-axis. We will rotate this region about the (vertical) line $x = -1$. Since $y = \ln x$, then $x = e^y$ and we will integrate with respect to y. We have $R_{outer} = e - (-1) = e + 1$ and $R_{inner} = e^y - (-1) = e^y + 1$. The volume is

$$\begin{aligned}
V &= \int_0^1 \pi \left((e + 1)^2 - (e^y + 1)^2\right)\, dy \\
&= \pi \int_0^1 (e^2 + 2e + 1) - (e^{2y} + 2e^y + 1)\, dy \\
&= \pi \int_0^1 e^2 + 2e - e^{2y} - 2e^y\, dy = \pi \left[y(e^2 + 2e) - \frac{1}{2}e^{2y} - 2e^y\right]_0^1 \\
&= \pi \left[\left((e^2 + 2e) - \frac{1}{2}e^2 - 2e\right) - \left(-\frac{1}{2} - 2\right)\right] \\
&= \frac{\pi}{2}\left(e^2 + 5\right).
\end{aligned}$$

Example. Consider a right circular cone with base of radius 4 and height 10. We imagine this cone lying on its side, so the horizontal slices will be cylinders. Using similar triangles, we find that the radius of the base of the i^{th} slice is $r_i = \frac{2}{5}x_i$. Hence, the area of base of the i^{th} slice is $A(x_i) = \pi(2x_i/5)^2$. Hence, the volume of a cylindrical slice is

$$V_i = \frac{4\pi}{25}x_i^2 \Delta x.$$

Working from the bottom to the top, we find that the volume is

$$V = \int_0^{10} \frac{4\pi}{25}x^2 dx = \frac{4\pi}{25}\left[\frac{1}{3}x^3\right]_0^{10} = \frac{4\pi}{25} \cdot \frac{1000}{3} = \frac{160\pi}{3}$$

Compare this with the result using the formula $V = \frac{1}{3}\pi r^2 h$.

6.4 Work

Let $s(t)$ be the position function of an object moving on a straight line. Then $v(t) = s'(t)$ is the *velocity* of that object and $a(t) = v'(t) = s''(t)$ is the *acceleration.* Newton's Second Law of Motion tells us that the *force* on an object is the product of its mass and its acceleration,

$$F = ma = m\frac{d^2s}{dt^2}.$$

For example, we can use this to measure the effect of Earth's gravity on a falling object.

If acceleration is assumed to be constant, then force F is also constant. In this case, the work done is the product of the force F and the distance d the objects moves, so

$$W = Fd.$$

If F is measured in newtons and d in meters, then the unit for W is newton-meter, which is called a *joule* (J). If F is measured in pounds and d in feet, then the unit for W is foot-pound (ft-lb). (1 foot-pound is approximately $1.26J$.)

Example. Suppose a 30-lb weight is lifted 5 feet off the ground. Then the work done is

$$W = Fd = (30 \text{ lb})(6 \text{ ft}) = 150 \text{ ft-lb}.$$

Now suppose we have a 1.4 kg book lying on the floor and we want to place it on a desk that is 0.5m high. In this case, we are given the mass but not the weight (which is a force). So to compute force we note that acceleration due to gravity is $g = 9.8\ m/s^2$, so

$$F = mg = (1.4)(9.8) = 13.72N.$$

Now the work done is

$$W = Fd = (13.72N)(0.5m) \approx 6.86J.$$

In the above we assumed that force was constant. Now suppose force is described by a function $f(x)$. We can approximate the work by assuming f is constant on small intervals and taking the sum of the approximations on each interval.

Work

Suppose an object moves along the x-axis in the positive direction, from $x = a$ to $x = b$, and at each point x between a and b a force $f(x)$ acts on the object, where f is a continuous function. Take a partition of $[a, b]$ with regular subintervals of length Δx. Then

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\ dx.$$

Example. Consider a particle that is located a distance of x meters from the origin such that a force of $\cos(\pi x/3)$ Newtons acts on it. Suppose the particle is moved from $x = 1$ to $x = 2$. Then we have

$$W = \int_1^2 \cos(\pi x/3)\ dx = \frac{3}{\pi}\left[\sin(\pi x/3)\right]_1^2 = \frac{3}{\pi}(\sin(2\pi/3) - \sin(\pi/3)) = 0.$$

This may seem like a strange answer, but consider that

$$W = \int_1^{1.5} \cos(\pi x/3)\ dx = \frac{3}{\pi}\left(1 - \frac{\sqrt{3}}{2}\right) \quad \text{and} \quad W = \int_{1.5}^{2} \cos(\pi x/3)\ dx = -\frac{3}{\pi}\left(1 - \frac{\sqrt{3}}{2}\right).$$

According to *Hooke's Law*, the force required to *maintain* a spring stretched x units beyond its natural length is proportional to x: $f(x) = kx$, where k is a positive constant called the *spring constant*. (Hooke's Law holds provided x is not too large.)

Example. Suppose a 25 N force to keep a spring of (natural) length 20 cm stretched to a length of 30 cm. The spring is stretched $10cm = 0.1m$, so $f(0.1) = 25$. Then $0.1k = 25$, which implies that $k = 250$. This means that $f(x) = 250x$ and the work done in stretching the spring from 30 cm to 40 cm is

$$W = \int_{0.1}^{0.2} 250x\ dx = \left[125x^2\right]_{0.1}^{0.2} = 5 - 1.25 = 3.75J.$$

Suppose instead we were told that it takes a 25 N force to *stretch* the spring from its natural length of 20 cm to a length of 30 cm. Then instead we would have

$$25 = \int_0^{.1} kx\ dx = \frac{k}{2}\left[x^2\right]_0^{.1} = 0.005k \Rightarrow k = 5000.$$

Example. A circular swimming pool has a diameter of 24 ft. The sides are 5 ft high, and the depth of the water is 4 ft. We want to measure the work required to pump all of the water out over the side. It will be helpful to recall that water weighs 62.5 lb/ft^3.)

We divide the pool into cylindrical segments of height Δx. These segments have radius 12 ft and so the volume of one of these segments is $144\pi\Delta x$. The segment weights about

$$(62.5 \text{ lb/ft}^3)(144\pi\Delta x \text{ ft}^3) = 9000\pi\Delta x \text{ lb}.$$

The slice lies x_i^* feet below the edge of the pool, where $1 \leq x_i^* \leq 5$, then the work needed to pump it out is about $9000\pi x_i^*\Delta x$. So we have

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 9000\pi x_i^*\Delta x = \int_1^5 9000\pi x dx = 9000\pi\left[\frac{1}{2}x^2\right]_1^5 = 108,000\pi \text{ ft-lb}.$$

8.1 Arc Length

Imagine we are trying to measure the length of a curve C. We could approximate the length by marking points $\{P_i = (x_i, y_i)\}$ on the lines and connecting them with line segments. We let $|P_{i-1}P_i|$ denote the length of the line segment from P_{i-1} to P_i. Then if L is the length of the curve C, and we assume we have a regular partition, we get

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

This is not very easy to compute, so we will try to evaluate it another way. Note that,

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

The Mean Value Theorem on $[x_{i-1}, x_i]$ gives,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

Or, in other terms, $\Delta y_i = f'(x_i^*)\Delta x$. Hence,

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + (f'(x_i^*)\Delta x)^2} = \sqrt{1 + (f'(x_i^*))^2}\Delta x.$$

This gives,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\ dx.$$

Arc Length

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx.$$

Alternatively, if f is a function of y then the length of curve $x = f(y)$, $a \le y \le b$, is

$$L = \int_a^b \sqrt{1 + (f'(y))^2}\ dy.$$

Example. Consider the curve $y^2 = 4(x + 4)^3$ on the interval $0 \le x \le 2$ and $y \ge 0$. Taking square roots (we need only consider the positive part since $y \ge 0$) we have $y = f(x) = 2(x + 4)^{3/2}$. Then $f'(x) = 3(x + 4)^{1/2}$. Using the arc length formula we have

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\ dx = \int_0^2 \sqrt{1 + 9(x + 4)}\ dx \\ &= \int_0^2 \sqrt{9x + 37}\ dx \quad \text{set } u = 9x + 37 \text{ so } du = 9\ dx \\ &= \frac{1}{9}\int_{37}^{55} \sqrt{u}\ du = \frac{1}{9}\left[\frac{2}{3}x^{3/2}\right]_{37}^{55} = \frac{2}{27}\left(55^{3/2} - 37^{3/2}\right). \end{aligned}$$

Example. Consider the curve $y^2 = 4x$ on the interval $0 \le y \le 2$. Set $x = f(y) = \frac{1}{4}y^2$. Then $f'(y) = \frac{1}{2}y$. Using the arc length formula we have

$$\begin{aligned}
L &= \int_a^b \sqrt{1 + (f'(y))^2}\ dy \\
&= \int_0^2 \sqrt{1 + \frac{1}{4}y^2}\ dx \quad \text{set } y = 2\tan\theta \text{ so } dy = 2\sec^2\theta\ d\theta \\
&= 2\int_0^{\pi/4} \sqrt{1 + \tan^2\theta}\,\sec^2\theta\ d\theta \\
&= 2\int_0^{\pi/4} \sec^3\theta\ d\theta \\
&= 2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} \\
&= \sqrt{2} + \ln(\sqrt{2} + 1).
\end{aligned}$$

Sequences and Series

11.1 Sequences

A *sequence* is a list of numbers written in a definite order,

$$\{a_1, a_2, a_3, \dots\} = \{a_n\}_{n=1}^{\infty}.$$

We call a_n the *general term* of the sequence.

Example. Assuming that the pattern of the first few terms continues, find a formula for the general term a_n of each sequence.

$$\text{(1)} \quad \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\text{(2)} \quad \left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\right\} = \left\{(-1)^n \frac{n}{(n+1)^2}\right\}_{n=1}^{\infty}$$

$$\text{(3)} \quad \left\{\frac{1}{4}, -\frac{2}{9}, \frac{3}{16}, -\frac{4}{25}, \dots\right\} = \left\{(-1)^{n+1} \frac{n}{(n+1)^2}\right\}_{n=1}^{\infty}$$

Some sequences (actually, many sequences) do not have a simple formula. Consider the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$. This is an example of a *recursive sequence*. There is a closed formula expression for the nth Fibonacci number $F(n)$ but it is harder to write down than the previous problem. Let $\phi = (1+\sqrt{5})/2$ (the golden ratio) and $\psi = (1-\sqrt{5})/2$, then

$$F(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

Informally, a sequence $\{a_n\}$ has the *limit* L if we can make the terms a_n as close to L as we like by taking n sufficiently large. We write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

If the limit exists, we say the sequences *converges*. Otherwise, we say the sequence *diverges*.

Definition: Limit (formal)

A sequence $\{a_n\}$ has the *limit* L if for every $\varepsilon > 0$ there is a corresponding number $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n > N$. The expression $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that $a_n > M$ for all $n > N$.

These notes are derived primarily from *Calculus, Early Transcendentals* by James Stewart (8ed). Most of this material is drawn from Chapter 11. Last Updated: April 12, 2023

Example (Geometric sequences)**.** A sequence of the form $\{r^n\}$ is called *geometric.* If $r = 1$, then this is just the sequence of 1's, and so it converges to 1. When $-1 < r < 1$, the sequence converges to 0. The sequence diverges otherwise.(When $r > 1$, the sequence diverges to ∞.)

Let $\{a_n\}$ be a sequence. If f is a function such that $f(n) = a_n$ for each integer n, then we say the function f *models* the sequence $\{a_n\}$. We can think of the function f as *tracing* the ordered pairs (n, a_n). If a function f models the sequence $\{a_n\}$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.

We need to be careful however, as this rule does not apply when f diverges. Consider the sequence $\{\sin(\pi n)\}$. This sequence converges to 0, in fact, every term in the sequence is 0. However, while $f(x) = \sin(\pi x)$ does model $\{a_n\}$, the limit of $f(x)$ as $x \to \infty$ does not exist.

Example. We can model the sequence $a_n = \frac{n}{\sqrt{n^2+1}}$ with the function $f(x) = \frac{x}{\sqrt{x^2+1}}$. Now,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2+1}} = \sqrt{\lim_{x\to\infty} \frac{x^2}{x^2+1}} = \sqrt{1} = 1.$$

Example. We can model the sequence $a_n = \left(1+\frac{1}{n}\right)^n$ with the function $f(x) = \left(1+\frac{1}{x}\right)^x$. Set $y = f(x)$. Since

$$\ln y = x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

then,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/\left(1+\frac{1}{x}\right)\cdot\frac{-1}{x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

Now,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e.$$

The next theorem is useful in determining the rations of rational functions

Theorem 1: Limits of rational functions

Let $a_n = \frac{p(n)}{q(n)}$ where $p(n)$ and $q(n)$ are polynomials in n with leading coefficients b and c, respectively. Then

$$\lim_{n\to\infty} a_n = \begin{cases} 0 & \text{if } \deg q(n) > \deg p(n) \\ \infty & \text{if } \deg p(n) > \deg q(n) \\ \frac{b}{c} & \text{if } \deg p(n) = \deg q(n). \end{cases}$$

Example. Consider the sequence $\left\{\dfrac{n!}{(n+2)!}\right\}$. We rewrite the general term to evaluate,

$$\frac{n!}{(n+2)!} = \frac{n!}{(n+2)(n+1)n!} = \frac{1}{(n+2)(n+1)} \to 0.$$

We'll now discuss several theorems that help us to evaluate limits of sequences. Many of these should remind you of corresponding theorems for limits of functions.

Theorem 2: Limit Laws for sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

(1) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

(2) $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} c = c$

(3) $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(4) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} b_n \neq 0$

Theorem 3: Squeeze Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that for some number M,

$$b_n \leq a_n \leq c_n \text{ for } n > M \quad \text{and} \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L,$$

then $\lim\limits_{n\to\infty} a_n = L$.

Example. Consider the sequence with general term $a_n = 1/\sqrt{n^4+n^8}$. We can bound a_n by

$$\frac{1}{\sqrt{2}n^4} = \frac{1}{\sqrt{n^8+n^8}} \leq a_n \leq \frac{1}{\sqrt{n^4+n^4}} = \frac{1}{\sqrt{2}n^2}.$$

Each of these sequences converges to 0 and so by the Squeeze Theorem, so does $\{a_n\}$.

For a sequence $\{a_n\}$, we can consider the sequence of absolute values $\{|a_n|\}$. Suppose $|a_n| \to 0$. Then of course $-|a_n| \to 0$. Since $-|a_n| \leq a_n \leq |a_n|$, then by the Squeeze Theorem, $a_n \to 0$.

Theorem 4: Absolute value sequence

If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

The previous theorem *only* works if the limit is 0. The sequence $\{(-1)^n\} = \{1, -1, 1, -1, 1, -1, \ldots\}$ diverges, but its absolute value sequence converges to 1.

Theorem 5: Limits and continuous functions

If $f(x)$ is continuous and $\lim\limits_{n\to\infty} a_n = L$ then $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right) = f(L)$.

Example. Since $\sin(x)$ is continuous and $\lim_{n\to\infty} 1/n = 0$, then

$$\lim_{n\to\infty} \sin(1/n) = \sin\left(\lim_{n\to\infty} 1/n\right) = \sin(0) = 0.$$

Consider the process of adding all of the positive integers. We could do this by hand,

$$1 + 2 + 3 + \cdots,$$

but that would take a while, and we're pretty sure its infinity. Thankfully, there is a formula that tells us the sum of the first n integers

$$S_n = \frac{n(n+1)}{2}.$$

The total sum is just the limit of these partial sums, so

$$\sum a_n = \lim_{n\to\infty} S_n = \infty.$$

On the other hand, consider the sequence given by $\{\frac{1}{2^n}\}$. Summing the terms gives us

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

We can think of this like taking a square, adding half, then half of the remainder, then half of the remainder, etc. The point is that with each term we get closer and closer to 1. If we look at the partial sums, we see the sequence

$$\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\} = \left\{\frac{2^n - 1}{2^n}\right\} = \left\{1 - \frac{1}{2^n}\right\}$$

which clearly converges to 1.

Definition: Series, partial sum, sum, convergent, divergent

A *series* (or infinite series) is the infinite sum of the terms in a sequence $\{a_n\}$. We denote it

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum a_n.$$

Let S_N denote its Nth *partial sum*, that is,

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

If the sequence $\{S_N\}$ is convergent and $\lim_{N\to\infty} s_N = S$ exists, we say the series $\sum a_n$ is *convergent* and write $\sum a_n = S$. The number S is called the *sum* of the series. If the sequence $\{S_n\}$ is divergent, then the series is called *divergent.*

We will often use the notation,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

We can add and subtract series termwise (see page 551). Moreover, a scalar multiple of an infinite series is the same as the infinite series of the terms multiplied by that scalar.

Example. Geometric series Consider a geometric sequence $\{cr^n\}$. The corresponding series is called a *geometric series*: $\displaystyle\sum_{n=1}^{\infty} cr^{n-1}$.

We will assume $c \neq 0$. If $r = 1$ then the series clearly diverges, so we suppose that $r \neq 1$. Let S_N be the Nth partial sum of the series. Then

$$\begin{aligned}
S_N &= c + cr + cr^2 + \cdots + cr^{N-1} \\
rS_N &= cr + cr^2 + \cdots + cr^N \\
S_N - rS_N &= c - cr^N \\
S_N(1-r) &= c(1-r^N).
\end{aligned}$$

This shows that the Nth partial sum of a geometric series is given by

$$S_N = \frac{c(1-r^N)}{1-r}.$$

Then we have

$$S = \lim_{n\to\infty} S_N = \lim_{n\to\infty} \frac{c\left(1-r^N\right)}{1-r} = \frac{c}{1-r} - \frac{c}{1-r}\lim_{n\to\infty} r^N.$$

By our work in the last section, we know that this series converges if $|r| < 1$ and the sum is

$$\sum_{n=1}^{\infty} cr^{n-1} = \frac{c}{1-r}.$$

If $|r| \geq 1$, the geometric series diverges.

Example. The series $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ is geometric with $r = -3/2$. Since $|r| > 1$, this series diverges.

Example. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$. We rewrite the series as

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} &= 4 + \sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n} = 4 + \sum_{n=1}^{\infty} \frac{4^2 \cdot 4^{n-1}}{5 \cdot 5^{n-1}} = 4 + \sum_{n=1}^{\infty} \frac{16}{5} \cdot \left(\frac{4}{5}\right)^{n-1} \\
&= 4 + \frac{16/5}{1-(4/5)} = 4 + 16 = 20.
\end{aligned}$$

Example. Consider the real number $6.2\overline{54} = 6.254545454\cdots$, which we can write as

$$\begin{aligned}
6.2\overline{54} &= 6.2 + \frac{54}{1000} + \frac{54}{100000} + \cdots = 6.2 + \frac{54}{10^3} + \frac{54}{10^5} + \frac{54}{10^7} + \cdots \\
&= 6.2 + \frac{54}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) = 6.2 + \frac{54}{10^3}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right) \\
&= 6.2 + \sum_{n=1}^{\infty} \frac{54}{10^3}\left(\frac{1}{100}\right)^{n-1}.
\end{aligned}$$

This is a geometric series with $c = 54/10^3$ and $|r| = 1/100 < 0$. Hence, the series converges to $\frac{54/10^3}{1-1/100} = \frac{54}{1000-10} = \frac{54}{990} = \frac{6}{110}$. Thus, the value of the number is $\frac{62}{10} + \frac{6}{110} = \frac{344}{55}$.

The following fact is useful and not too hard to see: If $\{a_n\}$ is a sequence containing a subsequence $\{b_n\}$ which diverges, then $\{a_n\}$ diverges.

Example. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is known as the *harmonic series*. Instead of looking for each partial sum, we will bound some of them. We have

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}.
\end{aligned}$$

This continues and we find $s_{2^n} > 1 + \frac{n}{2}$ and so $s_{2^n} \to \infty$ as $n \to \infty$. Hence $\{s_n\}$ diverges. That is, the harmonic series diverges.

Later we will use the harmonic series to compare to other series to show they diverge,

First we remark that if the series $\sum a_n$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$. The contrapositive of this statement is useful for determining *divergence*.

Theorem 6: Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

The reader is cautioned *not* to try and use the divergence test to determine *convergence*. For example, the terms of the harmonic series tend to zero, but the harmonic series diverges.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10n+12}$. The general term is $a_n = \frac{n}{10n+12}$. Since $a_n \to \frac{1}{10}$ as $n \to \infty$, then the series diverges by the Divergence Test.

We now turn to the question of when certain series converge or diverge. At present we will focus exclusively on series of the form $\sum a_n$ with $a_n \geq 0$ for all n.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. We treat each summand as the area of a rectangle with base 1 and height $\frac{1}{\sqrt{n}}$. All of these rectangles can be positioned so that their areas exceed that of the area under the curve $\frac{1}{\sqrt{x}}$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx.$$

Since this integral diverges, then so does the series.

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Starting with $n = 2$, we proceed as above by treating each summand as a rectangle with base 1 and height $\frac{1}{n^2}$. then we see that these rectangles lie below the curve $\frac{1}{x^2}$. Hence, it is easy to see that

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2}\, dx.$$

Since this integral converges, then so does the series.

Theorem 7: The Integral Test

Suppose $f(x)$ is continuous, positive, and decreasing function on $[1, \infty)$ that models the sequence $\{a_n\}$.

(1) If $\displaystyle\int_1^{\infty} f(x)\, dx$ converges, then so does $\displaystyle\sum_{n=1}^{\infty} a_n$.

(2) If $\displaystyle\int_1^{\infty} f(x)\, dx$ diverges, then so does $\displaystyle\sum_{n=1}^{\infty} a_n$.

In practice, it is not necessary that $f(x)$ satisfy all of those conditions, only that it satisfies them eventually, that is, there is some a such that $f(x)$ satisfies the conditions on $[a, \infty)$. This is because convergence or divergence of a series does not depend on the initial terms.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$. We model $\{n^{-1/4}\}$ with the function $f(x) = x^{-1/4}$. This function clearly satisfies the conditions of the Integral Test: it is continuous, positive, and decreasing on $[1, \infty)$. Since $\int_1^{\infty} \frac{1}{x^{1/4}}\, dx$ diverges (p-integral type I, $p = 1/4$), then so does the given series.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5n^4}$. This is similar to the previous example. Since $\displaystyle\int_1^{\infty} \frac{3}{5x^4}\, dx = \frac{3}{5}\int_1^{\infty} \frac{1}{x^4}\, dx$ converges (p-integral type I, $p = 4$), the so does the given series.

We summarize the previous two examples below.

Theorem 8: p-test for series

The infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$. We model the term by $f(x) = \frac{\ln x}{x^2}$ and apply the integral test.

It is clear that $f(x)$ is continuous and positive (at least non-negative). To show that it is decreasing, we consider

$$f'(x) = x^{-2}(1/x) + (-2)x^{-3}\ln(x) = x^{-3}(1 - 2\ln(x)) < 0 \quad \text{for } x > 1.$$

Thus, $f(x)$ satisfies the conditions of the Integral Test.

Consider the integral, $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\, dx$. To integrate, we apply IBP. Let $u = \ln x$ and $dv = x^{-2}\, dx$ so $du = x^{-1}\, dx$ and $v = -x^{-1}$. Then

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x^2}\, dx &= \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty} \left(\left[-\frac{\ln x}{x} \right]_1^b + \int_1^b \frac{1}{x^2}\, dx \right) \\ &= \lim_{b\to\infty} \left(-\frac{\ln b}{b} + 0 + \left[-\frac{1}{x} \right]_1^b \right) = 0 + \lim_{b\to\infty} \left[-\frac{1}{b} + 1 \right] = 1.\end{aligned}$$

Hence, by the integral test, the given series converges.

The above argument *does not* tell us that the sum of the series, only that it converges. The actual value of the series is a bit less than 1 (approx .9375) but this is difficult to determine.

The Integral Test is also useful in determining the error in approximating the sum of a (convergent) series with its nth partial sum.

Theorem 9: Remainder Estimate for the Integral Test

Suppose $f(x)$ is continuous, positive, and decreasing function on $[n, \infty)$ that models the sequence $\{a_n\}$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\, dx \le R_n \le \int_n^{\infty} f(x)\, dx.$$

We call R_n the *remainder*. In general one should take $R_n = |s - s_n|$, but in this case the function is positive so that $s > s_n$ always.

Example. We will approximate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the sum of the first 10 terms.

Let $f(x) = 1/x^2$. It is clear that $f(x)$ satisfies the conditions of the Integral Test. Note that

$$\int_n^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_n^t = \lim_{t\to\infty} \left[-\frac{1}{t} + \frac{1}{n}\right] = \frac{1}{n}.$$

Now,

$$\sum_{n=1}^{10} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{10^2} \approx 1.54976773.$$

By the Remainder Estimate for the Integral Test, $R_{10} = S - S_{10}$ is bounded between $\int_{11}^\infty f(x)\,dx$ and $\int_{10}^\infty f(x)\,dx$. Thus, $\frac{1}{11} \le R_{10} \le \frac{1}{10}$.

Suppose we want to ensure that the approximation is correct to three decimal places. We have seen that $R_n \le \frac{1}{n}$, so to ensure that $R_n \le \frac{5}{10^4}$, we just need to determine $\frac{1}{n} \le \frac{5}{10^4}$, so we set $n = \frac{10^4}{5}$.

11.4 Comparison Tests

In this section we will consider two different comparison tests. The first should seem familiar.

Theorem 10: The Comparison Test for Series

Assume there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$.

- If $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

The condition $n \geq M$ in the Comparison Test is there because it only matters what happens *eventually* in the sequence, not what happens in the early terms. Note that we compare the *terms* of the series, but conclude that the *series* converges or diverges.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$. For $n > e$, $\frac{\ln(n)}{n} > \frac{1}{n}$. Since the harmonic series diverges, then the given series also converges by the Comparison Test.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$. We have $n^2+n+1 > n^2$ for $n > 0$. Hence, $\frac{1}{n^2+n+1} < \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges (p-series, $p > 1$), then the given series does also by the Comparison Test.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$. One could use the integral test on this, but is quite difficult. Since $\sqrt{n^2+1} > \sqrt{n^2} = n$, then $\frac{1}{\sqrt{n^2+1}} < \frac{1}{n}$. Thus, our series is smaller than a divergent series and so the Comparison Test does not apply. We will see this series again below.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(3n)}{1+(1.2)^n}$. First note that $0 \leq \cos^2(3n) \leq 1$. Hence,

$$\frac{\cos^2(3n)}{1+(1.2)^n} \leq \frac{1}{1+(1.2)^n} \leq \frac{1}{(1.2)^n}.$$

The sequence $\sum_{n=1}^{\infty} \frac{1}{(1.2)^n}$ is geometric with $|r| = \frac{5}{6} < 1$ and so it converges. Thus, by the Comparison Test, the given series converges.

The Comparison Test gives us a way of comparing series in which the terms in the series are bigger or smaller than those in a convergent or divergent series. The next test is similar, but compares *growth rates* of series.

Theorem 11: The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number with $c > 0$, then either both series converge or both diverge.

Example. We will compare the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ the the harmonic series with terms $b_n = 1/n$,

$$\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = \sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = 1 > 0.$$

Since the harmonic series diverges, then so does the given series by the Limit Comparison Test.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3+2n-1}{5n^5-2n^3+3}$. Set $a_n = \frac{3n^3+2n-1}{5n^5-2n^3+3}$. To determine the comparing series, look for the highest power of n in the numerator and denominator. We will compare to the series $\sum_{n=1}^{\infty} b_n$ where $b_n = \frac{n^3}{n^5} = \frac{1}{n^2}$. This is a p-series with $p = 2 > 1$, and so it converges. Now

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n^3+2n-1)/(5n^5-2n^3+3)}{(1/n^2)} = \lim_{n\to\infty} \frac{3n^5+2n^3-n^2}{5n^5-2n^3+3} = \frac{3}{5} > 0.$$

Since $\sum b_n$ converges and $c > 0$, then $\sum a_n$ also converges by the Limit Comparison Test.

The next example illustrates that there may be times when it is advantageous to use both theorems in conjunction.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$. Since $n!$ grows fast, we will conjecture that the series converges. For $n > 1$ we have

$$n! > n(n-1) \text{ so } \frac{1}{n!} < \frac{1}{n(n-1)}.$$

By the Comparison Test, our given series will converge if the series $\sum \frac{1}{n(n-1)}$ converges.

Now we use the Limit Comparison Test. Let $a_n = \frac{1}{n(n-1)}$ and $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n(n-1)} = 1.$$

Thus, by the LCT, $\sum \frac{1}{n(n-1)}$ converges.

There is a much quicker way to show convergence of this series in Section 6.

11.5 Alternating series

Definition: Alternating series

An *alternating series* is a series whose terms are alternately positive and negative.

Example. Consider the *alternating harmonic series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}.$$

We saw previously that the alternating harmonic series converges. It turns out that the sum is $\ln(2)$ for the first one (and $-\ln(2)$ for the second). The reason for this will come later.

What we will consider in this section is a test for convergence of alternating series.

Theorem 12: The Alternating Series Test

If the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies

$$\text{(i) } b_{n+1} \leq b_n \text{ for all } n \text{ (decreasing) and (ii) } \lim_{n\to\infty} b_n = 0.$$

then the series is convergent.

The Alternating Series Test works similarly if $(-1)^{n-1}$ is replaced by $(-1)^n$. It is also not affected by the starting index. *However*, the AST does *not* give any information on divergence.

We give a sketch of why this argument works. A more complete argument is in your textbook. Start with b_1, drawn out on a line segment. We then subtract from that b_2, which is less than b_1 by hypothesis. Next, we add on b_3, which is less than b_2 and subtract b_4. This difference is completely contained in b_2. Continuing in this process, we see that the limit of partial sums is bounded by b_1. Hence, the sequence of partial sums is increasing (monotonic) and bounded, therefore convergent.

Example. Consider the alternating harmonic series converges and set $b_n = \frac{1}{n}$ (the absolute value of the n term). Clearly $\frac{1}{n+1} \leq \frac{1}{n}$ for all n, so $b_{n+1} \leq b_n$ and the series is decreasing. Since $\lim_{n\to\infty}\frac{1}{n} = 0$, then the alternating harmonic series converges by the Alternating Series Test.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+1}}$. Set $b_n = \frac{1}{\sqrt{n^2+1}}$. Then

$$b_{n+1} = \frac{1}{\sqrt{(n+1)^2+1}} \leq \frac{1}{\sqrt{n^2+1}} = b_n,$$

so the sequence is decreasing. Since $\lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}} = 0$, then the series converges by the Alternating Series Test.

Example. Consider the series $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{n!}$. This is an alternating series in disguise since n even implies $\cos(\pi n) = -1$ and n odd implies $\cos(\pi n) = 1$. We can then apply the alternating series test to show the series converges.

Example. Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{5n+1}$. In checking condition (ii), note that

$$\lim_{n\to\infty} \frac{2n}{5n+1} = \frac{2}{5} \neq 0.$$

The AST does not tell us (automatically) that the series diverges. However, since the given limit is not zero, this implies that

$$\lim_{n\to\infty}(-1)^{n-1}\frac{2n}{5n+1}$$

does not exist. Hence, this series diverges by the Divergence Test. (Note that one can also show that this series is not decreasing.)

Example. Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin^2\left(\frac{\pi}{2}n\right)}{n^2}$. One can show that condition (ii) holds by the Squeeze Theorem, but the series is not decreasing. In particular, if n is odd, then $b_n = \frac{1}{n^2}$ but if n is even then $b_n = 0$. Thus, this series fails the Alternating Series Test.

However, this series converges. Part of the reason is that this series is not really an alternating series. The 'negative terms' are all 0, and so this is in fact a positive series. It is actually the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

which converges by the Limit Comparison Test (compare to the series with terms $1/n^2$).

Example. Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$. Condition (ii) is easy to check and we omit that here. The real challenge is to check whether it is decreasing. It is enough, however, to show that the function $f(x) = \frac{x^2}{x^3+1}$ is decreasing. To determine intervals of increase and decrease, we look for critical points. Note that $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$. This has critical points at $x = -1, 0, \sqrt[3]{2}$. We only care about $x \geq 1$, so we check that $f'(x) < 0$ for $x > \sqrt[3]{2}$. Hence, f is decreasing for $x \geq 2$.

The statement of the theorem indicates that the decreasing condition must hold for all n, but as with most of our theorems, it is enough that it holds *eventually*.

Example. Consider the series $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$. Let $b_n = \sqrt{n+1}-\sqrt{n}$, then

$$b_n = \sqrt{n+1}-\sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Now it is clear that $\lim_{n\to\infty} b_n = 0$ and the b_n are decreasing. Thus, the series converges by the Alternating Series Test.

Alternating series are particularly nice because there is an easy formula to determine a bound for the error of the Nth partial sum S_N.

Theorem 13: Alternating series estimation theorem

If $S = \sum_{n=1}(-1)^{n-1}b_n$ is the sum of a series satisfying the AST, then

$$|R_N| = |S - S_N| \leq b_{N+1}.$$

(Equivalently, $S_N - b_{N+1} \leq S \leq S_N - b_N$.)

Example. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. First we verify that this series converges (the fact that the summation starts at $n = 0$ does not affect the AST). Set $b_n = \frac{1}{n!}$. Then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!} = 0$$

and $\frac{1}{(n+1)!} \leq \frac{1}{n!}$, so the b_n are decreasing. Hence, the given series converges by the Alternating Series Test.

Let S be the sum of the series. We will approximate S correct to three decimal places. Using the theorem, we must find a b_n such that the difference does not affect the third decimal place of the sum. This is a bit of trial and error.

We compute the b_n and find that $b_7 = 0.0002$. Hence, $|S - S_6| \leq b_7 = 0.0002$. We compute $S_6 \approx 0.368056$. Hence,

$$0.367856 \approx S_6 - 0.0002 \leq S \leq S_6 + 0.002 \approx 0.368256.$$

Round both sides to three decimal places gives $0.368 \leq S \leq 0.368$, so $S \approx 0.368$.

11.6 Absolute convergence, ratio and root tests

In the previous section, we limited ourselves to only series with positive terms. We now consider series with negative terms.

Definition: Absolutely convergent, conditionally convergent

A series $\sum a_n$ is *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent. We say $\sum a_n$ is *conditionally convergent* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent. It is also convergent (by the alternating series test). On the other hand, the alternating harmonic series converges but it is not absolutely convergent.

Theorem 14: Absolute convergence implies convergence

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

The converse of this theorem is *not true* in general. By the previous example, the alternating harmonic series is convergent but not absolutely convergent.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$. Since $|\cos n| \leq 1$, then $0 \leq \left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), and so the given series is absolutely convergent by the Comparison Test, and therefore convergent.

We will now consider two tests, the Ratio Test and the Root test, which can in certain cases determine whether a series is absolutely convergent.

Theorem 15: The Ratio Test

Assume the limit $L = \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

(1) If $L < 1$, then the series $\sum a_n$ is absolutely convergent.
(2) If $L > 1$ or $L = \infty$, then the series $\sum a_n$ is divergent.
(3) If $L = 1$, then the Ratio Test is inconclusive.

Inconclusive means inconclusive. If $L = 1$ in the Ratio Test then we cannot assume anything (like, say, conditionally convergent). All it means is that we need to consider another method.

Example. (1) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$. Set $a_n = \frac{n}{2^n}$. Then

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)/2^{n+1}}{n/2^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)2^n}{n2^{n+1}}\right| = \frac{1}{2}.$$

Since $L < 1$, then the series $\sum a_n$ converges absolutely by the Ratio Test.

(2) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. Set $a_n = \frac{1}{n^3}$. Now we compute

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty} \left|\frac{n^3}{(n+1)^3}\right| = 1.$$

Thus, the Ratio Test is inconclusive in this case. Thankfully, since the terms are always positive, we know already that this series converges absolutely by the p-test.

(3) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{6^n}$. Set $a_n = \frac{n!}{6^n}$. Then

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!/6^{n+1}}{n!/6^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!6^n}{n!6^{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{6} = \infty.$$

Since $L = \infty$, then the series $\sum a_n$ diverges by the Ratio Test.

Theorem 16: The Root Test

Assume the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists.
(1) If $L < 1$, then the series $\sum a_n$ is converges absolutely.
(2) If $L > 1$ or $L = \infty$, then the series $\sum a_n$ diverges.
(3) If $L = 1$, then the Root Test is inconclusive.

Example. (1) Consider the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$. Set $a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$. Then

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2}.$$

Since $L < 1$, then the series $\sum a_n$ converges absolutely by the Root Test.

(2) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{n^2}$. Set $a_n = \frac{(-5)^n}{n^2}$. Then

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-5)^n}{n^2}\right|} = \lim_{n\to\infty} \frac{5}{n^{2/n}} = 5.$$

Since $L > 1$, then the series $\sum a_n$ diverges by the Root Test.

For the last one, it helps to remember that

$$\lim_{n\to\infty} (n^p)^{1/n} = \lim_{n\to\infty} n^{p/n} = 1.$$

One can check this using L'Hospital's Rule.

11.7 Strategy for testing series

This section is a review of series convergence testing. There is no algorithm for deciding on a particular method in each convergence problem, but we will develop some useful strategies.

p-series Recall that a p-series is one of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$. A p-series converges if $p > 1$ and diverges is $p \leq 1$. The p-series with $p = 1$ is called the *harmonic series*.

Geometric series A geometric series is of the form $\sum_{n=1}^{\infty} cr^{n-1}$. This converges if and only if $|r| < 1$. When it converges, the sum is $c/(1-r)$. Keep in mind that the series must be in the above form for this sum formula to work.

Comparison Tests When a series is similar to one of the above forms, then it may be useful to apply a comparison test (comparison test or limit comparison test). The comparison tests *only* apply to series with (eventually) non-negative terms. However, if $\sum a_n$ has some negative terms, one may be able to apply a comparison test and check for *absolute convergence*.

Divergence Test If $\displaystyle\lim_{n\to\infty} a_n \neq 0$ then the divergence test tells us immediately that the series diverges. If the divergence test fails then we cannot make any conclusions.

Alternating series A series of the form $\sum(-1)^n b_n$ or $\sum(-1)^{n-1} b_n$, with $b_n \geq 0$ is an alternating series and the Alternating Series Test should be considered. Note that the AST does not tell us (directly) when a series diverges, but often the Divergence Test can be used when the AST fails.

Ratio Test This is useful for series with nth powers and factorials. It is never useful when terms involve rational or algebraic functions of n (inconclusive).

Root Test Sometimes this is interchangeable with the Ratio Test. This is most often useful when nth powers are involved.

Integral Test When a_n can be modeled by a continuous function $f(x)$ (i.e. $f(n) = a_n$ for all n) then the Integral Test is often useful (provided $f(x)$ is a function that we can easily integrate).

Example. (1) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$. Comparing degrees of the numerator and denominator, this series *should* behave like $\sum \frac{1}{n^2}$, a convergent p-series ($p = 2 > 1$). Note that

$$0 \leq \frac{n-1}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

Hence, the given series converges by the Comparison Test.

Alternatively, set $a_n = \frac{n-1}{n^3+1}$ and $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n^3-n^2}{n^3+1} = 1 > 0.$$

Again, since $\sum \frac{1}{n^2}$ converges, the given series converges by the Limit Comparison Test.

(2) Consider the series $\sum_{n=1}^{\infty}(-1)^n\dfrac{n^2-1}{n^2+1}$. This is an alternating series. However,

$$\lim_{n\to\infty}\left|(-1)^n\frac{n^2-1}{n^2+1}\right| = \lim_{n\to\infty}\frac{n^2-1}{n^2+1} = 1 \neq 0.$$

Hence, by the Divergence Test, the series diverges.

(3) Consider the series $\sum_{n=1}^{\infty}\dfrac{n^{2n}}{(1+n)^{3n}}$. Since the series involves nth powers, we use the Root Test (the Ratio Test is also a possibility here). Set $a_n = \frac{n^{2n}}{(1+n)^{3n}}$, then

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{n^{2n}}{(1+n)^{3n}}\right|} = \frac{n^2}{(1+n)^3} = 0,$$

because the degree of the denominator is greater than that of the numerator. As the limit we obtained is less than 1, then the series converges *absolutely* by the Root Test.

(4) Consider the series $\sum_{n=0}^{\infty}\dfrac{\pi^{2n}}{(2n)!}$. Since factorials are involved, we use the Ratio Test. Set $a_n = \frac{\pi^{2n}}{(2n)!}$ so $a_{n+1} = \frac{\pi^{2(n+1)}}{(2(n+1))!}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\pi^{2(n+1)}/(2(n+1))!}{\pi^{2n}/(2n)!}\right| = \pi^2\lim_{n\to\infty}\frac{(2n)!}{(2n+2)!} = \pi^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0.$$

Since this limit is less than 1, then the series converges *absolutely* by the Ratio Test.

(5) Consider the series $\sum_{n=1}^{\infty}(-1)^n\dfrac{\ln n}{\sqrt{n}}$. This is an alternating series. Set $b_n = \frac{\ln n}{\sqrt{n}}$. The function $f(x) = \frac{\ln x}{\sqrt{x}}$, which is continuous for $x > 0$, models the b_n. To show that $\{b_n\}$ is decreasing it is sufficient to show that $f(x)$ is decreasing. Note,

$$f'(x) = \frac{(\sqrt{x})(1/x) - (\ln x)(1/2)(x^{-1/2})}{x} = \frac{2\sqrt{x}(1-\ln x)}{2x^2}.$$

The critical points of this function are $x = 0, e$ and for $x > e$ we have $f'(x) < 0$. Thus, $f(x)$ is decreasing for $x > e$ and so b_n is decreasing for $n > e$. Moreover,

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \overset{\text{LR}}{=} \lim_{x\to\infty}\frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0.$$

Thus, the given series converges by the Alternating Series Test.

(6) Consider the series $\sum_{n=1}^{\infty}\dfrac{\ln n}{\sqrt{n}}$. Unlike the previous example, we cannot use the AST. Instead, we model $a_n = \frac{\ln n}{\sqrt{n}}$ with $f(x) = \frac{\ln x}{\sqrt{x}}$ and use the Integral Test. Note that f is continuous, positive,

and decreasing. We compute

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{\sqrt{x}}\ dx &= \lim_{t\to\infty}\int_1^t x^{-1/2}\ln x\ dx \quad \text{set } u=\ln x,\ dv=x^{-1/2}\ dx, \text{ so } du=(1/x)dx,\ v=2x^{1/2} \\
&= \lim_{t\to\infty}\left(\left[(\ln x)(2x^{1/2})\right]_1^t - 2\int_1^t x^{-1/2}\ dx\right) \\
&= \lim_{t\to\infty}\left(\left[(\ln x)(2x^{1/2}) - 4x^{1/2}\right]_1^t\right) \\
&= \lim_{t\to\infty}\Big((\ln t)(2t^{1/2}) - 4t^{1/2} + 4)\Big) \\
&= 4 + \lim_{t\to\infty}\Big(t^{1/2}(2\ln t - 4)\Big) = \infty.
\end{aligned}$$

Since the integral converges then so does the given series by the Integral Test.

We now turn our study to *power series*. The general motivation is to be able to approximate functions, like $\sin(x)$ by polynomials in which computations are easier. In fact, this is the primary tool that calculators and computer algebra systems use to approximate values of such functions.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}$ where x is an indeterminate. We have

$$\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\frac{(x+2)^n}{n2^n}\right|}=\lim_{n\to\infty}\frac{|x+2|}{\sqrt[n]{n}2}=\frac{|x+2|}{2}\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}=\frac{|x+2|}{2}.$$

By the Root Test, the series converges (absolutely) when $\frac{|x+2|}{2}<1$ and diverges when $\frac{|x+2|}{2}>1$. Equivalently, $|x+2|<2$ and $|x+2|>2$, respectively.

The condition $|x+2|<2$ implies $-2<x+2<2$ or $-4<x<0$. The series converges on this interval and diverges when $x>0$ or $x<-4$. It remains only to check what happens when $x=-4$ and when $x=0$. Substituting $x=0$ into the given series gives the alternating harmonic series so the series converges when $x=0$. On the other hand, substituting $x=-4$ reveals the harmonic series, so the series diverges when $x=-4$. Hence, the series converges on the interval $(-4,0]$.

Definition: Power series

A *power series with center a* is a series of the form

$$F(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+\cdots.$$

The c_n are constants called the **coefficients** of the series.

Theorem 17: Radius of covergence

Every power series $F(x)$ has a radius of convergence R, which is either a nonnegative number of infinity. If R is finite, $F(x)$ converges absolutely when $|x-c|<R$ and diverges when $|x-c|>R$. If $R=\infty$, then $F(x)$ converges absolutely for all x.

To find the radius of convergence, we apply the Ratio or Root Test to a given power series. We then check the endpoints individually (these are the values where the Ratio or Root Test are inconclusive). The set of all values on which the power series converges is called the *interval of convergence*. Note that indexing does not affect the radius or interval of convergence.

Example. Geometric series are power series with $c_n=c$ for all n and $r=x$. A geometric series converges, that is, when $|r|<1$. Hence, the radius of convergence is 1 and the interval of convergence is $(-1,1)$.

Example. Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$. The center is 0. We apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n x^n}{n+1}\right|} = |x| \lim_{n\to\infty} \frac{1}{\sqrt[n]{n+1}} = |x|.$$

This power series converges when $|x| < 1$ and diverges when $|x| > 1$, so the radius of convergence is 1. Now we need to check the endpoints $x = -1$ and $x = 1$.

When $x = 1$, this series is just the alternating harmonic series. On the other hand, when $x = -1$, this is the harmonic series. Hence, the interval of convergence is $(-1, 1]$.

Example. Consider the power series $\sum_{n=0}^{\infty} n!x^n$. Again we apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty} |(n+1)x| = |x| \lim_{n\to\infty} (n+1).$$

This limit is infinite unless $x = 0$. Thus, the radius of convergence of the series is 0.

Example. Consider the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$. Applying the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{2(n+1)}/(2(n+1))!}{(-1)^n x^{2n}/(2n)!}\right| = \lim_{n\to\infty} \frac{x^{2(n+1)}(2n)!}{x^{2n}(2(n+1))!} = x^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Hence, the series converges for all x and the radius of convergence is $R = \infty$.

Example. Consider the power series $\sum_{n=3}^{\infty} \frac{3^n(x+7)^n}{\sqrt{n^2-4}}$. Applying the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{3^{n+1}(x+7)^{n+1}}{\sqrt{(n+1)^2-4}} \cdot \frac{\sqrt{n^2-4}}{3^n(x+7)^n}\right| = 3|x+7| \lim_{n\to\infty} \frac{\sqrt{n^2-4}}{\sqrt{(n+1)^2-4}} = 3|x+7|.$$

Solving $3|x+7| < 1$ gives $|x+7| < 1/3$. Hence, $R = 1/3$ and the endpoints are $x = -7 \pm (1/3) = -22/3, -20/3$.

When $x = -20/3$, the series becomes $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2-4}}$. Since $\frac{1}{\sqrt{n^2-4}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$ and $\sum \frac{1}{n}$ is the harmonic series, which diverges, then the series $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2-4}}$ diverges by the Comparison Theorem.

On the other hand, when $x = -22/3$, the series becomes $\sum_{n=3}^{\infty} \frac{(-1)^n}{\sqrt{n^2-4}}$. This series converges by the Alternating Series Test. Thus, the interval of convergence is $[-22/3, -20/3)$.

Suppose $|x| < 1$. Then
$$\frac{1}{1-x} = \sum_{n=0} x^n = 1 + x + x^2 + x^3 + \cdots.$$
Said another way, the function $1/(1-x)$ may be represented by the power series. This has many practical applications. A function like e^x can be represented by a power series and so we can approximate it by cutting off the power series at some term. In this section we will study multiple techniques for representing certain functions by power series.

Example. Consider the rational function $1/(1+x^2)$. Since we already have the power series form for $1/(1-x)$, substitute $-x^2$ for x to get
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}(-x^2)^n = \sum(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots.$$
Because this is a geometric series, it converges when $|-x^2| < 1$, that is, when $x^2 < 1$. But this is just equivalent to $|x| < 1$. So the interval of convergence is $(-1, 1)$.

Example. Consider the rational function $1/(x-3)$. We write this in the form of the sum of a geometric series by factoring out a -3 from the denominator,
$$\frac{1}{x-3} = \frac{1}{(-3)(1-(x/3))} = -\frac{1}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}x^n.$$
This series converges when $|x/3| < 1$ or $|x| < 3$. Thus, the interval of convergence is $(-3, 3)$.

Example. Consider the rational function $x^2/(x-3)$. Using our work from the previous example,
$$\frac{x^2}{x-3} = x^2 \cdot \frac{1}{x-3} = x^2 \cdot \left(-\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}x^n\right) = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}x^{n+2} = -\frac{1}{3}\sum_{n=2}^{\infty}\frac{1}{3^{n-2}}x^n.$$
The interval of convergence is $(-3, 3)$.

Theorem 18: Term-by-term differentiation and integration

If the power series $F(x) = \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then $F(x)$ is differentiable (and therefore continuous) on $(a-R, a+R)$ (or all x if $R = \infty$). Furthermore, we can integrate and differentiate term by term. For $x \in (a-R, a+R)$,

(1) $F'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(2) $\displaystyle\int F(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \cdots = C + \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}$

Moreover, these series have the same radius of convergence R.

The radius of convergence will remain the same when we differentiate and integrate, but the interval of convergence may change. Check endpoints!

Example. Consider the rational function $\frac{1}{(1-x)^2}$. Using term-by-term differentiation,

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1+x+x^2+x^3+\cdots) = 0+1+2x+3x^2+\cdots = \sum_{n=1}^{\infty} nx^{n-1}.$$

Since term-by-term differentiation does not affect the radius of convergence, then we get radius of convergence $R = 1$.

Example. Consider the function $\arctan(x)$. By a previous example

$$(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$$

when $|x| < 1$. Then

$$\begin{aligned}\arctan x &= \int (1+x^2)^{-1}\ dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\ dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .\end{aligned}$$

When $x = 0$, $C = \arctan(0) = 0$. Hence,

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

This expansion is valid for $-1 < x < 1$ but not at the endpoints.

Example. Consider the function $\ln(1+x)$. We have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$$

and this series had radius of convergence 1. Now

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\ dx = \int (1 - x + x^2 - x^3 + x^4 + \cdots)\ dx \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right) + C = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.\end{aligned}$$

When $x = 0$, $\ln(1+x) = \ln(1) = 0$. Hence, $C = 0$. This has radius of convergence $R = 1$.

For a function $f(x)$, the linearization at $x = a$ is the function $L(x) = f'(x)(x - a) + f(a)$. The linearization approximates the function near $x = a$. More than that, $f(x)$ and $L(x)$ agree at $x = a$ *and* their derivatives agree at $x = a$. That is, $f(a) = L(a)$ and $f'(a) = L'(a)$. In this section we construct polynomials of higher degree that approximation functions. Extending this idea, we find power series representations of functions.

Suppose f can be represented by a power series centered at a, that is

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R.$$

Can we determine a formula for the c_n in general? First note that $f(a) = c_0$ and

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}.$$

Thus, $f'(a) = c_1$. Similarly,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2},$$

so $f''(a) = 2c_2$. Continuing in this way, we find that $f^{(n)}(a) = n! c_n$. Said another way, $c_n = \frac{f^{(n)}(a)}{n!}$.

Theorem 19: Taylor's Formula

If f has a power series representation centered at a, that is,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a| < R,$$

then its coefficients are given by the formula,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This is called the *Taylor Series* of f at a. When $a = 0$, it is called the *Maclaurin series.*

Example. Consider the function $f(x) = e^x$. Since $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$, for all n, then the Maclaurin series of e^x is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. By the ratio test, the radius of convergence is $R = \infty$.

Example. Consider the function $f(x) = \sin x$. Then

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)} = \sin x.$$

This cycle repeats indefinitely, so we have $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -1$. Thus, the Maclaurin series for $\sin x$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

We can use substitution, derivatives, and integration to obtain new Taylor series.

Example. By the previous example, $\sin x$ equals the sum of its Maclaurin series. Hence, $x \sin x$ is also equal to the sum of its Maclaurin series and

$$x \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(n+1)}}{(2n+1)!}.$$

Example. Consider the function $f(x) = \frac{1}{x}$ and set $a = -3$. We have $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$. We conclude that $f^{(n)}(x) = (-1)^n (n!) x^{-(n+1)}$. Hence,

$$f^{(n)}(-3) = (-1)^n (n!)(-3)^{-(n+1)} = -(n!)3^{-(n+1)}.$$

Therefore, the Taylor series expansion of $f(x) = \frac{1}{x}$ at $a = -3$ is

$$\sum_{n=0}^{\infty} \frac{-(n!)3^{-(n+1)}}{n!}(x - (-3))^n = \sum_{n=0}^{\infty} \frac{-1}{3^{n+1}}(x+3)^n.$$

Our understanding of Taylor series gives us a way to realize a very important series which you are probably already familiar with.

Example (The Binomial Series)**.** Let k be any real number and consider $f(x) = (1+x)^k$. We can show that

$$\begin{aligned} f^{(n)}(x) &= k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n} \\ f^{(n)}(0) &= k(k-1)(k-2)\cdots(k-n+1). \end{aligned}$$

Hence, the Maclaurin series of $(1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n.$$

The coefficients in this series are known as *binomial coefficients*,

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

We can write the series in shorthand as,

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

We turn to the question of whether a function is truly represented by its Taylor/Maclaurin series. One way to do this is to find the radius of convergence. In the first two examples, the radius of convergence is $R = \infty$. On the other hand, in the previous example, the radius of convergence is 3. There is a shortcut which can save time in certain instances.

Definition: Taylor polynomials, remainder

We call the partial sums of the Taylor series the *Taylor polynomials of f at a*. Let $T_n(x)$ represent the nth Taylor polynomial of a function f. Then $R_n(x) = f(x) - T_n(x)$ is called the *remainder* of the Taylor series. If $f(x) = T_n(x) + R_n(x)$ and

$$\lim_{n\to\infty} R_n(x) = 0 \text{ for } |x-a| < d,$$

then f is equal to the sum of its Taylor series on the interval $|x-a| < d$.

Theorem 20: Convergence criteria for Taylor series

Suppose there exists $M > 0$ such that $|f^{(n+1)}(x)| \leq M$ for all n and all x such that $|x-a| \leq d$, then $f(x)$ is equal to the sum of its Taylor series on $(a-d, a+d)$.

Generally in this theorem, a is the center of the Taylor series and d is the radius of convergence. Showing the condition in the theorem that $|f^{(n+1)}(x)| \leq M$ is, in general, quite hard. The next result is useful in many instances.

Taylor's Inequality

If $|f^{(n+1)}| \leq M$ for all $|x-a| \leq d$, then the remainder R_n of the Taylor series satisfies

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Another useful fact is that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real number x.

Example. Recall that the Maclaurin series for $f(x) = e^x$ is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. If d is any positive number and $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$. Now Taylor's inequality with $a = 0$ and $M = e^d$ gives

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{n+1} \quad \text{for } |x| \leq d$$

But

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{1}{(n+1)!}|x|^{n+1} = 0$$

so by the Squeeze Theorem

$$\lim_{n\to\infty} |R_n(x)| = 0.$$

Thus, e^x is equal to the sum of its Maclaurin series.

Taylor's Inequality helps us determine how well a Taylor polynomial approximates the function.

Example. Previously we computed the Maclaurin series for $\sin x$:

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Of course, $|f^{(k)}(x)| \leq 1$ for all x. Then by Taylor's Inequality,

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1} = \frac{|x|^{n+1}}{(n+1)!} = 0.$$

So by the Squeeze Theorem,

$$\lim_{n\to\infty} |R_n(x)| = 0.$$

Thus, $\sin x$ equals the sum of its Maclaurin series.

Say we want to estimate $\sin(1)$ within 0.0001. By Taylor's Inequality with $x = 1$, we want n such that

$$|R_n(x)| \leq \frac{1}{(n+1)!} < 0.0001.$$

By trial-and-error we find that $1/8! < 0.0001$, so we take $n = 7$. The degree 7 Taylor polynomial is

$$T_7(x) = \sum_{n=0}^{7} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7.$$

Now,

$$\sin(1) \approx T_7(1) = 4241/5040 \approx .8414682540.$$

Maple computes $\sin(1)$ as $\sin(1) \approx .8414709848$ so this is a good approximation.

Example. Consider $f(x) = \sqrt[3]{x}$. One can show that the Taylor polynomial of degree 2 at $a = 8$ is

$$T_2(x) = 2 + \frac{1}{12}(x-8) + \frac{1}{288}(x-8)^2.$$

Now we can ask how good this approximation is on some interval around 8, say $7 \leq x \leq 9$. Using Taylor's Inequality we have

$$|R_2(x)| \leq \frac{M}{3!}|x-8|^3$$

where $|f'''(x)| \leq M$. Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$ so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leq \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021.$$

This means that we can take $M = 0.0021$. Since $7 \leq x \leq 9$, then $-1 \leq x \leq 1$ and $|x - 8| \leq 1$. Taylor's inequality now gives

$$|R_2(x)| \leq \frac{0.0021}{3!} \cdot 1^3 < 0.0004.$$

Example. Consider the indefinite integral $\int e^{-x^2}\, dx$. Our tools for integration did not previously allow us to evaluate this integral. On the other hand, recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. By substitution and term-by-term integration,

$$\int e^{-x^2}\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}.$$

Thus, we are able to evaluate the integral as an infinite series.

Now if we have the definite integral $\int_0^1 e^{-x^2}\ dx$, then

$$\int_0^1 e^{-x^2}\ dx = \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{n!(2n+1)}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.$$

Finally, suppose we want this sum correct to three decimal places. We note that the series is alternating, so set $b_n = \frac{1}{n!(2n+1)}$. A calculation shows that $b_6 < 0.0005$. Thus, by the Alternating Series Estimation Theorem, we need 5 terms. Computing the fifth partial sum we have

$$\int_0^1 e^{-x^2}\ dx \approx \sum_{n=0}^{5} \frac{(-1)^n}{n!(2n+1)}. = .7467291967.$$

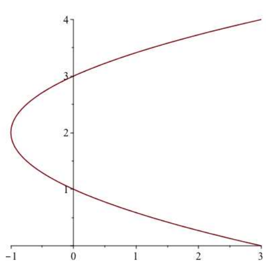
Parametric Equations and Polar Coordinates

10.1 Curves defined by parametric equations

Until this point, we have focused almost exclusively on curves given $y = f(x)$. This is limiting in many ways. For another approach, we turn to parametric curves, which describe a curve as it moves through space by a third parameter, time.

Definition: Parameter, parametric curve

Suppose that x and y are both given as functions of a third variable t, called the *parameter*, by the equations $x = f(t)$, $y = g(t)$. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a *parametric curve*.



Example. Consider the curve defined by the parametric equations

$$x = t^2 - 2t, \quad y = t + 1.$$

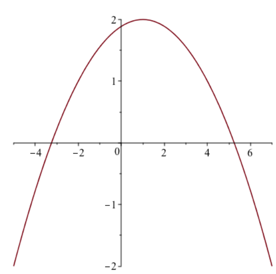
The second equation tells us that $t = y - 1$. Substituting this into the first equation gives the equation

$$x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3.$$

This is the parabola the left.

Example. Consider the curve given by the parametric equations

$$x = 1 + 3t, y = 2 - t^2.$$

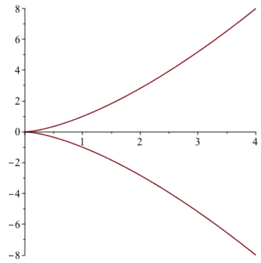


The curve moves to the right as t increases. Since $x = 1 + 3t$, then $t = \frac{1}{3}(x - 1)$. Thus,

$$y = 2 - t^2 = 2 - \left(\frac{1}{3}(x-1)\right)^2 = 2 - \frac{1}{9}(x-1)^2.$$

This is the parabola to the right.

These notes are derived primarily from *Calculus, Early Transcendentals* by James Stewart (8ed). Most of this material is drawn from Chapter 10. Last Updated: May 2, 2023



Example. Consider the curve given by the parametric equations

$$x = t^2, \quad y = t^3.$$

Since $x = t^2$, then $t = \pm x^{1/2}$ so $y = \pm x^{3/2}$. Alternatively, since $y = t^3$, then $t = y^{1/3}$ and so $x = y^{2/3}$. This is the curve to the left.

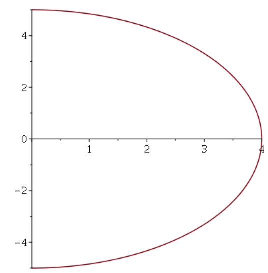
We will often restrict t to a finite interval, i.e., we will say $a \leq t \leq b$. In this case, if $x = f(t), y = g(t)$, then we say the curve has *initial point* $(f(a), g(a))$ and *terminal point* $(f(b), g(b))$.

Example. Consider the curve given by the parametric equations

$$x = 4\cos\theta, \quad y = 5\sin\theta, \quad -\pi/2 \leq \theta \leq \pi/2.$$

Since $x = 4\cos\theta$, then $\cos\theta = x/4$. Thus, we have a right triangle with adjacent side x and hypotenuse 4, so the opposite side is $\pm\sqrt{16 - x^2}$. It follows that

$$y = 5\sin\theta = \pm\frac{5\sqrt{16 - x^2}}{4}.$$



10.2 Calculus with parametric curves

We now see how to apply our methods of calculus to curves given by parametric curves. To find tangents, we use the chain rule.

First and second derivatives for parametric curves

Suppose x and y are functions of a variable t and $\frac{dx}{dt} \neq 0$. Then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}.$$

Example. Let C be the curve defined by the parametric equations $x = t^3 - 12t$ and $y = t^2 - 1$. Using the above formula we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 - 12} = \frac{2}{3}\left(\frac{t}{t^2 - 4}\right).$$

This has critical points at $t = -2, 0, 2$. Hence, it is increasing on the interval $(-2, 0) \cup (2, \infty)$ and decreasing on the interval $(-\infty, -2) \cup (0, 2)$. Now,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{2}{3}\left(\frac{t}{t^2-4}\right)}{3t^2 - 12} = \frac{\frac{2}{3}\left(\frac{(t^2-4)-t(2t)}{(t^2-4)^2}\right)}{3(t^2 - 4)} = \frac{-2(t^2 + 4)}{9(t^2 - 4)^3}$$

The possible points of inflection are at $t = \pm 2$. The graph is concave down when $t > 2$ or $t < -2$ and the graph is concave up when $-2 < t < 2$.

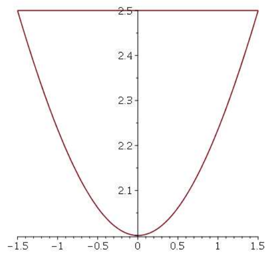
To compute areas under a curve $y = F(x)$ as $a \leq x \leq b$, we use $A = \int_a^b F(x)\, dx$. Now we consider this for parametric curves.

Area under parametric curves

Suppose we can represent y using parametric curves, say $x = f(t)$ and $y = g(t)$ as $\alpha \leq t \leq \beta$, so $dx = f'(t)\, dt$ and $y = F(x) = F(f(t)) = g(t)$. Then

$$A = \int_a^b y\, dx = \int_\alpha^\beta g(t) f'(t)\, dt.$$

Example. Consider the area bounded by the curve $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, and the line $y = 2.5$.



We will compute the area under the line $y = 2.5$ and subtract the area under the given parametric curve. First we need to identify the bounds. That is, when is $y = t + \frac{1}{t} = 2.5$. Solving for t we find that $t = \frac{1}{2}, 2$. Thus, we can compute area under the parametric curve as follows.

$$\begin{aligned}
A = \int_{\alpha}^{\beta} g(t)f'(t)\ dt &= \int_{1/2}^{2} \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right)\ dt \\
&= \int_{1/2}^{2} t + \frac{2}{t} + \frac{1}{t^3}\ dt = \left[\frac{t^2}{2} + 2\ln|t| - \frac{1}{2t^2}\right]_{1/2}^{2} \\
&= \left(2 + 2\ln(2) - \frac{1}{8}\right) - \left(\frac{1}{8} + 2\ln(1/2) - 2\right) \\
&= 4\ln(2) + \frac{15}{4} \approx 6.52259.
\end{aligned}$$

The points $t = \frac{1}{2}$ and $t = 2$ correspond to $x = \pm\frac{3}{2}$. Thus, the area under the line $y = 2.5$ is $3 \cdot 2.5 = 7.5$. Hence, the area between the two curves is $7.5 - (4\ln(2) - 3.75) \approx 8.477$.

Now we study arc length for parametric curves.

Arc length for parametric curves

If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \le t \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

Example. Consider the curve given by $x = \sin(2t)$, $y = \cos(2t)$, $0 \le t \le 2\pi$. Using the formula,

$$\begin{aligned}
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt &= \int_{0}^{2\pi} \sqrt{(2\cos(2t))^2 + (-2\sin(2t))^2}\ dt \\
&= \int_{0}^{2\pi} \sqrt{4\cos^2(2t) + \sin^2(2t)}\ dt = 2\int_{0}^{2\pi}\ dt = 4\pi.
\end{aligned}$$

Example. Consider the curve given by $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \leq t \leq 3$. By the formula,

$$\begin{aligned}
L &= \int_\alpha^\beta \sqrt{(e^t - e^{-t})^2 + (-2)^2}\ dt = \int_0^3 \sqrt{e^{2t} - 2 - e^{-2t} + 4}\ dt \\
&= \int_0^3 \sqrt{(e^t + e^{-t})^2}\ dt = \int_0^3 e^t + e^{-t}\ dt \\
&= \left[e^t - e^{-t}\right]_0^3 = \left(e^3 - e^{-3}\right) - (1 - 1) = e^3 - e^{-3}.
\end{aligned}$$

10.3 Polar Coordinates

Now we switch gears and discuss another way of writing equations in the plane. In the Cartesian system we write coordinates using rectangular coordinates (x, y). We could do something similar by writing points in terms of an angle from the x-axis and a radius from the origin.

Converting to/from polar coordinates

Let (x, y) be coordinates in the Cartesian system. In polar coordinates, the point (r, θ) is determined by

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$

Let (r, θ) be coordiantes in the polar system. In Cartesian coordinates, the point (x, y) is determined by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Example. Consider the rectangular coordinates $(-1, \sqrt{3})$. We have $\tan\theta = y/x = -\sqrt{3}$ and $r = x^2 + y^2 = (-1)^2 + (\sqrt{3})^2 = 4$. There are two possible solutions to this: either $\theta = 2\pi/3$ or $\theta = 8\pi/3$. Since our given point is in the second quadrant, then $\theta = 2\pi/3$.

Example. Consider the polar coordinates $(-1, -\pi/2)$. We have

$$x = r\cos\theta = (-1)\cos(-\pi/2) = 0,$$
$$y = r\sin\theta = (-1)\sin(-\pi/2) = (-1)(-1) = 1.$$

So, in rectangular coordinates the point is $(0, 1)$.

The *graph of a polar equation* $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

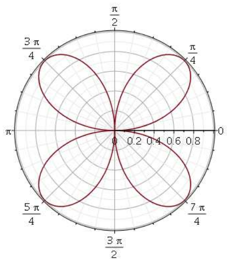
Example. The curve with polar equation $r = 2$ is the set of all polar points with radius 2. This is just a circle of radius 2.

Example. The curve with polar equation $\theta = \pi/4$ is the set of all polar points that form a right triangle with the origin and the x-axis, and have acute angle $\pi/4$. This is just a line through the origin and the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Example. Consider the curve with polar equation $r = 2\cos\theta$. Plotting points, we find that this is a circle of radius 1 centered at $(1, 0)$.

Let's see how to directly convert this to a Cartesian equation. Since $x = r\cos\theta$, then $2\cos\theta = 2x/r$. Thus, $r = 2x/r$, or $2x = r^2 = x^2 + y^2$. Completing the square gives $(x-1)^2 + y^2 = 1$.

Example. Consider the curve with polar equation $r = \sin 2\theta$. We will plot some points to get an idea of the shape. When $\theta = \pi/4$ or $5\pi/4$, then $r = 1$. When $\theta = 3\pi/4$ or $\theta = 7\pi/4$, then $r = -1$. Finally, when $\theta = 0, \pi/2, \pi, 3\pi/2$, we have $r = 0$. This is known as a *four-petaled rose.*



To find tangents to polar curves we can use our techniques from parametric equations.

Derivatives for polar equations

Suppose $r = f(\theta)$, so then $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$. Then

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Example. Consider the cardiod $r = 1 + \cos\theta$. We will find the equation of the tangent line when $\theta = \frac{\pi}{3}$. Note that $\frac{dr}{d\theta} = -\sin\theta$. Using the formula above, we have

$$\frac{dy}{dx} = \frac{-\sin^2\theta + (1+\cos\theta)\cos\theta}{-\sin\theta\cos\theta - (1+\cos\theta)\sin\theta} = \frac{-\sin^2\theta + \cos\theta + \cos^2\theta}{-2\sin\theta\cos\theta - \sin\theta}.$$

Evaluating at $\theta = \frac{pi}{3}$ we have

$$\frac{dy}{dx} = \frac{-(\sqrt{3}/2)^2 + (1/2) + (1/4)}{\sqrt{3}/2 + \sqrt{3}/2} = \frac{-(3/4) + (2/4) + (1/4)}{2\sqrt{3}} = 0.$$

Hence, the tangent line is horizontal at $\theta = \frac{pi}{3}$.

When $\theta = \frac{\pi}{3}$, then $r = \frac{3}{2}$. So in cartesian coordinates, we have $y = \frac{3}{2}\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{4}$.

Consider a sector of a circle and let θ be the angle determined by the arc of the sector. Since this sector represents $\frac{\theta}{2\pi}$ of the circle, its area is $A = \frac{1}{2}r^2\theta$.

Area of a polar graph

The area determined by the polar function $r = f(\theta)$ on the interval $\alpha \leq \theta \leq \beta$ is

$$\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

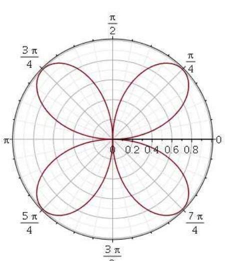
To see why this is true, we divide the interval in to n regular subintervals of length $\Delta\theta = (\beta - \alpha)/n$. Taking a sample point θ_i^* in each subinterval $[\theta_{i-1}, \theta_i]$, the area inside the polar graph on this interval is approximated by

$$A \approx \sum_{i=1}^{n} \frac{1}{2} \left[f(\theta_i^*)\right]^2 \Delta\theta.$$

Thus, we can take a limit to find the exact area.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} \left[f(\theta_i^*)\right] \Delta\theta = \int_{\alpha}^{\beta} \frac{1}{2} [f(\theta)]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

Example. Recall the four-petaled rose $r = \sin 2\theta$. We will find the area in one of the petals, which in turn will tell us the area inside the entire shape.



For the petal in the first quadrant, we have $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq 1$. Then

$$\begin{aligned} \frac{1}{4}A &= \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta) \, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta) \, d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} \frac{1}{2}(1 - \cos(4\theta)) \, d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} = \frac{1}{4}\left[\frac{\pi}{2}\right] = \frac{\pi}{8}. \end{aligned}$$

Thus, the total area is $\pi/2$.

Arc length of a polar graph

The arc length determined by the polar function $r = f(\theta)$ on the interval $\alpha \leq \theta \leq \beta$ is

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

We could use our formula for arc length, or we could derive this using the (Cartesian) formula for the arc length and the chain rule, along with some trig identities:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 + \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2}\ d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2\frac{dr}{d\theta} r\sin\theta\cos\theta + r^2\cos^2\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2\frac{dr}{d\theta} r\cos\theta\sin\theta + r^2\sin^2\theta}\ d\theta$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 (\sin^2\theta + \cos^2\theta) + r^2(\cos^2\theta + \sin^2\theta)}\ d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta.$$

For the next example, we will need a version of one of the *double-angle identities*. Namely,

$$2\cos^2(x/2) = 1 + \cos x.$$

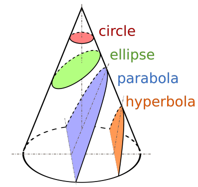
Example. Consider the curve $r = 2(1 + \cos\theta)$. Tracing this curve, we see that the interval is $0 \le \theta \le \pi$. So we have **add a picture**

$$\begin{aligned}
L &= 2\int_0^\pi \sqrt{(2(1+\cos\theta))^2 + (-2\sin\theta)^2}\ d\theta \\
&= 2\int_0^\pi \sqrt{4(1 + 2\cos\theta + \cos^2\theta) + 4\sin^2\theta}\ d\theta \\
&= 2\int_0^\pi \sqrt{4(1 + 2\cos\theta) + 4(\cos^2\theta + 4\sin^2\theta)}\ d\theta \\
&= 4\sqrt{2}\int_0^\pi \sqrt{1+\cos\theta}\ d\theta = 4\sqrt{2}\int_0^\pi \sqrt{2\cos^2(\theta/2)}\ d\theta \\
&= 8\int_0^\pi \cos(\theta/2)\ d\theta = 8\left[2\sin(\theta/2)\right]_0^\pi = 16\left[\sin(\pi/2) - \sin(0)\right] = 16.
\end{aligned}$$

As a final note/example, consider the four-petal rose $r = \sin(2\theta)$. To find the arc length of one petal (so the interval is $0 \le \theta \le \pi/2$ we have

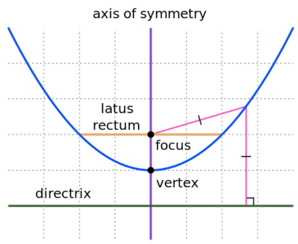
$$\begin{aligned}
L &= \int_0^{\pi/2} \sqrt{\sin^2(2\theta) + (2\sin(2\theta)\cos(2\theta))^2}\ d\theta \\
&= \int_0^{\pi/2} \sin(2\theta)\sqrt{1 + 4\cos^2(2\theta)}\ d\theta \quad \text{let } u = 2\cos(2\theta) \text{ so } du = -4\sin(2\theta) \\
&= -\frac{1}{4}\int_1^0 \sqrt{1+u^2}\ du = \frac{1}{4}\int_0^1 \sqrt{1+u^2}\ du \quad \text{let } u = \sin(\psi) \text{ so } du = \cos(\psi) \\
&= \frac{1}{4}\int_0^{\pi/2} \sqrt{1+\sin^2(\psi)}\cos(\psi)\ d\psi = \frac{1}{4}\int_0^{\pi/2} \cos^2\psi\ d\psi \\
&= \frac{1}{8}\int_0^{\pi/2} 1 + \cos(2\psi)\ d\psi = \frac{1}{8}\left[\psi - \frac{1}{2}\sin(2\psi)\right]_0^{\pi/2} = \frac{\pi}{16}.
\end{aligned}$$

In this section we define three types of *conic sections* (or just *conics*). These shapes are so-called because they can be obtained by intersecting a plane with a cone.



Definition: Parabola, focus, directrix, vertex, axis

A *parabola* is the set of points in a plane that are equidistant from a fixed point F, called the *focus* and a fixed line, called the *directrix*. The point halfway between the focus and the directrix is called the *vertex*. The line through the focus perpendicular to the directrix is the *axis* of the parabola.



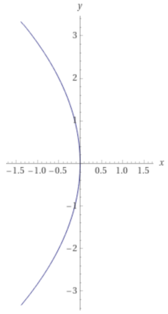
Suppose a parabola has its vertex at the origin O and the directrix is parallel to the x-axis. Let $(0, p)$ denote the focus. Then the directrix has equation $y = -p$. If $P(x, y)$ is *any* point on the parabola, then the distance from P to the focus is $|PF| = \sqrt{x^2 + (y-p)^2}$ and the distance from P to the directrix is $|y + p|$.

Now the defining equation of the parabola is

$$\begin{aligned}
\sqrt{x^2 + (y-p)^2} &= |y+p| \\
x^2 + (y-p)^2 &= (y+p)^2 \\
x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\
x^2 &= 4py.
\end{aligned}$$

One can similarly switch the roles of x and y to obtain a parabola with focus $(p, 0)$, directrix $x = -p$, and equation $y^2 = 4px$.

Example. Consider the parabola $y^2 + 8x = 0$. We rewrite in the form $y^2 = -8x = 4(-2)x$, so $p = -2$. Comparing to the above formula, we see that the the focus is $(-2, 0)$ and directrix $x = 2$.



Definition: Ellipse, foci

An *ellipse* is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2, called *foci* is a constant

If the foci are $(\pm c, 0)$, so that the origin is halfway between, then the points of intersection of the ellipse with the x-axis are called the *vertices*.

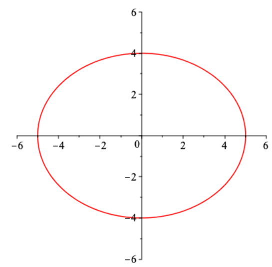
The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $a \geq b > 0$ has foci $(\pm c, 0)$ where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$. The line segment from $(-a, 0)$ to $(a, 0)$ is called the *major axis* and the line segment from $(0, -b)$ to $(0, b)$ is called the *minor axis*.

The ellipse $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ with $a \geq b > 0$ has foci $(0, \pm c)$ where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$. The line segment from $(0, -a)$ to $(0, a)$ is called the *major axis* and the line segment from $(-b, 0)$ to $(b, 0)$ is called the *minor axis*.

Example. Consider the ellipse $16x^2 + 25y^2 = 400$. We divide both sides by 400 to get

$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1.$$

Then $c^2 = 5^2 - 4^2 = 1$ so the foci are $(\pm 1, 0)$ and the vertices are $(\pm 5, 0)$.



Definition: Hyperbola, foci

A *hyperbola* is the set of points in a plane the difference of whose distances from two fixed points F_1 and F_2, called *foci* is a constant

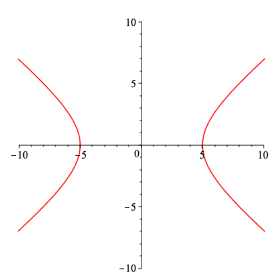
The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has foci $(\pm c, 0)$ where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

The hyperbola $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ has foci $(0, \pm c)$ where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

Example. Consider the hyperbola $16x^2 - 25y^2 = 400$. We divide both sides by 400 to get

$$\frac{x^2}{5^2} - \frac{y^2}{4^2} = 1.$$

Then $c^2 = 5^2 + 4^2 = 41$ so the foci are $(\pm\sqrt{41}, 0)$, the vertices are $(\pm 5, 0)$, and the asymptotes are $y = \pm(4/5)x$.

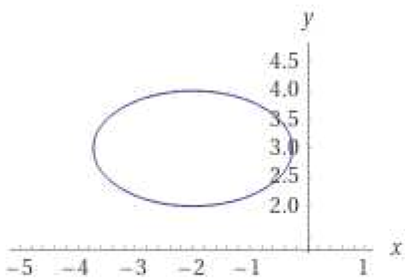


We can take shifted conics by replacing x and y in the above equations with $x - h$ and $y - k$.

Example. Consider the conic $x^2 + 3y^2 + 4x - 18y + 28 = 0$. We complete the square twice to get $(x + 2)^2 + 3(y - 3)^2 = 3$. Dividing both sides by 3 gives the equation of an ellipse:

$$\frac{(x+2)^2}{3} + \frac{(y-3)^2}{1} = 1.$$

This is (still) an ellipse. As before $a^2 = 3$, $b^2 = 1$, and $c^2 = 2$. The ellipse is shifted 2 units to the left and 3 units up. Thus, the foci are $(-2 \pm \sqrt{2}, 3)$. The vertices are $(-2 \pm \sqrt{3}, 3)$.



Here we give a unified description of conic sections.

Let F be a fixed point, called the *focus*, and ℓ be a fixed line, called the *directrix*, in a plane. Let e be a fixed positive number, called the *eccentricity*. The set of all points P in the plane such that

$$\frac{|PF|}{|P\ell|} = e$$

is a conic section.

- If $e < 1$, the conic is an ellipse.
- If $e = 1$, the conic is a parabola.
- If $e > 1$, the conic is a hyperbola.

Suppose a conic has focus F at the origin and directrix $x = \pm d$ where $d > 0$ (so in the positive case appears to the right of the conic and in the negative case it appears to the left). In polar form, the equation of a conic is

$$r = \frac{ed}{1 \pm e\cos\theta}$$

where e is the eccentricity (the above criteria still hold).

Similarly, if the focus is still at the origin and the directrix is $y = \pm d$ where $d > 0$ (replace right with above and left with below), then the equation of the conic in polar form is

$$r = \frac{ed}{1 \pm e\sin\theta}.$$

Example. A parabola that has focus at the origin and whose directrix is the line $y = -6$ has polar equation

$$r = \frac{6}{1 - \sin\theta}.$$

Example. Suppose a conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}.$$

Rewriting we have

$$r = \frac{10/3}{1 - (2/3)\cos\theta}.$$

Thus, $e = 2/3$, so the conic is a parabola. Since $ed = 10/3$, then $d = 5$. Thus, the directrix has Cartesian equation $x = -5$.

Example. Suppose a conic is given by the polar equation

$$r = \frac{12}{2 + 4\sin\theta}.$$