Isomorphisms of twisted graded Calabi—Yau algebras

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Background

We will be dealing with algebras over the field $\mathbb{C}$.

Algebras are vector spaces that also have multiplication that "plays nice" with the vector space operations.

This means they possess the right and left distributive properties and associative properties.

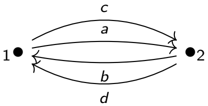
Example (Isomorphism problem for skew polynomial rings)

The skew polynomial ring $\mathbb{C}_q[x, y]$ is the same as the polynomial ring $\mathbb{C}[x, y]$ except $xy = qyx$ for some scalar $q \in \mathbb{C}^\times$.

For $p, q \in \mathbb{C}^\times$, $\mathbb{C}_q[x, y] \cong \mathbb{C}_p[x, y]$ if and only if $q = p^{\pm 1}$.

Background

Quivers are directed graphs consisting of vertices and arrows. Each arrow, a, has a source, s(a) and a target, t(a).



Given a quiver Q, the path algebra $\mathbb{C}Q$ has a $\mathbb{C}$-vector space basis consisting of paths in Q and multiplication of paths p and q is defined by the concatenation pq assuming $t(p) = s(q)$, and otherwise the product is set to 0.

Example ($n = 2$)

Let Q be the quiver below:

$$\begin{tikzcd}
1 \bullet \arrow[r, bend left=40, "c"] \arrow[r, bend left=10, "a"] & \bullet 2 \arrow[l, bend left=10, "b"] \arrow[l, bend left=40, "d"]
\end{tikzcd}$$

For $q_1, q_2 \in \mathbb{C}^\times$, the algebra $A(q_1, q_2)$ is the quotient of the path algebra $\mathbb{C}Q$ by the relations

$$ab - q_1 cd \qquad ba - q_2 dc$$

These are examples of Calabi-Yau algebras, which arise from the study of string theory. They play a similar role as polynomial rings do in (classical) algebraic geometry.

Goal

Study the isomorphism problem for the family of algebras $A(q_1, q_2)$.

More specifically, we want to determine for another set $p_1, p_2 \in \mathbb{C}^\times$, when is

$$A(q_1, q_2) \cong A(p_1, p_2)?$$

Some isomorphisms

We will restrict to the case that our isomorphisms are graded, which means that vertices are sent to vertices, and arrows "follow" the vertices.

We can categorize each isomorphism into one of three types or a composition of the three types presented below.

(1) $\phi_\alpha : A(q_1, q_2) \to A(\alpha q_1, \alpha q_2) \qquad \alpha \in \mathbb{C}^\times$

$$\phi_\alpha(e_1) = e_1 \qquad \phi_\alpha(e_2) = e_2$$
$$\phi_\alpha(a) = a \qquad \phi_\alpha(b) = b \qquad \phi_\alpha(c) = \alpha c \qquad \phi_\alpha(d) = d$$

(2) $\psi : A(q_1, q_2) \to A(q_2, q_1)$

$$\psi(e_1) = e_2 \qquad \psi(e_2) = e_1$$
$$\psi(a) = b \qquad \psi(b) = a \qquad \psi(c) = d \qquad \psi(d) = c$$

(3) $\pi : A(q_1, q_2) \to A(q_1^{-1}, q_2^{-1})$

$$\pi(e_1) = e_1 \qquad \pi(e_2) = e_2$$
$$\pi(a) = c \qquad \pi(b) = d \qquad \pi(c) = a \qquad \pi(d) = b$$

Some isomorphisms

To show that ϕ_α is an isomorphism, we must first check that it extends to a homomorphism. In order to do this, we must show that it respects the relations on the algebra:

$$\phi_\alpha(a)\phi_\alpha(b) - q_1\phi_\alpha(c)\phi_\alpha(d) = ab - \alpha q_1 cd = 0$$
$$\phi_\alpha(b)\phi_\alpha(a) - q_2\phi_\alpha(d)\phi_\alpha(c) = ba - \alpha q_2 dc = 0$$

Because the map ϕ_α is a linear map on the set of arrows, to check bijectivity it suffices to verify that ϕ_α is bijective as a linear transformation, which is clear.

The other maps, ψ and π, are checked similarly.

Main theorem

Theorem (GZ)

Let $(q_1, q_2), (p_1, p_2) \in (\mathbb{C}^\times)^2$. *Then* $A(q_1, q_2) \cong A(p_1, p_2)$ *if and only if there is some* $\alpha \in \mathbb{C}^\times$ *such that*

1. $p_1 = \alpha q_1$ *and* $p_2 = \alpha q_2$,
2. $p_1 = \alpha q_2$ *and* $p_2 = \alpha q_1$,
3. $p_1 = \alpha q_1^{-1}$ *and* $p_2 = \alpha q_2^{-1}$, *or*
4. $p_1 = \alpha q_2^{-1}$ *and* $p_2 = \alpha q_1^{-1}$.

Proof.

$(\Leftarrow)$ For each case, we need to exhibit an isomorphism:

1. ϕ_α
2. $\phi_\alpha \circ \psi$
3. $\phi_\alpha \circ \pi$
4. $\phi_\alpha \circ \psi \circ \pi$

$\square$

Main theorem

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4. $p_1 = \alpha q_2^{-1}$ *and* $p_2 = \alpha q_1^{-1}$*.*

Proof.

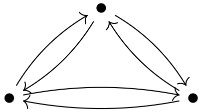
($\Rightarrow$) Let $\Phi : A(q_1, q_2) \to A(p_1, p_2)$ be an isomorphism. First we show that if Φ fixes the vertices then, up to composition with π, there must be some $\alpha \in \mathbb{C}^\times$ such that $p_1 = \alpha q_1$ and $p_2 = \alpha q_2$.

If Φ flips the vertices, then $\psi \circ \Phi$ fixes the vertices, and we reduce to the case above.

$\square$

Future work/directions

We are currently working on solving the isomorphism problem for quivers with the same rules, but with n-vertices. For example:



General idea: there is an action of the dihedral group on the quiver, and then arrows can be scaled in consistent way.

Thank You!