# Taft algebra actions on preprojective algebras

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(Joint work with Amrei Oswald)

# The Taft algebra

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero.

Let $H$ be a semisimple Hopf algebra that acts inner faithfully on the polynomial ring $\mathbb{k}[u, v]$. By Chan-Walton-Zhang (later generalized by Etingof-Walton), $H$ is a group algebra ($\mathbb{k}[u, v]$ has no semisimple quantum symmetries). However, $\mathbb{k}[u, v]$ does have finite-dimensional quantum symmetries through the Taft algebra.

Let $r, m \in \mathbb{N}$ such that $r > 1$ and $r \mid m$, and let $\lambda \in \mathbb{k}$ be a primitive $r^{\text{th}}$ root of unity. The (generalized) Taft algebra corresponding to this data is

$$T_\lambda(r, m) := \mathbb{k}\langle g, x : gx - \lambda xg, g^m - 1, x^r \rangle.$$

Then $T_\lambda(r, m)$ is a Hopf algebra with coalgebra structure is given by

$$\Delta(g) = g \otimes g \quad \Delta(x) = 1 \otimes x + x \otimes g \qquad \varepsilon(g) = 1 \quad \varepsilon(x) = 0$$

The Taft algebra is noncommutative, noncocommutative, and nonsemisimple.

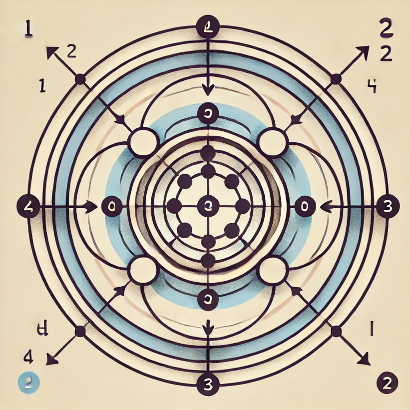
It is an example of a pointed Hopf algebra: every simple comodule is 1-dimensional.

They're points!

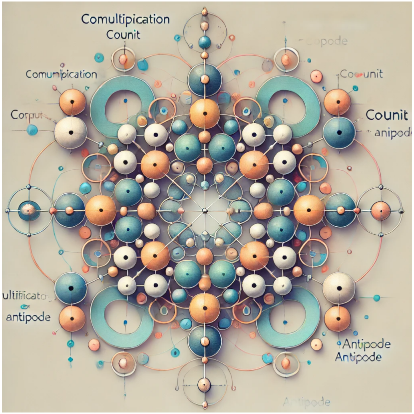
# The Taft algebra

What does the Taft algebra look like?

According to ChatGPT:



Make it Hopfier:



# Taft algebra actions

Allman classified (graded) Taft actions on polynomial rings. On $\mathbb{k}[u, v]$, the actions have the form (up to a change of variable)

$$g(u) = u, \quad g(v) = \lambda v, \quad x(u) = 0, \quad x(v) = u.$$

Actions of Taft algebras have been classified on a host of other families of algebras.

Actions on Artin–Schelter regular algebras:

- quantum planes (w/ Won, Yee)
- skew polynomial rings and quantum matrix algebras (w/ Cline)
- down-up algebras (w/ Crawford, Won)

On the other hand, actions of Taft algebras have been considered on path algebras of quivers by Kinser–Walton and Kinser–Oswald. (Also see work of Berrizbeitia on invariants of these actions.)

### Goal

Merge the above lines of study.

# Hopf actions on quivers

Let $Q$ be a finite quiver. There is a natural filtration on the path algebra $\Bbbk Q$ where $(\Bbbk Q)_k$ denotes the span of paths of length at most $k$. We consider actions that respect this filtration.

## Proposition (Kinser–Oswald)

*Let $\lambda$ be a primitive $r^{th}$ root of unity. Let $Q_0$ be the vertex set of a quiver.*

(A) *The following data determines a Hopf action of $T_\lambda(r, m)$ on $\Bbbk Q_0$.*

(i) *A permutation action of $G$ on the set $Q_0$ such that $\#(G \cdot i) \mid m$ for all $i \in Q_0$.*

(ii) *A collection of scalars $(\gamma_i \in \Bbbk)_{i \in Q_0}$ such that, for all $i \in Q_0$, $\gamma_{g \cdot i} = \lambda^{-1}\gamma_i$ and either $\#(G \cdot i) = r$ or $\gamma_i = 0$ for all $i$.*

*Given this data, the $x$-action is given by*

$$x \cdot e_i = \gamma_i(e_i - \lambda^{-1}e_{g \cdot i}) \quad \text{for all } i \in Q_0.$$

(B) *Every action of $T_\lambda(r, m)$ on $\Bbbk Q_0$ is of the form above.*

## Hopf actions on quivers

Let $Q$ be a quiver. We say a $\Bbbk$-linear endomorphism $\sigma : \Bbbk Q_0 \oplus \Bbbk Q_1 \to \Bbbk Q_0 \oplus \Bbbk Q_1$ is a quiver-Taft map for $Q$ if it satisfies (for all $a \in Q_1$):

$$\sigma(\Bbbk Q_0) = 0 \qquad \sigma(a) = e_{s(a)}\sigma(a)e_{g\cdot t(a)} \qquad \sigma(g \cdot a) = \lambda^{-1} g \cdot \sigma(a)$$

### Theorem (Kinser–Oswald)

*Let $\lambda$ be a primitive $r^{th}$ root of unity. Let $Q$ be the vertex set of a quiver.*

(A) *The following data determines a Hopf action of $T_\lambda(r, m)$ on $\Bbbk Q$.*

(i) *A Hopf action of $T_\lambda(r, m)$ on $\Bbbk Q_0$ as above;*

(ii) *A representation of $G$ on $\Bbbk Q_1$ satisfying (for all $a \in Q_1$):*

$$s(g \cdot a) = g \cdot s(a) \qquad t(g \cdot a) = g \cdot t(a),$$

*and $g^m$ acts as the identity on all of $\Bbbk Q$.*

(iii) *A quiver-Taft map $\sigma$ for $Q$ satisfying (for all $a \in \Bbbk Q_1$):*

$$\gamma^r_{s(a)} g^r(a) - \gamma^r_{t(a)} a = \sigma^r(a). \tag{1}$$

*Given this data, the $x$-action is given on $a \in Q_1$ by*

$$x \cdot a = \gamma_{t(a)} a - \gamma_{s(a)} \lambda^{-1} (g \cdot a) + \sigma(a).$$

(B) *Every action of $T_\lambda(r, m)$ on $\Bbbk Q$ is of the form above.*

# Preprojective algebras

The double $\overline{Q}$ of a quiver $Q$ is defined by setting $\overline{Q}_0 = Q_0$ and for every $a \in Q_1$ with $s(a) = e_i$ and $t(a) = e_j$ we add an arrow $a^*$ with $s(a^*) = e_j$ and $t(a^*) = e_i$.

The preprojective algebra associated to a quiver $Q$ is the quotient

$$\Bbbk\overline{Q}/\left(\sum_{a\in Q_1} aa^* - a^*a \in \overline{Q}\right)$$

For the remainder, let $Q = \widetilde{A_{n-1}}$ with $n > 2$:

$$\begin{tikzcd}
 & & e_0 \arrow[dll, "a_0"'] & & \\
e_1 \arrow[r, "a_1"'] & e_2 \arrow[r, "a_2"'] & \cdots \arrow[r, "a_{n-3}"'] & e_{n-2} \arrow[r, "a_{n-2}"'] & e_{n-1} \arrow[ull, "a_{n-1}"']
\end{tikzcd}$$

If $n = 1$, then $\Pi_Q \cong \Bbbk[x, y]$.

In general, $\Pi_Q$ is Calabi–Yau, noetherian, has global and GK dimension 2, and is PI.

## Preprojective algebras of type $A$

Let $g$ be a graded automorphism of $\Bbbk\overline{Q}$. The action of $g$ on $\Bbbk\overline{Q}$ is described by one of the following cases where $d$ is an integer satisfying $0 \leq d \leq n-1$:

- (rotation) There exists $\mu_i, \mu_i^* \in \Bbbk^\times$ such that for all $0 \leq i \leq n-1$,
$$g \cdot e_i = e_{i+d}, \qquad g \cdot a_i = \mu_i a_{i+d}, \qquad g \cdot a_i^* = \mu_i^* a_{i+d}^*.$$
- (reflection) There exists $\mu_i, \mu_i^* \in \Bbbk^\times$ such that for all $0 \leq i \leq n-1$,
$$g \cdot e_i = e_{n-(d+i)}, \qquad g \cdot a_i = \mu_i a_{n-(d+i+1)}^*, \qquad g \cdot a_i^* = \mu_i^* a_{n-(d+i+1)}.$$

Given an action of $g$ as above, the action of $T_\lambda(r, m)$ on $\Bbbk\overline{Q}$ is given by

$$x \cdot e_i = \gamma_i e_i - \gamma_i \lambda^{-1} e_{g \cdot i} \qquad x \cdot a_i = \gamma_{i+1} a_i - \gamma_i \lambda^{-1} g \cdot a_i + \sigma(a_i)$$
$$x \cdot a_i^* = \gamma_i a_i^* - \gamma_{i+1} \lambda^{-1} g \cdot a_i^* + \sigma(a_i^*).$$

### Lemma

*An inner faithful action of $T_\lambda(r, m)$ on $\Bbbk\overline{Q}$ descends to an action on $\Pi_Q$ if and only if*

$$\begin{aligned} 0 &= \mu_i \mu_i^* - \mu_{i+1} \mu_{i+1}^* \\ 0 &= a_i^* \sigma(a_i) + \sigma(a_i^*)(g \cdot a_i) - a_{i+1} \sigma(a_{i+1}^*) - \sigma(a_{i+1})(g \cdot a_{i+1}^*) \end{aligned}$$

# Rotations

## Theorem

*Suppose $T_\lambda(r,m)$ acts linearly and inner faithfully on $\Bbbk\overline{Q}$ so that $g \cdot e_i = e_{i+d}$ for some $0 \leq d < n$, and this action descends to an action on $\Pi_Q$.*

*(I) For each $i \in Q_0$, there exist scalars $\mu_i, \mu_i^* \in \Bbbk^\times$ and $\gamma_i \in \Bbbk$ so that*

$$g \cdot a_i = \mu_i a_{i+d}, \quad g \cdot a_i^* = \mu_i^* a_{i+d}^*, \quad x \cdot e_i = \gamma_i(e_i - \lambda^{-1} e_{i+d}), \quad \gamma_{i+d} = \lambda^{-1}\gamma_i.$$

*(II) If $\gamma_i = 0$ for all $i \in Q_0$, then $d \in \{1, 2, n-2, n-1\}$ and there exist $c, c^* \in \Bbbk^\times$ so that*

$$x \cdot a_i = \begin{cases} \lambda^i \mu_1\mu_2 \dots \mu_i c e_i & \text{if } d = n-1 \\ \lambda^i \mu_1\mu_2 \dots \mu_i c a_{i-1}^* & \text{if } d = n-2 \\ 0 & \text{if } d \neq n-1, n-2 \end{cases}$$

$$x \cdot a_i^* = \begin{cases} (\lambda^i \mu_0^*\mu_1^* \dots \mu_{i-1}^*)^{-1} c^* e_{i+1} & \text{if } d = 1 \\ (\lambda^i \mu_0^*\mu_1^* \dots \mu_{i-1}^*)^{-1} c^* a_{i+1} & \text{if } d = 2 \\ 0 & \text{if } d \neq 1, 2 \end{cases}$$

*where $\mu_i\mu_i^* = \lambda^{-1}$ and if $n$ or $d$ is odd, $m = \mathrm{lcm}(r,n)$.*

## Rotations

### Theorem

*Suppose $T_\lambda(r,m)$ acts linearly and inner faithfully on $\Bbbk\overline{Q}$ so that $g \cdot e_i = e_{i+d}$ for some $0 \leq d < n$, and this action descends to an action on $\Pi_Q$.*

*(III) Suppose there exists an $i \in Q_0$ so that $\gamma_i \neq 0$.*

- *($2 < d < n-2$ and $n > 4$) If $i \equiv k \mod d$ for $0 \leq k < d$, we have*

$$x \cdot a_i = \lambda^{\frac{k-i}{d}} \xi_0 \xi_1 \dots \xi_{k-1} \gamma_0 \left( \xi_k a_i - \lambda^{-1} \mu_i a_{i+d} \right),$$
$$x \cdot a_i^* = \lambda^{\frac{k-i}{d}} \xi_0 \xi_1 \dots \xi_{k-1} \gamma_0 \left( a_i^* - \lambda^{z-1} \xi_k \mu_i^{-1} a_{i+d}^* \right),$$

*for some $z \in \mathbb{Z}$, $\gamma_0, \xi_0, \xi_1, \dots \xi_{d-1} \in \Bbbk^\times$ where the $\xi_i$ are $m^{\text{th}}$ roots of unity so that*

$$\xi_0 \xi_1 \dots \xi_{d-1} = \lambda^{-1} \quad \textit{and} \quad \mu_k \mu_{k+d} \dots \mu_{k+(r-1)d} = \xi_k^r.$$

*This case also implies that $\gamma_i$ is nonzero.*

## Rotations

### Theorem

*Suppose $T_\lambda(r, m)$ acts linearly and inner faithfully on $\Bbbk\overline{Q}$ so that $g \cdot e_i = e_{i+d}$ for some $0 \leq d < n$, and this action descends to an action on $\Pi_Q$.*

*(III) Suppose there exists an $i \in Q_0$ so that $\gamma_i \neq 0$.*

- *($d = n - 1$ and $n \neq 3$) We have $r = m = n$. The $\gamma_i$ satisfy $\gamma_i = \lambda^i \gamma_0$. There exists $c \in \Bbbk$ such that the $x$-action is given by*

$$x \cdot a_i = \lambda^i \gamma_0(\lambda a_i - \lambda^{-1}\mu_i a_{i-1}) + \lambda^i \mu_i \mu_{i-1} \cdots \mu_1 c e_i$$
$$x \cdot a_i^* = \lambda^i \gamma_0(a_i^* - \mu_i^* a_{i-1}^*)$$

  *If $c \neq 0$, then $\mu_i^* = (\lambda \mu_i)^{-1}$.*

- *($d = 1$ and $n \neq 3$) Similar*
- *($d = 2$ or $d = n - 2$, and $n \neq 4$) NSFW*

### Theorem (G-Oswald)

*If $T = T_\lambda(r, m)$ acts on $\Pi_Q$ with $n > 3$, $d = 1$ or $n - 1$, and $\sigma = 0$, then $(\Pi_Q)^T = \mathcal{Z}(\Pi_Q)$.*

# Reflections

## Theorem

*Suppose $T_\lambda(r, m)$ acts linearly and inner faithfully on $\Bbbk\overline{Q}$ so that $g \cdot e_i = e_{n-(d+i)}$ for some $0 \le d \le n-1$, and this action descends to an action on $\Pi_Q$. Then $r = 2$ and the action is described as follows:*

- *There exists scalars $\mu_i \in \Bbbk^\times$ such that*

$$g \cdot e_i = e_{n-(d+i)}, \qquad g \cdot a_i = \mu_i a^*_{n-(d+i+1)}, \qquad g \cdot a^*_i = \mu_i^{-1} a_{n-(d+i+1)}$$

  *where $(\mu_i \mu^{-1}_{n-(d+i+1)})^{m/2} = 1$ for all $i$.*

- *There exist scalars $\gamma_i \in \Bbbk$ so that*

$$x \cdot e_i = \gamma_i(e_i + e_{n-(d+i)})$$

  *and the $\gamma_i$ satisfy $\gamma_i = -\gamma_{n-(d+i)}$ (so $\gamma_j = 0$ when $g \cdot j = j$) and*

$$\gamma^2_{i+1} = \mu_i \mu^{-1}_{n-(d+i+1)} \gamma^2_i.$$

  *for $i \neq j-1, j$.*

# Reflections

## Theorem

*Suppose $T_\lambda(r, m)$ acts linearly and inner faithfully on $\Bbbk\overline{Q}$ so that $g \cdot e_i = e_{n-(d+i)}$ for some $0 \le d \le n-1$, and this action descends to an action on $\Pi_Q$. Then $r = 2$ and the action is described as follows:*

- *If $j$ is a vertex such that $g \cdot j = j$, then we let $c_j = \pm\gamma_{j-1}\sqrt{\mu_j\mu_{j-1}}$ and*

$$x \cdot a_{j-1} = \gamma_{j-1}\mu_{j-1}a_j^* + \mu_j^{-1}c_j a_{j-1} \qquad x \cdot a_j = \gamma_{j+1}a_j + c_j a_{j-1}^*$$

$$x \cdot a_{j-1}^* = \gamma_{j-1}a_{j-1}^* - (\mu_j\mu_{j-1})^{-1}c_j a_j \qquad x \cdot a_j^* = \gamma_{j+1}\mu_i^{-1}a_{j-1} - \mu_j^{-1}c_j a_j^*$$

- *If $k$ is such that $g \cdot k = k+1$ and $g \cdot (k+1) = k$, then there exists $c_k \in \Bbbk$ such that*

$$x \cdot a_k = \gamma_{k+1}a_k + \gamma_k\mu_i a_{k+1}^* + c_k e_k$$

$$x \cdot a_k^* = \gamma_k a_k^* + \gamma_{k+1}\mu_i^{-1}a_{k+1} - \mu_k^{-1}c_k e_{k+1}$$

  *If $\gamma_i = 0$ for all $i$, then $c_k \neq 0$ for some such $k$.*

- *If $i$ satisfies neither of the above conditions, then*

$$x \cdot a_i = \gamma_{i+1}a_i + \gamma_i\mu_i a_{n-(d+i+1)}^* \qquad x \cdot a_i^* = \gamma_i a_i^* + \gamma_{i+1}\mu_i^{-1}a_{n-(d+i+1)}$$

## Future work

Some directions for future research:

- Compute invariants in some of the other cases including reflection actions.
- One can consider quantized versions of the preprojective algebra $\Pi_Q$ where the relations are
$$a_i^* a_i = q_i a_{i+1} a_{i+1}^* \qquad \text{for each } i,\ q_i \in \mathbb{k}^\times.$$
- Actions of other pointed Hopf algebras, e.g., bosonizations of quantum linear spaces.

# Thank You!