Four-Vertex Quivers Supporting Graded Calabi-Yau Algebras

Ke Liang

Advisor: Jason Gaddis

Department of Mathematics and Statistics, Wake Forest University

The Big Idea

Study Calabi-Yau algebras, which are algebraic objects that encode information about Calabi-Yau manifolds arising in string theory.

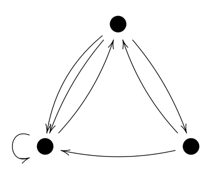
Background

If A is a graded Calabi-Yau algebra of global dimension 3 (3CY), then A is determined by a quiver and a superpotential.

Definition

A **quiver** is a directed graph $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 is the set of arrows. The quiver is strongly connected if for every pair of vertices, v_1,v_2, there is a path from v_1 to v_2.

The adjacency matrix M of a quiver is obtained by assigning M_{ij} to be the number of arrows from vertex i to vertex j. For example, the quiver



has adjacency matrix $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

If A is 3CY, then A is isomorphic to $\mathbb{C}Q/\{\delta_a \omega : a \in Q_0\}$ where ω is a homogeneous superpotential on a strongly connected quiver Q [Boc08]. We are interested in classifying the quivers Q that can support such a superpotential and have "nice" growth properties.

Definition

A graded algebra R has **Gelfand-Kirillov (GK) dimension** n if its graded components "grow" like a polynomial ring in n variables.

The GK dimension of a graded algebra can be read from its Hilbert series. The Hilbert series for the polynomial ring over $\mathbb{C}$, $\mathbb{C}[x_1, \dots, x_n]$, is $(1-t)^{-n}$. The Hilbert series of a 3CY algebra is

$$H(t) = 1/q(t) \text{ where } q(t) = 1 - M_Q t + M_Q^T t^{d-1} - t^d,$$

M_Q is the adjacency matrix of Q, and $d = \deg(\omega)$. The GK dimension of A is then the multiplicity of 1 as a root of $H_A(t)$.

Goal

Classify quivers with four vertices that support a 3CY algebra with finite GK dimension. This extends work on two and three vertices by Gaddis and Rogalski [GR16].

Good Matrices: Circulant

Circulant matrices have the two forms:

$$M_{\text{right}} = \begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix}, M_{\text{left}} = \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

where $a, b, c, d \in \mathbb{N}$ and all matrices are strongly connected.

Suppose M is a circulant adjacency matrix of a 3CY algebra. Then A has finite GK dimension only if the eigenvalues satisfy certain criteria [RR16]. We list all such tuples (a, b, c, d) and let $\rho(M)$ denote the largest absolute value of the eigenvalues. In the case of circulant matrices, $\rho(M)$ is the sum of the entries in any row.

For $\deg(w) = 3$, all eigenvalues in $[-1, 3]$ and $\rho(M) = 3$.

- Right: (0,0,1,2), (0,1,1,1),(0,2,1,0),(1,0,1,1),(1,1,0,1),(1,1,1,0);
- Left: (0,1,1,1),(1,0,1,1),(1,1,0,1),(1,1,1,0).

For $\deg(w) = 4$, all eigenvalues in $[-2, 2]$ and $\rho(M) = 2$

- Right: (0,0,0,2), (0,1,0,1),(0,2,0,0);
- Left: (0,0,1,1),(0,1,0,1),(0,1,1,0),(1,0,0,1),(1,1,0,0).

Good Matrices: McKay Quiver

McKay quivers are defined in terms of representations of groups and are known to support CY algebras. The number of vertices in the McKay quiver is the number of finite-dimensional irreducible representations of the group. Hence, the groups we consider are the alternating group A_4, the cyclic group $\mathbb{Z}_4$, and the dihedral groups D_4 and D_{10}. We list all 4-vertex McKay quivers for each group that correspond to a 2- or 3-dimensional representation.

	dim 2	dim 3
A_4	none	$\begin{pmatrix} 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \\ 1&1&1&2 \end{pmatrix}$
$\mathbb{Z}_4$	$\begin{pmatrix} 0&0&1&1 \\ 0&0&1&1 \\ 1&1&0&0 \\ 1&1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&2 \\ 0&0&2&0 \\ 2&0&0&0 \\ 0&2&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&1&1&0 \\ 1&1&0&1 \\ 0&1&1&1 \\ 1&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{pmatrix}, \begin{pmatrix} 0&1&2&0 \\ 1&0&0&2 \\ 0&2&0&1 \\ 2&0&1&0 \end{pmatrix}$
D_4	$\begin{pmatrix} 0&1&1&0 \\ 1&0&0&1 \\ 1&0&0&1 \\ 0&1&1&0 \end{pmatrix}$	$\begin{pmatrix} 1&1&1&0 \\ 1&1&0&1 \\ 1&0&1&1 \\ 0&1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&1 \\ 1&1&1&0 \\ 0&1&1&1 \\ 1&0&1&1 \end{pmatrix}, \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&1 \\ 1&0&1&1 \\ 1&1&0&1 \\ 1&1&1&0 \end{pmatrix}$
D_{10}	$\begin{pmatrix} 0&0&1&0 \\ 0&0&1&0 \\ 1&1&0&1 \\ 0&0&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&1&0 \\ 0&1&1&0 \\ 1&1&1&1 \\ 0&0&1&2 \end{pmatrix}, \begin{pmatrix} 1&0&0&1 \\ 0&1&0&1 \\ 0&0&2&1 \\ 1&1&1&1 \end{pmatrix}, \begin{pmatrix} 2&1&0&0 \\ 1&2&0&0 \\ 0&0&3&0 \\ 0&0&0&3 \end{pmatrix}$

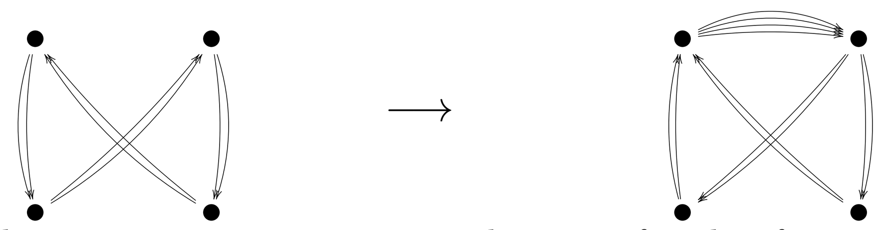
Good Matrices: Mutation

A quiver Q can be mutated if it has no loops and no 2-cycles.

- Fix a vertex v.
- Reverse every arrow that starts or ends at v.
- Remove any maximum collection of disjoint 2-cycle.

If Q supports a CY algebra A and is mutateable, then any mutation also supports a CY algebra but the mutation may not preserve GK dimension.

The second matrix in the dimension 2 list for $\mathbb{Z}_4$ can be mutated.



Through successive mutations one obtains a family of quivers of the form

$$Q_{b,c} = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & b & b \\ c & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix}, \qquad b^2 + c^2 = 4bc - 8.$$

Mutating $Q_{b,c}$ at vertex 1 or 2 produces another quiver of this form. We list the rules for mutating at each vertex below.

- Mutating on v_1: $[b, c] \xrightarrow{v_1} [c, 4c - b] \xrightarrow{v_1} [b, c]$
- Mutating on v_2: $[b, c] \xrightarrow{v_2} [4b - c, b] \xrightarrow{v_2} [b, c]$
- Mutating on v_1 and v_2: $[b, c] \xrightarrow{v_1} [c, 4c - b] \xrightarrow{v_2} [4c - b, 15c - b]$
- Mutating on v_2 and v_1: $[b, c] \xrightarrow{v_2} [4b - c, b] \xrightarrow{v_1} [15b - 4c, 4b - c]$
- Mutating on v_3 and v_4 (supposing $bc \geq 4$):

$$[b, c] \xrightarrow[v_4, v_3, v_4, v_3]{v_3, v_4, v_3, v_4} [b, c]$$

References

[Boc08] Raf Bocklandt.
Graded Calabi Yau algebras of dimension 3.
J. Pure Appl. Algebra, 212(1):14–32, 2008.

[GR16] Jason Gaddis and Daniel Rogalski.
Three-vertex quivers supporting a graded calabi-yau algebra of global dimension 3.
(in preparation), 2016.

[RR16] Manuel Reyes and Daniel Rogalski.
Skew-CY algebras of finite GK dimension.
(in preparation), 2016.

Classifying equal-loop quivers

Beyond finding good quivers, we also wanted to rule out quivers that can not support a 3CY algebra of finite GK dimension. We focused on quivers which have the same number of loops at each vertex.

Let A be a 3CY algebra of GK dimension at least 3, Q the quiver corresponding to A, and $H_A(t) = 1/q(t)$. Set $p(t) = \det[q(t)]$.

Since the GK dimension of A is at least 3, $(1-t)^3$ is a factor of $p(t)$. Using Maple we verified that $p(t)$ is palindromic. Since $(1-t)^3$ is anti-palindromic, then $p(t)/(1-t)^3$ must be anti-palindromic and so 1 is a root. We conclude that $(1-t)^4$ is a factor of $p(t)$.

Since all roots of $p(t)$ must be roots of unity [GR16], it follows that

$$p(t) = \begin{cases} (1-t)^4 \prod_{i=1}^{4} (1 - k_i t + t^2) & \text{if } \deg(w) = 3 \\ (1-t)^4 \prod_{i=1}^{6} (1 - k_i t + t^2) & \text{if } \deg(w) = 4 \end{cases}$$

where $k_i \in \mathbb{R}$ and $|k_i| \leq 2$ for each i.

Our goal is to determine which matrices could have matrix polynomials of these forms. Write an arbitrary 4×4 matrix $M = (m_{ij})$ over $\mathbb{C}$. We define the following values for M,

$$\lambda = \sum_{i=1} m_{ii}, \quad \beta = \sum_{i \neq j} m_{ii} m_{jj}, \quad \gamma = \sum_{i \neq j} m_{ij} m_{ji}.$$

Suppose $\det(w) = 3$ and let $P(t) = \det(I - Mt + M^T t^2 - It^3)$. By comparing coefficients of $p(t)$ and $P(t)$ we have

$$\lambda = (k_1 + k_2 + k_3 + k_4) + 4$$
$$\gamma = -3\lambda + \beta + 6 - \sum_{i \neq j} k_{ij}.$$

Using Mathematica, we obtain a maximum value of gamma corresponding to each possible set of diagonal entries. This allows us to check a smaller list of matrices.

Bad Matrices

Ruling-out criteria:

- Adjacency matrices cannot fit the formula of $p(t)$ above.
- The quiver is not strongly connected.
- The adjacency matrix contains negative entries.

For example,

$$M = \begin{pmatrix} x & a & 0 & 0 \\ 0 & x & b & 0 \\ 0 & 0 & x & c \\ d & 0 & 0 & x \end{pmatrix}$$

The coefficient of t^3 in $P(t) - p(t)$ is $a^2 + b^2 + c^2 + d^2 = 0$. The only possible solution is $a = b = c = d = 0$. This implies that M is not strongly connected.

Theorem

Let Q be an equal-loop quiver on 4 vertices supporting a CY algebra of global dimension 3 and finite GK dimension. If the degree of the superpotential is 3, then Q is a McKay quiver or a mutation.

Future Research

- Continue working on the degree four superpotential case.
- Find more general rules to classify quivers on four vertices supporting CY algebras.
- Expand research into the 5-vertex cases or look for more general criteria for $n \times n$ matrices.

Acknowledgements

Thanks to the Wake Forest Undergraduate Research and Creative Activities Center for supporting this project.