Classifying Actions of $T_n \otimes T_n$ on Path Algebras of Quivers

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Definition

A (finite) quiver $Q = Q_0 \cup Q_1$ *is a directed graph with finite vertex set* Q_0 *and finite arrow set* Q_1*.* Q *is said to be nontrivial if* $Q_1 \neq \varnothing$*.*

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Definition

A quiver $Q' = Q'_0 \cup Q'_1$ is a subquiver of a quiver $Q = Q_0 \cup Q_1$ if $Q'_0 = Q_0$ and $Q'_1 \subseteq Q_1$. We call Q' a proper subquiver of Q if $Q'_1 \subsetneq Q_1$.

Definition

Let Q be a quiver. For an arrow $a \in Q_1$, define the source of a, denoted $s(a)$, to be the vertex from which a starts. Define the target of a, denoted $t(a)$, to be the vertex to which a points. (Note that for a trivial path e_i, we have that $s(e_i) = t(e_i) = i$.)

Example Quiver

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r, bend left] & \underset{3}{\bullet} \arrow[l, bend left] \end{tikzcd}$$

Example Quiver

$$Q = \begin{tikzcd} \overset{\color{blue}e_1}{\underset{1}{\bullet}} \arrow[r, "\color{red}a_1^2"] & \overset{\color{blue}e_2}{\underset{2}{\bullet}} \arrow[r, bend left, "\color{red}a_2^3"] & \overset{\color{blue}e_3}{\underset{3}{\bullet}} \arrow[l, bend left, "\color{red}a_3^2"] \end{tikzcd}$$

More Examples

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \underset{2}{\bullet} & \underset{3}{\bullet} \arrow[l] \end{tikzcd}$$

More Examples

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[loop above] & \underset{2}{\bullet} \arrow[r] & \underset{3}{\bullet} \end{tikzcd}$$

Definition

We define a path algebra $\Bbbk Q$ as follows. For a field $\Bbbk$, $\Bbbk Q$ has a $\Bbbk$-basis given by all nontrivial paths in Q together with the trivial loops. Multiplication is defined by

$$pq = \begin{cases} p \cdot q & \text{if } t(p) = s(q) \\ 0 & \text{otherwise} \end{cases}$$

for paths p and q. Furthermore, if exactly one of p or q is a trivial loop and $pq \neq 0$, then the trivial loop acts as a one-sided identity. The multiplicative identity is $1_{\Bbbk Q} = \sum_{i \in Q_0} e_i$.

Definition

A group G is said to act on Q if G acts on the sets Q_0 and Q_1 such that each element of G acts by quiver automorphism.

Hopf Algebra

$$
\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes \mathrm{id}"] & & H \otimes H \arrow[dr] & \\
H \arrow[ur] \arrow[rr, "\varepsilon"] \arrow[dr] & & K \arrow[rr, "\Delta"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id} \otimes S"'] & & H \otimes H \arrow[ur] &
\end{tikzcd}
$$

Definition (Taft 1971)

The Taft algebra T_n is an n^2 - dimensional Hopf algebra generated by a nonzero element g with $\Delta(g) = g \otimes g$ (that is, g is grouplike), and an element x such that $\Delta(x) = g \otimes x + x \otimes 1$ (i.e., x is $(g, 1)$-skew-primitive), subject to relations:

$$g^n = 1, \qquad x^n = 0, \qquad xg = \lambda gx$$

for a primitive n^{th} root of unity λ.

Definition

Given a Hopf algebra H and an algebra A, we say that H acts on A (from the left) if, for all $h \in H$ and $p, q \in A$,

1. *A is a left H-module;*
2. $h \cdot (pq) = \Sigma(h_1 \cdot p)(h_2 \cdot q)$; *and*
3. $h \cdot 1_A = \epsilon(h)1_A$.

Proposition (Kinser, Walton 2016)

Let $Q_0 = \{0, 1, \ldots, m-1\}$ *be the vertex set of a quiver, where* m *divides* n*, and* $\mathbb{Z}_n$ *acts on* $\Bbbk Q_0$ *by* $g \cdot e_i = e_{i+1}$*. Here, the subscripts are always interpreted modulo* m*.*

1. *If* $m < n$*, then* x *acts on* $\Bbbk Q_0$ *by 0.*
2. *If* $m = n$*, then the action of* x *on* $\Bbbk Q_0$ *is exactly of the form*

$$x \cdot e_i = \gamma \zeta^i (e_i - \zeta e_{i+1}) \qquad \text{for all } i,$$

   where $\gamma \in \Bbbk$ *can be any scalar.*

In particular, we can extend the action of $\mathbb{Z}_n$ *on* $\Bbbk Q_0$ *to an inner faithful action of* T_n *on* $\Bbbk Q_0$ *if and only if* $m = n$*.*

Proposition (Kinser, Walton 2016)

Suppose we have an action of T_n on $\Bbbk Q$, and let $a \in Q_1$ with $s(a) = i$ and $t(a) = j$. Then there exist scalars $\alpha, \beta, \lambda \in \Bbbk$ such that

$$x \cdot a = \alpha a + \beta(g \cdot a) + \lambda\sigma(a).$$

Moreover, the scalars can be determined in special cases depending on the relative configuration of a and $g \cdot a$.

Definition

A quiver Q is $\mathbb{Z}_n$-minimal if it is a $\mathbb{Z}_n$-stable subquiver of K_m or $K_{m,m'}$ where m, m' divide n. A $\mathbb{Z}_n$-component is a $\mathbb{Z}_n$-minimal subquiver of Q which is maximal under inclusion.

Theorem (Kinser, Walton 2016)

Let Q be a quiver with a $\mathbb{Z}_n$-action. The T_n-actions on the path algebra of Q extending the given $\mathbb{Z}_n$-action are in bijection with the compatible collections of T_n-actions on path algebras of the $\mathbb{Z}_n$-components of Q.

Definition

Let G be a group that acts on a nontrivial quiver Q and let $g \in G$. The quiver Q is said to be g-minimal if it consists of exactly one arrow orbit under the action of g.

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Let G be a group that acts on a nontrivial quiver Q and let $g \in G$. A g-component of Q is a g-minimal subquiver.

$T_2 \otimes T_2$

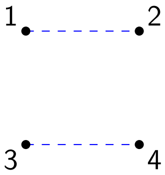
1. Elements g_1, g_2 generate $\mathbb{Z}_2 \times \mathbb{Z}_2$
2. Elements x_1, x_2 extend this to Taft action
3. Subject to relations

$$
\begin{aligned}
0 &= g_1 g_2 - g_2 g_1 = x_1 x_2 - x_2 x_1 \\
&= g_1 x_2 - x_2 g_1 = g_2 x_1 - x_1 g_2.
\end{aligned}
$$

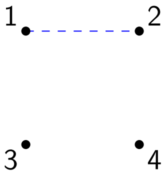
Theorem

There are three distinct nontrivial $\mathbb{Z}_2 \times \mathbb{Z}_2$ actions on a four-vertex quiver, and twenty-nine possibilities for a nontrivial quiver.

Proof Sketch

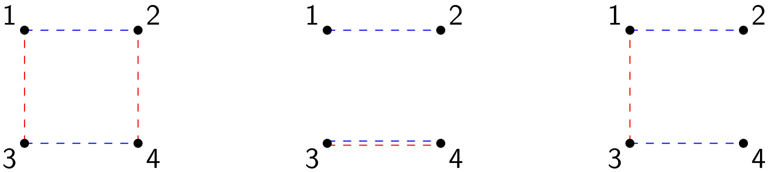


Case 1

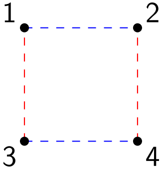


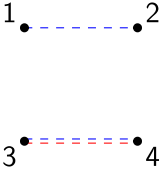
Case 2

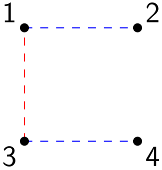
Proof Sketch - Case 1



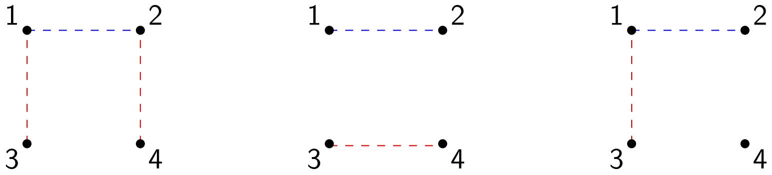
Proof Sketch - Case 1



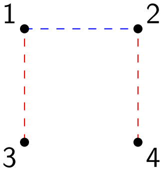




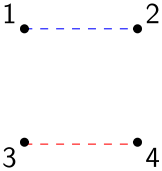
Proof Sketch - Case 2

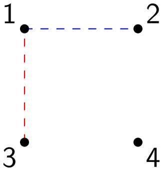


Proof Sketch - Case 2



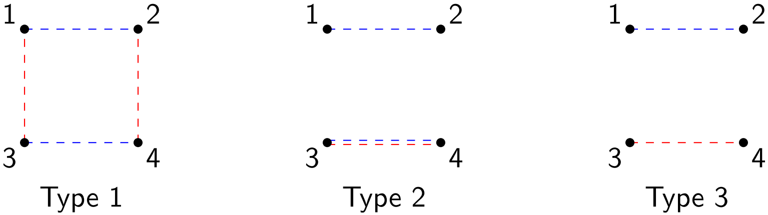
Fails



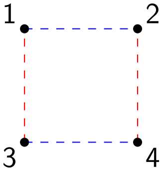


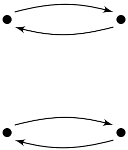
Fails

Types of Actions



Arrow Orbits of Type 1





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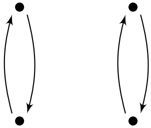
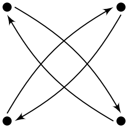


V



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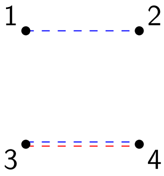
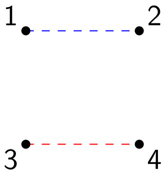
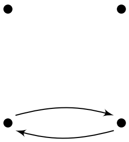
Arrow Orbits of Types 2 and 3



Type 2



Type 3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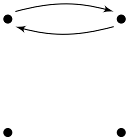
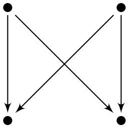

T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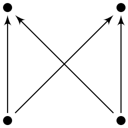
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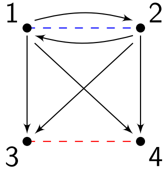
Example Quiver



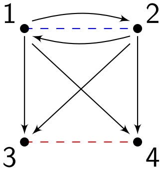
If the action of g_i on a particular vertex is trivial, then the corresponding x_i takes that vertex to 0. For example,

$$x_1(e_3) = \gamma_3^1 e_3,$$

and since

$$0 = (x_1 g_1 + g_1 x_1)(e_3) = x_1(e_3) + g_1(\gamma_3^1 e_3) = 2\gamma_3^1 e_3,$$

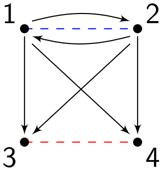
this forces $\gamma_3^1 = 0$.



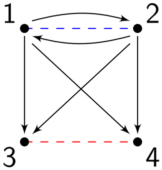
$g_1(e_1) = e_2$	$x_1(e_1) = \gamma_1^1 e_1 + \gamma_1^2 e_2$	$g_2(e_1) = e_1$	$x_2(e_1) = 0$
$g_1(e_2) = e_1$	$x_1(e_2) = \gamma_2^1 e_1 + \gamma_2^2 e_2$	$g_2(e_2) = e_2$	$x_2(e_2) = 0$
$g_1(e_3) = e_3$	$x_1(e_3) = 0$	$g_2(e_3) = e_4$	$x_2(e_3) = \delta_3^1 e_1 + \delta_3^2 e_3$
$g_1(e_4) = e_4$	$x_1(e_4) = 0$	$g_2(e_4) = e_3$	$x_2(e_4) = \delta_4^1 e_2 + \delta_4^2 e_4$

Because each x_i is $(g_i, 1)$-skew, we can simplify some of the results on the arrows. For example, by writing an arrow as an arrow composed with a vertex, then substituting in our previous results, we can get things like

$$\begin{aligned} x_2(a_1^2) &= \sigma_5^1 a_1^2 + \sigma_5^2 a_2^1 \\ &= x_2(e_1 a_1^2) = g_2(e_1) x_2(a_1^2) + x_2(e_1) a_1^2 \\ &= x_2(a_1^2 e_2) = g_2(a_1^2) x_2(e_2) + x_2(a_1^2) e_2 \\ &= \sigma_5^1 a_1^2. \end{aligned}$$

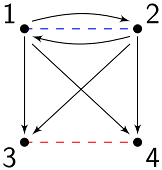


$g_1(a_1^2) = \mu_1^2 a_2^1$	$x_1(a_1^2) = \beta_1^1 a_1^2 + \beta_1^2 a_2^1$	$g_2(a_1^2) = \eta_1^2 a_1^2$	$x_2(a_1^2) = \sigma_1^1 a_1^2$
$g_1(a_2^1) = \mu_2^1 a_1^2$	$x_1(a_2^1) = \beta_2^1 a_1^2 + \beta_2^2 a_2^1$	$g_2(a_2^1) = \eta_2^1 a_2^1$	$x_2(a_2^1) = \sigma_2^1 a_2^1$
$g_1(a_1^3) = \mu_1^3 a_2^3$	$x_1(a_1^3) = \beta_3^1 a_1^3 + \beta_3^2 a_2^3$	$g_2(a_1^3) = \eta_1^3 a_1^4$	$x_2(a_1^3) = \sigma_3^1 a_1^3 + \sigma_3^2 a_1^4$
$g_1(a_1^4) = \mu_1^4 a_2^4$	$x_1(a_1^4) = \beta_4^1 a_1^4 + \beta_4^2 a_2^4$	$g_2(a_1^4) = \eta_1^4 a_1^3$	$x_2(a_1^4) = \sigma_4^1 a_1^3 + \sigma_4^2 a_1^4$
$g_1(a_2^3) = \mu_2^3 a_1^4$	$x_1(a_2^3) = \beta_5^1 a_1^3 + \beta_5^2 a_2^3$	$g_2(a_2^3) = \eta_2^3 a_2^4$	$x_2(a_2^3) = \sigma_5^1 a_2^3 + \sigma_5^2 a_2^4$
$g_1(a_2^4) = \mu_2^4 a_1^4$	$x_1(a_2^4) = \beta_6^1 a_1^4 + \beta_6^2 a_2^4$	$g_2(a_2^4) = \eta_2^4 a_2^3$	$x_2(a_2^4) = \sigma_6^1 a_2^3 + \sigma_6^2 a_2^4$

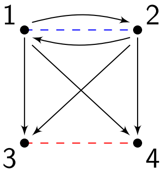


$(x_1g_1 + g_1x_1)(e_1) = (\gamma_2^1 + \gamma_1^2)e_1 + (\gamma_1^1 + \gamma_2^2)e_2$
$(x_1g_1 + g_1x_1)(e_2) = (\gamma_1^1 + \gamma_2^2)e_1 + (\gamma_1^2 + \gamma_2^1)e_2$
$(x_1g_1 + g_1x_1)(e_3) = 2\gamma_3^1 e_3$
$(x_1g_1 + g_1x_1)(e_4) = 2\gamma_4^1 e_4$
$(x_2g_2 + g_2x_2)(e_1) = 2\delta_1^1 e_1$
$(x_2g_2 + g_2x_2)(e_2) = 2\delta_2^1 e_2$
$(x_2g_2 + g_2x_2)(e_3) = (\delta_4^1 + \delta_3^2)e_3 + (\delta_4^2 + \delta_3^1)e_4$
$(x_2g_2 + g_2x_2)(e_4) = (\delta_3^1 + \delta_4^2)e_3 + (\delta_3^2 + \delta_4^1)e_4$

$$
\begin{aligned}
0 = (x_2 g_2 + g_2 x_2)(a_1^3) &= x_2(g_2(a_1^3)) + g_2(x_2(a_1^3)) \\
&= x_2(\eta_1^3 a_1^4) + g_2(\sigma_1^1 a_1^3 + \sigma_1^2 a_1^4) \\
&= (\eta_1^3 \sigma_2^1 + \eta_1^4 \sigma_1^2) a_1^3 + (\sigma_1^1 + \sigma_2^2) \eta_1^3 a_1^4
\end{aligned}
$$



$$
\begin{array}{|c|}
\hline
(x_1g_1 + g_1x_1)(a_1^2) = \mu_1^2(\mu_2^1\gamma_2^2 + \gamma_1^1)a_1^2 + (\mu_1^2\gamma_2^2 + \mu_2^1\beta_5^2)a_2^1 \\
(x_1g_1 + g_1x_1)(a_1^3) = (\gamma_1^1 + \gamma_2^2)\mu_1^3 a_2^3 + (\mu_1^3\beta_3^1 + \mu_2^3 + \beta_1^3)a_1^3 \\
(x_1g_1 + g_1x_1)(a_1^4) = (\mu_1^4\beta_4^2 + \mu_2^4\beta_2^4)a_1^4 + \mu_1^4(\gamma_2^2 + \gamma_1^1)a_2^4 \\
(x_1g_1 + g_1x_1)(a_2^1) = \mu_2^1(\beta_5^1 + \beta_6^2)a_1^2 + \mu_2^1\mu_1^2(\gamma_2^1 + \gamma_2^1)a_2^1 \\
(x_1g_1 + g_1x_1)(a_2^3) = \mu_2^3(\gamma_1^1 + \gamma_2^2)a_1^3 + (\mu_2^3\beta_1^3 + \mu_1^3\beta_3^1)a_2^3 \\
(x_1g_1 + g_1x_1)(a_2^4) = \mu_2^4(\gamma_1^1 + \gamma_2^2)a_1^4 + (\mu_2^4\beta_2^4 + \mu_1^4\beta_4^2)a_2^4 \\
\hline
(x_2g_2 + g_2x_2)(a_1^2) = 2\eta_1^2\sigma_5^1 a_1^2 \\
(x_2g_2 + g_2x_2)(a_1^3) = (\eta_1^3 + \sigma_2^1 + \eta_1^4\sigma_1^2)a_1^3 + \eta_1^3(\sigma_2^1 + \sigma_2^2)a_1^4 \\
(x_2g_2 + g_2x_2)(a_1^4) = \eta_1^4(\sigma_1^1 + \sigma_2^2)a_1^3 + \eta_2^3(\sigma_3^3 + \sigma_4^4)a_2^4 \\
(x_2g_2 + g_2x_2)(a_2^1) = 2\eta_2^1\sigma_6^2 a_2^1 \\
(x_2g_2 + g_2x_2)(a_2^3) = (\eta_2^3 + \sigma_4^3 + \eta_2^4\sigma_3^4)a_2^3 + \eta_2^3(\sigma_3^3 + \sigma_4^4)a_2^4 \\
(x_2g_2 + g_2x_2)(a_2^4) = \eta_2^4(\sigma_3^3 + \sigma_4^4)a_2^3 + \eta_2^4\sigma_3^4 + \eta_2^3\sigma_4^3)a_2^4 \\
\hline
\end{array}
$$



$0 = \gamma_3^1 = \gamma_4^1$	$\eta_2^4\delta_4^1 = \sigma_4^3$	$0 = \delta_1^1 = \delta_2^1$	$\beta_3^3 = \beta_4^4 = \gamma_2^2$
$\beta_1^1 = \beta_2^2 = \gamma_1^1$	$0 = \beta_1^4 = \beta_4^1$	$0 = \beta_2^3 = \beta_3^2$	$0 = \sigma_1^3 = \sigma_2^4$
$\sigma_1^2 = \eta_1^3\delta_3^2$	$\sigma_2^1 = \eta_1^4\delta_4^1$	$0 = \sigma_3^1 = \sigma_4^2$	$\sigma_3^4 = \eta_2^3\delta_3^2$
$\gamma_1^1 = -\gamma_2^2$	$\mu_1^3\beta_3^1 = -\mu_2^3\beta_1^3$	$\mu_1^4\beta_4^2 = -\mu_2^4\beta_2^4$	$\mu_2^3\beta_1^3 = -\mu_1^3\beta_3^1$
$\mu_2^4\beta_2^4 = -\mu_1^4\beta_4^2$	$\sigma_2^2 = -\sigma_1^1$	$\eta_1^3\sigma_2^1 = -\eta_1^4\sigma_1^2$	$\eta_1^4\sigma_1^2 = -\eta_1^3\sigma_2^1$
$\sigma_3^3 = -\sigma_4^4$	$\eta_2^3\sigma_4^3 = -\eta_2^4\sigma_3^4$	$\eta_2^4\sigma_3^4 = -\eta_2^3\sigma_4^3$	$0 = \eta_1^2\sigma_5^1 = \eta_2^1\sigma_6^2$
$\mu_1^2\mu_2^1\gamma_2^2 = -\mu_1^2\gamma_1^1$	$\mu_1^2\gamma_2^2 = -\mu_2^1\beta_5^2$	$\mu_2^1\beta_5^1 = -\mu_2^1\beta_6^2$	$\mu_2^1\mu_1^2\gamma_2^1 = -\mu_2^1\mu_1^2\gamma_2^2$

Next Steps

1. Classify $T_n \otimes T_n$ actions for arbitrary n
   1. Increase number of vertices
   2. Increase n
2. Classify actions of arbitrary bosonizations
3. Determine relationship to $\mathbb{Z}_n$ actions
4. Investigate inner faithfulness

Thank You for Hearing Meowt!