Centers and automorphisms of PI quantum matrix algebras

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Setup

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero.

Definition

Let $\mathbf{p} = (p_{ij}) \in M_n(\mathbb{k}^\times)$ be multiplicatively antisymmetric and let $\lambda \in \mathbb{k}^\times$.

The $(n \times n)$ multi-parameter quantum matrix algebra $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbb{k}))$ is the $\mathbb{k}$-algebra generated by $\{x_{ij}\}_{1 \leq i,j \leq n}$ subject to the relations

$$x_{lm}x_{ij} = \begin{cases} p_{li}p_{jm}x_{ij}x_{lm} + (\lambda - 1)p_{li}x_{im}x_{lj} & l > i, m > j \\ \lambda p_{li}p_{jm}x_{ij}x_{lm} & l > i, m \leq j \\ p_{jm}x_{ij}x_{lm} & l = i, m > j. \end{cases}$$

For $q \in \mathbb{k}^\times$, the single-parameter $\mathcal{O}_q(M_n(\mathbb{k}))$ is obtained from the above definition by setting $p_{ij} = q$ for $i > j$ and $\lambda = q^{-2}$.

It is well-known that $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbb{k}))$ is Artin-Schelter regular of global and GK dimension n^2. In particular, $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbb{k}))$ may be presented as an iterated Ore extension.

The quantum determinant

The quantum determinant of $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\Bbbk))$ is defined as

$$D_{\lambda,\mathbf{p}} = \sum_{\pi \in S_n} \left(\prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) x_{1,\pi(1)} x_{2,\pi(2)} \cdots x_{n,\pi(n)}.$$

The element $D_{\lambda,\mathbf{p}}$ is normal in $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\Bbbk))$. In the single-parameter case it is central.

Automorphisms

The group of scalar automorphisms of $\mathcal{O}_q(M_n(\Bbbk))$ is denoted

$$G = \{\sigma \in \mathrm{Aut}(\mathcal{O}_q(M_n(\Bbbk))) : \sigma(x_{ij}) = \alpha_{ij}x_{ij} \text{ for some } \alpha \in \Bbbk^\times\}.$$

Define $\tau \in \mathrm{Aut}(\mathcal{O}_q(M_n(\Bbbk)))$ by $\tau(x_{ij}) = x_{ji}$.

The Launois-Lenagan conjecture

Let $q \in \Bbbk^\times$ be a nonroot of unity. Then $\mathrm{Aut}(\mathcal{O}_q(M_n(\Bbbk))) = G \rtimes \{\tau\}$.

- The case $n = 2$ was first proved by Alev and Chamarie.
- Launois and Lenagan proved the case $n = 3$, motivating the above conjecture.
- The conjecture was resolved by Yakimov.

Motivation

Does the Launois-Lenagan conjecture hold in the case that q is a root of unity? More generally, does it hold for PI multi-parameter quantum matrix algebras?

Discriminants

Let A be a $\Bbbk$-algebra and C a central subalgebra which is a domain and such that A is finitely generated over C. Let F be a localization of C such that $A_F := A \otimes_C F$ is free (and finitely generated) over F. Set $w = \operatorname{rank}_F(A_F) < \infty$.

Left-multiplication gives a natural embedding

$$\mathrm{lm} : A \to A_F \to \operatorname{End}_F(A_F) \cong M_w(F).$$

The regular trace is then the composition

$$\mathrm{tr}_{\mathrm{reg}} : A \xrightarrow{\mathrm{lm}} M_w(F) \xrightarrow{\mathrm{tr}_{\mathrm{int}}} F.$$

If A is free over C of rank $w < \infty$, then the image of $\mathrm{tr}_{\mathrm{reg}}$ is in C. Let $\{a_1, \dots, a_w\}$ be a basis for A over C. The discriminant of A over C is

$$d(A/C) = \det(\mathrm{tr}_{\mathrm{reg}}(a_i a_j)_{i,j=1}^{w}) \in C.$$

Discriminants

Reflexive hull discriminant

Let A be a prime $\Bbbk$-algebra that is free over its center Z and such that the image of tr is in Z. Suppose that $X = \operatorname{Spec} Z$ is an affine normal $\Bbbk$-variety.

Let U be an open subset of X such that $\operatorname{codim}(X \backslash U) = 2$. If if there exists an element $d \in Z$ such that the principal ideal (d) of Z agrees with $d(A/C)$ on U, then

$$d(A/Z) =_{Z^\times} d.$$

$\mathcal{P}$-discriminants

Let A be a $\Bbbk$-algebra with center Z. Let $\mathcal{P}$ be a property defined for $\Bbbk$-algebras that is invariant under algebra isomorphisms. The $\mathcal{P}$-locus of A is

$$L_{\mathcal{P}}(A) := \{\mathfrak{m} \in \operatorname{Maxspec}(Z) \mid Z_{\mathfrak{m}} \text{ has property } \mathcal{P}\}.$$

The $\mathcal{P}$-discriminant ideal is

$$I_{\mathcal{P}}(A) := \bigcap_{\mathfrak{m} \in L_{\mathcal{P}}(A)} \mathfrak{m} \subseteq Z.$$

The 2×2 multi-parameter case

The relations in $M = \mathcal{O}_{\lambda,\mathbf{p}}(M_2(\Bbbk))$ are

$$\begin{aligned} x_{12}x_{11} &= p_{12}x_{11}x_{12} & x_{21}x_{11} &= (\lambda p_{21})x_{11}x_{21} \\ x_{22}x_{12} &= (\lambda p_{21})x_{12}x_{22} & x_{22}x_{21} &= p_{12}x_{21}x_{22} \\ x_{21}x_{12} &= (\lambda p_{21})p_{21}x_{12}x_{21} & x_{22}x_{11} &= x_{11}x_{22} + (\lambda - 1)p_{21}x_{12}x_{21}. \end{aligned}$$

Theorem (G-Lamkin)

Suppose p_{12} and λp_{12} are roots of unity whose orders are relatively prime. Set $\ell = \operatorname{lcm}(\operatorname{ord}(p_{12}), \operatorname{ord}(\lambda p_{21}))$.

- *The algebra M is free over its center and*

$$Z = \mathcal{Z}(M) = \Bbbk[x_{11}^{\ell}, x_{12}^{\ell}, x_{21}^{\ell}, x_{22}^{\ell}].$$

- *The discriminant of M over Z is*

$$d(M/Z) =_{\Bbbk^\times} (x_{12}x_{21}D_{\lambda,\mathbf{p}})^{\ell^4(\ell-1)}.$$

- *The automorphism group of M satisfies the Launois-Lenagan conjecture. That is,*

$$\operatorname{Aut}(M) = G \rtimes \{\tau\}.$$

The center of $\mathcal{O}_q(M_n(\Bbbk))$

Recall that $\mathcal{O}_q(M_n(\Bbbk))$ is generated by $\{x_{ij}\}_{1\leq i,j\leq n}$ subject to the relations

$$x_{ij}x_{kl} = \begin{cases} x_{kl}x_{ij} + (q-q^{-1})x_{il}x_{kj} & k>i, l>j \\ qx_{kl}x_{ij} & k>i, l=j \\ qx_{kl}x_{ij} & k=i, l>j \\ x_{ij}x_{lm} & k>i, l<j. \end{cases}$$

We study $\mathcal{O}_q(M_n(\Bbbk))$ under the hypothesis that $\mathrm{ord}(q) = m \geq 3$ is odd.

Our goal will be to give a full presentation for $Z = \mathcal{Z}(\mathcal{O}_q(M_n(\Bbbk)))$.

The center of $\mathcal{O}_q(M_n(\Bbbk))$

In the single-parameter case, the quantum determinant is central and

$$D := D_q = \sum_{\pi \in S_n} (-q)^{l(\pi)} x_{1,\pi(1)} x_{2,\pi(2)} \cdots x_{n,\pi(n)}.$$

Let $I, J \subset \{1, \ldots, n\}$ with $|I| = |J| = k \geq 1$. The (I, J)-quantum minor $D(I, J)$ is the quantum determinant of the subalgebra generated by $\{x_{ij} \mid i \in I, j \in J\}$.

For $1 \leq t \leq n$, let

$$D(t) := D(\{1, \ldots, t\}, \{n - t + 1, \ldots, n\}).$$

Theorem (Jakobsen, Zhang)

Under the above hypotheses, Z is generated by the elements

$$x_{ij}^m, \quad D, \quad Y_{tr} := D(t)^r \tau(D(n-t)^{m-r})$$

for $1 \leq i, j, t \leq n$ and $0 \leq r \leq m$.

The center of $\mathcal{O}_q(M_n(\mathbb{k}))$

Let $A_t = (x_{ij}^m)_{1 \leq i \leq t, n-t+1 \leq j \leq n}$ and $B_t = (x_{ij}^m)_{n-t+1 \leq i \leq n, 1 \leq j \leq t}$ for $1 \leq t \leq n$.

With respect to a certain monomial ordering $\leq$, the following families of elements form a Gröbner basis of the ideal I that they generate:

- $D^m - \det(x_{ij}^m)$,
- $Y_{ti} Y_{tj} - Y_{t,i+j} \det(B_{n-t})$ if $i + j < m$,
- $Y_{ti} Y_{tj} - \det(A_t) \det(B_{n-t})$ if $i + j = m$,
- $Y_{ti} Y_{tj} - Y_{t,i+j-m} \det(A_t)$ if $i + j > m$,

for $1 \leq t \leq n-1$.

The center of $\mathcal{O}_q(M_n(\mathbb{k}))$

Theorem (G-Lamkin)

Assume $m \geq 3$ is odd. Let

$$T = \mathbb{k}[x_{ij}^m, Y_{tr}, D \mid 1 \leq i,j \leq n, 1 \leq t \leq n-1, 1 \leq r \leq m-1]$$

and let $R = T/I$. Then $Z \cong R$. Additionally,

1. *Z is not Gorenstein.*
2. *$\mathcal{O}_q(M_n(\mathbb{k}))$ is not projective over Z.*
3. *$\mathcal{O}_q(M_n(\mathbb{k}))$ is not Azumaya over Z.*

Automorphisms of $\mathcal{O}_q(M_n(\mathbb{k}))$

Theorem (G-Lamkin)

There are automorphisms ϕ and ψ of $\mathcal{O}_q(M_n(\mathbb{k}))$ given by

$$\phi : x_{ij} \mapsto \begin{cases} x_{11} + D(\{2,\dots,n\},\{2,\dots,n\})^{m-1} & i = j = 1 \\ x_{ij} & (i,j) \neq (1,1), \end{cases}$$

$$\psi : x_{ij} \mapsto \begin{cases} x_{ij} & (i,j) \neq (n,n) \\ x_{nn} + D(\{1,\dots,n-1\},\{1,\dots,n-1\})^{m-1} & i = j = n. \end{cases}$$

The proper subgroup of $\mathrm{Aut}(\mathcal{O}_q(M_n(\mathbb{k})))$ generated by ϕ and ψ is isomorphic to a free group on two generators.

Automorphisms of $\mathcal{O}_q(M_2(\Bbbk))$

An automorphism of a polynomial ring $A = F[x_1, \ldots, x_t]$ over a field F is elementary if it is of the form

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n),$$

for some $0 \neq \alpha \in F$ and $f \in F[x_1, \ldots, \hat{x}_i, \ldots, x_n]$. An automorphism of A is tame if it is a composition of elementary automorphisms. An automorphism which is not tame is called wild.

- If $n = 1, 2$, then every automorphism of A is tame (Jung, van der Kulk)
- When $n = 3$, Umirbaev and Shestakov proved the Nagata automorphism is wild.

Theorem (G-Lamkin)

There is an automorphism of $\mathcal{O}_q(M_2(\Bbbk))$ which reduces to a wild automorphism of $\Bbbk[x_{12}, x_{21}, x_{11}^m]$.

Automorphisms of $\mathcal{O}_q(M_2(\Bbbk))$

Theorem (G-Lamkin)

If $\sigma \in \mathrm{Aut}(\mathcal{O}_q(M_2(\Bbbk)))$*, then* $\sigma((x_{12}, x_{21})) = (x_{12}, x_{21})$*.*

Sketch of proof

• Let $R = \Bbbk[x_1, \ldots, x_r]/I$. and let $\mathfrak{m}$ be a maximal ideal in R. The embedding dimension of $R_\mathfrak{m}$ is minimal number of generators for $\mathfrak{m}$. Hence,

$$\mathrm{edim}(R_\mathfrak{m}) = \dim_\Bbbk(m/m^2).$$

• If $J_\mathfrak{m}$ denotes the Jacobian matrix of I evaluated mod $\mathfrak{m}$, then

$$\mathrm{rank}(J_\mathfrak{m}) = r - \dim_\Bbbk(\mathfrak{m}/\mathfrak{m}^2) = r - \mathrm{edim}(R_\mathfrak{m}),$$

• Let $\mathcal{P}$ be the property of being a local ring with embedding dimension $\mathrm{ord}(q) + 3$. We compute the $\mathcal{P}$-discriminant ideal of $\mathcal{O}_q(M_2(\Bbbk))$.

Automorphisms of certain subalgebras of $\mathcal{O}_q(M_3(\Bbbk))$

Using techniques similar to the above, we can establish the automorphism groups of certain subalgebras of $\mathcal{O}_q(M_3(\Bbbk))$.

The subalgebra $B_1 = \langle x_{ij} \rangle_{1 \le i \le 2, 1 \le j \le 3}$ is isomorphic to $\mathcal{O}_q(M_{2,3}(\Bbbk))$, i.e., the 2×3 quantum matrix algebra. The center of B_1 is polynomial and $\mathrm{Aut}(B_1) = \mathrm{Aut}_{\mathrm{gr}}(B_1)$.

Like B_1, the subalgebra $C = \langle x_{ij} \rangle_{i+j \ge 4}$ is another 6-generated subalgebra of $\mathcal{O}_q(M_3(\Bbbk))$. The center of C is polynomial and $\mathrm{Aut}(C) = \mathrm{Aut}_{\mathrm{gr}}(C)$.

We do not know if there are any 7- or 8-generated subalgebras of $\mathcal{O}_q(M_3(\Bbbk))$ which have polynomial centers, though we suspect not.

On the other hand, the subalgebras

$$B_2 = \langle B_1 \cup x_{31} \rangle \quad \text{and} \quad B_3 = \langle B_2 \cup x_{32} \rangle$$

have non-graded automorphisms. In fact, if q is a root of unity then $\mathrm{Aut}(B_2)$ (properly) contains a contains a free group on two generator.

Thank You!