Stabilizers of canonical matrices arising from matrix congruence

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The Goal

Canonical forms under non-homogeneous congruence ⟷ Noncommutative, nonhomogeneous polynomials ⟷ Noncommutative algebras

Background

Denote the set of $n \times n$ matrices with entries in $\mathbb{C}$ by $\mathcal{M}_n(\mathbb{C})$ and the set of invertible $n \times n$ matrices with entries in $\mathbb{C}$ by $\mathrm{GL}_n(\mathbb{C})$.

A degree two homogeneous polynomial in n variables $p = \sum_{ij} a_{ij} x_i x_j$ may be represented by $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$ in the following way.

$$p = \mathbf{x} A \mathbf{x}^T \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}.$$

Definition

Two matrices M and M' in $\mathcal{M}_n(\mathbb{C})$ are said to be **congruent** if there exists a matrix $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P^T M P = M'$.

Two homogeneous degree two polynomials are related by a linear transformation if and only if their corresponding matrices are congruent. Congruence is an equivalence relation and a canonical form is a representative of an equivalence class that has a standard presentation in all dimensions. Horn and Sergeichuk defined certain *blocks* and proved that every $n \times n$ matrix is congruent to a direct sum of the blocks listed below [HS06].

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}, \qquad \Gamma_n = \begin{pmatrix} 0 & & & & & (-1)^{n+1} \\ & & & & \iddots & (-1)^n \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{pmatrix}.$$

$$H_{2n}(\mu) = \begin{pmatrix} 0 & I_n \\ J_n(\mu) & 0 \end{pmatrix},$$

Objectives

- To determine the stabilizer group for each canonical form under matrix congruence in $\mathcal{M}_3(\mathbb{C})$.
- Use the stabilizer group to classify nonhomogeneous polynomials in three variables up to affine transformation.
- Verify the minimal list by hand using linear algebra, often there are some addition forms

Methods

We utilized routines in the computer algebra package Maple to determine the stabilizer groups and to find potential canonical forms under nonhomogeneous congruence. The final list had to be checked by hand using linear algebra, often resulting in some additional forms.

Stabilizer Groups

The stabilizer group of $M \in \mathcal{M}_n(\mathbb{C})$ is defined as the set of $P \in \mathrm{GL}_n(\mathbb{C})$ such that $P^T M P = M$. We denote this group by $\mathrm{Stab}(M)$. To find the stabilizer group of each matrix M, we found conditions on a matrix P so that $P^T M P = M$. This process can be divided in to three steps.

1. Choose an arbitrary matrix P with variables representing the different entries.
2. Put all the matrices on one side to form the equation $P^T M P - M = 0$.
3. Create a list of all the individual equations and solve using Maple.

Stabilizer Groups in High Dimension

Part of our goal was to study relationships between the stabilizer groups of various dimensions. Our strongest result is below.

Theorem

There exists an embedding of groups

$$\mathrm{Stab}(\Gamma_{2n}) \hookrightarrow \mathrm{Stab}(\Gamma_{2(n+1)}).$$

Standard-Form Congruent Matrices

Nonhomogeneous polynomials in n variables can also be represented by a matrix. Write $p = \left(\sum_{ij} a_{ij} x_i x_j\right) + \left(\sum b_k x_k\right) + c$. If $A = (a_{ij})$, $B = (b_i)$, and $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n & 1 \end{pmatrix}$. Then $p = \mathbf{x} \begin{pmatrix} A & B \\ 0 & c \end{pmatrix} \mathbf{x}^T$.

Standard-form congruence can be used to classify nonhomogeneous degree-two polynominals in n variables [Gad15]. First we change a matrix to one in standard form. Let $M_1 \in \mathcal{M}_n$, $M_2, M_3 \in \mathbb{C}^n$, and $m \in M$.

$$\mathrm{sf} : \mathcal{M}_{n+1} \to \mathcal{M}_{n+1}$$
$$\begin{pmatrix} M_1 & M_2 \\ M_3^T & m \end{pmatrix} \to \begin{pmatrix} M_1 & M_2 + M_3 \\ 0 & m \end{pmatrix}$$

For each form, we compute the equivalence classes for the pairs (M_2, M) under a modified form of congruence. Define the group

$$\mathcal{G} = \left\{ \begin{pmatrix} Q & V \\ 0 & 1 \end{pmatrix} : Q \in \mathrm{GL}_n(\mathbb{C}), V \in \mathbb{C}^{n-1} \right\}.$$

Definition

$M, N \in \mathcal{M}_{n+1}(\mathbb{C})$ are standard form congruent if there exist $P \in \mathcal{G}$ and $\alpha \in \mathbb{C}$ such that $\mathrm{sf}(M) = \alpha \cdot \mathrm{sf}(P^T N P)$.

Minimal Lists

- We first perform a congruence so that the upper-left block is in HS form. The matrix is then sf-congruent to one whose right-hand vector is one in the minimal list.
- For each 3×3 HS form, the minimal list contains the canonical forms under standard form congruence. In order to find the minimal list, we first chose several candidates and used Maple routines to determine which ones were sf-congruent. We then verified the list and checked whether there were additional forms.

References

[Gad15] Jason Gaddis.
Two-generated algebras and standard-form congruence.
Comm. Algebra, 43(4):1668–1686, 2015.

[HS06] Roger A. Horn and Vladimir V. Sergeichuk.
Canonical forms for complex matrix congruence and *congruence.
Linear Algebra Appl., 416(2-3):1010–1032, 2006.

Theorem

Let $M \in \mathcal{M}_4(\mathbb{C})$ be nonzero. Then M is sf-congruent to exactly one matrix in the following list.

HS Matrix	Stabilizer Group	Minimal List
$\Gamma_2 \oplus J_2(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & a & 0 \\ 0 & \pm 1 & 0 \\ b & c & d \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
$J_2(0) \oplus J_1(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ b & c & d \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
$J_3(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & 0 \\ 0 & a^{-1} & 0 \\ 0 & -b & a \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
$\Gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & a & \pm a^2/2 \\ 0 & \pm 1 & a \\ 0 & 0 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
$J_2(0) \oplus \Gamma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
$\Gamma_2 \oplus J_1(0) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \pm 1 & a & 0 \\ 0 & \pm 1 & 0 \\ b & c & d \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
$\Gamma_1 \oplus \Gamma_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1/4 \end{pmatrix}$
$H_2(q) \oplus \Gamma_1 = \begin{pmatrix} 0 & 1 & 0 \\ q & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
$J_1(0) \oplus J_1(0) \oplus \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & \pm 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1/4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
$J_1(0) \oplus \Gamma_1 \oplus \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} a & b & c \\ 0 & d^{-1} & x \\ 0 & \pm x & d \end{pmatrix}$ $x^2 + d^2 = 1$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
$\Gamma_1 \oplus \Gamma_1 \oplus \Gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$O(3)$	?

Further Research

There are still some aspects that deserve further research.

- We wish to better understand the stabilizer group for each canonical form. For example, is there any rule to follow to find out the stabilizer group for $J_n(0)$?
- Is there a general rule for finding minimal lists in higher dimension.
- For HS blocks A and B, under what conditions is

$$\mathrm{Stab}(A \oplus B) = \mathrm{Stab}(A) \oplus \mathrm{Stab}(B)?$$

Even though this rule does not work in general, it works for most of cases. We would like to know under what conditions this rule applies.