Reflexive hull discriminants and applications

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Automorphism groups

A fundamental problem for any algebraic object is to determine its symmetries (automorphism group).

- Automorphisms of $\mathbb{C}[x]$ are triangular.

$$x \longmapsto ax + b \qquad a, b \in \mathbb{C} \qquad a \neq 0$$

affine

tame

- The automorphism group of $\mathbb{C}[x, y]$ is generated by *elementary automorphisms*.

not affine

1) $x \longmapsto x \qquad y \longmapsto y - \underline{\mu x^n} \qquad \mu \in \mathbb{C} \quad n \in \mathbb{N}$

2) $x \longmapsto y \qquad y \longmapsto x$

- The full automorphism group of $\mathbb{C}[x, y, z]$ is unknown. auto group is wild

The principle of *quantum rigidity* says that automorphism groups of quantum/noncommutative algebras should be small (relative to their commmutative counterparts).

Automorphism groups

(−1) - skew poly ring

We denote by $\mathbb{C}_{-1}[x_1, \ldots, x_n]$ the algebra generated over $\mathbb{C}$ by $x_1, \ldots, x_n$ subject to the relations $x_i x_j + x_j x_i = 0$ for $i \neq j$.

- Automorphisms of $\mathbb{C}_{-1}[x, y]$ are graded of the form

  1) $x \longmapsto ax \qquad y \longmapsto by \qquad a, b \in \mathbb{C}^\times$

  2) $x \longmapsto cy \qquad y \longmapsto dx \qquad c, d \in \mathbb{C}^\times$

- Automorphisms of $\mathbb{C}_{-1}[x, y, z]$ are not yet fully understood. It is *not* known even if the automorphism group is tame or wild. It does contain non-affine automorphisms.

The discriminant

One approach to determining automorphism groups is to find an *invariant* of the algebra that is fixed under automorphisms.

L Galois ext of $\mathbb{Q}$

$\mathcal{O}_L = \mathbb{Z}[\alpha] \cong \mathbb{Z}[x]/(f)$ $\quad f$ min poly of α

$$\Delta_{L/\mathbb{Q}} = \prod_{i \neq j} (r_i - r_j) \qquad r_1, \ldots, r_n \text{ roots of } f$$

adapted to n.c. algebra

Ceken, Palmieri, Wang, and Zhang ('15)

The (noncommutative) discriminant

often center of B

Let $\Bbbk$ be a field. Let B be a prime $\Bbbk$-algebra containing R as a central subalgebra such that B is a finitely generated (f.g.) R-module. Let F be a localization of R such that $B_F := B \otimes_R F$ is f.g. and free over F with $w = \mathrm{rk}_F(B_F) < \infty$.

The *regular trace* is the composition

$$tr : B \longrightarrow B_F \xrightarrow{\ell m} \mathrm{End}(B_F) \cong M_w(F) \xrightarrow{tr_{int}} F$$

(assume image is in R)

If B is f.g. free over R with basis $\{z_1, \ldots, z_n\}$, then the *discriminant of B over R* is

$$d(B/R) =_{R^\times} \det\left(tr(z_i z_j)_{i,j=1}^{n} \right) \in R$$

up to a unit in R

Theorem (Ceken-Palmieri-Wang-Zhang '15)

If B is f.g. free over R and $\sigma(R) = R$ for all $\sigma \in \mathrm{Aut}(B)$, then every automorphism fixes the ideal generated by $d(B/R)$.

An example: $\mathbb{C}_{-1}[x, y]$

$B = \mathbb{C}_{-1}[x, y]$ is f.g free over $R = \mathbb{C}[x^2, y^2]$ with basis $\{1, x, y, xy\}$.

center of B

Trace Computations

$tr(1) = 4$

	1	x	y	xy
1	1			
x		1		
y			1	
xy				1

$tr(x) = 0$

$x \cdot x = x^2 = x^2 \cdot 1$

$tr(y) = 0$

	1	x	y	xy
1		1		
x	x^2			
y				1
xy			x^2	

$tr(x^2) = 4x^2$

	1	x	y	xy
1	x^2			
x		x^2		
y			x^2	
xy				x^2

$tr(xy) = 0$

	1	x	y	xy
1				1
x			$-x^2$	
y		x^2		
xy	$-x^2y^2$			

c central

$tr(ca) = c \cdot tr(a)$

$xy \cdot x = -x^2 \cdot y$

$xy \cdot xy = -x^2 y^2$

An example: $\mathbb{C}_{-1}[x, y]$

$xy + yx = 0$

Discriminant Computation

	1	x	y	xy
1	4			
x		$4x^2$		
y			$4y^2$	
xy				$-4x^2y^2$

$xy \cdot xy = -x^2y^2$

$d(B/R) = \det(\text{this matrix})$

$= -256\, x^4 y^4$

$=_{R^\times} x^4 y^4$

use that B is a domain to get that every auto is affine

$x \mapsto ax + by + c$

$y \mapsto ux + vy + d$

use defining rel'n to get that every auto is graded

Problems

- computation
- freeness

Discriminants - some history

- Chan, Young, and Zhang ('16) supplied several tools for computing discriminants, such as through localization and filtrations. They also introduce the p-power discriminant in the study of discriminants of Veronese subrings ('18).

- Nguyen, Trampel, and Yakimov ('17) demonstrated a connection between discriminants and Poisson algebras. This method is used by Levitt and Yakimov ('18) to study automorphisms and isomorphisms of quantized Weyl algebras. G, Won, and Yee ('19) used this method in computing discriminants of Taft algebra smash products.

- Brown and Yakimov ('18) showed that the discriminant can be obtained through representation theory and the Azumaya locus. This is applied in the study of the representation theory of Sklyanin algebras by Walton, Wang, and Yakimov ('18).

- G, Kirkman, and Moore ('19) provide techniques for computing discriminants of twisted tensor products, including Ore extensions.

- Nguyen, Trampel, and Yakimov ('20) discovered a connection between discriminants and quantum cluster algebras.

(Modified) Discriminants

k-alg

Definition

For a positive integer v, let $\mathcal{U} = \{u_i\}_{i=1}^{v}$ and $\mathcal{U}' = \{u_i'\}_{i=1}^{v}$ be v-element subsets of B.

R central subalg

(1) The *discriminant* of the pair $(\mathcal{U}, \mathcal{U}')$ is defined to be

$$d_v(\mathcal{U}, \mathcal{U}') = \det\left(tr(u_i u_j')_{i,j=1}^{v} \right) \in R$$

(2) The *v-discriminant ideal* $D_v(B/R)$ is the ideal in R generated by the set of elements $d_v(\mathcal{U}, \mathcal{U})$ where $\mathcal{U}$ ranges over all v-element subsets of B.

(3) The *modified v-discriminant ideal* $\mathrm{MD}_v(B/R)$ is the ideal in R generated by the set of elements $d_v(\mathcal{U}, \mathcal{U}')$ where $\mathcal{U}, \mathcal{U}'$ range over all v-element subsets of B. If B is a f.g. R-module of rank w, we write $\mathrm{MD}(B/R) := \mathrm{MD}_w(B/R)$.

(4) The *v-discriminant* $d_v(B/R)$ is the gcd in B, if it exists, of the elements in $\mathrm{MD}_v(B/R)$.

If B is free over R of rank w, $D_w(B/R) = \mathrm{MD}_w(B/R)$ is generated by a single element $d(B/R) := d_w(B/R)$, which we call the *discriminant of B over R*.

Reflexive Hull Discriminants

The *reflexive hull* of a module M over a commutative domain R is

$$M^{\vee\vee} = \mathrm{Hom}_R(\underbrace{\mathrm{Hom}_R(M, R)}_{M^\vee}, R)$$

There is a natural R-morphism $j : M \to M^{\vee\vee}$ defined by

$$j(x)(f) = f(x) \quad \text{for all } x \in M, f \in M^\vee$$

Definition

(1) The $\mathcal{R}$*-discriminant ideal* of B over R is defined to be

$$\mathcal{R}(B/R) = (MD(B/R))^{\vee\vee} \subset R$$

(2) If further $\mathcal{R}(B/R)$ is a principal ideal of R generated by an element d, then d is called an $\mathcal{R}$*-discriminant* of B over R and denoted by $\underline{\varrho(B/R)}$.

We also call the $\mathcal{R}$-discriminant the *reflexive hull discriminant*. If $\varrho(B/R)$ exists, it is unique up to a unit in R. Under suitably nice circumstances, one can show that $\varrho(B/R)$ is preserved up to automorphism.

Reflexive Hull Discriminants

Definition

Let A be an algebra.

(1) We say an ideal $I \subseteq A$ satisfies the *principal closure condition* (PCC) if there exists a normal element $d \in A$ such that

(a) $I \subset dA = Ad$ (b) $\mathrm{GKdim}(dA/I) \leq \mathrm{GKdim}\, A - 2$

(2) We say B/R satisfies the *reflexive discriminant condition* (RDC) if $\mathrm{MD}(B/R) \subseteq R$ satisfies PCC for some nonzero element $d \in R$. In this case d is called a weak $\mathcal{R}$-discriminant of B over R.

The element d in either part (1) or (2), if it exists, may not be unique (even up to a unit) in general, unless R is CM.

Lemma

Let B be a prime $\Bbbk$-algebra containing R as a central subalgebra such that B is a f.g. R-module. Suppose that R is an affine CM domain and that B is a CM reflexive module over R. If B/R satisfies the RDC with respect to $d \in R$, then

$$\mathrm{MD}(B/R)^{\vee\vee} = dA \qquad \mathcal{R}(B/R) =_{R^\times} d$$

Reflexive Hull Discriminants

Lemma

Let A be a prime $\mathbb{k}$-algebra with center Z. Assume

- *Z is affine and CM,*
- *$X = \operatorname{Spec} Z$ is an affine integral normal $\mathbb{k}$-variety, and*
- *there exists an open subset U of X such that $X \setminus U$ has codimension ≥ 2.*

If there exists an element $d \in Z$ such that the principal ideal (d) of Z agrees with $\mathrm{MD}(A/Z)$ on U, then $\varrho(A/Z) = d$.

Moreover, if A is a $\mathcal{O}_X$-order, then we can compute the discriminant locally at a smooth closed point $\mathfrak{m} \in X$.

An example

Example

Suppose $\operatorname{char} \Bbbk \neq 2$. Let A be the skew polynomial ring

$$\Bbbk\langle x_1, x_2, x_3\rangle / (\underbrace{x_1x_2 + x_2x_1}, \underbrace{x_1x_3 + x_3x_1}, \underbrace{x_2x_3 - x_3x_2}).$$

$Z(A) = \Bbbk[a,b,c,d]/(bc - d^2)$ $\quad a = x_1^2 \quad b = x_2^2 \quad c = x_3^2 \quad d = x_2x_3$

$\operatorname{rank}(A/Z) = 4 \qquad X = \operatorname{Spec} Z$

U_b open subset of X w/ $b \neq 0$

U_c " " $c \neq 0$

Over U_b, $d(A_b/Z_b) = (ab)^2 = a^2$

U_c $\quad d(A_c/Z_c) = (ac)^2 = a^2$

$U = U_b \cup U_c \qquad \operatorname{codim}(X \setminus U) = 2$ in X

$\cancel{a^2 = x_1^4}$ $\quad$ so $\quad e(A/Z) = x_1^4$.

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Generalized Weyl Algebras

Our results apply in particular to quantum GWAs at roots of unity. Here we will consider only the rank one case but our results apply also to higher rank GWAs.

Definition

Let R be a $\mathbb{k}$-algebra, $\sigma \in \mathrm{Aut}(R)$, and h a nonzero central element in R. The (rank one) *generalized Weyl algebra* (GWA) $R(x, y, \sigma, h)$ is the $\mathbb{k}$ algebra generated over R by x, y subject to the relations

We say $R(x, y, \sigma, h)$ is a *quantum GWA* if

Quantum GWAs

Lemma

Let $W = \Bbbk[t](x, y, \sigma, h)$ *be a quantum GWA with* $\operatorname{ord}(\sigma) = n < \infty$. *Set*

$$a = x^n, \quad b = y^n, \quad c = t^n, \quad p(c) = \prod_{j=0}^{n-1} h(q^j t).$$

Then

$$Z(W) = \Bbbk[a, b, c] / (ab - p(c)).$$

Consequently, $Z(W)$ *is an affine normal CM domain.*

The quantum GWA W is a Z-algebra with presentation

$$W = \frac{Z\langle x, y, t\rangle}{(xt - qtx, yt - q^{-1}ty, xy - h(t), x^n - a, y^n - b, t^n - c)}.$$

Then W is generated as a Z-module by $\{x^i t^j, y^i t^j \mid i, j = 0, \ldots, n-1\}$.

Quantum GWAs

Theorem (CGWZ)

$\varrho(W/Z) =_{Z^\times} c^{n(n-1)} =_{Z^\times} t^{n^2(n-1)}$.

Proof.

Quantum GWAs

Other results:

- Compute the $\mathcal{R}$-discriminant for higher rank quantum GWAs (with some restrictions).
- Apply the $\mathcal{R}$-discriminant to compute the automorphism group for quantum GWAs. This recovers results of Suárez-Alvarez and Vivas ('15) in the rank one case.
- Study Zariski cancellation for quantum GWAs.
- Methods for computing the $\mathcal{R}$-discriminant for tensor products of algebras.

Thank You!

$G = \{\text{scalar autos}\}$

$\tau = \{\text{switching } x \longleftrightarrow y\}$

$G \rtimes \{\tau\}$