Cancellation and skew cancellation of Poisson algebras

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Zariski cancellation

The Zariski Cancellation Problem

Let Y be an affine variety. Does an isomorphism $Y \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ imply an isomorphism $Y \cong \mathbb{A}^n$?

Generalizations

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & \\
\begin{array}{c} Y \text{ a noncommutative} \\ \text{algebraic variety} \end{array} & & \begin{array}{c} Y \text{ an affine} \\ \text{Poisson variety} \end{array}
\end{tikzcd}$$

Zariski cancellation

Let $\Bbbk$ be a field. All algebras are $\Bbbk$-algebras.

Definition

An algebra A is *cancellative* if an isomorphism $A[x] \cong B[x]$ implies $A \cong B$ for any algebra B.

Cancellation Question

Under what conditions is an algebra A cancellative?

Example

(1) The polynomial rings $\Bbbk[x]$ and $\Bbbk[x, y]$ are cancellative over *any* field. (Abhynankar, Eakin, Heinzer) (Fujita) (Russell) (Crachiola, Makar-Limanov) (Bhatwadekar, Gupta)

(2) The polynomial ring $\Bbbk[x, y, z]$ is *not* cancellative for a field $\Bbbk$ of positive characteristic. (Asanuma) (Gupta)

The question in the case of zero characteristic is still open.

Noncommutative Zariski cancellation

Bell and Zhang initiated a study of the Zariski cancellation problem in the noncommutative setting.

Example

The following are cancellative assuming the algebra is affine and $\Bbbk$ has characteristic zero.

(1) Noncommutative domains of Gelfand-Kirillov (GK) dimension two assuming $\Bbbk$ is algebraically closed. (Bell, Zhang)

(2) Domains of GK dimension one over $\Bbbk$. (Bell, Hamidizadeh, Huang, Venegas)

(3) Path algebras of finite quivers. (Lezama, Wang, Zhang)

(4) Noetherian prime algebras of GK dimension three that are not PI. (Tang, Venegas, Zhang)

Poisson algebras

Definition

A *Poisson algebra* is a commutative algebra A equipped with a bilinear map $\{-,-\} : A \times A \to A$, called a *Poisson bracket*, which is both a Lie bracket and a biderivation. Hence,

$$\{a, bc\} = b\{a, c\} + \{a, b\}c \quad \text{for all } a, b, c \in A.$$

The *Poisson center* of a Poisson algebra A is the set

$$\mathcal{Z}_P(A) = \{z \in A : \{z, a\} = 0 \text{ for all } a \in A\}.$$

A Poisson algebra A is *Poisson cancellative* if an isomorphism of Poisson algebras $A[x] \cong B[x]$ implies $A \cong B$ as Poisson algebras.

Poisson Cancellation Question

Under what conditions is a Poisson algebra A cancellative?

Noncommutative Poisson Zariski cancellation

Many of the results for noncommutative algebras have analogues in the Poisson setting. However, there is no formal mechanism to translate between the two.

Example

Assume $\Bbbk$ has characteristic zero. The following Poisson algebras are cancellative.

(1) Poisson algebras with trivial Poisson center. This includes affine Poisson domains of Krull dimension two.

(2) Affine Poisson domains with finite Krull dimension and no nontrivial locally nilpotent Poisson derivations.

(3) Let A be a Poisson algebra with trivial Poisson center and let R be a Poisson algebra with trivial bracket that is cancellative. Then $A \otimes R$ is Poisson cancellative.

Question

Which Poisson algebras on $\Bbbk[x_1, \ldots, x_n]$ are cancellative?

Quadratic Poisson algebras

We assume for the remainder that $\mathbb{k}$ is algebraically closed and $\mathrm{char}(\mathbb{k}) = 0$.

Definition

A Poisson algebra on $\mathbb{k}[x, y, z]$ is said to have *Jacobian bracket* if there exists a homogeneous polynomial $f \in \mathbb{k}[x, y, z]$ such that

$$\{x, y\} = f_z, \quad \{y, z\} = f_x, \quad \{z, x\} = f_y.$$

Theorem (G, Wang)

If $f \in \mathbb{k}[x, y, z]$ has isolated singularities, then the Poisson algebra $\mathbb{k}[x, y, z]$ with corresponding Jacobian bracket is cancellative.

Goal

Show that *all* Poisson algebras on $\mathbb{k}[x, y, z]$ with nontrivial bracket are cancellative.

Quadratic Poisson algebras

Definition

A Poisson bracket on $A = \mathbb{k}[x_1, \ldots, x_n]$ is *quadratic* if $\{x_i, x_j\} \in A_2$.

New Goal

Show that all *quadratic* Poisson algebras on $\mathbb{k}[x, y, z]$ with nontrivial bracket are cancellative.

Lemma

If A is an affine Poisson domain with nontrivial bracket and $\operatorname{Kdim} A = 3$*, then* $\operatorname{Kdim} \mathcal{Z}_P(A) \leq 1$*. If in addition, $\mathcal{Z}_P(A)$ is connected graded and $\mathcal{Z}_P(A) \ncong \mathbb{k}[t]$, then A is Poisson cancellative.*

A consequence of this lemma is that we need only consider the case $\mathcal{Z}_P(A) = \mathbb{k}[t]$.

Poisson discriminants

Definition

For a property $\mathcal{P}$, the *$\mathcal{P}$-discriminant ideal* of A, denoted $I_{\mathcal{P}}(A)$ is the intersection of all $\mathfrak{m} \in \operatorname{Maxspec}(\mathcal{Z}_P(A))$ such that $A/\mathfrak{m}\, A$ *does not* have property $\mathcal{P}$.

In the special case such that $I_{\mathcal{P}}(A)$ is a principal ideal of $\mathcal{Z}_P(A)$, say generated by some element $d \in \mathcal{Z}_P(A)$, then we call d the *$\mathcal{P}$-discriminant* of A, denoted by $d_{\mathcal{P}}(A)$.

When $\mathcal{Z}_P(A)$ is a domain, $d_{\mathcal{P}}(A)$ is unique up to some unit element of $\mathcal{Z}_P(A)$.

Lemma

Let A be a noetherian connected graded Poisson domain that is generated in degree 1. Assume $\mathcal{Z}_P(A) = \Bbbk[t]$ for some homogeneous element $t \in A$ of positive degree. Then A is Poisson cancellative in either of the following cases:

(1) The $\mathcal{P}$-Poisson discriminant is t for some property $\mathcal{P}$ of A.

(2) We have $\operatorname{gldim} A/(t) = \infty$ *and either* $\operatorname{gldim} A/(t-1) < \infty$ *or* $\operatorname{gldim} A < \infty$.

We may now assume $\mathcal{Z}_P(A) = \Bbbk[t]$ and $\operatorname{gldim} A/(t) < \infty$, so $A = \Bbbk[x, y, t]$ with $t \in \mathcal{Z}_P(A)$ and $\{x, y\} = f \in A_2$.

Some Poisson classification problems

Theorem (-,Wang,Yee)

(I) Let $A = \Bbbk[x, y]$ be a Poisson algebra such that $\{x, y\} = f$ with $f \in A_{\leq 2}$. Then up to a change of variables, the possibilities for f are

(1) $f = 0$,
(2) $f = 1$,
(3) $f = x$,
(4a) $f = x^2$,
(4b) $f = x^2 + 1$,
(5a) $f = \lambda xy$ *with* $\lambda \in \Bbbk^\times$,
(5b) $f = \lambda xy + 1$ *with* $\lambda \in \Bbbk^\times$.

Moreover, the Poisson algebras determined by f above are pairwisely nonisomorphic with the exception of replacing λ by $-\lambda$ in (5a) and (5b).

(II) Let $A = \Bbbk[x, y, t]$ be an $\mathbb{N}$-graded Poisson algebra with t Poisson central and $\{x, y\} = f \in A_2$. Then up to a change of variables, the possibilities for f are

(1) $f = 0$,
(2) $f = t^2$,
(3) $f = xt$,
(4a) $f = x^2$,
(4b) $f = x^2 + t^2$,
(4c) $f = x^2 + yt$,
(5a) $f = \lambda xy$ *with* $\lambda \in \Bbbk^\times$,
(5b) $f = \lambda xy + t^2$ *with* $\lambda \in \Bbbk^\times$.

Moreover, the Poisson algebras determined by f above are pairwisely nonisomorphic with the exception of replacing λ by $-\lambda$ in (5a) and (5b).

Quadratic Poisson algebras

We now consider the cases (2)-(5) from part (II) of the previous theorem. Throughout, we set

$$\mathfrak{m}_\alpha = (t-\alpha) \in \mathrm{Maxspec}(\mathcal{Z}_P(A)).$$

In cases (2,3,4b), we find a property $\mathcal{P}$ such that the $\mathcal{P}$-discriminant is t.

(2) $f = t^2$. Let $\mathcal{P}$ be the property that $A/\mathfrak{m}_\alpha$ is Poisson simple.

(3) $f = xt$. Let $\mathcal{P}$ be the property that $A/\mathfrak{m}_\alpha A$ does not have trivial Poisson bracket.

(4b) $f = x^2 + t^2$. Let $\mathcal{P}$ be the property that $A/\mathfrak{m}_\alpha A$ is not isomorphic to $A/\mathfrak{m}_0 A$.

We handle case (4a) differently.

(4a) $f = x^2$. In this case, $A \cong A' \otimes \Bbbk[t]$ where $A' = \Bbbk[x,y]$ with Poisson bracket $\{x,y\} = x^2$. Since A' is cancellative and $\Bbbk[t]$ is cancellative, then A is Poisson cancellative.

Cases (4c) and (5b) are similar to (4b), while (5a) is similar to (4a).

Theorem (G, Wang, Yee)

Let A be a quadratic polynomial Poisson algebra on $\Bbbk[x,y,z]$ with nontrivial bracket, then A is Poisson cancellative.

Additional cancellation results

Theorem (G,Wang,Yee)

Let A be a noetherian connected graded Poisson domain generated in degree 1. Suppose A is a graded isolated singularity and $\operatorname{Kdim} \mathcal{Z}_P(A) \leq 1$. *Then A is Poisson cancellative.*

Corollary

Let A be a quadratic polynomial Poisson algebra with three variables. If A has nontrivial bracket, then the dth Veronese algebra $A^{(d)}$ is Poisson cancellative for every $d \geq 1$.

Theorem (G,Wang,Yee)

Let $\mathfrak{g}$ be a non-abelian Lie algebra of dimension ≤ 3. Then $PS(\mathfrak{g})$ is strongly Poisson cancellative.

Skew cancellation

In the associative setting, the *skew cancellation problem* asks when an isomorphism of Ore extensions $A[x;\sigma,\delta] \cong B[x';\sigma',\delta']$ implies $A \cong B$. This problem has been studied by (Bergen), (Bell, Hamidizadeh, Huang, Venegas), and (Tang, Zhang, Zhao). We consider a Poisson version of this problem.

Definition

Let A be a Poisson algebra.

(1) A derivation α of a Poisson algebra A is called a *Poisson derivation* if

$$\alpha(\{a,b\}) = \{\alpha(a),b\} + \{a,\alpha(b)\} \quad \text{for all } a,b \in A.$$

(2) Given a Poisson derivation α of A, a derivation δ of A is a *Poisson α-derivation* if

$$\delta(\{a,b\}) = \{\delta(a),b\} + \{a,\delta(b)\} + \alpha(a)\delta(b) - \delta(a)\alpha(b) \quad \text{for all } a,b \in A.$$

(3) Given a Poisson derivation α and a Poisson α-derivation δ of A, the *Poisson-Ore extension* $A[t;\alpha,\delta]_P$ is the polynomial ring $A[t]$ together with the Poisson bracket

$$\{a,b\} = \{a,b\}_A, \qquad \{a,t\} = \alpha(a)t + \delta(a) \quad \text{for all } a,b \in A.$$

Skew cancellation

Definition

Let A be a Poisson algebra. We say A is *Poisson skew cancellative* if any Poisson isomorphism

$$A[t;\alpha,\delta]_P \cong B[t';\alpha',\delta']_P$$

implies a Poisson isomorphism $A \cong B$.

If the above holds whenever $\delta = \delta' = 0$ (resp. $\alpha = \alpha' = 0$), we say A is *Poisson α-cancellative* (resp. *Poisson δ-cancellative*).

Theorem

Let A be a noetherian Poisson domain of finite Krull dimension. Suppose either

(1) *A is Poisson simple and $A^\times = \Bbbk^\times$, or*

(2) *A is affine of Krull dimension 1.*

Then A is Poisson α-cancellative.

Theorem

Let A be an affine Poisson domain such that $\mathrm{PML}(A) = A$. *Then A is Poisson δ-cancellative.*

Poisson stratiform algebras

Definition

Let S be a simple artinian Poisson algebra. We say that S is *Poisson stratiform* over $\mathbb{k}$ if there is a chain of artinian Poisson simple Poisson algebras

$$S = S_n \supseteq S_{n-1} \supseteq \cdots \supseteq S_1 \supseteq S_0 = \mathbb{k}$$

where, for every i, either

(i) S_{i+1} is finite over S_i; or

(ii) $S_{i+1} = S_i(t_i; \alpha_i, \delta_i)_P$ for an appropriate choice of α_i, δ_i.

A Poisson domain A is said to be *Poisson stratiform* if $Q(A)$ is Poisson stratiform. We define the Poisson *stratiform length* of S to be the number of extensions of type (ii).

It can be shown, using the notion of *Gelfand-Kirillov transcendence degree* as introduced by Zhang, that the stratiform length of S is independent of the filtration.

Theorem (G,Wang,Yee)

Let A be a noetherian Poisson stratiform domain such that $\mathbb{D}(1) = A$. Then A is strongly Poisson skew cancellative in the category of noetherian Poisson stratiform domains.