Weight modules over Bell–Rogalski algebras

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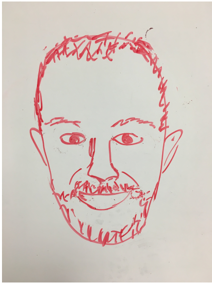

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Generalized Weyl algebras

In his 1990 thesis, Vladimir Bavula coined the term *generalized Weyl algebra* (GWA). This appears in his 1992 (in Russian) and its English translation is published in 1993.

In 1989, Rosenberg defines similar objects in a preprint what he calls *hyperbolic rings*. His 1995 monograph is the first widely circulated instance of these algebras.

The history is in fact even more complicated, with contributions from Joseph (1977), Stafford (1982), Smith (1990), Hodges (1990), and Jordan (1992).

Throughout, let $\mathbb{k}$ be a field.

Definition

Let R be a $\mathbb{k}$-algebra, σ an automorphism of R, and $a \in R$ be a central non-zero divisor. The GWA $R(\sigma, a)$ is generated over R by indeterminates x and y satisfying

$$\begin{aligned} &xr = \sigma(r)x, \quad yr = \sigma^{-1}(r)y, \quad \text{for all } r \in R, \\ &yx = a, \quad xy = \sigma(a). \end{aligned}$$

Examples of GWAs

Definition

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The GWAs $\mathbb{k}[t](\sigma, a)$ with $\sigma(t) = t - 1$ are called *classical GWAs*.

Example

Suppose $\mathbb{k}[t](\sigma, a)$ is a classical GWA with $a = t$. Then the relations become

$$xt = (t-1)x \qquad yt = (t+1)y \qquad yx = t \qquad xy = t - 1.$$

So we obtain the first Weyl algebra

$$\mathbb{k}[t](\sigma, a) \cong A_1(\mathbb{k}) = \mathbb{k}\langle x, \delta \rangle / (\delta x - x\delta - 1)$$

Example

A classical GWA with a quadratic is (isomorphic to) a primitive quotient of $U(\mathfrak{sl}_2)$.

Examples of GWAs

Definition

Let R be a $\Bbbk$-algebra, σ an automorphism of R, and a central non-zero divisor $a \in R$. The GWA $R(\sigma, a)$ is generated over R by indeterminates x and y satisfying

$$
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&xr = \sigma(r)x, \quad yr = \sigma^{-1}(r)y, \quad \text{for all } r \in R,\\
&yx = a, \quad xy = \sigma(a).
\end{aligned}
$$

Example

The GWA $\Bbbk[t](\sigma, a)$ with $a = t$ and $\sigma(t) = q^{-1}(t-1)$ for $q \in \Bbbk^{\times}$ is isomorphic to the quantum Weyl algebra $A_1^q(\Bbbk) = \Bbbk\langle x, y : yx - qxy - 1\rangle$.

Example

The GWA $\Bbbk[t_1, t_2](\sigma, a)$ with $a = t_2 - t_1(t_1 + 1)$ and $\sigma(t_1) = t_1 - 1$, $\sigma(t_2) = t_2$, is isomorphic to $U(\mathfrak{s}l_2)$.

Properties of GWAs

Let R be a commutative ring. GWAs over “nice” rings R are “nice” (in a ring-theoretic/homological sense).

- $R(\sigma, a)$ is a domain if R is a domain.
- $R(\sigma, a)$ is noetherian if R is noetherian.
- $R(\sigma, a)$ is simple if and only if R is σ-simple, σ has infinite order, and $R = aR + \sigma^n(a)R$ for all $n \geq 1$.

For a classical GWA $A = \Bbbk[t](\sigma, a)$ we can say even more.

- A is Auslander-Gorenstein (associated graded ring is Gorenstein).
- $$\operatorname{gldim} A = \begin{cases} \infty & \text{if } a \text{ has a multiple root} \\ 2 & \text{if } a \text{ has a congruent root and no multiple roots} \\ 1 & \text{if } a \text{ has no congruent roots and no multiple roots.} \end{cases}$$

Weight modules for GWAs

Here we assume that R is a commutative ring.

Definition

Let M be a (left) module for $R(\sigma, a)$. We say M is a *weight module* if

$$M = \bigoplus_{\mathfrak{m} \in \mathrm{Maxspec}(R)} M_{\mathfrak{m}}$$

where

$$M_{\mathfrak{m}} = \{z \in M \mid \mathfrak{m} \cdot z = 0\}.$$

We say that $M_{\mathfrak{m}}$ is a *weight space* of M and the *support* of M is

$$\mathrm{Supp}(M) = \{\mathfrak{m} \in \mathrm{Maxspec}(R) \mid M_{\mathfrak{m}} \neq 0\}.$$

Analogy

R plays the role of the Cartan subalgebra of a Lie algebra.

Weight modules for GWAs

Let $R(\sigma, a)$ be a GWA with R commutative. Let M be a weight module for $R(\sigma, a)$.

There is an action of $\mathbb{Z}$ on $\mathrm{Maxspec}(R)$ given by

$$n \cdot \mathfrak{m} = \sigma^n(\mathfrak{m}).$$

Then

$$x(M_{\mathfrak{m}}) \subset M_{\sigma(\mathfrak{m})} \quad \text{and} \quad y(M_{\mathfrak{m}}) \subset M_{\sigma^{-1}(\mathfrak{m})}$$

In fact, M splits as

$$M = \bigoplus_{\mathcal{O} \in \mathrm{Maxspec}(R)/\mathbb{Z}} M_{\mathcal{O}} \quad \text{where} \quad \mathrm{Supp}(M_{\mathcal{O}}) \subset \mathcal{O}.$$

So, we can consider weight modules supported on one orbit at a time.

The classification of simple and indecomposable weight modules for GWAs is due to Bavula (1992) and Drozd-Guzner-Ovsienko (1996).

Infinite orbit case

Let $R(\sigma, a)$ be a GWA with R commutative.

Definition

We say $\mathfrak{m} \in \mathcal{O}$ is a *break* if $a \in \mathfrak{m}$.

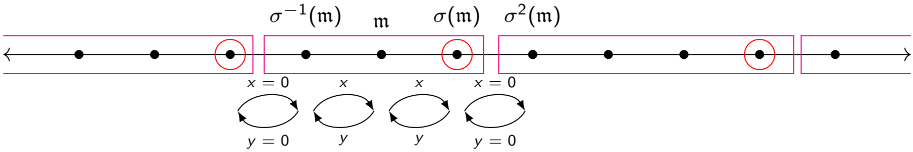
In the classical case ($R = \mathbb{k}[t]$ and $\sigma(t) = t - 1$), breaks correspond to roots of the polynomial a.

If $\mathfrak{m}$ is a break, then

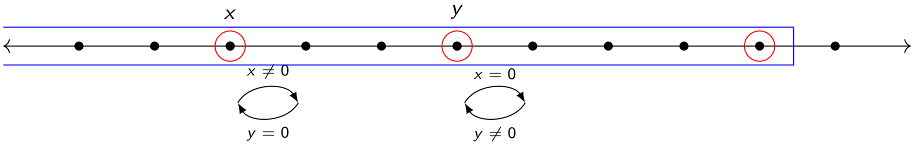
$$0 = \sigma(a)M_{\sigma(\mathfrak{m})} = (xy)M_{\sigma(\mathfrak{m})} = x(yM_{\sigma(\mathfrak{m})}) = xM_{\mathfrak{m}}$$

$$0 = aM_{\mathfrak{m}} = (yx)M_{\mathfrak{m}} = y(xM_{\mathfrak{m}}) = yM_{\sigma(m)}.$$

Infinite orbit case



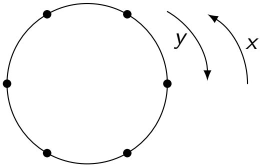
$\boxed{\text{Simples}}$ $\leftrightarrow$ Intervals between breaks (weight spaces $\cong R/\sigma^i(\mathfrak{m})$ are 1-dimensional)



$\boxed{\text{Indecomposables}}$ $\leftrightarrow$ Interval of simples with breaks labeled by x or y (weight spaces are 1-dimensional).

Finite orbit case

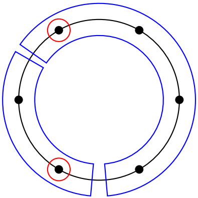
Let $\mathcal{O}$ be a finite orbit, so $\sigma^n(\mathfrak{m}) = \mathfrak{m}$ for some $n \in \mathbb{N}$.



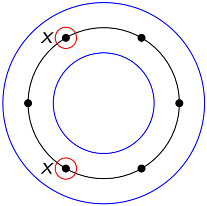
<u>No breaks</u>

$\boxed{\text{Simples}}$ $\leftrightarrow$ Linear transformations (up to similarity) with no invariant subspaces

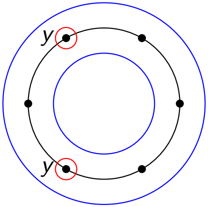
<u>With breaks</u>



Simples $\leftrightarrow$ intervals btw breaks



x^n linear transformation



y^n linear transformation

BR algebras

(2015 Bell and Rogalski)
Classify $\mathbb{Z}$-graded simple rings with graded quotient ring isomorphic to $R[t, t^{-1}; \sigma]$.

Definition

Let R be a commutative $\Bbbk$-algebra, $\sigma \in \mathrm{Aut}(R)$, and ideals H and J of R. Set

$$I^{(n)} = \begin{cases} R & n = 0 \\ J\sigma(J) \dots \sigma^{n-1}(J) & n \geq 1 \\ \sigma^{-1}(H)\sigma^{-2}(H) \dots \sigma^{n}(H) & n \leq -1. \end{cases}$$

We assume $I^{(n)} \neq 0$ for all n. The corresponding *Bell–Rogalski (BR) algebra* is

$$B = R(t, \sigma, H, J) = \bigoplus_{n \in \mathbb{Z}} I^{(n)} t^n \subset R[t, t^{-1}; \sigma].$$

The $\mathbb{Z}$-grading on $R[t, t^{-1}; \sigma]$ ($\deg(t^{\pm 1}) = \pm 1$, $\deg(r) = 0$ for all r) induces a grading on $B = R(t, \sigma, H, J)$. Then B is then generated by R, B_1 and B_{-1} where

$$B_1 = \{jt : j \in J\} \qquad B_{-1} = \{\sigma^{-1}(h)t^{-1} : h \in H\}.$$

BR algebras

If $H = (h)$ and $J = (j)$, then $R(\sigma, a) \cong B$ through the map

$$r \mapsto r \text{ for all } r \in R, \quad x \mapsto jt, \quad y \mapsto \sigma^{-1}(h)t^{-1}.$$

In this case, $a \mapsto \sigma^{-1}(hj)$.

Proposition (Bell-Rogalski, G-Rosso-Won)

As $\mathbb{Z}$-graded algebras, the (rank one) GWAs over R are exactly the BR algebras $R(t, \sigma, H, J)$ in which H and J are nonzero principal ideals.

Example

Let $R = \mathbb{k}[z_1, z_2]$. Define $\sigma \in \mathrm{Aut}(R)$ by

$$\sigma(z_1) = z_1 - 1 \quad \text{and} \quad \sigma(z_2) = z_2 + 1.$$

Set

$$H = (z_1 - 1, z_2 + 1) \quad \text{and} \quad J = R.$$

Since B_{-1} is not a free module of rank 1 over B_0, then B is not isomorphic to a GWA as $\mathbb{Z}$-graded algebras.

BR algebras

Let R be a commutative noetherian domain and let $B = R(t, \sigma, H, J)$.

Proposition (Bell-Rogalski, G-Rosso-Won)

The BR algebra B is simple if and only if the following hold:

1. *R is σ-simple (R has no non-trivial σ-invariant ideals), and*
2. *the set $\mathcal{S}(B) = \{\mathbf{p} \in \operatorname{Spec}(R) : HJ \subset \mathbf{p}\}$ is σ-lonely ($\mathcal{S}(B) \cap \sigma^i(\mathcal{S}(B)) = \emptyset$ for all $i \neq 0$).*

In case B is isomorphic to the GWA $R(\sigma, a)$, these criteria reduce to those for GWAs: R is σ-simple, σ has infinite order, and $R = aR + \sigma^n(a)R$ for all $n \geq 1$.

Example

Let $R = \Bbbk[z_1, z_2]$. Define $\sigma \in \operatorname{Aut}(R)$ by $\sigma(z_1) = z_1 - 1$ and $\sigma(z_2) = z_2 + 1$. Set $H = (z_1 - 1, z_2 + 1)$ and $J = R$.

Since the ideal $(z_1 + z_2)$ is fixed by σ, then R is not σ-simple and hence $B = R(t, \sigma, H, J)$ is not simple.

Weight modules for BR algebras

Let $B = R(\sigma, t, H, J)$ with R a commutative noetherian domain.

We define a weight module $M = \bigoplus_{\mathfrak{m} \in \mathrm{Maxspec}(R)} M_{\mathfrak{m}}$ for B as in the GWA case where

$$B_1(M_{\mathfrak{m}}) \subset M_{\sigma(\mathfrak{m})} \quad \text{and} \quad B_{-1}(M_{\mathfrak{m}}) \subset M_{\sigma^{-1}(\mathfrak{m})}.$$

Let (B, R)-wmod denote the full subcategory of (left) B-modules which are R-weight modules.

We have the same $\mathbb{Z}$-action on $\mathrm{Maxspec}(R)$: $n \cdot \mathfrak{m} = \sigma^n(\mathfrak{m})$.

For $\mathcal{O} \in \mathrm{Maxspec}(R)/\mathbb{Z}$, let (B, R)-wmod$_{\mathcal{O}}$ denote the full subcategory of modules $M \in \mathrm{Maxspec}(R)$ with $\mathrm{Supp}_R(M) \subset \mathcal{O}$.

Weight modules for BR algebras

Definition

We say $\mathfrak{m} \in \mathcal{O}$ is a *break* if $\sigma(\mathfrak{m}) \supset HJ$. (That is, $\sigma(\mathfrak{m}) \in \mathcal{S}(B)$.)

Theorem (G-Rosso-Won)

Let $\mathcal{O}$ be an infinite orbit and let $\beta \subset \mathcal{O}$ be the set of breaks.

- *If $\beta = \emptyset$, then there is a unique simple module in $(B, R)\text{-wmod}_{\mathcal{O}}$.*
- *If $\beta \neq \emptyset$, then the isomorphism classes of simples modules in $(B, R)\text{-wmod}_{\mathcal{O}}$ are parameterized by β ($\beta \cup \{\infty\}$ in case β contains a maximal element).*

$\boxed{\text{Simples}}$ $\leftrightarrow$ Same statement as for GWAs
(intervals between breaks, weight modules are 1-dimensional)

Theorem (G-Rosso-Won)

Fix $n \in \mathbb{N}$ and suppose that $\sigma^n = \mathrm{id}_R$. Let $\mathcal{O}$ be an orbit of size n.

- *No breaks on $\mathcal{O}$: the simple weight modules are parameterized by triples $(\mathfrak{m}, N, \theta)$ where $\mathfrak{m} \in \mathcal{O}$, N is a vector space over $R/\mathfrak{m}$, and $\theta \in \mathrm{Aut}_{R/\mathfrak{m}}(N)$ is an invertible linear transformation that leaves no nontrivial subspace invariant.*
- *Breaks on $\mathcal{O}$: each simple module belongs to one of three families.*

Indecomposable weight modules

Example

Assume $\operatorname{char}\mathbb{k} = 0$. Let $R = \mathbb{k}[z_1, z_2]$. Define $\sigma \in \mathrm{Aut}(R)$ by $\sigma(z_1) = z_1 - 1$ and $\sigma(z_2) = z_2 + 1$. Set $H = (z_1 - 1, z_2 + 1)$ and $J = R$. Consider the infinite orbit

$$\mathcal{O} = \mathbb{Z} \cdot \mathfrak{m} = \{(z_1 - k, z_2 + k) \in \mathrm{Maxspec}(R) : k \in \mathbb{Z}\}$$

which contains the only break $\mathfrak{m} = (z_1, z_2)$.

(1) Consider

$$M = \bigoplus_{k \in \mathbb{Z}} (R/(z_1 - k, z_2 + k))v_k$$

as an R-module. Define a B-action by

$$t \cdot v_k = v_{k+1} \qquad z_i t^{-1} \cdot v_k = \begin{cases} z_i v_{k-1} & \text{if } k \neq 1 \\ 0 & \text{if } k = 1. \end{cases}$$

Then M is indecomposable but not simple. This is similar to the GWA case.

Indecomposable weight modules

Example

Assume $\operatorname{char}\Bbbk = 0$. Let $R = \Bbbk[z_1, z_2]$. Define $\sigma \in \mathrm{Aut}(R)$ by $\sigma(z_1) = z_1 - 1$ and $\sigma(z_2) = z_2 + 1$. Set $H = (z_1 - 1, z_2 + 1)$ and $J = R$. Consider the infinite orbit

$$\mathcal{O} = \mathbb{Z} \cdot \mathfrak{m} = \{(z_1 - k, z_2 + k) \in \mathrm{Maxspec}(R) : k \in \mathbb{Z}\}$$

which contains the only break $\mathfrak{m} = (z_1, z_2)$.

(2) Let $M(\alpha_1, \alpha_2) = \bigoplus_{k \in \mathbb{Z}} (R/(z_1 - k, z_2 + k))v_k$, for $\alpha_1, \alpha_2 \in \Bbbk$, with

$$t \cdot v_k = \begin{cases} v_{k+1} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases} \qquad z_i t^{-1} \cdot v_k = \begin{cases} z_i v_{k-1} & \text{if } k \neq 1 \\ \alpha_i v_0 & \text{if } k = 1. \end{cases}$$

- $M(\alpha_1, \alpha_2)$ is indecomposable so long as $(\alpha_1, \alpha_2) \neq (0, 0)$, but not simple.
- $M(\alpha_1, \alpha_2) \simeq M(\alpha_1', \alpha_2')$ if and only if $(\alpha_1, \alpha_2) = c(\alpha_1', \alpha_2')$ for some $c \in \Bbbk$.
- This gives as $\mathbb{P}^1$-family of nonisomorphic, nonsimple indecomposable modules that have the same composition series of length two.

Indecomposable weight modules

Example

Assume $\operatorname{char}\Bbbk = 0$. Let $R = \Bbbk[z_1, z_2]$. Define $\sigma \in \mathrm{Aut}(R)$ by $\sigma(z_1) = z_1 - 1$ and $\sigma(z_2) = z_2 + 1$. Set $H = (z_1 - 1, z_2 + 1)$ and $J = R$. Consider the infinite orbit

$$\mathcal{O} = \mathbb{Z} \cdot \mathfrak{m} = \{(z_1 - k, z_2 + k) \in \mathrm{Maxspec}(R) : k \in \mathbb{Z}\}$$

which contains the only break $\mathfrak{m} = (z_1, z_2)$.

(3) We also have an example of an indecomposable module where the weight spaces are not all one-dimensional, which again does not happen in the GWA case.

Thank You!