Calculus II

Chapter 7 - Techniques of Integration

1. Integration by parts

Remark. How do we evaluate an integral like $\int x \sin x \, dx$? Apart from recognizable antiderivatives, the only tool we have is the substitution method. A moment's thought will show that there is no such substitution here. In particular, there is no composition of functions. Instead, we have a product of functions so we would like some analog of the product rule from differentiation.

Let f and g be differentiable functions.

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives

$$f(x)g(x) = \int f'(x)g(x) \ dx + \int f(x)g'(x) \ dx.$$

After rearranging we have the integration by parts formula,

$$\int f(x)g'(x) \ dx = f(x)g(x) - \int g(x)f'(x) \ dx.$$

Remark (Integration-by-Parts)**.** Let f and g be differentiable functions. Set $u = f(x), v = g(x)$, then the differentials are du $= f'(x) \ dx$ and dv $= g'(x) \ dx$. We then have the following formula,

$$\int u \ dv = uv - \int v \ du.$$

Example 1. Evaluate $\int x \sin x \ dx$.

We let $u = x$ and $dv = \sin x \ dx$. The rationale behind this choice is that we will differentiate u, because we want u to be *simpler*, and and integrate dv. This choice gives $du = dx$ and $v = -\cos x$. Thus, the IBP formula gives,

$$\int u \ dv = -x\cos x - \int -\cos x \ dx = -x\cos x + \sin x + C.$$

If we had chosen $u = \sin x$ and $dv = x \ dx$ instead, then we would have $du = \cos x \ dx$ and $v = \frac{1}{2}x^2$. The resulting integral would have been more complicated, instead of simpler, than the original.

These notes are derived primarily from *Calculus, Early Transcendentals* by James Stewart (8ed). Most of this material is drawn from Chapters 7. Last Updated: September 6, 2019

Example 2. Evaluate $\int x^2 \sin 4x \; dx$.

For reasons similar to the previous example, we make the choice $u = x^2$ and $dv = \sin 4x \; dx$. Then $du = 2x \; dx$ and $v = -\frac{1}{4}\cos 4x$. The IBP formula gives,

$$\int x^2 \sin 4x \; dx = x^2\left(-\frac{1}{4}\cos 4x\right) - \int -\frac{1}{4}\cos 4x \cdot 2x \; dx = -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\int x\cos 4x \; dx.$$

To evaluate the remaining integral we must use IBP again. Set $u = x$ and $dv = \cos 4x \; dx$, so $du = \; dx$ and $v = \frac{1}{4}\sin 4x$. So we have

$$\begin{aligned}
\int x^2 \sin 4x \; dx &= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\int x\cos 4x \; dx \\
&= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\left(x\cos 4x - \int \frac{1}{4}\sin 4x \; dx\right) \\
&= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}x\cos 4x - \frac{1}{8}\int \sin 4x \; dx \\
&= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}x\cos 4x - \frac{1}{8}\left(-\frac{1}{4}\cos 4x\right) + C \\
&= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}x\cos 4x + \frac{1}{24}\cos 4x + C.
\end{aligned}$$

Example 3. Evaluate $\int \ln x \; dx$.

Given $\int \ln x \; dx$, set $u = \ln x$ and $dv = \; dx$, so $du = \frac{1}{x} \; dx$ and $v = x$. By the IBP formula,

$$\int \ln x \; dx = x\ln x - \int x\frac{1}{x} \; dx = x\ln x - \int \; dx = x\ln x - x + C.$$

Remark. How do we choose u? So far we have relied on trial-and-error but there is a little pnuemonic which is very helpful: LIPET. This stands for

L(ogarithms)I(nverse trig)P(olynomials)E(xponentials)T(trig).

Remark (Integration by Parts for definite integrals)**.** Let f and g be differentiable functions. Then

$$\int_a^b f(x)g'(x) \; dx = f(x)g(x) \mid_a^b - \int_a^b g(x)f'(x) \; dx.$$

Note that, unlike with u-substitution, there is no need to change the bounds.

Example 4. Evaluate $\int_1^4 \sqrt{t}\ln t \; dt$.

Set $u = \ln t$ and $dv = \sqrt{t} \; dt$. Then $du = \frac{1}{t} \; dt$ and $v = \frac{2}{3}t^{3/2}$. By the IBP formula,

$$\begin{aligned}
\int_1^4 \sqrt{t}\ln t \; dt &= \frac{2}{3}t^{3/2}\ln t\Big|_1^4 - \int_1^4 \frac{2}{3}t^{3/2}\cdot\frac{1}{t} \; dt = \frac{2}{3}(8\ln 4 - 1\cdot 0) - \frac{2}{3}\int_1^4 t^{1/2} \; dt \\
&= \frac{18}{3}\ln 4 - \frac{2}{3}\left[\frac{2}{3}t^{3/2}\right]_1^4 = \frac{18}{3}\ln 4 - \frac{4}{9}[8-1] = \frac{18}{3}\ln 4 - \frac{28}{9}.
\end{aligned}$$

Example 5. Evaluate $\int x^5 e^{x^2}\ dx$.

First we make a substitution. Let $y = x^2$ so $dy = 2x\ dx$. Then

$$\int x^5 e^{x^2}\ dx = \int x^4 e^{x^2}(x\ dx) = \frac{1}{2}\int y^2 e^y\ dy.$$

Set $u = y^2$ and $dv = e^y$. We will apply IBP twice in order to reduce u to a constant and we can use *tabular integration* to organize our work. Our table consists of one column to keep track of signs (alternating +,-,+,-,...), one column for the derivatives of u, and one column for integrals of v.

sign	u	dv
+	y^2	e^y
-	$2y$	e^y
+	2	e^y
-	0	e^y

Thus,

$$\frac{1}{2}\int y^2 e^y\ dy = \frac{1}{2}\left(y^2 e^y - 2ye^y + 2e^y\right) + C = \frac{1}{2}\left(x^4 e^{x^2} - 2x^2 e^{x^2} + 2e^{x^2}\right) + C.$$

Example 6. Evaluate $\int e^{2x}\cos x\ dx$.

We could try the tabular method in this case, but we find ourselves going in circles. In particular, setting $u = \cos 2x$ and $dv = e^{-x}$ gives

sign	u	dv
+	e^{2x}	$\cos x$
-	$2e^{2x}$	$\sin x$
+	$4e^{2x}$	$-\cos x$

However, we find that

$$\begin{aligned}\int e^{2x}\cos x\ dx &= e^{2x}\sin x - \int 2e^{2x}\sin x\ dx \\ &= e^{2x}\sin x - 2\left(-e^{2x}\cos x + \int 2e^{2x}\cos x\ dx\right) \\ &= e^{2x}\sin x + 2e^{2x}\cos x - 4\int e^{2x}\cos x\ dx.\end{aligned}$$

Now we observe that the remaining integral is the same as the initial integral but with a coefficient of -4. Adding $4\int e^{2x}\cos x\ dx$ to both sides gives

$$5\int e^{2x}\cos x\ dx = e^{2x}\sin x + 2e^{2x}\cos x + C.$$

Thus,

$$\int e^{2x}\cos x\ dx = \frac{1}{5}e^{2x}\sin x + \frac{2}{5}e^{2x}\cos x + C.$$

This is known as *Roundin' the Corner*.

2. Trigonometric Integrals

Example 7. Evaluate $\int \sin^2 x \; dx$.

It turns out that we don't need integration by parts for this problem. It's easier to solve with a trig identity. Recall that $\sin^2 x = \frac{1}{2}(1-\cos(2x))$. Then

$$\begin{aligned}\int \sin^2 x \; dx &= \int \frac{1}{2}(1-\cos(2x)) \; dx = \frac{1}{2}\int 1-\cos(2x) \; dx \\ &= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + C = \frac{x}{2} - \frac{1}{4}\sin(2x) + C.\end{aligned}$$

Remark. For this section, it is worth reviewing some basic trig identities. We will use these, along with substitution and integration by parts, to evaluate some new integrals.

First, we want to evaluate integrals of the form $\int \sin^m x \cos^n x \; dx$. We may have that one of them is zero. We have the following rules,

(1) If n is odd, save one factor of $\cos x$ and rewrite the rest using

$$\cos^2 x = 1 - \sin^2 x.$$

(2) If m is odd, save one factor of $\sin x$ and rewrite the rest using

$$\sin^2 x = 1 - \cos^2 x.$$

(3) If both are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \text{ or } \cos^2 x = \frac{1}{2}(1+\cos 2x).$$

(4) It is sometimes helpful to use,

$$\sin x \cos x = \frac{1}{2}\sin 2x \text{ or } \cos^2(2x) = \frac{1}{2}(1+\cos 4x).$$

Example 8. Evaluate $\int \sin^3 x \; dx$.

We save one factor of $\sin x$ and rewrite the rest using the Pythagorean identity:

$$\int \sin^3 x \; dx = \int \sin x(\sin^2 x) \; dx = \int \sin x(1-\cos^2 x) \; dx.$$

Now let $u = \cos x$ so $du = -\cos x \; dx$. Then

$$\begin{aligned}\int \sin^3 x \; dx &= \int \sin x(1-\cos^2 x) \; dx = -\int 1-u^2 \; du \\ &= -\left(u - \frac{1}{3}u^3\right) + C = \frac{1}{3}\cos^3 x - \cos x + C.\end{aligned}$$

Example 9. Evaluate $\int_0^\pi \cos^4\theta\ d\theta$.

We will apply the half-angle identity.

$$\begin{aligned}
\int_0^\pi \cos^4\theta\ d\theta = \int_0^\pi \left[\frac{1}{2}(1+\cos 2\theta)\right]^2\ d\theta &= \int_0^\pi \frac{1}{4}\left[1+2\cos 2\theta+\cos^4(2\theta)\right]\ d\theta \\
&= \frac{1}{4}\int_0^\pi 1+2\cos(2\theta)+\frac{1}{2}(1+\cos(4\theta))\ d\theta \\
&= \frac{1}{4}\int_0^\pi \frac{3}{2}+2\cos(2\theta)+\frac{1}{2}\cos(4\theta)\ d\theta \\
&= \frac{1}{4}\left[\frac{3}{2}\theta+\sin(2\theta)+\frac{1}{8}\sin(4\theta)\right]_0^\pi = \frac{3}{8}\pi.
\end{aligned}$$

Remark. Next we want to evaluate $\int \tan^m x \sec^n x\ dx$.

(1) If n is even, save a factor of $\sec^2 x$ and use $\sec^2 x = 1+\tan^2 x$;

(2) If m is odd, save a factor of $\sec x\tan x$ and use $\tan^2 x = \sec^2 x - 1$.

Example 10. Evaluate $\int \sec^4 x\tan^4 x\ dx$.

We rewrite using the guidelines above,

$$\int \sec^4 x\tan^4 x\ dx = \int \sec^2 x(1+\tan^2 x)\tan^4 x\ dx.$$

Now let $u=\tan x$ so $du = \sec^2 x$. Then

$$\int \sec^2 x(1+\tan^2 x)\tan^4 x\ dx = \int (1+u^2)u^4\ du.$$

Example 11. Evaluate $\int_0^{\pi/3} \sec^3 x\tan x\ dx$.

Rewrite then let $u=\sec x$ so $du = \sec x\tan x\ dx$.

$$\begin{aligned}
\int_0^{\pi/3} \sec^3 x\tan x\ dx &= \int_0^{\pi/3} \sec^2 x(\sec x\tan x)\ dx \\
&= \int_1^2 u^2\ du = \left[\frac{u^3}{3}\right]_1^2 = \frac{8}{3}-\frac{1}{3} = \frac{7}{3}.
\end{aligned}$$

Remark. At times we will also need the following integrals,

$$\int \tan x\ dx = \ln|\sec x| + C \text{ and } \int \sec x\ dx = \ln|\sec x+\tan x| + C.$$

3. Trigonometric substitution

Remark. The key examples to think about are

$$\int \frac{x}{1+x^2}\,dx \text{ and } \int \frac{x^2}{1+x^2}\,dx.$$

The first one we can evaluate using the substitution method. On the other hand, our current methods don't seem adequate to handle the second. The trick is to use a sort of 'reverse substitution'. Instead of choosing a function to be $u(x)$ and then determining du, we instead assign x to be a function. A convenient choice will be certain trig functions which allows us to simplify the expression using the trigonometric identities.

Example 12. Evaluate

$$\int \frac{x^2}{1+x^2}\,dx$$

by making the substitution $x = \tan\theta$.

By making this change of variable, we have dx $= \sec^2\theta\; d\theta$. Hence,

$$\begin{aligned}\int \frac{x^2}{1+x^2}\,dx &= \int \frac{\tan^2\theta}{1+\tan^2\theta}(\sec^2\theta)\;d\theta = \int \frac{\tan^2\theta}{\sec^2\theta}(\sec^2\theta)\;d\theta\\ &= \int \tan^2\theta\;d\theta = \int \sec^2\theta - 1\;d\theta\\ &= \tan\theta - \theta + C = x - \arctan x + C.\end{aligned}$$

Remark (Trig substitution)**.** Let a be a nonzero constant. We make the following substitution for each expression.

(1) $\sqrt{a^2-x^2}$, let $x = a\sin\theta, -\pi/2 \le \theta \le \pi/2$;

(2) $\sqrt{a^2+x^2}$, let $x = a\tan\theta, -\pi/2 < \theta < \pi/2$;

(3) $\sqrt{x^2-a^2}$, let $x = a\sec\theta, 0 \le \theta < \pi/2$ or $\pi \le \theta < 3\pi/2$.

The bounds are necessary to ensure that we can take the inverse, e.g., $\theta = \arcsin(x/a)$.

Example 13. Evaluate

$$\int x^3\sqrt{9-x^2}\;dx.$$

Let $x = 3\sin\theta$, $-\pi/2 \le \theta \le \pi/2$. Then dx $= 3\cos\theta\; d\theta$ so we have

$$\begin{aligned}\int x^3\sqrt{9-x^2}\;dx &= \int 3^3\sin^3\theta\sqrt{9-9\sin^2\theta}(3\cos\theta\;d\theta) = 3^5\int \sin^3\theta\cos\theta\sqrt{1-\sin^2\theta}\;d\theta\\ &= 3^5\int \sin^3\theta\cos^2\theta\;d\theta = 3^5\int \sin\theta(1-\cos^2\theta)\cos^2\theta\;d\theta\end{aligned}$$

Now let $u = \cos\theta$ so $du = -\sin\theta\;d\theta$. Then

$$\int x^3\sqrt{9-x^2}\;dx = 3^5\int \sin\theta(\cos^4\theta - \cos^2\theta)\;d\theta = 3^5\int u^4 - u^2\;du = 3^5\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right] + C.$$

Since $x = 3\sin\theta$, then $\theta = \arcsin(x/3)$. Then

$$u = \cos\theta = \cos\left(\arcsin\left(\frac{x}{3}\right)\right) = \frac{\sqrt{9-x^2}}{3}.$$

Hence,

$$\begin{aligned}
\int x^3\sqrt{9-x^2}\ dx &= 3^5\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right] + C \\
&= 3^5\left[\frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3\right] + C \\
&= \frac{1}{5}(9-x^2)^{5/2} - 3(9-x^2)^{3/2} + C.
\end{aligned}$$

Example 14. Evaluate

$$\int_0^2 x^3\sqrt{x^2+4}\ dx.$$

Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$ so dx $= 2\sec^2\theta\ d\theta$. We must adjust the bounds as well. When $x = 0$, $\theta = 0$ and when $x = 2$, $\theta = \pi/4$. Then

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^2+4}\ dx &= \int_0^{\pi/4} (2\tan\theta)^3\sqrt{4\tan^2\theta + 4}(2\sec^2\theta\ d\theta) \\
&= 2^5\int_0^{\pi/4} \tan^3\theta(\sec^3\theta)\ d\theta \\
&= 2^5\int_0^{\pi/4} (\sec^2\theta - 1)\sec^2\theta(\tan\theta\sec\theta)\ d\theta.
\end{aligned}$$

Let $u = \sec\theta$ so $du = \tan\theta\sec\theta\ d\theta$. Now the bounds become $\sec(0) = 1$ and $\sec(\pi/4) = \sqrt{2}$. Hence,

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^2+4}\ dx &= 2^5\int_0^{\pi/4} (\sec^4\theta - \sec^2\theta)(\tan\theta\sec\theta)\ d\theta \\
&= 2^5\int_1^{\sqrt{2}} u^4 - u^2\ du = 2^5\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]_1^{\sqrt{2}} \\
&= 2^5\left[\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3}\right] = 2^5\left[\frac{2}{15}\sqrt{2} + \frac{2}{15}\right] = \frac{64}{15}(\sqrt{2}+1)
\end{aligned}$$

Example 15. Evaluate

$$\int \frac{dx}{x^2\sqrt{x^2-1}}.$$

Let $x = \sec\theta$, $0 \le \theta \le \pi/2$ or $\pi \le \theta \le 3\pi/2$, so dx $= \sec\theta\tan\theta\ d\theta$. We have

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\ d\theta}{\sec^2\theta\sqrt{\sec^2\theta - 1}} = \int \frac{\sec\theta\tan\theta\ d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \int \cos\theta\ d\theta = \sin\theta + C.$$

Now $\theta = \operatorname{arcsec}(x)$ and so,

$$\sin\theta + C = \sin(\operatorname{arcsec}(x)) + C = \frac{\sqrt{x^2-1}}{x} + C.$$

4. Partial Fractions

We begin with an integral we could have done in Calculus I using long division of fractions.

Example 16. Evaluate
$$\int \frac{x^3 - 1}{x + 3}\ dx.$$

Using polynomial long division, we get,
$$\frac{x^3 - 1}{x + 3} = x^2 - 3x + 3 - \frac{10}{x + 3}.$$

Hence,
$$\int \frac{x^3 - 1}{x + 3}\ dx = \int x^2 - 3x + 3 - \frac{10}{x + 3}\ dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 3x - 10 \ln |x + 3| + C.$$

Remark. Another way to evaluate integrals involving quotients of rational functions is to use partial fraction decomposition. To do this, we factor the denominator and then try to find numerators that, if we were to find a common denominator, would give us the original numerator.

Our first case is when the denominator is a product of distinct linear factors. Here we write,
$$\frac{P}{(x - a_1)(x - a_2) \cdots (x - a_n)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}.$$

Example 17. Evaluate
$$\int \frac{x + 4}{2x^2 - 5x - 3}\ dx.$$

The denominator factors as $2x^2 - 5x - 3 = (2x + 1)(x - 3)$. Hence, we need A and B such that
$$\frac{x + 4}{2x^2 - 5x - 3} = \frac{A}{2x + 1} + \frac{B}{x - 3}.$$

This implies that $A(x - 3) + B(2x + 1) = x + 4$. That is, $A + 2B = 1$ and $-3A + B = 4$. Solving gives $A = -1$ and $B = 1$. Thus,
$$\int \frac{x + 4}{2x^2 - 5x - 3}\ dx = \int \frac{-1}{2x + 1} - \frac{1}{x - 3}\ dx = -\frac{1}{2} \ln |2x - 1| - \ln |x + 3| + C.$$

Remark. Here is a shortcut to finding A and B in the previous example. This method is effective when all factors are distinct.

The equation $A(x-3)+B(2x+1) = x+4$ holds for *all* x. In particular, it holds for $x = 3$. Making this substitution gives
$$A(3 - 3) + B(2 \cdot (3) + 1) = 3 + 4 \Rightarrow 7B = 7 \Rightarrow B = 1.$$

Similarly, choosing $x = -1/2$ gives
$$A(-1/2 - 3) + B(2(-1/2) - 1) = (-1/2) + 4 \Rightarrow (-7/2)A = 7/2 \Rightarrow A = -1.$$

Remark. The next case is when the denominator is a product of linear factors but some are repeated. We treat each power of that factor as an individual factor, as below,

$$\frac{P}{(x-a_1)^r(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_r}{(x-a_1)^r} + \frac{B_2}{x-a_2} + \cdots + \frac{B_n}{x-a_n}.$$

Example 18. Evaluate

$$\int \frac{x^2-2x-2}{x^3+2x^2+x}\, dx.$$

The denominator factors as $x^3+2x^2+x = x(x^2+2x+1) = x(x+1)^2$. Thus, we need to find A, B, C such that

$$\frac{x^2-2x-2}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

This leads to the equation

$$A(x+1)^2 + Bx(x+1) + Cx = x^2-2x-2.$$

If we set $x=0$, then $A(1)^2 = -2$ so $A=-2$. If we set $x=-1$, then $C(-1) = (-1)^2 - 2(-1) - 2$, so $C=-1$. However, we cannot use this trick to find B. We can sub in A and C and then solve for B.

$$-2(x^2+2x+1) + B(x^2+x) - 1(x) = x^2-2x-2.$$

Comparing the coefficient of x^2 gives $-2+B=1$, so $B=3$. Now

$$\begin{aligned}\int \frac{x^2-2x-2}{x^3+2x^2+x}dx &= \int \frac{-2}{x} + \frac{3}{x+1} - \frac{1}{(x+1)^2}\, dx \\ &= -2\ln|x| + 3\ln|x+1| + \frac{1}{x+1} + C.\end{aligned}$$

Remark. Next, we consider when the denominator contains an irreducible quadratic factor, but none are repeated. Here we write,

$$\frac{Cx+D}{ax^2+bx+c}$$

and use the integration,

$$\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Example 19. Evaluate

$$\int \frac{3x^3+3x^2-3x-1}{x^4+x^2}\, dx.$$

Write

$$\frac{3x^3+3x^2-3x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}.$$

This implies the equation

$$Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 = 3x^3+3x^2-3x-1.$$

We can at least figure out one of the coefficients with the trick above. Let $x = 0$, then this implies $B = -1$. Now we have

$$Ax(x^2+1) - (x^2+1) + (Cx+D)x^2 = 3x^3 + 3x^2 - 3x - 1$$
$$Ax(x^2+1) + (Cx+D)x^2 = 3x^3 + 4x^2 - 3x.$$

Now we see that the only coefficient of x^2 on the left is D, so $D = 4$. The only coefficient of x on the left is A, so $A = -3$. Comparing coefficients of x^3 gives $A + C = 3$, so $C = 6$.

Now we have

$$\begin{aligned}\int \frac{3x^3+3x^2-3x-1}{x^2(x^2+1)}\,dx &= \int -\frac{3}{x} - \frac{1}{x^2} + \frac{6x+4}{x^2+1}\,dx \\ &= -3\ln|x| + \frac{1}{x} + \int \frac{6x}{x^2+1} + \frac{4}{x^2+1}\,dx \\ &= -3\ln|x| + \frac{1}{x} + 3\ln|x^2+1| + 4\arctan(x) + C.\end{aligned}$$

7. Approximate integration

Remark. Some integrals are still beyond algebraic methods for integration (at this time). Hence, we still may need to resort to the definition of integration at times to compute an integral. Recall that we can divide the domain of integration $[a, b]$ into a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of regular subintervals, so $\Delta x = (b - a)/n$. Then we can approximate the integral using left or right-hand endpoints,

$$\int_a^b f(x)\ dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x,$$
$$\int_a^b f(x)\ dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

However, there are other methods which may at times give us a better result.

Theorem 20 (Midpoint Rule)**.** Let P be a regular partition of $[a, b]$. Let $\overline{x_i} = \frac{1}{2}(x_{i-1} + x_i)$ be the midpoint of the interval $[x_{i-1}, x_i]$. Then

$$\int_a^b f(x)\ dx \approx M_n = \Delta x \sum_{i=1}^{n} f(\overline{x_i}).$$

Remark. Recall that the actual value of an integral lies between L_n and R_n. Hence, we can get a reasonable approximation of the actual value by averaging the two. This is the Trapezoidal Rule. Note that some integrals are better approximated with trapezoids instead of rectangles.

At points $y_i = f(x_i)$ and $y_j = f(x_j)$, we can find the area of the trapezoid formed by connecting y_i and y_j. The area of this trapezoid is

$$A = \frac{1}{2}(x_j - x_i)(y_i + y_j).$$

Thus, if we have N regular intervals of length Δx, then we can approximate the area under f by

$$\begin{aligned}
T_N &= \frac{1}{2}\Delta x(f(x_0) + f(x_1)) + \frac{1}{2}\Delta x(f(x_1) + f(x_2)) + \frac{1}{2}\Delta x(f(x_2) + f(x_3)) \\
&\quad + \cdots + \frac{1}{2}\Delta x(f(x_{N-1}) + f(x_N)) \\
&= \frac{1}{2}\Delta x\left((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + (f(x_2) + f(x_3))\right. \\
&\quad \left. + \cdots + (f(x_{N-1}) + f(x_N))\right) \\
&= \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{N-1}) + f(x_N)\right).
\end{aligned}$$

Theorem 21 (Trapezoidal Rule)**.** Let P be a regular partition of $[a, b]$ and $x_i = a + i\Delta x$. Then

$$\int_a^b f(x)\ dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

Example 22. Approximate the integral $\int_1^4 \frac{dx}{x}$ using the midpoint rule and trapezoidal rule with $N = 6$.

The partition is

$$\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$$

The midpoints of the partition are

$$\left\{\frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}\right\}.$$

We have $\Delta x = \frac{4-1}{6} = \frac{1}{2}$.

Thus, the midpoint rule gives,

$$\begin{aligned} M_6 &= \frac{1}{2}\left(f(5/4) + f(7/4) + f(9/4) + f(11/4) + f(13/4) + f(15/4)\right) \\ &= \frac{1}{2}\left(\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \frac{4}{13} + \frac{4}{15}\right) = \frac{62024}{45045} \approx 1.37693. \end{aligned}$$

The trapezoidal rule gives

$$\begin{aligned} T_6 &= \frac{1/2}{2}\left(f(1) + 2f(3/2) + 2f(2) + 2f(5/2) + 2f(3) + 2f(7/2) + f(4)\right) \\ &= \frac{1}{4}\left(1 + 2\left(\frac{2}{3}\right) + 2\left(\frac{1}{2}\right) + 2\left(\frac{2}{5}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{7}\right) + \frac{1}{4}\right) \\ &= \frac{1}{4}\left(1 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{4}\right) = \frac{787}{560} \approx 1.40536 \end{aligned}$$

Note that the actual value, obtained by using the Fundamental Theorem of Calculus, is $\ln(4) \approx 1.38629$, so the midpoint rule gives an underestimate (in this case) and the trapezoidal rule gives an overestimate.

Remark. The trapezoidal rule and the midpoint rule are comparable in certain circumstances.

- If $f(x)$ is concave up then $M_n \leq \displaystyle\int_a^b f(x)\, dx \leq T_n$.
- If $f(x)$ is concave down then $T_n \leq \displaystyle\int_a^b f(x)\, dx \leq M_n$.

Note that this matches our last example since $\frac{1}{x}$ is concave up on $[1, 4]$.

Remark. If we take the average of the two, then we might hope to reduce our error. This is the idea behind Simpson's rule.

Theorem 23 (Simpson's Rule)**.** Let P be a regular partition of $[a, b]$ and suppose n is even and $x_i = a + i\Delta x$. Then

$$\begin{aligned} \int_a^b f(x)\, dx \approx S_n = \frac{\Delta x}{3}\,[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \\ +2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]\,. \end{aligned}$$

Example 24. Approximate the integral $\int_1^4 \frac{dx}{x}$ using Simpson's rule with $N = 6$.

As before, we have $\Delta x = \frac{4-1}{6} = \frac{1}{2}$. Thus,

$$\begin{aligned} S_6 &= \frac{1/2}{3}\left(f(1) + 4f(3/2) + 2f(2) + 4f(5/2) + 2f(3) + 4f(7/2) + f(4)\right) \\ &= \frac{1}{6}\left(1 + 4\left(\frac{2}{3}\right) + 2\left(\frac{1}{2}\right) + 4\left(\frac{2}{5}\right) + 2\left(\frac{1}{3}\right) + 4\left(\frac{2}{7}\right) + \frac{1}{4}\right) \\ &= \frac{3497}{2520} \approx 1.38770. \end{aligned}$$

Remark. We can determine the value in Simpson's Rule with the other two values,

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n.$$

Example 25. Approximate the integral $\int_1^4 \frac{dx}{x}$ using Simpson's rule with $N = 6$.

We can use our approximations from before to determine,

$$S_{12} = \frac{1}{3}T_6 + \frac{2}{3}M_6 = \frac{2997637}{2162160} = 1.38641.$$

8. Improper Integrals

Consider the two integrals,

$$\int_1^\infty \frac{1}{x}\ dx \text{ and } \int_1^\infty \frac{1}{x^2}\ dx.$$

We can evaluate these integrals by rewriting the integral as a limit. We will see that, though they look similar, the two have very different answers,

$$\int_1^\infty \frac{1}{x}\ dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\ dx = \lim_{t\to\infty} (\ln t - \ln 1) = \infty$$
$$\int_1^\infty \frac{1}{x^2}\ dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\ dx = \lim_{t\to\infty} (1 - \frac{1}{t}) = 1$$

Definition 1 (Improper Integrals, Type I)**.** (1) If $\int_a^t f(x)\ dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$

provided this limit exists.

(2) If $\int_t^b f(x)\ dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$$

provided this limit exists.

The improper integrals $\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^b f(x)\ dx$ are called convergent if the corresponding limit exists and divergent if the limit does not exist or is infinite.

Example 26. Determine whether $\displaystyle\int_{-\infty}^{-1} e^{2x}\ dx$ converges or diverges. Evaluate if it converges.

$$\int_{-\infty}^{-1} e^{2x}\ dx = \lim_{t\to-\infty} \int_a^{-1} e^{2x}\ dx = \lim_{t\to-\infty} \left[\frac{1}{2}e^{2x}\right]_t^{-1} = \frac{1}{2} \lim_{t\to-\infty} \left[e^{-2} - e^{2t}\right] = \frac{1}{2}e^{-2}.$$

Theorem 27 (p-integral Type I)**.**

$$\int_1^\infty \frac{\mathrm{dx}}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1. \end{cases}.$$

Proof. As above, we write,

$$\int_1^\infty \frac{\mathrm{dx}}{x^p} = \lim_{t\to\infty} \int_1^t \frac{\mathrm{dx}}{x^p}.$$

We have already consider the case of $p = 1$, so suppose $p \neq 1$. Then,

$$\lim_{t\to\infty} \int_1^t \frac{\mathrm{dx}}{x^p} = \lim_{t\to\infty} \left[\frac{1}{1-p}x^{1-p}\right]_1^t = \frac{1}{1-p} \lim_{t\to\infty} \left[t^{1-p} - 1\right] = \frac{1}{p-1} + \lim_{t\to\infty} \frac{1}{t^{p-1}}.$$

If $p - 1 > 0$ (so $p > 1$), then the limit goes to 0. Otherwise, $p - 1 < 0$ so the limit goes to ∞. $\square$

Example 28. Determine whether $\displaystyle\int_1^\infty \frac{\ln x}{x^3}\ dx$ converges or diverges. Evaluate if it converges.

First (always first!) we rewrite as a limit,

$$\int_1^\infty \frac{\ln x}{x^3}\ dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\ dx.$$

We will employ integration by parts. Let $u = \ln x$, $dv = x^{-3}\ dx$ so $du = x^{-1}\ dx$ and $v = -\frac{1}{2}x^{-2}$. Then,

$$\begin{aligned}\lim_{t\to\infty}\int_1^b \frac{\ln x}{x^3}\ dx &= \lim_{t\to\infty}\left[(\ln x)\left(-\frac{1}{2}x^{-2}\right)\right]_1^t + \frac{1}{2}\int_1^t x^{-3}\ dx\\ &= \lim_{t\to\infty} -\frac{1}{2}\frac{\ln t}{t^2} + \frac{1}{2}\left[-\frac{1}{2}x^{-2}\right]_1^t = \lim_{t\to\infty} -\frac{1}{2}\frac{\ln t}{t^2} - \frac{1}{4}\left[t^{-2}-1\right] = \frac{1}{4}.\end{aligned}$$

Remark. If both $\int_a^\infty f(x)\ dx$ and $\int_{-\infty}^b f(x)\ dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\ dx = \int_a^\infty f(x)\ dx + \int_{-\infty}^b f(x)\ dx.$$

Example 29. Determine whether $\displaystyle\int_{-\infty}^\infty xe^{-x^2}\ dx$ converges or diverges. Evaluate if it converges.

$$\begin{aligned}\int_{-\infty}^\infty xe^{-x^2}\ dx &= \int_{-\infty}^0 xe^{-x^2}\ dx + \int_0^\infty xe^{-x^2}\ dx = \lim_{s\to-\infty}\int_s^0 xe^{-x^2}\ dx + \lim_{t\to\infty}\int_0^t xe^{-x^2}\ dx\\ &= \lim_{s\to-\infty}\left[-\frac{1}{2}e^{-x^2}\right]_s^0 + \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^t = -\frac{1}{2}\left[\lim_{s\to-\infty}\left(1-e^{-s^2}\right) + \lim_{t\to\infty}\left(e^{-t^2}-1\right)\right]\\ &= -\frac{1}{2}\left[(1-0)+(0-1)\right] = 0.\end{aligned}$$

Remark. Suppose a function $f(x)$ is continuous except at a point $x = b$. Can we evaluate the integral $\int_a^b f(x)\ dx$? We can by evaluating the limit as the integral approaches that point.

Definition 2 (Improper Integrals, Type II)**.** (1) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\ dx = \lim_{t\to b^-}\int_a^t f(x)\ dx$$

provided this limit exists.

(2) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\ dx = \lim_{t\to a^+}\int_t^b f(x)\ dx$$

provided this limit exists.

The improper integrals $\int_a^b f(x)\ dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist or is infinite.

Example 30. Determine whether $\displaystyle\int_2^3 \frac{dx}{\sqrt{3-x}}$ converges or diverges. Evaluate if it converges.

We have

$$\int_2^3 \frac{dx}{\sqrt{3-x}} = \lim_{t\to 3^-} \int_2^t \frac{dx}{\sqrt{3-x}} = \lim_{t\to 3^-} \left[-2\sqrt{3-x}\right]_2^t = \lim_{t\to 3^-} (-2\sqrt{3-t}+2) = 2.$$

Example 31. Determine whether the following integral converges or diverges. Evaluate the integral if it converges.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\ dx$$

$$\begin{aligned}
\int_0^1 \frac{\ln x}{\sqrt{x}}\ dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}}\ dx &= \lim_{t\to 0^+} \left[\left[(\ln x)(2\sqrt{x})\right]_t^1 - \int_t^1 2\frac{\sqrt{x}}{x}\ dx\right] \\
&= \lim_{t\to 0^+} \left[-2\sqrt{t}\ln t - 2\int_t^1 \frac{1}{\sqrt{x}}\ dx\right] \\
&= \lim_{t\to 0^+} \left[-2\sqrt{t}\ln t - 4\left[\sqrt{x}\right]_t^1\right] \\
&= \lim_{t\to 0^+} \left[-2\sqrt{t}\ln t - 4(1-\sqrt{t})\right] \\
&= -4 - 2\lim_{t\to 0^+} \sqrt{t}\ln t.
\end{aligned}$$

To evaluate the remaining integral, we must use L'Hospital's rule.

$$\lim_{t\to 0^+} \sqrt{t}\ln t = \lim_{t\to 0^+} \frac{\ln t}{t^{-1/2}} = \lim_{t\to 0^+} \frac{1/t}{-\frac{1}{2}t^{-3/2}} = -2\lim_{t\to 0^+} \sqrt{t} = 0.$$

Hence,

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\ dx = -4.$$

Theorem 32 (p-integral Type II)**.**

$$\int_0^1 \frac{\mathrm{dx}}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1. \end{cases}.$$

Proof. If $p \neq 1$,

$$\int_0^1 \frac{\mathrm{dx}}{x^p} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \left[\frac{1}{-p+1}x^{-p+1}\right]_t^1 = \frac{1}{1-p}\lim_{t\to 0^+} \left[1 - t^{-p+1}\right]_t^1.$$

The limit is 0 if $p < 1$ and otherwise it is infinite. Thus, the result follows. $\square$

Remark. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$ are convergent, then we define

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

Remark. Another way to evaluate improper integrals is to compare them to other functions in which we know the convergence.

Theorem 33 (Comparison Theorem)**.** Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(1) If $\int_a^\infty f(x)\ dx$ converges, then $\int_a^\infty g(x)\ dx$.

(2) If $\int_a^\infty g(x)\ dx$ diverges, then $\int_a^\infty f(x)\ dx$ diverges.

(3) If $\int_a^\infty g(x)\ dx$ converges or $\int_a^\infty f(x)\ dx$ diverges, then the theorem gives us no information.

Remark. A similar theorem holds for Type II integrals.

Example 34. Determine whether the following integral converges or diverges.

$$\int_1^\infty \frac{2+e^{-x}}{x}\ dx$$

Since $2+e^{-x} \geq 1$ for all $x \geq 1$, then

$$\frac{2+e^{-x}}{x} \geq \frac{1}{x} \text{ for all } x \geq 1.$$

Since $\int_1^\infty \frac{1}{x}\ dx$ diverges, then by the Comparison Theorem so does the given integral.

Example 35. Determine whether the following integral converges or diverges.

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\ dx.$$

For $0 \leq x \leq 1$, $0 \leq e^{-x} \leq 1$. Thus,

$$0 \leq \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

on the given interval. Since $\int_0^1 \frac{1}{\sqrt{x}}\ dx$ converges, then by the Comparison Theorem so does the given integral.

Calculus II

Applications of integration

8.1 Arc Length

The purpose of this section is to use our methods of integration to measure the length of a curve.

Imagine we are trying to measure the length of a curve C. We could approximate the length by marking points $\{P_i = (x_i, y_i)\}$ on the lines and connecting them with line segments. We let $|P_{i-1}P_i|$ denote the length of the line segment from P_{i-1} to P_i. Then if L is the length of the curve C, and we assume we have a regular partition, we get

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

This is not very easy to compute, so we will try to evaluate it another way. Note that,

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

The MVT on $[x_{i-1}, x_i]$ gives,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$

Or, in other terms, $\Delta y_i = f'(x_i^*)\Delta x$. Hence,

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \cdots = \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

This gives,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Remark (The Arc Length Formula)**.** If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Alternatively, if f is a function of y then the length of curve $x = f(y)$, $a \leq y \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(y)]^2}\, dy.$$

Example 1. Find the length of the curve: $y^2 = 4(x + 4)^3$, $0 \leq x \leq 2$, and $y \geq 0$.

Taking square roots (we need only consider the positive part since $y \geq 0$) we have $y = f(x) = 2(x+4)^{3/2}$. Then $f'(x) = 3(x+4)^{1/2}$. Using the arc length formula we have

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+[f'(x)]^2}\ dx = \int_0^2 \sqrt{1+9(x+4)}\ dx \\
&= \int_0^2 \sqrt{9x+37}\ dx \quad \text{set } u = 9x+37 \text{ so } \ du = 9\ dx \\
&= \frac{1}{9}\int_{37}^{55} \sqrt{u}\ du = \frac{1}{9}\left[\frac{2}{3}x^{3/2}\right]_{37}^{55} = \frac{2}{27}\left(55^{3/2} - 37^{3/2}\right)
\end{aligned}
$$

Example 2. Find the length of the curve: $y^2 = 4x$, $0 \leq y \leq 2$.

Set $x = f(y) = \frac{1}{4}y^2$. Then $f'(y) = \frac{1}{2}y$. Using the arc length formula we have

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+[f'(y)]^2}\ dy = \int_0^2 \sqrt{1+\frac{1}{4}y^2}\ dx \quad \text{set } x = 2\tan\theta \text{ so } \ dx = 2\sec^2\theta\ d\theta \\
&= 2\int_0^{\pi/4} \sqrt{1+\tan^2\theta}\sec^2\theta\ d\theta = 2\int_0^{\pi/4} \sec^3\theta \\
&= 2\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} \\
&= \sqrt{2} + \ln(\sqrt{2}+1).
\end{aligned}
$$

8.2 Surface Area

Recall that the lateral surface area of a cylinder is $A = 2\pi rh$, where r is the radius of the cylinder and h is the height. Similarly, the lateral surface area of a cone is $A = \pi r\ell$, where r is again the radius and ℓ is the diagonal (slant) height. Now we'll study more complicated objects.

Recall that a surface of revolution is obtained when a curve (usually represented by a function $y = f(x)$) is rotate about a line (maybe the x-axis, the y-axis, or some other horizontal or vertical line). In Calculus I you studied volume of these objects. Here we will derive a formula for the surface area. To do this, we follow our strategy from arc length.

We want to approximate the area using *bands*. Consider the frustrum with upper and lower radii r_1 and r_2, and upper and lower slant heights ℓ_1 and ℓ. Then the surface area of the frustrum is

$$A = \pi r_2(\ell_1 + \ell) - \pi r_2 \ell_1 = \pi[(r_2 - r_1)\ell_1 + r_2\ell].$$

From similar triangles,

$$\frac{\ell_1}{r_1} = \frac{\ell_1 + \ell}{r_2}$$

which gives $(r_2 - r_1)\ell_1 = r_1\ell$. Substituting into A gives

$$A = \pi[(r_2 - r_1)\ell_1 + r_2\ell] = \pi[r_1\ell + r_2\ell] = 2\pi\left(\frac{r_1 + r_2}{2}\right)\ell.$$

Consider the curve $y = f(x)$ on the interval $[a, b]$ and suppose that the curve is rotated about the x-axis to obtain a surface of revolution. We partition our interval into regular subintervals of length Δx and denote the endpoints by $x_0, x_1, \ldots, x_n$. Set $y_i = f(x_i)$. We approximate the curve on the interval $[x_{i-1}, x_i]$ by taking the line segment $P_{i-1}P_i$. and set $\ell = |P_{i-1}P_i|$ to be the slant height. The average radius is then $r = \frac{1}{2}(y_{i-1} + y_i)$. Now the area of the band is

$$2\pi\frac{y_{i-1} + y_i}{2}|P_{i-1}P_i|.$$

Recall from the previous section that

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

Taking the sum of all of these, and then taking the limit as $n \to \infty$ gives us the formula for surface area.

The formula we obtain for the surface area of a surface of revolution formed by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2]}\, dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Similarly, the surface area of a surface of revolution formed by rotating the curve $x = g(y)$, $c \le y \le d$, about the y-axis is

$$S = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}]\ dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy.$$

Example 3. Find the surface area obtained by rotation the curve $y = \cos(x/2)$, $0 \le x \le \pi$, about the x-axis.

Using the formula above, with $f(x) = \cos(x/2)$ and $f'(x) = -\frac{1}{2}\sin(x/2)$ we have

$$S = \int_0^\pi 2\pi\left(\cos(x/2)\right)\sqrt{1 + \left[-\frac{1}{2}\sin(x/2)\right]^2}\ dx = 2\pi\int_0^\pi \cos(x/2)\sqrt{1 + \frac{1}{4}\sin^2(x/2)}\ dx.$$

Let $u = \frac{1}{2}\sin(x/2)$, so $du = \frac{1}{4}\cos(x/2)\ dx$. Then

$$S = 2\pi\int_0^\pi \cos(x/2)\sqrt{1 + \frac{1}{4}\sin^2(x/2)}\ dx = 8\pi\int_0^{1/2}\sqrt{1+u^2}\ du.$$

Now let $u = \tan\theta$, so $du = \sec^2\theta\ d\theta$. Then

$$\begin{aligned}
S &= 8\pi\int_0^{1/2}\sqrt{1+u^2}\ du = 8\pi\int_0^{\arctan(1/2)}\sqrt{1+\tan^2\theta}(\sec^2\theta)\ d\theta = 8\pi\int_0^{\arctan(1/2)}\sec^3\theta\ d\theta \\
&= 8\pi\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\arctan(1/2)} \\
&= 4\pi\left[\sec(\arctan(1/2))\cdot\frac{1}{2} + \ln\left|\sec(\arctan(1/2)) + \frac{1}{2}\right|\right] \\
&= 4\pi\left[\frac{\sqrt{5}}{2}\cdot\frac{1}{2} + \ln\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)\right] = \pi\left[\sqrt{5} + 4\ln\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)\right].
\end{aligned}$$

8.4 Blood flow

The law of laminar flow states that

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where v is the velocity that blood flows along a blood vessel, R is the radius (of the blood vessel), l is the length, r the distance from the central axis, P the pressure difference between the ends of the vessel, and η the viscosity of the blood.

To compute the rate of blood flow (or *flux*, volume per unit time), we use the disk-and-washer method. This results in Poiseuille's Law,

$$F = \frac{\pi P R^4}{8\eta l}.$$

The cardiac output of the heart is the volume of blood pumped by the heart per unit time, that is, the rate of flow into the aorta. This is measured using the dye dilation method. Let $c(t)$ be the concentration of the dye at time t over an interval $[0, T]$ until the dye has cleared. This is divided into subintervals of length Δt so the amount of dye flowing past the measuring point is

$$(\text{concentration})(\text{volume}) = c(t_i)(F\Delta t)$$

where F is the rate of flow we are trying to determine.

The total amount of dye is then approximately

$$\sum_{i=1}^{n} c(t_i) F \Delta t = F \sum_{i=1}^{n} c(t_i)\Delta t.$$

Integrating (letting $t \to \infty$) we have that the amount of dye is

$$A = F \int_0^T c(t)\ dt.$$

Cardiac output is therefore given by

$$F = \frac{A}{\int_0^T c(t)}\ dt.$$

Example 4. The dye dilation method is used to measure cardiac output with 3 mg of dye. The dye concentrations, in mg/L, are modeled by $c(t) = 13t(12 - t), 0 \le t \le 12$, where t is measured in seconds. Find the cardiac output.

We have

$$\int_0^{12} 13t(12 - t)\ dt = 13 \int_0^{12} 12t - t^2\ dt = 13 \left[6t^2 - \frac{1}{3}t^3\right]_0^{12} = 13(288) = 3744.$$

Thus, cardiac output is given by $F = 3/3744$.

In science, we wish to be able to model systems, either mathematically, algorithmically, or experimentally. Differential equations are a tool for modeling dynamical systems, that is, systems that change over time. Consider a familiar example of population growth. Let P represent the population at time t (this is short-hand for just writing $P(t)$. Then the rate that P changes is proportional to the population at any time. This gives the differential equation,

$$\frac{dP}{dt} = kP, k > 0,$$

where k is known as the "growth constant." In calculus I, you learned to solve this equation. One goal of this course is to generalize your ability to solve various types of differential equations and use them to model dynamical systems.

Definition 1. A differential equation (DE) is an equations that contains derivatives of one or more dependent variables with respect to one or more independent variables. An ordinary differential equation (ODE) contains only ordinary derivatives, whereas a partial differential equation (PDE) contains partial derivatives. The order of a differential equations refers to the highest-order derivative that appears in the equation.

A first-order DE can always be written in the form one of the form

$$\frac{dy}{dt} = f(t, y) \text{ or } y' = f(t, y) \quad (\star).$$

The purpose of this section is to introduce what it means to find a solution to a DE.

Definition 2. Analytically, $y(t)$ is a solution of $(\star)$ if substituting $y(t)$ for t reduces the equation to an identity

$$y'(t) \equiv f(t, y(t))$$

on an appropriate domain.

Example 5. Verify that $y = 3t + t^2$ is a solution to the DE

$$y' = \frac{1}{t}y + t, \ (t > 0).$$

Taking the derivative we get $y' = 3 + 2t$. Now we substitute y' in the left-hand side of the DE and y into the right hand side. We reduce each side and check equality.

$$\begin{aligned} y' &= \frac{1}{t}y + t \\ (3 + 2t) &= \frac{1}{t}(3t + t^2) + t \\ 3 + 2t &= 3 + t + t \\ 3 + 2t &= 3 + 2t. \end{aligned}$$

It seems reasonable that a DE may have many solutions. In fact, most have infinitely many solutions. Consider the solutions to a DE of the form, dy/dt $= f(t)$. We have a family of solutions corresponding to different values of the constant c. Such a family is called the general solution while an individual member of the family is called a particular solution.

Definition 3. The combination of a first-order differential equation and an initial condition

$$\frac{dy}{dt} = f(t, y),\ \ y(t_0) = y_0$$

is called an initial-value problem (IVP). Its solution will pass through (t_0, y_0).

Example 6. Verify that $y = t + t^2 + 2e^{2t}$ is a solution to the IVP

$$y' = 2y + 1 - 2t^2, y(0) = 2.$$

First we check whether the given function is a solution to the given DE. Again, we compute the derivative and substitute in the left-hand side.

$$\begin{aligned} y' &= 2y + 1 - 2t^2 \\ (1 + 2t + 4e^{2t}) &= 2(t + t^2 + 2e^{2t}) + 1 - 2t^2 \\ 1 + 2t + 4e^{2t} &= 2t + 2t^2 + 4e^{2t} + 1 - 2t^2 \\ 1 + 2t + 4e^{2t} &= 2t + 4e^{2t} + 1 \end{aligned}$$

Whereas a DE will generally have an infinite number of solutions, an IVP will usually have just one.

To separate a DE means to use algebraic manipulation to put all y terms on the left and t terms on the right (including dy and dt). Not all DEs are separable (most are not). A separable DE has the form $y' = f(t)g(y)$.

If we can separate, then we can integrate both sides and then solve for y, giving us our solution. Here are the steps:

(1) Set $g(y) = 0$ and find equilibrium solutions.
(2) Assume $g(y) \neq 0$ and rewrite as $\dfrac{dy}{g(y)} = f(t)dt$.
(3) Integrate each side.
(4) Solve for y in terms of t (if possible).

Example 7. Solve by separation, $xy' = 4y$.

$$\begin{aligned}
x\frac{dy}{dx} &= 4y \\
\frac{1}{y}\ dy &= \frac{4}{x}\ dx \\
\int \frac{1}{y}\ dy &= \int \frac{4}{x}\ dx \\
\ln|y| &= 4\ln|x| + C \\
y &= e^{4\ln|x|+C} \\
y &= e^{C}x^4 \\
y &= Kx^4.
\end{aligned}$$

Example 8. Solve by separation of variables $ty' = 1 + y^2$. Check your answer.

$$\begin{aligned}
\frac{dy}{dt}t &= 1 + y^2 \\
\frac{1}{1+y^2}\ dy &= \frac{1}{t}\ dt \\
\tan^{-1} v(y) &= \ln|t| + C \\
y &= \tan(\ln|t| + C).
\end{aligned}$$

Example 9. Solve the IVP by separation, $y' = y^2 - y, y(0) = 2$.

$$\begin{aligned}
\frac{dy}{dx} &= y^2 - y \\
\frac{dy}{y(y-1)} &= \ dx \\
\int \frac{-1}{y} + \frac{1}{y-1}\ dy &= \int\ dx \\
-\ln|y| + \ln|y-1| &= x + C \\
\ln\left|\frac{y-1}{y}\right| &= x + C \\
\frac{y-1}{y} &= Ke^x
\end{aligned}$$

Setting $x = 0$ and $y = 2$ gives $\frac{1}{2} = K$.

Calculus II

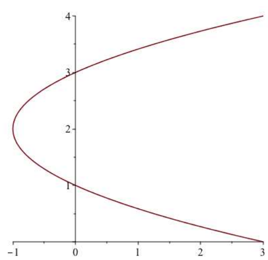
Parametric Equations and Polar Coordinates

10.1 Curves defined by parametric equations

Until this point, we have focused almost exclusively on curves given $y = f(x)$. This is limiting in many ways. For another approach, we turn to parametric curves, which describe a curve as it moves through space by a third parameter, time.

Definition 1. Suppose that x and y are both given as functions of a third variable t, called the parameter, by the equations $x = f(t)$, $y = g(t)$. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

Example 1. Sketch and identify the curve defined by the parametric equations $x = t^2 - 2t$, $y = t + 1$.

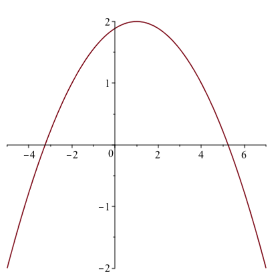


This is a parabola. In particular, it is the parabola with equation

$$x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3.$$

We obtained this by observing that $t = y - 1$ and substituting this into the equation $x = y^2 - 2t$.

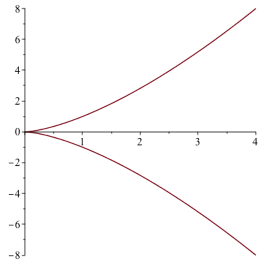
Example 2. Sketch the curve given by the parametric equations $x = 1 + 3t$, $y = 2 - t^2$. Indicate with an arrow the direction in which the curve is traced as t increases. Then eliminate the parameter to find a Cartesian equation of the curve.



The curve moves to the right as t increases.
Since $x = 1 + 3t$, then $t = \frac{1}{3}(x - 1)$. Thus,

$$y = 2 - t^2 = 2 - \left(\frac{1}{3}(x-1)\right)^2 = 2 - \frac{1}{9}(x-1)^2.$$

Example 3. Sketch the curve given by the parametric equations $x = t^2$, $y = t^3$. Indicate with an arrow the direction in which the curve is traced as t increases. Eliminate the parameter to find a Cartesian equation of the curve.

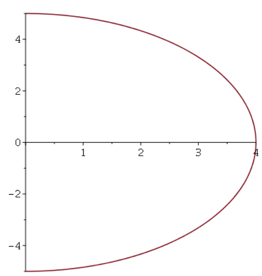


Since $x = t^2$, then $t = \pm x^{1/2}$ so $y = \pm x^{3/2}$.
Alternatively, since $y = t^3$, then $t = y^{1/3}$ and so $x = y^{2/3}$.

Definition 2. We will often restrict t to a finite interval, i.e., we will say $a \leq t \leq b$. In this case, if $x = f(t), y = g(t)$, then we say the curve has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

Example 4. Eliminate the parameter to find a Cartesian equation of the curve. Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

$$x = 4\cos\theta, \quad y = 5\sin\theta, \quad -\pi/2 \leq \theta \leq \pi/2.$$



Since $x = 4\cos\theta$, then $\cos\theta = x/4$. Thus, we have a right triangle with adjacent side x and hypotenuse 4, so the opposite side is $\pm\sqrt{16 - x^2}$. It follows that

$$y = 5\sin\theta = \pm\frac{5\sqrt{16 - x^2}}{4}.$$

Remark. We now see how to apply our methods of calculus to curves given by parametric curves.

Remark. To find tangents, we use the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0.$$

Also, to find the second derivative we replace y with $\frac{dy}{dx}$ to get,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0.$$

Example 5. Let C be the curve defined by the parametric equations $x = t^3 - 12t$ and $y = t^2 - 1$. Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

Using the above formula we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 - 12} = \frac{2}{3}\left(\frac{t}{t^2 - 4}\right).$$

Now,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{2}{3}\left(\frac{t}{t^2-4}\right)}{3t^2 - 12} = \frac{\frac{2}{3}\left(\frac{(t^2-4)-t(2t)}{(t^2-4)^2}\right)}{3(t^2-4)} = \frac{-2(t^2+4)}{9(t^2-4)^3}$$

The possible points of inflection are at $t = \pm 2$. The graph is concave down when $t > 2$ or $t < -2$ and the graph is concave up when $-2 < t < 2$.

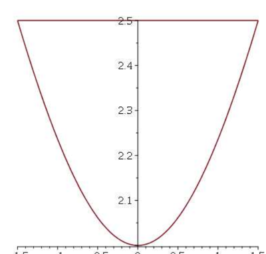
Remark. To compute areas under a curve $y = F(x)$ as $a \leq x \leq b$, we use $A = \int_a^b F(x)\ dx$. Suppose we can represent y using parametric curves, say $x = f(t)$ and $y = g(t)$ as $\alpha \leq t \leq \beta$. Then $dx = f'(t)\ dt$ and $y = F(x) = F(f(t)) = g(t)$. Thus, by the substitution rule for definite integrals,

$$A = \int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt.$$

If we simply write $x = x(t)$ and $y = y(t)$, then this is

$$A = \int_\alpha^\beta g(t)f'(t)\ dt.$$

Example 6. Find the area bounded by the curve $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, and the line $y = 2.5$.



We will compute the area under the line $y = 2.5$ and subtract the area under the given parametric curve. First we need to identify the bounds. That is, when is $y = t + \frac{1}{t} = 2.5$. Solving for t we find that $t = \frac{1}{2}, 2$. Thus, we can compute area under the parametric curve as follows.

$$\begin{aligned}
A &= \int_{\alpha}^{\beta} g(t)f'(t)\ dt = \int_{1/2}^{2} \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right)\ dt \\
&= \int_{1/2}^{2} t + \frac{2}{t} + \frac{1}{t^3}\ dt = \left[\frac{t^2}{2} + 2\ln|t| - \frac{1}{2t^2}\right]_{1/2}^{2} \\
&= \left(2 + 2\ln(2) - \frac{1}{8}\right) - \left(\frac{1}{8} + 2\ln(1/2) - 2\right) \\
&= 4\ln(2) + \frac{15}{4} \approx 6.52259.
\end{aligned}$$

The points $t = \frac{1}{2}$ and $t = 2$ correspond to $t = \pm\frac{3}{2}$. Thus, the area under the line $y = 2.5$ is $3 \cdot 2.5 = 7.5$. Hence, the area between the two curves is $7.5 - (4\ln(2) - 3.75) \approx 8.477$.

Now we study arc length for parametric curves.

Theorem 7. If a curve C is described by the parametric equations $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

Example 8. Find the length of the curve given by $x = \sin(2t)$, $y = \cos(2t)$, $0 \leq t \leq 2\pi$.

Using the above formula,

$$\begin{aligned}
L &= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_{0}^{2\pi} \sqrt{(2\cos(2t))^2 + (-2\sin(2t))^2}\ dt \\
&= \int_{0}^{2\pi} \sqrt{4\cos^2(2t) + \sin^2(2t)}\ dt = 2\int_{0}^{2\pi}\ dt = 4\pi.
\end{aligned}$$

Example 9. Find the length of the curve given by $x = e^t + e^{-t}$, $y = 5 - 2t$, $0 \leq t \leq 3$.

As in the previous example,

$$\begin{aligned}
L &= \int_{\alpha}^{\beta} \sqrt{(e^t - e^{-t})^2 + (-2)^2}\ dt = \int_{0}^{3} \sqrt{e^{2t} - 2 - e^{-2t} + 4}\ dt \\
&= \int_{0}^{3} \sqrt{(e^t + e^{-t})^2}\ dt = \int_{0}^{3} e^t + e^{-t}\ dt \\
&= \left[e^t - e^{-t}\right]_0^3 = \left(e^3 - e^{-3}\right) - (1 - 1) = e^3 - e^{-3}.
\end{aligned}$$

Remark. Now we switch gears and discuss another way of writing equations in the plane. In the Cartesian coordinate system we write coordinates using rectangular coordinates (x, y). We could do something similar by writing points in terms of an angle from the x-axis and a radius from the origin. That is,

$$x = r\cos\theta, y = r\sin\theta \quad \text{and} \quad r^2 = x^2 + y^2, \tan\theta = y/x.$$

Example 10. Convert the points $(-1, \sqrt{3})$ to polar coordinates.

We have $r = x^2 + y^2 = (-1)^2 + (\sqrt{3})^2 = 4$ and $\tan\theta = y/x = -\sqrt{3}$. There are two possible solutions to this: either $\theta = 2\pi/3$ or $\theta = 8\pi/3$. Since our given point is in the second quadrant, then $\theta = 2\pi/3$.

Example 11. Plot the point whose polar coordinates are given by $(-1, -\pi/2)$ and convert to rectangular coordinates.

Converting, we have

$$x = r\cos\theta = (-1)\cos(-\pi/2) = 0,$$
$$y = r\sin\theta = (-1)\sin(-\pi/2) = (-1)(-1) = 1.$$

Thus, this is the point $(0, 1)$.

Definition 3. The graph of a polar equation $r = f(\theta)$ consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example 12. Draw the curve represented by the polar equation $r = 2$.

This is the set of all polar points with radius 2. Thus, this is just a circle of radius 2.

Example 13. Draw the curve represented by the polar curve $\theta = \pi/4$?

This is the set of all polar points that form a right triangle with the origin and the x-axis, and have acute angle $\pi/4$. Thus, this is just a line through any such point.

Example 14. Sketch the curve with the polar equation $r = 2\cos\theta$. Find the Cartesian equation for this curve.

Plotting points, we find that this is a circle of radius 1 centered at $(1, 0)$. To convert, note that $x = r\cos\theta$ so $2\cos\theta = 2x/r$. Thus, $r = 2x/r$, or

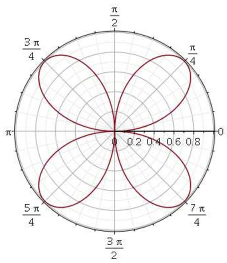
$$2x = r^2 = x^2 + y^2.$$

Completing the square gives

$$(x-1)^2 + y^2 = 1.$$

Example 15. Sketch the curve of each polar equation $r = \sin 2\theta$.

The graph is a four-petaled rose. Note that $r = 1$ when $\theta = \pi/4$ and $5\pi/4$ whereas $r = -1$ when $\theta = 3\pi/4$ and $\theta = 7\pi/4$. Finally $r = 0$ when $\theta = 0, \pi/2, \pi, 3\pi/2$.



To find tangents to polar curves we can use our techniques from parametric equations. Suppose $r = f(\theta)$, so then $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$. Thus,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Example 16. Find the slope of the tangent line to the cardiod $r = 1 + \cos\theta$ when $\theta = \pi/3$.

Note that $\frac{dr}{d\theta} = -\sin\theta$. Using the formula above, we have

$$\frac{dy}{dx} = \frac{-\sin^2\theta + (1+\cos\theta)\cos\theta}{-\sin\theta\cos\theta - (1+\cos\theta)\sin\theta} = \frac{-\sin^2\theta + \cos\theta + \cos^2\theta}{-2\sin\theta\cos\theta - \sin\theta}.$$

Evaluating at $\pi/3$ we have

$$\frac{dy}{dx} = \frac{-(\sqrt{3}/2)^2 + (1/2) + (1/4)}{\sqrt{3}/2 + \sqrt{3}/2} = \frac{-(3/4) + (2/4) + (1/4)}{2\sqrt{3}} = 0.$$

Remark. We continue with the calculus of equations in polar form. For computing area, we use the formula for the area of the sector of a circle, $A = \frac{1}{2}r^2\theta$. This comes from noting that a sector represents $\frac{\theta}{2\pi}$ of the circle. Thus, the area of the sector is that percentage of the total area of the circle. It follows that

$$A = (\pi r^2)\left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta.$$

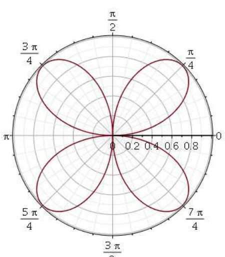
Suppose that a polar function $r = f(\theta)$ is defined on the interval $\alpha \leq \theta \leq \beta$. We divide the interval in to n regular subintervals of length $\Delta\theta = (\beta - \alpha)/n$. Taking a sample point θ_i^* in each subinterval $[\theta_{i-1}, \theta_i]$, the area inside the polar graph on this interval is approximated by

$$A = \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right]^2 \Delta\theta.$$

Thus, we can take a limit to find the exact area as,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\left[f(\theta_i^*)\right] \Delta\theta = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 \, d\theta = \int_\alpha^\beta \frac{1}{2}r^2 \, d\theta.$$

Example 17. Find the area enclosed by one loop of the four-leaved rose $r = \sin 2\theta$.



It suffices to find the area of one petal. In this case, $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq 1$. Then

$$\frac{1}{4}A = \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta) \, d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta) \, d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{2}(1 - \cos(4\theta)) \, d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} = \frac{1}{4}\left[\frac{\pi}{2}\right] = \frac{\pi}{8}.$$

Thus, the total area is $\pi/2$.

From our discussion of parametric equations we know how to find the length of an arc in polar coordinates. However, the formula simplifies as we see below.

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 + \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2} \, d\theta$$
$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2\frac{dr}{d\theta}r\sin\theta\cos\theta + r^2\cos^2\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2\frac{dr}{d\theta}r\cos\theta\sin\theta + r^2\sin^2\theta} \, d\theta$$
$$= \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\theta}\right)^2(\sin^2\theta + \cos^2\theta) + r^2(\cos^2\theta + \sin^2\theta)} \, d\theta.$$

Thus, using the Pythagorean identity we have

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.$$

For the next example, we will need a version of one of the *double-angle identities*. Namely,

$$2\cos^2(x/2) = 1 + \cos x.$$

Example 18. Find the length of the curve $r = 2(1 + \cos\theta)$.

$$\begin{aligned}
L &= 2\int_0^{\pi} \sqrt{(2(1+\cos\theta))^2 + (-2\sin\theta)^2}\ d\theta \\
&= 2\int_0^{\pi} \sqrt{4(1+2\cos\theta+\cos^2\theta) + 4\sin^2\theta}\ d\theta \\
&= 2\int_0^{\pi} \sqrt{4(1+2\cos\theta) + 4(\cos^2\theta + 4\sin^2\theta)}\ d\theta \\
&= 4\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\ d\theta \\
&= 4\sqrt{2}\int_0^{\pi} \sqrt{2\cos^2(\theta/2)}\ d\theta \\
&= 8\int_0^{\pi} \cos(\theta/2)\ d\theta \\
&= 8\left[2\sin(\theta/2)\right]_0^{\pi} \\
&= 16\left[\sin(\pi/2) - \sin(0)\right] = 16.
\end{aligned}$$

10.5 Conic sections

In this section we define three types of conic sections (or just conics). These shapes are so-called because they can be obtained by intersecting a plane with a cone.

Definition 4. A parabola is the set of points in a plane that are equidistant from a fixed point F, called the focus and a fixed line, called the directrix. The point halfway between the focus and the directrix is called the vertex. The line through the focus perpendicular to the directrix is the axis of the parabola.

Suppose a parabola has its vertex at the origin O and the directrix is parallel to the x-axis. Let $(0,p)$ denote the focus. Then the directrix has equation $y=-p$. Now if $P(x,y)$ is *any* point on the parabola, then the distance from P to the focus is $|PF|=\sqrt{x^2+(y-p)^2}$ and the distance from P to the directrix is $|y+p|$. Now the defining equation of the parabola is

$$\begin{aligned}\sqrt{x^2+(y-p)^2} &= |y+p| \\ x^2+(y-p)^2 &= (y+p)^2 \\ x^2+y^2-2py+p^2 &= y^2+2py+p^2 \\ x^2 &= 4py.\end{aligned}$$

Remark. The above computation shows that the equation of the parabola with focus $(0,p)$ and directrix $y=-p$ is

$$x^2=4py.$$

One can similarly switch the roles of x and y to obtain a parabola with focus $(p,0)$, directrix $x=-p$, and equation $y^2=4px$.

Example 19. Find the focus and directrix of the parabola $y^2+8x=0$ and sketch the graph.

We rewrite in the form $y^2=-8x=4(-2)x$, so $p=-2$. Comparing to the above formula, we see that the the focus is $(-2,0)$ and directrix $x=2$.

Definition 5. An ellipse is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2, called foci is a constant

If the foci are $(\pm c,0)$, so that the origin is halfway between, then the points of intersection of the ellipse with the x-axis are called the vertices.

Remark. The ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with $a\geq b>0$ has foci $(\pm c,0)$ where $c^2=a^2-b^2$, and vertices $(\pm a,0)$. The line segment from $(-a,0)$ to $(a,0)$ is called the major axis and the line segment from $(0,-b)$ to $(0,b)$ is called the minor axis.

The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

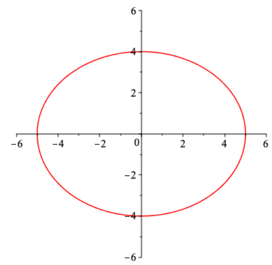
with $a \geq b > 0$ has foci $(0, \pm c)$ where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$. The line segment from $(0, -a)$ to $(0, a)$ is called the major axis and the line segment from $(-b, 0)$ to $(b, 0)$ is called the minor axis.

Example 20. Sketch the graph of $16x^2 + 25y^2 = 400$ and locate the foci.

First we divide both sides by 400 to get

$$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1.$$

Then $c^2 = 5^2 - 4^2 = 1$ so the foci are $(\pm 1, 0)$ and the vertices are $(\pm 5, 0)$.



Definition 6. A hyperbola is the set of points in a plane the difference of whose distances from two fixed points F_1 and F_2, called foci is a constant

Remark. The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$ where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

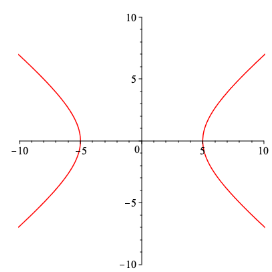
has foci $(0, \pm c)$ where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

Example 21. Sketch the graph of $16x^2 - 25y^2 = 400$. Locate the foci, vertices, and asymptotes.

First we divide both sides by 400 to get

$$\frac{x^2}{5^2} - \frac{y^2}{4^2} = 1.$$

Then $c^2 = 5^2 + 4^2 = 41$ so the foci are $(\pm\sqrt{41}, 0)$, the vertices are $(\pm 5, 0)$, and the asymptotes are $y = \pm(4/5)x$.



We can take shifted conics by replacing x and y in the above equations with $x - h$ and $y - k$.

Example 22. Consider the conic $x^2 + 3y^2 + 4x - 18y + 28 = 0$. We complete the square twice to get $(x+2)^2 + 3(y-3)^2 = 3$. Dividing both sides by 3 gives the equation of an ellipse:

$$\frac{(x+2)^2}{3} + \frac{(y-3)^2}{1} = 1.$$

This is (still) an ellipse. As before $a^2 = 3$, $b^2 = 1$, and $c^2 = 2$. The ellipse is shifted 2 units to the left and 3 units up. Thus, the foci are $(-2 \pm \sqrt{2}, 3)$. The vertices are $(-2 \pm \sqrt{3}, 3)$.

Calculus II

Chapter 11 - Sequences and Series

1. Sequences

Definition 1. A sequence is a list of numbers written in a definite order,

$$\{a_1, a_2, a_3, \ldots\} = \{a_n\}_{n=1}^{\infty}.$$

We call a_n the general term of the sequence.

Example. Assuming that the pattern of the first few terms continues, find a formula for the general term a_n of each sequence.

(1) $$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

(2) $$\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\right\} = \left\{(-1)^n \frac{n}{(n+1)^2}\right\}_{n=1}^{\infty}$$

(3) $$\left\{\frac{1}{4}, -\frac{2}{9}, \frac{3}{16}, -\frac{4}{25}, \ldots\right\} = \left\{(-1)^{n+1} \frac{n}{(n+1)^2}\right\}_{n=1}^{\infty}$$

Remark. Some sequences (actually, many sequences) do not have a simple formula. Consider the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. This is an example of a recursive sequence. There is a closed formula expression for the nth Fibonnaci number $F(n)$ but it is harder to write down than the previous problem. Let $\phi = (1+\sqrt{5})/2$ (the golden ratio) and $\psi = (1-\sqrt{5})/2$, then

$$F(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}.$$

Definition 2 (Informal)**.** A sequence $\{a_n\}$ has the limit L if we can make the terms a_n as close to L as we like by taking n sufficiently large. We write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

If the limit exists, we say the sequences converges. Otherwise, we say the sequence diverges.

Definition 3 (Formal)**.** A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$ there is a corresponding number $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n > N$.

Definition 4. The expression

$$\lim_{n\to\infty} a_n = \infty$$

means that for every positive number M there is an integer N such that $a_n > M$ for all $n > N$.

Example. Show that $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

Let $\varepsilon > 0$. We must find a number N such that $|(1/n) - 0| < \varepsilon$. This is equivalent to $\frac{1}{n} < \varepsilon$. Solving for n gives $n > \frac{1}{\varepsilon}$. Set $N = \frac{1}{\varepsilon}$. Then if $n > N$ then $\frac{1}{n} < \varepsilon$.

Example. Show that $\lim_{n\to\infty} \ln(n) = \infty$.

Let $M > 0$. To find N, set $\ln(n) > M$. This is equivalent to $n > e^M$. Set $N = e^M$. Now if $n > N$ we have $\ln(n) > M$.

Remark. Note that we can plot these points on a graph using the ordered pairs (n, a_n). If our sequence is determined by a function, then the long term behavior of the sequence is identical to that of the function. The following theorem formalizes that concept.

Definition 5. Let $\{a_n\}$ be a sequence. If f is a function such that $f(n) = a_n$ for each integer n, then we say the function f models the sequence $\{a_n\}$.

Theorem 1. If a function f models the sequence $\{a_n\}$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.

Remark. Note that the above theorem does not apply when f diverges. Consider the sequence $\{\sin(\pi n)\}$. This sequence converges to 0, in fact, every term in the sequence is 0. However, while $f(x) = \sin(\pi x)$ does model $\{a_n\}$, the limit of $f(x)$ as $x \to \infty$ does not exist.

Example. Use the previous theorem to evaluate the limit of the sequence $a_n = \dfrac{n}{\sqrt{n^2+1}}$.

Let $f(x) = \frac{x}{\sqrt{x^2+1}}$. Now,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2+1}} = \sqrt{\lim_{x\to\infty} \frac{x^2}{x^2+1}} = \sqrt{1} = 1.$$

Example. Use the previous theorem to evaluate the limit of the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Let $f(x) = \left(1 + \frac{1}{x}\right)^x$ and set $y = f(x)$. Then $\ln y = x \ln\left(1 + \frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$.

$$= \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{1}{x}\right)}{1/x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/\left(1+\frac{1}{x}\right)\cdot\frac{-1}{x^2}}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

Now,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e.$$

Theorem 2. Let $a_n = \frac{p(n)}{q(n)}$ where $p(n)$ and $q(n)$ are polynomials in n with leading coefficients b and c, respectively. Then

$$\lim_{n\to\infty} a_n = \begin{cases} 0 & \text{if } \deg q(n) > \deg p(n) \\ \infty & \text{if } \deg p(n) > \deg q(n) \\ \frac{b}{c} & \text{if } \deg p(n) = \deg q(n). \end{cases}$$

Example. Find the limit of the sequence $\left\{\dfrac{n!}{(n+2)!}\right\}$.

Notice that we could rewrite the general term

$$\frac{n!}{(n+2)!} = \frac{n!}{(n+2)(n+1)n!} = \frac{1}{(n+2)(n+1)} \to 0.$$

Remark. We'll now discuss several theorems that help us to evaluate limits of sequences. Many of these should remind you of corresponding theorems for limits of functions. The last one relates the two types of limits.

Theorem 3 (Limit Laws for sequences)**.** If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

(1) $\displaystyle\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

(2) $\displaystyle\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$ and $\displaystyle\lim_{n\to\infty} c = c$

(3) $\displaystyle\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

(4) $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\displaystyle\lim_{n\to\infty} b_n \neq 0$

Theorem 4 (Squeeze Theorem for Sequences)**.** Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that for some number M,

$$b_n \leq a_n \leq c_n \text{ for } n > M \quad \text{and} \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L,$$

then $\displaystyle\lim_{n\to\infty} a_n = L$.

Example. Evaluate the limit of the sequence with general term $a_n = 1/\sqrt{n^4+n^8}$.

We can bound a_n by

$$\frac{1}{\sqrt{2}n^4} \leq a_n \leq \frac{1}{\sqrt{2}n^2}.$$

Each of these sequences converges to 0 and then by the Squeeze Theorem, so does $\{a_n\}$.

Theorem 5. If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof. We have $-|a_n| \leq a_n \leq |a_n|$. Since $\lim_{n\to\infty}|a_n| = 0$, then so too do we have $\lim_{n\to\infty} -|a_n| = 0$. Hence, by the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$. $\square$

Remark. The previous theorem **only** works if the limit is 0. Consider the sequence $\{(-1)^n\} = \{1,-1,1,-1,1,-1,\ldots\}$. Then the limit of the absolute values is 1 but the sequence diverges.

Theorem 6. If $f(x)$ is continuous and $\displaystyle\lim_{n\to\infty} a_n = L$ then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Remark. The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and otherwise it is divergent.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \\ \infty & r > 1. \end{cases}$$

Definition 6. A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \leq M$ for all $n \geq 1$. It is bounded below if there is a number m such that $m \leq a_n$ for all $n \geq 1$. A sequence is bounded if it is bounded above and below.

Definition 7. A sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for all n and it is decreasing if $a_{n+1} \leq a_n$ for all n. A sequence is monotonic if it is increasing or decreasing. The sequence is eventually monotonic if there exists $N > 0$ such that the sequence is monotonic for all terms a_n with $n > N$.

Theorem 7 (Monotonic Sequence Theorem)**.** A bounded monotonic sequence is convergent.

Example. Write the first five terms of the recursive sequence $\{a_n\}$ defined by

$$a_1 = 2 \text{ and } a_n = \frac{1}{2}(a_{n-1} + 6).$$

Show that the sequence is increasing and bounded above. Conclude that the sequence is convergent and find its limit.

The first five terms are given by

$$a_1 = 2 \qquad\qquad a_4 = \frac{1}{2}(a_3 + 6) = \frac{11}{2}$$

$$a_2 = \frac{1}{2}(a_1 + 6) = \frac{1}{2}(8) = 4 \qquad a_5 = \frac{1}{2}(a_4 + 6) = \frac{23}{4}.a_3 \qquad = \frac{1}{2}(a_2 + 6) = \frac{1}{2}(10) = 5.$$

It is clear that $a_2 \geq a_1$. Suppose that $a_n \geq a_{n-1}$ for some n. Then $a_n + 6 \geq a_{n-1} + 6$ so $\frac{1}{2}(a_n + 6) \geq \frac{1}{2}(a_{n-1} + 6)$. Thus, $a_{n+1} \geq a_n$ for all n[1].

We show that the sequence is bounded above by induction. We claim that 6 is an upper bound for the sequence. Clearly $a_1 \leq 6$. Suppose $a_n \leq 6$ for some n. Then

$$a_{n+1} = \frac{1}{2}(a_n + 6) \leq \frac{1}{2}(6 + 6) = 6.$$

Thus, $a_n \leq 6$ for all n. Hence, the sequence $\{a_n\}$ is convergent by the Monotonic Sequence Theorem.

Since the sequence converges, then there exists a number L such that

$$\lim_{n\to\infty} a_n = L.$$

Convergence does not depend on the starting index (only the end behavior), then we have

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(L + 6).$$

Solving for L gives $L = 6$.

Warning. It does not always hold that the upper bound is actually the limit.

[1] This mathematical technique is known as *induction*.

2. Series

Consider the process of adding all of the integers. We could do this by hand,

$$1+2+3+\cdots,$$

but that would take a while, and we're pretty sure its infinity. Thankfully, there is a formula that tells us the sum of the first n integers

$$S_n = \frac{n(n+1)}{2},$$

and we can prove this with induction. Now we just increase n to get bigger and bigger sums. The total sum is just the limit of these partial sums, so

$$\sum a_n = \lim_{n\to\infty} S_n = \infty.$$

Now let's consider a similar problem, the sequence given by $\{1/2^n\}$. The more successive terms we add, the closer we get to 1. Hence, we might guess that $\sum \frac{1}{2^n} = 1$.

We can see this geometrically by drawing a square, cutting it in half and then cutting the remaining half in half...This forms a sequence,

$$\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\} = \left\{\frac{2^n-1}{2^n}\right\} = \left\{1-\frac{1}{2^n}\right\}.$$

This sequence clearly converges to 1.

Definition 8. A series (or infinite series) is the infinite sum of the terms in a sequence $\{a_n\}$. We denote it

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum a_n.$$

Let S_N denote its Nth partial sum, that is,

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

If the sequence $\{S_N\}$ is convergent and $\lim\limits_{N\to\infty} s_N = S$ exists, we say the series $\sum a_n$ is convergent and write

$$\sum a_n = S.$$

The number S is called the sum of the series. If the sequence $\{S_n\}$ is divergent, then the series is called divergent.

Remark. We will often use the notation,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

We can add and subtract series termwise (see page 551). Moreover, a scalar multiple of an infinite series is the same as the infinite series of the terms multiplied by that scalar.

Theorem 8. The geometric series $\sum_{n=1}^{\infty} cr^{n-1}$ is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} cr^{n-1} = \frac{c}{1-r}.$$

If $|r| \geq 1$, the geometric series diverges.

Proof. If $r = 1$ then the series clearly diverges. If $r \neq 1$, then observe that

$$\begin{aligned} S_N &= c + cr + cr^2 + \cdots + cr^{N-1} \\ rS_N &= cr + cr^2 + \cdots + cr^N \\ S_N - rS_N &= c - cr^N \\ S_N(1-r) &= c(1-r^N). \end{aligned}$$

Thus,

$$S = \lim_{n\to\infty} S_N = \lim_{n\to\infty} \frac{c\left(1-r^N\right)}{1-r} = \frac{c}{1-r} - \frac{c}{1-r}\lim_{n\to\infty} r^N.$$

Thus, the series converges if $|r| < 1$ and otherwise it diverges. $\square$

Example. The series $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$ is geometric with $r = -3/2$. Since $|r| > 1$, then this series diverges.

Example. Consider the series $\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$. We rewrite the series as

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} &= 4 + \sum_{n=1}^{\infty} \frac{4^{n+1}}{5^n} = 4 + \sum_{n=1}^{\infty} \frac{4^2 \cdot 4^{n-1}}{5 \cdot 5^{n-1}} = 4 + \sum_{n=1}^{\infty} \frac{16}{5} \cdot \left(\frac{4}{5}\right)^{n-1} \\ &= 4 + \frac{16/5}{1-(4/5)} = 4 + 16 = 20. \end{aligned}$$

Example. Express the number $6.2\overline{54} = 6.254545454\cdots$ as a ratio of integers. We can write this number as the series as

$$\begin{aligned} 6.2\overline{54} &= 6.2 + \frac{54}{1000} + \frac{54}{100000} + \cdots = 6.2 + \frac{54}{10^3} + \frac{54}{10^5} + \frac{54}{10^7} + \cdots \\ &= 6.2 + \frac{54}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) = 6.2 + \frac{54}{10^3}\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right) \\ &= 6.2 + \sum_{n=1}^{\infty} \frac{54}{10^3}\left(\frac{1}{100}\right)^{n-1}. \end{aligned}$$

This is a geometric series with $c = 54/10^3$ and $|r| = 1/100 < 0$. Hence, the series converges to $\frac{54/10^3}{1-1/100} = \frac{54}{1000-10} = \frac{54}{990} = \frac{6}{110}$. Thus, the value of the number is $\frac{62}{10} + \frac{6}{110} = \frac{344}{55}$.

Example. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is known as the **harmonic series**. The harmonic series diverges. We will show why below.

Instead of looking for each partial sum, we will bound some of them. We use the following fact: If $\{a_n\}$ is a sequence containing a subsequence $\{b_n\}$ which diverges, then $\{a_n\}$ diverges.

We have

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}.
\end{aligned}$$

This continues and we find $s_{2^n} > 1 + \frac{n}{2}$ and so $s_{2^n} \to \infty$ as $n \to \infty$. Hence $\{s_n\}$ diverges.

Example. Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. If it converges, find its sum.

First note that $\frac{1}{n+1} \leq \frac{1}{n^2}$. And so the given series should converge if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (We have not yet formalized a comparison theorem for series, but it should be rather intuitive.) This series should remind us of the indefinite integral $\int_1^{\infty} \frac{1}{x^2}\, dx$, which converges. Hence, our intuition should tells us that the given series converges.

We use the identity

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

which comes from partial fraction decomposition. Hence,

$$S_N = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) = 1 - \frac{1}{N+1}.$$

Since diverges $\displaystyle\lim_{N\to\infty} S_N = 1$, then the sum of the series is 1.

Remark. The series in Example is known as a telescoping series. The strategy for finding the sum of any telescoping series is the same as in that example.

Theorem 9. If the series $\sum a_n$ is convergent, then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Theorem 10 (Divergence Test)**.** If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges.

Example. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{10n+12}$ diverges.

Let $a_n = \frac{n}{10n+12}$, then $a_n \to \frac{1}{10}$ as $n \to \infty$. Hence, the series diverges by the Divergence Test.

Remark. Note that the converse of the Divergence Test does not hold. There are sequences such that the limit of the terms tends to zero but the series does not converge, e.g. the harmonic series.

3. The Integral Test

Remark. We now turn to the question of when certain series converge or diverge. At present we will focus exclusively on series of the form $\sum a_n$ with $a_n \geq 0$ for all n.

Remark. Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. We treat each summand as the area of a rectangle with base 1 and height $\frac{1}{\sqrt{n}}$. All of these rectangles can be positioned so that their areas exceed that of the area under the curve $\frac{1}{\sqrt{x}}$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}} \, dx.$$

Since this integral diverges, then so does the series.

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Starting with $n = 2$, we proceed as above by treating each summand as a rectangle with base 1 and height $\frac{1}{n^2}$. then we see that these rectangles lie below the curve $\frac{1}{x^2}$. Hence, it is easy to see that

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2} \, dx.$$

Since this integral converges, then so does the series.

Theorem 11 (The Integral Test)**.** Suppose $f(x)$ is continuous, positive, and decreasing function on $[1, \infty)$ that models the sequence $\{a_n\}$.

(1) If $\displaystyle\int_1^{\infty} f(x) \, dx$ converges, then so does $\displaystyle\sum_{n=1}^{\infty} a_n$.
(2) If $\displaystyle\int_1^{\infty} f(x) \, dx$ diverges, then so does $\displaystyle\sum_{n=1}^{\infty} a_n$.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.

We will consider the integral, $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2} \, dx$. To integrate, we apply IBP.

Let $u = \ln x$ and dv $= x^{-2} \, dx$ so du $= x^{-1} \, dx$ and $v = -x^{-1}$. Then

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x^2} \, dx &= \lim_{b \to \infty} \int_1^{b} \frac{\ln x}{x^2} \, dx = \lim_{b \to \infty} \left(\left[-\frac{\ln x}{x} \right]_1^b + \int_1^b \frac{1}{x^2} \, dx \right) \\ &= \lim_{b \to \infty} \left(-\frac{\ln b}{b} + 0 + \left[-\frac{1}{x} \right]_1^b \right) = 0 + \lim_{b \to \infty} \left[-\frac{1}{b} + 1 \right] = 1. \end{aligned}$$

Hence, by the integral test, the given series converges.

Caution! The above argument *does not* tell us that the sum of the series, only that it converges. The actual value of the series is a bit less than 1 (approx .9375) but this is difficult to determine.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ converges or diverges.

Let $f(x) = \frac{1}{x^{1/4}}$. Then f models the sequence $\left\{\frac{1}{\sqrt[4]{n}}\right\}$. Since

$$\int_1^\infty \frac{1}{x^{1/4}}\, dx$$

diverges (p-integral type I, $p = 1/4$), then so does the given series.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5n^4}$ converges or diverges.

Let $f(x) = \frac{3}{5x^4}$. Then f models the sequence $\left\{\frac{3}{5n^4}\right\}$. Since

$$\int_1^\infty \frac{3}{5x^4}\, dx = \frac{3}{5}\int_1^\infty \frac{1}{x^4}\, dx$$

converges (p-integral type I, $p = 4$), the so does the given series.

Theorem 12 (p-test for series)**.** The infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise.

The Integral Test is also useful in determining the error in approximating the sum of a (convergent) series with its nth partial sum.

Theorem 13 (Remainder Estimate for the Integral Test)**.** Suppose $f(x)$ is continuous, positive, and decreasing function on $[n, \infty)$ that models the sequence $\{a_n\}$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^\infty f(x)\, dx \le R_n \le \int_n^\infty f(x)\, dx.$$

We call R_n the remainder.

Example. Approximate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation. How many terms are necessary to ensure that the approximation is correct to three decimal places.

Let $f(x) = 1/x^2$. It is clear that $f(x)$ satisfies the conditions of the Integral Test. Note that

$$\int_n^\infty \frac{1}{x^2}\, dx = \lim_{t\to\infty}\int_n^t \frac{1}{x^2}\, dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_n^t = \lim_{t\to\infty}\left[-\frac{1}{t} + \frac{1}{n}\right] = \frac{1}{n}.$$

Now,

$$\sum_{n=1}^{10} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{10^2} \approx 1.54976773.$$

By the Remainder Estimate for the Integral Test, $R_{10} = S - S_{10}$ is bounded between $\int_{11}^\infty f(x)\, dx$ and $\int_{10}^\infty f(x)\, dx$. Thus, $\frac{1}{11} \le R_{10} \le \frac{1}{10}$.

We have seen that $R_n \le \frac{1}{n}$, so to ensure that $R_n \le \frac{5}{10^4}$, we just need to determine $\frac{1}{n} \le \frac{5}{10^4}$, so we set $n = \frac{10^4}{5}$.

4. Comparison Tests

In this section we will consider two different comparison tests. The first should seem familiar.

Theorem 14 (The Comparison Test for Series)**.** Assume there exists $M > 0$ such that $0 \leq a_n \leq b_n$ for $n \geq M$.

- If $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Remark. The condition $n \geq 4$ in the Comparison Test is there because it only matters what happens *eventually* in the sequence, not what happens in the early terms.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

For $n > e$, $\frac{\ln(n)}{n} > \frac{1}{n}$. Since the harmonic series diverges, then so too does our given series by the Comparison Test.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converges or diverges.

We have $n^2 + n + 1 > n^2$ for $n > 0$. Hence, $\frac{1}{n^2+n+1} < \frac{1}{n^2}$. Since the series $\sum \frac{1}{n^2}$ converges by the p-test, $p > 1$, then by the Comparison Test so too does the given series.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ converges or diverges.

Since $\sqrt{n^2+1} > \sqrt{n^2} = n$, then $\frac{1}{\sqrt{n^2+1}} < \frac{1}{n}$. Thus, our series is smaller than a divergent series and so the Comparison Test does not apply.

Remark. The Comparison Test gives us a way of comparing series in which the terms in the series are bigger or smaller than those in a convergent or divergent series. The next test is similar, but compares *growth rates* of series.

Theorem 15 (The Limit Comparison Test)**.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number with $c > 0$, then either both series converge or both diverge.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ converges or diverges.

We compare to the harmonic series, which has terms $b_n = 1/n$.

$$\lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = \sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = 1 > 0.$$

Thus, by the Limit Comparison Test, the given series diverges.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(3n)}{1+(1.2)^n}$ converges or diverges.

First note that $0 \leq \cos^2(3n) \leq 1$. Hence,

$$\frac{\cos^2(3n)}{1+(1.2)^n} \leq \frac{1}{1+(1.2)^n} \leq \frac{1}{(1.2)^n}.$$

The sequence $\sum_{n=1}^{\infty} \frac{1}{(1.2)^n}$ is geometric ($r = \frac{5}{6}$ so $|r| < 1$) and so it converges. Thus, by the Comparison Test, the given series converges.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3+2n-1}{5n^5-2n^3+3}$ converges or diverges.

It would be difficult to apply the Comparison Test (though not impossible perhaps). We will instead apply the Limit Comparison Test. Set $a_n = \frac{3n^3+2n-1}{5n^5-2n^3+3}$. To determine the comparing series, look for the highest power of n in the numerator and denominator. We will compare to the series $\sum_{n=1}^{\infty} b_n$ where $b_n = \frac{n^3}{n^5} = \frac{1}{n^2}$. By the p-test, this series converges ($p > 1$). Thus, if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where $c > 0$ and finite, then the given series converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n^3+2n-1)/(5n^5-2n^3+3)}{(1/n^2)} = \lim_{n\to\infty} \frac{3n^5+2n^3-n^2}{5n^5-2n^3+3} = \frac{3}{5} > 0.$$

Hence, the given series converges by the Limit Comparison Test.

Remark. The next example illustrates that there may be times when it is advantageous to use both theorems in conjunction.

Example. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Since $n!$ grows fast, we will conjecture that the series converges. For $n > 1$ we have

$$n! > n(n-1) \text{ so } \frac{1}{n!} < \frac{1}{n(n-1)}.$$

By the Comparison Test, our given series will converge if the series $\sum \frac{1}{n(n-1)}$ converges.

We will use the Limit Comparison Test. Let $a_n = \frac{1}{n(n-1)}$ and $b_n = \frac{1}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n(n-1)} = 1.$$

Thus, by the LCT, $\sum \frac{1}{n(n-1)}$ converges.

Remark. There is a much quicker way to show convergence of this series in Section 6.

5. Alternating series

Definition 9. An alternating series is a series whose terms are alternately positive and negative.

Example. Consider the alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}.$$

We saw previously that the alternating harmonic series converges. It turns out that the sum is $\ln(2)$ for the first one (and $-\ln(2)$ for the second). The reason for this will come later.

Remark. What we will consider in this section is a test for convergence of alternating series.

Theorem 16 (The Alternating Series Test)**.** If the alternating series,

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

satisfies

$$\text{(i) } b_{n+1} \leq b_n \text{ for all } n \text{ (decreasing) and (ii) } \lim_{n\to\infty} b_n = 0.$$

then the series is convergent.

Remark. The Alternating Series Test works similarly if $(-1)^{n-1}$ is replaced by $(-1)^n$. It is also not affected by the starting value.

Example. Use the alternating series test to give another explanation of why the alternating harmonic series converges.

We need to check two things: that the series is decreasing and the limit of the summands in zero. The second is obvious,

$$\lim_{n\to\infty}\frac{1}{n} = 0.$$

For the first, we need to show $\frac{1}{n+1} \leq \frac{1}{n}$ for all n. This is clear because the denominator on the left is always larger. Equivalently, we could cross multiply to be $n \leq n+1$, or $0 \leq 1$. Hence, by the Alternating Series Test, the series converges.

Remark. Why does this test work? We give a weak argument based on observation. A more complete argument is in your textbook.

Start with b_1, drawn out on a line segment. We then subtract from that b_2, which is less than b_1 by hypothesis. Next, we add on b_3, which is less than b_2 and subtract b_4. This difference is completely contained in b_2. Continuing in this process, we see that the limit of partial sums is bounded by b_1. Hence, the sequence of partial sums is increasing (monotonic) and bounded, therefore convergent. What issues are this argument ignoring?

Remark. Given a series as in the theorem above, we have $0 < S < b_1$ and $S_{2N} < S < S_{2N+1}$.

Example. Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$.

This is similar to the above. Note that the fact that the powers of -1 are off from the test does not affect the convergence.

Example. Determine the convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{n!}$.

This is an alternating series in disguise since n even implies $\cos(\pi n) = -1$ and n odd implies $\cos(\pi n) = 1$. We can then apply the alternating series test to show the series converges.

Example. Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{5n+1}$.

In checking condition (ii), note that

$$\lim_{n\to\infty} \frac{2n}{5n+1} = \frac{2}{5} \neq 0.$$

This implies that

$$\lim_{n\to\infty} (-1)^{n-1} \frac{2n}{5n+1}$$

does not exist. Hence, this series diverges by the Divergence Test. (Note that one can also show that this series is not decreasing.)

Example. Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin^2\left(\frac{\pi}{2} n\right)}{n^2}$.

In this case, condition (ii) holds (Squeeze Theorem). On the other hand, the series is not decreasing. Note that, if n is odd, then $b_n = \frac{1}{n^2}$ but if n is even then $b_n = 0$. Thus, this series fails the Alternating Series Test.

However, this series still converges. Part of the reason is that this series is not really an alternating series. The 'negative terms' are all 0, and so this is in fact a positive series. It is actually the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

which converges by the Limit Comparison Test.

Example. Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$.

Condition (ii) is easy to check and we omit that here. The real challenge is to check whether it is decreasing. It is enough, however, to show that the function $f(x) = \frac{x^2}{x^3+1}$ is decreasing (why?!). To determine intervals of increase and decrease, we look for critical points. Note that $f'(x) = \frac{x(2-x^3)}{(x^3+1)^2}$. This has critical points at $x = -1, 0, \sqrt[3]{2}$. We only care about $x \geq 1$, so we check that $f'(x) < 0$ for $x > \sqrt[3]{2}$. Hence, f is decreasing for $x \geq 2$.

The statement of the theorem indicates that the decreasing condition must hold for all n, but as with most of our theorems, it is enough that it holds *eventually*.

Example. Determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$.

Let $b_n = \sqrt{n+1}-\sqrt{n}$. Note that we can rationalize the numerator so that

$$b_n = \sqrt{n+1}-\sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Now it is clear that $\lim_{n\to\infty} b_n = 0$ and the b_n are decreasing. Thus, the series converges by the Alternating Series Test.

Remark. Alternating series are particularly nice because there is an easy formula to determine a bound for the error of the Nth partial sum S_N.

Theorem 17 (Alternating series estimation theorem)**.** If $S = \sum_{n=1}(-1)^{n-1}b_n$ is the sum of a series satisfying the AST, then

$$|R_N| = |S - S_N| \leq b_{N+1}.$$

(Equivalently, $S_N - b_{N+1} \leq S \leq S_N - b_N$.)

Example. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$ correct to three decimal places.

First check the Alternating Series Test. The fact that the summation starts at $n = 0$ does not affect the AST. Set $b_n = \frac{1}{n!}$. It is clear that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n!} = 0$$

and $\frac{1}{(n+1)!} \leq \frac{1}{n!}$, so the b_n are decreasing. Hence, the given series converges by the Alternating Series Test.

Let S be the sum of the series. The idea of the above theorem is that we must find a b_n such that the difference does not affect the third decimal place of the sum. This is a bit of trial and error.

We compute the b_n and find that $b_7 = 0.0002$. Hence, $|S - S_6| \leq b_7 = 0.0002$. We compute $S_6 \approx 0.368056$. Hence,

$$0.367856 \approx S_6 - 0.0002 \leq S \leq S_6 + 0.002 \approx 0.368256.$$

Round both sides to three decimal places gives $0.368 \leq S \leq 0.368$, so $S \approx 0.368$.

6. Absolute convergence, ratio and root tests

Remark. In the previous section, we limited ourselves to only series with positive terms. We now consider series with negative terms.

Definition 10. A series $\sum a_n$ is absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent. We say $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent. It is also convergent (by the alternating series test). On the other hand, the alternating harmonic series converges but it is not absolutely convergent.

Theorem 18. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. We know $-|a_n| \leq a_n \leq |a_n|$ so $0 \leq a_n + |a_n| \leq 2|a_n|$. Since $\sum |a_n|$ converges then so does $\sum 2|a_n|$. Now $0 \leq a_n + |a_n| \leq 2|a_n|$ so $\sum (a_n + |a_n|)$ converges by the Comparison Test. Since

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|,$$

is the difference of two convergent series, then $\sum a_n$ converges. □

Remark. The converse of this theorem is *not true* in general.

Example. Study the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

We have $|\cos n| \leq 1$, so

$$0 \leq \left| \frac{\cos n}{n^2} \right| \leq \frac{1}{n^2}.$$

Since $\sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$), then the given series is absolutely convergent by the Comparison Test, and therefore convergent.

Remark. We will now consider two tests, the Ratio Test and the Root test, which can in certain cases determine whether a series is absolutely convergent.

Theorem 19 (The Ratio Test)**.** Assume the following limit exists:

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- If $L < 1$, then the series $\sum a_n$ is absolutely convergent.
- If $L > 1$ or $L = \infty$, then the series $\sum a_n$ is divergent.
- If $L = 1$, then the Ratio Test is inconclusive.

Remark. Remember that inconclusive means inconclusive. If $L = 1$ in the Ratio Test then we cannot assume anything (like, say, conditionally convergent). All it means is that we need to consider another method.

Example. Study convergence in each example.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

Set $a_n = \frac{n}{2^n}$. Then

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)/2^{n+1}}{n/2^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)2^n}{n2^{n+1}}\right| = \frac{1}{2}.$$

Since $L < 1$, then the series $\sum a_n$ converges absolutely by the Ratio Test.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Set $a_n = \frac{1}{n^3}$. Now we compute

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty} \left|\frac{n^3}{(n+1)^3}\right| = 1.$$

Thus, the Ratio Test is inconclusive in this case. Thankfully, since the terms are always positive, we know already that this series converges absolutely by the p-test.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{6^n}$

Set $a_n = \frac{n!}{6^n}$. Then

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!/6^{n+1}}{n!/6^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!6^n}{n!6^{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{6} = \infty.$$

Since $L = \infty$, then the series $\sum a_n$ diverges by the Ratio Test.

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$

Set $a_n = \frac{(-3)^{n-1}}{\sqrt{n}}$. Then

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-3)^n/\sqrt{n+1}}{(-3)^{n-1}/\sqrt{n}}\right| = \lim_{n\to\infty} \frac{\sqrt{n}3^n}{\sqrt{n+1}3^{n-1}} = 3.$$

Since $L > 1$, then the series $\sum a_n$ diverges by the Ratio Test.

Remark. A quick reminder that for any finite number $p \geq 1$,

$$\lim_{n\to\infty} (n^p)^{1/n} = \lim_{n\to\infty} n^{p/n} = 1.$$

To see this, let $f(x) = (x^p)^{1/x}$, then $f(x)$ models $a_n = (n^p)^{1/n}$. Write $y = f(x) = (x^p)^{1/x}$. Then $\ln(y) = \frac{\ln(x^p)}{x} = \frac{p\ln x}{x}$. So, by L'Hospital's Rule,

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{p\ln x}{x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{p(1/x)}{1} = \lim_{x\to\infty} \frac{p}{x} = 0.$$

Because e^x is a continuous function we have

$$\lim_{n\to\infty} (n^p)^{1/n} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1.$$

Theorem 20 (The Root Test)**.** Assume the following limit exists:

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

- If $L < 1$, then the series $\sum a_n$ is converges absolutely.
- If $L > 1$ or $L = \infty$, then the series $\sum a_n$ diverges.
- If $L = 1$, then the Root Test is inconclusive.

Example. Study the convergence of each example.

(1) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

Set $a_n = \left(\frac{n^2+1}{2n^2+1}\right)^n$. Then

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2}.$$

Since $L < 1$, then the series $\sum a_n$ converges absolutely by the Root Test.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-5)^n}{n^2}$

This one could be done (fairly easily) with the Ratio Test. But we'll use the Root Test along with the fact mentioned above.

Set $a_n = \frac{(-5)^n}{n^2}$. Then

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(-5)^n}{n^2}\right|} = \lim_{n\to\infty} \frac{5}{n^{2/n}} = 5.$$

Since $L > 1$, then the series $\sum a_n$ diverges by the Root Test.

Here is one more example that is perhaps not obviously a Ratio Test problem.

Example. Let $a_n = \frac{2}{1!} \cdot \frac{3}{3!} \cdot \frac{4}{5!} \cdots \frac{n+1}{(2n-1)!}$. Study the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$.

The trick here is to recognize that a_n is a recursive sequence. We have $a_1 = 2$ and $a_{n+1} = a_n \cdot \frac{n+2}{(2(n+1)-1)!} = a_n \cdot \frac{n+2}{(2n+1)!}$. Now by the Ratio Test,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{a_n \cdot \frac{n+2}{(2n+1)!}}{a_n} = \lim_{n\to\infty} \frac{n+2}{(2n+1)!} = 0 < 1.$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely by the Ratio Test.

7. Strategy for testing series

This section is meant as a review of series convergence testing. There is not a particular algorithm for deciding on a particular method in each convergence problem, but we will develop some useful strategies.

p-series Recall that a p-series is one of the form $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$. A p-series converges if $p > 1$ and diverges is $p \leq 1$. The p-series with $p = 1$ is called the *harmonic series*.

geometric series A geometric series is of the form $\sum_{n=1}^{\infty} cr^{n-1}$. This converges if and only if $|r| < 1$. When it converges, the sum is $c/(1-r)$. Keep in mind that the series must be in the above form for this sum formula to work.

Comparison Tests When a series is similar to one of the above forms, then it may be useful to apply a comparison test (comparison test or limit comparison test). The comparison tests *only* apply to series with (eventually) non-negative terms. However, if $\sum a_n$ has some negative terms, one may be able to apply a comparison test and check for *absolute convergence*.

Divergence Test If $\displaystyle\lim_{n\to\infty} a_n \neq 0$ then the divergence test tells us immediately that the series diverges. I the divergence test fails then we cannot make any conclusions.

Alternating series A series of the form $\sum(-1)^n b_n$ or $\sum(-1)^{n-1} b_n$, with $b_n \geq 0$ is an alternating series and the Alternating Series Test should be considered. Note that the AST does not tell us (directly) when a series diverges, but often the Divergence Test can be used when the AST fails.

Ratio Test This is useful for series with nth powers and factorials. It is never useful when terms involve rational or algebraic functions of n (inconclusive).

Root Test Sometimes this is interchangeable with the Ratio Test. This is most often useful when nth powers are involved.

Integral Test When a_n can be modeled by a continuous function $f(x)$ (i.e. $f(n) = a_n$ for all n) then the Integral Test is often useful (provided $f(x)$ is a function that we can easily integrate).

Example. In each example, determine whether the given series converges or diverges.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

Comparing degrees of the numerator and denominator, we see that this series *should* behave like $\sum \frac{1}{n^2}$. One method is to note that

$$0 \leq \frac{n-1}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$), then the given series converges by the Comparison Test.

Alternatively, set $a_n = \frac{n-1}{n^3+1}$ and $b_n = 1/n^2$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{n^3 - n^2}{n^3+1} = 1 > 0.$$

Again, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p = 2 > 1$) and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite and greater than zero, then the given series converges by the Limit Comparison Test.

(2) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^2-1}{n^2+1}$

This is an alternating series. However,

$$\lim_{n\to\infty} \left|(-1)^n \frac{n^2-1}{n^2+1}\right| = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1 \neq 0.$$

Hence, by the Divergence Test, the series diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

Since the series involves nth powers, we use the Root Test (the Ratio Test is also a possibility here). Set $a_n = \frac{n^{2n}}{(1+n)^{3n}}$, then

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^{2n}}{(1+n)^{3n}}\right|} = \frac{n^2}{(1+n)^3} = 0,$$

because the degree of the denominator is greater than that of the numerator. As the limit we obtained is less than 1, then the series converges *absolutely* by the Root Test.

(4) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n)!}$

Since factorials are involved, we use the Ratio Test. Set $a_n = \frac{\pi^{2n}}{(2n)!}$ so $a_{n+1} = \frac{\pi^{2(n+1)}}{(2(n+1))!}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\pi^{2(n+1)}/(2(n+1))!}{\pi^{2n}/(2n)!}\right| = \pi^2 \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} = \pi^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Since this limit is less than 1, then the series converges *absolutely* by the Ratio Test.

(5) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{\sqrt{n}}$

This is an alternating series. Set $b_n = \frac{\ln n}{\sqrt{n}}$. The function $f(x) = \frac{\ln x}{\sqrt{x}}$, which is continuous for $x > 0$, models the b_n. To show that $\{b_n\}$ is decreasing it is sufficient to show that $f(x)$ is decreasing. Note,

$$f'(x) = \frac{(\sqrt{x})(1/x) - (\ln x)(1/2)(x^{-1/2})}{x} = \frac{2\sqrt{x}(1-\ln x)}{2x^2}.$$

The critical points of this function are $x = 0, e$ and for $x > e$ we have $f'(x) < 0$. Thus, $f(x)$ is decreasing for $x > e$ and so b_n is decreasing for $n > e$. Moreover,

$$\lim_{n\to\infty} b_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Thus, the given series converges by the Alternating Series Test.

(6) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$

Unlike the previous example, we cannot use the AST. Instead, we model $a_n = \frac{\ln n}{\sqrt{n}}$ with $f(x) = \frac{\ln x}{\sqrt{x}}$ and use the Integral Test. Note that f is continuous, positive, and decreasing. We compute

$$\begin{aligned}
\int_1^{\infty} \frac{\ln x}{\sqrt{x}}\, dx &= \lim_{t\to\infty} \int_1^t x^{-1/2} \ln x\, dx \quad \text{set } u = \ln x,\ dv = x^{-1/2}\, dx, \text{ so } du = (1/x)dx,\ v = 2x^{1/2} \\
&= \lim_{t\to\infty} \left(\Big[(\ln x)(2x^{1/2}) \Big]_1^t - 2\int_1^t x^{-1/2}\, dx \right) \\
&= \lim_{t\to\infty} \left(\Big[(\ln x)(2x^{1/2}) - 4x^{1/2} \Big]_1^t \right) \\
&= \lim_{t\to\infty} \Big((\ln t)(2t^{1/2}) - 4t^{1/2} + 4 \Big) \\
&= 4 + \lim_{t\to\infty} \Big(t^{1/2}(2\ln t - 4) \Big) = \infty.
\end{aligned}$$

Since the integral converges then so does the given series by the Integral Test.

8. Power Series

We now turn our study to *power series*. The general motivation is to be able to approximate functions, like $\sin(x)$ by polynomials in which computations are easier. In fact, this is the primary tool that calculators and computer algebra systems use to approximate values of such functions.

Example. Determine for what values of x the following series converges: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x+2)^n}{n2^n}$.

We apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{\left|(-1)^n\frac{(x+2)^n}{n2^n}\right|}=\lim_{n\to\infty}\frac{|x+2|}{\sqrt[n]{n}2}=\frac{|x+2|}{2}\lim_{n\to\infty}\frac{1}{\sqrt[n]{n}}=\frac{|x+2|}{2}.$$

By the Root Test, the series converges (absolutely) when $\frac{|x+2|}{2}<1$ and diverges when $\frac{|x+2|}{2}>1$. Equivalently, $|x+2|<2$ and $|x+2|>2$, respectively.

The condition $|x+2|<2$ implies $-2<x+2<2$ or $-4<x<0$. The series converges on this interval and diverges when $x>0$ or $x<-4$. It remains only to check what happens when $x=-4$ and when $x=0$. Substituting $x=0$ into the given series gives the alternating harmonic series so the series converges when $x=0$. On the other hand, substituting $x=-4$ reveals the harmonic series, so the series diverges when $x=-4$. Hence, the series converges on the interval $(-4,0]$.

Definition 11. A power series with center a is a series of the form

$$F(x)=\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+\cdots.$$

The c_n are constants called the **coefficients** of the series.

Theorem 21 (Radius of covergence)**.** Every power series $F(x)$ has a radius of convergence R, which is either a nonnegative number of infinity. If R is finite, $F(x)$ converges absolutely when $|x-c|<R$ and diverges when $|x-c|>R$. If $R=\infty$, then $F(x)$ converges absolutely for all x.

Remark. To find the radius of convergence, we apply the Ratio or Root Test to a given power series. We then check the endpoints individually (these are the values where the Ratio or Root Test are inconclusive). The set of all values on which the power series converges is called the interval of convergence. Note that indexing does not affect the radius or interval of convergence.

Example. Geometric series are power series with $c_n=c$ for all n and $r=x$. A geometric series converges, that is, when $|r|<1$. Hence, the radius of convergence is 1 and the interval of convergence is $(-1,1)$.

Example. Find the radius and interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^nx^n}{n+1}$.

The center here is 0. We apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n x^n}{n+1}\right|} = |x| \lim_{n\to\infty} \frac{1}{\sqrt[n]{n+1}} = |x|$$

This power series converges when $|x| < 1$ and diverges when $|x| > 1$, so the radius of convergence is 1. Now we need to check the endpoints $x = -1$ and $x = 1$.

When $x = 1$, this series is just the alternating harmonic series. On the other hand, when $x = -1$, this is the harmonic series. Hence, the interval of convergence is $(-1, 1]$.

Example. Find the radius and interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} n! x^n$.

Again we apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{(n+1)! x^{n+1}}{n! x^n}\right| = \lim_{n\to\infty} |(n+1)x| = |x| \lim_{n\to\infty} (n+1).$$

This limit is infinite unless $x = 0$. Thus, the radius of convergence of the series is 0.

Example. Find the radius and interval of convergence for the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

Applying the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1} x^{2(n+1)}/(2(n+1))!}{(-1)^n x^{2n}/(2n)!}\right| = \lim_{n\to\infty} \frac{x^{2(n+1)}(2n)!}{x^{2n}(2(n+1))!} = x^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Hence, the series converges for all x and the radius of convergence is $R = \infty$.

Example. Find the radius and interval of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^5 (x-2)^n}{5^n}$.

Again applying the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{(n+1)^5 (x-2)^{n+1}/5^{n+1}}{n^5 (x-2)^n/5^n}\right| = \frac{|x-2|}{5} \lim_{n\to\infty} \frac{(n+1)^5}{n^5} = \frac{|x-2|}{5}.$$

Hence, $R = 5$ and the endpoints are $x = -3, 7$. When $x = 7$, the series becomes $\sum n^5$ and when $x = -3$, the series becomes $\sum (-1)^n n^5$. Both diverge by the Divergence Test. Thus, the interval of convergence is $(-3, 7)$.

Example. Find the radius and interval of convergence for the series $\displaystyle\sum_{n=3}^{\infty} \frac{3^n (x+7)^n}{\sqrt{n^2-4}}$.

Applying the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{3^{n+1}(x+7)^{n+1}}{\sqrt{(n+1)^2-4}} \cdot \frac{\sqrt{n^2-4}}{3^n (x+7)^n}\right| = 3|x+7| \lim_{n\to\infty} \frac{\sqrt{n^2-4}}{\sqrt{(n+1)^2-4}} = 3|x+7|.$$

Solving $3|x+7| < 1$ gives $|x+7| < 1/3$. Hence, $R = 1/3$ and the endpoints are $x = -7 \pm (1/3) = -22/3, -20/3$.

When $x = -20/3$, the series becomes $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2 - 4}}$. Since

$$\frac{1}{\sqrt{n^2 - 4}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and $\sum \frac{1}{n}$ is the harmonic series, which diverges, then the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2 - 4}}$ diverges by the Comparison Theorem. On the other hand, when $x = -22/3$, the series becomes $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{\sqrt{n^2 - 4}}$. This series converges by the Alternating Series Test. Thus, the interval of convergence is $[-22/3, -20/3)$.

9. Representations of functions as power series

Remark. Suppose $|x| < 1$. Then

$$\frac{1}{1-x} = \sum_{n=0} x^n = 1 + x + x^2 + x^3 + \cdots .$$

Said another way, the function $1/(1-x)$ may be represented by the power series. This has many practical applications. In particular, power series are often better to work with because they resemble polynomials. Hence, if a function like e^x can be represented by a power series (and it can) then we can approximate e^5 by using the power series representation. In this section we will study multiple techniques for representing certain functions by power series.

Example. Express $1/(1+x^2)$ as the sum of a power series and find its interval of convergence.

We apply substitution to the power series form of $1/(1-x)$ by replacing x with $-x^2$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0} (-x^2)^n = \sum (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots .$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, when $x^2 < 1$. But this is just equivalent to $|x| < 1$. So the interval of convergence is $(-1, 1)$.

Example. Express $1/(x-3)$ as the sum of a power series and find its interval of convergence.

We first need to write this in the form of the sum of a geometric series. We do this by factoring out a -3 from the denominator.

$$\frac{1}{x-3} = \frac{1}{(-3)(1-(x/3))} = -\frac{1}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}x^n.$$

This series converges when $|x/3| < 1$ or $|x| < 3$. Thus, the interval of convergence is $(-3, 3)$.

Example. Express $x^2/(x-3)$ as the sum of a power series and find its interval of convergence.

We can use our work from the previous example.

$$\frac{x^2}{x-3} = x^2 \cdot \frac{1}{x-3} = x^2 \cdot \left(-\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}x^n\right) = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}x^{n+2} = -\frac{1}{3}\sum_{n=2}^{\infty}\frac{1}{3^{n-2}}x^n.$$

The interval of convergence is $(-3, 3)$.

Theorem 22 (Term-by-term differentiation and integration)**.** If the power series

$$F(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has radius of convergence $R > 0$, then $F(x)$ is differentiable (and therefore continuous) on $(a - R, a + R)$ (or all x if $R = \infty$). Furthermore, we can integrate and differentiate term by term. For

$x \in (a-R, a+R)$,

$$(1)\quad F'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$(2)\quad \int F(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + \cdots = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

Moreover, these series have the same radius of convergence R.

Example. Express $\dfrac{1}{(1-x)^2}$ as a power series.

Using term-by-term differentiation,

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) = 0 + 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}.$$

Since term-by-term differentiation does not affect the radius of convergence, then we get radius of convergence $R = 1$.

Remark. The radius of convergence will remain the same when we differentiate and integrate, but the interval of convergence may change. Check endpoints!

Example. Find a power series expansion for $\arctan(x)$.

By Example , $(1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$ when $|x| < 1$. Then

$$\begin{aligned}\arctan x &= \int (1+x^2)^{-1}\ dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\ dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .\end{aligned}$$

When $x = 0$, $C = \arctan(0) = 0$. Hence,

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

This expansion is valid for $-1 < x < 1$ but not at the endpoints.

Example. Find a power series expansion for $\ln(1+x)$.

We have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$$

and this series had radius of convergence 1. Now

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\ dx = \int (1 - x + x^2 - x^3 + x^4 + \cdots)\ dx \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right) + C = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}.\end{aligned}$$

When $x = 0$, $\ln(1+x) = \ln(1) = 0$. Hence, $C = 0$. This has radius of convergence $R = 1$.

10. Taylor Series

Remark. For a function $f(x)$, the linearization at $x = a$ is the function $L(x) = f'(x)(x-a)+f(a)$. The linearization approximates the function near $x = a$. More than that, $f(x)$ and $L(x)$ agree at $x = a$ *and* their derivatives agree at $x = a$. That is, $f(a) = L(a)$ and $f'(a) = L'(a)$. In this section we construct polynomials of higher degree that approximation functions. Extending this idea, we find power series representations of functions.

Remark. Suppose f can be represented by a power series centered at a, that is

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R.$$

Can we determine a formula for the c_n in general? First note that $f(a) = c_0$ and

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

Thus, $f'(a) = c_1$. Similarly,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

so $f''(a) = 2c_2$. Continuing in this way, we find that $f^{(n)}(a) = n!c_n$. Said another way, $c_n = \frac{f^{(n)}(a)}{n!}$.

Theorem 23 (Taylor's Formula)**.** If f has a power series representation centered at a, that is,

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R,$$

then its coefficients are given by the formula,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

This is called the Taylor Series of f at a. When $a = 0$, it is called the Maclaurin series.

Example. Find the Maclaurin Series of $f(x) = e^x$ and its radius of convergence.

First note that $f^{(n)}(x) = e^x$ for all n and so $f^{(n)}(0) = 1$ for all n. Hence,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Using the ratio test, we find the radius of convergence is $R = \infty$.

Example. Find the Maclaurin series for $\sin x$.

Let $f(x) = \sin x$, then $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$ and $f^{(4)} = \sin x$. Hence, this cycle repeats indefinitely. We have $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -1$. Thus, the

Maclaurin series for $\sin x$ is

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Example. Find the Taylor series of $f(x) = 1/x$ at $a = -3$.

We have $f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$. Thus, we conclude that $f^{(n)}(x) = (-1)^n(n!)x^{-(n+1)}$. Hence,

$$f^{(n)}(-3) = (-1)^n(n!)(-3)^{-(n+1)} = -(n!)3^{-(n+1)}.$$

Therefore, the Taylor series expansion of $f(x) = 1/x$ at $a = -3$ is

$$\sum_{n=0}^{\infty}\frac{-(n!)3^{-(n+1)}}{n!}(x-(-3))^n = \sum_{n=0}^{\infty}\frac{-1}{3^{n+1}}(x+3)^n.$$

Remark. Our understanding of Taylor series gives us a way to realize a very important series which you are probably already familiar with.

Example (The Binomial Series)**.** Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

We can show that

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n},$$

so that

$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1).$$

Hence, the Maclaurin series of $f(x)$ is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n.$$

The coefficients in this series are known as binomial coefficients,

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

Hence, we can write the series in shorthand as,

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n.$$

Remark. Now we turn to the question of whether a function is truly represented by its Taylor/Maclaurin series. One way to do this is to find the radius of convergence. In the first two examples, the radius of convergence is $R = \infty$. On the other hand, in the previous example, the radius of convergence is 3. There is a shortcut which can save time in certain instances.

Remark. We call the partial sums of the Taylor series the Taylor polynomials of f at a. Let $T_n(x)$ represent the nth Taylor polynomial of a function f. Then $R_n(x) = f(x) - T_n(x)$ is called the remainder of the Taylor series. If $f(x) = T_n(x) + R_n(x)$ and

$$\lim_{n\to\infty} R_n(x) = 0 \text{ for } |x-a| < d,$$

then f is equal to the sum of its Taylor series on the interval $|x - a| < d$.

Theorem 24. Suppose there exists $M > 0$ such that $|f^{(n+1)}(x)| \le M$ for all n and all x such that $|x - a| \le d$, then $f(x)$ is equal to the sum of its Taylor series on $(a - d, a + d)$.

Remark. This actually follows from two facts: The first is Taylor's Inequality:

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1},$$

and from the fact that $\lim_{n\to\infty} x^n/n! = 0$ for any real number x.

Example. Let $f(x) = e^x$. For all $x \in (a - R, a + R)$, $|f^{(k)}(x)| \le e^{a+R}$. Hence, e^x equals the sum of its Taylor Series at $x = a$ on this interval.

We can approximate functions using Taylor polynomials. Taylor's inequality gives a way of determining how well a given Taylor polynomial approximates the function.

Example. Use Taylor's Inequality to determine the number of terms of the Maclaurin series for $\sin(x)$ that should be used to estimate $\sin(1)$ to within 0.0001.

Let $f(x) = \sin x$. Previously we computed the Maclaurin series for $\sin x$:

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

First note that, since $|f^{(k)}(x)| \le 1$ for all x, then $\sin x$ equals the sum of its Maclaurin series.

Now we want to estimate $\sin(1)$ for within 0.0001. First we determine how many terms are necessary in Taylor polynomial approximation. By Taylor's Inequality, letting $x = 1$, we want n such that

$$|R_n(x)| \le \frac{1}{(n+1)!} < 0.0001.$$

By trial-and-error we find that $n = 7$. The degree 7 Taylor polynomial is

$$T_7(x) = \sum_{n=0}^{7}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7.$$

(Note the discrepancy between the n). Now,

$$\sin(1) \approx T_7(1) = 4241/5040 \approx .8414682540.$$

Note that Maple computes $\sin(1)$ as $\sin(1) \approx .8414709848$ so we indeed obtained a good approximation within the given error.

We can use substitution, derivatives, and integration to obtain new Taylor series.

Example. By the previous example, $\sin x$ equals the sum of its Maclaurin series. Hence, $x \sin x$ is also equal to the sum of its Maclaurin series and

$$x \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(n+1)}}{(2n+1)!}.$$

Example. The Maclaurin series for $\ln(1+x)$ is $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ with radius of convergence 1. Note that this matches what we found using substitution methods in previous sections.

We can approximate $\ln(1+x)$ with the degree three Taylor polynomial

$$T_3(x) = -x + \frac{x^2}{2} - \frac{x^3}{3}.$$

Using Taylor's inequality we can bound the error involved in this approximation on a given interval.

What is the bound on the interval $(-.5, .5)$? The third derivative is $2/(x+1)^3$. Thus, on $(-.5, .5)$, $|f'''(x)| \leq f'''(.5) = 16/27$. Take $M = 16/27$. Now, by Taylor's inequality,

$$|R_3(x)| \leq \frac{M}{(3+1)!} |.5 - 0|^{3+1} = \frac{1}{324} \approx 0.003086419753.$$

Example. (1) Evaluate $\displaystyle\int e^{-x^2}\ dx$ as an infinite series.

We have $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. By substitution and term-by-term integration,

$$\int e^{-x^2}\ dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\ dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}.$$

(2) Write $\displaystyle\int_0^1 e^{-x^2}\ dx$ as an alternating series.

Using part (1),

$$\int_0^1 e^{-x^2}\ dx = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)} \right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}.$$

(3) How many terms are necessary so that the sum in (2) is correct to three decimal places?

Let $b_n = \frac{1}{n!(2n+1)}$. A calculation shows that $b_6 < 0.0005$. Thus, by the Alternating Series Estimation Theorem, we need 5 terms. Computing the fifth partial sum we have

$$\int_0^1 e^{-x^2}\ dx \approx \sum_{n=0}^{5} \frac{(-1)^n}{n!(2n+1)}. = .7467291967.$$