Actions of Taft Algebras on Noetherian Down-Up Algebras

Jason Gaddis

(Joint work with Simon Crawford and Robert Won)

Classical Invariant Theory

Let $\Bbbk$ be an algebraically closed field of characteristic zero.

Theorem

Let G be a finite subgroup of $GL(n, \Bbbk)$ and $A = \Bbbk[x_1, \ldots, x_n]$ a polynomial ring.

- *(Shephard–Todd–Chevalley Theorem) The invariant ring A^G is a polynomial ring if and only if G is generated by quasi-reflections.*

- *(Watanabe's Theorem) If G contains no quasi-reflections, then A^G is Gorenstein if and only if $G \leqslant SL(n, \Bbbk)$.*

A major goal of noncommutative invariant theory is to find suitable generalizations of these results when

- A is replaced by a noncommutative polynomial ring, and/or

- G is replaced by a finite-dimensional Hopf algebra.

(Noncommutative) Invariant Theory

Definition
A connected $\mathbb{N}$-graded algebra A is called Artin–Schelter (AS) Gorenstein if A has injective dimension $d < \infty$ on the left and on the right, and there are isomorphisms

$$\mathrm{Ext}^i_A({}_A\mathbb{k}, {}_AA) \cong \mathrm{Ext}^i_A(\mathbb{k}_A, A_A) \cong \begin{cases} \mathbb{k}(\ell) & i = d \\ 0 & \text{otherwise.} \end{cases}$$

If, in addition, A has finite global dimension and finite Gelfand–Krillov (GK) dimension, then A is called Artin–Schelter (AS) regular of dimension d.

Some Tools
Let A be an AS Gorenstein algebra and G a finite group subgroup of $\mathrm{Aut}_{\mathrm{gr}}(A)$.

▶ Trace Series: $\mathrm{Tr}_A(g, t) = \displaystyle\sum_{k=0}^{\infty} \mathrm{trace}(g|_{A_k})t^k$

▶ Molien's Theorem: $H_{A^G} = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \mathrm{Tr}_A(g, t)$

▶ Homological determiniant: $\mathrm{hdet} : \mathrm{Aut}_{gr}(A) \to \mathbb{k}^\times$

$$\mathrm{Tr}_A(g, t) = (-1)^n (\mathrm{hdet}\, g)^{-1} t^{-\ell} + (\text{higher terms}).$$

(Noncommutative) Invariant Theory

Let A be an AS regular algebra with GKdim $A = n$. A graded automorphism g is a **reflection** if

$$\text{Tr}_A(g, t) = \frac{1}{(1-t)^{n-1}q(t)} \qquad \text{where } q(1) \neq 0$$

Theorem (Kirkman–Kuzmanovich–Zhang)

► Let $A = \mathbb{k}_{p_{ij}}[x_1, \ldots, x_n]$ be a skew polynomial ring and let G be a finite group of graded automorphisms of A. Then A^G has finite global dimension (and is again a skew polynomial ring) if and only if G is generated by reflections of A.

► Let A be AS regular and let H be a semisimple Hopf algebra acting on A with trivial homological determinant. Then A^H is AS Gorenstein.

The Taft algebra

Fix an integer $n \geqslant 2$ and let ω be a primitive nth root of unity. The corresponding Taft algebra of dimension n^2 is the algebra

$$T_n(\omega) = \Bbbk \langle x, g \mid g^n - 1, \ x^n, \ gx - \omega xg \rangle.$$

There is a Hopf structure on $T_n(\omega)$ with counit ϵ, comultiplication Δ, and antipode S defined on generators as follows:

$$\varepsilon(g) = 1, \qquad \Delta(g) = g \otimes g, \qquad S(g) = g^{n-1},$$
$$\varepsilon(x) = 0, \qquad \Delta(x) = g \otimes x + x \otimes 1, \qquad S(x) = -g^{n-1}x.$$

The group of grouplikes of $T_n(\omega)$ is $C_n = \langle g \rangle$.

The Taft algebras are non-semisimple, noncommutative, and noncocommutative.

They are pointed Hopf algebra: every simple comodule is 1-dimensional.

They're points!

Taft algebra actions

Actions of Taft algebras have been classified on a variety of families of algebras:

- ▶ finite-dimensional algebras (Montgomery–Schneider, Cline)
- ▶ polynomial rings (Allman)
- ▶ skew polynomial rings (G–Won–Yee, Cline–G)
- ▶ matrix algebras (Bahturin–Montgomery)
- ▶ quantum generalized Weyl algebras (G–Won)
- ▶ paths algebras of quivers (Kinser–Walton, Kinser–Oswald)
- ▶ preprojective algebras of type A (G–Oswald)

If H acts on an algebra A and I is a nonzero Hopf ideal such that $I \cdot A = 0$, then the action of H on A factors through, so H/I acts on A.

When there is no such ideal I, then we say the action is inner faithful.

A Taft algebra acts inner faithfully on A if and only if $x \cdot A \neq 0$.

If A is $\mathbb{N}$-graded, then we say the Taft action is homogeneous if $g(A_1), x(A_1) \subset A_1$.

Down-up algebras

The (graded) down-up algebra with parameters $(\alpha, \beta) \in \Bbbk^2$ is the algebra

$$A(\alpha, \beta) = \frac{\Bbbk \langle u, v \rangle}{\left\langle \begin{array}{c} v^2 u - \alpha v u v - \beta u v^2 \\ v u^2 - \alpha u v u - \beta u^2 v \end{array} \right\rangle}.$$

Assume $\beta \neq 0$. Then the down-up algebra $A = A(\alpha, \beta)$

- ▶ is a noetherian domain,

- ▶ is AS regular with global and GK dimension 3,

- ▶ has Hilbert series $H_A(t) = (1 - t)^{-2}(1 - t^2)^{-1}$,

- ▶ is PI if and only if the roots γ_1, γ_2 of $t^2 - \alpha t - \beta$ are roots of unity, and

- ▶ $\mathrm{Aut}_{\mathrm{gr}} A(\alpha, \beta) = \begin{cases} \mathrm{GL}(2, \Bbbk) & \text{if } (\alpha, \beta) \in \{(0, 1), (2, -1)\}, \\ \mathrm{diag}(2, \Bbbk) \rtimes S_2 & \text{if } \beta = -1, \alpha \neq 2, \\ \mathrm{diag}(2, \Bbbk) & \text{otherwise.} \end{cases}$

Taft actions on down-up algebras

Theorem (CGW)

Let $n \geqslant 2$ and suppose that $T_n(\omega)$ acts inner faithfully and homogeneously on a down-up algebra $A(\alpha, \beta)$. Then we have the following possibilities:

1. *For some $0 \leqslant k \leqslant n-1$ and $q \in \Bbbk^\times$,*

$$g = \begin{pmatrix} \omega^{k+1} & 0 \\ 0 & \omega^k \end{pmatrix}, \quad x = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix},$$

$$\alpha = \omega^{-(k+1)}(1 + \sqrt{\omega}), \quad \beta = -\omega^{-2(k+1)}\sqrt{\omega},$$

   where $\sqrt{\omega}$ denotes a choice of either one of the square roots of ω; or

2. *For some $0 \leqslant k \leqslant n-1$ and $r \in \Bbbk^\times$,*

$$g = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{k+1} \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix},$$

$$\alpha = \omega^{k+1}(1 + \sqrt{\omega^{-1}}), \quad \beta = -\omega^{2(k+1)}\sqrt{\omega^{-1}},$$

   where $\sqrt{\omega^{-1}}$ denotes a choice of either one of the square roots of ω^{-1}.

Conversely, each of the above parameter choices indeed gives rise to an inner faithful action of $T_n(\omega)$ on an appropriate down-up algebra.

Taft actions on down-up algebras

Corollary

Let $A(\alpha, \beta)$ be a down-up algebra where $(\alpha, \beta) \neq (2, -1), (0, 1)$.

There is an inner faithful, homogeneous action of some Taft algebra on $A(\alpha, \beta)$ if and only if γ_1 and γ_2 are roots of unity (so $A(\alpha, \beta)$ is PI), and either γ_1 or γ_2 is some power of $\gamma_1^{-2}\gamma_2^2$.

In this case, the only Taft algebra which acts is $T_n(\omega)$, where n is the multiplicative order of $\omega = \gamma_1^{-2}\gamma_2^2$.

For each $q \in \Bbbk^\times$, there are two actions of $T_n(\omega)$ on $A(\alpha, \beta)$: either

$$g = \begin{pmatrix} \gamma_1^{-1} & 0 \\ 0 & \gamma_1\gamma_2^{-2} \end{pmatrix}, \quad x = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix},$$

or

$$g = \begin{pmatrix} \gamma_1^{-1}\gamma_2^2 & 0 \\ 0 & \gamma_1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix},$$

where (if necessary) we have relabeled so that γ_1 is some power of $\gamma_1^{-2}\gamma_2^2$.

Invariants

For the remainder write $T = T_n(\omega)$, depending on some integer $n \geqslant 2$, which acts on a down-up algebra $A = A(\alpha, \beta)$, where both A and the action of T on A depend on some parameter $0 \leqslant k \leqslant n - 1$.

We will assume that g and x are represented by the following matrices:

$$g = \begin{pmatrix} \omega^{k+1} & 0 \\ 0 & \omega^k \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Define $z := vu - \omega^{-(k+1)}uv$. The element z is normal and satisfies

$$zu = \omega^{-(k+1)}\sqrt{\omega}\, uz, \qquad vz = \omega^{-(k+1)}\sqrt{\omega}\, zv.$$

Invariants

If H is a Hopf algebra and H acts on an algebra A, then the invariant ring A^H is

$$A^H = \{a \in A \mid h \cdot a = \varepsilon(h)a \text{ for all } h \in H\}.$$

In case of Taft actions, $A^T = A^x \cap A^{\langle g \rangle}$ where

$$A^x = \{a \in A \mid x \cdot a = 0\} \qquad A^{\langle g \rangle} = \{a \in A \mid g \cdot a = a\}.$$

Lemma

▶ If $\sqrt{\omega}$ has order n, then A^x is generated by u, z, and v^n, and is a skew polynomial ring where

$$v^n u = u v^n, \quad zu = \omega^{-(k+1)} \sqrt{\omega} \, uz, \quad v^n z = z v^n.$$

In particular, A^x is AS regular.

▶ If $\sqrt{\omega}$ has order $2n$, then A^x is generated by

$$u, \quad z, \quad v^{2n}, \quad x^{n-1} \cdot v^{2n-1}.$$

Then A^x is a quotient of a 4-dimensional AS regular algebra.

In particular, A^x is AS Gorenstein.

Invariants

Theorem (CGW)

If $\sqrt{\omega}$ has order n, then A^T is commutative.

Let $d = \gcd(k + 1, n)$ and $e = \gcd(2k + 1, n)$.

- *If $de = 1$, so that $k + 1$ and $2k + 1$ are both coprime to n, then*

$$A^T \cong \Bbbk[x, y]^G[t],$$

where

$$G = \left\langle \begin{pmatrix} \omega & 0 \\ 0 & \omega^r \end{pmatrix} \right\rangle, \quad r = (2k + 1)(k + 1)^{-1} \mod n.$$

In particular, A^T is (AS) Gorenstein if and only if $3k + 2 \equiv 0 \mod n$.

- *If $1 < de < n$, then A^T is (AS) Gorenstein if and only if*

$$e(k + 1) + d(2k + 1) \equiv 0 \mod n.$$

Invariants

Theorem (CGW)
If $\sqrt{\omega}$ has *order n*, then A^T is commutative.

Let $d = \gcd(k + 1, n)$ and $e = \gcd(2k + 1, n)$.

▶ If $de = n$ (equivalently, $(k + 1)(2k + 1) \equiv 0 \mod n$), then

$$A^T = \Bbbk[u^{n/d}, z^{n/e}, v^n]$$

is a polynomial ring. In particular, A^T is (AS) regular.

Some interesting aspects of this example:

▶ If H is a group algebra or its dual, then A^H is *not* AS regular.

▶ $A^T = Z(A)$, the center of A.

Corollary
The invariant ring A^T is (AS) Gorenstein if and only if

$$(k + 1)\gcd(2k + 1, n) + (2k + 1)\gcd(k + 1, n) \equiv 0 \mod n. \tag{1}$$

Invariants

Theorem (CGW)

If $\sqrt{\omega}$ has order $2n$, then A^T is not commutative and not AS regular.

The invariant ring A^T is AS Gorenstein if n and k satisfy the following relationships:

$$k = n - 1 > 1, \qquad 4k + 3 \equiv 0 \bmod n, \quad \text{or} \qquad n \geqslant 3 \text{ and } 4k + 2 \equiv 0 \bmod n.$$

n \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2		(✓)														
3	(✓)	(✓)	(✓)													
4				(✓)												
5			(✓)	(✓)	(✓)											
6		(✓)		(✓)	(✓)											
7		(✓)		(✓)			(✓)									
8								(✓)								
9		✓			(✓)	✓	(✓)		(✓)							
10		✓	(✓)					(✓)		(✓)						
11			(✓)			(✓)					(✓)					
12		✓			✓	✓			✓			(✓)				
13							(✓)			(✓)		(✓)				
14				(✓)							(✓)	✓		(✓)		
15			✓	(✓)	✓	✓		(✓)					✓		(✓)	
16																(✓)

Thank You!