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Automorphisms and Isomorphisms of Quantum Algebras

$\Bbbk$ an algebraically closed field, $\operatorname{char}\Bbbk = 0$.

Quantum Affine Space $\Bbbk_{p_{ij}}[x_1, \ldots, x_n]$

Generated by $x_1, \ldots, x_n$ with relations $x_i x_j = p_{ij} x_j x_i$ for $i < j$, $p_{ij} \in \Bbbk^\times$.

Automorphism group

1. $(\Bbbk^\times)^2$ when $n = 2$, $p_{12} \neq \pm 1$ [1].
2. $(\Bbbk^\times)^2 \rtimes \{\tau\}$ when $n = 2$, $p_{12} = -1$ [1].
3. Every p_{ij} is a root of unity for all $i < j$, $p_{ij} \neq 1$, the subgroup of $\Bbbk^\times$ generated by the p_{ij} is equal to $\langle q \rangle$ where ℓ is prime and q is a primitive ℓth root of unity. If the center is a polynomial ring, then $\operatorname{Aut}(\Bbbk_{p_{ij}}[x_1, \ldots, x_n])$ is affine [3].

Isomorphism Problem

$\Bbbk_{p_{ij}}[x_1, \ldots, x_n] \cong \Bbbk_{q_{ij}}[x_1, \ldots, x_n] \Leftrightarrow$ there exists $\sigma \in \mathcal{S}_n$ such that $q_{ij} = p_{\sigma(i)\sigma(j)}$ for all i, j [5].

Quantum $n \times n$ Matrix Algebra $\mathcal{O}_{\lambda,p_{ij}}(\mathcal{M}_n(\Bbbk))$

Generated by $\{X_{ij}\}$, $1 \leq i, j \leq n$, $p_{ij} \in \Bbbk^\times$, $p_{ij} = p_{ji}^{-1}$, $\lambda \in \Bbbk^\times \backslash \{\pm 1\}$, with relations

$$X_{lm} X_{ij} = \begin{cases} p_{li} p_{jm} X_{ij} X_{lm} + (\lambda - 1) p_{li} X_{im} X_{lj} & l > i, m > j \\ \lambda p_{li} p_{jm} X_{ij} X_{lm} & l > i, m \leq j \\ p_{jm} X_{ij} X_{lm} & l = i, m > j. \end{cases}$$

Single parameter case when $\lambda = q^{-2}$ and $p_{ij} = q$ for all $i > j$.

Automorphism group

1. $(\Bbbk^\times)^3 \rtimes \{\tau\}$ in the single parameter $n = 2$ case [1].
2. $(\Bbbk^\times)^5 \rtimes \{\tau\}$ in the single parameter $n = 3$ case, q not a root of unity [7].
3. $(\Bbbk^\times)^{2n-1} \rtimes \{\tau\}$ in the single parameter case, q not a root of unity [9].

Isomorphism Problem

If $\mathcal{O}_{\lambda,p_{ij}}(\mathcal{M}_n(\Bbbk)) \cong \mathcal{O}_{\mu,q_{ij}}(\mathcal{M}_n(\Bbbk))$, then $n = m$ and one of the following cases holds [5]:

1. $\lambda = \mu$ and $p_{ij} = q_{ij}$ for all i, j;
2. $\lambda = \mu$ and $p_{ij} = \lambda^{-1} q_{ji}$ for all i, j;
3. $\lambda = \mu^{-1}$ and $p_{ij} = q_{n+1-i,n+1-j}$;
4. $\lambda = \mu^{-1}$ and $p_{ij} = \lambda^{-1} q_{n+1-j,n+1-i}$.

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Quantized Weyl Algebra $A_n^{p_{ij},\gamma}$

Generated by $\{x_i, y_i\}$, $1 \leq i \leq n$, $p_{ij} \in \Bbbk^\times$, $p_{ij} = p_{ji}^{-1}$, $\gamma \in (\Bbbk^\times)^n$, with relations

$$y_i y_j = p_{ij} y_j y_i \quad \text{(all } i,j) \qquad x_i y_j = p_{ji} y_j x_i \quad (i < j)$$

$$x_i x_j = \gamma_i p_{ij} x_j x_i \quad (i < j) \qquad x_i y_j = \gamma_j p_{ji} y_j x_i \quad (i > j)$$

$$x_j y_j = 1 + \gamma_j y_j x_j + \sum_{l<j} (\gamma_l - 1) y_l x_l \qquad \text{(all } j).$$

Automorphism group

1. $\Bbbk^\times$ when $n = 1$, $\gamma_1 \neq \pm 1$ [2].
2. $\Bbbk^\times \rtimes \{\tau\}$ when $n = 1$, $\gamma_1 = -1$ [2].
3. $(\Bbbk^\times)^n$ when no γ_i is a root of unity [6].

Isomorphism Problem

1. $n = 1$ [4].
2. No γ_i is a root of unity [6].
3. All γ_i, p_{ij} are roots of unity distinct from 1 and $A_n^{p_{ij},\gamma}$ free over its center [8].
4. Homogenized $A_n^{p_{ij},\gamma}$ [5].

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