Welcome!

MTH 411/511

Foundations of Geometry

Your professor

(a little outdated)

A little bit about me:

- Went to undergrad at Indiana (Go Hoosiers!)
- Taught high school math in Baltimore, MD
- Earned my PhD from the University of Wisconsin-Milwaukee
- Did postdocs at the University of California, San Diego and Wake Forest University
- In my fourth year at Miami (halfway to tenure!)
- My research area is noncommutative algebra – more on this later in the course

Your professor

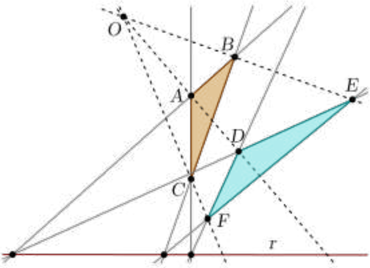
When I'm not doing math, or other academia-related stuff, I'm probably either

- spending time with my wife and 5-year-old daughter,
- playing with my pug Rocky,
- listening to music (Tool, Pearl Jam, Radiohead, too many to name),
- playing video games (this summer it was Zelda: BOTW),
- or exercising (I go early if you want to catch me at the rec).

Geometry

What is geometry? For most, geometry is probably one of the following:

- A description of the observable space around us.
- *Euclidean geometry*, i.e., the kind taught in high school.
- More precisely, it is the study of *points*, *lines*, *planes*, *circles*, etc., sitting in some two- or three-dimensional space.



(source: Wikipedia, public domain)

That is not what this course is about.

Geometry

Why do we study geometry?

- Provides a foundational understanding of geometry (especially important for future teachers).
- Good environment to study a field of mathematics "from the ground up" (axiomatically).
- Skills are applicable to a host of other fields of mathematics (and beyond).
- You will improve your skills at writing and analyzing proofs.
- You will learn to be more detail-oriented and learn to think carefully and deeply about mathematics, likely to a greater extent than you have in previous classes.

The Syllabus

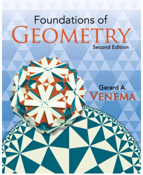
You should read the entire syllabus and post any questions on the discussion board. Here are some of the highlights.

Class Meetings: MWF 1:15pm-2:10pm.

Our class will meet synchronously on Zoom. You can find the link on Canvas. Attendance is not required but **highly encouraged**. Being able to ask questions and participate in the discussion is very useful in developing your skills. Students participating in the class asynchronously are assuming a great deal of responsibility for self-learning.

Office Hours: Thursdays from 10:00am-11:00am or by appointment.

Textbook: *Foundations of Geometry (2ed)* by Gerard A. Venema.



The Syllabus

Grading:

- Discussion Board (8%). Weekly discussion threads. Need to post twice weekly.
- Homework (12%). Assigned from the textbook each week and turned in on Canvas.
- Two midterm exams (25% each) and a cumulative final exam (30%). These are administered online using Proctorio. See the Canvas site for more information. It is your responsibility to be ready for the first exam.
- Grading scale is on the syllabus.

The Syllabus

Other stuff:

- Check Canvas regularly!
- Stay on top of work and don't get behind. If you're having trouble, ask for help.
- I take academic honesty very seriously. See the syllabus for more specifics.
- More on in-person instruction if we ever get to that.
- Have a great semester!

Chapter 1: Euclid's Elements

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Foundations of Geometry

It’s good to have goals

Goals for today:

- Discuss historical foundations for geometry.
- Examine Euclid’s axioms and their consequences.
- Give an example of a formal axiomatic proof in geometry.

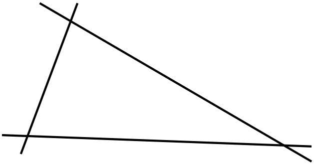
Figure: Euclid (source: Wikipedia, public domain)

Euclid is responsible for setting math on a path to formalism. His books consist of four main ingredients:

- Definitions (defined terms)
- Common notions (accepted facts)
- Postulates (assumed facts)
- Propositions (statements proved from the above).

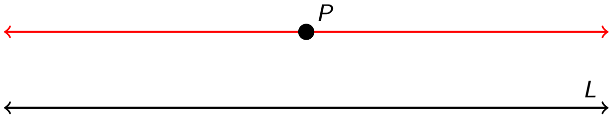
Euclid's Five Postulates

1. To draw a straight line from any point to another point.
2. To produce a straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. (Parallel Postulate) That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced infinitely, meet on that side which are the angles less than the two right angles.



Euclid's Five Postulates (translated)

1. There is a line joining any two points.
2. Any line segment can be extended to a line segment of any desired length.
3. For every line segment L, there is a circle that has L as its radius.
4. All right angles are congruent to each other.
5. (Parallel Postulate) If P is a point and L is a line not passing through P, then there is exactly one line through P and parallel to L.



Question

Does the Parallel Postulate follow from the other four postulates?

This question bothered geometers for centuries after Euclid. Euclid himself tried (unsuccessfully) to prove this. In the early-to-mid nineteenth century, this question was finally resolved.

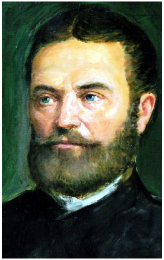


Figure: János Bolyai (source: Wikipedia, by Ferenc Márkos)

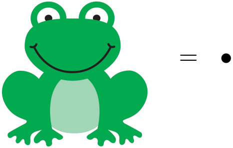
Figure: Carl Friedrich Gauss (source: Wikipedia, by Christian Albrecht Jensen)

Figure: Nikolai Lobachevsky (source: Wikipedia, by Lev Kryukov)

These men, independently, came to the realization that the parallel postulate was both necessary and unnecessary. On one hand, *Euclidean Geometry* requires the fifth postulate to accomplish much of what Euclid did. On the other hand, there are completely consistent geometries in which this postulate does not hold.

Figure: David Hilbert (source: Wikipedia, public domain)

Euclid's definitions are sometimes vague. For example, he defines a point as "that which has no part", which is difficult to interpret. Hilbert's goal was to reduce geometry to a level of abstraction where "point" could be replaced by "frog" and it would not matter.

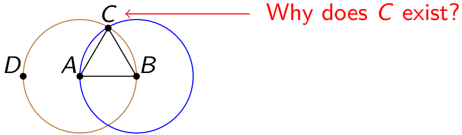


Our goal will be rigor and building geometry from its foundations. Let's now take a look at one of Euclid's arguments using his axioms.

An example

Proposition

On a given finite straight line to construct an equilateral triangle. (Given a line segment, there exists an equilateral triangle having that segment as a side.)



Proof.

Let AB be the given finite straight line. With center A and distance AB, let circle BCD be described (Post. 3). With center B and distance BA let the circle ACE be described (Post. 3). From the point C in which the circles cut one another, to the points A and B, let the straight lines CA and CB be joined (Post. 1).

Since the point A is the center of the circle CBD, AC is equal to AB (Defn 15: radii) Similarly, BC is equal to BA. As AC is equal to AB and AB is equal to BC, then AC is equal to BC (CN 1: transitivity). Thus, AC, AB, and BC are all equal to one another so the triangle ABC is equilateral and AB is a side. Q.E.D.

Next time

Before next class: Read Sections 2.1-2.2.

In the next lecture we will:

- Define an axiomatic system.
- Define incidence geometry and consequences of the axioms.
- Consider several examples (and non-examples) of incidence geometries.

Chapter 2: Axiomatic systems and incidence geometry
§2.1 The structure of an axiomatic system
§2.2 An example: incidence geometry

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Foundations of Geometry

It's good to have goals

Goals for today:

- Define an axiomatic system.
- Define incidence geometry and consequences of the axioms.
- Consider several examples (and non-examples) of incidence geometries.

Axiomatic systems

We'll begin by considering the different components of an axiomatic system.

- *Undefined terms* are fundamental objects in our system for which no good definition exists. We will take a minimal number of these and build from there.
- *Defined terms* are objects or notions that we can define using other parts of our system.
- *Axioms* (what Euclid called *Postulates*) will be accepted (without proof) as fundamental truths on which we build our system.
- *Theorems* (also called *propositions*) are logical consequences that follow from our terms and axioms.

- An *interpretation* of an axiomatic system is a particular way of giving meaning to the undefined terms in that system.
- A *model* is an interpretation in which the axioms are all true (and hence also all of the theorems).

A statement in an axiomatic system is *independent* of the axioms if cannot be proven or disproven as a logical consequence of the axioms. To demonstrate this, we may find model in which a particular statement is true and another model where it is false.

An axiomatic system is *consistent* if the axioms do not lead to a contradiction. To demonstrate this, we can show that there exists a model for the system.

Our first example of an axiomatic system will be *incidence geometry*. It is the axiomatic system with three undefined terms and three axioms (below). A model for incidence geometry is called *an incidence geometry*.

Undefined Terms: *point*, *line*, *lie on* (or *incident*)

Incidence Geometry Axioms

IA1) For every pair of distinct points P and Q, there exists exactly one line ℓ such that both P and Q lie on ℓ.

IA2) For every line ℓ, there exists at least two distinct points P and Q such that both P and Q lie on ℓ.

IA3) There exist three points that do not all lie on the same line.

Definition

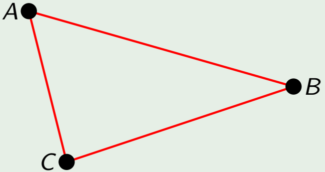
Three points A, B, and C are *collinear* if there exists one line ℓ such that all three points all line on ℓ. Three points are *noncollinear* if there is no such line.

Using this definition, we can restate IA3: There exist three noncollinear points.

Example 1 (Three-point geometry)

Our first interpretation of incidence geometry is the following:

- point: one of three symbols A, B, C
- line: a set of two points $\{A, B\}$, $\{A, C\}$, $\{B, C\}$
- lie on: "is an element of"



Under this interpretation, there is a distinct line for every pair of points (IA1), every line contains two distinct points that lie on it (IA2), and the three points do not lie on the same line (IA3).Thus, three-point geometry is a model for incidence geometry.

Example 2 (The three-point line)

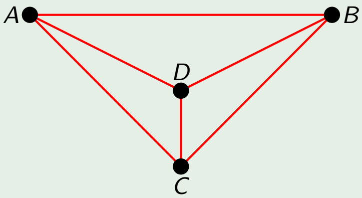
- point: one of three symbols A, B, C
- line: the set of all points $\{A, B, C\}$
- lie on: "is an element of"

Under this interpretation, there is a distinct line for every pair of points (IA1)and every line contains two distinct points that lie on it (IA2). However, there are not three noncollinear points (IA3).Thus, the three-point line is *not* a model for incidence geometry.

Example 3 (Four-point geometry)

This extends our first example (three-point geometry).

- point: one of four symbols A, B, C, D
- line: a set of two points $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, $\{C, D\}$
- lie on: "is an element of"



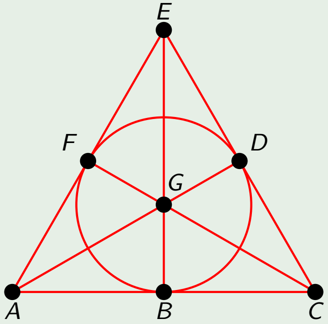
As with three-point geometry, four-point geometry is a model for incidence geometry.

Question

The idea behind three-point and four-point geometries can be extended to n-point geometry for any finite integer n. How many lines are there in n-point geometry?

Example 4 (Fano's geometry (also, the Fano plane))

Here, point is one of seven symbols A, B, C, D, E, F, G, the lines are the following sets: $\{A, B, C\}, \{C, D, E\}, \{E, F, A\}, \{A, G, D\}, \{C, G, F\}, \{E, G, B\}, \{B, D, F\}$, and lie on means "is an element of".



Exercise: Verify that Fano's geometry is a model for incidence geometry.

Definition 5

A one-to-one correspondence between models (for a particular axiomatic system) that maps points to points, lines to lines, and preserves incidence is an *isomorphism*. We say two models are *isomorphic* if there is an isomorphism between them.

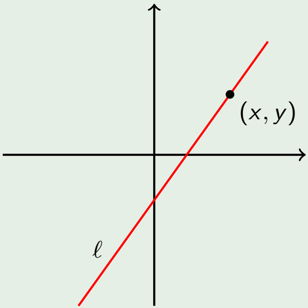
Example 6 (Interurban)

A point is one of three cities (Cincinnati, Oxford, Columbus) and a line is a railroad line between two cities. Here, "lie on" means to be on a particular railroad line. This is a model for incidence geometry that is isomorphic to three-point geometry.

Example 7 (The Cartesian plane $\mathbb{R}^2$)

This example should be familiar from high school algebra/geometry.

- point: an ordered pair (x, y) of real numbers
- line: the set of points satisfying a linear equation of the form $ax + by + c = 0$ where a, b, c are real numbers such that a and b are not both zero
- lie on: the point (x, y) satisfies the equation of a line ℓ.

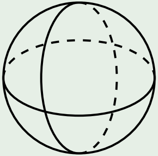


The Cartesian plane $(\mathbb{R}^2)$ is a model for incidence geometry with infinitely many points.

Example 8 (The sphere $\mathbb{S}^2$)

Geometry on the sphere is important in many applications such as navigation and astronomy.

- point: an ordered triple (x, y, z) of real numbers satisfying $x^2 + y^2 + z^2 = 1$
- line: a *great circle*, i.e., a circle on the sphere whose center is the center of the sphere
- lie on: "is an element of"

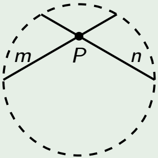


Two points on the sphere are *antipodal* if they lie on the same line through the origin. Antipodal points lie on an infinite number of distinct great circles, contradicting IA1. Hence, the sphere ($\mathbb{S}^2$) is *not* a model for incidence geometry.

Example 9 (The Klein disk)

This Klein disk seems innocent enough, but it is in fact one of the more exotic models we will consider this semester.

- point: an pair (x, y) of real numbers satisfying $x^2 + y^2 < 1$
- line: the part of a line ℓ lying inside the circle $x^2 + y^2 < 1$
- lie on: the point (x, y) satisfies the equation of a line ℓ



Exercise: The Klein disk is a model for incidence geometry with an infinite number of points.

Next time

Before next class: Read Sections 2.3-2.4.

In the next lecture we will:

- Define the notion of parallelism in incidence geometry.
- State different parallel postulates and investigate their validity in our various models.
- Discuss axiomatic systems as a global phenomenon.

Chapter 2: Axiomatic systems and incidence geometry
§2.3 The parallel postulates in incidence geometry
§2.4 Axiomatic systems and the real world

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Foundations of Geometry

It’s good to have goals

Goals for today:

- Define the notion of parallelism in incidence geometry.
- State different parallel postulates and investigate their validity in our various models.
- Discuss axiomatic systems as a global phenomenon.

We will state an initial definition of parallel. This may or may not agree with definition(s) you have seen previously in Euclidean geometry.

Definition 1

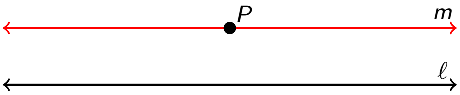
Two lines ℓ and m are said to be *parallel* if there is no point P such that P lies on (is incident with) both ℓ and m. The notation for parallelism is $\ell \parallel m$.

This definition is appropriate for incidence geometry. Note that, under this definition, a line is never parallel to itself.

Using this definition, we will now state three different parallel postulates that will be used in this course.

Euclidean Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there is exactly one line m such that P lies on m and $m \parallel \ell$.



The remaining two are not so easily drawn.

Elliptic Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there is no line m such that P lies on m and $m \parallel \ell$.

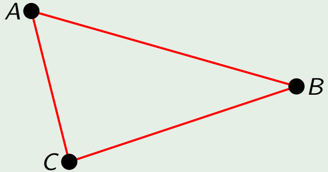
Hyperbolic Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there are at least two lines m and n such that P lies on both m and n and both m and n are parallel to ℓ.

Parallel postulates - Examples

Example 2 (Three-point geometry)

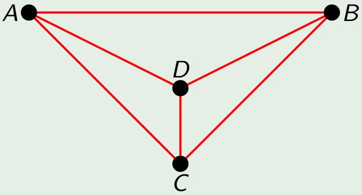
Recall three-point geometry, which is a model for incidence geometry.



Since there are no parallel lines, this model satisfies the Elliptic Parallel Postulate.

Example 3 (Four-point geometry)

Recall four-point geometry, which is a model for incidence geometry.



Given any line there are exactly two points not incident to that line. For example, C and D are not incident to $\{A, B\}$. Thus, $\{C, D\}$ is the unique line parallel to $\{A, B\}$. It follows that four-point geometry satisfies the Euclidean Parallel Postulate.

Example 4 (Five-point geometry)

The points in this geometry are one of the five symbols A, B, C, D, E, and the lines are any choice of two elements.

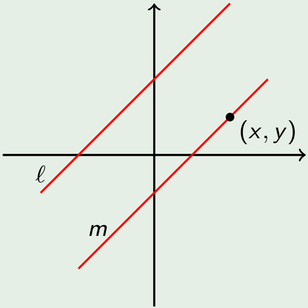
Given a line and a point not on that line, there are two lines parallel to that line. For example, take $\{A, B\}$ and the point C. The lines $\{C, D\}$ and $\{C, E\}$ are parallel to $\{A, B\}$. Thus, five-point geometry satisfies the Hyperbolic Parallel Postulate.

Question

Given an n-point geometry and a line ℓ, how many lines are parallel to ℓ?

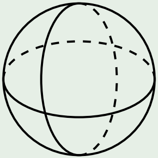
Example 5 (The Cartesian plane $\mathbb{R}^2$)

Given a line ℓ in $\mathbb{R}^2$ and a point (x, y) not on ℓ, there is a unique line m through (x, y) with the same slope as ℓ.



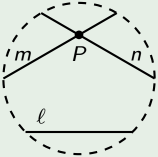
Thus, the Euclidean Parallel Postulate is true in the Cartesian plane.

Example 6 (The sphere $\mathbb{S}^2$)



Since any two great circles intersect on the sphere, then there are no parallel lines. Hence, $\mathbb{S}^2$ satisfies the Elliptic Parallel Postulate.

Example 7 (The Klein disk)



As seen in the picture above, the Klein disk satisfies the Hyperbolic Parallel Postulate.

Independence

Recall that a postulate is *independent* of the axioms in an axiomatic system if it holds in one model but not in another. This means that the postulate cannot be proved as a theorem from the other axioms.

- The Euclidean Parallel Postulate is true in $\mathbb{R}^2$ but not in three-point geometry. Hence, the Euclidean Parallel Postulate is independent of the axioms for incidence geometry.
- The Elliptic Parallel Postulate is true in the three-point geometry but not in four-point geometry. Hence, the Elliptic Parallel Postulate is independent of the axioms for incidence geometry.
- The Hyperbolic Parallel Postulate is true in the Klein disk but not in $\mathbb{R}^2$. Hence, the Hyperbolic Parallel Postulate is independent of the axioms for incidence geometry.

Axiomatic systems and the real world

The idea of an axiomatic system for geometry can be thought of as "abstractifying" the geometry we perceive in the physical world. The ancient Greeks were responsible for this shift from the study of the *physical* world to "ideal world of pure forms."

But making geometry, or any mathematical subject, abstract is not just an exercise in logic, it has real power. On one hand, it can make a subject easier to study by removing unnecessary information. There are many examples where applications developed long after the subject had been well-developed mathematically. The Egyptians and Babylonians certainly didn't have GPS in mind when they began to develop trigonometry.

Next time

Before next class: Read Section 2.5 and Chelsea Walton's proof tips.

In the next lecture we will:

- Review mathematical language and formality for proofs.
- Discuss the qualities of good proof writing.
- Prove some basic theorems in incidence geometry.

Chapter 2: Axiomatic systems and incidence geometry
§2.5 Theorems, proofs, and logic

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Foundations of Geometry

Goals for today:

- Review mathematical language and formality for proofs.
- Review components of logical arguments.
- Discuss the qualities of good proof writing.

Alongside learning the axiomatic foundations of geometry, this is also a course on good (and careful) proof writing. The goal in our writing is to be logically precise and so we want all statements to be clear and unambiguous.

Our first step is to make sure that our statements are written in a way to encourage a carefully laid out proof. Consider the following statement:

Proposition 1

Lines that are not parallel intersect in one point.

There is nothing particularly wrong with this statement, but it does not give us any guidance on how the proof should flow. The following statement is equivalent.

Restatement

If ℓ and m are two distinct lines that are not parallel, then there exists exactly one point P such that P lies on both ℓ and m.

This statement tells us how to start the proof ("Assume ℓ and m are two distinct lines that are not parallel.") and how to finish ("Hence, P lies on both ℓ and m.").

Statements

A *statement* is as assertion that is either true or false. This should be preceded by definitions so that all terms in the statement are defined.

Example 1

The statement "Math is hard" is not a valid statement because it is not clear what it means for a subject to be "hard". For this to be a valid statement, we would need a definition like "A subject X is said to be *hard* if I don't have the time to learn it."

Statements S and T can be combined to form a *compound statement* $S\square T$ where $\square$ is one of the following:

- *and*: S and T is true if both statements S and T are true.
- *or*: S or T is true if one (or both) of the statements is true.

The mathematical *or* should not be confused with the "exclusive or", which means *exactly one*.

The word *not* negates a statement. If statement S is true, then the statement *not* S is false. Conversely, if S is false, then *not* S is true. The logical rules for how negation plays with compound statements are known as *DeMorgan's Laws*.

DeMorgan's Laws

Let S and T be statements and let "$=$" denote logical equivalence. Then

- not (S and T) $=$ (not S) or (not T)
- not (S or T) $=$ (not S) and (not T)

Example 2

Consider the statements

$$S = \text{Dr. Gaddis is less than 40 years old}$$
$$T = \text{Dr. Gaddis is good at basketball.}$$

Statement S is true but T is false, so (S or T) is true while not(S or T) is false. Since (not S) is false and (not T) is true, then (not S) and (not T) is false.

A *propositional function* is an expression that becomes a statement after one or more unknowns (variables) have been replaced by values.

Example 3

- $P(x) = (x > 0)$. Then $P(0)$ is false but $P(1)$ is true.
- $Q(x, y) = (x > y)$. Then $Q(1, 0)$ is true but $Q(0, 1)$ is false.
- $P(\ell, m) = (\ell \parallel m)$. This is true if ℓ and m are parallel. It is false otherwise.

A *conditional statement* is a compound statement of the form "If P, then Q", where P is a statement called the *hypothesis* (antecedent) and Q is statement called the *conclusion* (consequent). The shorthand form is $P \Rightarrow Q$.

Notice!

All theorems should be stated as a conditional statement.

The statement "If P, then Q" means that Q is true if P is true.

Example 4

- If $x < 1$, then $x < 2$.
  Whenever $x < 1$, then it must be true that $x < 2$. Thus, this is a true conditional statement.
- If x is a real number and $x^2 < 0$, then $x = 3$.
  There are no x satisfying the hypothesis. Thus, the conditional statement is vacuously true.
- If x is irrational, then x^2 is irrational.
  This is false because $\sqrt{2}$ is irrational but $(\sqrt{2})^2 = 2$ is rational. Thus, this conditional statement is false. (The easiest way to show a conditional statement is false is by finding a counterexample.)

Example 5

Given a conditional statement $P \Rightarrow Q$ (ex. If $x = 2$, then $x^2 = 4$. *True!*),

- the *converse* is $Q \Rightarrow P$ (ex. If $x^2 = 4$, then $x = 2$. *False!*),
- the *inverse* is (not P) $\Rightarrow$ (not Q) (ex. If $x \neq 2$, then $x^2 \neq 4$. *False!*), and
- the *contrapositive* is (not Q) $\Rightarrow$ (not P) (ex. If $x^2 \neq 4$, then $x \neq 2$. *True!*).

The conditional statement and its contrapositive are always logically equivalent. The converse and inverses are always logically equivalent.

A true conditional statement $P \Rightarrow Q$ such that $Q \Rightarrow P$ is also true is called a *biconditional statement*. We write "P if and only if Q" (P iff Q, $P \Leftrightarrow Q$).

Example 6

The following conditional statement is true: "If $x = 0$, then $x^2 = 0$." Its converse is also true: "If $x^2 = 0$, then $x = 0$." This is a biconditional statement, so $x = 0$ if and only if $x^2 = 0$.

Truth tables

Truth tables are used to show logical equivalence between two conditional statements. If a conditional involves two propositional functions, then we first list all possibilities of true and false (T and F), the consider the outputs of the conditional statements. If their columns match up, then they are logically equivalent.

Example 7

Show $P \Rightarrow Q$ is equivalent to (not Q) $\Rightarrow$ (not P).

(Remember: a conditional statement is only false when a false implies a true.)

P	Q	$P \Rightarrow Q$	(not Q)	(not P)	(not Q) $\Rightarrow$ (not P)
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Quantifiers

The existential quantifier ($\exists$) asserts that something exists.

Example 8

There exists a point P such that P does not lie on ℓ. (Sufficient to give an example.)

The universal quantifier ($\forall$) asserts that some property holds for all objects in a certain class.

Example 9

For every point P not on ℓ, the distance from P to ℓ is positive. (This can be, and should be, rewritten as a conditional statement.)

If P is a point not on ℓ, then the distance from P to ℓ is positive.

(More) DeMorgan's Laws

Let $P(x)$ and $Q(x)$ be propositional functions and and let "$=$" denote logical equivalence. Then

- not $(\forall x \ P(x)) = \exists \ x(\text{not}P(x))$
- not $(\exists x \ P(x)) = \forall \ x(\text{not}P(x))$

By *unique*, we mean *exactly one*. This is used in conjunction with the existential quantifier: "There exists a unique..." The shorthand notation is $\exists!$.

Example 10

For every line ℓ and for every point P, there is (there exists) a unique line m such that P lies on m and m is perpendicular to ℓ.

A proof of such a statement will have two parts: existence and uniqueness. Uniqueness is commonly proved by assuming there are two objects with the property and showing they are the same.

Writing proofs

All proofs in this course should be written in **paragraph style** with justifications given in parentheses following each statement. Possible justifications include:

- by hypothesis
- by definition
- by axiom
- by rules of logic
- by previous theorem
- by algebra (see Appendix E)
- by properties of real numbers
- by earlier step in proof

Later, when we are comfortable with basic ideas of geometry, we can drop reasons for simple steps. Feel free to include additional steps that help the reader understand the structure, assumptions, and plan for the proof.

The proof should always begin with *Proof.* and end with $\square$ ("quod erat demonstrandum", translated: "that which was to be demonstrated"). Contrary to popular belief, QED *does not* stand for "quite easily done".

Direct vs Indirect Proof

In order to prove $P \Rightarrow Q$, we normally start by assuming P and deduce Q. On the other hand, we may assume P and not Q to arrive at a contradiction. This is known as *Proof by Contradiction*, but formally it is *reductio ad absurdum* (RAA).

Note, RAA is different than proving the contrapositive. That's just direct be the contrapositive is logically equivalent to the original statement. However, one should be clear when proving things this way, perhaps by including a line at the beginning of the proof indicating the strategy.

Warning: do not overuse RAA. Often it is not necessary. Also, be careful not to do a proof that is both direct and indirect. This often leads to contradictions.

Next time

Before next class: Read Section 2.6.

In the next lecture we will:

- We will begin proving basic theorems in incidence geometry.
- Practice formality in proofs and expectations in this class for proof writing.

Chapter 2: Axiomatic systems and incidence geometry
§2.6 Some theorems from incidence geometry

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- We will begin proving basic theorems in incidence geometry.
- Practice formality in proofs and expectations in this class for proof writing.

Let's begin by recalling the axioms of Incidence Geometry (IG).

Incidence Geometry Axioms

IA1) For every pair of distinct points P and Q, there exists exactly one line ℓ such that both P and Q lie on ℓ.

IA2) For every line ℓ, there exists at least two distinct points P and Q such that both P and Q lie on ℓ.

IA3) There exist three points that do not all lie on the same line.

An Intersection Theorem

We're almost ready to prove our first theorem in IG. First we need a definition.

Definition 1

Two lines are said to *intersect* if there exists a point that lies on both lines.

Theorem 2

Lines that are not parallel intersect in one point.

This theorem is not well-stated. We need to turn it into a conditional statement.

Theorem 2

If ℓ and m are distinct, non-parallel lines, then there exists a unique point P such that P lies on both ℓ and m.

This is better, but still not great because the fact that ℓ and m are lines is part of the hypothesis. Really, it is just part of the notation. So we'll try one more time.

Theorem 2

Let ℓ and m be two lines. If ℓ and m are distinct and non-parallel, then there exists a unique point P such that P lies on both ℓ and m.

Theorem 2

Let ℓ and m be two lines. If ℓ and m are distinct and non-parallel, then there exists a unique point P such that P lies on both ℓ and m.

Proof.

Let ℓ and m be two lines such that $\ell \neq m$ and $\ell \nparallel m$ (hypothesis). We must prove that there exists a point P that lies on both ℓ and m, and that it is unique.

There exists a point P that lies on ℓ and m (negation of the definition of parallel). It is left to prove uniqueness. Let Q be a different point that lies on ℓ and m (RAA hypothesis). Then ℓ is the unique line such that P and Q lie on ℓ (IA1). Similarly, m is the unique line such that P and Q lie on m (IA1). Hence, $\ell = m$ (definition of unique). This contradicts the hypothesis that ℓ and m are distinct so we reject the RAA hypothesis and conclude that no such point Q exists. $\square$

It is good practice to begin a proof by restating the hypotheses as well as to state what it is you intend to prove.

The converse

Another advantage of our final theorem statement is that it makes stating the hypothesis easier.

Converse to Theorem 2

Let ℓ and m be two lines. If there exists a unique point P such that P lies on both ℓ and m, then ℓ and m are distinct and non-parallel.

Proof.

Let ℓ and m be two lines such that there exists a unique point P such that P lies on both ℓ and m (hypothesis). We must prove that ℓ and m are distinct and non-parallel.

Suppose either $\ell = m$ or $\ell \parallel m$ (RAA hypothesis). If $\ell = m$, then there are at least two points lying on both ℓ and m (IA2). If $\ell \parallel m$, then there are no points lying on both ℓ and m (definition of parallel). In either case, we have a contradiction. Thus, we reject the RAA hypothesis and conclude that ℓ and m are distinct and non-parallel. □

Now we can combine our two results into a single biconditional statement.

Theorem 3

Let ℓ and m be two lines. Then ℓ and m are distinct and non-parallel if and only if there exists a unique point P such that P lies on both ℓ and m.

Points off lines

Theorem 4

If ℓ is any line, then there exists at least one point P such that P does not lie on ℓ.

It would be tempting here to just apply IA3. However, IA3 applies to *some* line and not the line ℓ in particular. However, we still want to use IA3 in our proof.

Proof.

Let ℓ be a line (hypothesis). We will prove that there exists a point P such that P does not lie on ℓ.

Suppose all points lie on ℓ (RAA hypothesis). Then all points are collinear (definition of collinear). This contradicts IA3. Thus we reject the RAA hypothesis and conclude that there exists a point P that does not lie on ℓ. □

We've used RAA multiple times so far and we will continue to do so because it is an effective strategy for proving distinctness. However, it's important to note that other parts of our proofs do not require this.

Theorem 5

If P is any point, then there are at least two distinct lines ℓ and m such that P lies on both ℓ and m.

Proof.

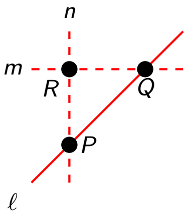
Let P be a point (hypothesis). We will show that there exists two distinct lines ℓ and m such that P lies on both ℓ and m.

Let Q be a point such that $Q \neq P$ (IA3) and let ℓ be the line through P and Q (IA1). Let R be a point not on ℓ (Theorem 4) and let m be the line through P and R (IA1). Then P lies on ℓ and m (definition of "lies on"). Since R lies on m but not on ℓ, then $\ell \neq m$ (definition of distinct). Thus, we have produced two distinct lines ℓ and m such that P lies on both ℓ and m. $\square$

Theorem 6

If ℓ is any line, then there exists lines m and n such that ℓ, m, and n are distinct.

Before proving this theorem, let's develop a strategy. First, we pick two distinct points on ℓ: P and Q. Then, using Theorem 4, we can choose a point R not on ℓ. Using IA1, this gives us the lines m and n that we need. The diagram is below:



Theorem 6

If ℓ is any line, then there exists lines m and n such that ℓ, m, and n are distinct.

Proof.

Let ℓ be a line (hypothesis). We will show that there exists lines m and n such that ℓ, m, and n are distinct.

There are two points lying on ℓ, say P and Q (IA2). There is a point, say R, not lying on ℓ (Theorem 4). Let m be the unique line through P and R and let n be the unique line through Q and R (IA1). It remains to prove that ℓ, m, and n are distinct.

Since R is not on ℓ, then $\ell \neq m$ and $\ell \neq n$ (definition of distinct). Suppose $m = n$ (RAA hypothesis). Then P, Q, and R are collinear, which contradicts the choice of R. Hence, we reject the RAA hypothesis and conclude that $m \neq n$. That is, the lines ℓ, m, and n are distinct. □

More Theorems (i.e., homework!)

The proofs of the following theorems are assigned as homework.

Theorem 7

If P is any point, the there exists at least one line ℓ such that P does not lie on ℓ.

Theorem 8

There exist three distinct lines such that no point lies on all three lines.

Theorem 9

If P is any point, then there exist points Q and R such that P, Q, and R are noncollinear.

Theorem 10

If P and Q are two points such that $P \neq Q$, then there exists a point R such that P, Q, and R are noncollinear.

Bonus theorem

Assuming the Euclidean parallel postulate, there are lots more things that we can prove. This next result is *transitivity of parallelism*.

Proposition 1

Let G be a model for IG that satisfies the Euclidean Parallel Postulate. If ℓ, m, and n are distinct lines such that $\ell \parallel m$, and $m \parallel n$, then $\ell \parallel n$.

Proof.

Let ℓ, m, and n be distinct lines such that $\ell \parallel m$ and $m \parallel n$ (hypothesis). We claim $\ell \parallel n$.

Suppose $\ell \nparallel n$ (RAA hypothesis). Then there exists a point P such that P lies on both ℓ and n (definition of parallel). Since both ℓ and n are parallel to m (hypothesis) and P does not lie on m (definition of parallel), then there exist 2 lines parallel to m through P, contradicting the Euclidean Parallel Postulate. Thus, we reject the RAA hypothesis and conclude $\ell \parallel n$. □

Next time

Before next class: Read Section 2.6.

In the next lecture we will:

- Introduce plane (neutral) geometry.
- Establish our undefined terms.
- State and discuss the Existence Postulate and Incidence Postulates.

Chapter 3: Axioms for plane geometry
§3.1 The undefined terms and two fundamental axioms
§3.2 Distance and Ruler Postulate

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Introduce plane (neutral) geometry.
- Establish our undefined terms.
- State and discuss the Existence Postulate and Incidence Postulates.

The goal of this chapter is to introduce the axioms necessary to define *neutral geometry*. Neutral geometry is, essentially, Euclidean geometry without assuming a particular parallel postulate. Though we are "starting over" in a sense, the intuition we built up in incidence geometry will be useful here.

Note that everything in this chapter relates to *plane geometry* and is not sufficient to develop geometry in three-dimensional space, which would require additional axioms.

Undefined Terms

The undefined terms for *neutral geometry* are

- point
- line
- distance
- half-plane
- angle measure
- area (not used right now).

Each of the first five undefined terms has an axiom that describes its nature. There is one additional axiom relating distance and angle measure.

The first axiom rules out trivial interpretations of the other axioms and guarantees that there are at least two points.

Axiom I (The Existence Postulate)

The collection of all points forms a nonempty set. There is more than one point in that set.

Definition 1

The set of all points is called *the plane* and is denoted by $\mathbb{P}$.

One consequence of the Existence Postulate is that it allows us to appeal to basic set theory in developing geometry. This is partially the reason that we did not define *lie on*. We will treat everything (points, lines, etc.) as elements or subsets of a set. Moreover, we will often use set-theoretic notation to describe geometric relationship (more on this soon).

Throughout, by saying that two points A and B are distinct we simply mean that they are not the same element in the set.

The second postulate is essentially Euclid's first postulate. The key difference is that he did not explicitly state that the line between two points in unique, but he clearly meant it to be (based on his proofs).

Axiom II (The Incidence Postulate)

Every line is a set of points. For every pair of distinct points A and B there is exactly one line ℓ such that $A \in \ell$ and $B \in \ell$.

Our notation for the unique line between A and B will be $\overleftrightarrow{AB}$.

A bunch of definitions

Definition 2

A point P is said to *lie on* a line ℓ if $P \in \ell$. The statements "P is incident with ℓ" and "ℓ is incident with P" are also used to indicate the same relationship.

Definition 3

A point Q is called an *external point* for a line ℓ if Q does not lie on ℓ.

In set notation, the previous definition says that $Q \notin \ell$.

Definition 4

Two lines ℓ and m are said to be *parallel* if there is no point P such that P lies on both ℓ and m.

Our notation for parallel is still $\ell \parallel m$. Note in set notation, parallel means $\ell \cap m = \emptyset$.

Under this definition of parallel, a line *is not* parallel to itself.

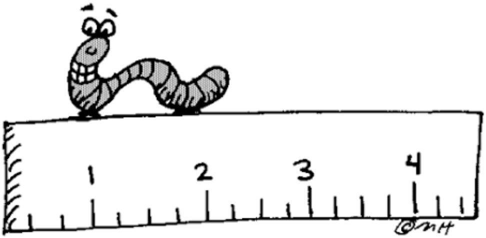
We've seen the following theorem before, in incidence geometry, and so the proof is still valid.

Theorem 5 (Theorem 3.1.7)

If ℓ and m are two distinct, nonparallel lines, then there exists exactly one point P such that P lies on both ℓ and m.

In light of this theorem, two arbitrary lines ℓ and m will always satisfy exactly one of the following:

- $\ell = m$,
- $\ell \parallel m$, or
- $\ell \cap m$ consists of exactly one point.



The third postulate explains what is meant by the undefined term *distance*.

Axiom III (The Ruler Postulate)

For every pair of points P and Q there exists a real number PQ, called the distance from P to Q. For each line ℓ there is a one-to-one correspondence from ℓ to $\mathbb{R}$ such that if P and Q are points on the line that correspond to the real numbers x and y, respectively, then $PQ = |x - y|$.

Whew! What this axiom says, essentially, is that on a line there is a way to put down a "ruler" and measure the distance from P to Q, and that distance will be a (nonnegative) real number. The axiom implies that we can match lines in our geometry to the real line $\mathbb{R}$ and that lines are continuous.

The next definition should be familiar.

Definition 6

Three points A, B, and C are *collinear* if there exists exactly one line ℓ such that A, B, and C all lie on ℓ. The points are *noncollinear* otherwise.

To say a point is between two other points is somewhat awkward right now. We will be able to simplify this next definition once we added more axioms.

Definition 7

Let A, B, and C be three distinct points. The point C is *between* A and B, written $A * C * B$, if A, B, and C are collinear and $AC + CB = AB$.

Note that this definition requires the three points to be distinct. Thus, we do not say that A is between A and B.

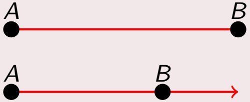
Using the notion of "between" we are now able to define line segments and rays in a set theoretic way.

Definition 8

Define the *segment* $\overline{AB}$ and the *ray* $\overrightarrow{AB}$ by

$$\overline{AB} = \{A, B\} \cup \{P : A * P * B\}$$

$$\overrightarrow{AB} = \overline{AB} \cup \{P : A * B * P\}$$



The points A and B are the *endpoints* of the segment $\overline{AB}$; all other points of $\overline{AB}$ are *interior* points. The point A is the *endpoint* of the ray $\overrightarrow{AB}$.

The *length* of the segment $\overline{AB}$ is AB, the distance from A to B. Two segments $\overline{AB}$ and $\overline{CD}$ are said to be *congruent*, written $\overline{AB} \cong \overline{CD}$, if they have the same length.

The next theorem details some of the properties of distance that are consequences of the Ruler Postulate. They may seem obvious, but keep in mind that they are not axioms of our geometry and cannot be taken for granted.

Theorem 9 (Theorem 3.2.7)

If P and Q are any two points, then (1) $PQ = QP$, (2) $PQ \geq 0$, and (3) $PQ = 0$ if and only if $P = Q$.

Proof.

Let P and Q be two points (hyp). We will show first that there exists a line ℓ such that P and Q both lie on ℓ. Either $P = Q$ or $P \neq Q$ (dichotomy). If $P \neq Q$, then there exists a (unique) line ℓ through P and Q (Incidence Postulate). If $P = Q$, then there exists a point $R \neq P$ (Existence Postulate) and we take ℓ to be the (unique) line through P and R (Incidence Postulate).

Next we prove $PQ = QP$ and $PQ \geq 0$. There exists a 1-1 correspondence from ℓ to $\mathbb{R}$ having the properties specified in the Ruler Postulate. In particular, P and Q correspond to real number x and y, respectively, such that $PQ = |x - y|$ and $QP = |y - x|$ (Ruler Postulate). As $|x - y| = |y - x|$ (algebra), then $PQ = QP$, proving (1). Since $|x - y| \geq 0$ (algebra), then $PQ \geq 0$, proving (2).

Finally, we prove (3). Suppose $PQ = 0$ (hypothesis), then $|x - y| = 0$, so $x = y$ (algebra) and hence $P = Q$ (correspondence is 1-1). Conversely, if $P = Q$, then $x = y$ (correspondence is a well-defined function), so $PQ = |x - y| = 0$. $\square$

Next time

Before next class: Finish reading Section 3.2.

In the next lecture we will:

- Use metrics to measure distance in various models.
- Use coordinate functions to restate the Ruler Postulate.

Chapter 3: Axioms for plane geometry
§3.2 Distance and Ruler Postulate

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Use metrics to measure distance in various models.
- Use coordinate functions to restate the Ruler Postulate.

First, let's recall a definition and theorem.

Definition 1

Let A, B, and C be three distinct points. The point C is *between* A and B, written $A * C * B$, if A, B, and C are collinear and $AC + CB = AB$.

Theorem 2 (Theorem 3.2.7)

If P and Q are any two points, then (1) $PQ = QP$, (2) $PQ \geq 0$, and (3) $PQ = 0$ if and only if $P = Q$.

Corollary 3 (Corollary 3.2.8)

Let A, B, and C be three points. Then $A * C * B$ if and only if $B * C * A$.

Proof.

Let A, B, and C be three points such that $A * C * B$ (hypothesis). Then $C \in \overleftrightarrow{AB}$ and $AC + CB = AB$ (definition of between). As $AB = BA$, $AC = CA$, and $BC = CB$ (Theorem 2), then $BC + CA = BA$ (properties of equality). Since $\overleftrightarrow{AB} = \overleftrightarrow{BA}$ (Incidence Postulate), then $B * C * A$ (definition of between). The proof of the converse is similar. $\square$

In the last lecture we discussed the concept of distance in neutral geometry (in the abstract). In this class we consider functions that measure distance in various models.

Definition 4

A *metric* is a function $D : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ such that

1. $D(P, Q) = D(Q, P)$ for every P and Q,
2. $D(P, Q) \geq 0$ for every P and Q, and
3. $D(P, Q) = 0$ if and only if $P = Q$.

The *triangle inequality*,

$$D(P, Q) \leq D(P, R) + D(R, Q),$$

is often included as part of the definition of a metric. In our context, however, it will be proved as a theorem from the other parts of the definition.

The first example should seem very familiar from high school algebra/geometry.

Example 5 (Euclidean metric)

Define the distance between points (x_1, y_1) and (x_2, y_2) in the Cartesian plane ($\mathbb{R}^2$) by the formula

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We will verify this is a metric.

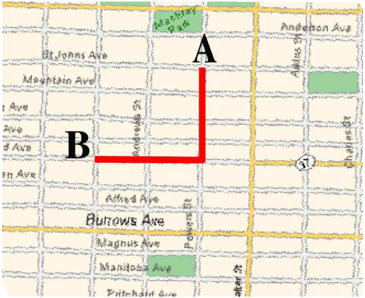
1. Since $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, then $d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$.

2. As the square of any real number is nonnegative, then the sum $(x_2 - x_1)^2 + (y_2 - y_1)^2$ is nonnegative. Hence, $d((x_1, y_1), (x_2, y_2)) \geq 0$.

3. If $d((x_1, y_1), (x_2, y_2)) = 0$, then $(x_2 - x_1)^2 + (y_2 - y_1)^2 = 0$, so $(x_2 - x_1)^2 = 0$ and $(y_2 - y_1)^2 = 0$. Thus, $x_1 = x_2$ and $y_1 = y_2$. Conversely, if $x_1 = x_2$ and $y_1 = y_2$ then clearly $d((x_1, y_1), (x_2, y_2)) = 0$.

It follows that d is a metric, called the *Euclidean metric*.

Imagine a taxicab traveling around Manhattan. The taxicab can travel in north-south or east-west, but not diagonally. Measuring distance in this way is the idea behind the next metric.



Imagine a taxicab traveling around Manhattan. The taxicab can travel in north-south or east-west, but not diagonally. Measuring distance in this way is the idea behind the next metric.

Example 6 (Taxicab metric)

Define the distance between points (x_1, y_1) and (x_2, y_2) in the Cartesian plane ($\mathbb{R}^2$) by the formula

$$\rho((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

This metric is called the *taxicab metric* (verifying this is indeed a metric is a homework exercise).

We can also measure distance on the sphere.

Example 7 (The spherical metric)

Let A and B be two points on the sphere ($\mathbb{S}^2$). Define $s(A, B)$ to be the length of the shortest arc of a great circle containing A and B (measured in radians). If A and B are antipodal, then $s(A, B) = \pi$. Note that $s(A, B) \leq \pi$ always.

By the Ruler Postulate, there exists a one-to-one correspondence $f : \ell \to \mathbb{R}$. (i.e., f is a function that is 1-1 and onto). Then for any two points P and Q on ℓ, $PQ = |f(P) - f(Q)|$. However, the postulate says nothing about what this function is, only that it exists.

Definition 8

Let ℓ be a line. A one-to-one correspondence $f : \ell \to \mathbb{R}$ such that $PQ = |f(P) - f(Q)|$ for every $P, Q \in \ell$ is called a *coordinate function* for the line ℓ. The number $f(P)$ is called the *coordinate* of the point P.

In $\mathbb{R}^2$, we can construct coordinate functions associated to our two metrics.

Example 9 (Euclidean metric)

Let ℓ be a non-vertical line in $\mathbb{R}^2$, so ℓ has the form $y = mx + b$. Define $f : \ell \to \mathbb{R}$ by

$$f(x, y) = x\sqrt{1 + m^2}.$$

We will verify that f is a coordinate function in this case. Let $P, Q \in \ell$ and write $P = (x_1, y_1)$, $Q = (x_2, y_2)$.

First we show that f is bijective. Suppose $f(P) = f(Q)$, then

$$x_1\sqrt{1 + m^2} = x_2\sqrt{1 + m^2}.$$

Since $\sqrt{1 + m^2} > 0$, then $x_1 = x_2$ (and hence $y_1 = y_2$). Thus, f is 1-1. Now if $r \in \mathbb{R}$, then choose the point

$$R = \left(\frac{r}{\sqrt{1 + m^2}}, \frac{mr}{\sqrt{1 + m^2}} + b \right).$$

We have $f(R) = r$, so f is onto.

Coordinate functions

In $\mathbb{R}^2$, we can construct coordinate functions associated to our two metrics.

Example 9 (Euclidean metric)

Let ℓ be a non-vertical line in $\mathbb{R}^2$, so ℓ has the form $y = mx + b$. Define $f : \ell \to \mathbb{R}$ by

$$f(x, y) = x\sqrt{1 + m^2}.$$

We will verify that f is a coordinate function in this case. Let $P, Q \in \ell$ and write $P = (x_1, y_1)$, $Q = (x_2, y_2)$.

Finally, we show that f is distance preserving according to the Euclidean metric. Note

$$1 + m^2 = 1 + \frac{(y_2 - y_1)^2}{(x_2 - x_1)^2} = \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{(x_2 - x_1)^2}$$

so we have

$$\begin{aligned} |f(P) - f(Q)| &= \left| x_1\sqrt{1 + m^2} - x_2\sqrt{1 + m^2} \right| = \left| (x_1 - x_2)\sqrt{1 + m^2} \right| \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d(P, Q). \end{aligned}$$

Coordinate functions

In $\mathbb{R}^2$, we can construct coordinate functions associated to our two metrics.

Example 10 (Euclidean metric)

Now suppose that ℓ is a vertical line in $\mathbb{R}^2$, so ℓ has the form $x = a$. Here we define the coordinate function $f : \ell \to \mathbb{R}$ by $f(a, y) = y$. Let $P, Q \in \ell$ and write $P = (a, y_1)$, $Q = (a, y_2)$.

First we show that f is bijective. Suppose $f(P) = f(Q)$, then $y_1 = y_2$, so $P = Q$. Thus, f is 1-1. Now if $r \in R$, then $f(a, r) = r$, so f is onto.

Finally, note that

$$|f(P) - f(Q)| = |y_1 - y_2| = |y_2 - y_1| = d(P, Q).$$

Example 11 (Taxicab metric)

Let ℓ be a non-vertical line in $\mathbb{R}^2$ with equation $y = mx + b$. Define $f : \ell \to \mathbb{R}$ by

$$f(x, y) = x(1 + |m|).$$

If ℓ is vertical with equation $x = a$, then f is given by $f(a, y) = y$. Verifying f is a coordinate function is a homework exercise.

There are no coordinate functions for lines in spherical geometry because distance has finite upper bound (π).

The second part of the Ruler Postulate can now be restated as: *For every line there exists a coordinate function.*

The Ruler Placement Postulate

When measuring an object with a ruler, we typically put one end of the object at 0 on the ruler rather than placing it arbitrarily and taking the difference between the two ruler marks. This is the idea behind the following theorem.

Theorem 12 (The Ruler Placement Postulate, Theorem 3.2.16)

For every pair of distinct points P and Q, there is a coordinate function $f : \overleftrightarrow{PQ} \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

Proof.

Fix two distinct points P and Q (hypothesis) and let $\ell = \overleftrightarrow{PQ}$ (Incidence Postulate). There exists a coordinate function $g : \ell \to \mathbb{R}$ (Ruler Postulate). Define $c = -g(P)$ and define $h : \ell \to \mathbb{R}$ by $h(X) = g(X) + c$ for every $X \in \ell$. Then h is a coordinate function (hw). Note that $h(P) = g(P) + (-g(P)) = 0$ (algebra). Now $h(Q) \neq 0$ (h is 1-1), so either $h(Q) > 0$ or $h(Q) < 0$ (trichotomy). If $h(Q) > 0$, then $h = f$ satisfies the conclusion. If $h(Q) < 0$, then set $f(X) = -h(X)$ for all $X \in \ell$. Then f is a coordinate function (hw). Then $f(P) = 0$ and $f(Q) > 0$, so the proof is complete. $\square$

Next time

Before next class: Read Section 3.3.

In the next lecture we will:

- Use coordinate functions to restate betweenness.
- State the Point Construction Postulate to choose points relative to fixed points.
- State the Plane Separation Postulate to give meaning to the term *half plane*.

Chapter 3: Axioms for plane geometry
§3.2 Distance and Ruler Postulate
§3.3 Plane Separation

MTH 411/511

Foundations of Geometry

Goals for today:

- Use coordinate functions to restate betweenness.
- State the Point Construction Postulate to choose points relative to fixed points.
- State the Plane Separation Postulate to give meaning to the term *half plane*.

Ruler Placement

First, we'll recall the Ruler Postulate and the Ruler Placement Postulate.

Axiom III (The Ruler Postulate)

For every pair of points P and Q there exists a real number PQ, called the distance from P to Q. For each line ℓ there is a one-to-one correspondence from ℓ to $\mathbb{R}$ such that if P and Q are points on the line that correspond to the real numbers x and y, respectively, then $PQ = |x - y|$.

Axiom III (The Ruler Postulate) *(simplified version)*

For every pair of points P and Q there exists a real number PQ, called the distance from P to Q. For each line ℓ, there exists a coordinate function $f : \ell \to \mathbb{R}$.

Theorem 1 (The Ruler Placement Postulate, Theorem 3.2.16)

For every pair of distinct points P and Q, there is a coordinate function $f : \overleftrightarrow{PQ} \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

This theorem says that coordinate functions should respect betweenness.

Theorem 2 (Betweenness Theorem for Points, Theorem 3.2.17)

Let ℓ be a line, let $A, B, C \in \ell$ be distinct, and let $f : \ell \to \mathbb{R}$ be a coordinate function. Then $A * C * B$ if and only if $f(A) < f(C) < f(B)$ or $f(B) < f(C) < f(A)$.

Proof.

Let ℓ be a line, let $A, B, C \in \ell$ be distinct, and let $f : \ell \to \mathbb{R}$ be a coordinate function (hypothesis). Suppose $f(A) < f(C) < f(B)$. Then

$$|f(C) - f(A)| + |f(B) - f(C)| = |f(B) - f(A)|. \qquad (\star)$$

Hence, $CA + BC = BA$, which implies $AC + CB = AB$. That is, C is between A and B. Now suppose C is between A and B. Then reversing the implications above, (**??**) holds. Moreover, it holds without the absolute values, for $f(C) - f(A)$ and $f(B) - f(A)$ have the same sign. Hence, the conclusion holds (algebra). $\square$

Corollary 3 (Corollary 3.2.18)

Let A, B, and C be three points such that B lies on $\overrightarrow{AC}$. Then $A * B * C$ if and only if $AB < AC$.

Proof.

This follows immediately from the previous theorem by noting that $AB < AC$ if and only if $f(A) < f(B) < f(C)$. □

Corollary 4 (Corollary 3.2.19)

If A, B, and C are three distinct collinear points, then exactly one of them lies between the other two.

Proof.

Let A, B, and C be three distinct collinear points (hypothesis). There is a coordinate function such that A, B, and C corresponding to real numbers x, y, and z, respectively (Ruler Postulate). These can be ordered from smallest to largest. By the Betweenness Theorem, this uniquely determines which point is between the other two. □

Corollary 5 (Corollary 3.2.20)

If A and B be two distinct points. If f is a coordinate function for $\ell = \overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B) > 0$, then $\overrightarrow{AB} = \{P \in \ell \mid f(P) \geq 0\}$.

Proof.

Let A and B be two distinct points and let f be a coordinate function for $\ell = \overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B) > 0$ (hypothesis). Note that f exists by the Ruler Placement Postulate. We will prove that $\overrightarrow{AB} = \{P \in \ell \mid f(P) \geq 0\}$ by showing that the two sets contain one another.

Let $P \in \ell$ such that $f(P) \geq 0$. If $f(P) = 0$, then $P = A$ (f is 1-1). Similarly, if $f(P) = f(B)$, then $P = B$. Otherwise, we either have $f(A) < f(P) < f(B)$ or $f(A) < f(B) < f(P)$. In the first case, $A * P * B$ (Betweenness Theorem for Points), so $P \in \overline{AB} \subset \overrightarrow{AB}$ (definition of segment/ray). Similarly, in the second case, $A * B * P$ so $P \in \overrightarrow{AB}$. Hence, $\{P \in \ell \mid f(P) \geq 0\} \subset \overrightarrow{AB}$. Showing $\overrightarrow{AB} \subset \{P \in \ell \mid f(P) \geq 0\}$ is similar. $\square$

Definition 6

Let A and B be two distinct points. The point M is call a *midpoint* of $\overline{AB}$ if M is between A and B and $AM = MB$.

The midpoint M is between A and B, so $AM + MB = AB$. So, $AM = \frac{1}{2}(AB) = MB$.

Theorem 7 (Existence and Uniqueness of Midpoints, Theorem 3.2.22)

If A and B are distinct points, then there exists a unique point M such that M is the midpoint of A and B.

Though this theorem can be proved independently, it will follow from a much more general fact that we prove in the next theorem. This next theorem will tell us that we can choose points of arbitrary length from any other point on our line.

Theorem 8 (Point Construction Postulate, 3.2.23)

If A and B are distinct points and d is any nonnegative real number, then there exists a unique point C such that C lies on $\overrightarrow{AB}$ and $AC = d$.

Proof.

Let A and B be distinct points and let $d \geq 0$ be a real number (hypothesis). Let $\ell = \overleftrightarrow{AB}$ (Incidence Postulate) and let $f : \ell \to \mathbb{R}$ be a coordinate function such that $f(A) = 0$ and $f(B) = x > 0$ (Ruler Placement Postulate).

First we prove existence. Choose $C \in \ell$ such that $f(C) = d$ (f is onto). Then $C \in \overrightarrow{AB}$ (Corollary 3.2.20). Then $AC = |f(A) - f(C)| = |0 - d| = d$ (f is a coordinate function).

Now we prove uniqueness. Let C' be a point on on $\overrightarrow{AB}$ and $AC' = d$ (hypothesis). Then

$$d = AC' = |f(A) - f(C')| = |0 - f(C')| = |f(C')|.$$

(f is a coordinate function). Then $f(C') = d$ (Corollary 3.2.20). Thus, $f(C') = f(C)$ and so $C' = C$ (f is 1-1). $\square$

Example 9 (The rational plane)

- point: (p, q) with $p, q \in \mathbb{Q}$.
- line: the set of points (p, q) with $p, q \in \mathbb{Q}$, that lie on a Cartesian line:

$$\ell = \{(r, s) : r, s \in \mathbb{Q}, ar + bs + c = 0\}.$$

We measure distance using the Euclidean metric. The rational plane satisfies the Existence and Incidence Postulates. It satisfies the first part of the Ruler Postulate, but not the second as there is no 1-1 correspondence from $\mathbb{Q}$ to $\mathbb{R}$.

Definition 10

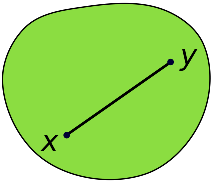
Give a point O and a positive real number r, the *circle* with *center O and radius r* is defined to be the set of all points P such that the distance from O to P is r.

Example 11

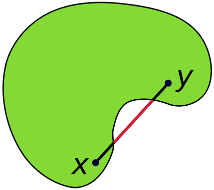
The rational plane satisfies all of Euclid's postulates, but fails his first proposition! Consider the rational points $A = (0, 0)$ and $B = (2, 0)$. The circles of radius 2 centered at A and B do not intersect in the rational plane. (Note: they do intersect at the points $(1, \pm\sqrt{3})$, but these points are not rational.)

Definition 12

A set of points S is sad to be a *convex set* if for every pair of points A and B in S, the entire segment $\overline{AB}$ is contained in S.



Convex Set



Not a convex set

Axiom IV (The Plane Separation Postulate)

For every line ℓ, the points that do not lie on ℓ form two disjoint, nonempty sets H_1 and H_2, called half-planes bounded by ℓ, such that the following conditions are satisfied:

1. Each of H_1 and H_2 is convex.
2. If $P \in H_1$ and $Q \in H_2$, then $\overline{PQ}$ intersects ℓ.

There's a lot baked into this postulate:

- $H_1 \cup H_2 = \mathbb{P} \backslash \ell$
- $H_1 \cap H_2 = \emptyset$
- $H_1 \neq \emptyset$ and $H_2 \neq \emptyset$
- If $A \in H_1$ and $B \in H_1$, then $\overline{AB} \subset H_1$ and $\overline{AB} \cap \ell = \emptyset$
- If $A \in H_2$ and $B \in H_2$, then $\overline{AB} \subset H_2$ and $\overline{AB} \cap \ell = \emptyset$
- If $A \in H_1$ and $B \in H_2$, then $\overline{AB} \cap \ell \neq \emptyset$

When given a line ℓ and an external point A, we denote the half-plane bounded by ℓ containing A by H_A.

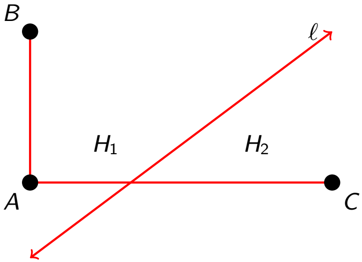
Sides

We can think of a line as dividing $\mathbb{P}$ into two "sides". We now define this formally.

Definition 13

Let ℓ be a line, let H_1 and H_2 be the two half-planes bounded by ℓ, and let A and B be two external points.

- We say that A and B are *on the same side of* ℓ if they are both in H_1 or both in H_2.
- We say that A and B are *on opposite sides of* ℓ if one is in H_1 and the other is in H_2.



The points A and B are on the same side of ℓ while A and C are on the opposite sides.

Next time

Before next class: Finish reading 3.3.

In the next lecture we will:

- Use half-planes to define angles.
- Discuss the Ray Theorem to explain betweenness for angles.
- Define triangles in plane geometry.

Chapter 3: Axioms for plane geometry
§3.3 Plane Separation

MTH 411/511

Foundations of Geometry

It's good to have goals

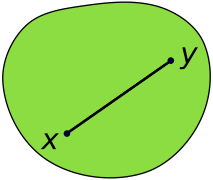
Goals for today:

- Use half-planes to define angles.
- Discuss the Ray Theorem to explain betweenness for angles.
- Define triangles in plane geometry.

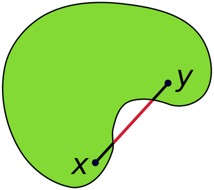
Convex Set

Definition 1

A set of points S is sad to be a *convex set* if for every pair of points A and B in S, the entire segment $\overline{AB}$ is contained in S.



Convex Set



Not a convex set

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There's a lot baked into this postulate:

- $H_1 \cup H_2 = \mathbb{P} \backslash \ell$
- $H_1 \cap H_2 = \emptyset$
- $H_1 \neq \emptyset$ and $H_2 \neq \emptyset$
- If $A \in H_1$ and $B \in H_1$, then $\overline{AB} \subset H_1$ and $\overline{AB} \cap \ell = \emptyset$
- If $A \in H_2$ and $B \in H_2$, then $\overline{AB} \subset H_2$ and $\overline{AB} \cap \ell = \emptyset$
- If $A \in H_1$ and $B \in H_2$, then $\overline{AB} \cap \ell \neq \emptyset$

When given a line ℓ and an external point A, we denote the half-plane bounded by ℓ containing A by H_A.

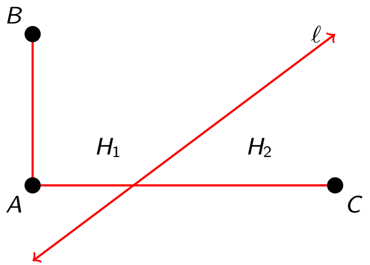
Sides

We can think of a line as dividing $\mathbb{P}$ into two "sides". We now define this formally.

Definition 2

Let ℓ be a line, let H_1 and H_2 be the two half-planes bounded by ℓ, and let A and B be two external points.

- We say that A and B are *on the same side of* ℓ if they are both in H_1 or both in H_2.
- We say that A and B are *on opposite sides of* ℓ if one is in H_1 and the other is in H_2.



The points A and B are on the same side of ℓ while A and C are on the opposite sides.

Opposite Rays

Using these terms we can restate the Plane Separation Postulate. The proof is immediate from the PSP.

Proposition 1 (Proposition 3.3.4)

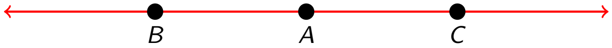
Let ℓ be a line and let A and B be points that do not lie on ℓ.

- *The points A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \emptyset$.*
- *The points A and B are on the opposite side of ℓ if and only if $\overline{AB} \cap \ell \neq \emptyset$.*

Definition 3

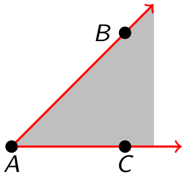
Two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with the same endpoint are *opposite rays* if the two rays are unequal but $\overleftrightarrow{AB} = \overleftrightarrow{AC}$. Otherwise they are *nonopposite*.

We restate this as: $\overrightarrow{AB}$ and $\overrightarrow{AB}$ are opposite if $B * A * C$.



Definition 4

An *angle* is the union of two nonopposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ sharing the same endpoint. The angle is denoted by either $\angle BAC$ or $\angle CAB$. The point A is called the *vertex* of the angle and rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called the *sides* of the angle.



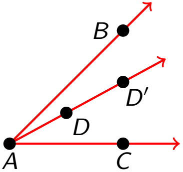
Definition 5

Let A, B, and C be three points such that the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are nonopposite. The *interior* of the angle $\angle BAC$ is defined as follows. If $\overrightarrow{AB} \neq \overrightarrow{AC}$, then the interior of $\angle BAC$ is defined to be the intersection of the half plane H_B (defined by B and $\overleftrightarrow{AC}$) and H_C (defined by C and $\overleftrightarrow{AB}$). If $\overrightarrow{AB} = \overrightarrow{AC}$, then the interior of $\angle BAC$ is defined to be $\emptyset$.

We can restate the definition of interior as follows: A point P is in the interior of $\angle BAC$ is it is on the same side of $\overleftrightarrow{AB}$ as C and on the same side of $\overleftrightarrow{AC}$ as B.

Definition 6

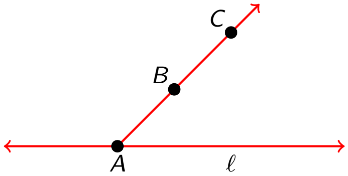
Ray $\overrightarrow{AD}$ is *between* $\overrightarrow{AB}$ and $\overrightarrow{AC}$ if D is in the interior of $\angle BAC$.



Our definition of betweenness depends on the choice of point D to define $\overrightarrow{AD}$. However, the definition should be independent of such a choice.

Theorem 7 (The Ray Theorem, Theorem 3.3.9)

Let ℓ be a line, A a point on ℓ and B an external point for ℓ. If C is a point on $\overrightarrow{AB}$ and $C \neq A$, then B and C are on the same side of ℓ.

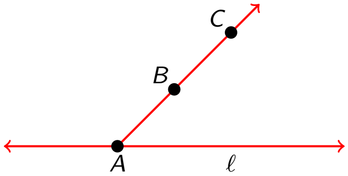


Proof.

Let ℓ be a line, let A be a point on ℓ, let B be an external point for ℓ, and let C be a point of $\overrightarrow{AB}$ that is different from A (hypothesis). We must prove that $\overline{BC} \cap \ell = \emptyset$ (definition of same side). There are two cases: either $A * C * B$ or $A * B * C$ (definition of ray). We will prove the first case. The second is similar.

Theorem 7 (The Ray Theorem, Theorem 3.3.9)

Let ℓ be a line, A a point on ℓ and B an external point for ℓ. If C is a point on $\overrightarrow{AB}$ and $C \neq A$, then B and C are on the same side of ℓ.



Proof.

Assume $A * C * B$. Then A is not between B and C (Corollary 3.19), so A is not on the segment $\overline{BC}$. The lines ℓ and $\overleftrightarrow{AB}$ have only one point in common (Theorem 3.1.7), and that point must be A, which is not on $\overline{BC}$, so $\overline{BC} \cap \ell = \emptyset$. □

Betweenness for rays

Theorem 8 (Theorem 3.3.10)

Let A, B, and C be three noncollinear points and let D be a point on the line $\overleftrightarrow{BC}$. The point D is between B and C if and only if the ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Proof.

Let A, B, and C be three noncollinear points and let D be a point on the line $\overleftrightarrow{BC}$ (hypothesis). Suppose that D is between B and C (hypothesis). Then C and D are on the same side of $\overleftrightarrow{AB}$ and B and D are on the same side of $\overleftrightarrow{AC}$ (The Ray Theorem). Hence, D is in the interior of $\angle BAC$ (definition of interior of angle) and $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (definition of betweenness for rays).

For the opposite implication, assume ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (hypothesis). Then D is in the interior of $\angle BAC$ (betweenness of rays). Therefore, B and D are on the same side of $\overleftrightarrow{AC}$ (definition of interior angle), so C is not on segment $\overline{BD}$ (Plane Separation Postulate). Similarly, B is not on segment $\overline{CD}$. Thus, B, C, and D are three collinear points such that B is not between C and D and C is not between B and D. It follows that D is between B and C (Corollary 3.2.19). □

Definition 9

Let A, B, and C be three noncollinear points. The *triangle* $\triangle ABC$ consists of the union of the three segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, that is,

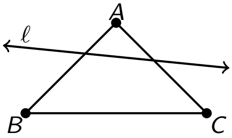
$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

The points A, B, and C are called the *vertices* of the triangle and the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called the *sides*.

The next result is called an axiom, but is really just a restatement of the Plane Separation Postulate.

Theorem 10 (Pasch's Axiom, Theorem 3.3.12)

Let $\triangle ABC$ be a triangle and let ℓ be a line such that none of A, B, and C lies on ℓ. If ℓ interesects $\overline{AB}$, then ℓ also intersects either $\overline{AC}$ or $\overline{BC}$.



Proof.

Let $\triangle ABC$ be a triangle and let ℓ be a line that intersects $\overline{AB}$ such that none of A, B, and C lies on ℓ (hypothesis). Let H_1 and H_2 be the two half-planes determined by ℓ (PSP). The points A and B are in opposite half-planes (Proposition 3.3.4). WLOG, we can assume $A \in H_1$ and $B \in H_2$. Then either $C \in H_1$ or $C \in H_2$ (PSP). In the first case, $\overline{BC} \cap \ell \neq \emptyset$, and in the second case, $\overline{AC} \cap \ell \neq \emptyset$ (PSP). $\square$

Next time

Before next class: Read Section 3.4.

In the next lecture we will:

- State the Protractor Postulate to define angle measure.
- State the Betweenness Theorem for Rays using angle measure.
- Define angle bisectors.

Chapter 3: Axioms for plane geometry
§3.4 Angle measure and the Protractor Postulate

MTH 411/511

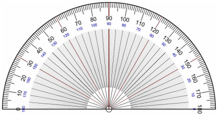
Foundations of Geometry

Goals for today:

- State the Protractor Postulate to define angle measure.
- State the Betweenness Theorem for Rays using angle measure.
- Define angle bisectors.

The Protractor Postulate

The Ruler Postulate gives meaning to the term *distance*. Similarly, the Protractor Postulate gives meaning the undefined term *angle measure*.



Axiom V (The Protractor Postulate)

For every angle $\angle BAC$ there is a real number $\mu(\angle BAC)$, called the *measure* of $\angle BAC$, such that the following conditions are satisfied.

1. $0^\circ \leq \mu(\angle BAC) < 180^\circ$ for every angle $\angle BAC$.
2. $\mu(\angle BAC) = 0^\circ$ if and only if $\overrightarrow{AB} = \overrightarrow{AC}$.
3. (Angle Construction Postulate) For each real number r, $0 < r < 180$, and for each half-plane H bounded by $\overleftrightarrow{AB}$ there eixsts a unique ray $\overrightarrow{AE}$ such that E is in H and $\mu(\angle BAE) = r^\circ$.
4. (Angle Addition Postulate) If the ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then
$$\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC).$$

Just as with segments, we have a notion of congruence

Definition 1

Two angles $\angle BAC$ and $\angle EDF$ are said to be *congruent*, written $\angle BAC \cong \angle EDF$ if $\mu(\angle BAC) = \mu(\angle EDF)$.

The following definitions should be familiar. Remember by definition an angle may not have angle measure greater than or equal to 180°.

Definition 2

Angle $\angle BAC$ is a

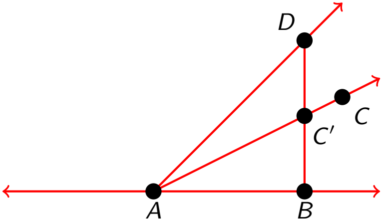
- *right angle* if $\mu(\angle BAC) = 90^\circ$,
- *acute angle* if $\mu(\angle BAC) < 90^\circ$, or
- *obtuse angle* if $\mu(\angle BAC) > 90^\circ$.

Just as we defined betweenness for points in terms of distance, we would like to establish betweenness for rays in terms of angle measure. The next major theorem does this, but we need some setup first.

Lemma 3 (Lemma 3.4.4)

If A, B, C, and D are four distinct points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overrightarrow{AC}$, then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.

This diagram demonstrates that if D is not in the interior of $\angle BAC$, then C is in the interior of $\angle BAD$.



Lemma 3 (Lemma 3.4.4)

If A, B, C, and D are four distinct points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overrightarrow{AC}$, then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.

Proof.

Let A, B, C, and D are four distinct points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overrightarrow{AC}$ (hypothesis). Assume D is not in the interior of $\angle BAC$ (hypothesis). We will prove that C is in the interior of $\angle BAD$.

Since C and D are on the same side of $\overleftrightarrow{AB}$ and D is not in the interior of $\angle BAC$ (hypothesis), then B and D lie on opposite sides of $\overleftrightarrow{AC}$ (negation of definition of angle interior). Thus, $\overline{BD} \cap \overleftrightarrow{AC} \neq \emptyset$ (Plane Separation Postulate). Let C' be the unique point at which $\overline{BD}$ intersects $\overleftrightarrow{AC}$. Since A, C, and C' are collinear, then either $\overrightarrow{AC} = \overrightarrow{AC'}$ or else they are opposite rays (dichotomy). We will show that they are not opposite rays.

Lemma 3 (Lemma 3.4.4)

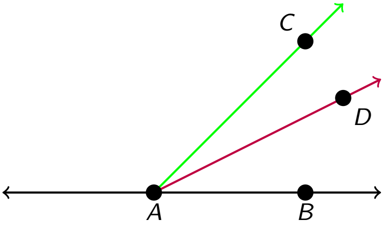
If A, B, C, and D are four distinct points such that C and D are on the same side of $\overleftrightarrow{AB}$ and D is not on $\overrightarrow{AC}$, then either C is in the interior of $\angle BAD$ or D is in the interior of $\angle BAC$.

Proof.

Since $C' \in \overline{BD}$, then C' is in the interior of $\angle BAD$ (Theorem 3.3.10). That is, D and C' lie on the same side of $\overleftrightarrow{AB}$. Since C and C' lie on the same side of $\overleftrightarrow{AB}$ as D, then A cannot be between C and C' (definition of between, PSP), so $\overrightarrow{AC}$ and $\overrightarrow{AC'}$ cannot be opposite rays (definition of opposite rays). Hence, $\overrightarrow{AC} = \overrightarrow{AC'}$. It follows that C is in the interior of $\angle BAD$ (The Ray Theorem). □

Theorem 4 (Betweenness Theorem for Rays, Theorem 3.4.5)

Let A, B, C, and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$. Then $\mu(\angle BAD) < \mu(\angle BAC)$ if and only if $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



Theorem 4 (Betweenness Theorem for Rays, Theorem 3.4.5)

Let A, B, C, and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$. Then $\mu(\angle BAD) < \mu(\angle BAC)$ if and only if $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Proof.

Let A, B, C, and D be four distinct points such that C and D lie on the same side of $\overleftrightarrow{AB}$ (hypothesis). Assume that $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (hypothesis). Then $\mu(\angle BAD) + \mu(\angle DAC) = \mu(\angle BAC)$ (Protractor Postulate pt 4) and $\mu(\angle DAC) > 0$ (Protractor Postulate pts 1 and 2), so $\mu(\angle BAD) < \mu(\angle BAC)$ (algebra).

For the reverse direction, we prove the contrapositive. Suppose $\overrightarrow{AD}$ is not between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (hypothesis). We claim $\mu(\angle BAD) \geq \mu(\angle BAC)$. If D lies on $\overrightarrow{AC}$, then $\mu(\angle BAD) = \mu(\angle BAC)$. Otherwise, C is in the interior of $\angle BAD$ (Lemma 3.4.4). Therefore, $\mu(\angle BAD) > \mu(\angle BAC)$ (by the above argument). □

Angle bisectors are essentially the equivalent of a midpoint but for angles in place of segments.

Definition 5

Let A, B, and C be three noncollinear points. A ray $\overrightarrow{AD}$ is an *angle bisector* of $\angle BAC$ if D is in the interior of $\angle BAC$ and $\mu(\angle BAD) = \mu(\angle DAC)$.

The proof of the next theorem is left as homework. To prove it, it might help to recall our discussion on midpoints.

Theorem 6 (Existence and uniqueness of angle bisectors, Theorem 3.4.7)

If A, B, and C are three noncollinear points, then there exists a unique angle bisector for $\angle BAC$.

Next time

Note: Exam 1 will cover material only through 3.4.

Before next class: Read Section 3.5.

In the next lecture we will:

- Study some of the consequences of the axioms we have so far defined and consider the relationship between points, lines, rays, and angles.
- Prove the Z-Theorem and Crossbar Theorem.
- Define a linear pair.

Chapter 3: Axioms for plane geometry
§3.5 The Crossbar Theorem and Linear Pair Theorem

MTH 411/511

Foundations of Geometry

It's good to have goals

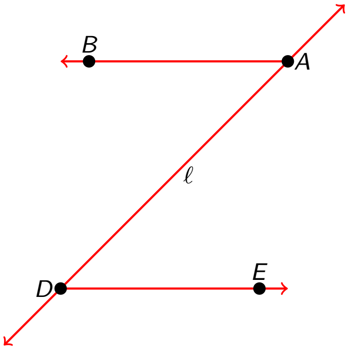
Goals for today:

- Study some of the consequences of the axioms we have so far defined and consider the relationship between points, lines, rays, and angles.
- Prove the Z-Theorem and Crossbar Theorem.

Remember: Exam 1 will cover material only through 3.4.

We come now to our first truly non-trivial results. These results are sometimes taken to be axioms but can in fact be proven based on our axioms so far.

The first result is the Z-Theorem. It *basically* says that rays in different half planes never intersect. This is reasonably obvious but only because we have already proven the Ray Theorem.



Theorem 1 (The Z-Theorem, Theorem 3.5.1)

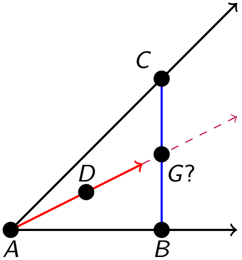
Let ℓ be a line and let A and D be distinct points on ℓ. If B and E are points on opposite sides of ℓ, then $\overrightarrow{AB} \cap \overrightarrow{DE} = \emptyset$.

Proof.

Let ℓ be a line and let A and D be distinct points on ℓ (hypothesis). Denote the two half-planes formed by ℓ by H_B and H_E. The points $P \in \overrightarrow{AB}$ such that $P \neq A$ lie in H_B and the points $Q \in \overrightarrow{DE}$ such that $Q \neq D$ lie in H_E (The Ray Theorem). Since $H_B \cap H_E = \emptyset$ (PSP), then the rays could only intersect at the endpoints. But these points are distinct by hypothesis. □

The Z-Theorem is essentially a lemma for the Crossbar Theorem.

The Crossbar Theorem is really a result about triangles. It says that a ray that is interior to an angle intersects the opposite side of the triangle "formed" by that angle.



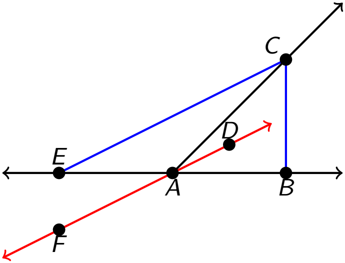
Given $\angle BAC$ and $\overrightarrow{AD}$, does the point G exist?

Theorem 2 (The Crossbar Theorem, Theorem 3.5.2)

Let $\triangle ABC$ be a triangle. If D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.

Proof.

Let $\triangle ABC$ be a triangle and let D be a point in the interior of $\angle BAC$ (hypothesis). We claim there exists a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$. Choose points E and F such that $E * A * B$ and $F * A * D$ (Ruler Postulate) and let $\ell = \overleftrightarrow{AD}$.



Theorem 2 (The Crossbar Theorem, Theorem 3.5.2)

Let $\triangle ABC$ be a triangle. If D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.

Proof.

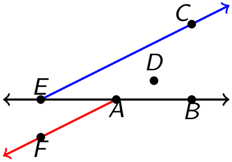
Since D is in the interior of $\angle BAC$, then $B, C \notin \ell$ (definition of angle interior and Incidence Postulate). Then ℓ intersects $\overline{EC}$ or $\overline{BC}$ (Pasch's Axiom applied to $\triangle EBC$). We claim $\overrightarrow{AD}$ (and not $\overrightarrow{AF}$) intersects $\overline{BC}$. It then suffices to show that

- $\overrightarrow{AF} \cap \overline{EC} = \emptyset$,
- $\overrightarrow{AF} \cap \overline{BC} = \emptyset$, and
- $\overrightarrow{AD} \cap \overline{EC} = \emptyset$.

We accomplish each of these by applying the Z-Theorem.

Theorem 2 (The Crossbar Theorem, Theorem 3.5.2)

Let $\triangle ABC$ be a triangle. If D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.

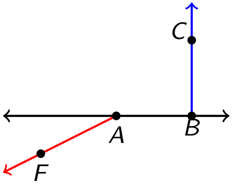


Proof.

Because A is between F and D, then F and D lie on opposite sides of $\overleftrightarrow{AB}$ (Plane Separation Postulate). Moreover, C and D are on the same side of $\overleftrightarrow{AB}$ (definition of interior angles), so C and F are on opposite sides of $\overleftrightarrow{AB}$ (Plane Separation Postulate). Thus, $\overrightarrow{EC} \cap \overrightarrow{AF} = \emptyset$ (Z-Theorem). Since $\overline{EC} \subset \overrightarrow{EC}$ (definition of ray), then $\overline{EC} \cap \overrightarrow{AF} = \emptyset$.

Theorem 2 (The Crossbar Theorem, Theorem 3.5.2)

Let $\triangle ABC$ be a triangle. If D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.

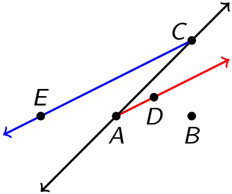


Proof.

By the above, C and F are on opposite sides of $\overleftrightarrow{AB}$. Thus, $\overrightarrow{BC} \cap \overrightarrow{AF} = \emptyset$ (Z-Theorem), so $\overline{BC} \cap \overrightarrow{AF} = \emptyset$.

Theorem 2 (The Crossbar Theorem, Theorem 3.5.2)

Let $\triangle ABC$ be a triangle. If D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.



Proof.

Since A is between E and B, then E and B lie on opposite sides of $\overleftrightarrow{AC}$ (Plane Separation Postulate). But B and D are on the same side of $\overleftrightarrow{AC}$ (definition of angle interior), so E and D are on opposite sides of $\overleftrightarrow{AC}$ (Plane Separation Postulate). Thus, $\overrightarrow{CE} \cap \overrightarrow{AD} = \emptyset$ (Z-Theorem), so $\overline{CE} \cap \overrightarrow{AD} = \emptyset$.

We conclude that $\overrightarrow{AD} \cap \overline{BC} \neq \emptyset$. □

The next result combines the Crossbar Theorem and Theorem 3.3.10 into one result.

Theorem 3

A point D is in the interior of the angle $\angle BAC$ if and only if the ray $\overrightarrow{AD}$ is in the interior of the segment $\overline{BC}$.

Proof.

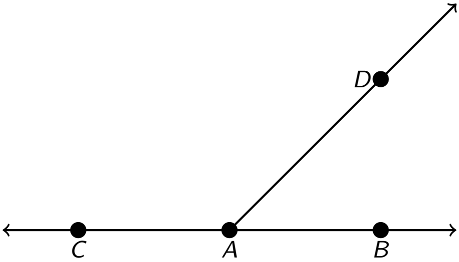
Suppose D is in the interior of $\angle BAC$ (hypothesis). Then $\overrightarrow{AD}$ intersects the interior of $\overline{BC}$ (Crossbar Theorem). Next, suppose $\overrightarrow{AD}$ intersects the interior of $\overline{BC}$, say at point E (hypothesis). Then E is in the interior of $\angle BAC$ (Theorem 3.3.10). Hence, D is in the interior of $\angle BAC$ (Ray Theorem). □

It is possible to take the above result as an axiom in place of the Plane Separation Postulate.

The next result could also be taken as an axiom.

Definition 4

Two angles $\angle BAD$ and $\angle DAC$ form a *linear pair* if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.



Theorem 5 (Linear Pair Theorem, Theorem 3.5.6)

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180^\circ$.

Next time

Before next class: Finishing reading Section 3.5.

In the next lecture we will:

- Define a linear pair and prove the Linear Pair Theorem.
- Define perpendicular lines and perpendicular bisectors.
- Define vertical angles and state the Vertical Angles Theorem.

Chapter 3: Axioms for plane geometry
§3.5 The Crossbar Theorem and Linear Pair Theorem

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Define a linear pair and prove the Linear Pair Theorem.
- Define perpendicular lines and perpendicular bisectors.
- Define vertical angles and state the Vertical Angles Theorem.

Remember: Exam 1 will cover material only through 3.4.

Last time

Last time we proved two important theorems.

Theorem 1 (The Z-Theorem, Theorem 3.5.1)

Let ℓ be a line and let A and D be distinct points on ℓ. If B and E are points on opposite sides of ℓ, then $\overrightarrow{AB} \cap \overrightarrow{DE} = \emptyset$.

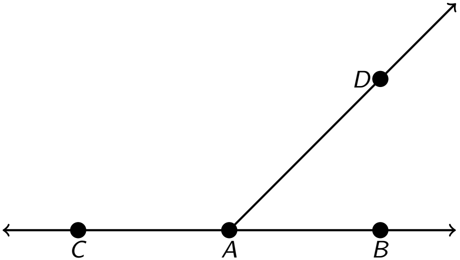
Theorem 2 (The Crossbar Theorem, Theorem 3.5.2)

Let $\triangle ABC$ be a triangle. If D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.

Linear pairs

Definition 3

Two angles $\angle BAD$ and $\angle DAC$ form a *linear pair* if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.



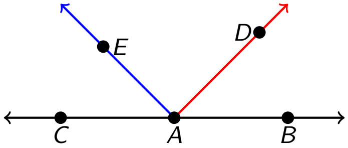
Linear Pair Theorem, Theorem 3.5.6

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180^\circ$.

Before proving the Linear Pair Theorem, we first need a lemma.

Lemma 4 (Lemma 3.5.7)

If $C * A * B$ and D is in the interior of $\angle BAE$, then E is in the interior of $\angle DAC$.



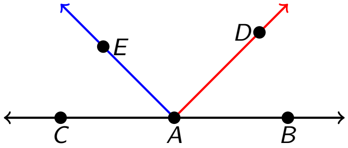
Proof.

Let A, B, C, D, and E be five points such that $C * A * B$ and D is in the interior of $\angle BAE$ (hypothesis). We claim that E is in the interior of $\angle DAC$. It suffices to show that E and D are on the same side of $\overleftrightarrow{AC}$, and that E and C are on the same side of $\overleftrightarrow{AD}$ (definition of angle interior).

Before proving the Linear Pair Theorem, we first need a lemma.

Lemma 4 (Lemma 3.5.7)

If $C * A * B$ and D is in the interior of $\angle BAE$, then E is in the interior of $\angle DAC$.



Proof.

Since D is in the interior of $\angle BAE$, then E and D are on the same side of $\overleftrightarrow{AB}$ (definition of angle interior). But $\overleftrightarrow{AB} = \overleftrightarrow{AC}$ (definition of between and collinear), so E and D are on the same side of $\overleftrightarrow{AC}$. Moreover, $\overrightarrow{AD}$ intersects $\overline{BE}$ (Crossbar Theorem), so E and B lie on opposite sides of $\overleftrightarrow{AD}$ (PSP). Since A is between C and B, then C and B are on opposite sides of $\overleftrightarrow{AD}$. Hence, C and E are on the same side of $\overleftrightarrow{AD}$. $\square$

The Linear Pair Theorem

Linear Pair Theorem, Theorem 3.5.6

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180^\circ$.

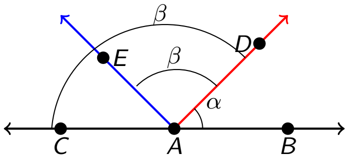
Proof.

Let $\angle BAD$ and $\angle DAC$ be a linear pair. Then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays (definition of linear pair). Set $\alpha = \mu(\angle BAD)$ and $beta = \mu(\angle DAC)$. Then either $\alpha + \beta < 180$, $\alpha + \beta > 180$, or $\alpha + \beta = 180$ (trichotomy). We show that the first two possibilities are impossible.

Linear Pair Theorem, Theorem 3.5.6

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180°$.

Strategy for the case $\alpha + \beta < 180$: Construct a point E such that $\mu(\angle BAE) = \alpha + \beta$. Show that this forces $\mu(\angle EAC) = 0$.



Linear Pair Theorem, Theorem 3.5.6

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180°$.

Proof.

Suppose $\alpha + \beta < 180$ (hypothesis). There is a point E on the same side of $\overleftrightarrow{AB}$ as D, such that $\mu(\angle BAE) = \alpha + \beta$ (Angle Construction Postulate). Hence, D is in the interior of $\angle BAE$ (Betweenness Theorem for Rays). Therefore,

$$\mu(\angle BAD) + \mu(\angle DAE) = \mu(\angle BAE)$$

(Angle Addition Postulate) and so $\mu(\angle DAE) = \beta$ (algebra). It follows that E is in the interior of $\angle DAC$ (Lemma 3.5.7), so

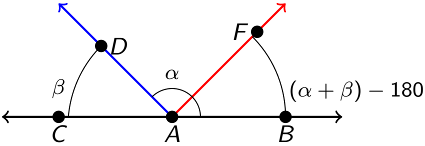
$$\mu(\angle DAE) + \mu(\angle EAC) = \mu(\angle DAC)$$

(Angle Addition Postulate). Thus, $\mu(\angle EAC) = 0$ (algebra), which contradicts the Protractor Postulate.

Linear Pair Theorem, Theorem 3.5.6

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180^\circ$.

Strategy for the case $\alpha + \beta > 180$: Construct a point F such that $\mu(\angle BAF) = \alpha + \beta - 180$. Show that this forces $\mu(\angle FAC) = 180$.



The Linear Pair Theorem

Linear Pair Theorem, Theorem 3.5.6

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180^\circ$.

Proof.

Suppose $\alpha + \beta > 180$ (hypothesis). There is a point F on the same side of $\overleftrightarrow{AB}$ as D, such that $\mu(\angle BAF) = (\alpha + \beta) - 180$ (Angle Construction Postulate). Since $\beta < 180$ (Protractor Postulate), then $\alpha + \beta - 180 < \alpha$ (algebra). Hence, F is in the interior of $\angle BAD$ (Betweenness Theorem for Rays). Therefore,

$$\mu(\angle BAF) + \mu(\angle FAD) = \mu(\angle BAD)$$

(Angle Addition Postulate) and so $\mu(\angle FAD) = 180 - \beta$ (algebra). It follows that D is in the interior of $\angle FAC$ (Lemma 3.5.7), so

$$\mu(\angle FAD) + \mu(\angle DAC) = \mu(\angle FAC)$$

(Angle Addition Postulate). Thus, $\mu(\angle FAC) = 180$ (algebra), which contradicts the Protractor Postulate. $\square$

Definition 5

Two angles $\angle BAC$ and $\angle DEF$ are *supplementary* (or *supplements*) if $\mu(\angle BAD) + \mu(\angle DAC) = 180^\circ$.

Hence, we could restate the Linear Pair Theorem as the following.

Theorem 6 (Linear Pair Theorem, Theorem 3.5.6)

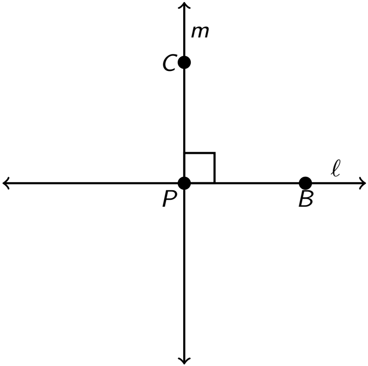
If two angles form a linear pair, then they are supplements.

Perpendicular Lines

Definition 7

Two lines ℓ and m are *perpendicular* if there exists a point A that lies on both ℓ and m and there exist points $B \in \ell$ and $C \in m$ such that $\angle BAC$ is a right angle.

Our notation for perpendicular lines is $\ell \perp m$.



Theorem 8 (Theorem 3.5.9)

If ℓ is a line and P is a point on ℓ, then there exists exactly one line m such that P lies on m and $m \perp \ell$.

Proof.

Let ℓ be a line and P a point on ℓ (hypothesis). We claim there exists exactly one line m such that P lies on m and $m \perp \ell$.

First we prove existence. Let B be point on ℓ distinct from P (Ruler Postulate). Fix a half-plane H bounded by ℓ. There exists a (unique) ray $\overrightarrow{PC}$ in H such that $\mu(\angle BPC) = 90^\circ$ (Angle Construction Postulate). That is, $\angle BPC$ is a right angle (definition of a right angle). Then ℓ and $\overleftrightarrow{PC}$ are perpendicular (definition of perpendicular).

Now we prove uniqueness. Let m be a line such that $\ell \perp m$. Choose a point $D \in m$ in H (Plane Separation Postulate). Then $\angle BPD$ is a right angle (definition of perpendicular) so $\mu(\angle BPD) = 90^\circ$ (definition of a right angle). Hence, $\overrightarrow{PD} = \overrightarrow{PC}$ (Angle Construction Postulate) so $m = \overleftrightarrow{PD} = \overleftrightarrow{PC}$ (Incidence Postulate).

Definition 9

Let D and E be two distinct points. A *perpendicular bisector* of $\overline{DE}$ is a line n such that the midpoint of $\overline{DE}$ lies on n and $n \perp \overleftrightarrow{DE}$.

Theorem 10 (Existence and Uniqueness of Perpendicular Bisectors, Theorem 3.5.11)

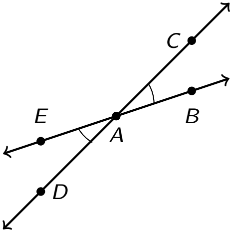
If D and E are two distinct points, then there exists a unique perpendicular bisector for $\overline{DE}$.

Vertical Angles

Definition 11

Angles $\angle BAC$ and $\angle DAE$ form a *vertical pair* (or are *vertical angles*) if one of the following is satisfied:

- rays $\overrightarrow{AB}$ and $\overrightarrow{AE}$ are opposite rays, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are opposite rays,
- rays $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays, $\overrightarrow{AC}$ and $\overrightarrow{AE}$ are opposite rays.



Theorem 12 (Vertical Angles Theorem, Theorem 3.5.13)

Vertical angles are congruent.

Next time

Before next class: Reading Section 3.6.

In the next lecture we will:

- Briefly discuss the Continuity Axiom.
- State the Side-Angle-Side Postulate.
- Define isosceles triangles and prove the Isosceles Triangle Theorem.

Chapter 3: Axioms for plane geometry
§3.6 Side-Angle-Side Postulate

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Briefly discuss the Continuity Axiom.
- State the Side-Angle-Side Postulate.
- Define isosceles triangles and prove the Isosceles Triangle Theorem.

Last time

In Section 3.5 we proved two important theorems.

The Crossbar Theorem (Theorem 3.5.2)

Let $\triangle ABC$ be a triangle. If D is a point in the interior of $\angle BAC$, then there is a point G such that G lies on both $\overrightarrow{AD}$ and $\overline{BC}$.

Definition 1

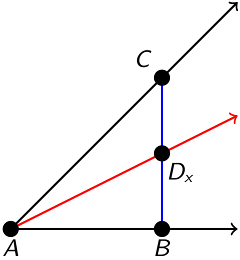
Two angles $\angle BAD$ and $\angle DAC$ form a *linear pair* if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.

Linear Pair Theorem (Theorem 3.5.6)

If angles $\angle BAD$ and $\angle DAC$ form a linear pair, then $\mu(\angle BAD) + \mu(\angle DAC) = 180^\circ$.

The next result relates angle measure and distance. We will continue studying this relationship in Section 3.6.

Let A, B, and C be three nonocollinear points. For each point D on $\overline{BC}$ there is an angle $\angle CAD$ and there is a distance CD. Let $d = BC$. By the Ruler Placement Postulate, there is a 1-1 correspondence (bijection) from the interval $[0, d]$ to points on $\overline{BC}$ such that C corresponds to 0 and B corresponds to d. Let D_x be the point on $\overline{BC}$ corresponding to x. Define a function $f : [0, d] \to [0, \mu(\angle CAB)]$ by $f(x) = \mu(\angle CAD_x)$.



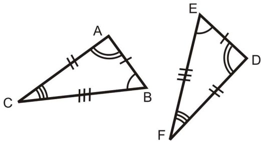
The Continuity Axiom (Theorem 3.5.15)

The function f described above is continuous, as is the inverse of f.

Now that we had developed our undefined terms through the first five postulates, we will explore the relationships between them. This will allow us to state our final postulate: *The Side-Angle-Side Postulate*.

Definition 2

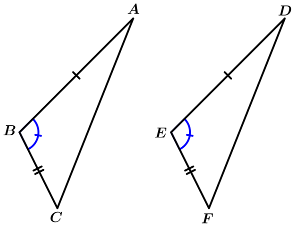
Two triangles are *congruent* if there is a correspondence between the vertices of the first triangle and the second triangle such that the corresponding angles are congruent and the corresponding sides are congruent.



Notation

We write $\triangle ABC \cong \triangle DEF$ to mean that the two triangles are congruent *and* the correspondence is given by $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$. This implies, for example, that $\overline{AB} \cong \overline{DE}$ and $\angle ACB \cong \angle DFE$.

In high school geometry you almost certainly learned about the Side-Angle-Side Congruence Condition.



Axiom 6 (The Side-Angle Side Postulate)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

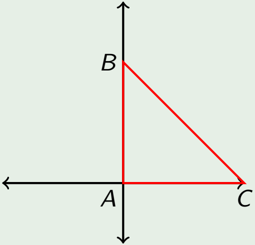
We typically refer to this axiom simply as SAS.

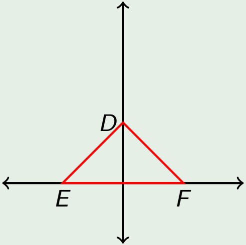
SAS - an example

The SAS Postulate is independent of the other axioms.

Example 3

Consider triangle $\triangle ABC$ in the Cartesian plane with coordinates $A = (0, 0)$, $B = (0, 2)$, and $C = (2, 0)$. Consider triangle $\triangle DEF$ with coordinates $D = (0, 1)$, $E = (-1, 0)$, and $F = (1, 0)$.



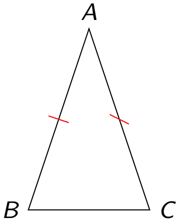


With the taxicab metric, $AB = AC = DE = DF = 2$. Moreover, $\angle BAC$ and $\angle EDF$ are right angles. That is, these triangles satisfy the hypotheses of SAS. However, $\triangle ABC \ncong \triangle DEF$ because $BC = 4$ while $EF = 2$.

Now we illustrate a theorem we can prove using SAS.

Definition 4

A triangle is called *isosceles* if it has a pair of congruent sides. The two angles not included between the congruent sides are called *base angles*.



The angles $\angle ABC$ and $\angle ACB$ are the base angles.

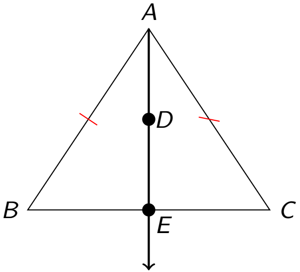
Isosceles Triangle Theorem (Theorem 3.6.5)

The base angles of an isosceles triangle are congruent.

Isosceles Triangle Theorem (restatement)

If $\triangle ABC$ is a triangle and $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.

One way to prove this theorem is to apply the Crossbar Theorem.



Isosceles Triangle Theorem (restatement)

If $\triangle ABC$ is a triangle and $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.

Proof.

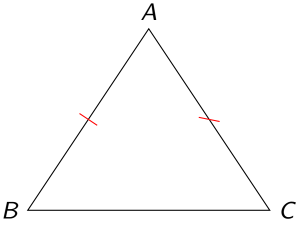
Let $\triangle ABC$ be a triangle such that $\overline{AB} \cong \overline{AC}$. We claim $\angle ABC \cong \angle ACB$.

Let $\overrightarrow{AD}$ be the angle bisector of $\triangle ABC$, so D is in the interior of $\angle BAC$ (existence and uniqueness of angle bisectors). There is a point E at which $\overrightarrow{AD}$ itersects $\overline{BC}$ (Crossbar Theorem). Then $\mu(\angle BAE) = \mu(\angle CAE)$ (definition of angle bisector), so $\angle BAE \cong \angle CAE$ (definition of angle congruence). Moreover, $\overline{AE} \cong \overline{AE}$ (tautology). Hence, $\triangle BAE \cong \triangle CAE$ (SAS). Thus, $\angle ABE \cong \angle ACE$ (definition of congruent triangle). Since $\angle ABE = \angle ABC$ and $\angle ACE = \angle ACB$ (definition of angles), then $\angle ABC \cong \angle ACB$. $\square$

Isosceles Triangle Theorem (restatement)

If $\triangle ABC$ is a triangle and $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.

Or we could prove this more simply using just SAS.



Proof.

Let $\triangle ABC$ be a triangle such that $\overline{AB} \cong \overline{AC}$. We claim $\angle ABC \cong \angle ACB$. Since $\overline{BA} \cong \overline{CA}$ (hyp), $\angle BAC \cong \angle CAB$ (tautology), and $\overline{AC} \cong \overline{AB}$ (hyp), then $\triangle BAC \cong \triangle CAB$ (SAS). Thus, $\angle ABC \cong \angle ACB$ (definition of congruent triangle). $\square$

Next time

Before next class: Read intro to Chapter 4 and Section 4.1.

In the next lecture we will:

- Introduce neutral geometry.
- Define exterior angles and prove the Exterior Angles Theorem.

Chapter 4: Neutral geometry
§3.7 The Parallel Postulates and Models
§4.1 The Exterior Angle Theorem and Existence of Perpendiculars

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Introduce neutral geometry.
- Define exterior angles and prove the Exterior Angles Theorem.

Let's recall our three parallel postulates.

Euclidean Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there is exactly one line m such that P lies on m and $m \parallel \ell$.

Elliptic Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there is no line m such that P lies on m and $m \parallel \ell$.

Hyperbolic Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there are at least two lines m and n such that P lies on both m and n and both m and n are parallel to ℓ.

In neutral geometry (essentially what we have defined so far), we will prove that parallel lines exist, so the Elliptic Parallel Postulate is inconsistent with the axioms of neutral geometry. However, we will show that both the Euclidean and the Hyperbolic Parallel Postulates are consistent.

The parallel postulates and models

Which models are consistent with our axioms?

- finite geometries (fail the Ruler Postulate)
- the sphere $\mathbb{S}^2$ (fails the Incidence and Ruler Postulates)
- the rational plane (fails the Ruler Postulate)
- the Cartesian plane $\mathbb{R}^2$ with the taxicab metric (fails SAS)
- the Cartesian plane $\mathbb{R}^2$ with the Euclidean metric (model with the Euclidean Parallel Postulate)
- the Klein disk (model with the Hyperbolic Parallel Postulate) (what's the metric???)

In the next chapter we will begin to develop neutral geometry. In later chapters we develop Euclidean Geometry ($\mathbb{R}^2$ + Euclidean metric) and Hyperbolic Geometry (Klein disk + some metric).

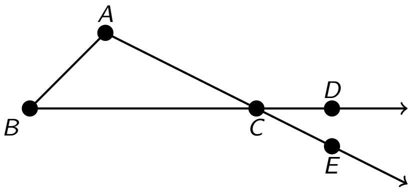
Neutral geometry is the geometry (axiomatic system) with five undefined terms: *point*, *line*, *distance*, *half-plane*, and *angle measure*, together with the following axioms:

- The Existence Postulate
- The Incidence Postulate
- The Ruler Postulate
- The Plane Separation Postulate
- The Protractor Postulate
- The Side-Angle-Side Postulate

We will not assume any model and we will remain *neutral* on the parallel postulates (Euclidean vs Hyperbolic). Our goal in this chapter will be to prove the propositions in Book I of Euclid's *Elements* (those that do not rely on the Euclidean Parallel Postulate.)

Definition 1

Let $\triangle ABC$ be a triangle. The angles $\angle CAB$, $\angle ABC$, and $\angle BCA$ are called *interior angles* of the triangle. An angle that forms a linear pair with one of the interior angles is called an *exterior angle* for the triangle. If the exterior angle forms a linear pair with the interior angle at one vertex, then the interior angles at the other vertices are referred to as *remote interior angles*.

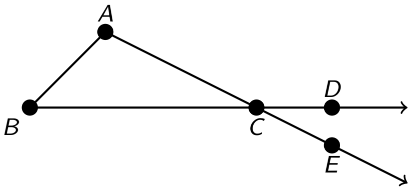


In the diagram, angles $\angle DCA$ and $\angle ECB$ are exterior angles for the interior angle $\angle BCA$. The remote interior angles for these exterior angles are $\angle ABC$ and $\angle BAC$. Note that $\angle DCA$ and $\angle ECB$ form a vertical pair.

Exterior Angle Theorem (Theorem 4.1.2)

The measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.

We'll restate this so it's easier to set up our proof.



Exterior Angle Theorem (Theorem 4.1.2) – Restatement

If $\triangle ABC$ is a triangle and D is a point such that $\overrightarrow{CD}$ is opposite to $\overrightarrow{CB}$, then $\mu(\angle DCA) > \mu(\angle BAC)$ and $\mu(\angle DCA) > \mu(\angle ABC)$.

Exterior Angle Theorem (Theorem 4.1.2)

If $\triangle ABC$ is a triangle and D is a point such that $\overrightarrow{CD}$ is opposite to $\overrightarrow{CB}$, then $\mu(\angle DCA) > \mu(\angle BAC)$ and $\mu(\angle DCA) > \mu(\angle ABC)$.

Proof.

Let $\triangle ABC$ be a triangle and D a point such that $\overrightarrow{CD}$ is opposite to $\overrightarrow{CB}$. We will prove that that $\mu(\angle DCA) > \mu(\angle BAC)$. The proof that $\mu(\angle DCA) > \mu(\angle ABC)$ is similar.

Let E be the midpoint of $\overline{AC}$ (Existence of Midpoints) and choose F to be the point on $\overrightarrow{BE}$ such that $\overline{BE} \cong \overline{EF}$ (PCP). Then $\angle BEA \cong \angle FEC$ (Vertical Angles Theorem). Hence, $\triangle BEA \cong \triangle FEC$ (SAS) and so $\angle FCA \cong \angle BAC$ (definition of congruent $\triangle$s).

Exterior Angle Theorem (Theorem 4.1.2)

If $\triangle ABC$ is a triangle and D is a point such that $\overrightarrow{CD}$ is opposite to $\overrightarrow{CB}$, then $\mu(\angle DCA) > \mu(\angle BAC)$ and $\mu(\angle DCA) > \mu(\angle ABC)$.

Proof.

Next we claim that F is in the interior of $\angle DCA$. Note that F and B are on opposite sides of $\overleftrightarrow{AC}$ and B and D are on opposite sides of $\overleftrightarrow{AC}$, so F and D are on the same side of $\overleftrightarrow{AC}$ (PSP). Also, A and E are on the same side of $\overleftrightarrow{CD}$, and E and F are on the same side $\overleftrightarrow{CD}$ (The Ray Theorem). Thus, A and F are on the same side of $\overleftrightarrow{CD}$ (PSP). It follows that F is in the interior of $\angle DCA$ as claimed (definition of angle interior).

Now $\mu(\angle DCA) > \mu(\angle FCA)$ (Btw Thm for Rays). Since $\angle FCA \cong \angle BAC$ (above), then $\mu(\angle DCA) > \mu(\angle BAC)$ as claimed. $\square$

Note that this theorem fails on the sphere $\mathbb{S}^2$.

Next time

Before next class: Finish reading Section 4.1. Start reading Section 4.2.

In the next lecture we will:

- Review definition of perpendicular and uniqueness of perpendiculars through a point on a line.
- Prove existence and uniqueness of perpendiculars (in general).
- Discuss various triangle congruence conditions

Chapter 4: Neutral geometry
§4.1 The Exterior Angle Theorem and Existence of Perpendiculars

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

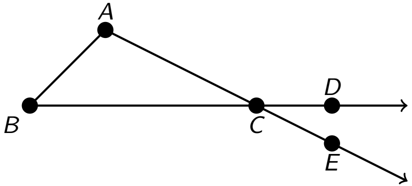
- Review definition of perpendicular and uniqueness of perpendiculars through a point on a line.
- Prove existence and uniqueness of perpendiculars (in general).
- Discuss various triangle congruence conditions

Last Time

Last time we proved the following theorem:

Exterior Angle Theorem (Theorem 4.1.2)

The measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.



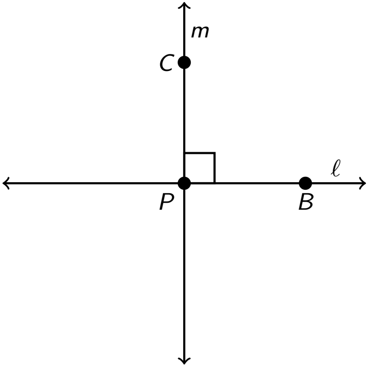
Today we'll prove another fundamental result in neutral geometry.

We've previously defined perpendicular lines. Since they are critical to our discussion today, we'll review the definition briefly.

Definition 1

Two lines ℓ and m are *perpendicular* if there exists a point A that lies on both ℓ and m and there exist points $B \in \ell$ and $C \in m$ such that $\angle BAC$ is a right angle.

Our notation for perpendicular lines is $\ell \perp m$.



(Theorem 3.5.9)

If ℓ is a line and P is a point on ℓ, then there exists exactly one line m such that P lies on m and $m \perp \ell$.

The proof of the above theorem was essentially just an application of the ACP. Now we prove the same result but without the assumption that $P \in \ell$. This is called *dropping a perpendicular* from P to ℓ. The point where m intersects ℓ is the *foot* of the perpendicular.

Existence and Uniqueness of Perpendiculars (Theorem 4.1.3)

For every line ℓ and for every point P, there exists a unique line m such that P lies on m and $m \perp \ell$.

Existence and Uniqueness of Perpendiculars (Theorem 4.1.3)

For every line ℓ and for every point P, there exists a unique line m such that P lies on m and $m \perp \ell$.

Proof.

First we prove existence. There exist distinct points $Q, Q' \in \ell$ (Ruler Postulate). There exists a point R, on the opposite side of ℓ from P, such that $\angle Q'QP \cong \angle Q'QR$ (ACP). Choose a point P' on $\overrightarrow{QR}$ such that $\overline{QP} \cong \overline{QP'}$ (PCP). Let $m = \overleftrightarrow{PP'}$. We claim that $m \perp \ell$.

Let $F \in \overline{PP'} \cap \ell$. There are three cases: $F = Q$, $F \neq Q$ and $F \in \overrightarrow{QQ'}$, or $F \neq Q$ and F is on the ray opposite $\overrightarrow{QQ'}$.

Suppose $F = Q$ (PSP). Then $\overrightarrow{QP}$ and $\overrightarrow{QP'}$ are opposite rays so $\angle Q'FP$ and $\angle Q'FP'$ form a linear pair (definition of linear pair). Since they are congruent, then they are each right angles (Linear Pair Theorem). Hence, $m \perp \ell$.

Existence and Uniqueness of Perpendiculars (Theorem 4.1.3)

For every line ℓ and for every point P, there exists a unique line m such that P lies on m and $m \perp \ell$.

Proof.

Suppose $F \neq Q$ but $F \in \overrightarrow{QQ'}$. Note that $\angle PQF = \angle PQQ'$, $\angle P'QF = \angle P'QQ'$. By our choice of P', $\angle PQF \cong \angle P'QF$. Hence, $\triangle FQP \cong FQP'$ (SAS), and so $\angle QFP \cong \angle QFP'$ (definition of congruent triangles). Since $\angle QFP$ and $\angle QFP'$ form a linear pair, then they are right angles by the same logic as above.

Finally, suppose $F \neq Q$ and F lies on the ray opposite $\overrightarrow{QQ'}$. Then $\angle PQF$ and $\angle PQQ'$ are supplements, as are $\angle P'QF$ and $\angle P'QQ'$. Thus, $\triangle FQP \cong FQP'$ (SAS) and the proof proceeds as above.

Existence and Uniqueness of Perpendiculars (Theorem 4.1.3)

For every line ℓ and for every point P, there exists a unique line m such that P lies on m and $m \perp \ell$.

Proof.

It is left to prove uniqueness. Let m' be a line such that P lies on m' and $m' \perp \ell$. Let Q' be the foot of the perpendicular. If $Q' = Q$, then $m = m'$ (Incidence Postulate). Suppose $Q' \neq Q$ (RAA hypothesis). Let G be a point on $\overrightarrow{QQ'}$ such that $Q * Q' * G$ (Ruler Postulate). Then $\angle PQQ'$ and $\angle PQ'G$ are right angles (definition of perpendicular). But $\angle PQ'G$ is an exterior angle for $\triangle PQQ'$ with remote interior angle $\angle PQQ'$. This contradicts the Exterior Angle Theorem. Thus, we reject the RAA hypothesis and conclude that $m = m'$. $\square$

Triangle Congruence Conditions

The triangle congruence conditions allow us to check whether two triangles are congruent without going through the tedious exercise of verifying that all three pairs of angles are congruent and all three pairs of sides are congruent. We have seen one of these already, it fact is part of our axiomatic system. We will prove several more in the next section:

- Side-Angle-Side (SAS) – Axiom VI
- Angle-Side-Angle (ASA) – Theorem 4.2.1
- Angle-Angle-Side (AAS) – Theorem 4.2.3
- Hypotenuse-Leg (HL) – Theorem 4.2.5, for right triangles only
- Side-Side-Side (SSS) – Theorem 4.2.7
- There is no Angle-Side-Side – Exercise 4.2.3

The ASA condition allows us to prove the following.

Converse to the Isosceles Triangle Theorem (Theorem 4.2.2)

Let $\triangle ABC$ is a triangle such that $\angle ABC \cong \angle ACB$, then $\overline{AB} \cong \overline{AC}$.

Next time

Before next class: Read Section 4.2.

In the next lecture we will:

- Prove several triangle congruence conditions.
- Prove the converse to the Isosceles Triangle Theorem.

Chapter 4: Neutral geometry
§4.2 Triangle Congruence Conditions

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Foundations of Geometry

It's good to have goals

Goals for today:

- Prove several triangle congruence conditions.
- Prove the converse to the Isosceles Triangle Theorem.

Last Time

So far in neutral geometry we have proved the following two fundamental results:

Exterior Angle Theorem (Theorem 4.1.2)

The measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.

Existence and Uniqueness of Perpendiculars (Theorem 4.1.3)

For every line ℓ and for every point P, there exists a unique line m such that P lies on m and $m \perp \ell$.

Triangle Congruence Conditions

The triangle congruence conditions allow us to check whether two triangles are congruent without going through the tedious exercise of verifying that all three pairs of angles are congruent and all three pairs of sides are congruent. We have seen one of these already, it fact is part of our axiomatic system. We will prove several more in this section:

- Side-Angle-Side (SAS) – Axiom VI
- Angle-Side-Angle (ASA) – Theorem 4.2.1
- Angle-Angle-Side (AAS) – Theorem 4.2.3
- Hypotenuse-Leg (HL) – Theorem 4.2.5, for right triangles only
- Side-Side-Side (SSS) – Theorem 4.2.7
- There is no Angle-Side-Side – Exercise 4.2.3

Angle-Side-Angle

ASA (Theorem 4.2.1)

If two angles and the included side of one triangle are congruent to the corresponding parts of a second triangle, then the two triangles are congruent.

ASA (Theorem 4.2.1) – Restatement

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$, $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.

Proof.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\angle CAB \cong \angle FDE$, $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$. We claim that $\triangle ABC \cong \triangle DEF$.

There exists a point C' on $\overrightarrow{AC}$ such that $\overline{AC'} \cong \overline{DF}$ (PCP). Then $\triangle ABC' \cong \triangle DEF$ (SAS) and so $\angle ABC' \cong \angle DEF$ (definition of congruent triangles). Since $\angle ABC \cong \angle DEF$ (hyp.), then $\angle ABC \cong \angle ABC'$. Thus, $\overrightarrow{BC} = \overrightarrow{BC'}$ (ACP). But $\overrightarrow{BC}$ intersects $\overleftrightarrow{AC}$ in at most one point (Theorem 3.1.7), so $C = C'$. $\square$

Recall that we used SAS to prove the Isosceles Triangle Theorem in a slick way. Similarly, we can use ASA to prove its converse.

Converse to the Isosceles Triangle Theorem (Theorem 4.2.2)

Let $\triangle ABC$ is a triangle such that $\angle ABC \cong \angle ACB$, then $\overline{AB} \cong \overline{AC}$.

Proof.

Let $\triangle ABC$ be a triangle such that $\angle ABC \cong \angle ACB$. Then $\angle ACB \cong \angle ABC$ (symmetric property) and $\overline{BC} \cong \overline{CB}$ (tautology), so $\triangle ABC \cong \triangle ACB$ (ASA). Thus, $\overline{AB} \cong \overline{AC}$ (definition of congruent triangles). $\square$

Proofs of the next two conditions are part of your homework exercises.

Angle-Angle-Side and Hypotenuse-Leg

AAS (Theorem 4.2.3)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.

The next one is special to triangles containing a right angle.

Definition 1

A triangle is a *right* triangle if one of the interior angles is a right angle. The side opposite the right angle is called the *hypotenuse* and the two sides adjacent to the right angle are called legs.

Hypotenuse-Leg Theorem (Theorem 4.2.5)

If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and a leg of a second right triangle, then the two triangles are congruent.

Hypotenuse-Leg Theorem (Theorem 4.2.5) – Restatement

If $\triangle ABC$ and $\triangle DEF$ are two right triangles with right angles at the vertices C and F, respectively, $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

Side-Side-Side

We need a preliminary result for SSS that is important in its own right. In "Euclidean" language, this result says *to construct a congruent copy of a triangle on a given base*.

(Theorem 4.2.6)

If $\triangle ABC$ is a triangle, $\overline{DE}$ is a segment such that $\overline{DE} \cong \overline{AB}$, and H is a half-plane bounded by $\overleftrightarrow{DE}$, then there is a unique point $F \in H$ such that $\triangle DEF \cong \triangle ABC$.

Proof.

Let $\triangle ABC$ be a triangle, $\overline{DE}$ a segment such that $\overline{DE} \cong \overline{AB}$, and H a half-plane bounded by $\overleftrightarrow{DE}$. We claim first that there is a unique point $F \in H$ such that $\triangle DEF \cong \triangle ABC$.

There exists a point $F' \in H$ such that $\angle EDF' \cong \angle BAC$ (ACP). Similarly, there exists a point F on $\overrightarrow{DF'}$ such that $\overline{DF} \cong \overline{AC}$ (PCP). Since $\angle EDF = \angle EDF'$, then $\triangle DEF \cong \triangle ABC$ (SAS).

For uniqueness, let $F'' \in H$ be such that $\triangle DEF'' \cong \triangle ABC$. Then $\angle BAC \cong \angle EDF''$ (defn of congruent triangles). Since $F, F'' \in H$, then $\overrightarrow{DF} = \overrightarrow{DF''}$ (ACP). Similarly, because $\overline{DF} \cong \overline{AC} \cong \overline{DF''}$ (defn of congruent triangles), then $F = F''$ (PCP). $\square$

Next time

Before next class: Read Section 4.3.

In the next lecture we will:

- Prove SSS.
- Consider further the relationship between sides and angles in triangles.
- Prove the Scalene Inequality and Hinge Theorem.

Chapter 4: Neutral geometry
§4.2 Triangle Congruence Conditions
§4.3 Three Inequalities for Triangles

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Prove SSS.
- Consider further the relationship between sides and angles in triangles.
- Prove the Scalene Inequality.

Triangle Congruence Conditions

The triangle congruence conditions allow us to check whether two triangles are congruent without going through the tedious exercise of verifying that all three pairs of angles are congruent and all three pairs of sides are congruent. We have seen one of these already, it fact is part of our axiomatic system. We will prove several more in this section:

- Side-Angle-Side (SAS) – Axiom VI
- Angle-Side-Angle (ASA) – Theorem 4.2.1 (last time)
- Angle-Angle-Side (AAS) – Theorem 4.2.3 (hw)
- Hypotenuse-Leg (HL) – Theorem 4.2.5, for right triangles only (hw)
- Side-Side-Side (SSS) – Theorem 4.2.7 (today!)
- There is no Angle-Side-Side – Exercise 4.2.3

Recall the following preliminary result that we need in the proof of SSS.

(Theorem 4.2.6)

If $\triangle ABC$ is a triangle, $\overline{DE}$ is a segment such that $\overline{DE} \cong \overline{AB}$, and H is a half-plane bounded by $\overleftrightarrow{DE}$, then there is a unique point $F \in H$ such that $\triangle DEF \cong \triangle ABC$.

SSS (Theorem 4.2.7)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.

Proof.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$. We claim $\triangle ABC \cong \triangle DEF$.

There exists a point G, on the opposite side of $\overleftrightarrow{AB}$ from C, such that $\triangle ABG \cong \triangle DEF$ (Theorem 4.2.6). Then there is a point H such that H is between C and G and H lies on $\overleftrightarrow{AB}$ (PSP). There are five cases: $H = A$, $H = B$, $H * A * B$, $A * H * B$, or $A * B * H$. We will show in each case that $\triangle ABC \cong \triangle ABG$. This is sufficient to complete the proof since $\triangle ABG \cong \triangle DEF$.

Case 1 ($H = A$). Then $\angle ACB = \angle GCB$ and $\angle AGB = \angle CGB$. Since $BC = BG$, then $\angle GCB \cong \angle CGB$ (Isosceles Triangle Theorem). Therefore, $\triangle ABC \cong \triangle ABG$ (SAS).

Case 1 ($H = B$). This is similar to the previous case.

SSS (Theorem 4.2.7)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.

Proof.

Case 3 ($A * H * B$). Since H is between A and B, H is in the interior of $\angle ACB$ and in the interior of $\angle AGB$. Then

$$\mu(\angle ACB) = \mu(\angle ACH) + \mu(\angle HCB), \quad \text{and}$$
$$\mu(\angle AGB) = \mu(\angle AGH) + \mu(\angle HGB)$$

(AAP). But $\mu(\angle ACH) = \mu(\angle ACG) = \mu(\angle AGC) = \mu(\angle AGH)$ (Isosceles Triangle Theorem). Similarly, $\mu(\angle HCB) = \mu(\angle HGB)$. Therefore, $\angle ACB \cong \angle AGB$ and so $\triangle ABC \cong \triangle ABG$ (SAS).

SSS (Theorem 4.2.7)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.

Proof.

Case 4 ($H * A * B$). Since A is between H and B, A is in the interior of $\angle HCB$ and in the interior of $\angle HGB$. Then

$$\mu(\angle ACB) = \mu(\angle HCB) - \mu(\angle HCA), \quad \text{and}$$
$$\mu(\angle AGB) = \mu(\angle HGB) - \mu(\angle HGA)$$

(AAP). But $\mu(\angle ACH) = \mu(\angle AGH)$ and $\mu(\angle HCB) = \mu(\angle HGB)$ (Isosceles Triangle Theorem). Similarly, $\mu(\angle HCB) = \mu(\angle HGB)$. Therefore, $\angle ACB \cong \angle AGB$ and so $\triangle ABC \cong \triangle ABG$ (SAS).

Case 5 ($A * B * H$). This is similar to case 4. $\square$

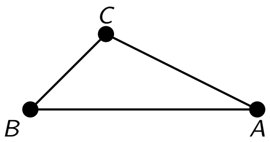
The next result extends the Isosceles Triangle Theorem.

Scalene Inequality (Theorem 4.3.1)

In any triangle, the greater side lies opposite the greater angle and the greater angle lies opposite the greater side.

Scalene Inequality (Theorem 4.3.1) – Restatement

Let $\triangle ABC$ be a triangle. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.



Scalene Inequality (Theorem 4.3.1) – Restatement

Let $\triangle ABC$ be a triangle. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.

Proof.

Let A, B, and C be three noncollinear points.

Assume first that $AB > BC$. We claim that $\mu(\angle ACB) > \mu(\angle BAC)$. Since $AB > BC$, then there exists a point D between A and B such that $\overline{BD} \cong \overline{BC}$ (Ruler Postulate). Then $\angle BCD \cong \angle CDB$ (Isosceles Triangle Theorem) and so $\mu(\angle ACB) > \mu(\angle CDB)$ (Theorem 3.3.10 and Angle Addition Postulate). But $\angle CDB$ is an exterior angle for $\triangle ADC$, so $\mu(\angle CDB) > \mu(\angle BAC)$ (Exterior Angle Theorem). Thus, $\mu(\angle ACB) > \mu(\angle BAC)$ as claimed.

Scalene Inequality (Theorem 4.3.1) – Restatement

Let $\triangle ABC$ be a triangle. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.

Proof.

Assume now that $\mu(\angle ACB) > \mu(\angle BAC)$. We claim that $AB > BC$.

Let D' be a point on $\overrightarrow{BA}$ such that $BC = BD'$ (PCP). There are three cases: $B * D' * A$, $D' = A$, and $B * A * D'$. In the first case, $AB > BD' = BC$ (definition of between), and there is nothing to prove. We claim the other two are impossible.

Suppose $D' = A$. Then $BC = BD' = BA$, so $\mu(\angle ACB) = \mu(\angle BAC)$ (Isosceles Triangle Theorem), a contradiction.

Suppose $B * A * D'$. Since $BC = BD'$, then $\mu(\angle BD'C) = \mu(\angle BCD')$ (Isosceles Triangle Theorem). As $B * A * D'$, then $\mu(\angle BCD') > \mu(\angle ACB)$ (Theorem 3.3.10 and Betweenness Theorem for Rays). But $\angle BAC$ is an external angle for $\triangle BD'C$, so $\mu(\angle BAC) > \mu(\angle BD'C)$. Hence, $\mu(\angle BAC) > \mu(\angle ACB)$, a contradiction. □

Next time

Before next class: Finish reading Section 4.3.

In the next lecture we will:

- Prove the Hinge Theorem.
- Define distance from a point to a line. Use this to reframe the notion of angle and perpendicular bisectors.

Chapter 4: Neutral geometry
§4.3 Three Inequalities for Triangles

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Prove the Hinge Theorem.
- Define distance from a point to a line. Use this to reframe the notion of angle and perpendicular bisectors.

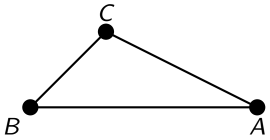
The next result extends the Isosceles Triangle Theorem.

Scalene Inequality (Theorem 4.3.1)

In any triangle, the greater side lies opposite the greater angle and the greater angle lies opposite the greater side.

Scalene Inequality (Theorem 4.3.1) – Restatement

Let $\triangle ABC$ be a triangle. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.



Scalene Inequality (Theorem 4.3.1) – Restatement

Let $\triangle ABC$ be a triangle. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.

Proof.

Let A, B, and C be three noncollinear points.

Assume first that $AB > BC$. We claim that $\mu(\angle ACB) > \mu(\angle BAC)$. Since $AB > BC$, then there exists a point D between A and B such that $\overline{BD} \cong \overline{BC}$ (Ruler Postulate). Then $\angle BCD \cong \angle CDB$ (Isosceles Triangle Theorem) and so $\mu(\angle ACB) > \mu(\angle CDB)$ (Theorem 3.3.10 and Angle Addition Postulate). But $\angle CDB$ is an exterior angle for $\triangle ADC$, so $\mu(\angle CDB) > \mu(\angle BAC)$ (Exterior Angle Theorem). Thus, $\mu(\angle ACB) > \mu(\angle BAC)$ as claimed.

Scalene Inequality (Theorem 4.3.1) – Restatement

Let $\triangle ABC$ be a triangle. Then $AB > BC$ if and only if $\mu(\angle ACB) > \mu(\angle BAC)$.

Proof.

Assume now that $\mu(\angle ACB) > \mu(\angle BAC)$. We claim that $AB > BC$.

Let D' be a point on $\overrightarrow{BA}$ such that $BC = BD'$ (PCP). There are three cases: $B * D' * A$, $D' = A$, and $B * A * D'$. In the first case, $AB > BD' = BC$ (definition of between), and there is nothing to prove. We claim the other two are impossible.

Suppose $D' = A$. Then $BC = BD' = BA$, so $\mu(\angle ACB) = \mu(\angle BAC)$ (Isosceles Triangle Theorem), a contradiction.

Suppose $B * A * D'$. Since $BC = BD'$, then $\mu(\angle BD'C) = \mu(\angle BCD')$ (Isosceles Triangle Theorem). As $B * A * D'$, then $\mu(\angle BCD') > \mu(\angle ACB)$ (Theorem 3.3.10 and Betweenneess Theorem for Rays). But $\angle BAC$ is an external angle for $\triangle BD'C$, so $\mu(\angle BAC) > \mu(\angle BD'C)$. Hence, $\mu(\angle BAC) > \mu(\angle ACB)$, a contradiction. □

The next result is left as a homework exercise.

Triangle Inequality (Theorem 4.3.3)

If A, B, and C are three noncollinear points, then $AC < AB + BC$.

Our last inequality generalizes SAS.

Hinge Theorem (Theorem 4.3.3)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $AB = DE$, $AC = DF$, and $\mu(\angle BAC) < \mu(EDF)$, then $BC < EF$.

The reason it is called the Hinge Theorem is that we should think of $\angle BAC$ as a hinge which, as it opens further, increases the length of the segment BC.

Hinge Theorem (Theorem 4.3.3)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $AB = DE$, $AC = DF$, and $\mu(\angle BAC) < \mu(EDF)$, then $BC < EF$.

Proof.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $AB = DE$, $AC = DF$, and $\mu(\angle BAC) < \mu(EDF)$. We claim $BC < EF$. Let G be a point on the same side of $\overleftrightarrow{AB}$ such that $\triangle ABG \cong \triangle DEF$ (Theorem 4.2.9). It suffices to prove that $BG > BC$.

Since C is in the interior of $\angle BAG$ (Betw. Thm. for Rays), then $\overrightarrow{AC}$ intersects $\overline{BG}$ in a point J (Crossbar Theorem). If $J = C$, then C is between B and G and the result follows. We assume henceforth that $J \neq C$. That is, $C \notin \overleftrightarrow{BG}$ (Thm 3.1.7).

Let $\overrightarrow{AH}$ be the bisector of $\angle CAG$ (existence of $\angle$ bisectors). Then H is in the interior of $\angle CAG = \angle JAG$ (def of $\angle$ bisector), so $\overrightarrow{AH}$ must intersect $\overline{JG}$ at a point H' (Crossbar Theorem). Since $\overline{JH} \subset \overline{BG}$, then H' is between B and G. Note that $\triangle AH'G \cong \triangle AHC$ (SAS), so $H'G = H'C$. Hence, $BG = BH' + H'G = BH' + H'C$. Since $C \notin \overleftrightarrow{BH'}$, then $BC < BH' + H'C$ (Triangle Inequality). The result follows. $\square$

These inequalities allow us to establish several fundamental results regarding perpendicular lines. In particular, the length of a perpendicular segment is the shortest distance from a line to an external point. The proof of this result is left as a homework.

(Theorem 4.3.4)

Let ℓ be a line, let P be an external point, and let F be the foot of the perpendicular from P to ℓ. If R is any point on ℓ that is different from F, then $PR > PF$.

We now formalize the notion of distance from a point to a line.

Definition 1

If ℓ is a line and P is a point, the *distance from P to ℓ*, denoted $d(P, \ell)$, is defined to be the distance from P to the foot of the perpendicular from P to ℓ.

This definition allows us to reframe two prior results in terms of distance.

Pointwise Characterization of Angle Bisectors (Theorem 4.3.6)

Let A, B, and C be three noncollinear points and let P be a point in the interior of $\angle BAC$. Then P lies on the angle bisector of $\angle BAC$ if and only if $d(P, \overleftrightarrow{AB}) = d(P, \overleftrightarrow{AC})$.

Pointwise Characterization of Perpendicular Bisectors (Theorem 4.3.7)

Let A and B be distinct points. A point P lies on the perpendicular bisector of $\overline{AB}$ if and only if $PA = PB$.

We won't prove the following theorem, but it is important in connecting axiomatic geometry with calculus.

Let A, B, and C be three noncollinear points. Let $d = AB$. For each $x \in [0, d]$ there exists a unique point $D_x \in \overline{AB}$ such that $AD_x = x$ (Ruler Postulate). Define a function $f : [0, d] \to [0, \infty)$ by $f(x) = CD_x$.

Continuity of Distance (Theorem 4.3.8)

The function f defined above is continuous.

Next time

Before next class: Read Section 4.4.

In the next lecture we will:

- Begin a systematic study of parallel lines.
- Prove the Alternate Interior Angles Theorem.
- Use the AIAT to prove the existence of parallels.

Chapter 4: Neutral geometry
§4.4 The Alternate Interior Angles Theorem

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Begin a systematic study of parallel lines.
- Prove the Alternate Interior Angles Theorem.
- Use the AIAT to prove the existence of parallels.

Definition 1

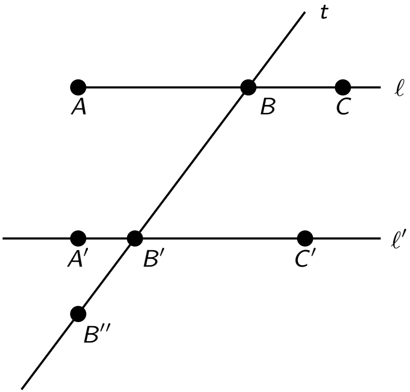
Let ℓ and ℓ' be two distinct lines. A third line t is called a *transversal* for ℓ and ℓ' if t intersects ℓ in one point B and t intersects ℓ' in one point B' with $B' \neq B$.

Let A and C be points of ℓ such that $A * B * C$, and let A' and C" be points on ℓ' such that $A' * B' * C'$. Let B'' be a point on t such that $B * B' * B''$.

- The four angles $\angle ABB'$, $\angle A'B'B$, $\angle CBB'$, and $\angle C'B'B$ are called *interior angles* for ℓ and ℓ' with transversal t.
- The two pairs $\{\angle ABB', \angle C'B'B\}$ and $\{\angle A'B'B, \angle CBB'\}$ are called *alternate interior angles*.
- The pairs $\{\angle CBB', \angle C'B'B''\}$ and $\{\angle ABB', \angle A'B'B''\}$ are called *corresponding angles*.

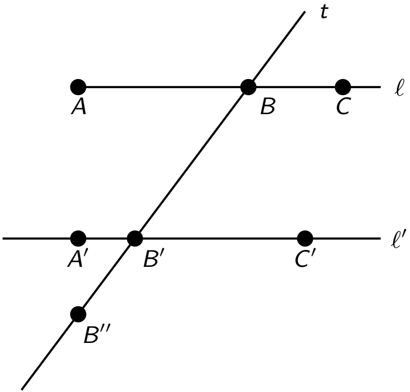
Given the above setup, we say that ℓ and ℓ' are cut by the transversal t.

Angles created by a transversal

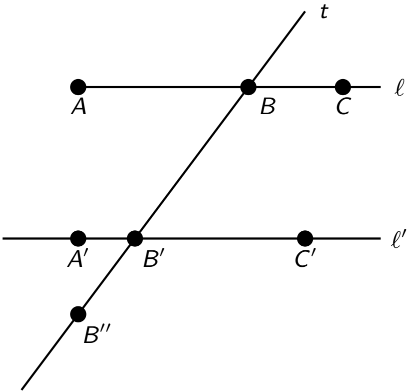


interior angles: $\angle ABB'$, $\angle A'B'B$, $\angle CBB'$, $\angle C'B'B$

Angles created by a transversal



alternate interior angles: $\{\angle ABB', \angle C'B'B\}$ and $\{\angle A'B'B, \angle CBB'\}$



corresponding angles: $\{\angle CBB', \angle C'B'B''\}$ and $\{\angle ABB', \angle A'B'B''\}$

Alternate Interior Angles Theorem (Theorem 4.4.2)

If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ'.

Proof.

Let ℓ and ℓ' be two lines cut by a transversal t and let B, B' be the respective points of intersection. Choose points $A, C \in \ell$ and $A', C' \in \ell'$ such that $A * B * C$ and $A' * B' * C'$. Assume $\angle A'B'B \cong \angle CBB'$ (the other cases are similar). We claim that ℓ is parallel to ℓ'.

Suppose there exists a point D such that D lies on ℓ and ℓ' (RAA hypothesis). Then either D lies on the same side of t as C or else D lies on the same side of t as A (PSP). Assume the first case. The second case is similar.

Since D lies on the same side of t as C, then $\angle A'B'B$ is an exterior angle for $\triangle BB'D$ with remote interior angle $\angle CBB'$. But this contradicts the Exterior Angles Theorem. Hence, we reject the RAA hypothesis and conclude that that $\ell \parallel \ell'$.

The next results are left as homework.

Corresponding Angles Theorem (Corollary 4.4.4)

If ℓ and ℓ' are two lines cut by a transversal t in such a way that two corresponding angles are congruent, then ℓ is parallel to ℓ'.

(Corollary 4.4.5)

If ℓ and ℓ' are two lines cut by a transversal t in such a way that two nonalternating interior angles on the same side of t are supplementary, then ℓ is parallel to ℓ'.

We now apply existence of perpendiculars and the AIAT to prove existence of parallel lines in neutral geometry.

Existence of Parallels (Corollary 4.4.6)

If ℓ is a line and P is an external point, then there is a line m such that P lies on m and m is parallel to ℓ.

Proof.

Let ℓ be a line and let P be an external point. Drop a perpendicular from P to ℓ (Existence of Perpendiculars) and call the foot of that perpendicular Q. Let $t = \overleftrightarrow{PQ}$. Let m be the line through P that is perpendicular to t (Theorem 3.5.9). Then $\ell \parallel m$ by the AIAT. $\square$

The construction used in the previous proof will be used often.

The Double Perpendicular Construction

Given a line ℓ and an external point P, drop a perpendicular t from P to ℓ. Let m be the line through P that is perpendicular to t. The P lies on both t and m, m is parallel to ℓ, and t is a transversal for ℓ and m that is perpendicular to both ℓ and m.

Existence of Parallels

(Corollary 4.4.7)

The Elliptic Parallel Postulate is false in any model for neutral geometry.

Note, however, that the Double Perpendicular Construction does not make any statement as to the uniqueness of the parallel line through P.

(Corollary 4.4.8)

If ℓ, m, and n are three lines such that $m \perp \ell$ and $n \perp \ell$, then either $m = n$ or $m \parallel n$.

Next time

Before next class: Read Section 4.5.

In the next lecture we will:

- Discuss angle sums of triangles.
- Prove the Saccheri-Legendre Theorem (the angle sum of a triangle is at most 180°).

Chapter 4: Neutral geometry
§4.5 The Saccheri-Legendre Theorem

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Discuss angle sums of triangles.
- Prove the Saccheri-Legendre Theorem (the angle sum of a triangle is at most 180°).

This is the first set of results that diverge from Euclid's presentation.

Definition 1

Let A, B, and C be three noncollinear points. The *angle sum* for $\triangle ABC$ is the sum of the measures of the three interior angles of $\triangle ABC$. More specifically, the angle sum is defined by the equation

$$\sigma(\triangle ABC) = \mu(\angle CAB) + \mu(\angle ABC) + \mu(\angle BCA).$$

Note that congruent triangles have equal angle sums.

Saccheri-Legendre Theorem (Theorem 4.5.2)

If $\triangle ABC$ is any triangle, then $\sigma(\triangle ABC) \leq 180^\circ$.

Before proving this theorem, we will need some setup.

The first lemma tells us that the sum of any two angles of a triangle is strictly less than 180°.

(Lemma 4.5.3)

If $\triangle ABC$ is any triangle, then $\mu(\angle CAB) + \mu(\angle ABC) < 180^\circ$.

Proof.

Let $\triangle ABC$ be a triangle. We claim $\mu(\angle CAB) + \mu(\angle ABC) < 180^\circ$.

Let D be a point on $\overleftrightarrow{AB}$ such that $A * B * D$. Then $\mu(\angle ABC) + \mu(\angle CBD) = 180^\circ$ (LPT). Moreover, $\mu(\angle CAB) < \mu(\angle CBD)$ (Exterior Angle Theorem). Therefore, $\mu(\angle CAB) + \mu(\angle ABC) < 180^\circ$ (algebra). $\square$

The second lemma tells us how angle sums behave when a triangle is subdivided.

(Lemma 4.5.4)

If $\triangle ABC$ is a triangle and E is a point in the interior of $\overline{BC}$, then

$$\sigma(\triangle ABE) + \sigma(\triangle ECA) = \sigma(\triangle ABC) + 180^\circ.$$

Proof.

Let $\triangle ABC$ be a triangle and let E be a point in the interior of $\overline{BC}$. We claim $\sigma(\triangle ABE) + \sigma(\triangle ECA) = \sigma(\triangle ABC) + 180^\circ$. We have

$$\begin{aligned}\sigma(\triangle ABE) + \sigma(\triangle ECA) = {} & \mu(\angle EAB) + \mu(\angle ABE) + \mu(\angle BEA) \\ & + \mu(\angle CAE) + \mu(\angle ECA) + \mu(\angle AEC).\end{aligned}$$

Then $\mu(\angle EAB) + \mu(\angle CAE) = \mu(\angle BAC)$ (Theorem 3.3.10, AAP). Furthermore, $\mu(\angle BEA) + \mu(\angle AEC) = 180^\circ$ (LPT). Then

$$\begin{aligned}\sigma(\triangle ABE) + \sigma(\triangle ECA) &= \mu(\angle BAC) + \mu(\angle ABE) + \mu(\angle ECA) + 180^\circ \\ &= \sigma(\triangle ABC) + 180^\circ\end{aligned}$$

(algebra and definition of angle sum). □

(Lemma 4.5.5)

If A, B, and C are three noncollinear points, then there exists a point D that does not lie on $\overleftrightarrow{AB}$ such that $\sigma(\triangle ABD) = \sigma(\triangle ABC)$ and the angle measure of one of the interior angles in $\triangle ABD$ is less than or equal to $\frac{1}{2}\mu(\angle CAB)$.

Proof.

Let A, B, and C be three noncollinear points. Let E be the midpoint of $\overline{BC}$ (Existence of Midpoints). Let D be a point on $\overrightarrow{AE}$ such that $A * E * D$ and $AE = ED$ (Ruler Postulate). We will show that $\sigma(\triangle ABD) = \sigma(\triangle ABC)$ and either $\mu(\angle BAD) \leq \frac{1}{2}\mu(\angle BAC)$ or $\mu(\angle ADB) \leq \frac{1}{2}\mu(\angle BAC)$.

Since $\angle AEC \cong \angle DEB$ (Vertical Angles Theorem), then $\triangle AEC \cong \triangle DEB$ (SAS) and so $\sigma(\triangle AEC) = \sigma(\triangle DEB)$. Then

$$\sigma(\triangle ABC) = \sigma(\triangle ABE) + \sigma(\triangle AEC) - 180^\circ$$
$$\sigma(\triangle ABD) = \sigma(\triangle ABE) + \sigma(\triangle DEB) - 180^\circ$$

(Lemma 4.5.4). It follows that $\sigma(\triangle ABD) = \sigma(\triangle ABC)$.

(Lemma 4.5.5)

If A, B, and C are three noncollinear points, then there exists a point D that does not lie on $\overleftrightarrow{AB}$ such that $\sigma(\triangle ABD) = \sigma(\triangle ABC)$ and the angle measure of one of the interior angles in $\triangle ABD$ is less than or equal to $\frac{1}{2}\mu(\angle CAB)$.

Proof.

Now $\mu(\angle BAE) + \mu(\angle EAC) = \mu(\angle BAC)$ (Protractor Postulate). Hence, either $\mu(\angle BAE) \leq \frac{1}{2}\mu(\angle BAC)$ or $\mu(\angle EAC) \leq \frac{1}{2}\mu(\angle BAC)$. Since $\mu(\angle EAC) = \mu(\angle ADB)$, the proof is complete. $\square$

We take a short detour to recall and important property of real numbers needed in the proof of the Saccheri-Legendre Theorem.

Definition 2

Let A be a set of real numbers. A number b is called an *upper bound* for A is $x \leq b$ for every $x \in A$. The number b_0 is called the *least upper bound* for A if b_0 is an upper bound for A and $b_0 \leq b$ for every b that is an upper bound for A.

The Least Upper Bound Postulate

If A is any nonempty set of real numbers that has an upper bound, then A has a least upper bound.

Archimedean Property of Real Numbers

Archimedean Property of Real Numbers (Theorem E.3.4)

If M and ϵ are any two positive real numbers, then there exists a positive integer n such that $n\epsilon > M$.

Proof.

Let M and ϵ be two positive real numbers. Suppose $n\epsilon \leq M$ for every positive integer n (RAA hypothesis). Then M is an upper bound for the set

$$A = \{n\epsilon : n \text{ is a positive integer}\}.$$

Then $A \neq \emptyset$, so A has a least upper bound b (LUB Postulate). Since b is an upper bound for A, then $(n+1)\epsilon \leq b$ for every positive integer n. Thus $n\epsilon \leq b - \epsilon$ for every positive integer n. This contradicts the minimality of b. Hence we reject the RAA hypothesis and conclude that there exists a positive integer n such that $n\epsilon > M$. $\square$

Saccheri-Legendre Theorem (Theorem 4.5.2)

If $\triangle ABC$ is any triangle, then $\sigma(\triangle ABC) \leq 180^\circ$.

Proof.

Let $\triangle ABC$ be a triangle. Suppose $\sigma(\triangle ABC) > 180^\circ$ (RAA hypothesis). Write $\sigma(\triangle ABC) = 180^\circ + \epsilon^\circ$ where ϵ is a positive real number. Choose a positive integer n large enough so that $2^n\epsilon^\circ > \mu(\angle CAB)$ (Archimedean Property of Real Numbers).

There is a triangle $\triangle A_1B_1C_1$ such that $\sigma(\triangle A_1B_1C_1) = \sigma(\triangle ABC)$ and one of the angles in $\triangle A_1B_1C_1$ has angle measure $\leq \frac{1}{2}\mu(\angle CAB)$ (Lemma 4.5.5). Similarly, there is a triangle $\triangle A_2B_2C_2$ such that $\sigma(\triangle A_2B_2C_2) = \sigma(\triangle ABC)$ and one of the angles in $\triangle A_2B_2C_2$ has angle measure $\leq \frac{1}{4}\mu(\angle CAB)$ (Lemma 4.5.5). Repeating this n times gives a triangle $\triangle A_nB_nC_n$ such that $\sigma(\triangle A_nB_nC_n) = \sigma(\triangle ABC) = 180^\circ + \epsilon^\circ$ and one of the angles in $\triangle A_nB_nC_n$ has angle measure $\leq \frac{1}{2^n}\mu(\angle CAB) < \epsilon^\circ$ (Lemma 4.5.5). This contradicts Lemma 4.5.3. Hence, we reject the RAA hypothesis and the conclusion follows. $\square$

There are two immediate consequences of the Saccheri-Legendre Theorem.

(Corollary 4.5.6)

The sum of the measures of two interior angles of a triangle is less than or equal to the measure of their remote exterior angle.

Converse to Euclid's Fifth Postulate (Corollary 4.5.7)

Let ℓ and ℓ' be two lines cut by a transversal t. If ℓ and ℓ' meet on one side of t, then the sum of the measures of the two interior angles on that side of t is strictly less than 180°.

Next time

Before next class: Read Section 4.6.

In the next lecture we will:

- Define quadrilaterals.
- Discuss angle sum of quadrilaterals.
- Study convexity of quadrilaterals.

Chapter 4: Neutral geometry
§4.6 Quadrilaterals

MTH 411/511

Foundations of Geometry

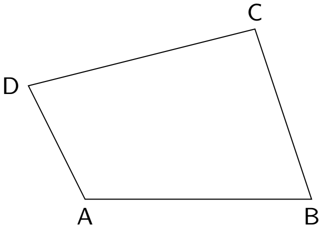
It's good to have goals

Goals for today:

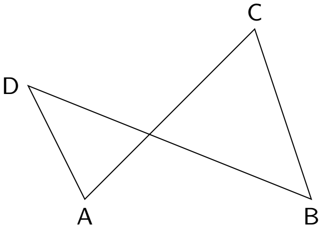
- Define quadrilaterals.
- Discuss angle sum of quadrilaterals.
- Study convexity of quadrilaterals.

Definition 1

A *quadrilateral* is determined by four points A, B, C, and D (called *vertices*), no three of which are collinear, and is defined as the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ (called *sides*) assuming these segments either have no point in common or have on an endpoint in common. The sides $\overline{AB}$ and $\overline{CD}$ are called *opposite sides*, as are $\overline{BC}$ and $\overline{DA}$.



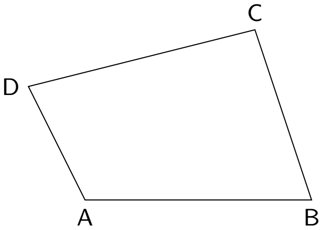
$\square ABCD$ is a quadrilateral



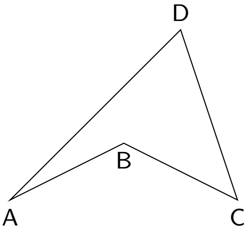
$\square ACBD$ is not a quadrilateral

Definition 2

The *diagonals* of the quadrilateral $\square ABCD$ are the segments $\overline{AC}$ and $\overline{BD}$. The *angles* of the quadrilateral $\square ABCD$ are the angles $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$. A quadrilateral is said to be *convex* if each vertex of the quadrilateral is contained in the interior of the angles formed by the other three vertices.



$\square ABCD$ is convex



$\square ACBD$ is not convex

Definition 3

Two quadrilaterals are *congruent* if there is a correspondence between their vertices so that all four corresponding sides are congruent and all four corresponding angles are congruent.

Angle sum

Definition 4

If $\square ABCD$ is a convex quadrilateral, then the *angle sum* is defined by

$$\sigma(\square ABCD) = \mu(\angle ABC) + \mu(\angle BCD) + \mu(\angle CDA) + \mu(\angle DAB).$$

The next result is a homework exercise.

(Theorem 4.6.4)

If $\square ABCD$ is a convex quadrilateral, then $\sigma(\square ABCD) \leq 360^\circ$.

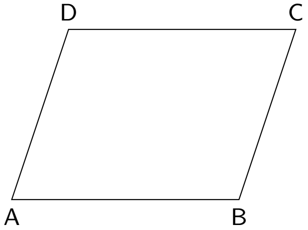
Definition 5

The quadrilateral $\square ABCD$ is called a parallelogram if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

The next result is a homework exercise.

(Theorem 4.6.6)

Every parallelogram is a convex quadrilateral.



Some convex quadrilaterals

(Theorem 4.6.7)

If $\triangle ABC$ is a triangle, D is between A and B, and E is between A and C, then $\square BCED$ is a convex quadrilateral.

Proof.

Let $\triangle ABC$ be a triangle. Let D and E be points such that $A * D * B$ and $A * E * C$. We claim that $\square BCED$ is a convex quadrilateral.

First we prove that $\square BCED$ is a quadrilateral. To do this, we need to show that no three points are collinear and the sides have at most one point in common.

Suppose B, C, E are collinear (RAA hypothesis). Then $\overleftrightarrow{BC} = \overleftrightarrow{AC}$ (Incidence Postulate). This contradicts the definition of $\triangle ABC$. A similar argument shows that the other triples of points are non-collinear.

Moreover, the same argument shows that $\overline{DE}$ and $\overline{EC}$ intersect in one point, as do $\overline{DE}$ and $\overline{BD}$. Suppose $\overline{BC}$ intersects $\overline{DE}$ at a point P. Then $B * P * C$ and $D * P * E$. Thus, D and E are on opposite sides of $\overleftrightarrow{BC}$. This implies that $A * C * E$, contradicting the choice of E. A similar argument shows that $\overline{BD}$ and $\overline{EC}$ do not intersect.

(Theorem 4.6.7)

If $\triangle ABC$ is a triangle, D is between A and B, and E is between A and C, then $\square BCED$ is a convex quadrilateral.

Proof.

Next we claim that $\square BCED$ is convex.

Since $A * E * C$, then E is in the interior of $\angle DBC$ (Theorem 3.3.10). Similarly, D is in the interior of $\angle BCE$. Since $A * D * B$ and $A * E * C$, then B is on the same side of $\overleftrightarrow{DE}$ as C and the same side of $\overleftrightarrow{EC}$ as D. Thus, B is in the interior of $\angle DEC$. A similar argument shows that C is in the interior of $\angle BDE$. $\square$

The next result gives a different way of checking convexity.

(Theorem 4.6.8)

The quadrilateral $\square ABCD$ is convex if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ have an interior point in common.

Proof.

Assume that $\square ABCD$ is convex. We claim $\overline{AC}$ and $\overline{BD}$ have an interior point in common.

Note that C is in the interior of $\angle DAB$ (definition of convex quadrilaterals). Hence, $\overrightarrow{AC}$ intersects $\overline{BD}$ in a point, say E (Crossbar Theorem). Similarly, $\overrightarrow{BD}$ intersects $\overline{AC}$ in a point E'. But $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect in at most one point (Incidence Postulate), so $E = E'$ and E lies on both $\overline{AC}$ and $\overline{BD}$.

Some convex quadrilaterals

(Theorem 4.6.8)

The quadrilateral $\square ABCD$ is convex if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ have an interior point in common.

Proof.

Assume that $\overline{AC}$ and $\overline{BD}$ have an interior point in common. We claim $\square ABCD$ is convex.

We will show that C is in the interior of $\angle DAB$. The other vertices are similar. By hypothesis, $\overrightarrow{AC}$ intersects $\overline{BD}$ at a point E in the interior of $\square ABCD$, so E is between B and D. Thus, $\overrightarrow{AC}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ (Theorem 3.3.10). Thus, C is in the interior of $\angle DAB$ (definition of angle interior). $\square$

The following corollary is left as homework.

(Corollary 4.6.9)

If $\square ABCD$ and $\square ACBD$ are both quadrilaterals, then $\square ABCD$ is not convex. If $\square ABCD$ is a nonconvex quadrtilateral, then $\square ACBD$ is a quadrilateral.

Next time

Before next class: Read Section 4.7.

In the next lecture we will:

- Consider statements equivalent to the Euclidean Parallel Postulate.
- Prove equivalences between several of the statements.

Chapter 4: Neutral geometry
§4.6 Quadrilaterals
§4.7 Statements equivalent to the Euclidean Parallel Postulate

MTH 411/511

Foundations of Geometry

It's good to have goals

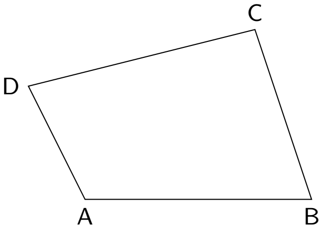
Goals for today:

- Continue our study of convexity of quadrilaterals.
- Consider statements equivalent to the Euclidean Parallel Postulate.
- Prove equivalences between several of the statements.

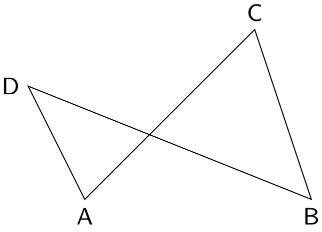
First we recall our definition of quadrilaterals.

Definition 1

A *quadrilateral* is determined by four points A, B, C, and D (called *vertices*), no three of which are collinear, and is defined as the union of the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ (called *sides*) assuming these segments either have no point in common or have on an endpoint in common.



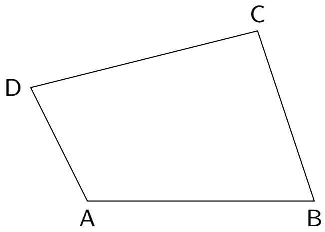
$\square ABCD$ is a quadrilateral



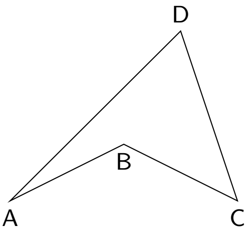
$\square ACBD$ is not a quadrilateral

Definition 2

The *diagonals* of the quadrilateral $\square ABCD$ are the segments $\overline{AC}$ and $\overline{BD}$. The *angles* of the quadrilateral $\square ABCD$ are the angles $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$. A quadrilateral is said to be *convex* if each vertex of the quadrilateral is contained in the interior of the angles formed by the other three vertices.



$\square ABCD$ is convex



$\square ACBD$ is not convex

Last time we proved the following result on convex quadrilaterals.

(Theorem 4.6.7)

If $\triangle ABC$ is a triangle, D is between A and B, and E is between A and C, then $\square BCED$ is a convex quadrilateral.

The next result gives a different way of checking convexity.

(Theorem 4.6.8)

The quadrilateral $\square ABCD$ is convex if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ have an interior point in common.

Proof.

Assume that $\square ABCD$ is convex. We claim $\overline{AC}$ and $\overline{BD}$ have an interior point in common.

Note that C is in the interior of $\angle DAB$ (definition of convex quadrilaterals). Hence, $\overrightarrow{AC}$ intersects $\overline{BD}$ in a point, say E (Crossbar Theorem). Similarly, $\overrightarrow{BD}$ intersects $\overline{AC}$ in a point E'. But $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ intersect in at most one point (Incidence Postulate), so $E = E'$ and E lies on both $\overline{AC}$ and $\overline{BD}$.

(Theorem 4.6.8)

The quadrilateral $\square ABCD$ is convex if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ have an interior point in common.

Proof.

Assume that $\overline{AC}$ and $\overline{BD}$ have an interior point in common. We claim $\square ABCD$ is convex.

We will show that C is in the interior of $\angle DAB$. The other vertices are similar. By hypothesis, $\overrightarrow{AC}$ intersects $\overline{BD}$ at a point E in the interior of $\square ABCD$, so E is between B and D. Thus, $\overrightarrow{AC}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AD}$ (Theorem 3.3.10). Thus, C is in the interior of $\angle DAB$ (definition of angle interior). $\square$

The following corollary is left as homework.

(Corollary 4.6.9)

If $\square ABCD$ and $\square ACBD$ are both quadrilaterals, then $\square ABCD$ is not convex. If $\square ABCD$ is a nonconvex quadrilateral, then $\square ACBD$ is a quadrilateral.

Euclid put off using his fifth postulate until Proposition 29 of Elements. This result is the converse to the AIAT. Today we will show that this statement is actually equivalent to the Euclidean Parallel Postulate. We will also consider several other statements that are equivalent. These will not be theorems in neutral geometry, instead they will be theorems in any model in which we assume the Euclidean Parallel Postulate. On the other hand, we will also show that assuming any one of these statements is equivalent to assuming the Euclidean Parallel Postulate.

First we recall some statements:

Euclidean Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there is exactly one line m such that P lies on m and $m \parallel \ell$.

Alternate Interior Angles Theorem (Theorem 4.4.2)

If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ'.

Converse to the AIAT

If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.

(Theorem 4.7.1)

The Converse to the AIAT is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume the Converse to the AIAT (hypothesis). We will prove the Euclidean Parallel Postulate holds. That is, let ℓ be a line and let P be an external point. We must show that there exists a unique line m such that P lies on m and $m \parallel \ell$.

Using the Double Perpendicular Construction, we obtain a parallel line m and a transversal t that is perpendicular to ℓ and m. This proves existence and it is left only to prove uniqueness.

Let m' be any line such that P lies on m' and $m' \parallel \ell$. Then t is a transversal for ℓ and m'. Hence the alternate interior angles made by ℓ, m', and t must be congruent (Converse to AIAT). But this implies that $m' \perp t$, so $m' = m$ (ACP).

Converse to AIAT

Converse to the AIAT

If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.

(Theorem 4.7.1)

The Converse to the AIAT is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume the Euclidean Parallel Postulate (hypothesis). We will prove the Converse to the AIAT. That is, let ℓ and ℓ' be parallel lines cut by a transversal t. We must show that both pairs of alternate interior angles are congruent.

Let B' be the point where ℓ' intersects t. Let ℓ'' be a line through B' for which the alternate interior angles formed with ℓ are congruent (ACP). Then $\ell'' \parallel \ell$ (AIAT) and so $\ell'' = \ell'$ (Euclidean Parallel Postulate). Hence, the alternate interior angles formed by ℓ and ℓ' with t are congruent. □

Next time

Before next class: Continue reading Section 4.7.

In the next lecture we will:

- Consider additional statements equivalent to the Euclidean Parallel Postulate.
- Prove Euclid's Postulate V is equivalent to the Euclidean Parallel Postulate.
- Prove that Transitivity of Parallelism is equivalent to the Euclidean Parallel Postulate.

Chapter 4: Neutral geometry
§4.7 Statements equivalent to the Euclidean Parallel Postulate

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Consider additional statements equivalent to the Euclidean Parallel Postulate.
- Prove Euclid's Postulate V is equivalent to the Euclidean Parallel Postulate.
- Prove that Transitivity of Parallelism is equivalent to the Euclidean Parallel Postulate.

The statement we call the Euclidean Parallel Postulate is not the formulation given by Euclid himself, but instead is sometimes called *Playfair's Postulate* (after John Playfair)). It is logically equivalent to Euclid's Fifth Postulate.

Euclid's Postulate V

If ℓ and ℓ' are two lines cut by a transversal t in such a way that the sum of the measures of the two interior angles on one side of t is less that 180°, then ℓ and ℓ' intersect on that side of t.

(Theorem 4.7.2)

Euclid's Postulate V is equivalent to the Euclidean Parallel Postulate.

(Theorem 4.7.2)

Euclid's Postulate V is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume the Euclidean Parallel Postulate. We will show that Euclid's Postulate V holds. That is, let ℓ and ℓ' be two lines cut by a transversal t such that the sum of the measures of the two interior angles on one side of t is less than 180°. We will show that that ℓ and ℓ' intersect on that side of t.

Let B and B' be points where t cuts ℓ and ℓ', respectively. There is a line ℓ'' such that B' lies on ℓ'' and both pairs of nonalternating interior angles form by ℓ and ℓ'' with transversal t have measures whose sum is 180° and $\ell'' \neq \ell'$ (ACP). Then $\ell'' \parallel \ell$ (AIAT) so $\ell'' \nparallel \ell'$ (Euclidean Parallel Postulate). Thus there exists a point C that lies on both ℓ and ℓ' (negation of definition of parallel). Since $\triangle BB'C$ is a triangle, then the measure of two of the interior angles is strictly less than 180° (Lemma 4.5.3). Thus, C must be on the side of t where the interior angles formed by ℓ and ℓ with transversal t have measures whose sum is less than 180°.

(Theorem 4.7.2)

Euclid's Postulate V is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume Euclid's Postulate V. We will show that the Euclidean Parallel Postulate holds. That is, let ℓ be a line and P and external point. We will show that there exists a unique line parallel to ℓ and through P.

Using the Double Perpendicular Construction, we obtain a parallel line m and a transversal t that is perpendicular to ℓ and m. This proves existence and it is left only to prove uniqueness.

Let m' be any line such that P lies on m' and $m' \parallel \ell$. Then t is a transversal for ℓ and m' and the sum of the interior angles made by ℓ and m' is 180° on both sides (Euclid's Postulate V). But then the $m' \perp t$ and so $m = m'$ (Uniqueness of Perpendiculars). $\square$

Hilbert's Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there is at most one line m such that P lies on m and $m \parallel \ell$.

Since we have already proven existence of parallels in neutral geometry, it is clear that Hilbert's Parallel Postulate is equivalent to the Euclidean Parallel Postulate in neutral geometry.

We now consider several other statements equivalent to the Euclidean Parallel Postulate.

Equivalent statement

(Theorem 4.7.3)

Each of the following statements is equivalent to the Euclidean Parallel Postulate.

1. (Proclus's Axiom) If ℓ and ℓ' are parallel lines and $t \neq \ell$ is a line such that t intersects ℓ, then t also intersects ℓ'.
2. If ℓ and ℓ' are parallel lines and t is a transversal such that $t \perp \ell$, then $t \perp \ell'$.
3. If ℓ, m, n and k are lines such that $k \parallel \ell$, $m \perp k$, and $n \perp \ell$, then either $m = n$ or $m \parallel n$.
4. (Transitivity of Parallelism) If ℓ is parallel to m and m is parallel to n, then either $\ell = n$ or $\ell \parallel n$.

Equivalences for statements 1-3 are left as homework exercises. We will prove statement 4 is equivalent to the Euclidean Parallel Postulate.

Transitivity of Parallelism

(Theorem 4.7.3 (4))

Transitivity of Parallelism is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume Transitivity of Parallelism. We will show the Euclidean Parallel Postulate holds. That is, assume that ℓ is a line and P an exterior point. We claim there exists a unique line m such that P lies on m and $m \parallel \ell$.

Using the Double Perpendicular Construction, we obtain a line m such that $P \in m$ and $m \parallel \ell$. Let n be a line through P such that $n \parallel \ell$. Since $P \in n \cap m$, then $m \nparallel n$ and so $m = n$ (Transitivity of Parallelism).

(Theorem 4.7.3 (4))

Transitivity of Parallelism is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume the Euclidean Parallel Postulate. We will show Transitivity of Parallelism holds. That is, assume ℓ, m, and n are lines such that $\ell \parallel m$ and $m \parallel n$. Suppose $\ell \neq n$. We claim $\ell \parallel n$.

Let t be a transversal through ℓ, m, and n. Note that t exists by Proclus's Axiom. Then the alternate interior angles formed by ℓ, m, and t are congruent (Converse to the AIAT). Similarly, then alternate interior angles formed by m, n, and t are congruent (Converse to the AIAT). It follows that the alternate interior angles formed by ℓ, n, and t are congruent (Vertical Angles Theorem) and so $\ell \parallel n$ (AIAT). $\square$

Angle Sum Postulate

The next equivalence we will prove is with the Angle Sum Postulate.

Angle Sum Postulate

If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180^\circ$.

(Theorem 4.7.4)

The Euclidean Parallel Postulate is equivalent to the Angle Sum Postulate.

That the Euclidean Parallel Postulate implies the Angle Sum Postulate is fairly elementary. We need some setup in order to prove the converse.

Next time

Before next class: Finish reading Section 4.7.

In the next lecture we will:

- Prove the Angle Sum Postulate is equivalent to the Euclidean Parallel Postulate.
- State Wallis' Postulate and prove it is equivalent to the Euclidean Parallel Postulate.

Chapter 4: Neutral geometry
§4.7 Statements equivalent to the Euclidean Parallel Postulate

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Prove the Angle Sum Postulate is equivalent to the Euclidean Parallel Postulate.
- State Wallis' Postulate and prove it is equivalent to the Euclidean Parallel Postulate.

The next equivalence we will prove is with the Angle Sum Postulate.

Angle Sum Postulate

If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180^\circ$.

(Theorem 4.7.4)

The Euclidean Parallel Postulate is equivalent to the Angle Sum Postulate.

That the Euclidean Parallel Postulate implies the Angle Sum Postulate is fairly elementary. We need some setup in order to prove the converse.

(Lemma 4.7.5)

Suppose $\overline{PQ}$ is a segment and Q' is a point such that $\angle PQQ'$ is a right angle. For every $\epsilon > 0$ there exists a point T on $\overrightarrow{QQ'}$ such that $\mu(\angle PTQ) < \epsilon^\circ$.

Proof.

Let $\overline{PQ}$ be a segment and Q' a point such that $\angle PQQ'$ is a right angle and let $\epsilon > 0$. We claim there exists a point T on $\overrightarrow{QQ'}$ such that $\mu(\angle PTQ) < \epsilon^\circ$.

Choose P' on the same side of $\overleftrightarrow{PQ}$ as Q' such that $\overleftrightarrow{PP'} \perp \overleftrightarrow{PQ}$. Then every point T on $\overrightarrow{QQ'}$ is in the interior of $\angle QPP'$ (definition of interior).

Choose T_1 on $\overrightarrow{QQ'}$ such that $\overline{PQ} \cong \overline{QT_1}$ (Ruler Postulate). Then choose T_2 on $\overrightarrow{T_1 Q'}$ such that $\overline{PT_1} \cong \overline{T_1 T_2}$. Continuing this way we choose T_n such that T_{n-1} is between Q and T_n such that $\overline{PT_{n-1}} \cong \overline{T_{n-1} T_n}$. In this way we obtain a sequence of points $T_1, T_2, \ldots$ on $\overrightarrow{QQ'}$.

(Lemma 4.7.5)

Suppose $\overline{PQ}$ is a segment and Q' is a point such that $\angle PQQ'$ is a right angle. For every $\epsilon > 0$ there exists a point T on $\overrightarrow{QQ'}$ such that $\mu(\angle PTQ) < \epsilon^\circ$.

Proof.

Now $\angle QT_nP \cong \angle T_{n-1}PT_n$ for every n (Isosceles Triangle Theorem). For each n,

$$\mu(\angle QPT_1) + \mu(\angle T_1PT_2) + \cdots + \mu(\angle T_{n-1}PT_n) = \mu(\angle QPT_n) < \mu(\angle QPP') = 90^\circ. \quad (1)$$

(AAP). Suppose $\mu(\angle T_{i-1}PT_i) \geq \epsilon$ for every i (RAA hypothesis). There is an n for which $n\epsilon > 90$ (Archimedean Property of Real Numbers). It follows that

$$\mu(\angle QPT_1) + \mu(\angle T_1PT_2) + \cdots + \mu(\angle T_{n-1}PT_n) \geq n\epsilon > 90^\circ.$$

This contradicts (1). Hence, we reject the RAA hypothesis and conclude that there exists an i such that $\mu(\angle T_{i-1}PT_i) < \epsilon^\circ$. Hence, $\mu(\angle QT_iP) < \epsilon^\circ$ and so $T = T_i$ satisfies the conclusion of the lemma. □

(Theorem 4.7.4)

The Euclidean Parallel Postulate is equivalent to the Angle Sum Postulate.

Proof.

Assume the Euclidean Parallel Postulate. We will show the Angle Sum Postulate holds. That is, assume $\triangle ABC$ is a triangle. We must show that $\sigma(\triangle 180^\circ)$.

Using the Double Perpendicular Construction we obtain a line m such that B lies on m and $m \parallel \overleftrightarrow{AC}$. Let D and E be points on m such that $D * B * E$. By the AAP and the Linear Pair Theorem,

$$180^\circ = \mu(\angle ABD) + \mu(\angle ABC) + \mu(\angle CBE).$$

Note that $\overleftrightarrow{AB}$ is a transversal for m and $\overleftrightarrow{AC}$, so $\mu(\angle ABD) = \mu(\angle BAC)$ (Converse to AIAT). Similarly, $\mu(\angle CBE) = \mu(\angle BCA)$. Hence,

$$180^\circ = \mu(\angle BAC) + \mu(\angle ABC) + \mu(\angle BCA) = \sigma(\triangle ABC).$$

(Theorem 4.7.4)

The Euclidean Parallel Postulate is equivalent to the Angle Sum Postulate.

Proof.

Assume the Angle Sum Postulate. We will show the Euclidean Parallel Postulate holds. That is, assume that ℓ is a line and P an exterior point. Let m be the line through P obtained through the Double Perpendicular Construction. We claim m is the unique line parallel through P and parallel to ℓ.

Suppose there exists a line m' distinct from m such that P lies on m' and $m' \parallel \ell$. (RAA hypothesis). We will show that there exists a triangle whose angle sum is not 180°, contradicting the Angle Sum Postulate.

Let Q be the foot of the perpendicular from P to ℓ. Choose a point S on m' such that S is on the same side of m as Q. Choose a point R on m such that R is on the same side of $\overleftrightarrow{PQ}$ as S. Choose a point T on ℓ such that T lies on the same side of $\overleftrightarrow{PQ}$ as S and $\mu(\angle QTP) < \mu(\angle SPR)$ (Lemma 4.7.5). We claim $\sigma(\triangle PQT) < 180^\circ$.

(Theorem 4.7.4)

The Euclidean Parallel Postulate is equivalent to the Angle Sum Postulate.

Proof.

Since $\overrightarrow{PS}$ does not intersect $\overline{QT}$, then T is in the interior of $\angle QPS$. Thus, $\mu(\angle QPT) < \mu(\angle QPS)$ (AAP). We chose S such that S is in the interior of $\angle QPR$, so $\mu(\angle SPR) + \mu(\angle SPQ) = \mu(\angle RPQ)$ (AAP). Thus,

$$\begin{aligned}\sigma(\triangle QTP) &= \mu(\angle PQT) + \mu(\angle QTP) + \mu(\angle TPQ)\\ &< \mu(\angle PQT) + \mu(\angle SPR) + \mu(\angle QPS)\\ &= \mu(\angle PQT) + \mu(\angle RPQ)\\ &= 180^\circ\end{aligned}$$

because angle $\angle PQT$ and $\angle RPQ$ are right angles. □

Definition 1

Triangle $\triangle ABC$ and $\triangle DEF$ are *similar* if $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \angle FDE$.

Our notation for similar triangles is $\triangle ABC \sim \triangle DEF$.

Wallis' Postulate

If $\triangle ABC$ is a triangle and $\overline{DE}$ is a segment, then there exists a point F such that $\triangle ABC \sim \triangle DEF$

(Theorem 4.7.7)

Wallis' Postulate is equivalent to the Euclidean Parallel Postulate.

(Theorem 4.7.7)

Wallis' Postulate is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume the Euclidean Parallel Postulate. We will prove that Wallis' Postulate holds. That is, assume that $\triangle ABC$ is a triangle and $\overline{DE}$ is a segment. We claim there exists a point F such that $\triangle ABC \sim \triangle DEF$.

There exists a ray $\overrightarrow{DG}$ such that angle $\angle EDG \cong \angle BAC$ and there exists a ray $\overrightarrow{EH}$, with H on the same side of $\overleftrightarrow{DE}$ as G, such that $\angle DEH \cong \angle ABC$ (ACP). Then

$$\mu(\angle EDG) + \mu(\angle DEH) = \mu(\angle CAB) + \mu(\angle ABC) < 180^\circ$$

(definition of congruence, Lemma 4.5.3). Hence, there exists a point F where $\overrightarrow{DG}$ and $\overrightarrow{EH}$ intersect (Euclid's Postulate V). Since $\sigma(\triangle ABC) = \sigma(\triangle DEF) = 180^\circ$ (Angle Sum Postulate), then $\mu(\angle EFD) = \mu(\angle BCA)$, so $\sigma(\triangle ABC) \sim \sigma(\triangle DEF)$.

(Theorem 4.7.7)

Wallis' Postulate is equivalent to the Euclidean Parallel Postulate.

Proof.

Assume the Wallis' Postulate. We will prove that holds Euclidean Parallel Postulate. That is, assume that ℓ is a line and P an exterior point. Let m be the line through P obtained through the Double Perpendicular Construction. We claim m is the unique line parallel through P and parallel to ℓ.

Let $m' \neq m$ be a line through P that is parallel to ℓ (RAA hypothesis). Let Q be the foot of the perpendicular from P to ℓ. Choose a point S on m' such that S and Q are on the same side of m. Drop a perpendicular from S to $\overleftrightarrow{PQ}$ and call the foot of that perpendicular R. Then there exists a point T such that $\triangle PRS \sim \triangle PQT$ (Wallis' Postulate). Since $\angle PQT$ is a right angle, then T must lie on ℓ (ACP). This contradicts the choice of m', so we reject the RAA hypothesis and conclude that $m' \neq m$. $\square$

Next time

Before next class: Read Section 4.8.

In the next lecture we will:

- Define defect of a triangle and quadrilateral and prove additivity of defect.
- Prove that the Euclidean Parallel Postulate is equivalent to the existence of a rectangle.

Chapter 4: Neutral geometry
§4.8 Rectangles and defect

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Define defect of a triangle and quadrilateral and prove additivity of defect.
- Prove that the Euclidean Parallel Postulate is equivalent to the existence of a rectangle.

Defect

We have shown already that the Euclidean Parallel Postulate is equivalent to every triangle having angle sum 180°. Here we show that in any model, if one triangle has angle sum 180°, then *every* triangle has angle sum 180°. We will also show that existence of a rectangle is equivalent to the Euclidean Parallel Postulate.

Definition 1

The *defect* of a triangle $\triangle ABC$ is defined by

$$\delta(\triangle ABC) = 180 - \sigma(\triangle ABC).$$

(By Saccheri-Legendre, $\delta(\triangle ABC) \geq 0$. We call a triangle *defective* if its defect is positive.

Similarly, the *defect* of a convex quadrilateral $\square ABCD$ defined by

$$\delta(\square ABCD) = 360 - \sigma(\square ABCD).$$

Additivity of Defect (Theorem 4.8.2)

1. If $\triangle ABC$ is a triangle and E is a point in the interior of $\overline{BC}$, then

$$\delta(\triangle ABC) = \delta(\triangle ABE) + \delta(\triangle ECA).$$

2. If $\square ABCD$ is a convex quadrilateral, then

$$\delta(\square ABCD) = \delta(\triangle ABC) + \delta(\triangle ACD).$$

The proof of these statements is left as a homework exercise.

Definition 2

A *rectangle* is a quadrilateral each of whose angles is a right angle.

It is easy to prove that a rectangle is a parallelogram (by AIAT) and hence convex (by Theorem 4.6.6). In general, rectangles *do not* exist in models for neutral geometry. In fact, they exist if and only if the Euclidean Parallel Postulate is true in that model.

Zero-defect objects

The goal for the remainder of this section will be to prove the following set of equivalences regarding zero-defect objects.

(Theorem 4.8.4)

The following statements are equivalent.

1. There exists a triangle whose defect is 0°.
2. There exists a right triangle whose defect is 0°.
3. There exists a rectangle.
4. There exist arbitrarily large rectangles.
5. The defect of every right triangle is 0°.
6. The defect of every triangle is 0°.

(Corollary 4.8.5)

In any model for neutral geometry, there exists one triangle whose defect is 0° if and only if every triangle in that model has defect 0°.

Zero-defect objects

It follows immediately from the Saccheri-Legendre theorem that at least two angles in any triangle are acute.

(Lemma 4.8.6)

Let $\triangle ABC$ be a triangle. If the interior angles at vertices A and B are acute, then the foot of the perpendicular from C to $\overleftrightarrow{AB}$ is between A and B.

Proof.

Let $\triangle ABC$ be a triangle with acute interior angles at vertices A and B. Let D be the foot of the perpendicular from C to $\overleftrightarrow{AB}$. We claim D is between A and B.

Suppose D is not between A and B (RAA hypothesis). If $D = A$, then $\angle CAB$ is a right angle contradicting our hypothesis. Hence, $D \neq A$ and similarly $D \neq B$. Then either $D * A * B$ or $A * B * D$. The two cases are similar so assume $D * A * B$. Then $\angle CAB$ is an exterior angle for $\triangle ADC$ with remote interior angle $\angle ADC$. This contradicts the Exterior Angle Theorem. Thus we reject the RAA hypothesis and conclude that D is between A and B. □

Zero-defect objects

Proof of Theorem 4.8.4.

(1) $\Rightarrow$ (2) Let $\triangle ABC$ be a triangle such that $\delta(\triangle ABC) = 0^\circ$. We claim that there exists a right triangle whose defect is 0°.

After relabeling vertices, we may assume that the angles at vertices A and B are acute. Drop a perpendicular from C to $\overleftrightarrow{AB}$, then $A * D * B$ (Lemma 4.8.4). Hence, triangles $\triangle ADC$ and $\triangle CDB$ are right triangles. Moreover, $\delta(\triangle ABC) = \delta(\triangle ADC) + \delta(\triangle CDB)$ (Additivity of Defect). Since $\delta(\triangle ABC) = 0^\circ$, then $\delta(\triangle ADC) = \delta(\triangle CDB) = 0^\circ$.

(2) $\Rightarrow$ (3) Assume there exists a triangle $\triangle ABC$ with right angle $\angle ABC$ such that $\delta(\triangle ABC) = 0^\circ$. We claim that there exists a rectangle.

There exists a point D on the opposite side of $\overleftrightarrow{AC}$ from B such that $\triangle CDA \cong \triangle ABC$. Since $\delta(\triangle ABC) = 0^\circ$, then $\mu(\angle BAC) + \mu(\angle BCA) = 90^\circ$ (definition of defect). Thus, $\mu(\angle DCA) + \mu(\angle DAC) = 90^\circ$ (definition of congruent triangles). Thus, $\mu(\angle DAB) + \mu(\angle BCD) = 180^\circ$. But $\angle DAB \cong \angle BCD$ (definition of congruent triangles) so $\mu(\angle BAD) = \mu(\angle BCD) = 90^\circ$ (AAP). Hence, $\square ABCD$ is a rectangle (definition of a rectangle).

Zero-defect objects

Proof of Theorem 4.8.4.

(3) $\Rightarrow$ (4) Let $\square ABCD$ be a rectangle. Choose a point E on $\overrightarrow{AB}$ such that $A * B * E$ and $AB = BE$ and choose a point F on $\overrightarrow{DC}$ such that $D * C * F$ and $DC = CD$ (Ruler Postulate). Then $\triangle ABC \cong \triangle EBC$ (SAS) so $\mu(\angle BCE) = \mu(\angle BCA) < 90^\circ$. Hence, E is in the interior of angle BCF (Btw Thm for Rays) and so $\angle ACD \cong \angle ECF$ (AAP). Thus, $\triangle ADC \cong \triangle ECF$ (SAS).

Now $0^\circ = \delta(\square ABCD) = \delta(\triangle ABC) + \delta(\triangle ACD)$ (Additivity of Defect) so $\delta(\triangle ABC) = \delta(\triangle ACD) = 0^\circ$. Hence, $\mu(\angle BAC) + \mu(\angle BCA) = 90^\circ$. The rectangle $\square ABCD$ is convex so C is in the interior of $\angle BAD$. Hence, $\mu(\angle BAC) + \mu(\angle CAD) = 90^\circ$. Thus, $\angle CAD \cong \angle BCA$ and similarly $\angle ACD \cong \angle CAB$. It follows that $\triangle ABC \cong \angle CDA$ (SAS).

Since $\triangle ABC \cong \triangle CDA \cong \triangle CFE$, then $\angle CFE$ is a right angle. Similarly we can show that $\angle BEF$ is a right angle. Moreover, $AE = 2AB = DF$. Repeating this process we can produce a rectangle $\square AEGH$ such that $AH = 2AD = EG$. Iterating this process (and through the Archimedean Property of Real Numbers) will produce an arbitrarily large rectangle.

Proof of Theorem 4.8.4.

(4) $\Rightarrow$ (5) Let $\triangle ABC$ be a right triangle with a right angle at vertex C. We claim $\delta(\triangle ABC) = 0^\circ$.

By (4), there exists a rectangle $\square DEFG$ such that $DG > AC$ and $FG > BC$. Choose a point B' on $\overline{GF}$ such that $GB' = CB$ and a point A' on $\overline{GD}$ such that $GA' = CA$. Then $\triangle ABC \cong \triangle A'B'G$ (SAS). To simplify notation, let $A = A'$ and $B = B'$.

The rectangle $\square CDEF$ is convex (Theorem 4.6.6) and has defect 0°. Hence, $\delta(\triangle DEF) = \delta(\triangle CDF) = 0^\circ$ (Additivity of Defect). Then

$$0^\circ = \delta(\triangle CDF) = \delta(\triangle ADF) + \delta(\triangle AFC)$$

(Additivity of Defect) and so $\delta(\triangle AFC) = 0^\circ$. Applying one more time gives

$$0^\circ = \delta(\triangle AFC) = \delta(\triangle AFB) + \delta(\triangle ABC)$$

(Additivity of Defect) and so $\delta(\triangle ABC) = 0^\circ$.

Proof of Theorem 4.8.4.

(5) $\Rightarrow$ (6) Let $\triangle ABC$ be a triangle such that the interior angles at vertices A and B are acute. Drop a perpendicular from C to $\overleftrightarrow{AB}$ and denote the foot of the perpendicular by D. Then $A * D * B$ (Lemma 4.8.6) and triangles $\triangle ADC$ and $\triangle BDC$ are right triangles. Thus, $\delta(\triangle ABC) = \delta(\triangle ADC) + \delta(\triangle BDC) = 0^\circ$ by (5).

(6) $\Rightarrow$ (1) This is obvious. $\square$

Our work so far has led to the following.

(Clairaut's Axiom)

There exists a rectangle.

(Corollary 4.8.7)

Clairaut's Axiom is equivalent to the Euclidean Parallel Postulate.

Next time

Before next class: Read Section 4.9.

In the next lecture we will:

- Define Saccheri and Lambert quadrilaterals.
- Study properties of these quadrilaterals.
- State and prove the Universal Hyperbolic Theorem.

Chapter 4: Neutral geometry
§4.8 Rectangles and defect
§4.9 The Universal Hyperbolic Theorem

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Foundations of Geometry

It’s good to have goals

Goals for today:

- Define Saccheri and Lambert quadrilaterals.
- Study properties of these quadrilaterals.
- State and prove the Universal Hyperbolic Theorem.

Rectangles

In the previous lecture we considered existence of rectangles and proved that they only exist in models where the Euclidean Parallel Postulate holds.

(Clairaut's Axiom)

There exists a rectangle.

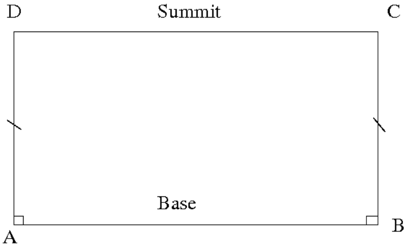
(Corollary 4.8.7)

Clairaut's Axiom is equivalent to the Euclidean Parallel Postulate.

However, other similar objects exist in ubiquity in neutral geometry.

Definition 1

A *Saccheri quadrilateral* is a quadrilateral $\square ABCD$ such that $\angle ABC$ and $\angle DAB$ are right angles and $\overline{AD} \cong \overline{BC}$. The segment $\overline{AB}$ is called the *base* and $\overline{CD}$ the *summit*. The two right angles are the *base angles* and the other two the *summit angles*.



Saccheri quadrilaterals

Properties of Saccheri quadrilaterals (Theorem 4.8.10)

If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$, then

1. the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent
2. the summit angles $\angle BCD$ and $\angle ADC$ are congruent,
3. the segment joining the midpoint of $\overline{AB}$ to the midpoint of $\overline{CD}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$,
4. $\square ABCD$ is a parallelogram,
5. $\square ABCD$ is a convex quadrilateral, and
6. the summit angles $\angle BCD$ and $\angle ADC$ are either right or acute.

Properties of Saccheri quadrilaterals (Theorem 4.8.10)

If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$, then

1. the diagonals $\overline{AC}$ and $\overline{BD}$ are congruent,
2. the summit angles $\angle BCD$ and $\angle ADC$ are congruent,

Proof.

(1) Consider the triangles $\triangle ACB$ and $\triangle BDA$. Since $\overline{AD} \cong \overline{CB}$ and $\angle DAB \cong \angle CBA$ (definition of Saccheri quadrilaterals), then $\triangle ACB \cong \triangle BDA$ (SAS). Thus, $\overline{AC} \cong \overline{BD}$ (definition of congruent triangles).

(2) Consider the triangles $\triangle ADC$ and $\triangle BCD$. By (1), $\overline{AC} \cong \overline{BD}$. Since $\overline{AD} \cong \overline{BC}$ (definition of Saccheri quadrilaterals), then $\triangle ADC \cong \triangle BCD$ (SSS). Thus, $\angle ADC \cong \angle BCD$ (definition of congruent triangles).

Properties of Saccheri quadrilaterals (Theorem 4.8.10)

If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$, then

3. the segment joining the midpoint of $\overline{AB}$ to the midpoint of $\overline{CD}$ is perpendicular to both $\overline{AB}$ and $\overline{CD}$,

Proof.

(3) Let M be the midpoint of $\overline{AB}$ and N the midpoint of $\overline{CD}$ (Existence of Midpoints). Then $\overline{CN} \cong \overline{DN}$ (definition of the midpoint). Thus, $\triangle AND \cong \triangle BNC$ (part (2) and SSS). Now $\overline{AN} \cong \overline{BN}$ (definition of congruent triangles). Since $\overline{AM} \cong \overline{BM}$ (definition of the midpoint) then $\triangle ANM \cong \triangle BNM$ (SSS). It follows that $\angle AMN \cong \angle BMN$ and they form a linear pair. Thus, they are right angles and one can show similarly that $\angle DNM$ and $\angle CNM$ are right angles.

Properties of Saccheri quadrilaterals (Theorem 4.8.10)

If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$, then

4. $\square ABCD$ is a parallelogram,
5. $\square ABCD$ is a convex quadrilateral, and

Proof.

(4) Let A' be a point on $\overleftrightarrow{AD}$ such that $A' * A * D$ and $A' \neq A$ (Ruler Postulate). Then $\angle A'AB$ is a right angle and angles $\angle A'AB$ and $\angle ABC$ are a pair of alternate interior angles for the lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ with transversal $\overleftrightarrow{AB}$ (definition of Saccheri quadrilaterals). Thus, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ (AIAT). Similarly, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$.

(5) We proved on a previous homework assignment that parallelograms are convex.

Properties of Saccheri quadrilaterals (Theorem 4.8.10)

If $\square ABCD$ is a Saccheri quadrilateral with base $\overline{AB}$, then

6. the summit angles $\angle BCD$ and $\angle ADC$ are either right or acute

Proof.

(6) By (2), $\angle BCD \cong \angle ADC$. Set $x = \mu(\angle BCD) = \mu(\angle ADC)$. Because $\square ABCD$ is convex,

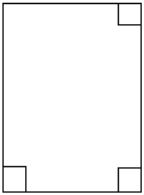
$$\begin{aligned} 360^\circ &\geq \sigma(\square ABCD) \\ &= \mu(\angle ABC) + \mu(\angle BCD) + \mu(\angle CDA) + \mu(\angle DAB) \\ &= 90^\circ + x + x + 90^\circ. \end{aligned}$$

Thus, $180^\circ \geq 2x$, so $x \leq 90^\circ$. $\square$

Lambert quadrilaterals

Definition 2

A *Lambert quadrilateral* is a quadrilateral in which three of the angles are right angles.



Properties of Lambert quadrilaterals (Theorem 4.8.11)

If $\square ABCD$ is a Lambert quadrilateral with right angles are vertices A, B and C, then

1. $\square ABCD$ is a parallelogram,
2. $\square ABCD$ is a convex quadrilateral,
3. angle $\angle ADC$ is either right or acute, and
4. $BC \leq AD$.

Proof.

(1) This is similar to the proof of part (4) in the previous theorem.

(2) This follows because $\square ABCD$ is a parallelogram by part (1).

(3) This is similar to part (6) in the previous theorem.

Properties of Lambert quadrilaterals (Theorem 4.8.11)

If $\square ABCD$ is a Lambert quadrilateral with right angles are vertices A, B and C, then

1. $\square ABCD$ is a parallelogram,
2. $\square ABCD$ is a convex quadrilateral,
3. angle $\angle ADC$ is either right or acute, and
4. $BC \leq AD$.

Proof.

(4) Suppose $AD < BC$ (RAA hypothesis). Choose a point D' on $\overrightarrow{AD}$ such that $AD' = BC$ (PCP), so $D \neq D'$. Hence, $\square AD'CB$ is a Saccheri quadrilateral, so $\angle D'CB$ is either right or acute (Properties of Saccheri quadrilaterals (6)). Since D is between A and D', then $\overrightarrow{CD}$ is between $\overrightarrow{CD'}$ and $\overrightarrow{CB}$. Thus,

$$90 \geq \mu(\angle D'CB) = \mu(\angle D'CD) + \mu(\angle DCB) = \mu(\angle D'CD) + 90$$

(ACP). Then $0 \geq \mu(\angle D'CD)$, a contradiction. Thus, we reject the RAA hypothesis and conclude that $AD \geq BC$. $\square$

Just as existence of triangles with defect 0° is all or nothing in neutral geometry, so two are unique parallels. That is, as the next theorem asserts, uniqueness of a parallel for one line and external point guarantees unique parallels for *all* lines and all possible external points.

The Universal Hyperbolic Theorem (Theorem 4.9.1)

If there exists one line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0, then for every line ℓ and for every external point P there exist at least two lines that pass through P and are parallel to ℓ.

(Corollary 4.9.2)

The Universal Hyperbolic Theorem is equivalent to the negation of the Euclidean Parallel Postulate.

(Corollary 4.9.3)

In any model for neutral geometry either the Euclidean Parallel Postulate or the Hyperbolic Parallel Postulate will hold.

The Universal Hyperbolic Theorem (Theorem 4.9.1)

If there exists one line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0, then for every line ℓ and for every external point P there exist at least two lines that pass through P and are parallel to ℓ.

Proof.

Assume there exists a line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0. Let ℓ be a line and P an external point. We claim there exist at least two lines through P that are parallel to ℓ.

Let m be the line through P parallel to m obtained by the Double Perpendicular Construction and let Q be the foot of that perpendicular. Choose a point R on ℓ that is different for Q and let t be the line through R that is perpendicular to ℓ. Drop a perpendicular from P to t and call the foot of that perpendicular S.

Now $\square PQRS$ is a Lambert quadrilateral. By our hypothesis, the Euclidean Parallel Postulate fails and so $\square PQRS$ is not a rectangle (Corollary 4.8.7). Hence, $\angle QPS$ is not a right angle (ACP) and so $\overleftrightarrow{PS} \neq m$. But $\overleftrightarrow{PS}$ is parallel to ℓ (AIAT). $\square$

Next time

Before next class: Read Sections 5.1 and 5.2.

In the next lecture we will:

- Begin our study of Euclidean geometry.
- State and prove the Parallel Projection Theorem.

Chapter 5: Euclidean geometry
§5.1 Basic Theorems of Euclidean Geometry
§5.2 The Parallel Projection Theorem

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Foundations of Geometry

It's good to have goals

Goals for today:

- Begin our study of Euclidean geometry.
- State and prove the Parallel Projection Theorem.

We now add the Euclidean Parallel Postulate to our list of axioms. Our goal in this chapter will be to develop the theory sufficiently so as to prove the following fundamental results.

Fundamental Theorem on Similar Triangles (Theorem 5.3.1)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then

$$\frac{AB}{AC} = \frac{DE}{DF}.$$

Pythagorean Theorem (Theorem 5.4.1)

If $\triangle ABC$ is a right triangle with right angle at vertex C, then

$$(AC)^2 + (BC)^2 = (AB)^2.$$

Since we are assuming the Euclidean Parallel Postulate, the following statements which we proved equivalent are now all theorems in Euclidean Geometry.

Converse to the AIAT (Theorem 5.1.1)

If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.

Euclid's Postulate V (Theorem 5.1.2)

If ℓ and ℓ' are two lines cut by a transversal t in such a way that the sum of the measures of the two interior angles on one side of t is less that 180°, then ℓ and ℓ' intersect on that side of t.

Angle Sum Theorem (5.1.3)

If $\triangle ABC$ is a triangle, then $\sigma(\triangle ABC) = 180^\circ$.

Wallis' Postulate

If $\triangle ABC$ is a triangle and $\overline{DE}$ is a segment, then there exists a point F such that $\triangle ABC \sim \triangle DEF$.

Since we are assuming the Euclidean Parallel Postulate, the following statements which we proved equivalent are now all theorems in Euclidean Geometry.

Proclus's Axiom (Theorem 5.1.5)

If ℓ and ℓ' are parallel lines and $t \neq \ell$ is a line such that t intersects ℓ, then t also intersects ℓ'.

(Theorem 5.1.6)

If ℓ and ℓ' are parallel lines and t is a transversal such that $t \perp \ell$, then $t \perp \ell'$.

(Theorem 5.1.7)

If ℓ, m, n and k are lines such that $k \parallel \ell$, $m \perp k$, and $n \perp \ell$, then either $m = n$ or $m \parallel n$.

Transitivity of Parallelism (Theorem 5.1.8)

If ℓ is parallel to m and m is parallel to n, then either $\ell = n$ or $\ell \parallel n$.

Since we are assuming the Euclidean Parallel Postulate, the following statements which we proved equivalent are now all theorems in Euclidean Geometry.

Clairaut's Axiom (Theorem 5.1.9)

There exists a rectangle.

Proofs of the next results are left as an exercise.

Properties of Euclidean Parallelograms (Theorem 5.1.10)

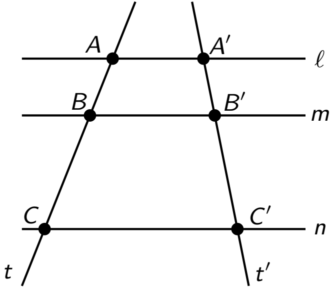
If $\square ABCD$ is a parallelogram, then

1. the diagonals divide the quadrilateral into congruent triangles (i.e., $\triangle ABC \cong \triangle CDA$ and $\triangle ABD \cong \triangle CDB$),
2. the opposite sides are congruent (i.e., $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$),
3. the opposite angles are congruent (I.e. $\angle DAB \cong \angle BCD$ and $\angle ABC \cong \angle CDA$),
4. the diagonals bisect each other (i.e., $\overline{AC}$ and $\overline{BD}$ intersect in a point E that is the midpoint of each).

The Parallel Projection Theorem (Theorem 5.2.1)

Let ℓ, m and n be distinct parallel lines. Let t be transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume $A * B * C$, then

$$\frac{AB}{AC} = \frac{A'B'}{A'C'}.$$



First we need a lemma.

(Lemma 5.2.2)

Let ℓ, m and n be distinct parallel lines. Let t be transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume $A * B * C$. If $\overline{AB} \cong \overline{BC}$, then $\overline{A'B'} \cong \overline{B'C'}$.

Proof.

Let ℓ, m and n be distinct parallel lines. Let t be transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume $A * B * C$ and $\overline{AB} \cong \overline{BC}$. We claim $\overline{A'B'} \cong \overline{B'C'}$.

If $A' \neq A$, let t'' be the line through A' such that $t'' \parallel t$ (EPP). If $A' = A$, then let $t'' = t$. Similarly, if $B' \neq B$, let t'' be the line through B' that is parallel to t; otherwise let $t''' = t$. Define B'' to be the point at which t'' crosses m and C''' to be the point at which t''' crosses n. Note that all of these points exist by Proclus's Axiom.

(Lemma 5.2.2)

Let ℓ, m and n be distinct parallel lines. Let t be transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume $A * B * C$. If $\overline{AB} \cong \overline{BC}$, then $\overline{A'B'} \cong \overline{B'C'}$.

Proof.

If $A' = A$, then $B'' = B$ and $\overline{AB} = \overline{A'B''}$. Similarly, if $B' = B$, then $C''' = C$ and $\overline{BC} = \overline{B'C'''}$. If $A' \neq A$ and $B' \neq B$, then $\overline{A'B''} \cong \overline{AB}$ and $\overline{B'C'''} \cong \overline{BC}$ (Properties of Euclidean Parallelograms). Since $\overline{AB} \cong \overline{BC}$ (hypothesis), then $\overline{A'B''} \cong \overline{B'C'''}$ (in all cases).

If $B'' = B'$, then $C''' = C$ and the proof is complete. Otherwise, $t'' \parallel t'''$ or $t'' = t'''$ (Transitivity of Parallelism). Hence, $\angle B''A'B' \cong \angle C'''B'C'$ and $\angle A'B''B' \cong \angle B'C'''C'$ (Converse to Corresponding Angles Theorem). Thus, $\triangle B''A'B' \cong \triangle C'''B'C'$ (ASA). It follows that $\overline{A'B'} \cong \overline{B'C'}$ (definition of congruent triangles). □

We will also need the following property of real numbers.

Comparison Theorem (Theorem E.3.3)

If x and y are any real numbers such that

(1) every rational number that is less that x is also less than y, and

(2) every rational number that is less than y is also less than x,

then $x = y$.

We will not prove this result here, though it is proved in the appendix of the book and makes use of the Density Theorem.

Density Theorem (Theorem E.3.2)

If a and b are real numbers such that $a < b$, then there exists a rational number x such that $a < x < b$ and there exists an irrational number y such that $a < y < b$.

Proof of the Parallel Projection Theorem.

First consider the case that AB/AC is a rational number. That is, $AB/AC = p/q$ for positive integers p and q. Choose points $A_0, A_1, \ldots, A_q$ on t such that $A_0 = A$, $A_q = C$, and for each i, $A_iA_{i+1} = AC/q$ (Ruler Postulate). Then $A_p = B$ and for each i, $1 \leq i \leq q$, there exists a line ℓ_i such that A_i lies on ℓ_i and $\ell_i \parallel \ell$ (DPC). Let $A_0' = A'$ and let A_i', $i \geq 1$, be the point at which ℓ_i crosses t' (Proclus's Axiom). Then $A_i'A_{i+1}' = A'C'/q$ for each i (Lemma 5.2.2). Since $\ell_p = m$ and $\ell_q = n$, we must have $A_p' = B'$ and $A_q' = C'$. It follows that

$$\frac{A'B'}{A'C'} = \frac{A_0'A_p'}{A_0'A_q'} = \frac{(A'C'/q)p}{(A'C'/q)q} = \frac{p}{q} = \frac{(AC/q)p}{(AC/q)q} = \frac{A_0A_p}{A_0A_q} = \frac{AB}{AC}.$$

This completes the proof in the case AB/AC is rational.

Proof of the Parallel Projection Theorem.

Now consider the case that $AB/AC = x$ (a possibly irrational number) and $A'B'/A'C' = y$. Let t be a rational number such that $0 < r < x$. Choose a point D on t such that $AD/AC = r$. Let m' be the line such that D lies on m' and $m' \parallel m$ (DPC), and let D' be the point at which m' meets t'. (Proclus's Axiom) Then $A'D'/A'C' = r$ (by the previous part). Since ℓ, m, and m' are parallel, $A' * D' * B'$ and therefore

$$r = \frac{A'D'}{A'C'} < \frac{A'B'}{A'C'} = y.$$

A similar arguments shows that if r is rational number such that $0 < r < y$, then $r < x$. Hence, $x = y$ (Comparison Theorem). □

Next time

Before next class: Read Section 5.3.

In the next lecture we will:

- Prove the Fundamental Theorem on Similar Triangles.
- Consider its consequences, including the SAS Similarity Criterion.
- Use the Fundamental Theorem to prove the Pythagorean Theorem.

Chapter 5: Euclidean geometry
§5.3 Similar Triangles
§5.4 The Pythagorean Theorem
§5.5 Trigonometry

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Foundations of Geometry

It's good to have goals

Goals for today:

- Prove the Fundamental Theorem on Similar Triangles.
- Consider its consequences, including the SAS Similarity Criterion.
- Use the Fundamental Theorem to prove the Pythagorean Theorem.
- Discuss trigonometry.

Recall last time we proved the following theorem.

The Parallel Projection Theorem (Theorem 5.2.1)

Let ℓ, m and n be distinct parallel lines. Let t be transversal that cuts these lines at points A, B, and C, respectively, and let t' be a transversal that cuts these lines at points A', B', and C', respectively. Assume $A * B * C$, then

$$\frac{AB}{AC} = \frac{A'B'}{A'C'}.$$

With this result in hand, we are now ready to prove a major theorem in Euclidean geometry.

Fundamental Theorem on Similar Triangles (Theorem 5.3.1)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then

$$\frac{AB}{AC} = \frac{DE}{DF}.$$

Fundamental Theorem on Similar Triangles (Theorem 5.3.1)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then

$$\frac{AB}{AC} = \frac{DE}{DF}.$$

Proof.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\triangle ABC \sim \triangle DEF$. We claim $AB/AC = DE/DF$. There are three cases: either $AB = DE$, $AB > DE$, or $AB < DE$ (trichotomy). In the first case, the result is obvious. The second two cases are similar, so we will prove the case $AB > DE$.

Choose a point B' on $\overrightarrow{AB}$ such that $AB' = DE$ (PCP). Let m be the line through B' such that m is parallel to $\ell = \overleftrightarrow{BC}$ (EPP) and let C' be the point at which m intersects $\overline{AC}$ (Pasch's Axiom). Then $\angle AB'C' \cong \angle ABC \cong \angle DEF$ (Converse to AIAT) so $\triangle AB'C' \cong \triangle DEF$ (ASA). Let n be the line through A that is parallel to ℓ and m (EPP and ToP). Thus $AB'/AB = AC'/AC$ (Parallel Projection Theorem) and $DE/AB' = DF/AC'$ (definition of congruent triangles). Hence, $DE/DF = AB/AC$ (algebra). $\square$

The following is an alternate way to view the previous result.

(Corollary 5.3.2)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then there is a positive number r such that

$$DE = r \cdot AB, DF = r \cdot AC, \text{ and } EF = r \cdot BC.$$

We call r the *common ratio* of the sides of the similar triangles.

Just as with congruent triangles, there are more efficient ways to check for similarity than by using the definition. The following could be taken as an axiom because it is equivalent to the Euclidean Parallel Postulate.

SAS Similarity Criterion (Theorem 5.3.3)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$ and $AB/AC = DE/DF$, then $\triangle ABC \sim \triangle DEF$.

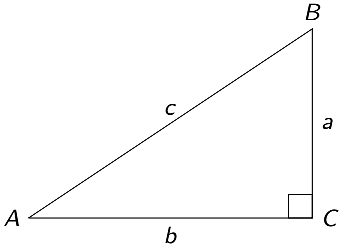
Converse to the Similar Triangles Theorem (Theorem 5.3.4)

If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $AB/DE = AC/DF = BC/EF$, then $\triangle ABC \sim \triangle DEF$.

The Pythagorean Theorem

We introduce some notation for triangles to simplify the exposition in this section.

Let $\triangle ABC$ be a triangle. Denote $\angle CAB$ by $\angle A$, $\angle ABC$ by $\angle B$, and $\angle ACB$ by $\angle C$. The corresponding lowercase letter is used to denote the length of the opposite side, so $a = BC$, $b = AC$, and $c = AB$. If $\triangle ABC$ is a right triangle, then the right angle is always located at C.



Pythagorean Theorem (Theorem 5.4.1)

If $\triangle ABC$ is a right triangle with right angle at vertex C, then $a^2 + b^2 = c^2$.

Pythagorean Theorem (Theorem 5.4.1)

If $\triangle ABC$ is a right triangle with right angle at vertex C, then $a^2 + b^2 = c^2$.

Proof.

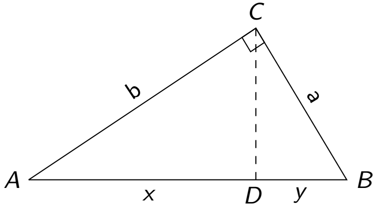
Drop a perpendicular from C to $\overleftrightarrow{AB}$ and call the foot of that perpendicular D. Then D is in the interior of $\overline{AB}$ (Lemma 4.8.6). Now $\mu(\angle A) + \mu(\angle B) = 90^\circ$ and $\mu(\angle A) + \mu(\angle ACD) = 90^\circ$ (Angle Sum Theorem). Thus, $\angle B \cong \angle ACD$ and similarly $\angle A \cong \angle DCB$. Hence, there are similar triangles

$$\triangle ABC \sim \triangle CBD \sim \triangle ACD.$$

Let $x = AD$, $y = BD$, and $h = CD$. Then $x/b = b/c$ and $y/a = a/c$ (Fund. Thm. of Similar Triangles). Thus, $b^2 = cx$ and $a^2 = cy$, so $a^2 + b^2 = c(x + y)$ (algebra). Since $x + y = c$, the proof is complete. $\square$

The Pythagorean Theorem

In the right triangle $\triangle ABC$, the segment $\overline{CD}$ is an *altitude* of the triangle and its length, $h = CD$, is called the height of the triangle. The segment $\overline{AD}$ is called the *projection* of $\overline{AC}$ onto $\overline{AB}$. Similarly, $\overline{BD}$ is the projection of $\overline{BC}$ onto $\overline{AB}$.



Definition 1

The *geometric mean* of two positive numbers x and y is defined to be $\sqrt{xy}$.

Using this terminology, we can rephrase the Pythagorean Theorem.

(Theorem 5.4.3)

The height of a right triangle is the geometric mean of the lengths of the projections of the legs.

The Pythagorean Theorem

(Theorem 5.4.4)

The length of one leg of a right triangles the geometric mean of the length of the hypothenuse and the length of the projection of that leg onto the hypotenuse.

Converse to the Pythagorean Theorem (Theorem 5.4.4)

If $\triangle ABC$ is a triangle such that $a^2 + b^2 = c^2$, then $\angle BCA$ is a right angle.

The Fundamental Theorem of Similar Triangles and the Pythagorean Theorem form the foundation for studying trigonometry.

Definition 2

Let θ be an acute angle with vertex A. Then θ consists of two rays with a common endpoint A. Choose a point B on one of the rays and drop a perpendicular to the other ray. Call the foot of that perpendicular C. Define the *sine* and *cosine* functions by

$$\sin\theta = \frac{BC}{AB} \quad \text{and} \quad \cos\theta = \frac{AC}{AB}.$$

If θ is an obtuse angle, let θ' denote its supplement. Define

$$\sin\theta = \sin\theta' \quad \text{and} \quad \cos\theta = -\cos\theta'.$$

If θ has measure 0°, define

$$\sin\theta = 0 \quad \text{and} \quad \cos\theta = 1.$$

If θ is a right angle, define

$$\sin\theta = 1 \quad \text{and} \quad \cos\theta = 0.$$

The trigonometric functions have domain the set of angles, but by the Fundamental Theorem of Similar Triangles, they only depend on the angle measure. Hence, we may regard then as functions on the interval $[0, 180)$. The range (image) of sine is $[0, 1]$ and the range (image) of cosine is $(-1, 1]$.

The following theorem follows almost directly from the Pythagorean Theorem.

Pythagorean Identity (Theorem 5.5.2)

For any angle θ, $\sin^2\theta + \cos^2\theta = 1$.

The next pair of theorems hold for *any* triangle. The proofs of these statements is part of your next homework.

Law of Sines (Theorem 5.5.3)

If $\triangle ABC$ is any triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Law of Cosines (Theorem 5.5.4)

If $\triangle ABC$ is any triangle, then

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Next time

Before next class: Read the Chapter 6 introduction and Section 6.1.

In the next lecture we will:

- Introduce hyperbolic geometry.
- Review basic theorems in hyperbolic geometry and properties of quadrilaterals.
- Prove the AAA congruence condition.

Chapter 6: Hyperbolic geometry
§6.1 Basic Theorems of Hyperbolic Geometry.

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Introduce hyperbolic geometry.
- Review basic theorems in hyperbolic geometry and properties of quadrilaterals.
- Prove the AAA congruence condition.

First we recall a result from neutral geometry.

The Universal Hyperbolic Theorem (Theorem 4.9.1)

If there exists one line ℓ_0, an external point P_0, and at least two lines that pass through P_0 and are parallel to ℓ_0, then for every line ℓ and for every external point P there exist at least two lines that pass through P and are parallel to ℓ.

(Corollary 4.9.2)

The Universal Hyperbolic Theorem is equivalent to the negation of the Euclidean Parallel Postulate.

(Corollary 4.9.3)

In any model for neutral geometry either the Euclidean Parallel Postulate or the Hyperbolic Parallel Postulate will hold.

In hyperbolic geometry we assume all of the axioms of neutral geometry and the Hyperbolic Parallel Postulate. Our goal in this section will be to develop enough of an understanding of Hyperbolic geometry so that we can outline a model (the Poincaré Disk Model).

Hyperbolic Parallel Postulate

For every line ℓ and for every point P that does not lie on ℓ, there are at least two lines m and n such that P lies on both m and n and both m and n are parallel to ℓ.

Hence, by (the corollary to) the Universal Hyperbolic Theorem, any statement equivalent to the Euclidean Parallel Postulate is *false* in hyperbolic geometry. Thus, the next results are immediate.

Basic theorems in hyperbolic geometry

(Theorem 6.1.1)

For every triangle $\triangle ABC$, $\sigma(\triangle ABC) < 180^\circ$.

(Corollary 6.1.2)

For every triangle $\triangle ABC$, $0^\circ < \delta(\triangle ABC) < 180^\circ$.

(Theorem 6.1.3)

For every convex quadrilateral $\square ABCD$, $\sigma(\square ABCD) < 360^\circ$.

Basic theorems in hyperbolic geometry

(Corollary 6.1.4)

The summit angles in a Saccheri quadrilateral are acute.

(Corollary 6.1.4)

The fourth angles in a Lamber quadrilateral is acute.

(Theorem 6.1.6)

There does not exist a rectangle.

(Theorem 6.1.7)

In a Lambert quadrilateral, the length of a side between two right angles is strictly less than the length of the opposite side.

Proof.

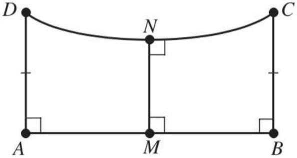
Let $\square ABCD$ be a Lambert quadrilateral. Assume that the angles at vertices A, B, and C are right angles. We claim $BC < AD$. Since $BC \leq AD$ (Properties of Lambert quadrilaterals), it suffices to prove $BC \neq AD$.

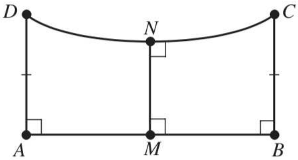
Suppose $BC = AD$ (RAA hypothesis). Then $\square ABCD$ is a Saccheri quadrilateral so $\angle BAD \cong \angle CDA$ (Properties of Saccheri quadrilaterals). Hence, $\square ABCD$ is a rectangle, contradicting Theorem 6.1.6. So we reject the RAA hypothesis and conclude that $BC \neq AD$. $\square$

Definition 1

Let $\square ABCD$ be a Saccheri quadrilateral with base $\overline{AB}$. The segment joining the midpoint of $\overline{AB}$ to the midpoint of $\overline{CD}$ is called the *altitude* of the quadrilateral. The length of the altitude is called the *height* of the quadrilateral.

By properties of Saccheri quadrilaterals, the altitude is perpendicular to the base and the summit.





(Corollary 6.1.9)

In a Saccheri quadrilateral, the length of the altitude is less than the length of a side.

(Corollary 6.1.10)

In a Saccheri quadrilateral, the length of the summit is greater than the length of the base.

The next result proves that in hyperbolic geometry, similarity implies congruence in triangles.

AAA (Theorem 6.1.1)

If $\triangle ABC$ is similar to $\triangle DEF$, then $\triangle ABC$ is congruent to $\triangle DEF$.

Proof.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\triangle ABC \sim \triangle DEF$. We claim $\triangle ABC \cong \triangle DEF$.

Suppose one pair of sides are congruent. Then $\triangle ABC \cong \triangle DEF$ (ASA). We now assume there are no pair of congruent sides (RAA hypothesis). WLOG, we assume $AB > DE$ and $AC > DF$.

Choose a point B' on $\overrightarrow{AB}$ such that $AB' = DE$ and a point C' on $\overrightarrow{AC}$ such that $AC' = DF$ (PCP). Then $\square BCC'B'$ is convex (Theorem 4.6.7) and $\triangle AB'C' \cong \triangle DEF$ (SAS). Thus $\angle AB'C' \cong \angle ABC$ and $\angle AC'B' \cong \angle ACB$ (definition of congruent triangle). Since $\mu(\angle CC'B') = 180 - \mu(\angle AC'B')$ and $\mu(\angle BB'C') = 180 - \mu(\angle AB'C')$ (LPT), then $\sigma(\square BCC'B') = 360^\circ$, contradicting Theorem 6.1.3. Therefore we reject the RAA hypothesis and conclude that $\triangle ABC \cong \triangle DEF$. $\square$

(Theorem 6.1.12)

If $\square ABCD$ and $\square A'B'C'D'$ are two Saccheri quadrilaterals such that $\delta(\square ABCD) = \delta(\square A'B'C'D')$ and $\overline{CD} \cong \overline{C'D'}$, then $\square ABCD \cong \square A'B'C'D'$.

Proof.

Let $\square ABCD$ and $\square A'B'C'D'$ be two Saccheri quadrilaterals such that $\delta(\square ABCD) = \delta(\square A'B'C'D')$ and $\overline{CD} \cong \overline{C'D'}$. We claim that $\square ABCD \cong \square A'B'C'D'$.

WLOG, suppose $AD \leq A'D'$, so also $BC \leq B'C'$. Choose a point E on $\overrightarrow{AD}$ such that $DE = D'A$" and a point F on $\overrightarrow{CB}$ such that $CF = C'B'$. We have $\angle ADC \cong \angle A'D'C' \cong \angle DCB \cong D'C'B'$ (Properties of Saccheri Triangles). Hence $\triangle EDC \cong \triangle A'D'C'$ (SAS). Since Saccheri quadrilaterals are convex, then $\angle ECF \cong \angle A'C'B'$ (AAP) so $\triangle ECF \cong \angle A'C'B'$ (SAS). Thus $\angle EFC$ is a right angle and a similar proof shows the same for $\angle FED$. It follows that $\square EFCD \cong \square A'B'C'D'$.

Suppose $E \neq A$ (RAA hypothesis). Then $\overleftrightarrow{AB} \parallel \overleftrightarrow{EF}$ (AIAT), so $F \neq B$ and $\square ABFE$ is a rectangle, contradicting Theorem 6.1.3. Thus we reject the RAA hypothesis and conclude that $A = E$ and $B = F$. □

Next time

Before next class: Read Section 6.2.

In the next lecture we will:

- Discuss parallel and perpendicular lines in hyperbolic geometry.
- Define common perpendiculars and use this to consider alternate interior angles.

Chapter 6: Hyperbolic geometry
§6.2 Common perpendiculars

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Discuss parallel and perpendicular lines in hyperbolic geometry.
- Define common perpendiculars and use this to consider alternate interior angles.

Distance between lines

In Euclidean geometry, parallel lines are lines that are the same distance apart. That is, if $\ell \parallel m$ and P and Q are distinct points on m, then $d(P, \ell) = d(Q, \ell)$. The next result shows that this is not a good characterization of parallel lines in hyperbolic geometry.

(Theorem 6.2.1)

If ℓ is a line, P is an external point, and m is a line such that P lies on m, then there exists at most one point Q such that $Q \neq P$, Q lies on m, and $d(Q, \ell) = d(P, \ell)$.

Note that this theorem does not guarantee the existence of the point Q. When the point $\ell \parallel m$ and Q does exist, then ℓ and m admit a *common perpendicular*. We will return to this after proving theorem.

Distance between lines

(Theorem 6.2.1)

If ℓ is a line, P is an external point, and m is a line such that P lies on m, then there exists at most one point Q such that $Q \neq P$, Q lies on m, and $d(Q, \ell) = d(P, \ell)$.

Proof.

Let ℓ and m be two lines. We claim there exists at most one point on m distinct from P that is equidistant from ℓ. Suppose there exist three distinct points P, Q, and R on m such that $d(P, \ell) = d(Q, \ell) = d(R, \ell)$ (RAA hypothesis). Denote the feet of their perpendiculars by P', Q', and R', respectively.

Since $d(P, \ell) > 0$, then none of the points P, Q, and R lie on ℓ. Hence, at least two points lie on the same side of ℓ (PSP). WLOG, assume P and Q lie on the same side of ℓ, so $\square PP'Q'Q$ is a Saccheri quadrilateral. It follows that $\ell \parallel m$ (properties of Saccheri quadrilaterals) and so all the points lie on the same side of ℓ.

WLOG assume that $P * Q * R$. Then $\square PP'Q'Q$ and $\square QQ'R'R$ are Saccheri quadrilaterals. Thus, the angles $\angle PQQ'$ and $\angle RQQ'$ are acute (Corollary 6.1.4). But $\angle PQQ'$ and $\angle RQQ'$ are supplements (LPT), a contradiction. Thus we reject the RAA hypothesis and conclude there cannot be three points on m equidistant from ℓ. $\square$

Definition 1

Lines ℓ and m *admit a common perpendicular* if there exists a line n such that $n \perp m$ and $n \perp \ell$. If ℓ and m admit a common perpendicular, then the line n intersects ℓ at a point P and intersects m at a point Q; the line n is called the common perpendicular (line) while the segment $\overline{PQ}$ is called a *common perpendicular segment*.

(Theorem 6.2.3)

If ℓ and m are parallel lines and there exist two points on m that are equidistant from ℓ, then ℓ and m admit a common perpendicular.

(Theorem 6.2.4)

If lines ℓ and m admit a common perpendicular, then that common perpendicular is unique.

The Converse to the AIAT is equivalent to the Euclidean Parallel Postulate and hence is not valid in hyperbolic geometry. *However*, the Converse to the AIAT asserts that parallel lines imply congruent alternate interior angles *for all* parallel lines. It can still happen in hyperbolic geometry that the conclusion holds for certain parallel lines and certain transversals.

(Theorem 6.2.5)

Let ℓ and m be parallel lines cut by a transversal t. Alternate interior angles formed by ℓ and m with transversal t are congruent if and only if ℓ and m admit a common perpendicular and t passes through the midpoint of the common perpendicular segment.

(Theorem 6.2.5)

Let ℓ and m be parallel lines cut by a transversal t. Alternate interior angles formed by ℓ and m with transversal t are congruent if and only if ℓ and m admit a common perpendicular and t passes through the midpoint of the common perpendicular seg.

Proof of Theorem 6.2.5.

Let ℓ and m be parallel lines cut by a transversal t. Let R be the point at which t crosses ℓ and let S be the point at which t crosses m.

Assume that ℓ and m admit a common perpendicular and t passes through the midpoint of common perpendicular segment. We claim the alternate interior angles formed by t are congruent. If t is the common perpendicular, then the proof is complete so we assume otherwise.

Let P and Q be the points where the common perpendicular intersects ℓ and m, respectively, and let M be the midpoint of $\overline{PQ}$. Then M is the unique point where t intersects $\overleftrightarrow{PQ}$. Thus, $\overrightarrow{MS}$ and $\overrightarrow{MR}$ are opposite rays and $\overrightarrow{MQ}$ and $\overrightarrow{MP}$ are opposite rays. Now $\angle SMP \cong \angle RMP$ (Vertical Angles Theorem), $\angle MPR \cong \angle MQS$ (definition of perpendicular), and $MS = MR$ (definition of midpoint). Thus, $\triangle MRP \cong \triangle MSQ$ (AAS), so $\angle MRP \cong \angle MSQ$ (definition of congruent triangles).

Proof of Theorem 6.2.5.

Assume both pairs of alt. int. angles formed by ℓ and m with transversal t are congruent. We claim ℓ and m admit a common perpendicular and that t passes through the midpoint of that segment. If the alt. int. angles are right angles, then t is the common perpendicular. We assume henceforth that they are not right angles.

Let M be the midpoint of $\overline{RS}$. Drop perpendiculars from M to ℓ and m and call the feet P and Q, respectively. Suppose P and Q lie on the same side of t (RAA hypothesis). The alt. int. angles formed by t are congruent and so one of $\triangle MRP$ and $\triangle MQS$ has an obtuse external angle, contradicting the Exterior Angle Theorem. Thus, we reject the RAA hypothesis and conclude that P and Q lie on opposite sides of t.

Now $\angle MPR \cong \angle MQS$ (definition of perpendicular), $\angle MRP \cong \angle MSQ$ (hypothesis), and $MS = MR$ (definition of midpoint). Thus, $\triangle SQM \cong \triangle RPM$ (AAS), so $\angle SMQ \cong \angle RMP$. It follows that $\overrightarrow{MP}$ and $\overrightarrow{MQ}$ are opposite rays so $\overline{PQ}$ is a common perpendicular segment for ℓ and m. $\square$

Next time

Before next class: Read Section 6.3.

In the next lecture we will:

- Discuss ways to construct additional parallel lines in hyperbolic geometry.
- Define the angle of parallelism.
- Define the critical function and consider its properties.

Chapter 6: Hyperbolic geometry
§6.3 The Angle of Parallelism

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Foundations of Geometry

It's good to have goals

Goals for today:

- Discuss ways to construct additional parallel lines in hyperbolic geometry.
- Define the angle of parallelism.

Our goal is to characterize, in some meaningful way, parallel lines in hyperbolic geometry.

Construction

Let ℓ be a line and let P be an external point. Drop a perpendicular from P to ℓ and call the foot of that perpendicular A. Let $B \neq A$ be a point on ℓ. For each real number r with $0 \leq r \leq 90$ there exists a point D_r, on the same side of $\overleftrightarrow{PA}$ as B, such that $\mu(\angle APD_r) = r^\circ$ (ACP). Define

$$K = \{r : \overrightarrow{PD_r} \cap \overrightarrow{AB} \neq \emptyset\}.$$

Note that $0 \in K$ (take $D_r = A$) and $90 \notin K$ (since $\overleftrightarrow{PD_{90}} \parallel \ell$). Hence, $K \subset [0, 90)$.

Definition 1

The set K is called the *intersecting set* for P and $\overrightarrow{AB}$.

Intersecting Set

(Theorem 6.3.2)

Let K be the intersecting set for P and $\overrightarrow{AB}$. If $r \in K$, then

1. $s \in K$ for every s with $0 < s < r$, and
2. there exists $t \in K$ such that $t > r$.

Proof.

Let K be the intersecting set for P and $\overrightarrow{AB}$ and fix a number $r \in K$.
1. Let R be the point at which $\overrightarrow{PD_r}$ intersects $\overrightarrow{AB}$ (definition of intersecting set). If $0 < s < r$, then $\overrightarrow{PD_s}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PD_r} = \overrightarrow{PR}$. Thus $\overrightarrow{PD_s}$ intersects $\overline{AR}$ at a point S (Crossbar Theorem) and $s \in K$ (definition of intersecting set).

2. Choose a point T such that $A * R * T$ (Ruler Postulate) and define $t = \mu(\angle APT)$. Since $\overrightarrow{PD_t} \cap \overrightarrow{AB} \neq \emptyset$, then $t \in K$ (definition of intersecting set) and $t > r$ (Protractor Postulate). $\square$

Critical Number

Definition 2

The number r_0 such that $K = [0, r_0)$ is called the *critical number* for P and $\overrightarrow{AB}$.

Note that the critical number is well-defined because of Theorem 6.3.2. That is, the theorem guarantees the existence of such a number. In particular, this is an application of the following.

Definition 3

Let A be a set of real numbers. A number b is called an *upper bound* for A if $x \leq b$ for every $x \in A$. The number b_0 is called the *least upper bound* for A if b_0 is an upper bound for A and $b_0 \leq b$ for every upper bound b of A.

The Least Upper Bound Postulate

If A is any nonempty set of real numbers that has an upper bound, then A has a least upper bound.

Angle of Parallelism

Definition 4

Suppose P, A, and B are as in the definition of the intersecting set and that r_0 is the critical number for P and $\overrightarrow{AB}$. Let D be a point on the same side of $\overleftrightarrow{PA}$ as B such that $\mu(\angle APD) = r_0$). The angle $\angle APD$ is called the *angle of parallelism* for P and $\overrightarrow{AB}$.

The angle of parallelism, and the critical number, do not depend in any meaningful way on the choices we made in our construction (e.g., the points A and B).

(Theorem 6.3.5)

The critical number depends only on $d(P, \ell)$.

Proof.

Let ℓ be a line, P an external point, A the foot of the perpendicular from P to ℓ, and B a point on ℓ with $B \neq A$. Let P', ℓ', A', and B' be another setup such that $PA = P'A'$. We claim the critical number for P and $\overrightarrow{AB}$ is equal to the critical number for P' and $\overrightarrow{A'B'}$. It suffices to show that the intersecting set K for P and $\overrightarrow{AB}$ is equal to the intersecting set for P' and $\overrightarrow{A'B'}$.

Suppose $r \in K$. The $\overrightarrow{PD_r}$ intersects $\overrightarrow{AB}$ at a point T. Choose a point T' on $\overrightarrow{A'B'}$ such that $A'T' = AT$. Then $\triangle PAT \cong \triangle P'A'T'$ (SAS) and so $r \in K'$. Hence $K \subset K'$. A similar argument shows that $K' \subset K$, so $K = K'$. □

It follows that if we had chosen B' so that $\overrightarrow{AB}$ and $\overrightarrow{AB'}$ are opposite rays, then the two angles of parallelism on either side of $\overleftrightarrow{PA}$ are congruent.

Next time

Before next class: Read Section 6.4.

In the next lecture we will:

- Define the critical function and consider its properties.
- Define limiting parallel rays.
- Prove existence and uniqueness of limiting parallel rays.

Chapter 6: Hyperbolic geometry
§6.3 The Angle of Parallelism
§6.4 Limiting Parallel Rays

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Foundations of Geometry

It's good to have goals

Goals for today:

- Define the critical function and consider its properties.
- Define limiting parallel rays.
- Prove existence and uniqueness of limiting parallel rays.

Critical Function

Definition 1

Given a positive real number x, locate a point P and a line ℓ such that $d(P, \ell) = x$. Then define $\kappa(x)$ to be the critical number associated with P and ℓ; that is, choose a point $B \neq A$ on ℓ and define $\kappa(x) = \mu(\angle APD)$, where $\angle APD$ is the angle of parallelism for P and $\overrightarrow{AB}$. The function $\kappa : (0, \infty) \to (0, 90]$ is called the *critical function*.

By Theorem 6.3.5, κ is a well-defined function and does not depend on the particular choice of P, ℓ, or B.

(Theorem 6.3.7)

The critical function $\kappa : (0, \infty) \to (0, 90]$ is nonincreasing.

Proof.

Let a and b be two positive numbers such that $a < b$. We claim $\kappa(a) \geq \kappa(b)$. Choose P, A, B, and D to be points such that $PA = a$ and $\angle APD$ is the angle of parallelism for P and $\overrightarrow{AB}$. Define $r = \mu(\angle APD)$. Then choose Q to be a point on $\overrightarrow{AP}$ such that $QA = b$. It suffices to show that the angle of parallelism at Q has angle measure at most r.

Choose a point E on the same side of $\overleftrightarrow{AP}$ as B such that $\mu(\angle AQE) = r$. Then $\overleftrightarrow{QE} \parallel \overleftrightarrow{PD}$ (Corresponding Angles Theorem). Hence, $\overrightarrow{QE} \cap \overrightarrow{AB} = \emptyset$. Thus, r is not in the intersecting set for Q and $\overrightarrow{AB}$, so r cannot be less than the critical number for Q and $\overrightarrow{AB}$ (Theorem 6.3.2). The result follows. $\square$

Critical Function

Note that the definitions and theorems we have stated are all valid in neutral geometry. In Euclidean geometry, the angle of parallelism is always 90°. The following is the hyperbolic version.

(Theorem 6.3.8)

Every angle of parallelism is acute and every critical number is less than 90.

Proof.

Let ℓ be a line and let P be an external point. Drop a perpendicular from P to ℓ and call the foot A. We claim $\kappa(PA) < 90$.

Let m be the line such that P lies on m and $m \perp \overleftrightarrow{PA}$. Hence, $m \parallel \ell$. There exists a line n such that $n \neq m$, P lies on n, and $n \parallel \ell$ (Hyperbolic Parallel Postulate). Since n is not perpendicular to $\overleftrightarrow{PA}$, then angle between n and $\overrightarrow{PA}$ must measure less than 90° on one side or the other. The measure of that angle is not in the intersecting set, so the critical number cannot be larger than this measure. Thus, the critical number is less than 90 and the angle of parallelism is acute. $\square$

Critical Function

There are many other properties of the critical function that we could prove (but won't for lack of time). Here is a summary.

Previously we proved that the critical function is nonincreasing and our proof was valid in neutral geometry. Here are some relevant results which hold in hyperbolic geometry. (See Section 6.7 for more details.)

Properties of the critical function

Let $\kappa : (0, \infty) \to (0, 90]$ be the critical function. The following hold:

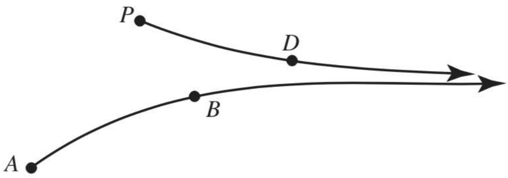
- (Theorem 6.7.1) The function κ is decreasing.
- (Theorem 6.7.3) $\lim_{x \to \infty} \kappa(x) = 0$.
- (Theorem 6.7.4) $\lim_{x \to 0^+} \kappa(x) = 90$.
- (Theorem 6.7.5) The function κ is surjective.
- (Corollary 6.7.6) The function κ is continuous.

To summarize, as P moves further away from the line ℓ, then the angle of parallelism decreases, tending towards 0.

Limiting parallel rays

Definition 2

Two rays $\overrightarrow{PD}$ and $\overrightarrow{AB}$ are called *limiting parallel rays*, written $\overrightarrow{PD} \mid \overrightarrow{AB}$, if B and D are on the same side of $\overleftrightarrow{PA}$, $\overrightarrow{PD} \cap \overrightarrow{AB} = \emptyset$, and every ray between $\overrightarrow{PD}$ and $\overrightarrow{PA}$ intersects $\overrightarrow{AB}$.



If $\angle APD$ is the angle of parallelism for P and $\overrightarrow{AB}$, then $\overrightarrow{PA} \mid \overrightarrow{AB}$ by definition.

(Theorem 6.4.2)

If $\overrightarrow{PD} \mid \overrightarrow{AB}$, then $\overleftrightarrow{PD} \parallel \overleftrightarrow{AB}$.

Proof.

Let $\overrightarrow{PD}$ and $\overrightarrow{AB}$ be two rays such that $\overrightarrow{PD} \mid \overrightarrow{AB}$. We claim that $\overleftrightarrow{PD} \parallel \overleftrightarrow{AB}$. Suppose there exists a point Q that lies on both $\overleftrightarrow{PD}$ and $\overleftrightarrow{AB}$ (RAA hypothesis).

Choose points D' and B' such that $\overrightarrow{PD'}$ is opposite $\overrightarrow{PD}$ and $\overrightarrow{AB'}$ is opposite $\overrightarrow{AB}$. Since B and D are on the same side of $\overleftrightarrow{AB}$ (defn of limiting parallel rays), then D' and B' are on the opposite side of $\overleftrightarrow{AB}$. Now $\overrightarrow{PD} \cap \overrightarrow{AB'} = \emptyset$ and $\overrightarrow{AB} \cap \overrightarrow{PD'} = \emptyset$ (Z-Theorem). Moreover, $\overrightarrow{PD} \cap \overrightarrow{AB} = \emptyset$ (defn of limiting parallel rays), so $Q \in \overrightarrow{PD'} \cap \overrightarrow{AB'}$.

Since $\mu(\angle APD) > \mu(\angle AQP)$ (Exterior Angle Theorem), then there exists a ray $\overrightarrow{PE}$ between $\overrightarrow{PA}$ and $\overrightarrow{PD}$ such that $\mu(\angle EPD) = \mu(\angle AQP)$ (ACP). Then $\overleftrightarrow{PE} \parallel \overleftrightarrow{QA}$ (Corr Angles Theorem), so $\overrightarrow{PE} \cap \overrightarrow{AB} = \emptyset$. This contradicts the definition of limiting parallel rays so we reject the RAA hypothesis and conclude that $\overleftrightarrow{PD} \parallel \overleftrightarrow{AB}$. $\square$

Symmetry of Limiting Parallelism (Theorem 6.4.3)

If $\overrightarrow{PD} \mid \overrightarrow{AB}$, then $ray AB \mid \overrightarrow{PD}$.

Proof.

Let $\overrightarrow{PD}$ and $\overrightarrow{AB}$ be two rays such that $\overrightarrow{PD} \mid \overrightarrow{AB}$. We claim $\overleftrightarrow{AB} \parallel \overleftrightarrow{PD}$. By definition, it suffices to prove that every ray between $\overrightarrow{AP}$ and $\overrightarrow{AB}$ intersects $\overrightarrow{PD}$.

Suppose there exists a ray $\overrightarrow{AE}$ between $\overrightarrow{AP}$ and $\overrightarrow{AB}$ such that $\overrightarrow{AE} \cap \overrightarrow{PD} = \emptyset$ (RAA hypothesis). There is a point T on $\overrightarrow{AB}$ such that $\mu(\angle PTA) < \mu(\angle EAB)$ (Lemma 4.7.5). Since the ray $\overrightarrow{AE}$ is between $\overrightarrow{AP}$ and $\overrightarrow{AB}$, then T is in the interior of $\angle PAB$ and so T and B are on the same side of $\overleftrightarrow{AP}$. Hence T and D are on the same side of $\overleftrightarrow{AP}$. As $\overline{AT} \subset \overrightarrow{AT}$, so $\overline{AT} \cap \overrightarrow{PD} = \emptyset$. Then $\overline{AT}$ does not intersect the ray opposite $\overrightarrow{PD}$ (Z-Theorem). Thus A and T are on the same side of $\overleftrightarrow{PD}$. Thus, T is in the interior of $\angle APD$ and so $\overrightarrow{AT}$ intersects $\overrightarrow{AB}$ in a point S (defn of limiting parallel ray).

Since T is in the interior of $\angle PAB = \angle PAS$, we have $P * T * S$ (Theorem 3.5.3). Hence $\angle PTA$ is an exterior angle for $\triangle TAS$ whose measure is smaller than $\mu(\angle TAS)$. This contradicts the Exterior Angle Theorem so we reject the RAA hypothesis. $\square$

Further results

For the sake of time, we omit many of the proofs of further results.

Endpoint Independence (Theorem 6.4.4)

If $\overrightarrow{AB}$ is a ray and P, Q, and D are points such that $Q * P * D$, then $\overrightarrow{PD} \mid \overrightarrow{AB}$ if and only if $\overrightarrow{QD} \mid \overrightarrow{AB}$.

Existence and Uniqueness of Limiting Parallel Rays (Theorem 6.4.5)

If $\overrightarrow{AB}$ is a ray and P is a point that does not lie on $\overleftrightarrow{AB}$, then there exists a unique ray $\overrightarrow{PD}$ such that $\overrightarrow{PD} \mid \overrightarrow{AB}$.

Definition 3

Two rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be *equivalent rays* if either $\overrightarrow{AB} \subset \overrightarrow{CD}$ or $\overrightarrow{CD} \subset \overrightarrow{AB}$.

Transititivity of Limiting Parallelism (Theorem 6.4.7)

If $\overrightarrow{AB}$, $\overrightarrow{CD}$, and $\overrightarrow{EF}$ are three rays such that $\overrightarrow{AB} \mid \overrightarrow{CD}$ and $\overrightarrow{CD} \mid \overrightarrow{EF}$, then either $\overrightarrow{AB} \mid \overrightarrow{EF}$ or $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are equivalent rays.

Next time

Before next class: Read Sections 6.5 and 6.6.

In the next lecture we will:

- Define asymptotic triangles in terms of limiting rays.
- Prove standard triangle results (such as the SAS) for asymptotic triangles.
- Classify parallel lines in hyperbolic geometry.

Chapter 6: Hyperbolic geometry
§6.5 Asymptotic Triangles

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Define asymptotic triangles in terms of limiting rays.
- Prove standard triangle results (such as the SAS) for asymptotic triangles.
- Classify parallel lines in hyperbolic geometry.

Asymptotic Triangles

Definition 1

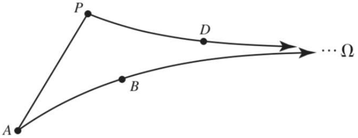
Two rays $\overrightarrow{PD}$ and $\overrightarrow{AB}$ are called *limiting parallel rays*, written $\overrightarrow{PD} \mid \overrightarrow{AB}$, if B and D are on the same side of $\overleftrightarrow{PA}$, $\overrightarrow{PD} \cap \overrightarrow{AB} = \emptyset$, and every ray between $\overrightarrow{PD}$ and $\overrightarrow{PA}$ intersects $\overrightarrow{AB}$.

We now use limiting rays to define a different type of "triangle". Since limiting rays do not intersect, we can imagine that there is a point at infinity, denoted Ω in which the triangles intersect.

Definition 2

An *asymptotic triangle* consists of two limiting parallel rays together with the line segment joining the endpoints of the ray. Specifically, if $\overrightarrow{PD} \mid \overrightarrow{AD}$, then

$$\triangle DPAB = \overrightarrow{PD} \cup \overline{PA} \cup \overrightarrow{AB}.$$



The next few results demonstrate that asymptotic triangles share many properties with ordinary triangles.

Exterior Angle Theorem (Theorem 6.5.2)

If $\triangle DPAB$ is an asymptotic triangle and $C * A * B$, then $\mu(\angle CAP) > \mu(\angle APD)$.

Proof.

Let $\triangle DPAB$ be an asymptotic triangle and let C be a point such that $C * A * B$. We claim $\mu(\angle CAP) > \mu(\angle APD)$. Choose a point G on the opposite side of $\overleftrightarrow{PA}$ from B and D such that $\angle GAP \cong \angle APD$ (ACP). Choose a point H so that $\overrightarrow{AH}$ is opposite $\overrightarrow{AG}$. It suffices then to prove that $\mu(\angle PAH) > \mu(\angle PAB)$ (LPT).

Let M be the midpoint of $\overline{AB}$ (Existence of Midpoints). The lines $\overleftrightarrow{PD}$ and $\overleftrightarrow{AG}$ admit a common perpendicular and that perpendicular passes through M (Theorem 6.2.5). Let E and F be the points at which the common perpendicular intersects $\overleftrightarrow{PD}$ and $\overleftrightarrow{AG}$, respectively.

Asymptotic Triangles

Exterior Angle Theorem (Theorem 6.5.2)

If $\triangle DPAB$ is an asymptotic triangle and $C * A * B$, then $\mu(\angle CAP) > \mu(\angle APD)$.

Proof.

Suppose $\overrightarrow{AH} \mid \overrightarrow{PD}$ (RAA hypothesis). Then $\overrightarrow{FH} \mid \overrightarrow{ED}$ (Endpoint Independence). But the angle of parallelism of E with $\overrightarrow{FH}$ is actue, while $\overleftrightarrow{ED} \perp \overleftrightarrow{EF}$, contradicting Theorem 6.3.8. Thus we reject the RAA hypothesis.

If H were in the interior of $\angle PAB$, then $\overrightarrow{AH}$ would intersect $\overrightarrow{PD}$ (definition of limiting parallel rays). Thus, H is not in the interior so B is in the interior of $\angle PAH$ (Lemma 3.4.4). Thus, $\mu(\angle PAH) > \mu(\angle PAB)$ (Btw Thm for Rays). □

The following corollary is left as a homework exercise.

Angle Sum Theorem (Corollary 6.5.3)

If $\triangle DPAB$ is an asymptotic triangle, then $\mu(\angle APD) + \mu(\angle PAB) < 180^\circ$.

The next two results give a version of triangle congruence for asymptotic triangles.

Side-Angle Congruence Condition (Theorem 6.5.4)

Let $\triangle EPAB$ and $\triangle FQCD$ be two asymptotic triangles. If $\angle APE \cong \angle CQF$ and $\overline{AP} \cong \overline{CQ}$, then $\angle PAB \cong \angle QCD$.

Proof.

Let $\triangle EPAB$ and $\triangle FQCD$ be two asymptotic triangles such that $\angle APE \cong \angle CQF$ and $\overline{AP} \cong \overline{CQ}$. We claim $\angle PAB \cong \angle QCD$.

Suppose $\mu(\angle PAB) > \mu(\angle QCD)$ (RAA hypothesis). Then there exists a ray $\overrightarrow{AG}$ between $\overrightarrow{AB}$ and $\overrightarrow{AP}$ such that $\angle PAG \cong \angle QCH$. Since $\overrightarrow{PE} \mid \overrightarrow{AB}$, then $\overrightarrow{AG}$ intersects $\overrightarrow{PE}$. To simplify notation we assume G is the point of intersection.

There exists a point H on $\overrightarrow{CD}$ such that $CH = AG$. By SAS, $\triangle QCH \cong \triangle PAG$. Hence, $\angle CQH \cong \angle APG \cong \angle APE \cong \angle CQF$ (definition of congruent triangles). Then $\overrightarrow{QF} = \overrightarrow{QH}$ (ACP), contradicting $\overrightarrow{QF} \mid \overrightarrow{CD}$. Thus we reject the RAA hypothesis.

A similar proof shows that we cannot have $\mu(\angle PAB) < \mu(\angle QCD)$. We conclude that $\mu(\angle PAB) = \mu(\angle QCD)$. $\square$

Angle-Angle Congruence Condition (Theorem 6.5.5)

Let $\triangle EPAB$ and $\triangle FQCD$ be two asymptotic triangles. If $\angle APE \cong \angle CQF$ and $\angle PAB \cong \angle QCD$, then $\overline{AB} \cong \overline{CQ}$.

One proof of this uses methods similar to Theorem 6.7.1.

Definition 3

Two lines are said to be *asymptotically parallel* if they contain asymptotically parallel rays.

In hyperbolic geometry, parallel lines fall into one of two families.

Classification of Parallels (Theorem 6.6.2)

Let ℓ and m be parallel lines.

Part 1 If ℓ and m are asymptotically parallel, then ℓ and m do not admit a common perpendicular.

Part 2 Either ℓ and m admit a common perpendicular, or they are asymptotically parallel.

Next time

Before next class: Read the Chapter 11 intro, Sections 11.1 and 11.2.

In the next lecture we will:

- Discuss models for Euclidean and hyperbolic geometry.
- Consider the postulates for the Ponicaré disk model.

Chapter 11: Models
§11.1 The Cartesian model for Euclidean geometry
§11.2 The Ponicaré disk model for hyperbolic geometry

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Discuss models for Euclidean and hyperbolic geometry.
- Consider the postulates for the Ponicaré disk model.

The two models we will discuss show that the Euclidean Parallel Postulate and the Hyperbolic Parallel Postulate are independent of the neutral axioms. These models are

- The Cartesian model for Euclidean Geometry
- The Ponicaré disk model for hyperbolic geometry

At times we will need to review some material that we skipped in order to have time to explain these models.

Cartesian model

First we give meaning to the undefined terms:

- Point: a point is an ordered pair of real numbers (x, y).
- Line: given real numbers a, b, c with $a, b \neq 0$, a line ℓ is the set
$$\ell = \{(x, y) : ax + by + c = 0\}.$$
- Distance: given two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, the distance from A to B is
$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$
- Half-plane: given a line ℓ as above, the half-planes determined by ℓ are
$$H_1 = \{(x, y) : ax + by + c > 0\} \quad \text{and} \quad H_2 = \{(x, y) : ax + by + c < 0\}.$$
- Angle measure: define the inverse tangent function as
$$\arctan x = \int_0^x \frac{1}{1 + t^2}\, dt.$$
Let y_1, y_2 be two non-vertical lines with slopes m_1, m_2, respectively. If $m_1 m_2 = -1$, then the lines are perpendicular. Otherwise, define the measure of the angle between them to be
$$\left(\frac{180}{\pi}\right) \left|\arctan\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)\right|.$$

Next we check the various postulates given the interpretation above.

Existence Postulate: The set of points is $\mathbb{R}^2$ and this set has an infinite number of points.

Incidence Postulate: Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points. If $x_1 = x_2$, then $x = x_1$ is the unique line between A and B. Otherwise the unique line is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

Ruler Postulate: Recall from Chapter 3 that there is a coordinate function for the Euclidean metric which satisfies the Ruler Postulate.

Plane Separation Postulate: Most of the PSP is easy to check. One can use coordinate functions to verify that H_1 and H_2 are convex, and to prove that a segment containing endpoints in different half-planes crosses the the line ℓ.

Protractor Postulate: This is left as a long and tedious exercise.

Next we check the various postulates given the interpretation above.

Side-Angle-Side: This is reasonably easy to check. Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $AB = DE$, $AC = DF$, and $\mu(\angle BAC) = \mu(\angle EDF)$. The angle at vertex A and then lengths of AB and AC uniquely determine the length BC. This in turn determines the remaining angles.

Euclidean Parallel Postulate: This uses the fact that parallel lines have the same slope, and a line is uniquely determined by a point and slope.

We will construct a model for hyperbolic geometry inside a model for Euclidean geometry. Recall that a circle O with center P and radius r is the set of points equidistant from P.

Ponicaré disk model

Let γ be a circle of radius 1 centered at the origin in the Cartesian model for the Euclidean plane.

- Point: a point inside of γ.
- Line: there are two types:
  (a) the points inside of γ of a Euclidean line through the origin,
  (b) the points inside of γ of a Euclidean circle β that is orthogonal to γ.
- Half-plane: this definition depends on the type of line:
  (a) the half-planes in this case are determined by the Euclidean half-planes inside of γ,
  (b) the half-planes are determined by the interior and exterior of β inside of γ
- Angle measure: this is defined as the angle determined by the tangent rays to the two Ponicaré rays.

A model in which angles are faithfully represented is called *conformal*.

It remains to interpret distance.

Ponicaré disk model

Let A and B be points in the model. Let P and Q be the points the corresponding Euclidean line (or circle) containing A and B intersects γ. Note that P and Q are not points in the model.

The cross ratio $[AB, PQ]$ is defined by

$$[AB, PQ] = \frac{(AB)(PQ)}{(AQ)(BP)}$$

where each of the individual distances is measured using Euclidean distances. Note that $[BA, PQ] = \frac{1}{[AB,PQ]} = [AB, QP]$.

Define the *distance* from A to B by

$$d(A, B) = |\ln([AB, PQ])|.$$

Note that

$$d(B, A) = |\ln([BA, PQ])| = |\ln([AB, PQ])| = d(A, B).$$

Next we check the various postulates given the interpretation above.

Existence Postulate: Clear.

Incidence Postulate: If points A and B lie on a diameter, then there is a unique Euclidean line through them and no Euclidean circle through them that is perpendicular to γ. Otherwise, there is a unique Ponicaré line of the second type.

Ruler Postulate: Given a Ponicaré line m, define a map $f(X) : m \to \mathbb{R}$ by $f(X) = \ln([AX, PQ])$. One verifies that this is a coordinate function (exercise) and that $d(B, C) = |f(B) - f(C)|$.

Plane Separation Postulate: This is easy to check.

Protractor Postulate: See Chapter 10.

Side-Angle-Side: See Chapter 10. Specifically the Reflection Postulate.

Next time

Before next class: Read the Section 11.3.

In the next lecture we will:

- Consider additional models for hyperbolic geometry.

Chapter 10: Transformations

MTH 411/511

Foundations of Geometry

It's good to have goals

Goals for today:

- Give an overview of transformations in the plane.
- Discuss the Reflection Postulate.

Transformations offer a different viewpoint on the geometric ideas we have discussed in this class. We will not cover these ideas rigorously. The main goal will be to introduce the Reflection Postulate, which is necessary to verify the SAS in the Poincaré disk model for hyperbolic geometry.

As usual, we denote the plane by $\mathbb{P}$.

Definition 1

A *transformation* is a function $T : \mathbb{P} \to \mathbb{P}$ that is both one-to-one and onto. A transformation is called an *isometry* if it preserves distances. That is, if $A, B \in \mathbb{P}$, then $T(A)T(B) = AB$.

Example 2

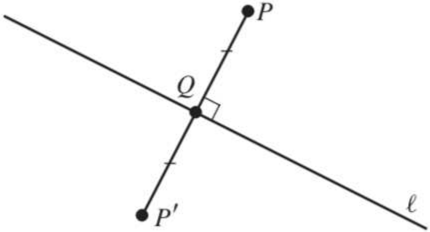
The *identity* function $\iota : \mathbb{P} \to \mathbb{P}$ defined by $\iota(P) = P$ for all $P \in \mathbb{P}$ is an isometry.

Example 3

Let ℓ be a line.

- If $P \in \ell$, then set $P = P'$.
- If $P \notin \ell$, then drop a perpendicular from P to ℓ and call the foot Q. Let P' be the point on $\overleftrightarrow{PQ}$ on the opposite side of ℓ as P such that $PQ = QP'$.

Define the *reflection* $\rho_\ell : \mathbb{P} \to \mathbb{P}$ by $\rho_\ell(P) = P'$.



Definition 4

A point P is called a *fixed point* for transformation T if $T(P) = P$.

(Theorem 10.1.6)

The composition of two isometries is an isometry. The inverse of an isometry is an isometry.

The previous theorem demonstrates that the set of isometries forms a group under composition.

We say a transformation T preserves an object S if $T(S)$ is an object of the same type.

Properties of isometries (Theorem 10.1.7)

Let $T : \mathbb{P} \to \mathbb{P}$ be an isometry. Then T preserves the following objects/properties/relationships:

- collinearity,
- betweenness of points,
- segments,
- lines,
- betweenness of rays,
- angles,
- triangles,
- circles, and
- areas.

Existence and uniqueness of isometries (Theorem 10.1.8)

If $\triangle ABC$ and $\triangle DEF$ are two triangles with $\triangle ABC \cong \triangle DEF$, then there exists a unique isometry T such that $T(A) = D$, $T(B) = E$, and $T(C)$. Furthermore, T is the composition of either two or three reflections.

(Corollary 10.1.11)

Every isometry of the plane can be expressed as a composition of reflections. The number of reflections required is either two or three.

Note that the decomposition in the previous corollary is not unique, but the parity is.

There are other isometries that we can now form. For example, the composition of two intersecting reflections is a rotation. One can also use reflections to define translations and glide reflections.

In order to define a reflection (and to prove that it is an isometry) we need the SAS postulate. An alternative approach is to assume the existence of reflections. We will show that this is equivalent to assuming the SAS postulate.

Reflection Postulate (Axiom 10.5.1)

For every line ℓ there exists a transformation $\rho_\ell : \mathbb{P} \to \mathbb{P}$, called the *reflection in* ℓ, that satisfied the following conditions.

1. If P lies on ℓ, then P is a fixed point for ρ_ℓ.
2. If P lies in one of the half-planes determined by ℓ, then $\rho_\ell(P)$ lies in the opposite half-plane.
3. ρ_ℓ preserves collinearity.
4. ρ_ℓ preserves distance.
5. ρ_ℓ preserves angle measure.

Note that SAS implies the Reflection Postulate by our earlier discussion (though this could be made more formal).

Before continuing we review some key definitions needed in the next argument.

Definition 5

- A *rigid motion* is a transformation of the plane that preserves collinearity.
- A *figure* is any subset of the plane.
- Two figures X and Y are *congruent* if there exists a rigid motion $f : \mathbb{P} \to \mathbb{P}$ such that $f(X) = Y$.

Note that this definition of congruence is more in line with Euclid's notion of congruence. In particular, two triangles $\triangle ABC$ and $\triangle DEF$ are congruent if there exists a rigid motion $f : \mathbb{P} \to \mathbb{P}$ such that $f(\triangle ABC) = \triangle DEF$. That is, $f(A) = D$, $f(B) = E$, and $f(C) = F$. This implies that corresponding parts of congruent triangles are congruent (CPCTC).

Reflection Postulate implies SAS

Theorem 10.5.5)

The Reflection Postulate implies the Side-Angle-Side triangle congruence condition.

Proof.

Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $\overline{AB} \cong \overline{DE}$, $\angle BAC \cong \angle EDF$, and $\overline{AC} \cong \overline{DF}$. We claim there exists a rigid motion T such that $T(\triangle ABC) = \triangle DEF$. It suffices to construct T such that $T(A) = D$, $T(B) = E$, and $T(C) = F$.

Let ℓ be the perpendicular bisector of $\overline{AD}$ (Existence of perpendicular bisectors) and let ρ_ℓ be the associated reflection (Reflection Postulate). Let G be the point of intersection of ℓ and $\overline{AD}$. Then $\rho_\ell(\overrightarrow{GA}) = \overrightarrow{GD}$ and $\rho_\ell(A) = D$ (Reflection Postulate). Set $B' = \rho_\ell(B)$ and $C' = \rho_\ell(C)$.

Theorem 10.5.5)

The Reflection Postulate implies the Side-Angle-Side triangle congruence condition.

Proof.

We have three cases based on the relationship between C', D, and F.

(1) If C', D, and F are noncollinear, then let m be the bisector of $\angle C'DF$ (Existence of angle bisectors). Let ρ_m be the associated reflection. Then $\rho_m(D) = D$ and as above $\rho_m(\overrightarrow{DC'}) = \overrightarrow{DF}$. It follows that $\rho_m(C') = F$. In this case, set $g = \rho_m$.

(2) If $\overrightarrow{DC'}$ and $\overrightarrow{DF}$ are opposite rays, then define m to be the line that is perpendicular to $\overleftrightarrow{FC'}$ at D. Then $\rho_m(D) = D$ and $\rho_m(C') = F$ so again we take $g = \rho_m$.

(3) If $\overrightarrow{DC'} = \overrightarrow{DF}$, then $C' = F$ so take $g = \iota$.

Note in all cases we have a rigid motion g such that $g(D) = D$.

Reflection Postulate implies SAS

Theorem 10.5.5)

The Reflection Postulate implies the Side-Angle-Side triangle congruence condition.

Proof.

We have either $g(B') = E$ or $g(B')$ is the reflection of E across $n = \overleftrightarrow{DF}$. If $\rho_m(B') = E$, define $f = \iota$. Otherwise, set $f = \rho_n$. Now we define $T = f \circ g \circ \rho_\ell$ and verify that $T(A) = D$, $T(B) = E$, and $T(C) = F$. □

The Reflection Postulate can be used to give transformational proofs of the Isosceles Triangle Theorem (Theorem 10.5.6) and (assuming the EPP) the Angle Sum Theorem (Theorem 10.5.7).

Next time

Before next class: Review the exam review on Canvas.

In the next lecture we will:

- Review for the Final Exam!