Reflexive hull discriminants and applications

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Discriminants - background and history

- Discriminants have long been used as a tool in algebraic number theory (Galois field extensions). This theory was adapted to the setting of noncommutative algebras by Ceken, Palmieri, Wang, and Zhang ('15). There are applications to a host of problems in noncommutative algebra, including automorphism groups, isomorphism problems, Zariski cancellation, and representation theory.

- Chan, Young, and Zhang ('16) supplied several tools for computing discriminants, such as through localization and filtrations. They also introduce the p-power discriminant in the study of discriminants of Veronese subrings ('18).

- Nguyen, Trampel, and Yakimov ('17) demonstrated a connection between discriminants and Poisson algebras.

- Brown and Yakimov ('18) showed that the discriminant can be obtained through representation theory and the Azumaya locus.

- G, Kirkman, and Moore ('19) provide techniques for computing discriminants of twisted tensor products, including Ore extensions.

The (noncommutative) discriminant

Let $\Bbbk$ be a field. Let B be a prime $\Bbbk$-algebra containing R as a central subalgebra such that B is a finitely generated (f.g.) R-module. Let F be a localization of R such that $B_F := B \otimes_R F$ is f.g. and free over F with $w = \mathrm{rk}_F(B_F) < \infty$.

The *regular trace* is the composition

$$\mathrm{tr} : B \to B_F \xrightarrow{\mathrm{lm}} \mathrm{End}_F(B_F) \cong M_w(F) \xrightarrow{\mathrm{tr_{int}}} F.$$

We will assume throughout that the image of tr is in R.

If $\mathcal{U} = \{u_i\}_{i=1}^w$ and $\mathcal{U}' = \{u_i'\}_{i=1}^w$ are w-element subsets of B, then the *discriminant* of the pair $(\mathcal{U}, \mathcal{U}')$ is defined to be

$$d_v(\mathcal{U}, \mathcal{U}') := \det(\mathrm{tr}(u_i u_j')_{i,j=1}^w) \in R.$$

The *modified discriminant ideal* of B over R, denoted $\mathrm{MD}(B/R)$, is generated by the elements $d_w(\mathcal{U}, \mathcal{U}')$ where $\mathcal{U}, \mathcal{U}'$ range over all w-element subsets of B.

If B is free over R of rank w, $\mathrm{MD}(B/R)$ is generated by a single element $d(B/R)$, which we call the *discriminant of B over R*.

An example: $\mathbb{C}_{-1}[x, y]$

$B = \mathbb{C}_{-1}[x, y]$ is f.g free over $R = \mathbb{C}[x^2, y^2]$ with basis $\{1, x, y, xy\}$.

	1	**x**	**y**	**xy**
1	4	0	0	0
x	0	$4x^2$	0	0
y	0	0	$4y^2$	0
xy	0	0	0	$-4x^2y^2$

$d(B/R) = \det(\text{matrix above}) = -256x^4y^4 =_{R^\times} x^4y^4$.

If $\sigma \in \mathrm{Aut}(B)$, then $\sigma(x^4y^4) = cx^4y^4$ for $c \in R^\times = \mathbb{k}^\times$.

Consequently, every automorphism of $\mathrm{Aut}(B)$ is *affine*.

We then use the defining relation to prove that all automorphisms are graded. It is then a simple matter to determine the graded automorphisms.

Reflexive Hull Discriminants

The *reflexive hull* of a module M over a commutative domain R is

$$M^{\vee\vee} = \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R).$$

There is a natural R-morphism $j : M \to M^{\vee\vee}$ defined by

$$j(x)(f) = f(x) \text{ for all } x \in M, f \in M^{\vee}.$$

The module M is reflexive if j is an isomorphism.

The $\mathcal{R}$-*discriminant ideal* of B over R is defined to be

$$\mathcal{R}(B/R) := (\mathrm{MD}(B/R))^{\vee\vee} \subseteq R.$$

If further $\mathcal{R}(B/R)$ is a principal ideal of R generated by an element d, then d is called an $\mathcal{R}$-*discriminant* of B over R and denoted by $\varrho(B/R)$.

We also call the $\mathcal{R}$-discriminant the *reflexive hull discriminant*. If $\varrho(B/R)$ exists, it is unique up to a unit in R.

Reflexive Hull Discriminants

Set $M = \mathrm{MD}(B/R) \subseteq R$. We say B/R satisfies the *reflexive discriminant condition* (RDC) if there exists a $d \in R$ such that

(a) $M \subseteq dR = Rd$, and

(b) $\mathrm{GKdim}\,(dR/M) \leq \mathrm{GKdim}\, R - 2$.

In this case d is called a *weak $\mathcal{R}$-discriminant* of B over R. The element d in either part (1) or (2), if it exists, may not be unique (even up to a unit) in general, unless R is CM.

Lemma

Let B be a prime $\Bbbk$-algebra containing R as a central subalgebra such that B is a f.g. R-module. Suppose that R is an affine CM domain and that B is a CM reflexive module over R. If B/R satisfies the RDC with respect to $d \in R$, then

$$\mathcal{R}(B/R) = dR \quad \textit{and} \quad \varrho(B/R) =_{R^\times} d.$$

Reflexive Hull Discriminants

Theorem

Let A be a prime $\Bbbk$-algebra. Let Z be the center of A and suppose Z is affine and CM. Suppose A is f.g. as a Z-module and that there is a quasi-basis with respect to a finite generating set of the Z-module A. If $d = \varrho(A/Z)$ and $g \in \mathrm{Aut}(A)$, then

$$g(d) =_{Z^\times} d.$$

Theorem

Let A and A' be prime PI $\Bbbk$-algebras with centers Z, Z', respectively. Assume that $A \otimes A'$ is prime, and that $Z \otimes Z'$ is affine and CM.

Suppose that A/Z (resp. A'/Z') satisfies RDC with respect to d (resp. d'), a weak $\mathcal{R}$-discriminant of A/Z (resp. A'/Z'). Then $(A \otimes A')/(Z \otimes Z')$ satisfies RDC with respect to $d^{w'} \otimes (d')^{w}$, which is a weak $\mathcal{R}$-discriminant of $(A \otimes A')/(Z \otimes Z')$. Then

$$\varrho((A \otimes A')/(Z \otimes Z')) =_{(Z \times Z')^\times} d^{w'} \otimes (d')^{w}$$

Reflexive Hull Discriminants

Lemma

Let A be a prime $\Bbbk$-algebra with center Z. Assume

- *Z is affine and CM,*
- *$X = \operatorname{Spec} Z$ is an affine integral normal $\Bbbk$-variety, and*
- *there exists an open subset U of X such that $X \setminus U$ has codimension ≥ 2.*

If there exists an element $d \in Z$ such that the principal ideal (d) of Z agrees with $\mathrm{MD}(A/Z)$ *on U, then $\varrho(A/Z) = d$. Moreover, if A is a $\mathcal{O}_X$-order, then we can compute the discriminant locally at a smooth closed point $\mathfrak{m} \in X$.*

Example

Suppose $\operatorname{char} \Bbbk \neq 2$. Let $A = \Bbbk\langle x_1, x_2, x_3\rangle/(x_1x_2 + x_2x_1, x_1x_3 + x_3x_1, x_2x_3 - x_3x_2)$.

$Z = Z(A) = \Bbbk[u, v, w, z]/(vw - z^2)$ where $u = x_1^2$, $v = x_2^2$, $w = x_3^2$ and $z = x_2x_3$. The rank of A over Z is 4. Let $X = \operatorname{Spec} Z$.

Let V be the open subset of X with $v \neq 0$. Over V, $d(A_v/Z_v) = (uv)^2 =_{(Z_v)^\times} u^2$.
Let W be the open subset of X with $w \neq 0$. Over W, $d(A_w/Z_w) = (uw)^2 =_{(Z_w)^\times} u^2$.

Since $U := V \cup W$ is an open subscheme of X, $\operatorname{codim}(X \backslash U) = 2$, and $d = u^2 = x_1^4$ defines a Cartier divisor D on U that extends to all of X, then $\varrho(A/Z) = d$.

Generalized Weyl Algebras

Our results apply in particular to quantum GWAs at roots of unity. Here we will consider only the rank one case but our results apply also to higher rank GWAs.

Definition

Let R be a $\mathbb{k}$-algebra, $\sigma \in \mathrm{Aut}(R)$, and h a nonzero central element in R. The (rank one) *generalized Weyl algebra* (GWA) $R(x, y, \sigma, h)$ is the $\mathbb{k}$ algebra generated over R by x, y subject to the relations

$$xr = \sigma(r)x \qquad yr = \sigma^{-1}(r)y \qquad \text{for all } r \in R$$

$$xy = h \qquad yx = \sigma^{-1}(h).$$

We say $R(x, y, \sigma, h)$ is a *quantum GWA* if $R = \mathbb{k}[t]$ and $\sigma(t) = qt$ for some $q \in \mathbb{k}^{\times}$.

Quantum GWAs

Lemma

Let $W = \Bbbk[t](x, y, \sigma, h)$ *be a quantum GWA with* $\operatorname{ord}(\sigma) = n < \infty$. *Set*

$$a = x^n, \quad b = y^n, \quad c = t^n, \quad p(c) = \prod_{j=0}^{n-1} h(q^j t).$$

Then

$$Z(W) = \Bbbk[a, b, c] / (ab - p(c)).$$

Consequently, $Z = Z(W)$ *is an affine normal CM domain.*

The quantum GWA W is a Z-algebra with presentation

$$W = \frac{Z\langle x, y, t\rangle}{(xt - qtx, yt - q^{-1}ty, xy - h(t), x^n - a, y^n - b, t^n - c)}.$$

Then W is generated as a Z-module by $\{x^i t^j, y^i t^j \mid i, j = 0, \ldots, n-1\}$.

Quantum GWAs

Theorem

$\varrho(W/Z) =_{Z^\times} c^{n(n-1)} =_{Z^\times} t^{n^2(n-1)}$.

Proof.

Let U_a and U_b denote the open subsets of $X = \operatorname{Spec} Z$ where $a \neq 0$ and $b \neq 0$, respectively, so that $X \backslash (U_a \cup U_b)$ has codimension 2.

Then $W_a := W\big|_{U_a} = W[a^{-1}]$ is free of rank n^2 with basis $\{x^i t^j \mid i, j = 0, \ldots, n-1\}$.

The discriminant of W_a is given by $(ac)^{n(n-1)} =_{Z(W_a)^\times} c^{n(n-1)}$.

Similarly the discriminant of W_b is given by $(bc)^{n(n-1)} =_{Z(W_b)^\times} c^{n(n-1)}$.

Then $d = c^{n(n-1)}$ defines a Cartier divisor on $U_a \cup U_b$, which extends to a Cartier divisor on all of X.

Indeed, the data $\{(U_a, c^{n(n-1)}), (U_b, c^{n(n-1)}), (X, c^{n(n-1)})\}$ defines a Cartier divisor on X which restricts to the above Cartier divisor on $U_a \cup U_b$. $\square$

Quantum GWAs

We apply the $\mathcal{R}$-discriminant to compute the automorphism group for quantum GWAs. This recovers a result of Suárez-Alvarez and Vivas ('15) in the rank one case. We also recover results on the isomorphism problem for quantum GWAs and Zariski cancellation.

Proposition

Let $W_1, \ldots, W_k$ be a collection of degree one quantum GWAs with canonical generators $\{x_i, y_i, t_i\}$ and parameters $\{q_i, h_i\}$. Set $A = W_1 \otimes \cdots \otimes W_k$. Assume each q_i is a root of unity with $q_i^2 \neq 1$. Also assume $\deg_{t_i} h_i \geq 2$ for all i. If $\phi \in \mathrm{Aut}(A)$, then the following hold:

1. *There exists $\tau \in \mathbb{S}_k$ such that $\phi(W_i) = W_{\tau(i)}$ for all $i = 1, \ldots, k$.*
2. *There exists scalars $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in \Bbbk^\times$ and a sequence $\{\epsilon_1, \ldots, \epsilon_k\} \in \{\pm 1\}^k$ such that for each $i = 1, \ldots, k$,*

$$\begin{aligned} \phi(x_i) &= \alpha_i x_{\tau(i)}, \quad \phi(y_i) = \beta_i y_{\tau(i)}, \quad \text{if } \epsilon_i = 1, \\ \phi(x_i) &= \alpha_i y_{\tau(i)}, \quad \phi(y_i) = \beta_i x_{\tau(i)}, \quad \text{if } \epsilon_i = -1. \end{aligned}$$

Moreover, there exists scalars $\gamma_1, \ldots, \gamma_k \in \Bbbk^\times$ such that

$$h_i(\gamma_i t) = \begin{cases} \alpha_i \beta_i h_{\tau(i)}(t) & \text{if } \epsilon_i = 1 \\ \alpha_i \beta_i h_{\tau(i)}(q^{-1} t) & \text{if } \epsilon_i = -1. \end{cases}$$

Thank You!