Invariant Theory of Twisted Generalized Weyl Algebras

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Shameless self promotion

This talk focuses primarily on several recent projects:

- *Fixed rings of generalized Weyl algebras*, J. Algebra 536 (2019). Joint with Rob Won.
- *Fixed rings of quantum generalized Weyl algebras*, Communications in Algebra, 2020, 48 (9), 4051-4064. Joint with Phuong Ho (undergraduate).
- *Fixed rings of twisted generalized Weyl algebras*, arXiv:2011.13029. Joint with Daniele Rosso.
- *Pointed Hopf actions on quantum generalized Weyl algebras*, forthcoming. Joint with Rob Won.
- *Reflexive hull discriminants and applications*, forthcoming. Joint with Kenneth Chan, Rob Won, and James Zhang. (More on this at JMM.)

Shephard-Todd-Chevalley

Throughout, $\Bbbk$ is an algebraically closed field of characteristic zero.

Theorem (Shephard-Todd-Chevalley)

Let G be a finite group of graded automorphisms on $A = \Bbbk[x_1, \ldots, x_n]$. The fixed ring A^G is a polynomial ring if and only if G is generated by reflections.

A major goal in noncommutative invariant theory is to generalize the STC theorem to noncommutative analogues of polynomial rings.

- Artin-Schelter regular algebras (Kirkman, Kuzmanovich, Zhang)
- Preprojective algebras (Weispfenning)
- Quadratic Poisson algebras (-, Veerapen, Wang)

To further generalize STC, we can relax the grading requirement and consider Calabi-Yau algebras.

- An algebra A is *homologically smooth* if it has a finitely generated projective resolution of finite length in $A^e - \mathrm{MOD}$.
- The algebra A is *twisted Calabi-Yau* of dimension d if it is homologically smooth and there exists an invertible bimodule U of A such that $\mathrm{Ext}^i_{A^e}(A, A^e) = \delta_{id} U$.
- If $U = A$, then A is said to be *Calabi-Yau*.

Question

When is the fixed ring of a (twisted) Calabi-Yau algebra again (twisted) Calabi-Yau?

This talk will focus on certain $\mathbb{Z}$-graded twisted Calabi-Yau algebras and their generalizations.

Definition

Let R be a unital $\Bbbk$-algebra and n a positive integer. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an n-tuple of commuting automorphisms of R and $t = (t_1, \ldots, t_n)$ be an n-tuple of nonzero central elements of R such that $\sigma_i(t_j) = t_j$ for $i \neq j$. The *rank n generalized Weyl algebra* (GWA) $R(\sigma, t)$ is generated over R by $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ subject to the relations

$$x_i r = \sigma_i(r) x_i, \quad y_i r = \sigma_i^{-1}(r) y_i \text{ for all } r \in R,$$
$$y_i x_i = t_i, \quad x_i y_i = \sigma_i(t_i), \quad [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 0 \text{ for } i \neq j.$$

Proposition

Let $R(\sigma, t)$ be a rank n GWA.

- *The algebra $R(\sigma, t)$ is $\mathbb{Z}^n$-graded by setting $\deg(x_i) = \mathbf{e}_i$, $\deg(y_i) = -\mathbf{e}_i$, and $\deg(r) = \mathbf{0}$ for all $r \in R$.*
- *(Bavula) If R is (left/right) noetherian, then so is $R(\sigma, t)$.*
- *(Bavula) If R is a domain, then so is $R(\sigma, t)$.*
- *(Ebrahim) If each σ_i is locally algebraic, then $\operatorname{GKdim} R(\sigma, t) = \operatorname{GKdim}(R) + n$.*
- *(Liu) A rank one GWA $\Bbbk[h](\sigma, t)$ is twisted Calabi-Yau if and only if it has finite global dimension.*

Theorem (Jordan, Wells)

Let $A = R(\sigma, t)$ be a rank one GWA and let α be a primitive ℓth root of unity. Define the automorphism Θ_α of R by

$$\Theta_\alpha(x) = \alpha x, \quad \Theta_\alpha(y) = \alpha^{-1} y, \text{ and } \Theta_\alpha(r) = r \text{ for all } r \in R.$$

The fixed ring $A^{\langle \Theta_\alpha \rangle}$ is the GWA $R(\sigma^\ell, T)$, generated over R by x^ℓ and y^ℓ with defining polynomial $T = \prod_{i=0}^{\ell-1} \sigma^{-i}(t)$.

The following proposition was inspired by work of Kirkman and Kuzmanovich.

Proposition (-, Won)

Let g be a finite filtered automorphism of $A_1(\Bbbk)$. Then g acts diagonally on a generating set $\{X, Y\}$ of $A_1(\Bbbk)$ such that $XY - YX = 1$. Thus, $A_1(\Bbbk)^{\langle g \rangle}$ is a (classical) GWA.

The classical case - automorphisms

A rank one GWA $A = \Bbbk[h](\sigma, t)$ is *classical* if $\sigma(h) = h - 1$. Examples of classical GWAs include the Weyl algebra, primitive quotients of $U(\mathfrak{sl}_2)$, and noncommutative deformations of type A Kleinian singularities. Automorphisms of classical GWAs were classified by Bavula and Jordan.

Let $n = \deg_h(t)$, $\lambda \in \Bbbk$, $\alpha \in \Bbbk^\times$, $m \in \mathbb{N}$, and let $\Delta_m : \Bbbk[h] \to \Bbbk[h]$ be the linear map given by $\sigma^m - 1$. Generically, $\mathrm{Aut}(A)$ is generated by the following maps:

$$\Theta_\alpha : x \mapsto \alpha x, y \mapsto \alpha^{-1} y, h \mapsto h,$$

$$\Psi_{m,\lambda} : x \mapsto x, y \mapsto y + \sum_{i=1}^{n} \frac{\lambda^i}{i!} \Delta_m^i(t) x^{im-1}, h \mapsto h - m\lambda x^m,$$

$$\Phi_{m,\lambda} : x \mapsto x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{i!} y^{im-1} \Delta_m^i(t), y \mapsto y, h \mapsto h + m\lambda y^m.$$

If t is *reflective* ($t(\rho - h) = (-1)^n t(h)$ for some $\rho \in \Bbbk$) there is an additional generator Ω given by

$$\Omega(x) = y, \quad \Omega(y) = (-1)^n x, \quad \Omega(h) = 1 + \rho - h.$$

The classical case - invariants

In the classical case, $A = \mathbb{k}[h](\sigma, t)$ is $\mathbb{N}$-filtered by setting $\deg(h) = 2$ and $\deg(x) = \deg(y) = \deg_h(t)$.

Generically, in the case $\deg_h(t) > 2$, the only filtered maps are the Θ_α.

In the case t is quadratic, g is a filtered automorphism of R if and only if

- $g = \tau_{\lambda,\mu,\alpha} = \Psi_{1,\lambda} \circ \Phi_{1,\mu} \circ \Theta_\alpha$ or
- $g = \tau_{\lambda,\mu,\alpha} \circ \Omega$.

Theorem (-, Won)

Let $A = \mathbb{k}[h](\sigma, t)$ be a classical GWA with t quadratic. Let g be a finite filtered automorphism of R. There exists a generating set $\{X, Y\}$ over $\mathbb{k}[h]$ with GWA structure such that g acts diagonally. Hence, $A^{\langle g \rangle}$ is again a classical GWA.

The classical case - global dimension of invariants

Let $A = \Bbbk[h](\sigma, t)$ be a classical GWA. Two roots c_1, c_2 of a are said to be *congruent* if there exists an $i \in \mathbb{Z}$ such that, as ideals of $\Bbbk[h]$, $\big(\sigma^i(h - c_1)\big) = (h - c_2)$. Work of Bavula, Hodges, Jordan, and Stafford implies that

$$\operatorname{gldim} A = \begin{cases} \infty & \text{if } t \text{ has a multiple root} \\ 2 & \text{if } t \text{ has a congruent root and no multiple roots} \\ 1 & \text{if } t \text{ has no congruent roots and no multiple roots.} \end{cases}$$

Corollary

Let $A = \Bbbk[h](\sigma, h(h-k))$ be a classical GWA. Let g be a finite filtered automorphism of A.

- *If* $\operatorname{gldim}(A) = 1$ *(resp. ∞), then* $\operatorname{gldim}(A^{\langle g \rangle}) = 1$ *(resp. ∞).*
- *If* $\operatorname{gldim}(A) = 2$*, then $k \in \mathbb{Z}$ and*

$$\operatorname{gldim}(A^{\langle g \rangle}) = \begin{cases} 2 & \textit{if } |k| \geq |g| \\ \infty & \textit{otherwise.} \end{cases}$$

The quantum case

To consider automorphisms on GWAs that do not act trivially on the base ring, we need the following generalization of Jordan and Wells' result.

Theorem (-,Ho)

Let R be an integral domain, let $A = R(\sigma, t)$ be a rank one GWA, and let $\phi \in \mathrm{Aut}(A)$ with $\mathrm{ord}(\phi) < \infty$. Suppose

- *ϕ restricts to an automorphism of R with $n = \mathrm{ord}(\phi\big|_R)$,*
- *$\phi(x) = \alpha x$ and $\phi(y) = \alpha^{-1} y$ for $\alpha \in \Bbbk^\times$ with $m = \mathrm{ord}(\alpha)$, and*
- *$\gcd(n, m) = 1$.*

Then $A^{\langle \phi \rangle} = R^{\langle \phi \rangle}(\sigma^m, T)$ with $T = \prod_{i=0}^{m-1} \sigma^{-i}(t)$.

The quantum case - automorphisms

A rank one GWA $R(\sigma, t)$ is *quantum* if $R = \Bbbk[h]$ or $\Bbbk[h^{\pm 1}]$ and $\sigma(h) = qh$ for $q \in \Bbbk \backslash \{0, 1\}$. Examples of quantum GWAs include quantum Weyl algebras, quantum planes, and primitive quotients of $U_q(\mathfrak{sl}_2)$. Suárez-Alverez and Vivas fully determined the automorphism group for quantum GWAs $A = R(\sigma, t)$:

- Write $t = \sum_{i \in I} t_i h^i$ where $I = \{i : t_i \neq 0\}$.
- Let $g = \gcd\{i - j : t_i t_j \neq 0\}$.
- If t is a monomial then let $C_g = \Bbbk^\times$ and otherwise let C_g be the subgroup of $\Bbbk^\times$ consisting of gth roots of unity.

For $(\gamma, \alpha) \in C_g \times \Bbbk^\times$, define $\eta_{\gamma,\alpha} \in \mathrm{Aut}(R)$ by

$$\eta_{\gamma,\alpha}(h) = \gamma h, \quad \eta_{\gamma,\alpha}(y) = \alpha^{-1} y, \quad \eta(x) = \gamma^{i_0} \alpha x, \quad i_0 \in I.$$

Corollary

Let $A = \Bbbk[h](\sigma, t)$ be a quantum GWA and $\eta = \eta_{\gamma,\alpha} \in \mathrm{Aut}(A)$. Set $n = \mathrm{ord}(\gamma)$ and $m = \mathrm{ord}(\alpha)$ with $n, m < \infty$. If $\gcd(n, m) = 1$ and $n \mid i_0$, then $A^{\langle \eta \rangle}$ again a quantum GWA.

The quantum case - global dimension of invaraiants

Global dimension for quantum GWAs is much more sensitive. The following criteria is due to Bavula and Jordan.

Let $A = R(\sigma, t)$ be a quantum GWA.

- A has infinite global dimension if and only if t has multiple roots.
- A has global dimension 2 if and only if t has no multiple roots and one of the following hold:
  - $R = \mathbb{k}[h]$,
  - q is a root of unity, or
  - t has a pair of congruent roots.
- A has global dimension 1 otherwise.

The quantum case - global dimension of invariants

Suppose $A = R(\sigma, t)$ is a quantum GWA. Let $\eta = \eta_{\gamma,\alpha} \in \text{Aut}(R)$ with $n = \text{ord}(\gamma) < \infty$ and $m = \text{ord}(\alpha) < \infty$ such that $\gcd(n, m) = 1$.

Theorem (-,Ho)

- *If* $\text{gldim}\, A = 1$, *then* $\text{gldim}\, A^{\langle\eta\rangle} = 1$.
- *If* $\text{gldim}\, A = \infty$, *then* $\text{gldim}\, A^{\langle\eta\rangle} = 2$ *if and only if* $m = 1$ *and* 0 *is a root of* t *with multiplicity* $k = n$. *Otherwise* $\text{gldim}\, A^{\langle\eta\rangle} = \infty$.
- *If* $\text{gldim}\, A = 2$ *and* q *is not a root of unity, then* $\text{gldim}\, A^{\langle\eta\rangle} = \infty$ *if and only if there exists roots* c_i, c_j *of* t *such that* $c_i = q^k c_j$ *for some* k *with* $0 \le k \le m-1$. *Otherwise* $\text{gldim}\, A^{\langle\eta\rangle} = 2$.
- *If* $\text{gldim}\, A = 2$ *and* q *is a root of unity,* $q \ne 1$, *then* $\text{gldim}\, A^{\langle\eta\rangle} = \infty$ *if and only if one of the following conditions is satisfied:*
  - t *has multiple roots,*
  - t *has congruent roots* c_i, c_j *such that* $c_i = q^k c_j$ *with* $0 \le k \le m-1$, *or*
  - *there exists* k *such that* $0 < k \le m-1$ *and* $\text{ord}(q)$ *divides* nk.

  Otherwise $\text{gldim}\, A^{\langle\eta\rangle} = 2$.

Definition

Let R be a unital $\Bbbk$-algebra and n a positive integer.

- A *twisted generalized Weyl datum (TGWD) of rank n* is the triple (R, σ, t) where $\sigma = (\sigma_1, \dots, \sigma_n)$ is an n-tuple of commuting automorphisms of R, and $t = (t_1, \dots, t_n)$ is an n-tuple of nonzero central elements of R.
- Given a TGWD (R, σ, t) and $\mu = (\mu_{ij}) \in M_n(\Bbbk^\times)$, the associated *twisted generalized Weyl construction (TGWC)*, $\mathcal{C}_\mu(R, \sigma, t)$, is the $\Bbbk$-algebra generated over R by the $2n$ indeterminates $X_1^\pm, \dots, X_n^\pm$ subject to the relations

$$
\begin{array}{lll}
X_i^\pm r - \sigma_i^{\pm 1}(r) X_i^\pm & & \text{for all } r \in R \text{ and all ,} \\
X_i^- X_i^+ - t_i, & X_i^+ X_i^- - \sigma_i(t_i) & \text{for all } i, \\
X_i^+ X_j^- - \mu_{ij} X_j^- X_i^+ & & \text{for all } i \neq j.
\end{array}
$$

  There is a natural $\mathbb{Z}^n$-grading on $\mathcal{C}_\mu(R, \sigma, t)$ obtained by setting $\deg(r) = \mathbf{0}$ for all $r \in R$ and $\deg(X_i^\pm) = \pm \mathbf{e}_i$ for all i.
- The associated *twisted generalized Weyl algebra (TGWA)*, $A = \mathcal{A}_\mu(R, \sigma, t)$, is the quotient $\mathcal{C}_\mu(R, \sigma, t)/\mathcal{J}$ where $\mathcal{J}$ is the sum of all graded ideals $J = \bigoplus_{g \in \mathbb{Z}^n} J_g$ such that $J_\mathbf{0} = \{0\}$.

Twisted Generalized Weyl algebras

Examples of TGWAs include

- GWAs of rank n,
- multiparameter quantized Weyl algebras, and
- certain primitive quotients of enveloping algebras of simple Lie algebras.

Definition

- A TGWD (R, σ, t) is *regular* if t_i is regular in R for all i.
- A TGWD is *μ-consistent* if the canonical map $R \to A_\mu(R, \sigma, t)$ is injective.
- A TGWA $A_\mu(R, \sigma, t)$ is regular (resp. μ-consistent) if the underlying TGWD is regular (resp. μ-consistent).

If (R, σ, t) is μ-consistent for some parameter matrix μ, then $A_\mu(R, \sigma, t)$ is necessarily non-trivial. If (R, σ, t) is a regular TGWD, then (R, σ, t) is μ-consistent if and only if the following equations hold:

$$\sigma_i \sigma_j(t_i t_j) = \mu_{ij}\mu_{ji}\sigma_i(t_i)\sigma_j(t_j) \text{ for all distinct } i, j, \tag{1}$$

$$t_j \sigma_i \sigma_k(t_j) = \sigma_i(t_j)\sigma_k(t_j) \text{ for all pairwise distinct } i, j, k. \tag{2}$$

Tensor products

Theorem (Futorny, Hartwig)

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a regular, μ-consistent TGWA. Then A is a domain if and only if R is a domain.

Theorem (-, Rosso)

Let $A = A_\mu(R, \sigma, t)$ (resp. $A' = A_{\mu'}(R', \sigma', t')$) be a regular, μ-consistent (resp. μ'-consistent) TGWAs of rank n (resp. m). Define $\eta = (\eta_{ij}) \in M_{m+n}(\Bbbk^\times)$ by

$$\eta_{ij} = \begin{cases} \mu_{ij} & \text{if } 1 \leq i, j \leq m \\ \mu'_{(i-m)(j-m)} & \text{if } m < i, j \leq m+n \\ 1 & \text{otherwise.} \end{cases}$$

Then $A \otimes A'$ is a regular, η-consistent TGWA of rank $m + n$

Question

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a regular, μ-consistent TGWA. If R is (left/right) noetherian, is A also (left/right) noetherian?

Let $C = \mathcal{C}_\mu(R, \sigma, t)$ be a TGWC of rank n. Assume $\phi \in \mathrm{Aut}(C)$ satisfies

- $\phi|_R$ is an automorphism of R with $\ell = \mathrm{ord}(\phi|_R) < \infty$;
- for each i, $\phi(X_i^\pm) = \alpha_i^{\pm 1} X_i^\pm$ for some $\alpha_i \in \Bbbk^\times$ with $m_i = \mathrm{ord}(\alpha_i) < \infty$;
- the integers $\ell, m_1, \dots, m_n$ are pairwise relatively prime;
- either R is a commutative domain or $\phi|_R = \mathrm{id}_R$.

We will call automorphisms satisfying the above *diagonal*.

Lemma

Let $C = \mathcal{C}_\mu(R, \sigma, t)$ be a TGWC and suppose $\phi \in \mathrm{Aut}(C)$ is diagonal. Then $\phi(\mathcal{J}) = \mathcal{J}$. Hence, ϕ descends to an automorphism of $\mathcal{A}_\mu(R, \sigma, t)$ (which we will denote again by ϕ).

The next theorem is the most general version of the Jordan-Wells theorem (thus far).

Theorem

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a TGWA of rank n and let $\phi \in \mathrm{Aut}(A)$ be diagonal. Let B be the subalgebra of $A^{\langle \phi \rangle}$ generated by $R^{\langle \phi \rangle}$ and the $(X_i^\pm)^{m_i}$. Then B is a TGWA of rank n. Moreover, B is regular and satisfies the first consistency equation if A does.

Let $C = \mathcal{C}_\mu(R, \sigma, t)$ be the TGWC inducing A. Let D be the subalgebra of $C^{\langle \phi \rangle}$ generated by $R^{\langle \phi \rangle}$ and the $(X_i^\pm)^{m_i}$. Then D is a TGWC. Let $\mathcal{J}'$ be canonical ideal associated to D so that $D/\mathcal{J}'$ is a TGWA. The proof requires that we show $B \cong D/\mathcal{J}'$ so that there is a commutative diagram:

$$\begin{tikzcd} D \arrow[r, hook, "\iota"] \arrow[d, dotted, two heads] & C \arrow[d, two heads, "\pi"] \\ B \arrow[r, hook] & A \end{tikzcd}$$

Question

When is $A^{\langle \phi \rangle} = B$ so that $A^{\langle \phi \rangle}$ is a TGWA?

Definition

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a regular, μ-consistent TGWA. For each i, j, define

$$V_{ij} = \mathrm{Span}_\Bbbk\{\sigma_i^k(t_j) : k \in \mathbb{Z}\}.$$

Then A is $\Bbbk$-*finitistic* if $\dim_\Bbbk V_{ij} < \infty$ for all i, j.

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGWA. For each i, j, let $p_{ij} \in \Bbbk[x]$ denote the minimal polynomial for σ_i acting on V_{ij}. Associated to this data we define the (generalized Cartan) matrix $C_A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 - \deg p_{ij} & \text{if } i \neq j. \end{cases}$$

$\Bbbk$-finitistic TGWAs - Type $(A_1)^n$

Let $A = A_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGWA of (Cartan) type $(A_1)^n$. Then for all $i \neq j$, $p_{ij}(x) = x - \gamma_{ij}$ for some $\gamma_{ij} \in \Bbbk^\times$. This condition is equivalent to

$$\sigma_i(t_j) = \gamma_{ij} t_j \quad \text{for all } i \neq j.$$

In this case, A is (isomorphic to) the $\Bbbk$-algebra generated over R by $X_1^\pm, \ldots, X_n^\pm$ with the following relations for all $r \in R$ and all i, j with $i \neq j$,

$$\begin{gathered} X_i^\pm r - \sigma_i^{\pm 1}(r) X_i^\pm, \quad X_i^- X_i^+ - t_i, \quad X_i^+ X_i^- - \sigma_i(t_i), \\ X_i^+ X_j^- - \mu_{ij} X_j^- X_i^+, \quad X_i^+ X_j^+ - \gamma_{ij} \mu_{ij}^{-1} X_j^+ X_i^+, \quad X_j^- X_i^- - \gamma_{ij} \mu_{ji}^{-1} X_i^- X_j^-. \end{gathered}$$

Theorem (-, Rosso)

Let $A = A_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGWA of type $(A_1)^n$.

- *If R is (left/right) noetherian, then so is A.*
- *If R is an Auslander-Gorenstein domain, then so is A.*
- *We have* $\operatorname{GKdim}(A) \geq \operatorname{GKdim}(R) + n$.
- *If* $\operatorname{lgld} R < \infty$ *and* $\operatorname{lgld} A < \infty$, *then* $\operatorname{lgld} R \leq \operatorname{lgld} A \leq \operatorname{lgld} R + n$.

Theorem (-, Rosso)

Let R be a domain and let $A = A_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGWA of type $(A_1)^n$. Let ϕ be diagonal. Set

$$s_i = \prod_{k=0}^{m_i-1} \sigma_i^{-k}(t_i),$$

$\tau = (\tau_1, \dots, \tau_n) = (\sigma_1^{m_1}, \dots, \sigma_n^{m_n})$, *and* $\nu = (\nu_{ij}) = (\mu_{ij}^{m_i m_j})$ *for* $i \neq j$.

Then $A^{\langle\phi\rangle} = A_\nu(R^{\langle\phi\rangle}, \tau, s)$ *is a TGWA of type* $(A_1)^n$.

Corollary

Let $A = A_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGWA of type $(A_1)^n$. Suppose ϕ is a diagonal automorphism.

- *If R is (left/right) noetherian, then so is $A^{\langle\phi\rangle}$.*
- *If $R^{\langle\phi\rangle}$ is an AG domain, then so is $A^{\langle\phi\rangle}$.*

Let q and β be indeterminates and let $a \in \mathbb{N}$. Define $S_a(q,\beta) \in \mathbb{Z}[q,\beta]$ by

$$S_a(q,\beta) := \sum_{i=0}^{\lfloor \frac{a}{2} \rfloor} (-1)^i \binom{a-i}{i} \beta^i q^{a-2i}.$$

The polynomials $S_a = S_a(q,\beta)$ can also be defined by the recurrence relation

$$S_{a+1} = qS_a - \beta S_{a-1}, \qquad S_0 = 1, \quad S_1 = q.$$

They are related to the Chebyshev polynomials of the second kind by

$$S_a(q,\beta) = \beta^{a/2} U_a \left(\frac{q}{2\beta^{1/2}} \right)$$

and they satisfy the identity

$$\beta S_{c-2} S_{a-1} + S_{a+c-1} = S_a S_{c-1}$$

for all $a \geq 1$, $c \geq 2$.

Let A be a $\mathbb{k}$-finitistic TGWA of type A_2. Fix $\lambda_1, \lambda_2, \eta_1, \eta_2 \in \mathbb{k}$ such that

$$p_{12}(x) = x^2 + \lambda_1 x + \lambda_2, \qquad p_{21}(x) = x^2 + \eta_1 x + \eta_2.$$

Then A is generated by R, $X_1^{\pm}$, $X_2^{\pm}$ with the usual TGWA relations plus

$$\begin{aligned}
&(X_1^+)^2 X_2^+ + \lambda_1 \mu_{12}^{-1} X_1^+ X_2^+ X_1^+ + \lambda_2 \mu_{12}^{-2} X_2^+ (X_1^+)^2 = 0,\\
&X_2^- (X_1^-)^2 + \lambda_1 \mu_{21}^{-1} X_1^- X_2^- X_1^- + \lambda_2 \mu_{21}^{-2} (X_1^-)^2 X_2^- = 0,\\
&(X_2^+)^2 X_1^+ + \eta_1 \mu_{21}^{-1} X_2^+ X_1^+ X_2^+ + \eta_2 \mu_{21}^{-2} X_1^+ (X_2^+)^2 = 0,\\
&X_1^- (X_2^-)^2 + \eta_1 \mu_{12}^{-1} X_2^- X_1^- X_2^- + \eta_2 \mu_{12}^{-2} (X_2^-)^2 X_1^- = 0.
\end{aligned}$$

Lemma

Let A be a $\mathbb{k}$-finitistic TGWA of type A_2.

- *If $S_a(-\lambda_1, \lambda_2) \neq 0$ for all $a \geq 0$, then the monomials*
$$(X_2^+)^a (X_1^+)^b (X_2^+)^c, \quad (X_1^+)^a (X_2^-)^b, \quad (X_1^-)^a (X_2^+)^b, \quad (X_2^-)^a (X_1^-)^b (X_2^-)^c$$
with $a, b, c \geq 0$ generate A as a left (and as a right) R-module.
- *If $S_a(-\eta_1, \eta_2) \neq 0$ for all $a \geq 0$, then the monomials*
$$(X_1^+)^a (X_2^+)^b (X_1^+)^c, \quad (X_1^+)^a (X_2^-)^b, \quad (X_1^-)^a (X_2^+)^b, \quad (X_1^-)^a (X_2^-)^b (X_1^-)^c$$
with $a, b, c \geq 0$ generate A as a left (and as a right) R-module.

Theorem

Let $A = \mathcal{A}_\mu(R, \sigma, t)$ be a $\Bbbk$-finitistic TGWA of type A_2.

- *Let ϕ_1 be a diagonal automorphism with $\alpha_1 = 1$. If $S_a(-\eta_1, \eta_2) \neq 0$ for all $a \geq 0$, then $A^{\langle \phi_1 \rangle}$ is a regular, ν-consistent rank 2 TGWA.*
- *Let ϕ_2 be a diagonal automorphism with $\alpha_2 = 1$. If $S_a(-\lambda_1, \lambda_2) \neq 0$ for all $a \geq 0$, then $A^{\langle \phi_2 \rangle}$ is a regular, ν-consistent rank 2 TGWA.*

In general, the fixed rings $A^{\langle \phi_1 \rangle}$ and $A^{\langle \phi_2 \rangle}$ will no longer be of type A_2.

Thank You!