Auslander's Theorem for permutation actions on noncommutative algebras

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Invariant Theory

Throughout, let $\mathbb{k}$ denote an algebraically closed, characteristic zero field. All algebras are $\mathbb{k}$-algebras.

An algebra A is connected $\mathbb{N}$-graded if it has a vector space decomposition

$$A = \bigoplus_{i=0}^{\infty} A_i$$

such that $A_i A_j \subset A_{i+j}$ and $A_0 = \mathbb{k}$. Throughout, a graded algebra is understood to be $\mathbb{N}$-graded.

Classically: Let G be a group acting on $A = \mathbb{k}[x_1, \ldots, x_n]$ and study A^G, the subring of invariants.

Invariant Theory

A $g \in G$ is a reflection if g fixes a codimension 1 subspace of $\Bbbk[x_1, \dots, x_n]$.

A group G acting on $A = \Bbbk[x_1, \dots, x_n]$ is small if G contains no reflections.

The skew group algebra $A \# G$ is defined with basis $A \otimes G$ and multiplication

$$(a \otimes g)(b \otimes h) = (ag(b)) \otimes gh \quad \text{for all } a, b \in A, g, h \in G.$$

Invariant Theory

Theorem (Shephard-Todd 1954, Chevalley 1955)

Let G be a finite group acting on $A = \Bbbk[x_1, \ldots, x_n]$. Then $A^G \cong A$ if and only if G is generated by reflections.

Theorem (Auslander 1962)

Let G be a finite group acting linearly on $A = \Bbbk[x_1, \ldots, x_n]$. Define the Auslander map:

$$\begin{aligned} \gamma_{A,G} : A\#G &\to \operatorname{End}_{A^G}(A) \\ a\#g &\mapsto (b \mapsto ag(b)). \end{aligned}$$

If G is small, then $\gamma_{A,G}$ is an isomorphism.

Noncommutative Invariant Theory

There are two (standard) ways to extend classical invariant theory to the noncommutative setting:

- One could replace A with a noncommutative algebra.

Example

Let $\Bbbk_{-1}[x_1, \dots, x_n] = \Bbbk\langle x_1, \dots, x_n : x_i x_j + x_j x_i = 0$ for $i \neq j\rangle$ denote the (-1)-skew polynomial ring in n variables. The symmetric group $\mathcal{S}_n$ acts linearly on $\Bbbk_{-1}[x_1, \dots, x_n]$ as permutations ($\sigma.x_i = x_{\sigma(i)}$)

- One could replace G with a Hopf algebra H.

Example

The Sweedler algebra $H_2 = \Bbbk\langle g, x : g^2 = 1, x^2 = 0, gx + xg = 0\rangle$ acts on $\Bbbk[u, v]$ by $g(u) = u$, $g(v) = -v$, $x(u) = 0$, $x(v) = u$.

AS regular algebras

Definition

A graded algebra A is Artin-Schelter (AS) regular if

- $\mathrm{gldim}(A) = d < \infty$;
- $\mathrm{GKdim}(A) < \infty$;
- $\mathrm{Ext}^i_A(\Bbbk) = \delta_{id}\Bbbk(\ell)$.

The constant ℓ is the Gorenstein parameter of A.

Definition

A quantum polynomial ring is a noetherian AS regular algebra A with Hilbert series $H_A(t) = (1-t)^{-n}$.

Question

In what context do the Shephard-Todd-Chevalley Theorem and Auslander's theorem hold when A is replaced by an AS regular algebra?

Reflections

But first...

Question

What do we mean by a reflection in the noncommutative setting?

In general, it is too much to ask that $A^G \cong A$.

Example

Let $A = \Bbbk_{-1}[x, y]$ *and* $G = \langle g \rangle$, $|g| = 4$, *where* $g(x) = y$ *and* $g(y) = -x$. *Then* $A^G = \Bbbk[x^2 + y^2, xy]$. *In the classical sense,* g *is not a reflection.*

Definition

A finite group G acting linearly on an AS regular algebra A is a reflection group if A^G is AS regular.

Reflections

Let A be a graded algebra. The trace function of a graded automorphism g acting on A is defined to be the formal power series

$$\mathrm{Tr}_A(g,t) = \sum_{j=0}^{\infty} \mathrm{tr}\left(g|_{A_j}\right) t^j.$$

Definition (Kirkman, Kuzmanovich, and Zhang 2008)

Let A be a graded algebra of GK dimension n. Then $g \in G$ is a quasi-reflection if its trace series is of the form

$$\mathrm{Tr}_A(g,t) := \frac{1}{(1-t)^{n-1}q(t)}, \quad q(1) \neq 0.$$

Conjecture (NC Shephard-Todd-Chevalley)

Let A be an AS regular algebra and G a finite group of graded automorphisms of A. Then A^G is AS regular if and only if G is generated by quasi-reflections of A.

Theorem (Kirkman, Kuzmanovich, and Zhang 2008)

The conjecture holds when A is a quantum polynomial ring and G is a finite abelian group of graded automorphisms of A.

The Auslander Map

Auslander's theorem first appeared in his 1962 paper, "On the purity of the branch locus". It is not actually stated as a theorem at all!

The result relates finitely generated projective modules over $\Bbbk[x, y]\#G$ to maximal Cohen-Macaulay modules over $\Bbbk[x, y]^G$.

Conjecture (NC Auslander's Theorem)

If A is an AS regular algebra and G a finite group acting linearly on A without quasi-reflections, then the Auslander map is an isomorphism.

The Auslander Map

One can try to adapt the original proof to the noncommutative setting.

In general, with the right initial conditions, injectivity is not a problem. On the other hand, to prove surjectivity, the original proof requires that every minimal prime ideal in A is unramified over A^G.

We have run into quantum rigidity, that is, noncommutative algebras have few prime ideals.

Ample Group Actions

If A is a right noetherian graded ring, then set $\operatorname{tails} A := \operatorname{grmod} A / \operatorname{tors} A$.

Definition

Let A be a right noetherian graded algebra and G a finite subgroup of graded automorphisms of A. Set $e = \frac{1}{|G|} \sum_{g \in G} 1 \# g \in A \# G$. The group G is said to be ample (for A) if

$$(-)e : \operatorname{tails} A \# G \to \operatorname{tails} A^G$$

is an equivalence functor.

Ample Group Actions

Theorem (Mori, Ueyama 2015)

Let A be a noetherian AS-regular algebra of dimension $d \geq 2$ and $G \subset \operatorname{GrAut} A$ a finite ample subgroup. Then $A \# G \cong \operatorname{End}_{A^G}(A)$ as graded algebras.

One problem with the ampleness condition, apart from being hard to check, is that it implies something stronger than Auslander's theorem. It actually implies that A^G is a graded isolated singularity (more on this later).

Pertinency

Let A be a graded algebra and write $A = \bigoplus_n A_n$. The Gelfand-Kirillov (GK) dimension of A is

$$\operatorname{GKdim}(A) := \limsup_{n\to\infty} \log_n(\dim_k A_n).$$

Definition (Bao, He, Zhang 2016)

Let A be an graded algebra with $\operatorname{GKdim} A < \infty$ and G a finite group acting linearly on A. The pertinency of the action of G on A is defined to be

$$\mathrm{p}(A, G) = \operatorname{GKdim} A - \operatorname{GKdim} A\#G/(f_G)$$

where (f_G) is the two-sided ideal generated by $f_G = \sum_{g\in G} 1\#g$.

Theorem (BHZ)

Given the above setup, $A\#G \cong \operatorname{End}_{A^G}(A)$ *if and only if* $\mathrm{p}(A, G) \geq 2$.

Pertinency

Example (BHZ)

Consider $W = \langle \sigma \rangle$, $|\sigma| = n \geq 2$, *acting on* $\Bbbk_{-1}[x_1, \ldots, x_n]$ *by* $\sigma : x_i \mapsto x_{i+1}, x_n \mapsto x_1$ *for* $1 \leq i \leq n-1$. *If* $n = 2^d$, $d \geq 2$, *then*

$$p(\Bbbk_{-1}[x_1, \ldots, x_n], W) = n \geq 2.$$

Hence, the Auslander map is an isomorphism in this case.

Question

Is the Auslander map an isomorphism for $\Bbbk_{-1}[x_1, \ldots, x_n]$ *and any subgroup of* $\mathcal{S}_n$*?*

Strategery

Thanks to Bao, He, Zhang, we just need to understand the ideal

$$(f_G) = \left(\sum_{g \in G} 1 \# g \right).$$

Theorem (BHZ)

Let A be finitely generated over a central subalgebra T. Let A' be the image of the map

$$A \hookrightarrow A \# G \to (A \# G)/(f_G)$$

and $T' \subseteq A'$ be the image of T. Then

$$\operatorname{GKdim} T' = \operatorname{GKdim} A' = \operatorname{GKdim}(A \# G)/(f_G).$$

So need only understand $(f_G) \cap A$ or even $(f_G) \cap T$.

Strategery

Let $J \subset (f_G) \cap T$ be an ideal.

Assuming we can show GKdim $T/J \leq n-2$ we have

$$\begin{aligned} p(A, G) &= \operatorname{GKdim} A - \operatorname{GKdim}(A\#G)/(f_G) \\ &\geq \operatorname{GKdim} A - \operatorname{GKdim} T/J \\ &\geq n - (n-2) \\ &= 2. \end{aligned}$$

Thus, under this assumption, the Auslander map is an isomorphism for A and G.

Producing elements

Let R be a commutative algebra and G a finite group acting on R.

For $g \in G$, let $I(g)$ be the ideal generated by $\{r - g.r : r \in R\}$.

Lemma (Brown, Lorenz 1994)

$$\prod_{\substack{g \in G \\ g \neq e}} I(g) \subset (f_G) \cap R$$

There are two problems with applying/adapting this lemma:

- The proof of this lemma is highly commutative.
- It produces elements of degree $|G| - 1$, often much higher than lowest degree element in (f_G).

However, the idea can be adapted for algebras with large centers.

Example: $\mathbb{k}_{-1}[x_1, x_2, x_3]$ and $\mathcal{S}_3$

Let $T = \mathbb{k}[x_1^2, x_2^2, x_3^2] \subset C(V_3)$ and $f = \sum_{\sigma \in \mathcal{S}_3} 1\#\sigma$. Define

$$\begin{aligned}
f_1 &= x_1^2 f - f x_2^2 \\
&= (x_1^2 - x_2^2)\#(1) + (x_1^2 - x_2^2)\#(13) \\
&\qquad + (x_1^2 - x_3^2)\#(23) + (x_1^2 - x_3^2)\#(123) \\
f_2 &= x_1^2 f_1 - f_1 x_3^2 \\
&= (x_1^2 - x_2^2)(x_1^2 - x_3^2)\#(1) + (x_1^2 - x_3^2)(x_1^2 - x_2^2)\#(23) \\
f_3 &= x_2^2 f_2 - f_2 x_3^2 \\
&= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)\#(1) \in (f) \cap C(A).
\end{aligned}$$

This provides only one of the elements we need. We must use noncommutativity to obtain the second element.

Example: $\Bbbk_{-1}[x_1, x_2, x_3]$ and $\mathcal{S}_3$

Recall

$$f_2 = (x_1^2 - x_2^2)(x_1^2 - x_3^2)\#(1) + (x_1^2 - x_3^2)(x_1^2 - x_2^2)\#(23).$$

Now

$$\begin{aligned} g_{23} &= (x_2 f_2 - f_2 x_3)(x_2 - x_3) \\ &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2 - x_3)^2\#(1) \\ &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 + x_3^2)\#(1) \in (f) \cap C(A). \end{aligned}$$

We can similarly construct g_{12} and g_{13}. Set $g = g_{12} + g_{13} + g_{23}$.

Example: $\Bbbk_{-1}[x_1, x_2, x_3]$ and $\mathcal{S}_3$

The elements

$$\begin{aligned} f_3 &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2) \\ g &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 + x_3^2) \\ &\quad + (x_1^2 - x_2^2)(x_1^2 + x_3^2)(x_2^2 - x_3^2) \\ &\quad + (x_1^2 + x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2) \end{aligned}$$

are relatively prime in $T = \Bbbk[x_1^2, x_2^2, x_3^2]$ and GKdim $T/(f_3, g) \leq 1$.

Theorem

Let G be any subgroup of $\mathcal{S}_n$ acting on $V_n = \Bbbk_{-1}[x_1, \dots x_n]$ as permutations. Then $\mathrm{p}(V_n, G) \geq 2$ so the Auslander map is an isomorphism.

Example: $S(a,b,c)$ and $\langle(1\ 2\ 3)\rangle$

Let $S(a,b,c)$ be the three-dimensional Sklyanin algebra

$$S(a,b,c) = \Bbbk\left\langle x_1, x_2, x_3 \;\middle|\; \begin{array}{l} ax_1x_2 + bx_2x_1 + cx_3^2 \\ ax_2x_3 + bx_3x_2 + cx_1^2 \\ ax_3x_1 + bx_1x_3 + cx_2^2 \end{array}\right\rangle$$

acted on by $\langle(1\ 2\ 3)\rangle$. Let $f = 1\#e + 1\#(1\ 2\ 3) + 1\#(1\ 3\ 2)$.

$$f_1 = x_1 f - f x_3 = (x_1 - x_3)\#e + (x_1 - x_2)\#(1\ 3\ 2).$$

Then

$$\begin{aligned}(x_1 - x_2)f_1 + f_1(x_2 - x_3) &= (x_1 - x_2)(x_1 - x_3) + (x_1 - x_3)(x_2 - x_3)\#e \\ &= (x_1^2 - x_3^2 - x_2x_1 - x_1x_2)\#e \in (f).\end{aligned}$$

Example: $S(a, b, c)$ and $\langle (1\ 2\ 3) \rangle$

So

$$x_1^2 - x_3^2 - x_2x_1 - x_1x_2 \in (f) \cap S(a, b, c)$$
$$x_2^2 - x_3^2 - x_1x_2 - x_2x_1 \in (f) \cap S(a, b, c).$$

Now a Gröbner basis argument implies

$$\dim_{\Bbbk} \frac{S(a, b, c)}{(f) \cap S(a, b, c)} < \infty.$$

Theorem

Let $G = \langle (1\ 2\ 3) \rangle$ acting on $A = S(a, b, c)$ for generic $(a : b : c) \in \mathbb{P}^2$.

Then $\mathrm{p}(A, G) = 3 \geq 2$ so the Auslander map is an isomorphism.

All the Auslanders!

Theorem

The Auslander map is an isomorphism for the following:

- *subgroups of* $\mathcal{S}_n$ *acting on* $\Bbbk_{-1}[x_1, \ldots, x_n]$,
- *subgroups of* $\mathcal{S}_n$ *acting on the* (-1)*-quantum Weyl algebra,*
- *subgroups of* $\mathcal{S}_3$ *acting on the three-dimensional Sklyanin algebra* $S(1, 1, -1)$,
- *the cyclic group* $\langle(1\ 2\ 3)\rangle$ *acting on a generic three-dimensional Sklyanin algebra* $S(a, b, c)$,
- *subgroups of weighted permutations acting on the down-up algebra* $A(2, -1)$,
- $\langle -I_n, (1\ 3)(2\ 4)\rangle$ *acting on* $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$.

Graded isolated singularities

Definition (Ueyama 2013)

A^G is a graded isolated singularity if $\operatorname{gldim} \operatorname{tails} A^G < \infty$.

Theorem (Ueyama 2016)

If A^G is a graded isolated singularity, then

- *A^G is an AS-Gorenstein algebra of dimension $d \geq 2$,*
- *$A \in \mathrm{CM}^{\mathrm{gr}}(A^G)$ is a $(d-1)$-cluster tilting module, and*
- *$\mathrm{Ext}^1_{A^G}(A, M)$ and $\mathrm{Ext}^1_{A^G}(M, A)$ are f.d. for $M \in \mathrm{CM}^{\mathrm{gr}}(A^G)$.*

Theorem (Mori and Ueyama 2016)

If $\operatorname{GKdim} A \geq 2$, A^G is a graded isolated singularity if and only if $\dim_{\Bbbk} A\#G/(f_G) < \infty$ if and only if $\mathrm{p}(A, G) = n$.

Graded isolated singularities

Theorem (BHZ)

Let $A = \Bbbk_{-1}[x_1, \ldots, x_{2^n}]$ and $G = \langle (1\ 2\ \cdots\ 2^n) \rangle$. Then $\mathrm{p}(A, G) = 2^n$ so A^G is a graded isolated singularity.

Theorem

For the following, A^G is a graded isolated singularity:

- $\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$ *acting on* $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$,
- $\langle (1\ 2)(3\ 4) \cdots (2n-1\ \ 2n) \rangle$ *acting on* $\Bbbk_{-1}[x_1, \ldots, x_{2n}]$,
- $\langle (1\ 2\ 3) \rangle$ *acting on a generic Sklyanin algebra* $S(a, b, c)$,
- $\langle -I_n, (1\ 3)(2\ 4) \rangle$ *acting on* $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$.

Whither the upper bounds?

Constructing elements of (f_G) gives lower bounds for $\mathrm{p}(A, G)$. Can we find upper bounds as well?

Theorem

If $G' \leq G$ then $\mathrm{p}(A, G) \leq \mathrm{p}(A, G')$.

This resolves a conjecture of Bao-He-Zhang for the group case.

Corollary

Let A be a noetherian connected graded algebra and suppose G contains a quasi-reflection g. If A and $A^{\langle g \rangle}$ have finite global dimension, then the Auslander map $\gamma_{A,G}$ is not an isomorphism.

Computing pertinency exactly

Lower bounds: constructing elements of (f_G)

Upper bounds: subgroup theorem

Subgroups of $\mathcal{S}_3$ acting on $\Bbbk_{-1}[x_1, x_2, x_3]$:

conjugacy class	$\mathrm{p}(A, G)$
$\langle (12) \rangle$	2
$\langle (123) \rangle$	2 or 3
$\langle (12), (23) \rangle$	2

Computing pertinency exactly

Subgroups of $\mathcal{S}_4$ acting on $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$:

conjugacy class	p(A, G)
$\langle (12) \rangle$	2
$\langle (12)(34) \rangle$	4
$\langle (123) \rangle$	2 or 3
$\langle (1234) \rangle$	4
$\langle (12), (34) \rangle$	2
$\langle (12)(34), (13)(24) \rangle$	4
$\langle (1234), (24) \rangle$	2
$\langle (123), (124) \rangle$	2 or 3
$\langle (123), (12) \rangle$	2
$\langle (1234), (12) \rangle$	2

Weighted Permutations

Question

In the main theorem, can the group G be replaced by a finite group of weighted permutations?

Here is one example of such a group that acts on V_3:

$$\mathcal{W} = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & ab \end{bmatrix}, \begin{bmatrix} 0 & c & 0 \\ d & 0 & 0 \\ 0 & 0 & cd \end{bmatrix} : a, b, c, d \in \Bbbk^\times \right\}.$$

A modification of our method shows that the Auslander map is an isomorphism for any finite subgroup G of $\mathcal{W}$ acting on V_3 as permutations.

Graded Down-up Algebras

Let A be the graded down-up algebra generated by x and y subject to the relations

$$\begin{aligned} x^2y + yx^2 + 2xyx &= 0 \\ xy^2 + y^2x + 2yxy &= 0. \end{aligned}$$

A result of Kirkman and Kuzmanovich gives

$$\mathrm{Aut}_{gr}(A) = \left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in \Bbbk^\times \right\}.$$

Graded Down-up Algebras

Let G be a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(A)$ acting linearly on A. Take the filtration $\mathcal{F} = \{F_n\}$ defined by

$$F_n A = (\Bbbk \oplus \Bbbk x \oplus \Bbbk y \oplus \Bbbk z) \subset A \text{ for all } n \geq 0.$$

$\mathcal{F}$ is G-stable and $R := \mathrm{gr}_{\mathcal{F}}(A) \cong V_3$.

Since R is a connected graded algebra with G-action, then $A \cong R$ as G-modules so the G-action is inner faithful and homogeneous.

Hence, for A and any finite subgroup G of $\mathrm{Aut}_{\mathrm{gr}}(A)$ acting linearly on A, the Auslander map is an isomorphism

Other questions

- What is the pertinency of $\langle(1\ 2\ 3)\rangle$ acting on $\Bbbk_{-1}[x_1, x_2, x_3]$ as permutations (computational evidence suggests it is 2)? In general, need more methods for constructing upper bounds.
- Direct connections between Tr and pertinency?
- Replace the relations in the down-up algebra A before with

$$x^2y + yx^2 - \alpha xyx = xy^2 + y^2x - \alpha yxy = 0.$$

  Is the Auslander map an isomorphism for any finite subgroup of graded automorphisms acting on A? Have some partial results in this direction.
- Hopf actions! A theorem of Chan, Kirkman, Walton, and Zhang (2016) shows that the Auslander map is an isomorphism for A an AS regular algebra of dimension 2 and H a semisimple Hopf algebra.

Thank You!