Ozone groups of AS regular algebras satisfying a polynomial identity

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Recent Developments in Noncommutative Algebra and Tensor Categories
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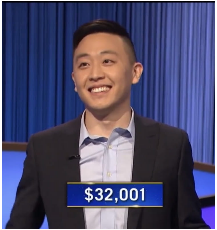

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Setup

Let $\mathbb{k}$ be a field of characteristic zero.

Definition

A connected graded algebra A is called Artin–Schelter (AS) Gorenstein if A has injective dimension $d < \infty$ on the left and on the right, and

$$\mathrm{Ext}^i_A({}_A\mathbb{k}, {}_AA) \cong \mathrm{Ext}^i_A(\mathbb{k}_A, A_A) \cong \delta_{id}\mathbb{k}(\ell)$$

where δ_{id} is the Kronecker-delta.

If, in addition, A has finite global dimension and finite Gelfand–Krillov (GK) dimension, then A is called Artin–Schelter (AS) regular of dimension d.

Skew polynomial rings

Given $\mathbf{p} = (p_{ij}) \in M_n(\Bbbk^\times)$ multiplicatively antisymmetric, the skew polynomial ring is

$$S_\mathbf{p} = \Bbbk_\mathbf{p}[x_1, \ldots, x_n] = \Bbbk\langle x_1, \ldots, x_n\rangle/(x_j x_i = p_{ij} x_i x_j).$$

Let $\phi_i \in \mathrm{Aut}_{\mathrm{gr}}(S_\mathbf{p})$ denote conjugation by x_i:

$$\phi_i(f) = x_i^{-1} f x_i \quad \text{for all } f \in S_\mathbf{p}.$$

Let $O = \langle \phi_1, \ldots, \phi_n \rangle$, which is a subgroup of $\mathrm{Aut}_{\mathrm{gr}}(S_\mathbf{p})$.

One can show that $O = \mathrm{Aut}_{Z(S_\mathbf{p})-\mathrm{alg}}(S)$.

Suppose the p_{ij} are all roots of unity (e.g., S is PI). Since $Z(S_\mathbf{p}) = S_\mathbf{p}^O$, we can study $Z(S_\mathbf{p})$ using tools from (noncommutative) invariant theory. For example...

Theorem (CGWZ)

Suppose $n = 3$ and $S_\mathbf{p}$ is PI. Then $Z(S_\mathbf{p})$ is regular if and only if the orders of p_{12}, p_{13}, and p_{23} are pairwise coprime.

The ozone group

Definition

Let A be an algebra and C be a subalgebra of $Z(A)$. The Galois group of A over C is

$$\mathrm{Gal}(A/C) := \{\sigma \in \mathrm{Aut}(A) \mid \sigma(c) = c \text{ for all } c \in C\}.$$

We call $\mathrm{Oz}(A) = \mathrm{Gal}(A/Z(A))$ the ozone group of A.

Example

Let $A = \Bbbk_q[x, y]$ where q is a primitive nth root of unity.

The automorphisms determined by conjugation by x and by y generate the ozone group and we have $\mathrm{Oz}(A) = \mathbb{Z}_n \times \mathbb{Z}_n$.

Lemma

Suppose A is $\mathbb{Z}^n$-graded domain which is prime and a finite module over its center.

If $\phi \in \mathrm{Oz}(A)$, then ϕ is given by conjugation by a normal homogeneous element.

In particular, $\mathrm{Oz}_{\mathrm{gr}}(A) = \mathrm{Oz}(A)$.

The ozone group

Example

Given a primitive ℓth root of unity q, $\ell \geq 2$ and $3 \nmid \ell$, the quantum Heisenberg algebra is

$$H_q = \Bbbk\langle x, y, z\rangle / (zx - qxz, yz - qzy, xy - qyx - z^2).$$

Set $\Omega = xy - q^{-2}yx$. The center of H_q is generated by x^ℓ, y^ℓ, z^ℓ, and Ωz.

Let $\phi \in \mathrm{Oz}(H_q)$. Then

$$\phi(x) = \epsilon_1 x, \quad \phi(y) = \epsilon_2 y, \quad \phi(z) = \epsilon_3 z$$

where each ϵ_i is an ℓth root of unity.

In order to fix Ωz and satisfy $0 = \phi(xy - qyx - z^2)$, we must have

$$\epsilon_3^{-1} = \epsilon_1 \epsilon_2 = \epsilon_3^2,$$

so $\epsilon_3^3 = 1$. Since $3 \nmid \ell$, then $\mathrm{Oz}(H_q) \cong \mathbb{Z}_\ell$.

Down-up algebras

Definition

For $\alpha, \beta \in \Bbbk$ with $\beta \neq 0$, the graded down-up algebra is defined as

$$A(\alpha, \beta) := \Bbbk\langle x, y\rangle/(x^2y - \alpha xyx - \beta yx^2, xy^2 - \alpha yxy - \beta y^2x).$$

The algebras $A(\alpha, \beta)$ are AS regular of global dimension three.

Let $A = A(\alpha, \beta)$ be PI. Let ω_1 and ω_2 be the roots of the characteristic equation

$$w^2 - \alpha w - \beta = 0.$$

Set $z = \Omega_1 = xy - \omega_1 yx$. Then

$$xz = \omega_2 zx, \quad \omega_2 yz = zy, \quad xy = \omega_1 yx + z.$$

That is, A is a filtered skew polynomial ring.

If $\phi \in \mathrm{Oz}(A)$, then $\phi(x) \in \Bbbk x$ and $\phi(y) \in \Bbbk y$. As a consequence, $\mathrm{Oz}(A)$ is abelian.

Ozone groups of extensions

Lemma

Let A be a noetherian PI domain and suppose $\sigma \in \mathrm{Aut}(A)$ has finite order n. Suppose further that σ^i is not inner for any $1 \leq i < n$. Then

$$\mathrm{Oz}(A[t;\sigma]) \cong \mathbb{Z}_n \times \{\tau \in \mathrm{Aut}(A) \mid \sigma\tau = \tau\sigma \text{ and } \tau(a) = a \text{ for all } a \in Z(A)^{\langle\sigma\rangle}\}$$

where $\mathbb{Z}_n$ is the cyclic group of automorphisms generated by an automorphism which fixes A and maps t to ξt for a primitive nth root of unity ξ.

Lemma

If A and B are noetherian PI AS regular algebras, then

$$\mathrm{Oz}(A \otimes B) = \mathrm{Oz}(A) \times \mathrm{Oz}(B).$$

In particular, $\mathrm{Oz}(A[t]) = \mathrm{Oz}(A)$.

Hence, every finite abelian group is realizable as the ozone group of a noetherian PI AS regular algebra.

The hole in the ozone group

Conjecture

If A is a PI AS regular algebra, then $\mathrm{Oz}(A)$ is abelian.

Example

Let A be the preprojective algebra of the extended Dynkin quiver $\widetilde{A}_2$. Then A is a quotient of the path algebra of the quiver.

$$\begin{tikzcd}
& e_0 \arrow[dl, shift left, "a_0"] \arrow[dr, shift left, "a_2^*"] & \\
e_1 \arrow[ur, shift left, "a_0^*"] \arrow[rr, shift left, "a_1"] & & e_2 \arrow[ul, shift left, "a_2"] \arrow[ll, shift left, "a_1^*"]
\end{tikzcd}$$

with the relation $\sum (a_i a_i^* - a_i^* a_i)$.

The Ozone group of A is infinite and non-abelian.

Trivial ozone

If A is PI AS regular, then the Nakayama automorphism μ_A is an element of $\mathrm{Oz}(A)$.

So, trivial ozone implies Calabi–Yau.

Example

Let $(a, b, c) \in (\Bbbk)^3$, the 3-dimensional Sklyanin algebra

$$S(a, b, c) = \Bbbk\langle x, y, z\rangle/(axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2).$$

Suppose $S(a, b, c)$ is AS regular and rank n^2 over its center. Then $S(a, b, c)$ has trivial ozone group if and only if $n > 1$ and $3 \nmid n$.

Example

Let $S_{\mathbf{p}} = \Bbbk_{\mathbf{p}}[x, y, z]$ be a Calabi-Yau PI skew polynomial ring. In this case, the relations are

$$yx = \xi xy, \qquad zy = \xi yz, \qquad xz = \xi zx.$$

Then $\mathrm{Oz}(S_{\mathbf{p}}) = \mathbb{Z}_n \times \mathbb{Z}_n$.

Trivial ozone

Using the classification of Itaba and Mori of quantum projective planes finite over their centers, we obtain the following classification result.

Theorem (CGWZ)

Let A be a PI quadratic AS regular algebra of global dimension 3 with trivial ozone group. Then A is isomorphic to one of the following:

- *a Skylanin algebra $S(a,b,c)$ which is rank n^2 over its center, $n>1$ and $3 \nmid n$, or*
- *the algebra*

$$B_q = \Bbbk\langle x,y,z\rangle/(xy - qyx, zx - qxz - y^2, zy - q^{-1}yz - x^2)$$

  where $q \neq 1$ is a root of unity and 3 does not divide the order of q.

Lemma

Let A be a $\mathbb{Z}$-graded domain which is a finite module over its center. Then $\mathrm{Oz}(A)$ is trivial if and only if every normal element is central.

Skew polynomial rings

The ozone group can be used to characterize skew polynomial rings.

Lemma

Let A be a noetherian connected graded algebra, generated in degree 1, with finite global dimension. If A is generated (as a $\Bbbk$-algebra) by normal elements, then A is isomorphic to $S_{\mathbf{p}}$ for some $\mathbf{p}$.

Lemma

Suppose $\Bbbk = \overline{\Bbbk}$. Let A be a noetherian PI AS regular algebra. If $\mathrm{Oz}(A)$ is abelian and $|\mathrm{Oz}(A)| = \mathrm{rk}_Z(A)$, then A is generated by normal elements.

Theorem (CGWZ)

Suppose $\Bbbk = \overline{\Bbbk}$ and A is generated in degree 1. Then A is a skew polynomial ring if and only if $\mathrm{Oz}(A)$ is abelian and $|\mathrm{Oz}(A)| = \mathrm{rk}_Z(A)$.

Iterated Fixed Rings

Our results imply that for a PI AS regular algebra A generated in degree one, if $A^{\mathrm{Oz}(A)} = Z(A)$ and $\mathrm{Oz}(A)$ is abelian, then A is a skew polynomial ring.

Can we use invariant theory and ozone groups to understand centers of other PI AS regular algebras?

Example

Let $A = A(0, -1) = \Bbbk\langle x, y\rangle/(x^2y + yx^2, xy^2 + y^2x)$.

Then $\mathrm{Oz}(A) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Here $B = A^{\mathrm{Oz}(A)}$ is generated by

$$a = x^4, \quad b = y^4, \quad c = x^2y^2, \quad d = xyxy, \quad e = yxyx.$$

Since $\mathrm{Oz}(A)$ acts by trivial homological determinant on A, then B is (AS) Gorenstein.

We have $G = \mathrm{Gal}(B/Z(A)) = \{1, \tau\}$ where

$$\tau(a) = a, \quad \tau(b) = b, \quad \tau(c) = -c, \quad \tau(d) = e, \quad \tau(e) = d.$$

Then $B^G = Z(A)$.

Since B is commutative and $\det \tau|_{B_1} = 1$, then $B^G = Z(A)$ is Gorenstein.

Quantum thickenings

For a Hopf algebra H, let $G(H)$ denote the group of grouplike elements of H.

Definition

Let G be a group acting faithfully on an algebra A. Suppose that H is a semisimple Hopf algebra acting inner-faithfully on A with group of grouplikes $G(H)$.

The H-action on A is a quantum thickening of the G-action on A if $G(H)$ if there is an isomorphism $G \to G(H)$ which preserves the actions on A, or H is a quantum thickening of G if the actions are clear.

Question

Given a PI AS regular algebra A, is there a quantum thickening H of $\mathrm{Oz}(A)$ such that $A^H = Z(A)$?

Quantum thickenings?

Example

Let $A = A(0,1) = \Bbbk\langle x, y\rangle/(x^2y - yx^2, xy^2 - y^2x)$.

Then $\mathrm{Oz}(A) = \langle \phi \rangle \cong \mathbb{Z}_2$ where $\phi(x) = -x$ and $\phi(y) = -y$.

Let H be a quantum thickening of $\mathrm{Oz}(A)$ with $\dim(H) = 4$. Then H is either a group algebra or a dual of a group algebra and hence $A^H \neq Z(A)$.

Example

Let $A = A(0,-1) = \Bbbk\langle x, y\rangle/(x^2y + yx^2, xy^2 + y^2x)$.

Then $\mathrm{Oz}(A) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

There is an action of the dimension 8 Kac–Palyutkin Hopf algebra H on A in which $G(H) = \mathrm{Oz}(A)$, but one can verify that $A^H \neq Z(A)$.

Similarly, there are actions when $\dim(H) = 16$ and $G(H) = \mathrm{Oz}(A)$, but again we find that $A^H \neq Z(A)$.

But wait! There's more!

Questions

Let A be a PI AS regular algebra.

- *Is there a connection between $Z(A)$ having an isolated singularity and $\mathrm{Oz}(A)$ being trivial?*

- *What role does the Nakayama automorphism play in the study of the ozone group and the center?*

- *Assume $\mathrm{Oz}(A)$ is abelian. If the $\mathrm{Oz}(A)$-action on A has trivial homological determinant, then is A CY? One can also ask the converse. If A is CY, does the action of $\mathrm{Oz}(A)$ on A necessarily have trivial homological determinant?*

- *Let Z' denote the center of $A^{\mathrm{Oz}(A)}$ and $\overline{Z}$ denote the center of $A\#\Bbbk Oz(A)$. Under what hypotheses on A are Z' and $\overline{Z}$ isomorphic as graded algebras? (They are when A is a skew polynomial ring.)*

Thank You!