Explorations in Mathematics - Spring 2016

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Welcome!

My name is Dr. Jason Gaddis. I am a Teacher-Scholar Postdoctoral Fellow in my second year at WFU.

You can often find me in my office, Manchester 338. My office hours are MW 10-11, T 1-2, F 3-4, or by appointment.

If you want to contact me, your best bet is email, gaddisjd@wfu.edu.

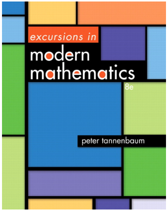
Class Meetings

Our class meets MWF 12:30pm - 1:45pm in Manchester 020. If you didn't know that already you probably wouldn't be here.

We will meet 75 minutes each day, which is 9/8 as much as we should. In other words, we will skip 5 total classes. Here are some days we will not meet:

March 4, April 8, April 18, and two days to be announced.

Text



Our text is *Excursions in Modern Mathematics (8e)* by Peter Christmas Tree~~Christmas Tree~~ Tannenbaum.

Additional readings will be assigned through the library's course reserve system.

Your first reading is "There is no such thing as public opinion" from Jordan Ellenberg's book, *How Not To Be Wrong*. You can access this from Sakai.

Grading

Your final grade will be computed using a weighted average on

- Classwork and Participation (10%)
- Homework (15%)
- Projects (25%)
- Exams (25%) (2/10, 3/23, 4/20)
- Final Exam (25%) (5/2)

Final Exemption

If you satisfy the three following requirements, the final exam is optional for you.

- Average on April 29 is at least 90.0%
- Test average is at least 85%
- No more than 2 absences

Bonus

Bonus points are available for attending math colloquiums (most Wednesdays, 4pm-5pm) and writing a summary.

Each colloquium is worth 2 points on the next exam.

There will be further class-wide bonus assignments but bonus points are not available for individual students.

Disabilities

If you have a disability which may require an accommodation for taking this course, please contact the Learning Assistance Center (758-5929), then contact me within the first 2 weeks of the semester.

Academic Integrity

Academic integrity is something I take seriously.

In this class I encourage you to talk to other students and myself about assignments. When completing assignments for a grade, however, you must write up/work out the assignment on your own.

During exams, it is expected that you are writing up your own work and do not have any outside help.

Technology

I ask that you silence your cell phone and put all devices away unless specifically instructed otherwise.

What is Math?

This class is about exploring mathematical concepts you might not realize even existed.

This class is about answering hard questions with math, or sometimes just figuring out what the question is in the first place.

There are three basic units to this course, each roughly 5 weeks long.

Voting

Question: Who won the state of Florida in the 2000 presidential election?

Answer: George W. Bush

Question: Who *should have* won the state of Florida in the 2000 presidential election?

Answer: ?

Statistics and Probability

Question: Will I win the lottery this week?

Answer: No (probably)

Question: Should I play the lottery?

Answer: Maybe.

Symmetry

Question: Is the Golden Ratio everywhere?

Answer: No, but it shows up a lot.

Question: Why?

Answer: Fibonacci.

The first unit

Our first topic will be Voting.

Some things to keep in mind:

- This is not a politics class (even if we use examples from politics).
- In this class, we will not decide who the next president will be, but we will study how it is decided.
- Our personal political views are irrelevant in this class. Our goal will be to study ways in which elections can be decided. We should be respectful of others at all times.

Your first project

You will conduct a survey and analyze the results using the various voting methods we discuss.

I will distribute more information in a later class. For now, you should start thinking about a question you would like to ask.

Your first project

Some ground rules:

- Your question may not pertain to the 2016 presidential election (or the primaries). Nor can you use questions which we use as examples in class. Sports-related questions are discouraged. Each student must have a different topic.
- There must be between 4-6 legitimate options, and respondents must be able to rank their options not just choose their favorite.
- Respondents are limited to Wake Forest students, faculty, and staff (anyone with a WFU email adress).
- Your topics are due to me by Friday, January 22.

Your first project

Example of an acceptable question.
Rank your prefered flavor of ice cream from among the following options.

- Chocolate
- Chocolate Chip Cookie Dough
- Cookies and Cream
- Mint Chocolate Chip
- Vanilla

Voting Is Complex

Who do Iowa voters want as the Republican Presidential nominee?

28% Donald Trump
26% Ted Cruz
13% Marco Rubio
8% Ben Carson
6% Jeb Bush
3% Chris Christie
3% Carly Fiorina
3% Mike Huckabee
3% John Kasich
3% Rand Paul
2% Rick Santorum
0% Jim Gilmore

Source:Public Policy Polling

Voting Is Complex

Who is the second choice of Iowa voters?

21% Ted Cruz
13% Marco Rubio
10% Donald Trump
9% Ben Carson
7% Carly Fiorina
5% Chris Christie
4% Jeb Bush
4% Mike Huckabee
4% Rand Paul
4% Rick Santorum
2% John Kasich
0% Jim Gilmore

Source:Public Policy Polling

1.1 The Basic Elements of an Election
1.2 The Plurality Method

Some announcements

- Math Center study sessions with Katie Greene (TA). Tuesday and Wednesday 7pm-9pm in Kirby 120.
- First Math colloquium this Thursday at 5pm in Manchester 016. Attend and write summary for 2 bonus points on next exam.
- Another bonus opportunity: volunteering for a math enrichment program at a local middle school. If you are interested please speak or email me and I will send you the information.

Today's Goals

- To examine different ways that voters can choose a candidate from a list of options.
- We will discuss advantages and disadvantages to each method.
- Discuss examples from various types of elections.

Terminology

- Candidates
- Voters
- The ballots
- The outcome
- The voting method

Our interest will be in discussing primarily the different types of ballots and outcomes that an election can have. We will only briefly dwell on the other points.

Types of Ballots: Single-Choice Ballot

Choose your favorite flavor of ice cream amongst the options below.

- ○ Chocolate
- ✓ Cookie Dough
- ○ Mint Chocolate Chip
- ○ Vanilla

The winner can be decided by the candidate with the most votes, or via a traditional runoff.

Types of Ballots: Preference Ballot (1)

List the choices of ice cream in order of preference.

1st Cookie Dough

2nd Vanilla

3rd Chocolate

4th Mint Chocolate Chip

Types of Ballots: Preference Ballot (2)

Rank your preferred flavor of ice cream.

3rd Chocolate

1st Cookie Dough

4th Mint Chocolate Chip

2nd Vanilla

Types of Ballots: Truncated Preference Ballot

Rank your top 3 preferred flavors of ice cream in order of preference.

1st Cookie Dough

2nd Vanilla

3rd Chocolate

Types of Ballots: Approval Voting

Indicate which ice cream flavors you enjoy eating. You may mark as many as you wish, or none at all.

- ✓ Chocolate
- ✓ Cookie Dough
- ○ Mint Chocolate Chip
- ✓ Vanilla

The winner is the candidate with the most votes.

Outcomes

What are some different outcomes that an election can have?

- Winner-only (e.g., modern presidential elections)
- Partial Ranking (e.g., presidential elections prior to the 12th amendment)
- Full Ranking (sometimes just called Ranking) (e.g. College Basketball rankings)

Creating a preference schedule

A preference schedule is an efficient way to record election results using a preference ballot.

Let's take a hypothetical outcome for our ice cream election. We'll abbreviate the names: Cookie Dough (CD), Vanilla (V), Chocolate (C), Mint Chocolate Chip (M).

Creating a preference schedule

Suppose we have the following number of standard preference ballots.

- 18: CD, C, M, V
- 8: V, C, M, CD
- 20: M, C, CD, V
- 15: V, CD, C, M
- 3: C, M, V, CD

Creating a preference schedule

The columns list the different kinds of ballots with the rows indicating the rankings (1st through 4th).

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	CD	V	M	V	C
2nd	C	C	C	CD	M
3rd	M	M	CD	C	V
4th	V	CD	V	M	CD

Creating a preference schedule (alternate)

Now suppose we have the same results but using the alternate preference ballot where voters list their rankings. The order is alphabetical.

- 18: C(2), CD(1), M(3), V(4)
- 8: C(2), CD(4), M(3), V(1)
- 20: C(2), CD(3), M(1), V(4)
- 15: C(3), CD(2), M(4), V(1)
- 3: C(1), CD(4), M(2), V(3)

Creating a preference schedule (alternate)

The columns list number of ballots. The rows list the the candidates in the presecribed order (in this case alphabetical).

Number of voters	**18**	**8**	**20**	**15**	**3**
Chocolate	2	2	2	3	1
Cookie Dough	1	4	3	2	4
Mint Chocolate Chip	3	3	1	4	2
Vanilla	4	1	4	1	3

The Plurality Method

This is the most common method used in US elections today. The winner is the candidate with the most votes, regardless of whether this number represents a majority of the voters.

In our ice cream example, Vanilla is the winner using this method. But Vanilla only has 23 out of the 64 1st place votes (about 36%). Note that Vanilla also has the most 4th place votes (38 out of 64 or about 59%).

The Plurality Method

Here's one particular problem which is very troubling. Let's recall first our preference schedule.

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	CD	V	M	V	C
2nd	C	C	C	CD	M
3rd	M	M	CD	C	V
4th	V	CD	V	M	CD

When we compare Vanilla alone to any other flavor, it loses. That is, more people prefer Chocolate to Vanilla; more people prefer Cookie Dough to Vanilla; more people prefer Mint Chocolate Chip to Vanilla. So why should Vanilla win?

Condorcet

Recall from the Ellenberg reading, the mathematician/social scientist/French revolutionist Marie-Jean-Antoine-Nicolas de Caritat, Marquis de Condorcet. He established the following axiom of voting:

If the majority of voters prefer candidate A to candidate B, then candidate B cannot be the people's choice

Clearly our example is a violation of this if we allow Vanilla to win.

Condorcet

We call a candidate a Condorcet Candidate if they beat every other candidate in a head-to-head matchup.

Are there any condorcet candidates in the ice cream example?

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	CD	V	M	V	C
2nd	C	C	C	CD	M
3rd	M	M	CD	C	V
4th	V	CD	V	M	CD

Ties

What if there is a tie? The text gives several different ways that a tie can be broken.

- The result is shared (e.g., with some awards)
- Chance (coin flip or card draw)
- Runoff
- Other criteria from the election
- Deferred to another electing body (a tie in the Electoral College results in the House of Representatives selecting the president)

1.3-1.5 Voting Methods

Some announcements

- Homework #1: Text (pages 28-33) 1, 4, 7, 10, 12, 19, 22, 29, 32, 38, 42, 50, 51, 56-60, 61, 65 (this is posted on Sakai)
- Math Center study sessions with Katie Greene (TA). Tuesday and Wednesday 7pm-9pm in Kirby 120.
- Second Math colloquium this afternoon at 4pm in Manchester 016. Attend and write summary for 2 bonus points on next exam.
- Another colloquium will be this Wednesday at 4pm in Manchester 016.

Today's Goals

- To examine some different methods for evaluating election results, including Borda Count, Plurality-with-Elimination (IRV), and Pairwise Comparison.
- Work through some examples.

Last Time

In our last class we learned some of the lingo for elections and saw the different types of ballots that we can use for an election.

We saw how the Plurality Method was a quick and easy way to pick a winner but didn't always represent the will of the voters.

Plurality Method

Of all the methods, the Plurality Method is the most susceptible to **insincere voting** (also called tactical voting).

Insincere voting is the practice of voting against one's own interest in order to sway the election.

Think: In 2000, some voters who would have preferred Nader voted for Gore because they were afraid of 'wasting their vote'.

Last Time

We had a running example last time: favorite ice cream flavor. The choices were Chocolate (C), Cookie Dough (CD), Mint Chocolate Chip (M), and Vanilla (V). Here's the preference schedule.

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	CD	V	M	V	C
2nd	C	C	C	CD	M
3rd	M	M	CD	C	V
4th	V	CD	V	M	CD

There were 64 total voters. Remember that Vanilla was the winner using the **Plurality Method** but it also had the most last-place votes.

Borda Count

The **Borda Count Method** requires a preference ballot.

For each ballot, the candidate ranked last gets 1 points. The candidate ranked second to last gets 2 points, and so on. If there are N candidates, the top candidate gets N points.

There are variations on this. For example, we may not give points to all candidates, or we might weight the points differently. This is a **Modified Borda Count**.

2015 Heismann Award

The Heismann Award is decided using a Borda Count Method.

Player	1st (3pts.)	2nd (2pts.)	3rd (1pt.)	Total Points
Derrick Henry	378	277	144	1,832
Christian McCaffrey	290	246	177	1,539
DeShaun Watson	148	240	241	1,165
Baker Mayfield	34	55	122	334
Keenan Reynolds	20	17	86	180

Borda Count Example

Number of voters	18	8	20	15	3
1st	CD	V	M	V	C
2nd	C	C	C	CD	M
3rd	M	M	CD	C	V
4th	V	CD	V	M	CD

Vanilla got 23 1st place votes (with 4 points each), 3 3rd place votes (worth 2 points each), and 38 4th place votes (worth 1 point each). Thus the total for Vanilla is 136.

Similarly, Chocolate's total is 180. Mint Chocolate Chip's total is 156 and Cookie Dough's total is 168.

Chocolate wins! But Chocolate came in last using the **Plurality Method**.

Plurality-With-Elimination

This is also what we call **Instant Runoff Voting (IRV)**.

The idea is that we eliminate the loser from preference ballot and retally the votes. Only 1st place votes are counted.

In the first round, we tally the 1st place votes. A candidate must have a majority to win. Otherwise we eliminate the candidate with the fewest 1st place votes and tally again. This continues until one candidate has a majority (or there is a tie).

IRV example

In our ice cream election, Chocolate has the fewest 1st place votes so we eliminate it.

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	CD	V	M	V	~~C~~
2nd	~~C~~	~~C~~	~~C~~	CD	M
3rd	M	M	CD	~~C~~	V
4th	V	CD	V	M	CD

The effect of this is that Mint Chocolate Chip now has 3 more 1st place votes.

The tally now is Vanilla (23), Mint Chocolate Chip (23), and Cookie Dough (18). No flavor has a majority.

IRV example

Cookie Dough had the fewest 1st place votes in our new tally, so we now eliminate it :-(

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	~~CD~~	V	M	V	~~C~~
2nd	~~C~~	~~C~~	~~C~~	~~CD~~	M
3rd	M	M	~~CD~~	~~C~~	V
4th	V	~~CD~~	V	M	~~CD~~

The effect of this is that Mint Chocolate Chip now has 18 more 1st place votes.

The tally now is Vanilla (23), Mint Chocolate Chip (41).

Mint Chocolate Chip wins!

Reality TV competitions

Your book claims Reality TV shows (such as American Idol or The Voice) use IRV.

In reality they are a hybrib between IRV and traditional runoffs.

The Method of Pairwise Comparisons

With a preference schedule, it is possible to compare any pair of candidates by eliminating all others.

In the **Method of Pairwise Comparisons** we compare each pair of candidates, the winner of each comparison is given a point (1/2 point for ties).

The winner is the candidate with the most points.

Pairwise Comparison Example

Number of voters	18	8	20	15	3
1st	**CD**	V	M	V	**C**
2nd	**C**	**C**	**C**	**CD**	M
3rd	M	M	**CD**	**C**	V
4th	V	**CD**	V	M	**CD**

Cookie Dough (33) beats Chocolate (31)

Pairwise Comparison Example

Number of voters	18	8	20	15	3
1st	CD	V	**M**	V	**C**
2nd	**C**	**C**	**C**	CD	**M**
3rd	**M**	**M**	CD	**C**	V
4th	V	CD	V	**M**	CD

Chocolate (44) beats Mint Chocolate Chip (20)

Pairwise Comparison Example

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	CD	**V**	M	**V**	**C**
2nd	**C**	**C**	**C**	CD	M
3rd	M	M	CD	**C**	**V**
4th	**V**	CD	**V**	M	CD

Chocolate (41) beats Vanilla (23)

Pairwise Comparison Example

Number of voters	18	8	20	15	3
1st	**CD**	V	**M**	V	C
2nd	C	C	C	**CD**	**M**
3rd	**M**	**M**	**CD**	C	V
4th	V	**CD**	V	**M**	**CD**

Cookie Dough (33) beats Mint Chocolate Chip (31)

Pairwise Comparison Example

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	**CD**	**V**	M	**V**	C
2nd	C	C	C	**CD**	M
3rd	M	M	**CD**	C	**V**
4th	**V**	**CD**	**V**	M	**CD**

Cookie Dough (38) beats Vanilla (26)

Pairwise Comparison Example

Number of voters	**18**	**8**	**20**	**15**	**3**
1st	CD	**V**	**M**	**V**	C
2nd	C	C	C	CD	**M**
3rd	**M**	**M**	CD	C	**V**
4th	**V**	CD	**V**	**M**	CD

Mint Chocolate Chip (41) beats Vanilla (23)

Pairwise Comparison Example

Our tally is then,

- Chocolate (2)
- Cookie Dough (3)
- Mint Chocolate Chip (1)
- Vanilla (0)

Cookie Dough Wins!

Condorcet Criterion

In an election, a **Condorcet Candidate** is one that beats all other candidates in a head-to-head (pairwise) comparison.

The **Condorcet criterion** states that such a candidate should be the winner.

In our ice cream election, Cookie Dough is a Condorcet candidate.

So who really won?

We saw 4 different voting methods (Plurality, Borda Count, IRV, Pairwise Comparison). Each gave a different winner.

Several mathematicians, including Condorcet, Borda, and Arrow, have tried to establish fairness criteria for elections. Unfortunately, every voting method violates some fairness criteria. This is the gist behind Arrow's Impossibility Theorem (Section 1.6).

Additional Methods

There are many additional methods which are not mentioned in your book (some are in the exercises). Here are three that we will discuss:

- approval voting
- least worst defeat
- ranked pairs

Approval Voting

We have already seen an approval ballot in which voters select all acceptable options.

Candidates are given 1 point each time they are selected on a ballot.

The winner is the candidate with the most points, that is, the candidate that is acceptable to the most voters.

Approval Voting Examples

Suppose we have 3 candidates (A,B,C). There are 8 different ballot types, but realistically there are only 6.

Results:

- 15: A,B
- 20: A,C
- 25: B,C
- 10: A
- 15: B
- 15: C

Final Tally: A(45), B(55), C(60). C wins!

Least Worst Defeat

- First candidates are compared using Pairwise Comparison.
- Each candidate's worst loss is recorded.
- From these losses, the candidate with the smallest margin in defeat is selected as the winner.

In essence, we are selecting the candidate who loses the best, or is the best of all bad options.

Note that a Condorcet Candidate always wins using LWD.

LWD example

Since Cookie Dough was a Condorcet Candidate (and never lost) it is the winner using LWD. Let's look at the other matchups. We will record each loser and by how much they lost.

- Chocolate v. Cookie Dough (C -2)
- Chocolate v. Mint Chocolate Chip (M -24)
- Chocolate v. Vanilla (V -18)
- Cookie Dough v. Mint Chocolate Chip (M -2)
- Cookie Dough v. Vanilla (V -12)
- Mint Chocolate Chip v. Vanilla (V -18)

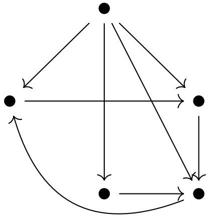
LWD example

- The worst defeat for Chocolate was -2.
- The worst defeat for Mint Chocolate Chip was -24.
- The worst defeat for Vanilla was -18.

Hence, Chocolate is second, Vanilla is third, and Mint Chocolate Chip is last.

Ranked Pairs

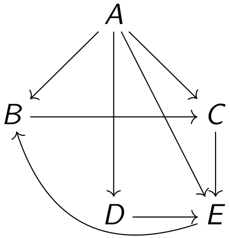
The method of **Ranked Pairs** uses a *directed graph* with vertices and arrows.



We will replace the vertices with the candidates.

Ranked Pairs

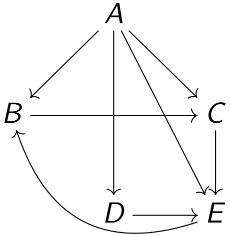
The vertex at the beginning of an arrow is called the *source* and the vertex at the end is called the *target*.



A is the source of four arrows but the target of no arrows. E is the target of three arrows and the source of one arrow.

Ranked Pairs

A *path* between two vertices is a collection of arrows that one may follow of successive sources and targets. A *cycle* is a path whose source and target are the same.



There is a path from A to E but no path from E to A. There is a cycle at vertex B.

Ranked Pairs

- First candidates are compared using Pairwise Comparison.
- We rank the various comparisons by margin of victory (largest to smallest).
- We begin by drawing an arrow from the winner to loser in the first victory.
- We continue *unless* a victory creates a cycle. In this case we skip the result.

The winner is the one candidate which is the source of all arrows connected to it.

Ranked Pairs Example

First we rank the outcomes of our ice cream election.

- Chocolate v. Mint Chocolate Chip (C $+24$)
- Chocolate v. Vanilla (C $+18$)
- Mint Chocolate Chip v. Vanilla (M $+18$)
- Cookie Dough v. Vanilla (CD $+12$)
- Chocolate v. Cookie Dough (CD $+ 2$)
- Cookie Dough v. Mint Chocolate Chip (CD $+ 2$)

Ranked Pairs Example

There are no cycles to worry about. Thus our ranked pairs graph is the following.

$$
\begin{tikzcd}
 & C \arrow[dr] \arrow[dd] & \\
CD \arrow[ur] \arrow[rr] \arrow[dr] & & V \\
 & M \arrow[ur] &
\end{tikzcd}
$$

Since Cookie Dough is the only vertex that is only a source (and never a target) it is the winner.

Next Time

We will discuss Arrow's Impossibility Theorem and practice with the different voting methods.

1.6 Arrow's Impossibility Theorem

Some announcements

- Homework #2: Text (pages 33-35) 51, 56-60, 61, 65, 71-75 (this is posted on Sakai)
- For Monday, read Chapter 2 (pages 36-57)

Today's Goals

- We will discuss various fairness criteria and how the different voting methods violate these criteria.

Last Time

We have so far discussed the following voting methods.

- Plurality Method (Majority Method)
- Borda Count Method
- Plurality-with-Elimination Method (IRV)
- Pairwise Comparison Method
- Method of Least Worst Defeat
- Ranked Pairs Method
- Approval Voting

Fairness Criteria

The following fairness criteria were developed by Kenneth Arrow, an economist in the 1940s. Economists are often interested in voting theories because of their impact on *game theory*.

The mathematician John Nash (the subject of A Beautiful Mind) won the Nobel Prize in economics for his contributions to game theory.

The Majority Criterion

A majority candidate should always be the winner.

Note that this does not say that a candidate must have a majority to win, only that such a candidate should not lose.

Plurality, IRV, Pairwise Comparison, LWD, and Ranked Pairs all satisfy the Majority Criterion.

The Majority Criterion

The Borda Count Method violates the Majority Criterion.

Here is the preference schedule given in the text.

Number of voters	6	2	3
1st	A	B	C
2nd	B	C	D
3rd	C	D	B
4th	D	A	A

Even though A had a majority of votes it only has 29 points. B is the winner with 32 points.

Moral: Borda Count punishes polarizing candidates.

The Condorcet Criterion

A Condorcet candidate should always be the winner.

Recall that a Condorcet Candidate is one that beats all other candidates in a head-to-head (pairwise) comparison.

The Condorcet Criterion

A voting method which satisfies the Condorcet Criterion is called a **Condorcet Method**.

Pairwise Comparison, LWD, and Ranked Pairs are all Condorcet Methods.

The Condorcet Criterion

The Plurality Method violates the Condorcet Criterion.
Here is the preference schedule given in the text.

Number of voters	49	48	3
1st	R	H	F
2nd	H	S	H
3rd	F	O	S
4th	O	F	O
5th	S	R	R

R wins by the Plurality Method but loses in a head-to-head with H.

IRV and Borda Count also violate the Condorcet Criterion.

The Monotonicity Criterion

If candidate X is the winner, then X would still be the winner had a voter ranked X higher in his preference ballot.

Plurality, Borda Count, Pairwise Comparison, LWD, and Ranked Pairs all satisfy the Monotonicity Criterion.

The Monotonicity Criterion

IRV violates the Monotonicity Criterion.

Here is the preference schedule given in the text.

Number of voters	7	8	10	2
1st	A	B	C	A
2nd	B	C	A	C
3rd	C	A	B	B

B has the fewest first place votes and is therefore eliminated. The result is that C wins.

Look at what happens if the 2 people in the last column actually rank C higher.

The Monotonicity Criterion

IRV violates the Monotonicity Criterion.

Here is the preference schedule given in the text.

Number of voters	7	8	10	2
1st	A	B	C	**C**
2nd	B	C	A	**A**
3rd	C	A	B	B

Now A has the fewest first place votes and so it is eliminated. The result is that B wins.

Moral: IRV is vulnerable to *insincere voting*.

The Independence-of-Irrelevant-Alternatives (IIA) Criterion

If candidate X is the winner, then X would still be the winner had one or more of the irrelevant alternatives not been in the race.

Plurality, Borda Count, IRV, Pairwise Comparison, and Ranked Pairs all violate the IIA Criterion.

The IIA Criterion

Pairwise comparison violates the IIA Criterion.

Here is the preference schedule given in the text.

Number of voters	2	6	4	1	1	4	4
1st	A	B	B	C	C	D	E
2nd	D	A	A	B	D	A	C
3rd	C	C	D	A	A	E	D
4th	B	D	E	D	B	C	B
5th	E	E	C	E	E	B	A

A has 3 wins and 1 loss. A wins with 3 points.

Now suppose C is removed (C is an irrelevant alternative).

The IIA Criterion

Pairwise comparison violates the IIA Criterion.

Here is the preference schedule given in the text.

Number of voters	2	6	4	1	1	4	4
1st	A	B	B	B	D	D	E
2nd	D	A	A	A	A	A	D
3rd	B	D	D	D	B	E	B
4th	E	E	E	E	E	B	A

Now B has 2 wins and 1 tie but A only has 2 wins. B wins.

Arrow's Impossibility Theorem

Theorem

It is mathematically impossible for a voting method to satisfy all four of these fairness criteria.

2.1 - Introduction to Weighted Voting

Some announcements

- Homework #2: Text (pages 33-35) 51, 56-60, 61, 65, 71-75 (this is posted on Sakai)
- Exam 1: schedule for Wednesday, Feb. 10. Should we move to Monday, Feb. 15?
- Keep working on projects. Be sure to review guidelines. You're welcome to show me your ballots ahead of time.

Today's Goals

- We're going to play a game!
- This game will introduce us to weighted voting and the different potential outcomes.

Weighted Voting Game

You are a player in a voting game. There are six rounds to this game.

For each round, your vote has been assigned a weight.

There is a quota for each round. Your goal is to form a coalition of other players so that your total weight is more than the quota.

Once coalitions have formed, we will take a vote to see who gets the points for that round.

Weighted Voting Game

However, the winnings from each round will be divided amongst the coalition equally.

Therefore, it is in your best interest to form the smallest possible coalition that reaches the quota.

Terminology

- Weighted Voting System
- Motions (yes/no)
- Players $(P_1, P_2, \ldots, P_N)$
- Weights $(w_1, \ldots, w_N)$, total weight $(V = w_1 + \cdots + w_N)$
- Quota q
- Coalition

Our notation for a system will be $[q : w_1, \ldots, w_N]$.

Types of systems

Anarchy: Multiple coalitions reach the quota so we have the possibility that both yes and no can pass.

Gridlock: The quota is greater than the total weight. Hence, no motion can ever pass.

We would prefer to avoid anarchy and gridlock. Hence, we will usually only consider systems satisfying

$$V/2 < q \leq V.$$

That is, the quota lies somewhere between the total number of votes and a simple majority.

Types of systems

One person-one vote: When $q = V$, any motion requires the unanimous support of all players. Thus, the system is equivalent to $[N : 1, 1, \ldots, 1]$.

Irrelevant player: This occurs when one player's vote cannot impact whether a motion passes.

Types of systems

Dictator: If $P_1 \geq q$, then P_1 is a dictator and the motion passes or fails depending on P_1's decision. Every other player is irrelevant.

Veto Power: A player has veto power if its weight w satisfies

$$\begin{aligned} w < q & \ \textbf{not a dictator} \\ V - w < q & \ \textbf{remaining votes are less than the quota} \end{aligned}$$

That is, the player cannot force a motion to pass, but no motion can pass without the player's support.

2.2, 2.3 - Power

Some announcements

- Homework #2: Text (pages 33-35) 51, 56-60, 61, 65, 71-75 (this is posted on Sakai)
- Exam 1: Moved to Monday, Feb. 15.
- Keep working on projects. Be sure to review guidelines. You're welcome to show me your ballots ahead of time.

Today's Goals

- To discuss different types of defining power in a weighted voting system.

Terminology

Recall our terminology.

- Weighted Voting System
- Motions (yes/no)
- Players $(P_1, P_2, \ldots, P_N)$
- Weights $(w_1, \ldots, w_N)$, total weight $(V = w_1 + \cdots + w_N)$
- Quota q
- Coalition

Our notation for a system will be $[q : w_1, \ldots, w_N]$.

Power

What is power?

Power is not just the weight that a certain player's vote has or the total weight of a coalition.

It is how much influence a player or coalition can exert compared to its total weight.

Types of coalitions

Any group of players who join forces and vote the same way is a **coalition**.

We write the coalition consisting of the first three players as $\{P_1, P_2, P_3\}$. Note that this is equivalent to any *permuation* of this set.

The coalition consisting of all players is called a **grand coalition**.
Winning/losing coaltion: A coalition which has enough votes to always win/lose.

Critical players

A **critical player** in a coalition is one who is necessary in order for that coalition to win.

A player P with weight w is critical in a coalition with total weight W if $W - w < q$.

The number of winning coalitions in which a player is critical is referred to as that player's **critical count**

Critical count

Consider the weighted voting system $[6 : 4, 3, 2, 1]$ and label the players A, B, C, D.

What are the possible winning coalitions?

Who are the critical players?

What is each player's critical count?

Banzhaf Power Index

The Banzhaf Power Index is a measure of the size of a player's power.

Let $P_1, \ldots, P_N$ be the players in a weighted voting system with critical counts $B_1, \ldots, B_N$. Set $T = B_1 + \cdots + B_N$ (the total critical count).

A player's Banzhaf Power Index (BPI) is

$$\beta_i = \frac{B_i}{T}.$$

Banzhaf Power Index example

Consider the weighted voting system $[6 : 4, 3, 2, 1]$ again.

What is each player's BPI?

What is the sum of all the BPI's?

Will this always be the sum?

Banzhaf Power Index

What is the BPI when one player is a dictator?

What is the BPI when one player has veto power?

Shapley-Shubik Power

In BPI, order did not matter. Coalitions were formed before votes were taken.

Shapley-Shubik power considers sequential voting.

We denote a sequential coalition by $\langle P_1, P_2, P_3 \rangle$ to mean P_1 casts a vote, then P_2, then finally P_3. A sequential coalition always consists of all players.

When listing a sequential coalition, the player that first votes to meet the quota is a **pivotal player**. Each sequential coalition has one and only one pivotal player. The number of coalitions in which a player is pivotal is called that player's **pivotal count**.

Shapley-Shubik Power

Consider the weighted voting system $[4 : 3, 2, 1]$.

List all sequential coalitions.

Determine each player's pivotal count. (This is recorded as SS_i for the ith player's pivotal count.)

Shapley-Shubik Power

The **Shapley-Shubik Power Index (SSPI)** of a player is denoted

$$\sigma_i = \frac{SS_i}{N!}$$

Determine the SSPIs for the players in the system $[4 : 3, 2, 1]$.

2.4 - Subsets and Permutations

Some announcements

- Homework #3: Text (59-66): 2, 4, 5, 7, 10, 12, 14, 20, 25, 28, 29, 34, 36, 58, 63, 73 (this is posted on Sakai)
- Exam 1: next Monday (Feb. 15).
- Keep working on projects. Be sure to review guidelines. You're welcome to show me your ballots ahead of time.

Today's Goals

- Homework questions?
- Discuss classwork activity from Friday.
- Discuss subsets and permutations. Questions arising from our discussions on coalitions. (This material will appear again in our study of symmetry - Unit 3.)

Coalitions

In general, a coalition is any collection of voters in a system.

Different measures of power use different types of coalitions.

BPI counts critical players in winning coalitions (coalitions that meet or exceed the quota). In these coalitions, order *does not* matter.

SSPI counts pivotal players in sequential coalitions. A sequential coalition always consists of all players but order *does* matter.

Sets

A **set** is a collection of objects (called elements). A coalition of players in a weighted voting system is an example of a set.

A set may have no elements, finitely many elements, or infinitely many elements. A set with no elements it is called the **empty set**.

A **subset** is any combination of elements from a set. The empty set is a subset of any set.

In sets written with the notation $\{,\}$, order does not matter. The notation $\langle,\rangle$ means order does matter.

Subsets

Consider a set with one element: $\{A\}$. How many subsets are there of this set?

There are two: the empty set $\{\}$, and the set itself $\{A\}$.

Subsets

Consider a set with two elements: $\{A, B\}$. How many subsets are there of this set? We want to reduce this problem to the previous one.

Say we remove B from the set. The sets that *do not* contain B are the same as the subsets of $\{A\}$. These are $\{\}$ and $\{A\}$.

Next we find the subsets that *do* contain B: $\{B\}$ and $\{A, B\}$.

Thus, there are a total of 4 subsets of $\{A, B\}$:

$$\{\}, \{A\}, \{B\}, \{A, B\}$$

Subsets

Consider a set with three elements: $\{A, B, C\}$. How many subsets are there of this set?

The sets that *do not* contain C are the same as the subsets of $\{A, B\}$:

$$\{\}, \{A\}, \{B\}, \{A, B\}$$

The subsets that *do* contain C:

$$\{C\}, \{A, C\}, \{B, C\}, \{A, B, C\}$$

Thus, there are a total of 8 subsets of $\{A, B, C\}$.

Subsets

Let's collect what we've done so far.

There is **1** set with zero elements.

There are **2** subsets of a set with one element.

There are **4** subsets of a set with two elements.

There are **8** subsets of a set with three elments.

How many subsets of a set with four elements? N elements?

Subsets

A set with N elements has 2^N subsets.

The reasoning behind this statement comes from a mathematical idea called *induction*.

Show the statement holds for the first case, then show that one can *step up* to the next case.

Coalitions

A weighted voting system with N players has $2^N - 1$ coalitions.

We remove the empty set because it is not a coalition.

If a system does not have a dictator, then we can remove all subsets consisting of 1 player. Thus, there are $2^N - N - 1$ coalitions of 2 or more players.

Subsets

We had a different formula before.

The number of subsets of k elements in a set of N elements

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}.$$

In fact, what we've shown is that

$$\binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{N-1} + \binom{N}{N} = 2^N.$$

Subsets

Wait a minute! Does that really work?

Yes! One way to see it is with Pascal's Triangle.

$$
\begin{array}{lccccccccc l}
N=0 & & & & & 1 & & & & & \text{Total} = 1 = 2^0 \\
N=1 & & & & 1 & & 1 & & & & \text{Total} = 2 = 2^1 \\
N=2 & & & 1 & & 2 & & 1 & & & \text{Total} = 4 = 2^2 \\
N=3 & & 1 & & 3 & & 3 & & 1 & & \text{Total} = 8 = 2^3 \\
N=4 & 1 & & 4 & & 6 & & 4 & & 1 & \text{Total} = 16 = 2^4 \\
 & \| & & \| & & \| & & \| & & \| & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} &
\end{array}
$$

Permutations

Here is a fact we've discussed previously.

There are $N!$ sequential coalitions of N players.

Mathematically one sees this in the study of *permutations*.

Permutations

A **permutation** of a set of objects is an ordered list of the objects. Different permutations correspond to different orders.

Given N objects, there are N choices for the first spot, $N-1$ choices for the second spot, $N-2$ choices for the second spot, etc. Thus, the total number of choices is

$$N \times (N-1) \times (N-2) \times \cdots \times 2 \times 1 = N!.$$

The Electoral College

Some announcements

- Homework #3: Text (59-66): 2, 4, 5, 7, 10, 12, 14, 20, 25, 28, 29, 34, 36, 58, 63, 73 (this is posted on Sakai)
- Exam 1: next Monday (Feb. 15).
- Keep working on projects. Be sure to review guidelines. You're welcome to show me your ballots ahead of time.
- Remember study sessions with Katie Greene (TA). Tuesday and Wednesday 7pm-9pm in Kirby 120.

Today's Goals

- Homework questions?
- The Electoral College as a weighted voting system.
- Prime numbers.

How we pick a president

On Tuesday, November 8, 2016, we will all go to the polls to elect a new president.

But, as you probably remember from civics class in high school, the process for picking a president is a bit convoluted.

How we pick a president

Instead of a majority or plurality method, the US uses a version of a weighted voting system called the Electoral College.

Essentially, each state is assigned a certain number of electoral votes. In general, the plurality winner of each state wins those electoral votes. Two states (Maine and Nebraska) assign electoral votes proportionally.

The number of electoral votes a state receives is determined by the number of representatives from that state plus 2 additional, with DC receiving 3.

Electoral College

How we pick a president

There are 538 total electoral votes.

The winner of the Electoral College is decided by the Majority Method. That is, a candidate must have 270 electoral votes to win.

Ties in the electoral college are decided by congress. The House of Representatives chooses the president, with each state's delegation casting one vote. The Senate chooses the vice president.

How we pick a president

Why do we use the Electoral College?

The original plan was for congress to elect the president. Others wanted states to pick and some wanted popular vote. The Electoral College (not called that at the time) was a compromise of sorts that maintained some power for the states.

The Electoral College

The Electoral College is a (very large) weighted voting system.

There are $2^{51} - 1$ coalitions and $51!$ sequential coalitions.

These are BIG numbers. Which one is bigger?

The Electoral College

To compute BPI and SSPI in the Electoral College, one needs mathematical software. Thankfully, someone already did it for us.

One caveat here. We are ignoring the fact that Maine and Nebraska distribute their votes proportionally.

The Electoral College

Some key observations:

- The SSPI and BPI are almost identical for each state. Why?
- Some states have significantly more power than others according to SSPI and BPI. Does that mean that the voters in those states have more power than voters in states with less power?

Voter Power

How do we compute the power of a given voter?

The Banzhaf power index of an individual voter in a state with population P is roughly proportional to $\frac{1}{\sqrt{P}}$ times the Banzhaf power index of the state.

Voter Power

The population of North Carolina is approximately 10 million and its BPI is 2.7%.

The BPI of an individual North Carolina voter is

$$\frac{1}{\sqrt{10,000,000}} \cdot 2.7 \approx .085\%$$

How does this compare to a state like California (population approximately 39 million)?

And now for something completely different...

We have recently discussed numbers of the form $2^N - 1$.

It turns out that certain numbers of this form are very special.

Prime numbers

A prime number is a positive integer (whole number) greater than 1 whose only integer divisors are 1 and itself.

The first few prime numbers are

$$2, 3, 5, 7, 11, 13, 17, \cdots$$

A famous theorem of Euclid proves that there are infinitely many prime numbers.

Prime numbers

Factoring is hard! In fact, it's so hard that the security of our credit cards depends on it.
No one will ever find the biggest prime number (because of Euclid's theorem) but some mathematicians are interested in how big of a prime one can find.

Mersenne Prime

A **Mersenne prime** is a prime number of the form $2^N - 1$.

Not every number of this form is prime, but many are so it's a good place to look.

Just last month, a new Mersenne Prime was found:

$$2^{74{,}207{,}281} - 1.$$

This number has $22,338,618$ digits. It is HUGE.

Probability

Some announcements

- Projects due today. Please turn in now.
- For Monday, read "What to expect when you're expecting to win the lottery", by Jordan Ellenberg (in Course Reserves on Sakai). We will discuss.
- Remember study sessions with Katie Greene (TA). Tuesday and Wednesday 7pm-9pm in Kirby 120.

Today's Goals

- Discuss terminology for our unit on Probability.
- In particular, we will study how mathematicians measure risk.
- We will look at several examples of probability experiments and games where one can gain an advantage by knowing the odds ahead of time.

Terminology

- **Random Experiment**: An activity or process whose outcome cannot be predicted ahead of time.
- **Sample Space**: The set of all possible outcomes of the experiment. (We will use S to represent this set and N to denote the number of possible outcomes.)

In all of our examples, we will assume N is a finite number, though it is certainly possible to consider examples where N is infinite.

Tossing a coin

Suppose we toss a coin. There are two possible outcomes: heads (H) or tails (T). The sample space for this experiment is

$$S = \{H, T\} \text{ and } N = 2.$$

Now suppose we toss a coin twice. How many possible outcomes are there?

$$S = \{HH, HT, TH, TT\} \text{ and } N = 4.$$

Suppose we toss a coin k times. What is the size of the sample space?

Probability

We will be interested in determining the *probability* or likelihood of a certain *event*.

The following questions don't need exact answers yet, but intuitively we might have an idea of the answers.

- If we toss a coin once, what is the probability that it lands on heads?
- If we toss a coin twice, what is the probability that it lands on heads at least once?

Rolling dice

This is like the coin toss example but with more options.Suppose we roll a six-sided die. The sample space looks like $S = \{1, 2, 3, 4, 5, 6\}$.

Now suppose we roll two six-sided die. We could represent the sample space in pairs: $S = \{(1, 1), (1, 2), (1, 3), \cdots\}$. But there are a lot of options! (How many?)

Instead, we might say we are interested in the *sum* of the two numbers, in which case our sample space will be $S = \{1, 2, 3, \ldots, 12\}$.

Now the questions get harder. What is the probability of rolling 12? What about 11?

Blackjack

The rules: You are dealt two cards and the dealer is dealt two cards, one face up. Cards 2-7 are worth their number, face cards are worth 10, an ace is worth 1 or 11.

You have the option to **stand** (keep the cards you are dealt) or **hit** (get another card). (There are other possibilities such as doubling down and splitting.)

Basic blackjack strategy involves knowing what events are possible if you were to hit. When playing at a casino, using basic strategy gives you are 49% chance of winning *in the long run*. When card counting is used to also take into account the cards left in the deck, your chance increases to about 52% *in the long run.*

Events

- An **event** is a subset of the sample space. (Recall: How many subsets are there of a set of size N, including the empty set?)
- The empty set $\{\ \}$ is called the **impossible event**.
- The set itself is called the **certain event**.
- The events containing a single outcome (sets of size 1) are called **simple events**.

What are all of the events from the experiment when we flipped a coin twice?

Probability

Now that we have the terminology, we can talk about probability a bit more concretely.

A **probability assignment** is a function that assigns to each event E a number between 0 and 1. We denote this $\Pr(E)$.

The impossible event should always be assigned a probability of 0 ($\Pr(\{\ \}) = 0$) and the certain event should always be assigned a probability of 1 ($\Pr(S) = 1$).

Since our sample spaces are finite, probability assignments are determined by assigning a probability to just the simple events.

The probability of an event is then obtained by adding the probabilities of the individual outcomes that make up that event.

A valid probability assignment satisfies two requirements:

- All probabilities are numbers between 0 and 1.
- The sum of the probabilities of the simple events is 1.

Probability

Suppose we have an experiment (like tossing a coin) where each *simple event* has the same probability.

Then each simple event occurs with probability $1/N$ where N is the size of the sample space.

This means that the probability of an event E of size k occurs with probability k/N (so $\Pr(E) = k/N$).

Such a sample space is called **equiprobable**.

Probability

Suppose we roll two six-sided dice. This is an equiprobable space *if* we consider the sample space as pairs of numbers 1-6.

In this case, the probability that we roll a 2 on the first die and a 3 on the second die is $1/36$.

However, if we write our sample space using sums then this is no longer equiprobable. That is, we are *more likely* to roll a 5 than a 4. Why is this?

Probability

What is the probability of rolling a sum of 5? How do we compute it?

In order to figure this out we should consider the number of ways that a 5 could occur:

$$(1,4),(2,3),(3,2),(4,1).$$

Hence, the probability of rolling a 5 is $4/36 = 1/9$.

Shooting Free Throws

This is Shaquille O'Neal, or Shaq.

Shaq was a 15-time NBA all-star and notoriously bad free throw shooter. He was so bad that other teams would employ the hack-a-shaq strategy to force him to shoot free throws.

During his career, he made approximately 52% of his free throws.

Shooting Free Throws

Suppose Shaq is fouled (hack-a-shaq!) and gets a free throw. The different outcomes are a success (s) or a failure (f).

What is the solution space? $S = \{s, f\}$.

This is not an equiprobable space. What are the probabilities of the events $\{s\}$ and $\{f\}$? $\Pr(\{s\}) = .52$ and $\Pr(\{f\}) = 1 - .52 = .48$.

In this case, the event $\{f\}$ is the **complement** of the event $\{s\}$ because $\Pr(\{f\}) = 1 - \Pr(\{s\})$. Said another way, they are complements because the sum of their probabilities is 1.

Shooting Free Throws

Was hack-a-shaq a good strategy? On first glance it's not because he still made free throws more often than he missed. But what about near the end of the game when a player might get 2 free throws.

Suppose Shaq is fouled and gets 2 free throws. What is the solution space? $S = \{ss, sf, fs, ff\}$.

What is the probability he makes both?

Shooting Free Throws

We will ignore any psychological pressure associated with this problem and assume that each free throw is independent.

The **multiplication principle for independent events** says that if E and F are independent (one event occurring does not effect the probability of the other) then

$$\Pr(E \text{ and } F) = \Pr(E) \times \Pr(F).$$

This means that the probability of Shaq making *both* free throws is $(.52) \times (.52) \approx .27$.

Shooting Free Throws

Let's continue with this and compute the other probabilities in the solution space for two free throws.

- $\Pr(\{sf\}) = (.52) \times (.48) \approx .25$
- $\Pr(\{fs\}) = (.48) \times (.52) \approx .25$
- $\Pr(\{ff\}) = (.48) \times (.48) \approx .23$

What is the probability that Shaq makes *at least* one free throw? Here, the even is $E = \{ss, sf, fs\}$, so we could get the probability by addition,

$$\Pr(E) = \Pr(\{sf\}) + \Pr(\{fs\}) + \Pr(\{ff\}) \approx .77.$$

Or we could recognize that this is the complement of the event $F = \{ff\}$ (he makes no free throws). In which case

$$\Pr(E) = 1 - \Pr(F) \approx .77$$

Shooting Free Throws

Here's a problem to think about: Say Shaq gets a 1-and-1, that is, he gets to shoot a free throw and if he makes it he can shoot another one.

What is the solution space? Compute the probabilities for the different events. How is this problem different from the last one?

Acknowledgement: This isn't really a thing in the NBA but it's still an interesting problem.

Shooting Free Throws

Another problem to think about, but harder: Shaq's average of making a shot without being fouled was about 58% in his career. Assuming such a shot was worth 2 points, was hack-a-shaq a good strategy in the long run?

What are some factors not taken into account here? How could we make this analysis more precise?

Permutations, Combinations, and Expected Value

Some announcements

- Homework 4: Chapter 16 (1, 9, 11, 17, 19, 22, 30, 33, 35, 39, 57, 61, 64).
- Homework is due (officially) on Monday, March 7th. You may turn it in by Thursday, March 3 at 4pm for 5 bonus points.
- Remember study sessions with Katie Greene (TA). Tuesday and Wednesday 7pm-9pm in Kirby 120.

Today's Goals

- Review our terminology for probability.
- Discuss/review counting ordered and unordered subsets.
- Define (formally) and discuss expected value.
- Compute several examples of expected value in sports, gambling, and investing.
- Discuss guidelines for probability projects!

Terminology

- **Random Experiment**: An activity or process whose outcome cannot be predicted ahead of time.
- **Sample Space**: The set of all possible outcomes of the experiment. (We will use S to represent this set and N to denote the number of possible outcomes.)

In all of our examples, we will assume N is a finite number, though it is certainly possible to consider examples where N is infinite.

Events

- An **event** is a subset of the sample space. (Recall: How many subsets are there of a set of size N, including the empty set?)
- The empty set $\{\ \}$ is called the **impossible event**.
- The set itself is called the **certain event**.
- The events containing a single outcome (sets of size 1) are called **simple events**.

The Multiplication Rule

In a given experiment, how many events do we expect? This depends somewhat on *how* you want to count.

Multiplication Rule: If there are m different ways to do X and n different ways to do Y, then X and Y together (in that order) can be done in $m \times n$ different ways.

Where have we seen this before?

Example: Frozen Custard

This is Kopps.

Kopps does not serve ice cream.

It serves **frozen custard** (and jumbo burgers).

Example: Frozen Custard

Every day at Kopps you can get chocolate or vanilla custard, plus there are two *flavor of the day* options. Tomorrow the options are Mint Chip and Drumstick.

You can choose between a cup, a cake cone, or a waffle cone.

Example: Frozen Custard

How many combinations are there of custard and container?

$$(\# \text{ of custard options }) \times (\# \text{ of container options }) = 4 \times 3 = 12.$$

Example: Frozen Custard

Suppose you want a double with two different flavors. How many flavor combinations are there?

There are 4 options for the first choice and 3 options for the second choice. So $4 \times 3 = 12$ options, right?

Chocolate and Vanilla is the same as Vanilla and Chocolate, so we have double counted. This means that there are really $12/2 = 6$ choices.

Unless order matters to you. Maybe you really want Mint Chip with Drumstick on top. Then there really are 12 choices.

Permutations and Combinations

Permutation: An ordered selection of r objects chosen from a set of n objects.

$$_nP_r = \frac{n!}{(n-r)!}$$

Combination: An unordered selection of r objects chosen from a set of n objects.

$$_nC_r = \frac{n!}{r!(n-r)!}$$

Previously, our notation for this was $\binom{n}{r}$.

Probability

A **probability assignment** is a function that assigns to each event E a number between 0 and 1. We denote this $\Pr(E)$.

A valid probability assignment satisfies two requirements:

- All probabilities are numbers between 0 and 1.
- The sum of the probabilities of the simple events is 1.

The impossible event should always be assigned a probability of 0 ($\Pr(\{\ \}) = 0$) and the certain event should always be assigned a probability of 1 ($\Pr(S) = 1$).

Weighted average

Let X be a variable that takes the values $x_1, x_2, \ldots, x_N$, and let $w_1, w_2, \ldots, w_N$ denote the respective weights for these values, with $w_1 + w_2 + \cdots + w_N = 1$.

The **weighted average** for X is given by

$$x_1 \cdot w_1 + \cdots + x_N \cdot w_N.$$

Example: Grades

This is Tom.

Tom spends too much time on Myspace and not enough time studying for Math 107.

Example: Grades

Here are Tom's grades:

- Classwork/Participation: 0 %
- Homework: 10 %
- Exams: 60 %
- Projects: 90 %
- Final: 70 %

In Math 107, Classwork is 10%, Homework is 15 %, Exams, Projects, and the Final are 25 % each. Thus, Tom's weighted average is,

$$(0 \times .1) + (10 \times .15) + (60 \times .25) + (90 \times .25) + (70 \times .25) = 56.5$$

Don't be like Tom.

Expected value

Expected value is just the probabilistic version of a weighted average.
Suppose that X is a variable that takes on the numerical values (outcomes) $x_1, x_2, \ldots, x_N$ with probabilities $p_1, p_2, \ldots, p_N$, respectively.
The **expected value** (or expectation) for X is given by

$$E = x_1 \cdot p_1 + \cdots + x_N \cdot p_N.$$

That is, it is the sum of each outcome times its probability.

Example: Going for 2

When a team in the NFL scores a touchdown, that team has two options: kick an extra point or go for a two-point conversion. New rules this past season made extra points less automatic.

During the 2015 season, the Jacksonville Jaguars made 82.1% of their extra point kicks. Thus, the expected value of an extra point is

$$(1 \times .821) + (0 \times .179) = .821$$

Jacksonville made only 20% of their two-point conversions. Thus, the expected value of a two-point conversion attempt is

$$(2 \times .2) + (0 \times .8) = .4$$

This means that it is better, on average, for Jacksonville to always kick an extra point.

Example: Going for 2

On the other hand, during the 2015 season, the Pittsburgh Steelers made 94.1% of their extra point kicks. Thus, the expected value of an extra point is

$$(1 \times .941) + (0 \times .059) = .941$$

Pittsburgh made 72.7% of their two-point conversions. Thus, the expected value of a two-point conversion attempt is

$$(2 \times .727) + (0 \times .273) = 1.454$$

This means that, on average, it is better for Pittsburgh to go for 2 on every attempt.

Birthday Problem

Some announcements

- Homework 4: Chapter 16 (1, 9, 11, 17, 19, 22, 30, 33, 35, 39, 57, 61, 64).
- Homework is due (officially) on Monday, March 7th. You may turn it in by Thursday, March 3 at 4pm for 5 bonus points.
- Homework 5: Chapter 16 (65, 66, 69, 71, 77). Read Chapter 17.
- Remember study sessions with Katie Greene (TA). Tuesday and Wednesday 7pm-9pm in Kirby 120.

Today's Goals

- Discuss classwork from Monday.
- The Birthday Game (fun with probability).

50 Cent

This is 50 Cent.

He's going to party like it's your birthday.

The Birthday Game

What is the probability that two people in the class have the same birthday?

We will go by date, ignoring year and also ignoring leap day (sorry leap babies).

The Birthday Game

It turns out it is easier to compute the complementary event:
What is the probability that no two people have the same birthday?

Say we start with just two people. What is the probability that their birthdays are on different days?

$$P = \frac{365}{365} \times \frac{364}{365} \approx 99.73\%$$

The Birthday Game

What is the probability that three people have their birthdays, all on different days?

$$P = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \approx 99.18\%$$

That's still pretty likely. What if we jump to 10 people?

$$P = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{356}{365} \approx 88.31\%$$

The Birthday Game

21 people?

$$P = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{345}{365} \approx 55.63\%$$

We're getting dangerously close to 50%.

22 people?

$$P = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{345}{365} \approx 52.43\%$$

So if everyone is here today, there's about a 52% chance two people *do not* share a birthday, so about a 48% chance that two people do share a birthday.

The Birthday Game

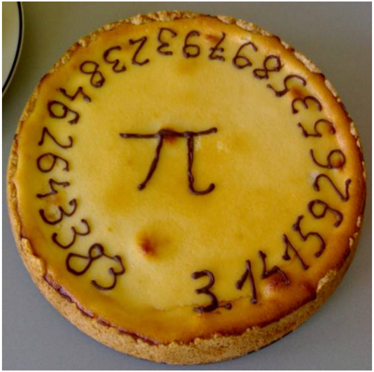
This game works best in a class of about 30 people.

$$P = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{336}{365} \approx 29.37\%$$

In this case, there's over a 70% chance that two people share a birthday.

Normal Curves

Happy Pi Day!



Some announcements

- Homework 5: Chapter 16 (65, 66, 69, 71, 77). Due this Friday (3/18)
- Homework 6: Chapter 17 (2, 4, 6, 8, 15, 21, 23). Due next Wednesday (3/23).
- Game announcements due next Monday (3/21) and Game Day will be Wednesday (3/23).
- Exam 2 will be March 28. Review on Monday (3/21).
- Math talk on Friday at 4pm. The speaker is Rob Won from UCSD.
- Additional test extra credit assignment posted on Sakai.
- I will post all of this on Sakai!

Today's Goals

- Normal curves!
- We will learn what a normal curve is and how to read information from them.
- Before this we need a basic review of statistical terms. I mean basic as in underlying, not easy.

Statistics vs Probability

Why do we study Statistics and Probability together? Here's one answer due to Persi Diaconis:

The problems considered by probability and statistics are inverse to each other. In probability theory we consider some underlying process which has some randomness or uncertainty modeled by random variables, and we figure out what happens. In statistics we observe something that has happened, and try to figure out what underlying process would explain those observations.

Statistics basics

Here's some terminology you should be familiar with (Chapter 15):

- **Mean/Average**: For a set of N numbers, $d_1, d_2, \ldots, d_N$, the mean is given by $\mu = (d_1 + d_2 + \cdots + d_N)/N$.
- **Median**: Sort the data set from smallest to largest: $d_1, d_2, \ldots, d_N$. The median is the *middle number*. If N is odd, the median is $d_{(N+1)/2}$. If N is even, the median is the average of $d_{N/2}$ and $d_{(N/2)+1}$.
- **Mode**: this is the *most common number(s)*. A data set can have more than one mode. (We won't really study mode. It was just feeling left out so I put it on the slide.)
- **Range**: this is difference between the highest and lowest values of the data ($R = Max - Min$).

Percentiles

The pth **percentiles** of a data set is a number X_p such that $p\%$ is smaller or equal to X_p and $(100 - p)\%$ of the data is bigger or equal to X_p.

To find the pth percentile of a *sorted* data set $d_1, d_2, \ldots, d_N$, first find the *locator* $L = (p/100)\,N$.

If L is a whole number, then $X_p = \frac{d_L + d_{L+1}}{2}$.

If L is not a whole number, then $X_p = d_{L^+}$ where L^+ is L rounded up.

Percentiles

This is Baby Evelyn.

Baby Evelyn is in the 40th percentile for height (40% of babies Evelyn's age weigh as much or less than she does while 60% weigh as much or more).

Quartiles

- The **first quartile** Q_1 is the 25th percentile of a data set.
- The **median** is the 50th percentile of a data set (also technically the *second quartile*).
- The **third quartile** Q_3 is the 75th percentile of a data set.
- The **fourth quartile** is d_N (the last number in the data set).

The **interquartile range (IQR)** is the difference between the third quartile and the first quartile ($IQR = Q_3 - Q_1$).

IQR tells us how spread out the middle 50% of the data values are.

Hold up!

Why aren't we doing any examples?

Because I'm not going to ask you to compute any of these things directly from a set of data. Instead, we will study visual representations of the data called *bell curves*.

But, I want you to be familiar with the terminology and how it's computed. So bear with me.

Standard deviation and variance

Standard deviation tells us how spread out a data set is *from the mean*.

Let A be the mean of a data set. For each value x in the data set, $x - A$ is the *deviation from the mean*. We want to average these values but for technical reasons we actually need to average their *squares*.

This average is called the **variance** V. The **standard deviation** is the square root of the variance, $\sigma = \sqrt{V}$.

Example

Scores (x)	Deviation (x-A)	(x-A)^2
40.00	-37.29	1390.22
41.00	-36.29	1316.65
48.00	-29.29	857.65
48.00	-29.29	857.65
70.00	-7.29	53.08
73.00	-4.29	18.37
73.00	-4.29	18.37
74.00	-3.29	10.80
77.00	-0.29	0.08
77.00	-0.29	0.08
82.00	4.71	22.22
85.00	7.71	59.51
85.00	7.71	59.51
88.00	10.71	114.80
90.00	12.71	161.65
90.00	12.71	161.65
94.00	16.71	279.37
95.00	17.71	313.80
96.00	18.71	350.22
98.00	20.71	429.08
99.00	21.71	471.51
77.29	0.00	330.78

The number 330.78 is the variance V, the average of squared devations.

The standard deviation is then

$$\sigma = \sqrt{V} \approx 18.19$$

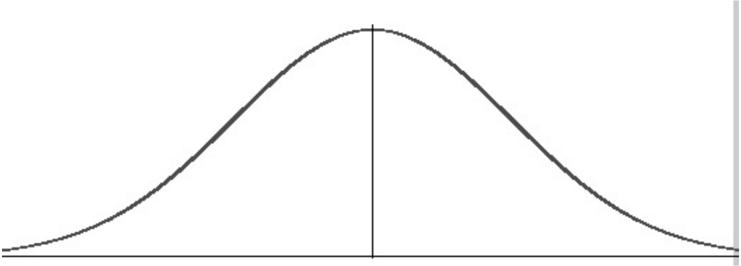
Normal curves

Say we flipped a coin 100 times? We *expect* to get heads 50 times and tails 50 times, but it's also very likely that we would not get this. (For a challenge, compute the probability of this event.)

When John Kerrich was a POW during World War II he wanted to test the probabilistic theory on coin flipping with a real life experiment. He flipped a coin 10,000 times and recorded the number of heads for each 100 trials.

What took Kerrich weeks (months?) we can do in a matter of seconds via computer software like Maple.

Bell curves



A set of data with **normal distribution** has a bar graph that is perfectly bell shaped.

Properties of normal curves

- **Symmetry**: Every normal curve has a vertical axis of symmetry.
- **Median and mean**: If a data set, then the median and mean are the same and they correspond to the point where the axis of symmetry intersects the horizontal axis.
- **Standard deviation**: The standard deviation is the horizontal distance between the mean and the **point of inflection**, where the graph changes the direction it is bending.

Normal Curves

Some announcements

- Homework 5: Chapter 16 (65, 66, 69, 71, 77). Due this Friday (3/18)
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- Additional test extra credit assignment posted on Sakai.
- All of this is on Sakai!

Today's Goals

- We will learn how to retrieve statistical data from normal curves.

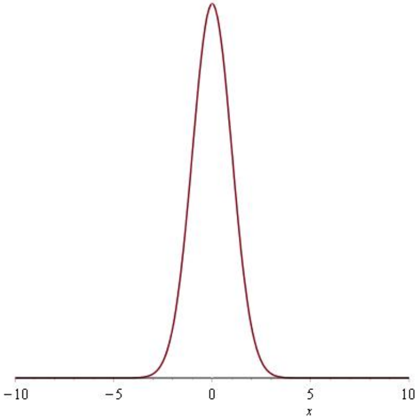
Terminology

Recall our terminology from last class:

- Mean, median, mode, and range
- Percentiles and quartiles
- Standard deviation and variance
- Normal curve (more on this today)

Normal distributions and curve

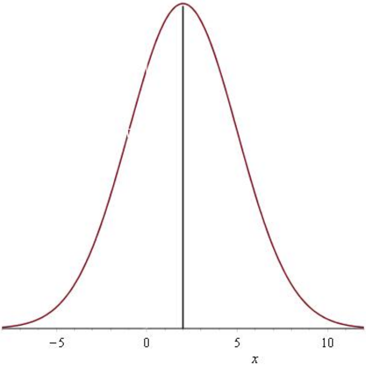
We say a distribution of data is **normal** if its bar graph is perfectly bell shaped.



This type of curve is called **normal**

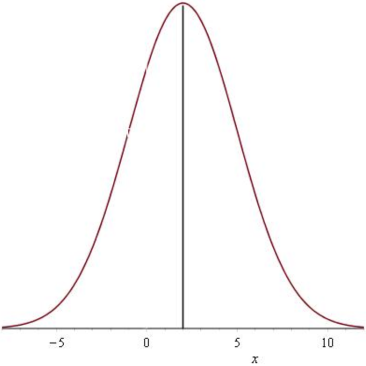
Properties of normal curves

Symmetry: Every normal curve has a vertical axis of symmetry.



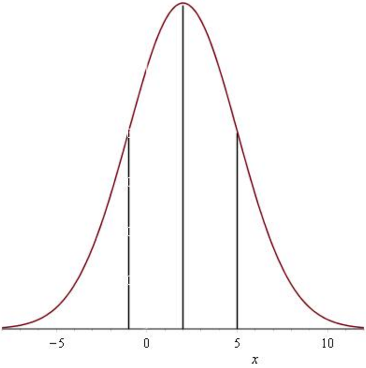
Properties of normal curves

Median and mean: If a data set is normal, then the median and mean are the same and they correspond to the point where the axis of symmetry intersects the horizontal axis.



Properties of normal curves

Standard deviation: The standard deviation is the horizontal distance between the mean and the **point of inflection**, where the graph changes the direction it is bending.



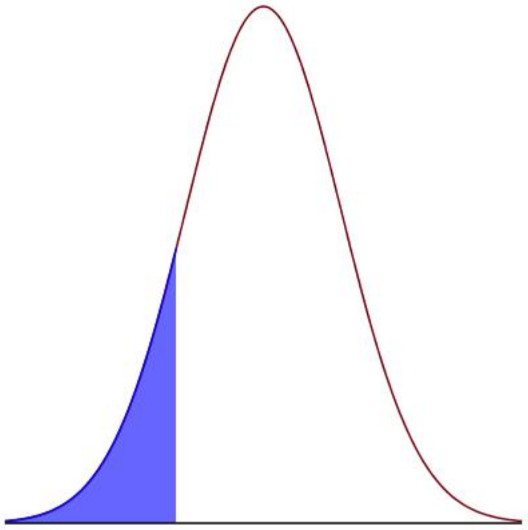
Properties of normal curves

Quartiles: The first and third quartiles can be found using the mean μ and the standard deviation σ.

$$Q_1 = \mu - (.675)\sigma \text{ and } Q_3 = \mu + (.675)\sigma.$$

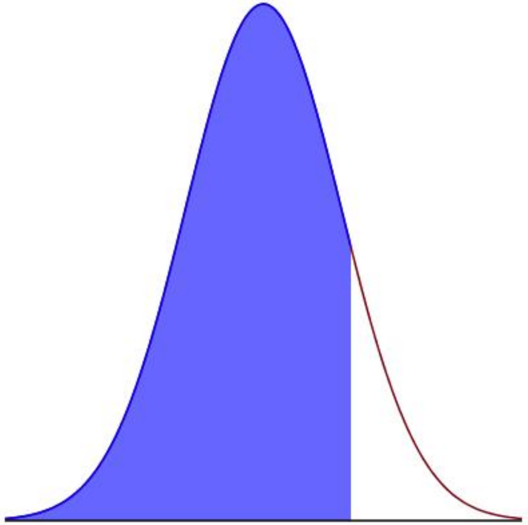
Properties of normal curves

$Q_1 = \mu - (.675)\sigma$



Properties of normal curves

$Q_3 = \mu + (.675)\sigma$



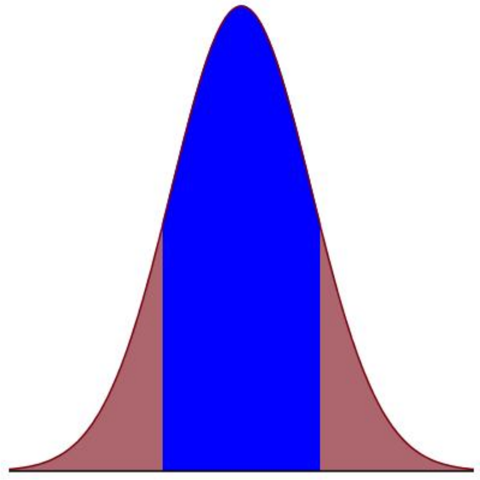
Properties of normal curves

The 68-95-99.7 Rule: In a normal data set,

- Approximately 68% of the data falls between one standard deviation of the mean ($\mu \pm \sigma$). This is the data between P_{16} and P_{84}.
- Approximately 95% of the data falls within two standard deviations of the mean ($\mu \pm 2\sigma$). This is the data between $P_{2.5}$ and $P_{97.5}$.
- Approximately 99.7% of the data falls within three standard deviations of the mean ($\mu \pm 3\sigma$). This is the data between $P_{0.15}$ and $P_{99.85}$.

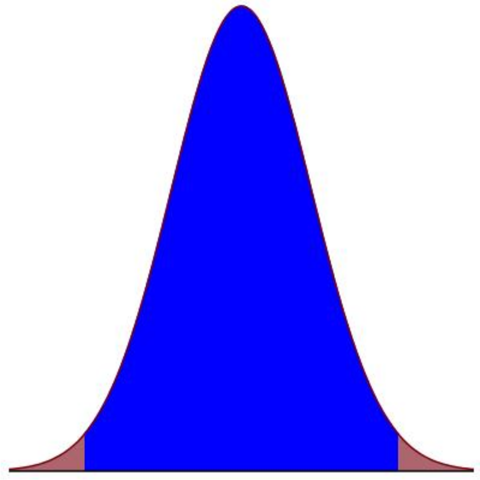
Properties of normal curves

68%



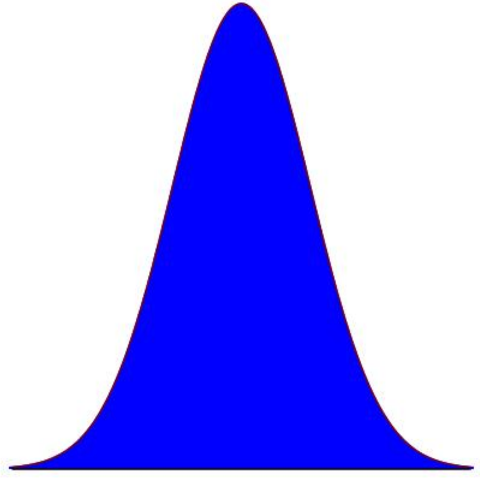
Properties of normal curves

95%



Properties of normal curves

99.7%



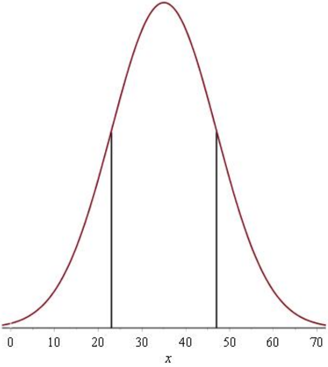
Example

Suppose we have a normal data set with mean $\mu = 500$ and standard deviation $\sigma = 150$. We have the following:

- $Q_1 = 500 - .675 \times 150 \approx 399$
- $Q_3 = 500 + .675 \times 150 \approx 601$
- Middle 68%: $P_{16} = 500 - 150 = 350$, $P_{84} = 500 + 150 = 650$.
- Middle 95%: $P_{2.5} = 500 - 2(150) = 200$, $P_{97.5} = 500 + 2(150) = 800$.
- Middle 99.7%: $P_{0.15} = 500 - 3(150) = 50$, $P_{99.85} = 500 + 3(150) = 850$.

Example

Consider a normal distribution represented by the normal curve with points of inflection at $x = 23$ and $x = 45$. Find the mean and standard deviation. Use them to compute Q_1, Q_3 and the middle 68%, 95%, and 99.7%.



Standardizing normal data

In essence, all normalized data sets are the same. They all have a mean μ and standard deviation σ. The same percentage of data is located in the same increments of σ from the mean. Thus, there is value in *standardizing normal data*.

Psychometry

This is Laura.

Laura is a *psychometrist*. She conducts psychological assessments.

Her patients are adults but their ages range from 18 and up. She uses z-values to standardize her patients' assessment scores.

Standardizing Rule

In a normal distribution with mean μ and standard deviation σ, the standardized value of a data point x is

$$z = \frac{x - \mu}{\sigma}.$$

The result of this is the **z-value** of the data point x.

Conversions

Suppose we have a normal data set with mean $\mu = 120$ and standard deviation $\sigma = 30$. If $x = 100$, then the z-value of x is

$$z = \frac{x - 120}{30} = -\frac{2}{3} \approx -.67.$$

Conversely, if a z-value of some x is .5, what is x (for the data above)?

Random Variables

Some announcements

- Homework 6: Chapter 17 (2, 4, 6, 8, 15, 21, 23). Due next Wednesday (3/23).
- Game announcements due next Monday (3/21) by 5pm and Game Day will be Wednesday (3/23).
- Exam 2 will be March 28. Review on Monday (3/21).
- Math talk on Friday at 4pm. The speaker is Rob Won from UCSD ($\mathbb{Z}$-graded algebras in noncommutative projective algebraic geometry).
- Additional test extra credit assignment posted on Sakai.
- All of this is on Sakai!

Today's Goals

- Discuss the intersection of statistics and probability.
- Practice practice practice.

Variables

In algebra, a variable typically is a placeholder for some type of solution or set of solutions.

Given the equation $x + 3 = 10$, then the variable x represents the number 7.

Given the equation $x^2 + 5 = 21$, then x represents a member of the set of solutions $\{-4, 4\}$.

Random variables

A variable representing a random (probabilistic) event is called a **random variable**.

For example, if we toss a coin 100 times and let X represent the number of times heads comes up, then X is a random variable.

Like an algebraic variable, X represents a number between 0 and 100, but the possible values for X are not equally likely.

The probability of $X = 0$ or $X = 100$ is $(1/2)^{100}$, which is a very small number.

The probability of $X = 50$ is about 8%.

Random variable

Continuing with the example, we know that X has an approximately normal distribution with mean $\mu = 50$ and standard deviation $\sigma = 5$ (for a sufficiently large number of repetitions).

What is the (approximate) probability that X will fall between 45 and 55? This is 1 standard deviation from the mean, so the probability is approximately 68%.

The Honest-Coin Principle

We can now generalize the previous example to a trial with n tosses.

Let X be a random variable representing the number of heads in n tosses of an honest (fair) coin (assume $n \geq 30$). Then X has an approximately normal distribution with mean $\mu = n/2$ and standard deviation $\sigma = \sqrt{n}/2$.

The Dishonest-Coin Principle

Let X be a random variable representing the number of heads in n tosses of a coin (assume $n \geq 30$), and let p denote the probability of heads on each toss of the coin. Then X has an approximately normal distribution with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$.

Note that when $p = \frac{1}{2}$ we recover the Honest-Coin Principle.

Margin of Error

In a poll conducted by Public Policy Polling before the recent Democratic primary in Missouri interviewed 839 likely voters.

Their poll found almost a tie between Hillary Clinton and Bernie Sanders.

Therefore, we can use the Honest-Coin Principle to compute the margin of error for the poll.

Margin of Error

According the Honest-Coin Principle, we have

$$\mu = \frac{839}{2} = 419.5 \text{ and } \sigma = \frac{\sqrt{839}}{2} = 14.48.$$

The standard deviation σ is approximately 1.72% of the sample.

This means that the pollsters could assume with 95% confidence that either candidate would get between $(50 \pm 2(1.72))\%$ of the vote. That is, between 46.55% and 53.45%.

The value 2σ is called the **margin of error**.

Margin of Error

On the other hand, in a poll conducted by Public Policy Polling before the recent Democratic primary in North Carolina interviewed 747 likely voters.

Their poll found Hillary Clinton with 60% support and Bernie Sanders with 40%.

Therefore, we can use the Dishonest-Coin Principle to compute the margin of error for the poll.

Margin of Error

According the Dishonest-Coin Principle, we have

$$\mu = 747 * .6 = 448.2 \text{ and } \sigma = \sqrt{747 * .6 * .4} = 13.39.$$

The standard deviation σ is approximately 1.79% of the sample.

This means that the pollsters could assume with 95% confidence that Clinton candidate would get between $(60 \pm 2(1.79))\%$ of the vote. That is, between 56.42% and 63.58%.

The margin of error in this example is $2\sigma = 3.58\%$.

Symmetry

Some announcements

- Read Chapter 11 in the text for next class.
- Game reflections deadline extended to Monday (4/4)
- No math talks this week. Talks on 4/6, 4/13, 4/19, and 4/20.

Unit 3: Symmetry

Our last unit is the least "numbers" based. We will study *symmetry*. We will define this rigorously later. For now we'll say that it is the study of shapes and their movement in space.

To some, mathematics is all about symmetry and everything is just a manifestation of some form of symmetry. We will not be so philosophical, but it's good to know that this philosophy is out there, and not all mathematicians think about numbers all day.

Unit 3: Symmetry

Our unit will be broken down into 3 pieces:

- Rigid motions and symmetry types (Chapter 11)
- Fibonacci numbers and the golden ratio (Chapter 13)
- Fractals Maple lab (Chapter 12) – Unit 3 "project"

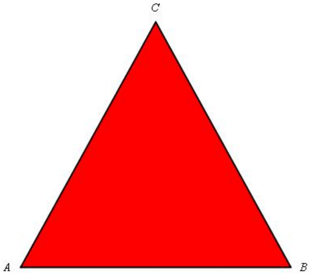
This will be our shortest unit. The exam is on April 20. The Maple labs will be on April 22 and April 25. (No class on 4/8 and 4/18).

Today's Goals

- Define a rigid motion.
- See examples of basic rigid motions: reflections, rotations, translations, and glide reflections.
- Define symmetry.

Triangle

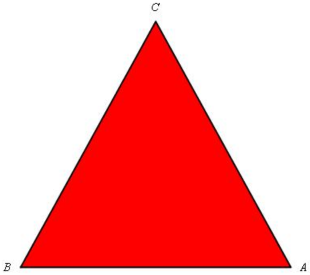
Here is an equilateral triangle. Remember that by *equilateral* we mean all sides are equal (and all angles).



Roughly, the question we will ask is: in what ways can we move the triangle so that it appears the same? The vertices can change, but the object should not appear any different.

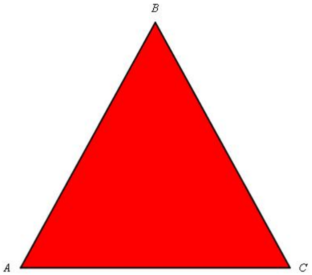
Triangle

For example, we could flip the triangle over a line going vertically through vertex C.



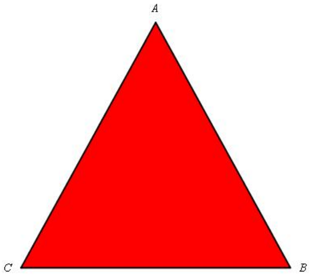
Triangle

We could do the same for a line through vertex A.



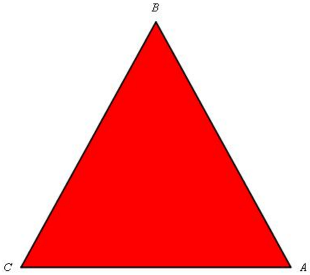
Triangle

Or a line through vertex B.



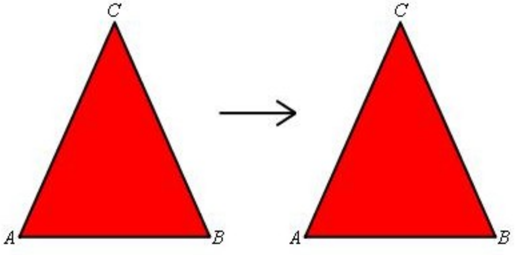
Triangle

Are there other such “motions” that preserve the shape of the triangle? What about rotations? Here is a 120° rotation counterclockwise,



Triangle

Or we could move the triangle to another location.



Rigid Motions

Everything we have described so far is a type of rigid motion (though we haven't described all of them).

A **rigid motion** is the act of taking an object and moving it from some starting position to some ending position without altering its shape or size.

If two rigid motions move an object from the same starting point to the same ending point, we call these rigid motions **equivalent**. It is irrelevant if the process of moving them is different.

Rigid Motions

Any rigid motion of the plane is equivalent to one of the following (basic) rigid motions:

- reflections
- rotations
- translations
- glide reflections

We will study each of these in more details. Only the last one we have not seen an example of yet.

Rigid Motions

If $\mathcal{M}$ is a rigid motion of the plane and P is a point on an object, then we call the ending position of P the **image of P** and denote it P'.

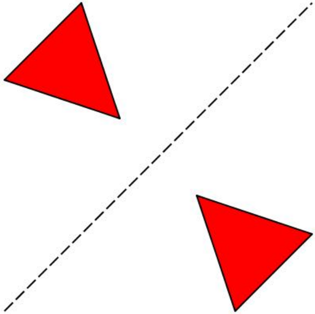
If $\mathcal{M}$ moves P to itself (so $P = P'$) then we call P a **fixed point**.

A rigid motion that is equivalent to not moving the object is called the **identity motion**.

A rigid motion which preserves the left-right and clockwise-counterclockwise orientation of an object is called **proper**. Otherwise it is called **improper**.

Reflections

A **reflection** is a rigid motion that moves an object into a new position that is a mirror image of the starting position. That is, it "flips" the object over a line, called the **axis of reflection**.



Properties of reflections

- A reflection is completely determined by its axis l.
- A reflection is completely determined by a single point-image pair P and P' (unless $P = P'$).
- A reflection has infinitely many fixed points (all points on l).
- A reflection is an improper rigid motion.
- When the reflection is applied twice, we get the identity motion.

Rotations

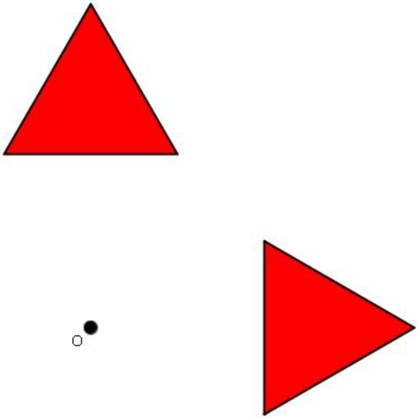
A **rotation** is a rigid motion that pivots an object around a fixed point O.

A rotation is described by three pieces of data:

- The **rotocenter** O.
- The **angle of rotation**.
- The direction of the rotation (clockwise or counterclockwise).

Rotations

Here is an equilateral triangle rotated 90° about the point O.



Properties of rotations

- A rotation is completely determined by two point-image pairs P, P' and Q, Q'.
- A rotation that is not the identity motion has only one fixed point (the rotocenter).
- A rotation is a proper rigid motion.
- A 360° rotation is equivalent to the identity motion.

Translations

A **translation** is a rigid motion in which all points are moved the same distance in the same direction.

A translation is described by a **vector of translation**, which gives the direction and distance the object or points are being moved.

Properties of translations

- A translation is completely determined by a single point-image pair P and P'.
- A translation has no fixed points.
- A translation is a proper rigid motion.
- A translation with vector v followed by the translation with vector $-v$ is equivalent to the identity motion.

Glide reflections

A **glide reflection** is a rigid motion obtained by combining a translation with a reflection.

As such, a glide reflection is described by the vector of translation v and the axis of the reflection l, *and* these two must be parallel.

Properties of glide reflections

- A glide reflections is completely determined by two point-image pairs.
- A glide reflections has no fixed points.
- A glide reflections is an improper rigid motion.
- When a glide reflections with vector v and axis of reflection l followed by the glide reflections with vector $-v$ and the same axis of reflection is equivalent to the identity motion.

Symmetry

Some announcements

- Game reflections deadline extended to Monday (4/4)
- Next math talk on Wednesday (4/6) at 4pm. Speaker is Caroline Turnage of Duke. (Future talks on 4/13, 4/19, and 4/20.

Next week

I have a visitor from the University of Washington (Paul Smith). Here's some things you need to know.

- No regular office hours (appt. only). I will not assign any homework to be due during the week.
- Monday will be a regular class day.
- On Wednesday, Kent Vashaw (TA) will run a classwork session.
- No class on Friday.

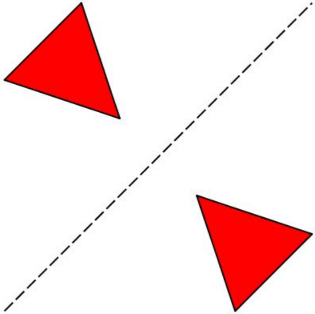
Finding symmetry

Last time we discussed basic rigid motions: reflections, rotations, translations, and glide reflections.

Today we will start with identifying axes of reflections, rotocenters, angles of rotation, and vectors of translation.

Reflections

Recall that a **reflection** is a rigid motion that moves an object into a new position that is a mirror image of the starting position. That is, it "flips" the object over a line, called the **axis of reflection**.



Properties of reflections

- A reflection is completely determined by its axis l.
- A reflection is completely determined by a single point-image pair P and P' (unless $P = P'$).
- A reflection has infinitely many fixed points (all points on l).
- A reflection is an improper rigid motion.
- When the reflection is applied twice, we get the identity motion.

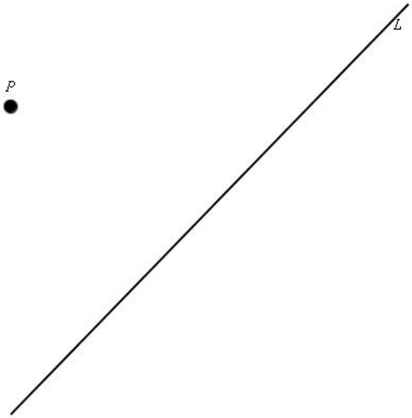
Drawing a reflection

If we are given the axis of symmetry ℓ and a point P, we can locate P' in the following way:

- Draw a line from P perpendicular to ℓ. Call the point of intersection R.
- The point P' lies on ℓ and is the same distance from R as P is from R (just in the opposite direction).

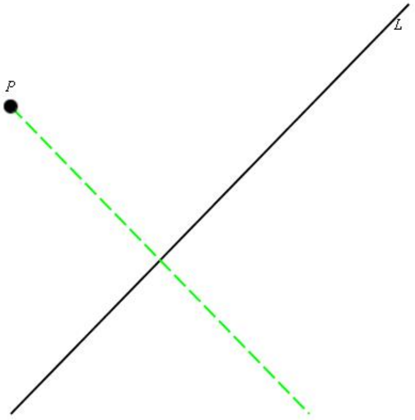
Drawing a reflection

Here we are given a point P and an axis of reflection L.



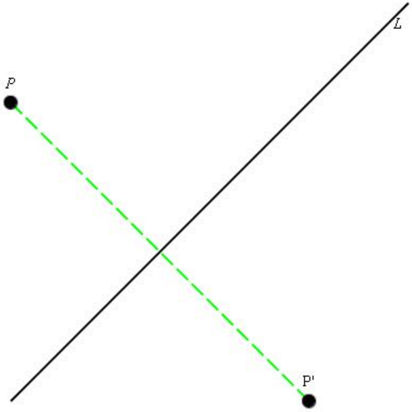
Drawing a reflection

We draw a line from P perpendicular to L.



Drawing a reflection

We measure the distance from P to the intersection. The same distance in the opposite direction gives P'.



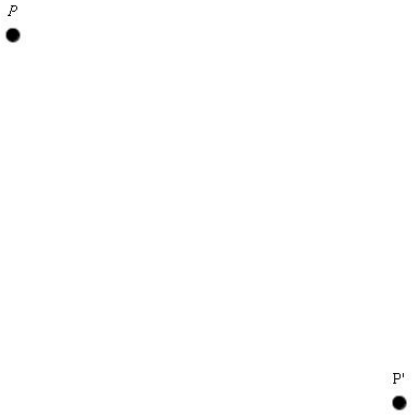
Finding the axis of reflection

Given a point P and its reflection P', we can identify the axis of reflection in the following way:

- Draw a line ℓ from P to P'.
- Draw the perpendicular bisector of ℓ (perpendicular to and splits in half). This is the axis of symmetry.

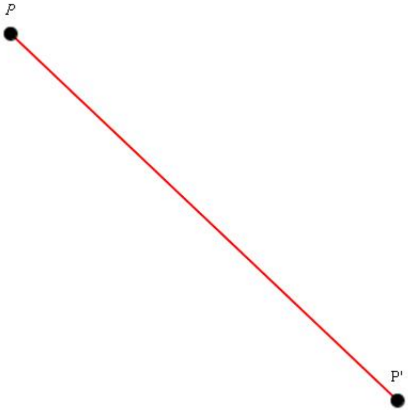
Drawing a reflection

Here we are given a point P its image P'.



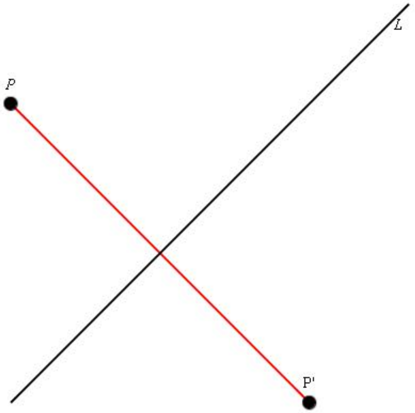
Drawing a reflection

Draw a line from P to P'.



Drawing a reflection

Draw the perpendicular bisector of that line.



Rotations

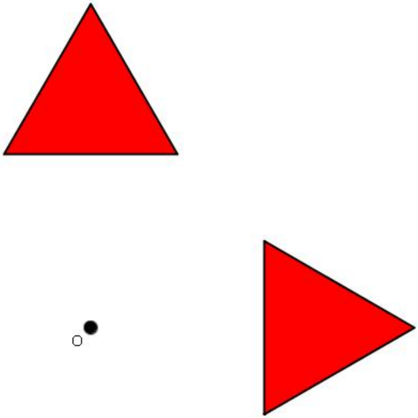
Recall that a **rotation** is a rigid motion that pivots an object around a fixed point O.

A rotation is described by three pieces of data:

- The **rotocenter** O.
- The **angle of rotation**.
- The direction of the rotation (clockwise or counterclockwise).

Rotations

Here is an equilateral triangle rotated 90° about the point O.



Properties of rotations

- A rotation is completely determined by two point-image pairs P, P' and Q, Q'.
- A rotation that is not the identity motion has only one fixed point (the rotocenter).
- A rotation is a proper rigid motion.
- A 360° rotation is equivalent to the identity motion.

Drawing a rotation

Given the rotocenter O and the angle of rotation α (and direction) we can find the image of a point P in the following way:

- Draw a line L from P to O.
- Draw a line L' through O such that the angle between L and L' is α (helps to use a protractor).
- The point P' is the point on L' that is the same distance from O as P is from O

Drawing a rotation

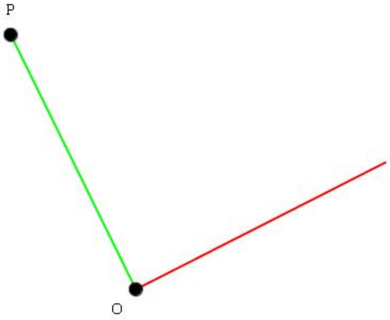
We are given a rotation with rotocenter O and angle of rotation 90° clockwise. We want to find the image of the point P.

Drawing a rotation

Draw a line from P to O.

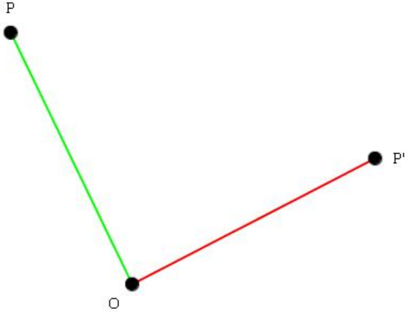
Drawing a rotation

Draw a new line through O. such that the two lines make a 90° angle.



Drawing a rotation

The point P' is the point on the new line that is the same distance from O as P is from O.



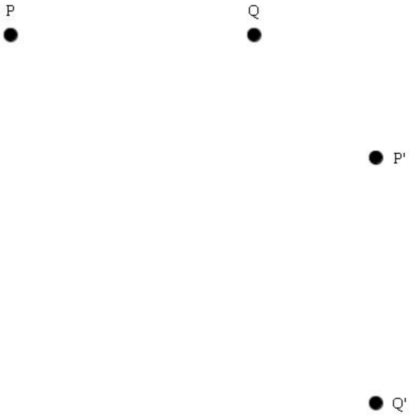
Finding the rotocenter

Given points P, Q and their images P', Q', we can identify the rotocenter in the following way:

- Draw the perpendicular bisector of the lines connecting P, P' and Q, Q'.
- The point where the bisectors meet is the rotocenter.

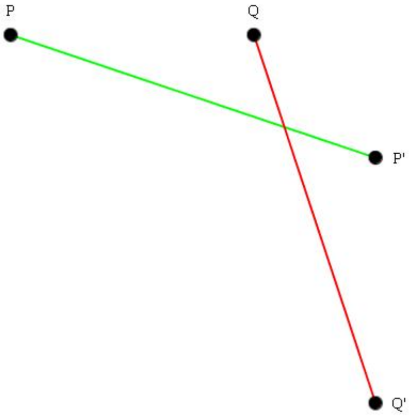
Drawing a rotation

Here we are given points P, Q and their images P', Q'.



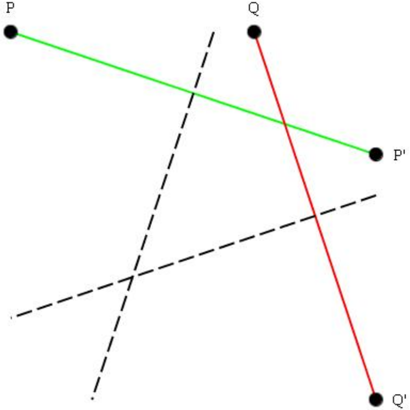
Drawing a rotation

We draw lines between P, P' and Q, Q'.



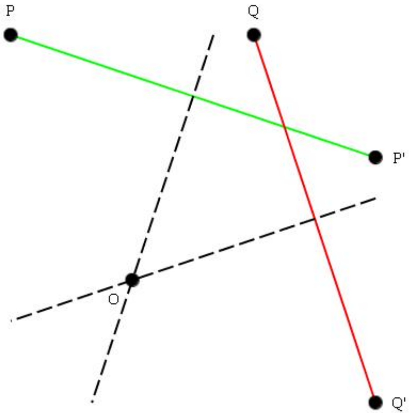
Drawing a rotation

Now we draw perpendicular bisectors for each of these lines.



Drawing a rotation

The intersection points of the bisectors is the rotocenter.



Symmetry

A **symmetry** of an object is a rigid motion that moves the object back onto itself.

Every object has one symmetry: the identity motion.

The idea is this: with no labels, an object should not appear any different.

Symmetries of regular polygon

- Triangle: a triangle has six symmetries.
- Square: a square has eight symmetries.
- Pentagon: a pentagon has ten symmetries.

These symmetries are known as **dihedral symmetries** and denoted D_3, D_4, and D_5, respectively.

Other objects can have these symmetry types. For example, a propeller has symmetry type D_4.

Symmetry types

Some announcements

- Next math talk on Wednesday (4/6) at 4pm. Speaker is Dr. Caroline Turnage-Butterbaugh of Duke. Title: "Distribution of the Primes". (Future talks on 4/13, 4/19, and 4/20.

This week

I have a visitor from the University of Washington (Paul Smith). Here's some things you need to know.

- No regular office hours (appt. only). I will not assign any homework to be due during the week.
- On Wednesday, Kent Vashaw (TA) will run a classwork session.
- No class on Friday.

Finding symmetry

Recall that we have 4 types of basic rigid motions:

- reflections
- rotations
- translations
- glide reflections

A **symmetry** of an object is a rigid motion that moves the object back onto itself.

Finding symmetry

Given an object, we will ask what the symmetries are (up to equivalence).Different objects can have the same symmetries, so we will study certain families of **symmetry types**.

Every object has one symmetry (the identity symmetry). If this is the only symmetry, then the object is said to have symmetry type Z_1 (more on this notation later).

Symmetries of regular polygons

- (Equilateral) Triangle: a triangle has six symmetries.
- Square: a square has eight symmetries.
- Pentagon: a pentagon has ten symmetries.

These symmetries are known as **dihedral symmetries** and denoted D_3, D_4, and D_5, respectively.

Other objects can have these symmetry types. For example, a propeller has symmetry type D_4.

Symmetry Type D_n

An object has symmetry type D_n if the only symmetries of the object (up to equivalence) are n reflections and n rotations. One of the rotations will be (equivalent to) the identity motion.

The sea star below has symmetry type D_5:

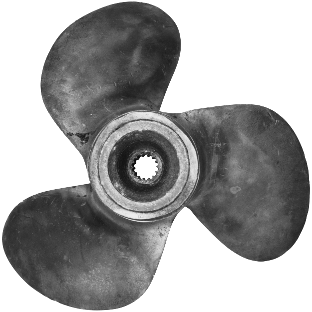
Symmetry Type D_∞

What sort of object would have symmetry type D_∞ (infinitely many rotations and reflections?)

Circles and circular objects are the only shapes with this type of symmetry.

Propeller

What is the symmetry type of the propeller below?



The propeller only has rotational symmetry (by 120°). We say it has symmetry type Z_3.

Symmetry type Z_n

An object has symmetry type Z_n if the only symmetries of the object (up to equivalence) are n rotations. One of the rotations will be (equivalent to) the identity motion.

A quick note on the notation: Z is the standard symbol for the integers (actually $\mathbb{Z}$). Composing rotations in Z_n is equivalent to adding numbers "mod n". That is, $a + b$ is equal to its remainder when you divide by n.

Translations

Recall that a **translation** is a rigid motion in which all points are moved the same distance in the same direction.

A translation is described by a **vector of translation**, which gives the direction and distance the object or points are being moved.

Properties of translations

- A translation is completely determined by a single point-image pair P and P'.
- A translation has no fixed points.
- A translation is a proper rigid motion.
- A translation with vector v followed by the translation with vector $-v$ is equivalent to the identity motion.

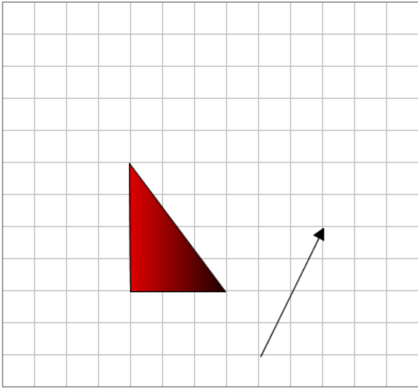
Drawing a translation

Given a point P and the vector of translation v, the image of P is obtained by tracing the vector v starting at P. Then P' is the point at the other end.

Given a point P and its image, the vector of translation is the vector from P to P'.

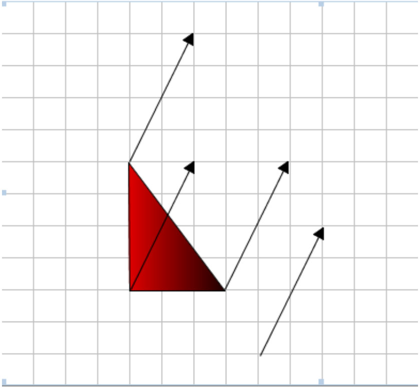
Drawing a translation

Here we are given a triangle and the vector of translation.



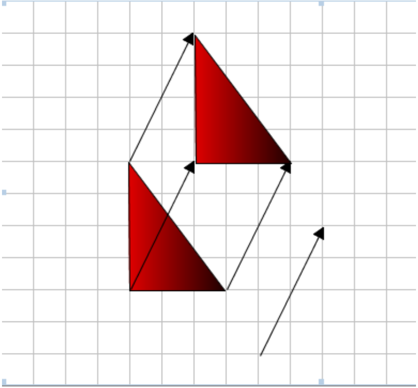
Drawing a translation

We draw a perpendicular vector of the same length from each vertex of the triangle.



Drawing a translation

The endpoints of each vector form the vertices of our new triangle.



Objects with translational symmetry

Repeating patters have translation symmetry.

Glide reflections

Recall that a **glide reflection** is a rigid motion obtained by combining a translation with a reflection.

As such, a glide reflection is described by the vector of translation v and the axis of the reflection l, *and* these two must be parallel.

Properties of glide reflections

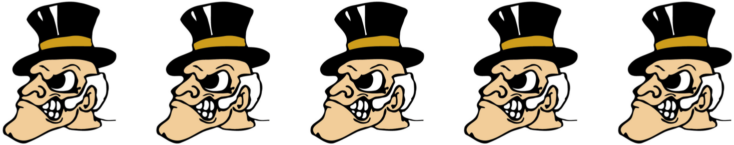
- A glide reflections is completely determined by two point-image pairs.
- A glide reflections has no fixed points.
- A glide reflections is an improper rigid motion.
- When a glide reflections with vector v and axis of reflection l followed by the glide reflections with vector $-v$ and the same axis of reflection is equivalent to the identity motion.

Drawing a glide reflection

Draw a translation using the vector v, **then** draw the reflection using the axis of reflection l.

Border patterns

A **border pattern** are patterns in which a basic motif repeats itself indefinitely in a single direction.

. . . 

. . .

Every border pattern has the identity symmetry and translation symmetry.

There are 7 different symmetry types for border patterns.

Border patterns

…JJJJJJJJ…

Type 11 The pattern has only the identity and translation symmetry.

…BBBBBBBB…

Type 1m The pattern has just horizontal reflection symmetry.

Border patterns

$$\ldots\mathbf{AAAAAAAA}\ldots$$

Type m1 The pattern has just vertical reflection symmetry.

$$\ldots\mathbf{XXXXXXXX}\ldots$$

Type mm The pattern has horizontal and vertical reflection symmetry. There is also a half-turn symmetry.

Border patterns

$\dots$ **ZZZZZZZZ** $\dots$

Type 12 The pattern has only half-turn symmetry.

Type 1g The pattern has only glide reflection symmetry.

Border patterns

A A A

... ...

∀ ∀

Type mg The pattern has vertical reflection and glide reflection symmetry. There is also half-turn symmetry.

Fibonacci Numbers

Some announcements

- Homework: Handout due Friday.
- Next math talk on Wednesday (4/13) at 4pm. Speaker is Kimberly Kaufield (NC State) Title: Assessing mountain pine beetle outbreaks.
- Gentry Lectures (4/19 and 4/20)

Breeding (magic) bunnies

Congratulations! You are now the owner of a farm of (magic) bunnies. Your farm only has one pair of bunnies right now.

These bunnies aren't old enough to reproduce yet. After 2 months they will produce another pair of bunnies.

We'll count bunnies by pairs each month. This is the start, so we say $P_0 = 1$.

Month 1

You have one pair of bunnies. Aren't they cute?

After 1 month, we have $P_1 = 1$.

Month 2

Your bunnies made new bunnies! You now have two pairs of bunnies.

After 2 months, we have $P_2 = 2$.

Month 3

Your original bunnies had new bunnies and the bunnies from last month are all grown up!

After 3 months, we have $P_3 = 3$.

Month 4

The two grown up pairs of bunnies each made a pair of bunnies.
The baby bunnies from last month are now mature.

After 4 months, we have $P_4 = 5$.

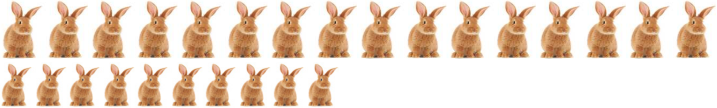
Month 5

Your bunny farm is starting to get a little out of control! The three mature pairs of bunnies each had new bunnies and the two pairs of baby bunnies are grown up.

After 4 months, we have $P_5 = 8$.

Month 6

Local animal control would like a word with you...



After 4 months, we have $P_8 = 13$.

Month N

How many bunny pairs will we have after N months assuming we know how many bunnies we have each month prior to the Nth month?

All of the bunny pairs that are 2 months old can reproduce, so that is P_{N-2} bunnies, plus all of the bunny pairs we had the previous month, P_{N-1}.

Thus, $P_N = P_{N-2} + P_{N-1}$.

The Fibonacci sequence

The sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ is known as the Fibonacci sequence.

The Nth Fibonacci number F_N is obtained *recursively* by adding the previous two numbers (with $F_1 = 1$ and $F_2 = 1$). So $F_N = F_{N-2} + F_{N-1}$.

Growth of Fibonacci numbers

Fibonacci numbers grow quickly.

$F_{10} = 55$

$F_{50} = 12,586,269,025$

$F_{100} = 354,224,848,179,261,915,075$

Binet's formula

The Nth Fibonacci number is given by Binet's formula:

$$F_N = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N}{\sqrt{5}}$$

A simplified version is

$$F_N = \left[\!\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^N}{\sqrt{5}} \right]\!\right]$$

where $[\![\quad]\!]$ means 'round to the nearest integer'.

The Golden ratio

The number $\phi = \dfrac{1+\sqrt{5}}{2}$ is known as the **golden ratio**, $\phi \approx 1.618$.

The golden ratio satisfies the **golden property**: $\phi^2 = \phi + 1$. This is because ϕ is a *root* of the equation $x^2 = x + 1$. What is the other root?

Using this notation, Binet's (simplified) formula simplifies further,

$$F_N = \left[\!\left[\phi^N / \sqrt{5} \right]\!\right]$$

where $[\![\quad]\!]$ means 'round to the nearest integer'.

The Golden ratio and Fibonacci numbers

Let's consider a sequence formed by taking ratios of consecutive Fibonacci numbers:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \ldots$$

In decimal form, this is,

$$1, 2, 1.5, 1.666667, 1.6, 1.625, 1.615385, 1.619048, 1.617647, \ldots$$

The numbers approach the golden ratio. In fact, no matter how many decimal places you go out, at some point the sequence will match the golden ratio.

We say this sequence *coverges* to the golden ratio ϕ.

The Divine Proportion

What is the most aesthetically pleasing way to divide an object? According to the ancient Greeks, it is the following:

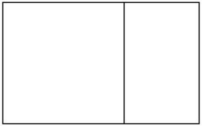
Split the object so that the ratio of the bigger piece to the smaller piece is equal to the ratio of the whole object to the bigger piece.

If we divide the object this way, then the ratio of the bigger piece to the smaller piece is the golden ratio!

The Golden Rectangle

Here is a rectangle with height 1 and base ϕ.

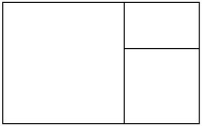
Say we divide it according to the divine proportion. This gives a square and a rectangle.



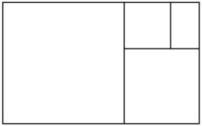
The ratio of the sides of the smaller rectangle is $1 : \phi - 1$. This is the golden ratio!

The Golden Rectangle

Let's divide it again.



The ratio of the sides of the smaller rectangle is the golden ratio.



Gnomons

Some announcements

- Homework: Handout due Friday.
- Next math talk today at 4pm. Speaker is Kimberly Kaufield (NC State) Title: Assessing mountain pine beetle outbreaks.
- Gentry Lectures (4/19 and 4/20)

Fibonacci review

The sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ is known as the Fibonacci sequence.

The Nth Fibonacci number F_N is obtained *recursively* by adding the previous two numbers (with $F_1 = 1$ and $F_2 = 1$). So $F_N = F_{N-2} + F_{N-1}$.

Fibonacci review

The number $\phi = \dfrac{1+\sqrt{5}}{2}$ is known as the **golden ratio**, $\phi \approx 1.618$.

The Nth Fibonacci number is given by Binet's formula:

$$F_N = \left[\!\left[\phi^N/\sqrt{5}\right]\!\right]$$

where $[\![\quad]\!]$ means 'round to the nearest integer'.

Fibonacci review

The sequence formed by taking ratios of consecutive Fibonacci numbers converges to to the golden ratio ϕ:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \ldots$$

Fibonacci review

The Divine Proportion:
Split the object so that the ratio of the bigger piece to the smaller piece is equal to the ratio of the whole object to the bigger piece.

A golden rectangle is one whose ratio of larger side to smaller side is the golden ratio *phi*.

When a square is cut from a golden rectangle using the shorter side, the remaining rectangle is again a golden rectangle.

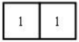
More on golden rectangles

We saw last time how ‘Fibonacci rectangles’ approximate golden rectangles. Let’s make this a little more precise.

We start with a rectangle of size 1 (so a square).

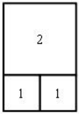
More on golden rectangles

We attach to this another rectangle of size 1.



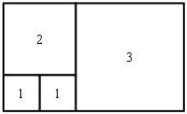
More on golden rectangles

Next we attach a rectangle of size 2.



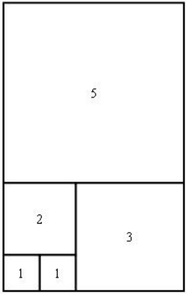
More on golden rectangles

Next we attach a rectangle of size 3.



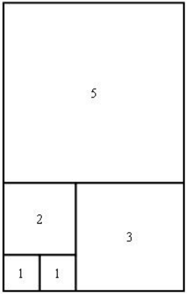
More on golden rectangles

Next we attach a rectangle of size 5.



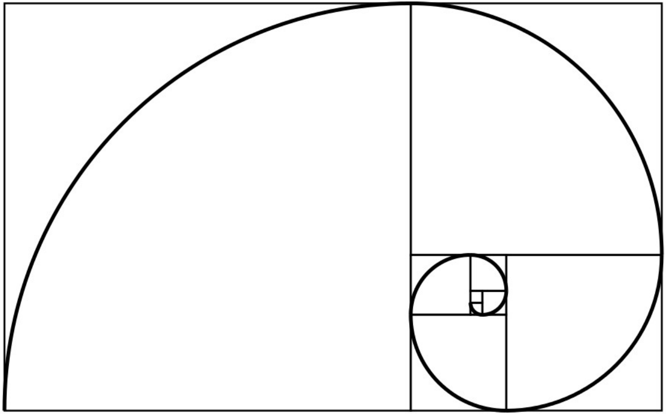
More on golden rectangles

With each iteration, the ratio of sides is closer to the golden ratio, and hence the rectangle is closer to being a golden rectangle.



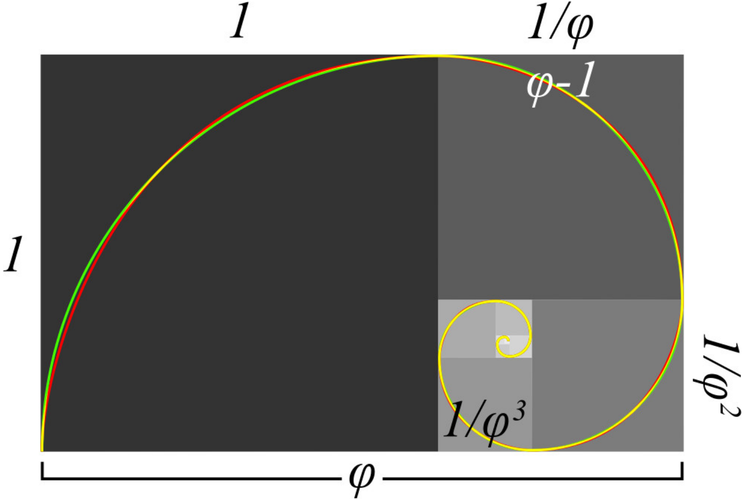
Fibonacci spiral

If we attached these Fibonacci rectangles in a certain way and drew a quarter circle through each one, the result is what is known as the **Fibonacci spiral**.



Golden spiral

The Fibonacci spiral approximates what the **Golden spiral** which is drawn through a true series of golden rectangles.



Fibonacci spiral

This type of growth appears in nature, for example in the growth of a nautilus shell. However, the growth rate may not always mirror the golden ratio but may be an approximation of it.

Similarity

In our discussion of symmetry, we focused on *rigid motions*, that is, we were not allowed to alter the shape of an object only its position. Essentially, we were studying *congruent* objects, but one can also study *similar* objects.

Similarity

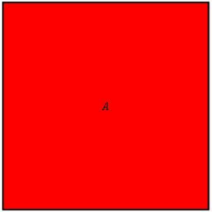
Two objects are said to be **similar** if one is obtained from the other by scaling.

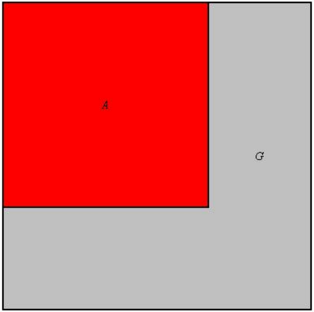
Similar polygons

- Triangles: two triangles are similar if and only if the measures of their respective angles are the same.
- Squares: any two squares are similar.
- Rectangles two rectangles are similar if their corresponding sides are proportional. (Hence, **any** two golden rectangles are similar.)
- Circles: any two circles are similar. (Same for any two circular disks).
- Circular rings: two circular rings are similar if and only if their inner and outer radii are proportional.

Gnomons

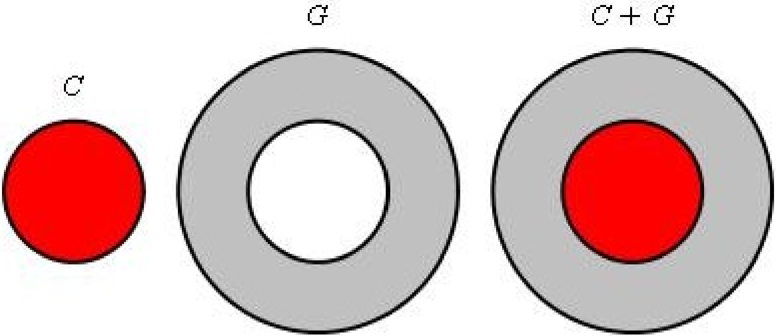
A **gnomon** G to an object A is a connected figure that, when attached to A (without overlapping) produces a new figure similar to A.





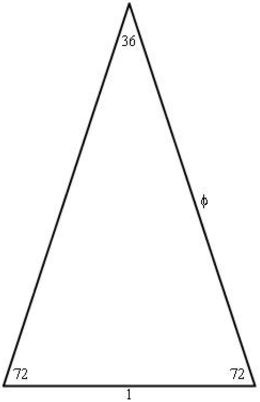
Gnomons

Gnomon to a circle.



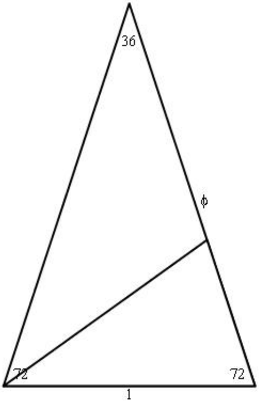
Golden Triangle

Let's start with an isosceles triangle such that the ratio of the length of the longer side to the shorter side is ϕ.



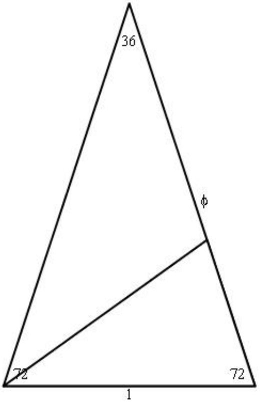
Golden Triangle

Suppose this triangle is the combination of a similar triangle and a gnomon. We can cut out a similar triangle using the shorter side.



Golden Triangle

We see that the gnomon to a 36-72-72 triangle is a 36-36-108 triangle.



Final Review

Some announcements

- Maple Labs due today.
- Final Exam is Monday, May 2 at 2pm in our normal classroom.
- Office hours:
  - Thursday (4/28) 12:30-2:30
  - Friday (4/29) 9am-10am and 11am-1pm
  - Monday (5/2) 9am-noon
  - by appointment.
- Grades on Sakai are now current for all of those that turned in the Maple Lab before class.

Final Exam

Some important information about the final exam:

- The final exam is cumulative but will focus on major themes.
- Problems will be similar to those on our three in-class exams. Hence, those exams are a good way to study (but shouldn't be your only way to study). I have posted additional practice problems from your textbook on Sakai.
- Problems on previous exams were broken down as 75% easy, 20% medium, and 5% hard. This exam is approximately 80% easy, 20% medium.
- There will be no problems on *self-symmetry* except *maybe* a couple bonus problems.

Final Exam

- The exam will be divided into three parts, one for each unit.
- Each part will be worth 50 points, so between 5-10 problems for each section depending on length.
- You may use a scientific or four-function calculator. Please let me know by Friday if you need to borrow one so I can make sure I have enough. You will not need a ruler.
- Questions?

Key Points - Unit 1

- Reading and creating a preference schedule.
- Voting methods: Plurality and Majority, Borda Count, IRV, Pairwise Comparison, Least Worst Defeat, Ranked Pairs.
- Fairness Criteria: Majority Criterion, Condorcet Criterion, Monotonicity Criterion, Independence-of-Irrelevant-Alternatives.
- Read and analyze a weighted voting system.
- Compute BPI and SSPI for a weighted voting system.

Key Points - Unit 2

- Compute basic probabilities given a sample space or determine probability assignments. You may need to use the Multiplication Principle for Independent Events.
- Compute permutations and combinations.
- Compute expected value and make decisions using expected value.
- Read a normal curve and relevant data which can be extracted from it.

Key Points - Unit 3

- Know the different rigid motions and their basic properties. Identify the axis of reflection, rotocenter of a rotation, or vector of translation given a figure and its image.
- Given a series of rigid motions on a object, determine the equivalent basic rigid motion.
- Identify symmetry types.
- Fibonacci numbers and how to construct the sequence. Identify connections between Fibonacci numbers and the golden ratio/golden rectangles.
- Use similarity and the concept of gnomons to find lengths in an object.

What you've learned (I hope)

Normally I teach classes like calculus or algebra, this was a new experience for me and one that I thoroughly enjoyed. I hope that, while you certainly had to put work into this course, that you enjoyed it as well. Here are some of the things I hope you will take away from this semester:

- Math is not just about numbers and computation, there is much more to Math than what you've seen in algebra or calculus classes.
- Math shows up in some weird places. Really, Math is just a way of thinking that can be applied to a vast array of situations.
- Math can be fun! It doesn't have to be mindless computation all the time (just maybe some of the time).