The discriminant of twisted tensor products

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0/ Setup/Notation

$k = \bar{k}$ field, $\operatorname{char} k = 0$ $\quad R$ k-alg

$\sigma \in \operatorname{Aut}(R)$, $R^\sigma = \{ r \in R : \sigma(r) = r \}$ $\quad C(R) =$ center of R

Idea: Study automorphism grps of n.c. algs that are f.g. as modules over their centers

Discriminant in n.c. setting due to work of Ceken, Palmieri Wang & Zhang (CPWZ)

Alt approach using Poisson geom: see work of Nguyen - Trampel - Yakimov & Levitt - Yakimov.

1/ Defns

S'pose B f.g./ $R \subset C(B)$ central subalg (free)

$l_m : B \longrightarrow \operatorname{End}_R(B) \cong M_n(R)$

$tr_{int} : M_n(R) \longrightarrow R$ internal trace

$tr_{reg} : B \xrightarrow{l_m} M_n(R) \xrightarrow{tr_{int}} R$ regular trace $\quad (tr = tr_{reg})$

$Z = \{z_i\} \subset B$ finite subset (basis B/R)

The <u>discriminant</u> of B over R is defn'd as

$$d(B/R) =_{R^\times} \det(\mathrm{tr}(z_i z_j)) \in R.$$

Key Result:

<u>Thm</u> (CPWZ) If $\phi \in \mathrm{Aut}(B)$ & ϕ preserves R, then ϕ preserves the ideal gen by $d(B/R)$.

(Problem: discriminant is computationally difficult)

(Our interest is in certain Ore extensions & skew group rings. Can unify these through twisted tensor products.)

A, B algs, τ a k-linear hom $\tau : B \otimes A \longrightarrow A \otimes B$ subject to

$$\tau(b \otimes 1_A) = 1_A \otimes b \qquad \tau(1_B \otimes a) = a \otimes 1_B \qquad \forall a \in A, b \in B.$$

This defines a twisted mult on $A \otimes B$ by

$$\mu_\tau = (\mu_A \otimes \mu_B) \circ (id_A \otimes \tau \otimes id_B)$$

(certain conds ensure μ_τ is assoc.)

The triple (A, B, μ_τ) is a <u>twisted tensor product</u> denoted $A \otimes_\tau B$.

Exs Let A be base alg and $B = k[M]$ for M a monoid along w/ a monoid hom $\rho: M \longrightarrow \mathrm{Aut}(A)$. Twisting map is

$$\tau: k[M] \otimes A \longrightarrow A \otimes k[M]$$
$$m \otimes a \longmapsto ma \otimes m$$

1) $\sigma \in \mathrm{Aut}(A)$, then $A \otimes_\tau k[\mathbb{N}] \cong A[t;\sigma]$ where $\rho: \mathbb{N} \longrightarrow \mathrm{Aut}(A)$, $1 \longmapsto \sigma$

2) G grp $\curvearrowright A$ as autos then $\rho: G \longrightarrow \mathrm{Aut}(A)$ gives usual skew grp ring.

3/ Results

Lemma $H = \ker \rho$. Suppose $H \subset C(M)$ & $\mathrm{im}\,\rho \cap \mathrm{Inn}(A) = \{id_A\}$. Then $C(A \otimes_\tau k[M]) = C(A)^M \otimes k[H]$.

Thm A k-alg, G grp, $\rho: G \longrightarrow \mathrm{Aut}(A)$ grp hom, M a submonoid of G s.t. $\mathrm{im}(\rho|_M) \cap \mathrm{Inn}(A) = \{id_A\}$ and $H := \ker\rho \cap M \subset C(M)$. Set $T = A \otimes_\tau k[M]$ & s'pose $R \subset C(T)$ is a subalg s.t.

(a) A free over $A \cap R$ of rank $n < \infty$
(b) $R = (A \cap R) \otimes k[H]$
(c) $\exists$ basis $\{x_1, \dots, x_\ell\}$ of $k[M]$ over $k[H]$ w/ $x_1 = e_M$ & $x_i \in M$.

Thm: $d(T/R) = d(A/A \cap R)^\ell \cdot d(k[M]/k[H])^n$

Key pt in proof:
under hypotheses,

$$tr(a \otimes m) = \begin{cases} tr(a) \otimes tr(m) & m \in H \\ 0 & \text{o/w} \end{cases}$$

Translating:

<u>Thm</u> A an algebra, $S = A[t;\sigma]$, $\sigma \in \mathrm{Aut}(A)$, $|\sigma| = m < \infty$ and no σ^i $(1 \le i < m)$ is inner, $R \subset C(S)$. Set $B = R \cap A^\sigma$. If A is free over B of rank n ~~there~~ and $R = B[t^m]$ then

$$d(S/R) =_{R^\times} d(A/B)^m \, (t^{m-1})^{mn}.$$

<u>Ex</u> $A = k_{-1}[x,y]$, $\sigma: x \leftrightarrow y$, $S = A[t;\sigma]$, $|\sigma| = 2 = m$
$R = C(S) = k[X,Y,T]$ where $X = x^2+y^2$, $Y = x^2y^2$, $T = t^2$.
A is free over $B = R \cap A^\sigma$ of rank 4. A computation shows
$d(A/B) =_{k^\times} (Y(X^2-4Y))^4$ so

$$d(S/R) =_{k^\times} Y^8 (X^2-4Y)^8 \, T^8$$

All automorphisms of S are graded.

<u>Thm</u> A an algebra, G a finite group acting on A as autos s.t. no non-id auto is inner. Set $S = A \# G$ (identify $a \longmapsto a \otimes e$)
Suppose A is free over $R \subset C(A)^G$. Then

$$d(S/R) =_{R^\times} d(A/R)^{|G|}.$$

<u>Ex</u> Same A and σ as before. Set $S = A \# \langle\sigma\rangle$. $C(S) = k[X,Y]$ where $X = x^2+y^2$, $Y = x^2y^2$. Then $d(A/C(S)) = (Y(X^2-4Y))^4$ (as before) and thm implies that

$$d(S/C(S)) =_{k^\times} \left[(Y(X^2-4Y))^4\right]^2 \otimes e$$

All automorphisms are "graded" in the sense that

$$\phi(x \otimes e) = \ell_1 \otimes e + \ell_2 \otimes \sigma, \quad \ell_1, \ell_2 \in A_1.$$