Invariant Theory of Generalized Weyl Algebras

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This talk focuses primarily on two recent projects:

- *Fixed rings of generalized Weyl algebras*, J. Algebra 536 (2019). Joint with Rob Won.
- *Fixed rings of quantum generalized Weyl algebras*, arXiv:1910.06847. Joint with Phuong Ho (undergraduate).

Shephard-Todd-Chevalley

Throughout, $\Bbbk$ is an algebraically closed field of characteristic zero.

Theorem (Shephard-Todd-Chevalley)

Let G be a finite group of graded automorphisms on $A = \Bbbk[x_1, \ldots, x_n]$. The fixed ring A^G is a polynomial ring if and only if G is generated by reflections.

A major goal in noncommutative invariant theory is to generalize the STC theorem to noncommutative analogues of polynomial rings.

- Artin-Schelter regular algebras (Kirkman-Kuzmanovich-Zhang)
- Preprojective algebras (Weispfenning)

Shephard-Todd-Chevalley

To further generalize STC, we can relax the grading requirement and consider Calabi-Yau algebras.

- An algebra A is *homologically smooth* if it has a finitely generated projective resolution of finite length in $A^e - \mathrm{MOD}$.
- The algebra A is *twisted Calabi-Yau* of dimension d if it is homologically smooth and there exists an invertible bimodule U of A such that $\mathrm{Ext}^i_{A^e}(A, A^e) = \delta_{id} U$.
- If $U = A$, then A is said to be *Calabi-Yau*.

Question: When is the fixed ring of a (twisted) Calabi-Yau algebra again (twisted) Calabi-Yau?

Generalized Weyl algebras

This talk will focus on certain $\mathbb{Z}$-graded (twisted) Calabi-Yau algebras.

Definition

Let D be a commutative algebra, $\sigma \in \text{Aut}(D)$, and $a \in D$, $a \neq 0$. The *generalized Weyl algebra* (GWA) $D(\sigma, a)$ is generated over D by x and y subject to the relations

$$xd = \sigma(d)x, \quad yx = \sigma^{-1}(d)y, \quad yx = a, \quad xy = \sigma(a).$$

There is a natural $\mathbb{Z}$-grading on a GWA $D(\sigma, a)$ defined by setting $\deg(D) = 0$, $\deg(x) = 1$, and $\deg(y) = -1$.

Generalized Weyl algebras

GWAs were named by Bavula, but are related to algebras studied by Hodges, Jordan, Joseph, Rosenberg, Smith, Stafford, and others...

Examples of GWAs include the Weyl algebra, the quantum Weyl algebras, quantum planes, primitive quotients of $U(\mathfrak{sl}_2)$, and primitive quotients of $U_q(\mathfrak{sl}_2)$.

A GWA $R = D(\sigma, a)$ is

- *classical* if $D = \mathbb{k}[h]$ and $\sigma(h) = h - 1$,
- *quantum* if $D = \mathbb{k}[h]$ or $\mathbb{k}[h^{\pm 1}]$ and $\sigma(h) = qh$, $q \in \mathbb{k} \backslash \{0, 1\}$.

Theorem (Liu)

A GWA $\mathbb{k}[h](\sigma, a)$ is twisted Calabi-Yau if and only if it is has finite global dimension.

Fixed rings of GWAs

Theorem (Jordan-Wells)

Let $R = D(\sigma, a)$ be a GWA and let β be a primitive ℓth root of unity. Define the automorphism Θ_β of R by

$$\Theta_\beta(x) = \beta x, \quad \Theta_\beta(y) = \beta^{-1} y, \text{ and } \Theta_\beta(d) = d \text{ for all } d \in D.$$

The fixed ring $R^{\langle \Theta_\beta \rangle}$ is the GWA $D(\sigma^\ell, A)$, generated over D by x^ℓ and y^ℓ with defining polynomial $A = \prod_{i=0}^{\ell-1} \sigma^{-i}(a)$.

Further examples of GWA fixed rings were considered by Kirkman and Kuzmanovich.

The classical case

Automorphisms of classical GWAs were classified by Bavula and Jordan.

Let $n = \deg_h(a)$, $\lambda \in \Bbbk$, $\beta \in \Bbbk^\times$, $m \in \mathbb{N}$, and let $\Delta_m : \Bbbk[h] \to \Bbbk[h]$ be the linear map given by $\sigma^m - 1$. Generically, $\mathrm{Aut}(R)$ is generated by the following maps:

$$\Theta_\beta : x \mapsto \beta x, y \mapsto \beta^{-1} y, h \mapsto h,$$

$$\Psi_{m,\lambda} : x \mapsto x, y \mapsto y + \sum_{i=1}^{n} \frac{\lambda^i}{i!} \Delta_m^i(a) x^{im-1}, h \mapsto h - m\lambda x^m,$$

$$\Phi_{m,\lambda} : x \mapsto x + \sum_{i=1}^{n} \frac{(-\lambda)^i}{i!} y^{im-1} \Delta_m^i(a), y \mapsto y, h \mapsto h + m\lambda y^m.$$

If a is *reflective* ($a(\rho - h) = (-1)^n a(h)$ for some $\rho \in \Bbbk$) there is an additional generator Ω given by

$$\Omega(x) = y, \quad \Omega(y) = (-1)^n x, \quad \Omega(z) = 1 + \rho - z.$$

The classical case

In the classical case, $R = \Bbbk[h](\sigma, a)$ is $\mathbb{N}$-filtered by setting $\deg(h) = 2$ and $\deg(x) = \deg(y) = \deg_h(a)$.

Generically, in the case $\deg_h(a) > 2$, the only filtered maps are the Θ_β.

In the case a is quadratic, g is a filtered automorphism of R if and only if

- $g = \tau_{\lambda,\mu,\beta} = \Psi_{1,\lambda} \circ \Phi_{1,\mu} \circ \Theta_\beta$ or
- $g = \tau_{\lambda,\mu,\beta} \circ \Omega$.

The classical case

The Weyl algebra is a classical GWA with $\deg_h(a) = 1$. The following proposition was inspired by work of Kirkman and Kuzmanovich.

Proposition (-, Won)

Let g be a finite filtered automorphism of $A_1(\mathbb{k})$. Then g acts diagonally on a generating set $\{X, Y\}$ of $A_1(\mathbb{k})$ such that $XY - YX = 1$. Thus, $A_1(\mathbb{k})^{\langle g \rangle}$ is a (classical) GWA.

This is true in the quadratic case as well.

Theorem (-, Won)

Let $R = \mathbb{k}[h](\sigma, a)$ be a classical GWA with a quadratic. Let g be a finite filtered automorphism of R. There exists a generating set $\{X, Y\}$ over $\mathbb{k}[h]$ with GWA structure so that $R^{\langle g \rangle}$ is again a (classical) GWA.

Global dimension

Let $R = \Bbbk[h](\sigma, a)$ be a classical GWA. Two roots α, β of a are said to be congruent if there exists an $i \in \mathbb{Z}$ such that, as ideals of $\Bbbk[h]$, $(\sigma^i(h-\alpha)) = (h-\beta)$. Work of Bavula and Hodges implies that

$$\operatorname{gldim} R = \begin{cases} \infty & \text{if } a \text{ has a multiple root} \\ 2 & \text{if } a \text{ has a congruent root and no multiple roots} \\ 1 & \text{if } a \text{ has no congruent roots and no multiple roots.} \end{cases}$$

Corollary

Let $R = \Bbbk[h](\sigma, h(h-t))$ be a classical GWA. Let g be a finite filtered automorphism of R.

- *If* $\operatorname{gldim}(R) = 1$ *(resp. ∞), then* $\operatorname{gldim}(R^{\langle g \rangle}) = 1$ *(resp. ∞).*
- *If* $\operatorname{gldim}(R) = 2$*, then $t \in \mathbb{Z}$ and*

$$\operatorname{gldim}(R^{\langle g \rangle}) = \begin{cases} 2 & \textit{if } |t| \geq |g| \\ \infty & \textit{otherwise.} \end{cases}$$

The quantum case

Let $R = D(\sigma, a)$ be a quantum GWA with a not a unit. Suárez-Alverez and Vivas fully determined $\operatorname{Aut}(R)$. We consider certain automorphisms defined as follows:

- Write $a = \Sigma_{i \in I} a_i h^i$ where $I = \{i : a_i \neq 0\}$.
- Let $g = \gcd\{i - j : a_i a_j \neq 0\}$.
- If a is a monomial then let $C_g = \mathbb{k}^\times$ and otherwise let C_g be the subgroup of $\mathbb{k}^\times$ consisting of gth roots of unity.

For $(\gamma, \mu) \in C_g \times \mathbb{k}^\times$, define $\eta_{\gamma,\mu} \in \operatorname{Aut}(R)$ by

$$\eta_{\gamma,\mu}(h) = \gamma h, \quad \eta_{\gamma,\mu}(y) = \mu y, \quad \eta(x) = \mu^{-1} x.$$

The quantum case

The following theorem generalizes the result of Jordan and Wells.

Theorem (-,Ho)

Let D be an integral domain, let $R = D(\sigma, a)$ be a GWA, and let $\phi \in \mathrm{Aut}(R)$ with $|\phi| < \infty$. Suppose $\phi\big|_D$ restricts to an automorphism of D, $\phi(x) = \mu^{-1}x$, and $\phi(y) = \mu y$ for $\mu \in \Bbbk^\times$. Set $n = |\phi\big|_D|$ and $m = |\mu|$. If $\gcd(n, m) = 1$, then $R^{\langle\phi\rangle} = D^{\langle\phi\rangle}(\sigma^m, A)$ with $A = \prod_{i=0}^{m-1} \sigma^{-i}(a)$.

Corollary

Let $R = \Bbbk[h](\sigma, a)$ be a quantum GWA and $\eta = \eta_{\gamma,\mu} \in \mathrm{Aut}(R)$. Set $n = |\gamma|$ and $m = |\mu|$ with $n, m < \infty$. If $\gcd(n, m) = 1$, then $R^{\langle\eta\rangle}$ again a quantum GWA.

Global dimension

Global dimension for quantum GWAs is much more sensitive. The following criteria is due to Bavula and Jordan.

Let $R = D(\sigma, a)$ be a quantum GWA.

- R has infinite global dimension if and only if a has multiple roots.
- R has global dimension 2 if and only if a has no multiple roots and one of the following hold:
    - $D = \mathbb{k}[h]$,
    - q is a root of unity, or
    - a has a pair of congruent roots.
- R has global dimension 1 otherwise.

Global dimension

Suppose $R = D(\sigma, a)$ is a quantum GWA. Let $\eta = \eta_{\gamma,\mu} \in \mathrm{Aut}(R)$ with $n = |\gamma| < \infty$ and $m = |\mu| < \infty$ such that $\gcd(n, m) = 1$.

Theorem (-,Ho)

- *If* $\operatorname{gldim} R = 1$, *then* $\operatorname{gldim} R^{\langle\eta\rangle} = 1$.
- *If* $\operatorname{gldim} R = \infty$, *then* $\operatorname{gldim} R^{\langle\eta\rangle} = 2$ *if and only if* $m = 1$ *and* 0 *is a root of* $a(h)$ *with multiplicity* $k = n$. *Otherwise* $\operatorname{gldim} R^{\langle\eta\rangle} = \infty$.
- *If* $\operatorname{gldim} R = 2$ *and* q *is not a root of unity, then* $\operatorname{gldim} R^{\langle\eta\rangle} = \infty$ *if and only if there exists roots* c_i, c_j *of* $a(h)$ *such that* $c_i = q^k c_j$ *for some* k *with* $0 \le k \le m-1$. *Otherwise* $\operatorname{gldim} R^{\langle\eta\rangle} = 2$.

Global dimension

Theorem (cont.)

- *If* $\operatorname{gldim} R = 2$ *and* q *is a root of unity,* $q \neq 1$*, then* $\operatorname{gldim} R^{\langle \eta \rangle} = \infty$ *if and only if one of the following conditions is satisfied:*
  - $a(h)$ *has multiple roots,*
  - $a(h)$ *has congruent roots* c_i, c_j *such that* $c_i = q^k c_j$ *with* $0 \leq k \leq m-1$*, or*
  - *there exists* k *such that* $0 < k \leq m-1$ *and* $|q|$ *divides* nk*.*

  Otherwise $\operatorname{gldim} R^{\langle \eta \rangle} = 2$*.*

Future work - Generalized generalized Weyl algebras

Definition (Bell, Rogalski)

Let D be a commutative noetherian $\Bbbk$-algebra with $\sigma \in \operatorname{Aut}(D)$. Let $X = \operatorname{Spec} D$ and Z a closed subset of X. Let H and J be ideals of D such that $\operatorname{Spec} D/H, \operatorname{Spec} D/J \subset Z$. Then we define the ring

$$B(Z, H, J) = \bigoplus_{n \in \mathbb{Z}} I_n t^n \subset D[t, t^{-1}; \sigma]$$

with $I_0 = D$, $I_n = J\sigma(J) \cdots \sigma^{n-1}(J)$ for $n \geq 1$, and $I_n = \sigma^{-1}(H)\sigma^{-2}(H) \cdots \sigma^{n}(H)$ for $n \leq 1$.

A GWA $\Bbbk[h](\sigma, a)$ is one such ring. Set $D = \Bbbk[h]$, $H = \sigma(a)D$, $J = D$, $x = t$, and $y = at^{-1}$. (Take $Z = \operatorname{Spec} D/H$).

Question

Is there an analogue of the Jordan-Wells theorem in this setting?

Future work - Rigidity

Unlike in the classical STC, it is too much to expect that $A^G \cong A$ for a noncommutative algebra A.

Theorem (Smith)

Let $A_1(\Bbbk)$ be the first Weyl algebra and $G \subset \operatorname{Aut}(A_1(\Bbbk))$ a finite group. Then $A_1(\Bbbk)^G \ncong A_1(\Bbbk)$.

This theorem was extended to the nth Weyl algebra by Alev and Polo.

Theorem (Tikaradze)

The Weyl algebra is in fact not the fixed ring of any domain.

Future work - Rigidity

Proposition (-,Won)

Let $R = \Bbbk[h](\sigma, a)$ be a classical GWA with a quadratic. Let g be finite filtered automorphisms of R. If $R^{\langle g \rangle} \cong R$, then $\langle g \rangle$ is trivial.

Proposition (-,Ho)

Suppose $R = D(\sigma, a)$ is a quantum GWA. Let $\eta = \eta_{\gamma,\mu} \in \mathrm{Aut}(R)$ with $n = |\gamma| < \infty$ and $m = |\mu| < \infty$ such that $\gcd(n, m) = 1$. If $R^{\langle \eta \rangle} \cong R$, then $\langle \eta \rangle$ is trivial.

Thank You!