# Fixed Rings of Quantum Generalized Weyl Algebras

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## The Big Idea

We study the invariant theory of Generalized Weyl Algebras (GWAs), which appear in diverse areas of mathematics including mathematical physics, noncommutative algebra, and representation theory.

## Background

In the study of quantum mechanics, Heisenberg's Uncertainty Principle states that one cannot observe the position and momentum of a particle at the same time.

### Fundamental Equation of Quantum Mechanics

$$PQ - QP = \frac{i\hbar}{2\pi} I$$

(P and Q - square matrices of observables position and momentum, $\hbar$ - Planck constant, $I$ - identity matrix of appropriate size)

We study this equation and its generalizations from the viewpoint of algebra. Algebraically, the Fundamental Equation of QM may be expressed by the operators of multiplication and differentiation by $x$ on $\mathbb{C}[x]$. So,

$$\left(\frac{d}{dx}x\right)(p) - \left(x\frac{d}{dx}\right)(p) = p \quad \text{for any } p \in \mathbb{C}[x].$$

### Definition

A $\mathbb{C}$-*algebra* is a $\mathbb{C}$-vector space with an associative multiplication operation that distributes over addition. An *automorphism* of a $\mathbb{C}$-algebra $A$ is a bijective, structure-preserving function from $A$ to itself. We denote the group of automorphisms of $A$ by $\mathrm{Aut}(A)$.

Examples include polynomial rings and matrix rings. Another example is the Weyl algebra, which is the $\mathbb{C}$-algebra generated by $x$ and $y$ with the relation $yx - xy = 1$.

## Invariant Theory

Invariant theory is the study of algebraic or geometric properties of an object that are preserved under symmetry. Its applications reach throughout mathematics and physics. For example, a famous theorem of Noether states that every symmetry of an action on a physical system has a corresponding conservation law [8].

If $A$ is a $\mathbb{C}$-algebra and $G$ a subgroup of $\mathrm{Aut}(A)$, then the *ring of invariants of $A$ by $G$* is

$$A^G = \{a \in A : g(a) = a \text{ for all } g \in G\}.$$

An important problem in invariant theory is to determine the properties of the fixed ring of an algebra under a group of automorphisms.

### Shephard-Todd-Chevalley Theorem [9, 3]

Let $A = \mathbb{C}[x_1, \dots, x_n]$ and let $G$ be a finite group of linear automorphisms of $A$. Then the fixed ring $A^G$ is again a polynomial ring if and only if $G$ is generated by reflections.

Example: Let $A = \mathbb{C}[x, y]$ and $g \in \mathrm{Aut}\, A$ be given by $g(x) = y$ and $g(y) = x$. Then $A^G = \mathbb{C}[x + y, xy]$.

## Generalized Weyl algebras

### Definition [1]

Let $D$ be a commutative algebra, $\sigma \in \mathrm{Aut}(D)$, and $a \in D$, $a \neq 0$. The *generalized Weyl algebra* (GWA) $D(\sigma, a)$ is generated over $D$ by $x$ and $y$ subject to the relations

$$xd = \sigma(d)x, \quad yx = \sigma^{-1}(d)y, \quad yx = a, \quad xy = \sigma(a).$$

A GWA $R = D(\sigma, a)$ is *quantum* if $D = \mathbb{C}[h]$ or $\mathbb{C}[h^{\pm 1}]$ and $\sigma(h) = qh$, $q \in \mathbb{C}\backslash\{0, 1\}$.

There are several important families of algebras that can be constructed as GWAs, such as the Weyl algebras and primitive quotients of $U(\mathfrak{sl}_2)$. Examples of quantum GWAs include quantum Weyl algebras and primitive quotients of $U_q(\mathfrak{sl}_2)$.

## Fixed rings of quantum GWAs

Our goal is to develop a STC-like theorem for (quantum) GWAs. We study the structure and properties of the ring of invariants of quantum GWAs under finite automorphisms. The following result generalizes a theorem of Jordan and Wells [6].

### Theorem [4]

Let $D$ be an integral domain, let $R = D(\sigma, a)$ be a GWA, and let $\phi \in \mathrm{Aut}(R)$ with $|\phi| < \infty$. Suppose $\phi|_D$ restricts to an automorphism of D, $\phi(x) = \mu^{-1}x$, and $\phi(y) = \mu y$ for $\mu \in \mathbb{C}^\times$. Set $n = \mathrm{ord}(\phi|_D)$ and $m = \mathrm{ord}(\mu)$. If $\gcd(n, m) = 1$, then

$$R^{\langle\phi\rangle} = D^{\langle\phi\rangle}(\sigma^m, A) \quad \text{with} \quad A = \prod_{i=0}^{m-1} \sigma^{-i}(a).$$

Let $R = D(\sigma, a)$ be a quantum GWA with $a$ not a unit. Let $\gamma, \mu \in \mathbb{C}^\times$ such that $a(\gamma h) = a(h)$. Set $n = \mathrm{ord}(\gamma) < \infty$, $m = \mathrm{ord}(\mu) < \infty$. There is an automorphism $\eta_{\gamma,\mu}$ of $R$ such that

$$\eta_{\gamma,\mu}(h) = \gamma h, \quad \eta_{\gamma,\mu}(y) = \mu y, \quad \eta_{\gamma,\mu}(x) = \mu^{-1}x.$$

### Corollary [4]

If $\gcd(n, m) = 1$, then $R^{\langle\eta_{\gamma,\mu}\rangle}$ is again a quantum GWA.

Global dimension is a measure of regularity in algebras. For example, the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ has global dimension $n$. It is known that a quantum GWA either has global dimension 1, 2, or $\infty$ [2].

### Theorem [4]

Let $R = D(\sigma, a)$ be a quantum GWA and assume $\gcd(n, m) = 1$. The fixed ring $R^{\langle\eta_{\gamma,\mu}\rangle}$ has global dimension 1 if and only if $R$ does. If $R$ has global dimension 2 or $\infty$, then the fixed ring may have either global dimension 2 or $\infty$ depending on the parameters $n, m$ and the roots of $a$.

## Future Research

- Determine the structure and properties of fixed rings for other automorphisms on quantum GWAs.
- Study fixed rings for higher degree (quantum) GWAs.

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