Rigidity of quadratic Poisson algebras

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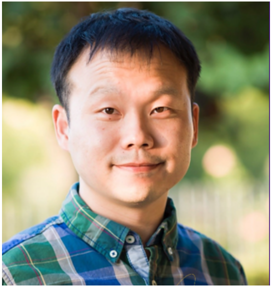

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This talk is based on work from three papers:

- "Reflection groups and rigidity of quadratic Poisson algebras" (G, Veerapen, Wang)
- "Cancellation and skew cancellation for Poisson algebras" (G, Wang, Yee)
- "The Zariski cancellation problem for Poisson algebras" (G, Wang)

Setup

Let $\mathbb{k}$ be a field which is algebraically closed and characteristic zero. All algebras are $\mathbb{k}$-algebras.

Definition

A Poisson algebra is an associative commutative algebra A equipped with a bracket $\{,\}$ such that

- $(A, \{,\})$ is a Lie algebra and,
- for each $a \in A$, $\{a, -\}$ is a derivation on A.

A Poisson homomorphism $\phi : A \to B$ between Poisson algebras A and B is an algebra homomorphism satisfying

$$\phi(\{a, a'\}_A) = \{\phi(a), \phi(a')\}_B \qquad \text{for all } a, a' \in A.$$

A Poisson automorphism is a bijective Poisson homomorphism $A \to A$.

Shephard-Todd-Chevalley

Recall that a reflection of a polynomial ring is a graded automorphism that fixes a codimension 1 subspace.

Theorem (Shephard-Todd-Chevalley Theorem)

The invariant ring $\Bbbk[x_1, \ldots, x_n]^G$ by a finite linear group G is polynomial if and only if G is generated by reflections.

If $A = \Bbbk[x_1, \ldots, x_n]$ has Poisson structure, then the invariant ring naturally has Poisson structure. Suppose $a, a' \in A^G$. If g is a Poisson automorphism of A, then

$$\{a, a'\} = \{g(a), g(a')\} = g(\{a, a'\}),$$

so $\{a, a'\} \in A^G$.

Question

Let $A = \Bbbk[x_1, \ldots, x_n]$ be a Poisson algebra. What properties/structures on A are preserved by taking invariants?

Shephard-Todd-Chevalley

Let $A = \Bbbk[x_1, \dots, x_n]$ be a Poisson algebra and consider the natural grading on A. We say A is quadratic if $\{x_i, x_j\} \in A_2$ for all i, j. A quadratic Poisson polynomial algebra is skew-symmetric if $\{x_i, x_j\} = q_{ij} x_i x_j$ for some scalars q_{ij} (with $q_{ij} = -q_{ji}$) for all i, j.

A Poisson reflection is simply a reflection that is also a (graded) Poisson automorphism.

Theorem (G, Veerapen, Wang)

Let $A = \Bbbk[x_1, \dots, x_n]$ be a skew-symmetric Poisson algebra and let G be a finite subgroup of graded Poisson automorphisms of A. Then A^G has skew-symmetric Poisson structure if and only if G is generated by Poisson reflections.

This is a natural analogue of a theorem for skew polynomial algebras by Kirkman, Kuzamonovich, and Zhang.

If A^G has skew-symmetric Poisson structure, then A^G is a polynomial ring and so by the STC, G is generated by reflections, which are necessarily Poisson reflections. The other direction is harder.

Shephard-Todd-Chevalley

Example

Let $A = \Bbbk[x, y]$ with $\{x, y\} = pxy$ $(p \neq 0)$. Let g be a Poisson automorphism. Then

$$g(x) = \mu x, g(y) = \nu y \qquad \text{or} \qquad g(x) = \mu y, g(y) = \nu x,$$

for some scalars $\mu, \nu \in \Bbbk^{\times}$. But the second one is impossible since it would give

$$\mu\nu pxy = g(pxy) = g(\{x, y\}) = \{g(x), g(y)\} = \{\mu y, \nu x\} = -\mu\nu pxy.$$

Now if g is a reflection, then either $\mu = 1$ or $\nu = 1$, and the other is an mth root of unity.

Suppose $\nu = 1$. Then the invariant ring is $\Bbbk[x^m, y]$ and the Poisson bracket is

$$\{x^m, y\} = mx^{m-1}\{x, y\} = mpx^m y.$$

So, the invariant ring has skew-symmetric Poisson structure.

Block decomposition

Let $A = \mathbb{k}[x_1, \ldots, x_n]$ be a skew-symmetric Poisson algebra such that $\{x_i, x_j\} = q_{ij} x_i x_j$. Set $[n] = \{1, 2, \ldots, n\}$.

For $i \in [n]$, the block of i is

$$B_i = \{i' \in [n] : q_{ik} = q_{i'k} \text{ for all } k \in [n]\}.$$

We define an equivalence relation on $[n]$ using blocks ($i \sim j$ if and only if $B_i = B_j$).

Let $W \subset [n]$ be a complete collection of distinct representatives of the relation. The corresponding block decomposition of $[n]$ is

$$[n] = \bigcup_{i \in W} B_i.$$

Block decomposition

Let $A = \Bbbk[x_1, \dots, x_n]$ be a skew-symmetric Poisson algebra such that $\{x_i, x_j\} = q_{ij}x_ix_j$.

Let $w \in W$. Let $\mathrm{GrAut}_P^w(A)$ be the subgroup of $\mathrm{GrAut}_P(A)$ consisting of (graded) automorphisms θ satisfying $\theta(x_s) = x_s$ for all $s \notin B_w$, and $\theta(x_s) \in \bigoplus_{i \in B_w} \Bbbk x_i$ for all $s \in B_w$. Every Poisson reflection belongs to $\mathrm{GrAut}_P^w(A)$ for some $w \in W$.

Suppose G is a subgroup of $\mathrm{GrAut}(A)$ generated by Poisson reflections. Set

$$G_w = G \cap \mathrm{GrAut}_P^w(A).$$

Each generator of G belongs to some G_w, each G_w is generated by Poisson reflections, and

$$G = \prod_{w \in W} G_w.$$

Consider $w = 1$ so $B_w = \{1, \dots, k\}$. Then $x_1, \dots, x_k$ generate a Poisson subalgebra of A, denoted A_w. We have $A^{G_w} = A_w^{G_w}[x_{k+1}, \dots, x_n]$. By STC, $A_w^{G_w} = \Bbbk[u_1, \dots, u_k]$ with u_i homogeneous in the $x_1, \dots, x_k$. If u_i has degree d_i, then

$$\{u_i, u_j\} = 0 \quad \text{and} \quad \{u_i, x_j\} = d_i q_{1j} u_i x_j \text{ for } j > k.$$

Now we use the fact that if $w' \in W$ with $B_{w'} \neq B_w$, then $G_{w'}$ and G_w commute so $G_{w'}$ acts on A^{G_w}.

Rigid Poisson algebras

Example

Let $A = \Bbbk[x, y]$ with $\{x, y\} = pxy$ $(p \neq 0)$.

Let g be a Poisson automorphism defined by $g(x) = \mu x$ and $g(y) = y$, where μ is a primitive mth root of unity, $m > 1$. The invariant ring is $A^{\langle g \rangle} = \Bbbk[X, Y]$ (where $X = x^m$ and $Y = y$) with bracket $\{X, Y\} = mpXY$.

Then $A \ncong A^{\langle g \rangle}$, which can be shown easily using the following theorem:

Theorem (G, Wang)

Let A and B be two connected graded Poisson algebras finitely generated in degree one. If $A \cong B$ as (ungraded) Poisson algebras, then $A \cong B$ as graded Poisson algebras.

The predecessor to this theorem is a result for associative algebras by Bell and Zhang.

Conjecture

Let $A = \Bbbk[x_1, \ldots, x_n]$ be a skew-symmetric Poisson algebra and G a finite subgroup of $\mathrm{GrAut}_P(A)$. *Then $A^G \cong A$ implies G is trivial.*

Rigid Poisson algebras

Definition

Let A be a (graded) Poisson algebra and let G be a finite subgroup of the (graded) Poisson automorphisms of A. We say A is (graded) rigid if $A^G \cong A$ as Poisson algebras implies that G is trivial.

We have several additional examples of rigid Poisson algebras.

Example

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. There is a natural Poisson structure on $S(\mathfrak{g})$ obtained by setting

$$\{x, y\} = [x, y] \qquad \text{for all } x, y \in \mathfrak{g}.$$

We call this the Kostant-Kirillov bracket. This is not a quadratic structure, but we can homogenize by introducing a Poisson central element variable t (so the associative structure is $S(\mathfrak{g})[t]$) with relations

$$\{x, y\} = [x, y]t \qquad \text{for all } x, y \in \mathfrak{g}.$$

If $\mathfrak{g}$ has no 1-dimensional Lie ideal, then this homogenized algebra is graded rigid.

Rigid Poisson algebras

Let M_n denote the ring of $n \times n$ matrices for some $n \geq 2$. The Poisson bracket on the polynomial ring $\mathcal{O}(M_n) = \Bbbk[x_{ij}]_{1 \leq i,j \leq n}$ is given by

$$\{x_{im}, x_{j\ell}\} = 0, \quad \{x_{i\ell}, x_{im}\} = x_{i\ell}x_{im}, \quad \{x_{i\ell}, x_{j\ell}\} = x_{i\ell}x_{j\ell}, \quad \{x_{i\ell}, x_{jm}\} = 2x_{im}x_{j\ell}$$

with $i < j$ and $\ell < m$. This Poisson bracket can be realized as the semiclassical limit of the family of $n \times n$ quantum matrices $\{\mathcal{O}_q(M_n)\}$ for $q \in \Bbbk^\times$.

When $n > 2$, one can show that the Poisson algebra $\mathcal{O}(M_n)$ has no Poisson reflections.

In the case $n = 2$, the Poisson reflections are certain scalar automorphisms and one can compute the invariant ring directly and show that the invariant ring is not isomorphic to $\mathcal{O}(M_n)$.

Hence, for all n, $\mathcal{O}(M_n)$ is graded rigid.

Rigid Poisson algebras

Consider the nth Weyl Poisson algebra $\mathcal{P}_n = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n]$ with Poisson bracket

$$\{x_i, y_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = \{y_i, y_j\} = 0.$$

(This is $\mathbb{Z}$-graded but not $\mathbb{N}$-graded.)

The Weyl Poisson algebra $\mathcal{P}_n$ is rigid by a result of Tikaradze.

Consider the homogenized Weyl Poisson algebra $\mathcal{H}_n = \Bbbk[x_1, \dots, x_n, y_1, \dots, y_n, z]$ with Poisson bracket

$$\{x_i, y_j\} = \delta_{ij} z^2, \qquad \{x_i, x_j\} = \{y_i, y_j\} = \{z, -\} = 0.$$

Again, one can compute the Poisson reflections of $\mathcal{H}_n$ directly. In this case, the only nontrivial Poisson reflection groups are given by $G = \langle g \rangle$ for some order 2 automorphism g. In this case, $\mathcal{H}_n^G \ncong \mathcal{H}_n$, so $\mathcal{H}_n$ is graded rigid.

Unimodular Poisson algebras

Let $A = \Bbbk[x_1, \ldots, x_n]$ be a Poisson algebra with bracket $\{-,-\}$. The modular derivation of A is given by

$$\phi_\eta(f) := \sum_{j=1}^{n} \frac{\partial\{f, x_j\}}{\partial x_j} \qquad \text{for all } f \in A.$$

We say A is unimodular if $\phi_\eta = 0$.

The unimodularity condition is closely connected with the Calabi–Yau condition for associative algebras. In particular, A is unimodular if and only if its Poisson enveloping algebra $U(A)$ is Calabi–Yau.

Let $A = \Bbbk[x, y]$ be a Poisson algebra. Then A is unimodular if and only if A is the Weyl Poisson algebra.

Let $A = \Bbbk[x, y, z]$ be a Poisson algebra. Then A is unimodular if and only if there exists a nonzero $f \in A$ (called the potential) such that the bracket is given by

$$\{x, y\} = \frac{\partial}{\partial z} f, \quad \{y, z\} = \frac{\partial}{\partial x} f, \quad \{z, x\} = \frac{\partial}{\partial y} f.$$

Unimodular Poisson algebras

Let $A = \Bbbk[x, y, z]$ be a Poisson algebra with potential

$$f_{p,q} := \frac{p}{3}(x^3 + y^3 + z^3) + qxyz, \quad p, q \in \Bbbk.$$

Generically, A has no Poisson reflections and is then trivially graded rigid.

If A has Poisson reflections, then A is Poisson isomorphic to a Poisson algebra with potential $f_{0,q}$. In this case, A has skew-symmetric structure and we can compute the invariant ring by Poisson reflections explicitly. Again, we obtain $A^G \cong A$ implies G is trivial, so A is graded rigid.

Theorem (Chengyuan Ma)

Let $A = \Bbbk[x_1, x_2, x_3]$ be a unimodular quadratic Poisson algebra and let G be a finite subgroup of Poisson automorphisms. Then A is a graded rigid.

Zariski cancellation

The Zariski Cancellation Problem

Let Y be an affine variety. Does an isomorphism $Y \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ imply an isomorphism $Y \cong \mathbb{A}^n$?

Generalizations

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ \begin{array}{c} Y \text{ a noncommutative} \\ \text{algebraic variety} \end{array} & & \begin{array}{c} Y \text{ an affine} \\ \text{Poisson variety} \end{array} \end{tikzcd}$$

Zariski cancellation

Now we consider a different type of rigidity.

Definition

An algebra A is cancellative if an isomorphism $A[x] \cong B[x]$ implies $A \cong B$ for any algebra B.

Cancellation Question

Under what conditions is an algebra A cancellative?

Example

(1) The polynomial rings $\mathbb{k}[x]$ and $\mathbb{k}[x, y]$ are cancellative over any field. (Abhynankar, Eakin, Heinzer) (Fujita) (Russell) (Crachiola, Makar-Limanov) (Bhatwadekar, Gupta)

(2) The polynomial ring $\mathbb{k}[x, y, z]$ is not cancellative for a field $\mathbb{k}$ of positive characteristic. (Asanuma) (Gupta)

The question in the case of zero characteristic is still open.

Noncommutative Zariski cancellation

Bell and Zhang initiated a study of the Zariski cancellation problem in the noncommutative setting.

Example

The following are cancellative assuming the algebra is affine.

(1) Noncommutative domains of Gelfand-Kirillov (GK) dimension two. (Bell, Zhang)

(2) Domains of GK dimension one over $\mathbb{k}$. (Bell, Hamidizadeh, Huang, Venegas)

(3) Path algebras of finite quivers. (Lezama, Wang, Zhang)

(4) Noetherian prime algebras of GK dimension three that are not PI. (Tang, Venegas, Zhang)

Poisson Zariski cancellation

Poisson Cancellation Question

Under what conditions is a Poisson algebra A Poisson cancellative?

Many of the results for noncommutative algebras have analogues in the Poisson setting. However, there is no formal mechanism to translate between the two.

Theorem (G, Wang)

Let A be a Poisson algebra with trivial Poisson center. Then A is Poisson cancellative.

Proof.

Suppose $\phi : A[t] \to B[t]$ is a Poisson algebra isomorphism for some Poisson algebra B. Since $\mathcal{Z}_P(A) = \Bbbk$, then $\mathcal{Z}_P(A[t]) = \Bbbk[t]$ and $\mathcal{Z}_P(B[t]) = \mathcal{Z}_P(B)[t]$. But ϕ restricts to an isomorphism of Poisson centers so $\Bbbk[t] \cong \mathcal{Z}_P(B)[t]$. Thus, $\mathcal{Z}_P(B)$ is a domain with $\mathrm{Kdim}\mathcal{Z}_P(B) = 0$. It follows that $\mathcal{Z}_P(B)$ is a field, so $\mathcal{Z}_P(B) = \Bbbk$. Now if $I = (t)$ in A, then ϕ maps I to $\phi(I)$ and so

$$A \cong A[t]/I \cong B[t]/\phi(I) \cong B \qquad \square.$$

(Actually more is true: A is universally Poisson cancellative.)

Poisson discriminants

Goal

Show that quadratic Poisson algebras on $\Bbbk[x, y, z]$ with nontrivial bracket are cancellative.

Lemma

If A is an affine Poisson domain with nontrivial bracket and $\operatorname{Kdim} A = 3$, *then* $\operatorname{Kdim} \mathcal{Z}_P(A) \leq 1$. *If in addition, $\mathcal{Z}_P(A)$ is connected graded and $\mathcal{Z}_P(A) \ncong \Bbbk[t]$, then A is Poisson cancellative.*

A consequence of this lemma is that we need only consider the case $\mathcal{Z}_P(A) = \Bbbk[t]$.

Definition

For a property $\mathcal{P}$, the $\mathcal{P}$-discriminant ideal of A, denoted $I_{\mathcal{P}}(A)$ is the intersection of all $\mathfrak{m} \in \operatorname{Maxspec}(\mathcal{Z}_P(A))$ such that $A/\mathfrak{m}A$ does not have property $\mathcal{P}$.

In the special case such that $I_{\mathcal{P}}(A) = (d)$ is a principal ideal of $\mathcal{Z}_P(A)$, we call d the $\mathcal{P}$-discriminant of A.

When $\mathcal{Z}_P(A)$ is a domain, the $\mathcal{P}$-discriminant is unique up to a unit of $\mathcal{Z}_P(A)$.

Poisson discriminants

Lemma

Let A be a noetherian connected graded Poisson domain that is generated in degree 1. Assume $\mathcal{Z}_P(A) = \Bbbk[t]$ for some homogeneous element $t \in A$ of positive degree. Then A is Poisson cancellative in either of the following cases:

(1) The $\mathcal{P}$-Poisson discriminant is t for some property $\mathcal{P}$ of A.

(2) We have $\operatorname{gldim} A/(t) = \infty$ and either $\operatorname{gldim} A/(t-1) < \infty$ or $\operatorname{gldim} A < \infty$.

We may now assume $\mathcal{Z}_P(A) = \Bbbk[t]$ and $\operatorname{gldim} A/(t) < \infty$, so $A = \Bbbk[x, y, t]$ with $t \in \mathcal{Z}_P(A)$ and $\{x, y\} = f \in A_2$.

Some Poisson classification problems

Theorem (G, Wang, Yee)

(I) Let $A = \Bbbk[x, y]$ be a Poisson algebra such that $\{x, y\} = f$ with $f \in A_{\leq 2}$. Then up to a change of variables, the possibilities for f are

(1) $f = 0$, *(4a)* $f = x^2$, *(5a)* $f = \lambda xy$ *with* $\lambda \in \Bbbk^\times$,

(2) $f = 1$, *(4b)* $f = x^2 + 1$, *(5b)* $f = \lambda xy + 1$ *with* $\lambda \in \Bbbk^\times$.

(3) $f = x$,

Moreover, the Poisson algebras determined by f above are pairwisely nonisomorphic with the exception of replacing λ by $-\lambda$ in (5a) and (5b).

(II) Let $A = \Bbbk[x, y, t]$ be an $\mathbb{N}$-graded Poisson algebra with t Poisson central and $\{x, y\} = f \in A_2$. Then up to a change of variables, the possibilities for f are

(1) $f = 0$, *(4a)* $f = x^2$, *(5a)* $f = \lambda xy$ *with* $\lambda \in \Bbbk^\times$,

(2) $f = t^2$, *(4b)* $f = x^2 + t^2$, *(5b)* $f = \lambda xy + t^2$ *with* $\lambda \in \Bbbk^\times$.

(3) $f = xt$, *(4c)* $f = x^2 + yt$,

Moreover, the Poisson algebras determined by f above are pairwisely nonisomorphic with the exception of replacing λ by $-\lambda$ in (5a) and (5b).

Poisson cancellation

We now consider the cases (2)-(5) from part (II) of the previous theorem. Set

$$\mathfrak{m}_\alpha = (t-\alpha) \in \mathrm{Maxspec}(\mathcal{Z}_P(A)).$$

In cases (2,3,4b), we find a property $\mathcal{P}$ such that the $\mathcal{P}$-discriminant is t.

(2) $f = t^2$. Let $\mathcal{P}$ be the property that $A/\mathfrak{m}_\alpha A$ is Poisson simple.

(3) $f = xt$. Let $\mathcal{P}$ be the property that $A/\mathfrak{m}_\alpha A$ does not have trivial Poisson bracket.

(4b) $f = x^2 + t^2$. Let $\mathcal{P}$ be the property that $A/\mathfrak{m}_\alpha A$ is not isomorphic to $A/\mathfrak{m}_0 A$.

We handle case (4a) differently.

(4a) $f = x^2$. In this case, $A \cong A' \otimes \Bbbk[t]$ where $A' = \Bbbk[x,y]$ with Poisson bracket $\{x,y\} = x^2$. Since A' is cancellative and $\Bbbk[t]$ is cancellative, then A is Poisson cancellative.

Cases (4c) and (5b) are similar to (4b), while (5a) is similar to (4a).

Theorem (G, Wang, Yee)

Let A be a quadratic polynomial Poisson algebra on $\Bbbk[x,y,z]$ with nontrivial bracket, then A is Poisson cancellative.

Poisson cancellation

We are able to show some additional families of Poisson algebras are Poisson cancellative:

• A is a noetherian connected graded Poisson domain generated in degree 1 and a graded isolated singularity with Kdim $\mathcal{Z}_P(A) \leq 1$.

• The dth Veronese algebra $A^{(d)}$, $d \geq 1$, of a quadratic polynomial Poisson algebra on three variables.

• The Poisson algebra $PS(\mathfrak{g})$ with Kostant-Kirillov bracket where $\mathfrak{g}$ is a non-abelian Lie algebra of dimension ≤ 3.

Thank You!