Calculus III
Chapter 12 - Vectors and the Geometry of Space

1. Three-Dimensional Coordinate System

Third-semester calculus is the study of functions of more than one variable. Much of what we do will be to generalize concepts from your previous two semesters. However, multivariable calculus will soon take on a life of its own and there are many instances where the picture is significantly more complicated than in single variable calculus. Nevertheless, you will often be able to rely on intuition you have built up in previous courses.

Initially, we will focus on building up the geometry of three-dimensional space. Much of what we do will extend beyond this, but pictures are harder to come by.

The three-dimensional coordinate system is formed by three coordinate axes labeled the x-axis, y-axis, and z-axis, all pairwise perpendicular. The origin, denoted O, is the intersection of all three axes. In order to standardize direction, we use the right-hand rule.

Right-hand rule: Extend your right hand, with fingers curled and thumb pointed up. Your fingers align with the positive direction of the x-axis, your arm with the positive direction of the y-axis, and your thumb with the positive direction of the z-axis.

These axes form three planes, labeled the xy-plane, xz-plane, and yz-plane. These planes in turn divide space into eight octants. We will not concern ourselves with numbering the eight octants, except for the first octant which is bounded by the positive part of each plane.

A point P in three-dimensional space can be represented by an ordered triple (a, b, c), indicating the distance from the origin O in the x, y, and z direction, respectively. Hence, O is represented by $(0, 0, 0)$. This gives a one-to-one bijection between points in space and ordered triples in the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) : x, y, z \in \mathbb{R}\}$.

Given a point $P(a, b, c)$, we can drop a perpendicular to the xy-plane[1]. This perpendicular intersects the xy-plane at the point $Q(a, b, 0)$. We call this point the projection of P onto the xy-plane. One can project similarly onto the xz- and yz-planes.

[1] What this means precisely will be defined later. Roughly, it means that we draw a line from P through the xy-plane such that it is perpendicular to every line that lies on that plane.

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ (i.e., a function of n-variables), the graph of f is the set of points P such that $f(P) = 0$. When $n = 3$, we call this graph a surface.

Example 1. (1) The surface represented by $z - 3$ is a plane parallel to the xy-plane through the point $(0, 0, 3)$.

(2) The surface represented by $x^2 + y^2 - 1$ is a cylinder of radius 1, perpendicular to the xy-plane.

The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, denoted $|P_1P_2|$, is the length of the line segment whose endpoints are P_1 and P_2. To find this distance, we form a box whose opposite endpoints are P_1 and P_2. Set two other points on this box, $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$. Applying the Pythagorean Theorem (twice) gives

$$\begin{aligned}
|P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2 \quad \text{(PT)} \\
&= |P_1A|^2 + |AB|^2 + |BP_2|^2 \quad \text{(PT)} \\
&= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \quad \text{(distance formula on real line).}
\end{aligned}$$

Hence, we have now derived the following.

Distance Formula in Three Dimensions: The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

A sphere of radius $r > 0$ centered at $C(h, k, \ell)$ is the set of points whose distance from (h, k, ℓ) is r. Thus, it follows from the distance formula that the equation of a sphere is

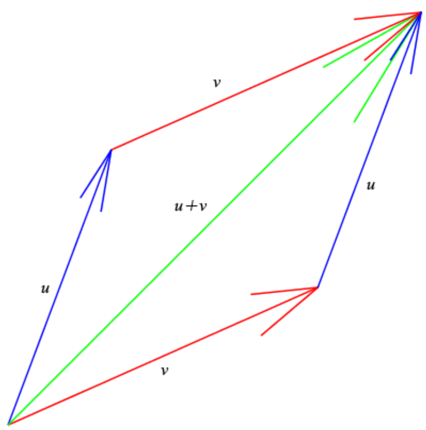
$$(x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2.$$

2. Vectors

A vector is, informally (for now), a quantity that has magnitude and direction. We will formalize this eventually, for now we think of a vector $\mathbf{v}$ as a ray, having an initial point A (the tail) and a terminal point B (the tip), denoted $\mathbf{v} = \overrightarrow{AB}$. The length of $\mathbf{v}$ is the distance $|AB|$. We say that two vectors $\mathbf{u}$ and $\mathbf{v}$ are equivalent if they have the same magnitude and direction, even if they are in different positions, and write $\mathbf{u} = \mathbf{v}$. The zero vector, denoted by $\mathbf{0}$, is the unique vector with length zero and no direction.

Definition of Vector Addition: If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so that the initial point of $\mathbf{v}$ is the terminal point of $\mathbf{u}$, then the sum of $\mathbf{u}$ and $\mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

Hence, if $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$, then $\mathbf{u}+\mathbf{v} = \overrightarrow{AC}$. Note that vector addition is commutative, as can be seen by the parallelogram law:



Definition of Scalar Multiplication: If c is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and opposite if $c < 0$. If $c = 0$, then $c\mathbf{v} = \mathbf{0}$.

Two vectors are said to be parallel if they are scalar multiples of one another. We call $-\mathbf{v} = (-1)\mathbf{v}$ the negative of $\mathbf{v}$. By the difference of vectors $\mathbf{u}$ and $\mathbf{v}$ we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}.$$

If we fix a coordinate system, we can denote a vector by its terminal point P, fixing its initial point to be the origin (since position doesn't matter). Of course, a vector may have many representations, i.e., different initial and terminal points. We call this one the position vector of the point P. The coordinates for this point are called the components of the vector. In two-dimensions, we write $\mathbf{a} = \langle a_1, a_2 \rangle$, and in three-dimensions, we write $\mathbf{a} =$

$\langle a_1, a_2, a_3\rangle$. We denote by V_n the set of all n-dimensional vectors. So if $\mathbf{a} \in V_n$, then $\mathbf{a} = \langle a_1, a_2, \ldots, a_n\rangle$. Given any representation $\overrightarrow{AB}$ of the vector $\mathbf{a}$, where $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the position vector has coordinates

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle.$$

The length (or magnitude) of the vector $\mathbf{v}$ is the length of any representation and is denoted by $|\mathbf{v}|$. Hence, the length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2\rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$. The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Given components, it is straightforward to find the sum, difference, and scalar multiple of vectors. Let $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{v} = \langle b_1, b_2\rangle$, and c a scalar. Then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle, \quad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2\rangle, \quad c\mathbf{a} = \langle ca_1, ca_2\rangle.$$

This extends in the obvious way to three or more dimensions. Note that, $|c\mathbf{a}| = |c||\mathbf{a}|$. This can be verified directly,

$$|c\mathbf{a}| = \sqrt{(ca_1)^2 + (ca_2)^2} = \sqrt{c^2(a_1^2 + a_2^2)} = |c||\mathbf{a}|.$$

Caution: It is **not** true that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$.

Properties of Vectors For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V_n$ and all scalars c, d,

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	(v) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
(ii) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$	(vi) $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
(iii) $\mathbf{a} + \mathbf{0} = \mathbf{a}$	(vii) $c(d\mathbf{a}) = (cd)\mathbf{a}$
(iv) $\mathbf{a} + (-\mathbf{a}) = \mathbf{a} + (-1)\mathbf{a} = 0$	(viii) $1\mathbf{a} = \mathbf{a}$

Another way to represent vectors is as sums (linear combinations) of the **standard basis vectors**. In V_2, these are the vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. In V_3, these are the vectors

$$\mathbf{i} = \langle 1, 0, 0\rangle, \quad \mathbf{j} = \langle 0, 1, 0\rangle, \quad \mathbf{k} = \langle 0, 0, 1\rangle.$$

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, then $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Of course, it is important to know whether you are working in V_2 or V_3.

A unit vector is a vector whose length is 1. Hence, the standard basis vectors are, by definition, unit vectors. In general, if $\mathbf{a} \neq 0$, the the unit vector in the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a}.$$

We can check this easily. Let $c = 1/|\mathbf{a}|$. Then $|\mathbf{u}| = |c\mathbf{a}| = |c||\mathbf{a}| = |\mathbf{a}|$.

Application. A 100-lb weight hangs from two wires as shown in Figure 19 on page 804. Find the tensions (forces) T_1 and T_2 in both wires and the magnitudes of the tensions.

This is left mostly as a reading exercise. We will walk through some of the initial steps.

We regard T_1 and T_2 as vectors in V_2. Draw a line through the common initial point parallel to the "top" of the triangle and drop perpendiculars from the terminal points of both vectors to this line. This gives two right triangles with hypotenuses $|T_1|$ and $|T_2|$, respectively. Hence, using right triangle trigonometry, we have

$$\begin{aligned} T_1 &= -|T_1|\cos 50°\mathbf{i} + |T_1|\sin 50°\mathbf{j} \\ T_2 &= |T_2|\cos 32°\mathbf{i} + |T_2|\sin 32°\mathbf{j}. \end{aligned}$$

The resultant force experience by an object is the vector sum of the forces acting on it. In this case, the resultant $T_1 + T_2$ of the tensions counterbalances the weight $\mathbf{w} = -100\mathbf{j}$ and so we have $T_1 + T_2 = -\mathbf{w} = 100\mathbf{j}$. This equation must hold *in each component.* Hence we have the system of equations,

$$\begin{aligned} -|T_1|\cos 50° + |T_2|\cos 32° &= 0 \\ |T_1|\sin 50° + |T_2|\sin 32° &= 100. \end{aligned}$$

We can solve this system using elementary means[2]. We obtain

$$|T_1| \approx 85.64\text{lb} \quad \text{and} \quad |T_2| = 64.9\text{lb}.$$

[2]Students who have taken Linear Algebra should be able to solve this very quickly.

3. The Dot Product

Our discussion of vectors so far included no multiplication-type operation. One way to multiply vectors is using the dot product that is related to the angle between two vectors.

Let $\mathbf{a}, \mathbf{b} \in V_n$ and write $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$, $\mathbf{b} = \langle b_1, \ldots, b_n \rangle$. The dot product of $\mathbf{a}$ and $\mathbf{b}$, denoted by $a \cdot b$, is a scalar[3] (not an element of V_n) given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + \cdots + a_n b_n.$$

Example 2. Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 5\mathbf{k}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \langle 2, -3, 1 \rangle \cdot \langle 1, 0, -5 \rangle = (2)(1) + (-3)(0) + (1)(-5) = -3.$$

The following properties are easy to check directly for V_2 and V_3.

Properties of the Dot Product For all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V_n$ and all scalars c,

(i) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ (iii) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) + (\mathbf{b} \cdot \mathbf{c})$ (v) $\mathbf{0} \cdot \mathbf{a} = 0$

(ii) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (iv) $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

Theorem 3. Let A, B be points such that the angle θ between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ satisfies $0 \leq \theta \leq \pi$. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors for A and B, respectively, then

$$\mathbf{a} \cdot \mathbf{b} = |a| \cdot |b| \cos\theta.$$

Proof. Apply the Law of Cosines to the triangle ΔOAB,

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta.$$

In terms of the position vectors, this is

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta. \tag{1}$$

Using the properties of the dot product, we have

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2. \tag{2}$$

Putting (1) and (2) together we arrive at the desired result. □

Example 4. Suppose $\mathbf{a} = \langle 2, 1 \rangle$ and $\mathbf{b} = \langle 1, 3 \rangle$. Then

$$5 = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = \sqrt{5}\sqrt{10}\cos\theta.$$

Hence, $\cos\theta = \frac{1}{\sqrt{2}}$, and so $\theta = \pi/4$.

[3]Formally, we would say that $\cdot$ is a function $V_n \times V_n \to \mathbb{R}$.

Two nonzero vectors are perpendicular (or orthogonal) if the angle between them is $\pi/2$, in which case $\cos(\pi/2) = 0$ and so $\mathbf{a} \cdot \mathbf{b} = 0$. In general, we say two vectors are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 5. The vectors $3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ are orthogonal.

The direction angles of a nonzero vector $\mathbf{a} \in V_3$ are the angles $\alpha, \beta, \gamma \in [0, \pi]$ that $\mathbf{a}$ makes with the positive x-, $y-$, and $z-$ axes, respectively. The cosines of these angles are called the direction cosines of the vector $\mathbf{a}$. We have

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}.$$

Similarly,

$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\mathbf{a}|}.$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ and $\mathbf{a} = |\mathbf{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$. It now follows that

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle.$$

Let $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ be representations of two vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. We drop a perpendicular from R to $\overrightarrow{PQ}$ and call the foot of the perpendicular S. The vector $\overrightarrow{PS}$ is called the vector projection of $\mathbf{b}$ onto $\mathbf{a}$, denoted $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is defined as the signed magnitude of $\operatorname{proj}_{\mathbf{a}} \mathbf{b}$, denoted $\operatorname{comp}_{\mathbf{a}} \mathbf{b}$ (for the component of $\mathbf{b}$ along $\mathbf{a}$).

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then using right triangle trigonometry we have

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = |\mathbf{b}|\cos\theta = |\mathbf{b}|\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$

The vector projection is now the vector of magnitude $\operatorname{comp}_{\mathbf{a}} \mathbf{b}$ in the direction of $\mathbf{a}$, so

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

Example 6. Find the scalar and vector projections of $\mathbf{b} = \langle 2, 3\rangle$ onto $\mathbf{a} = \langle 1, 4\rangle$.

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{14}{3} \qquad \operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} = \frac{14}{9}.$$

Application. Suppose a wagon is pulled a distance of 100m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.

When a constant force F moves an object through a distance d along the line of motion of the object, the work done by the force is defined as $W = Fd$. Now suppose the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$. If the force moves the object from P to Q, then the displacement vector is $\mathbf{D} = \overrightarrow{PQ}$ and the work done by this force is defined to be the component of the force along $\mathbf{D}$ and the distance moved:

$$W = (|\mathbf{F}|\cos\theta)|\mathbf{D}| = \mathbf{F}\cdot\mathbf{D}.$$

In the example, we have

$$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos 35^\circ = (70)(100)\cos 35^\circ \approx 5734N\cdot m = 5734J.$$

4. The Cross Product

The cross product of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is defined as the vector $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. Thus, $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$. Writing this out we have

$$\begin{aligned} a_1c_1 + a_2c_2 + a_3c_3 &= 0 \\ b_1c_1 + b_2c_2 + b_3c_3 &= 0. \end{aligned}$$

Solving this system for the c_i we obtain

$$\mathbf{a} \times \mathbf{b} = \langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

This definition, though correct, can be difficult to remember. To simplify, we use determinants.

The determinant of order 2 is defined as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of order 3 is define as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

In terms of the standard basis vectors, the cross product can then be expressed as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Here's another method for computing the determinant, called the *butterfly method.* We rewrite the first and second column to the right of the matrix. We draw diagonals going down (these are the positive summands) and then draw diagonals going up (these are the negative summands). The result is the determinant of the matrix.

Example 7. Find the cross product of $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$. We want the determinant of the matrix,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -5 \\ 1 & 6 & 3 \end{vmatrix}.$$

We will use the butterfly method to compute the determinant. The diagonals going up are colored below:

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 3 & 2 & \text{-}5 & 3 & 2 \\ 1 & 6 & 3 & 1 & 6 \end{array}\right. = 6\mathbf{i} - 5\mathbf{j} + 18\mathbf{k}.$$

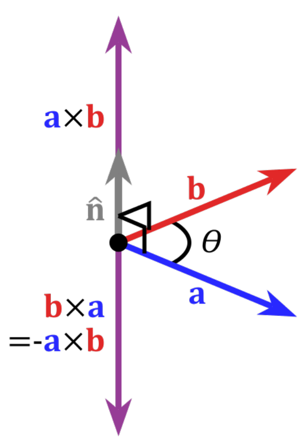
The diagonals going down are colored below:

$$\left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 3 & 2 & \text{-}5 & 3 & 2 \\ 1 & 6 & 3 & 1 & 6 \end{array}\right. = -30\mathbf{i} + 9\mathbf{j} + 2\mathbf{k}.$$

Thus, the cross product (determinant) is

$$(6\mathbf{i} - 5\mathbf{j} + 18\mathbf{k}) - (-30\mathbf{i} + 9\mathbf{j} + 2\mathbf{k}) = 36\mathbf{i} - 14\mathbf{j} + 16\mathbf{k}.$$

Suppose $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} \times \mathbf{b}$ are represented by vectors with the same endpoint. Then $\mathbf{a} \times \mathbf{b}$ points in a direction perpendicular to the plane through $\mathbf{a}$ and $\mathbf{b}$. The direction is determined by the right-hand rule. Extend your right hand, thumb extended, with your fingers curled in the direction of the *angle from* $\mathbf{a}$ *to* $\mathbf{b}$. Then your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.



Theorem 8. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors such that the angle θ between them satisfies $0 \leq \theta \leq \pi$. Then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$.

Proof. A tedious computation using the original definition of the cross product (see page 817 in the text) shows that

$$|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2.$$

Now

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 &= |\mathbf{a}|^2|\mathbf{b}|^2-(\mathbf{a}\cdot\mathbf{b})^2\\ &= |\mathbf{a}|^2|\mathbf{b}|^2-|\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta\\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta)\\ &= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta.\end{aligned}$$

The result now follows because $\sin\theta \geq 0$ for all $0 \leq \theta \leq \pi$, so $\sqrt{\sin^2\theta} = \sin\theta$. □

Corollary 9. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a}\times\mathbf{b}=\mathbf{0}$.

Proof. Two vectors are parallel if and only if $\theta = 0$ or π if and only if $\sin\theta = 0$. □

Let $\mathbf{a}$ and $\mathbf{b}$ represent two vectors with the same initial point and let θ, $0\leq\theta\leq\pi$ be the angle between them. Form a parallelogram with base $|\mathbf{a}|$ and altitude $|\mathbf{b}|\sin\theta$. Thus, the area of the parallelogram is

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a}\times\mathbf{b}|.$$

Thus, the magnitude of the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the area of the parallelogram determined by these vectors.

Properties of the Dot Product For all $\mathbf{a},\mathbf{b},\mathbf{c}\in V_3$ and all scalars c,

(i) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

(ii) $(c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(c\mathbf{b})$

(iii) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})$

(iv) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = (\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})$

(v) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

(vi) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

The product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is called the scalar triple product of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. This can be written as the determinant

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}.$$

Consider the parallelepiped determined by these vectors. The area of the base parallelogram is $A = |\mathbf{b}\times\mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, then the height of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. Thus, the volume of the parallelepiped is

$$V = Ah = |\mathbf{b}\times\mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|.$$

Note that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$ implies that the vectors are all coplanar.

Application. A bolt is tightened by applying a 40-N force to a 0.25-m wrench. Let θ be the angle between the position and force vectors, $\mathbf{r}$ and $\mathbf{F}$, respectively, and suppose θ measures $75°$. Let τ be the torque (relative to the origin) defined to be the cross product of the position and force vectors, so $\tau = \mathbf{r} \times \mathbf{F}$. Hence, in this example, we have

$$|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin 75° = (0.25)(40) \sin 75° \approx 9.66 N \cdot m.$$

5. Equations of lines and planes

For points in the Cartesian (xy-)plane, a line is determined by a point and slope. Really what the slope is telling us is the direction of the line. In three dimensions, we need the same information, but we express it in terms of vectors.

Let L be a line in three-dimensional space and let $\mathbf{v}$ be a vector parallel to L. Let $P_0(x_0, y_0, z_0)$ be a given point on L and let $P(x, y, z)$ be an arbitrary point on L. Let $\mathbf{r}_0$ and $\mathbf{r}$ denote the position vectors associated to P_0 and P, respectively. Denote by $\mathbf{a}$ the vector with representation $\overrightarrow{P_0P}$, so by the Triangle Law, $\mathbf{r} = \mathbf{r}_0 + \mathbf{a}$. Also, $\mathbf{a}$ is parallel to $\mathbf{v}$ so $\mathbf{a} = t\mathbf{v}$ for some scalar t. Thus, the vector equation for the line L (in vector form) is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

The values of the parameter t correspond to points on the line L, where $\mathbf{r}_0$ is the point with $t = 0$. To indicate only a portion of the line, we could restrict the values of t. For example, setting $t \geq 0$ would give one "half" of the line.

If we write $\langle a, b, c\rangle$ for the direction vector, the values a, b, and c are called the direction numbers of L. The parametric equations for the line L through the point (x_0, y_0, z_0) and parallel to $\langle a, b, c\rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct.$$

Note that we could also solve each of these equations for t to obtain

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}.$$

This gives the symmetric equations of L,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 10. Find the vector, parametric, and symmetric equations for the line through the points $A(1, 2, 5)$ and $B(3, 1, 0)$. At what point does this line intersect the xy-plane? Write the equation of the line segment from A to B.

The vector $\mathbf{v}$ with representation $\overrightarrow{AB}$ is parallel to the line and

$$\mathbf{v} = \langle 3 - 1, 1 - 2, 0 - 5\rangle = \langle 2, -1, -5\rangle.$$

Thus, taking $\mathbf{r}_0 = \langle 1, 2, 5\rangle$, we have that the vector equation for the line is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (1 + 2t)\mathbf{i} + (2 - t)\mathbf{j} + (5 - 5t)\mathbf{k}.$$

The parametric equations are

$$x = 1 + 2t \qquad y = 2 - t \qquad z = 5 - 5t.$$

The symmetric equations are then,

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-5}{-5}.$$

The line intersects the xy-plane when $z = 0$. Plugging this in the symmetric equations gives,

$$\frac{x-1}{2} = \frac{y-2}{-1} = 1.$$

Thus, this occurs when $x = 3$ and $y = 1$.

Recalling the vector equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, we note that A occurs (technically, its position vector) when $t = 0$. On the other hand, B occurs when $t = 1$. Hence, the equation for the line segment $\overline{AB}$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad 0 \leq t \leq 1.$$

Let r_0 and r_1 be points on the line L. Then the direction vector is $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$. Thus, in this case, the vector equation is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1.$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ corresponds to the values $0 \leq t \leq 1$.

Example 11. Consider the lines L_1 and L_2 with symmetric equations given below

$$L_1 : \frac{x}{1} = \frac{y-1}{-1} = \frac{z-2}{3} \qquad L_2 : \frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{7}.$$

These lines are *not parallel* because their respective direction vectors, $\langle 1, -1, 3\rangle$ and $\langle 2, -2, 7\rangle$, are not scalar multiples of one another.

One could then ask whether these lines intersect. To check this, we rewrite in parametric form, careful to use a different parameter for each:

$$\begin{aligned} &L_1 : x = t && y = 1 - t && z = 2 + 3t \\ &L_2 : x = 1 + 2s && y = 3 - 2s && z = 7s. \end{aligned}$$

We are asking whether there are values for t and s so that the respective equations match. The equations for x tell us that $t = 1+2s$. Then $y = 1-t = 1-(1+2s) = -2s \neq 3-2s$. Thus, there is no point of intersection and thus these lines do not intersect. Lines in three-space that do not intersect and are not parallel are called skew lines.

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ and a vector $\mathbf{n}$ that is orthogonal to the plane, called a normal vector to the plane. The normal vector determines the direction of the plane, while the point determines its position. Let $P(x, y, z)$ be an arbitrary point and let $\mathbf{r}_0$ and $\mathbf{r}$ be the position vectors of P_0 and P, respectively. The normal vector is orthogonal to every vector in the plane, including $\overrightarrow{P_0P}$, and so we have the vector equation of the plane

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$

Write $\mathbf{n} = \langle a, b, c\rangle$, $\mathbf{r} = \langle x, y, z\rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$. We can then rewrite the vector equation as the scalar equation of the plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

If we set $d = -(ax_0 + by_0 + cz_0)$, then the linear equation of the plane is

$$ax + by + cz + d = 0.$$

Example 12. The plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ is

$$2(x - 5) + 1(y - 3) - 1(z - 5) = 0.$$

This can be rewritten as a linear equation,

$$2x + y - z - 8 = 0.$$

Example 13. Find the equation of the plane through the points $P(3, 0, -1)$, $Q(-2, -2, 3)$, and $R(7, 1, -4)$.

The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, respectively, are

$$\mathbf{a} = \langle -5, -2, 4\rangle \qquad \mathbf{b} = \langle 4, 1, -3\rangle.$$

Both of these vectors lie in the plane, thus, their cross product is orthogonal to the plane. We take this as the normal vector,

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -2 & 4 \\ 4 & 1 & -3 \end{vmatrix} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

Thus, using P as our point, the equation of the plane is

$$2(x - 3) + 1(y - 0) + 3(z + 1) = 0.$$

or $2x + y + 3z = 3$.

The normal vector determines the direction of the plane, thus, given two planes, the angle between their respective normal vectors determines the angle between the planes, if they intersect. The planes are parallel if and only if the normal vectors are scalar multiples of one another.

Example 14. (1) Consider the planes with equations $9x - 3y + 6z = 2$ and $2y = 6x + 4z$. The normal vectors for these planes are $\mathbf{n}_1 = \langle 9, -3, 6 \rangle$ and $\mathbf{n}_2 = \langle 6, -2, 4 \rangle$. Note that $\mathbf{n}_2 = \frac{2}{3}\mathbf{n}_1$, and so the planes are parallel.

(2) Consider the planes with equations $x - y + 3z = 1$ and $3x + y - z = 2$. The normal vectors for these planes are $\mathbf{n}_1 = \langle 1, -1, 3 \rangle$ and $\mathbf{n}_2 = \langle 3, 1, -1 \rangle$. Then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{-1}{11}.$$

Thus, $\theta = \arccos(-1/11) \approx 95.22°$.

Let $P_1(x_1, y_1, z_1)$ be a point and $ax + by + cz + d = 0$ the equation of a plane. Given any point $P_0(x_0, y_0, z_0)$ on the plane, we let $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ denote $\overrightarrow{P_0P_1}$. The distance D from P_1 to the plane is the length of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$ of the plane. Hence,

$$\begin{aligned} D = |\operatorname{comp}_{\mathbf{n}} \mathbf{b}| &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}. \end{aligned}$$

As P_0 lies on the plane, its coordinates satisfy the equation of the plane. That is, $ax_0 + by_0 + cz_0 = -d$. Hence we have

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 15. Find the distance between the parallel planes with equations $6z = 4y - 2x$ and $9z = 1 - 3x + 6y$.

The point $P_1(0, 0, 0)$ lies on the first plane while $\mathbf{n} = \langle -3, 6, -9 \rangle$ is the normal vector of the second plane. Hence, using the above equation,

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{126}}.$$

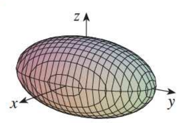
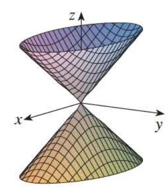
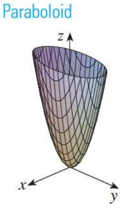
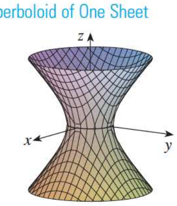
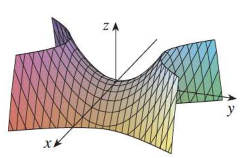
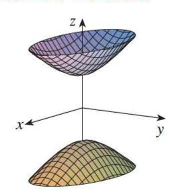
6. Cylinders and quadric surfaces

Quadric surfaces are the analogs in three dimensions of conic sections in the plane. They are represented by a degree two equation in the three variables x, y, and z. A somewhat tedious argument shows that they can be written in one of two forms:

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0.$$

The observant reader will recognize the first one as the equation of a sphere, centered at the origin, when $A = B = C = 1$. The sphere is one example of a quadric surface.

There are six families of quadric surfaces, detailed on page 837 of your text. You should familiarize yourself with these:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

How do we draw these? For that matter, how do we have any idea what these should look like? The key idea is to take traces.

Example 16. Let's sketch the surface $4x^2 + 9y^2 + 9z^2 = 36$.

Set $z = 0$, then the equation reduces to $4x^2 + 9y^2 = 36$, or

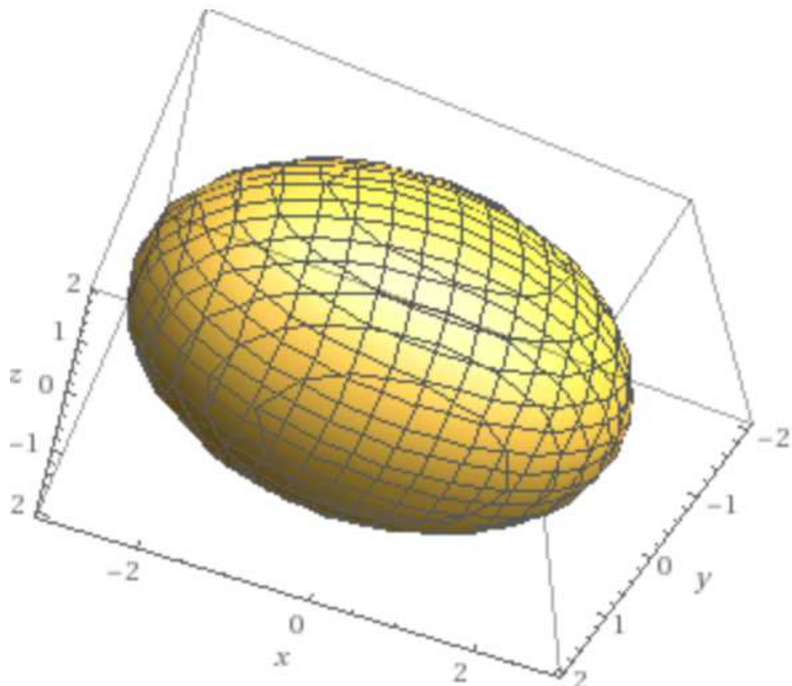
$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

We say the trace of the surface at $z = 0$ is an ellipse is the equation of an ellipse. So, we draw an ellipse in the xy-plane.

Now set $z = 1$, then the equation becomes $4x^2 + 9y^2 + 9 = 36$, or

$$\frac{x^2}{9} + \frac{y^2}{4} = \frac{25}{36}.$$

Again, we have an ellipse in the plane $z = 1$. Repeating this, we obtain the ellipsoid shown below:



Calculus III

Chapter 13 - Vector Functions

1. Vector Functions and Space Curves

A function $\mathbf{r} : \mathbb{R} \to V_n$ is called a vector-valued function. That is, the input is a real number and the output is a vector. For V_3, we write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

We call $f(t), g(t), h(t)$ the component functions of $\mathbf{r}$. The domain of $\mathbf{r}$ is the intersection of the domains of its component functions.

We find the limit of $\mathbf{r}$ as $t \to a$ by taking the limit of its component functions. Hence,

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle.$$

Then $\mathbf{r}$ is continuous at a if

$$\lim_{t\to a} r(t) = r(a).$$

Example 1. Consider the vector valued function

$$\mathbf{r}(t) = \frac{t^2 - t}{t-1}\mathbf{i} + \sqrt{t+8}\mathbf{j} + \frac{\sin \pi t}{\ln t}\mathbf{k}.$$

We identify the component functions as the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The domain of $\mathbf{r}(t)$ is then

$$\begin{aligned} D\left(\frac{t^2-t}{t-1}\right) \cap D\left(\sqrt{t+8}\right) \cap D\left(\frac{\sin \pi t}{\ln t}\right) &= \{t : t \neq 1\} \cap \{t : t \geq -8\} \cap \{t : t > 0 \text{ and } t \neq 1\} \\ &= (0,1) \cup (1, \infty). \end{aligned}$$

The function $\mathbf{r}(t)$ is not continuous at $t = 1$, however

$$\begin{aligned} \lim_{t\to 1} \mathbf{r}(t) &= \left(\lim_{t\to 1} \frac{t^2-t}{t-1}\right)\mathbf{i} + \left(\lim_{t\to 1}\sqrt{t+8}\right)\mathbf{j} + \left(\lim_{t\to 1}\frac{\sin \pi t}{\ln t}\right)\mathbf{k} \\ &= \mathbf{i} + 3\mathbf{j} - \pi\mathbf{k}. \end{aligned}$$

For the last limit, we applied L'Hospital's rule to get,

$$\lim_{t\to 1} \frac{\sin \pi t}{\ln t} \overset{\text{LR}}{=} \lim_{t\to 1} \frac{\pi \cos \pi t}{1/t} = -\pi.$$

Given a curve C in three-dimensional space, we can think of tracing that curve out over time (think of a particle moving in some direction along the curve). If we restrict to only recording the x coordinate of the particle at various intervals, we obtain a function $f(t)$ of time. We can then do this for y and z. This is the basic idea of *space curves*.

Let f, g, h be continuous, real-valued functions on an interval I. The set C of all points (x, y, z) with $x = f(t)$, $y = g(t)$, and $z = h(t)$ as t varies over I, is called a space curve. We call these functions the parametric equations of C and call t the parameter.

Example 2. The curve whose vector equation is $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ is a helix. Note that all of the points lie on the cylinder $x^2 + y^2 = 1$.

Example 3. Consider the intersection of the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $y = x^2$. Set $x = t$, then the equation for the parabolic cylinder gives $y = t^2$. The paraboloid's equation then says that $z = 4t^2 + t^4$. Thus, the intersection is the space curve,

$$\mathbf{r}(t) = \langle t, t^2, 4t^2 + t^4 \rangle.$$

2. Derivatives and integrals of vector functions

Let C be a space curve with equation $\mathbf{r}(t)$. Let P and Q be points on C corresponding to $r(t)$ (for some t) and $r(t+h)$ (for some $h>0$), respectively. We call $\overrightarrow{PQ}$ a secant vector of the curve and its equation is $\mathbf{r}(t+h)-\mathbf{r}(t)$. The scalar multiple $(1/h)(\mathbf{r}(t+h)-\mathbf{r}(t))$ has the same direction as $\overrightarrow{PQ}$. We let $h \to 0$ to obtain the tangent vector to the curve C defined by $\mathbf{r}(t)$ at the point P. That is, the derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h},$$

if the limit exists and $\mathbf{r}'(t) \neq 0$. The tangent line to C at P is the line through P parallel to $\mathbf{r}'(t)$. The unit tangent vector is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

The next theorem just says that, in nice cases, we can compute the derivative componentwise.

Theorem 4. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ where f, g, and h are differentiable functions, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

Example 5. Consider the vector function $\mathbf{r}(t) = \langle e^{-t}, \sin^2(t), \ln t\rangle$. All three components are differentiable functions, and

$$\mathbf{r}'(t) = \langle -e^{-t}, 2\sin(t)\cos(t), \frac{1}{t}\rangle.$$

Most of the differentiation rules you know from Calc I still apply to vector functions.

Differentiation Rules: Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function.

(i) $\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$

(ii) $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

(iii) $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f(t)\mathbf{u}'(t)+f'(t)\mathbf{u}(t)$

(iv) $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$

(v) $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

(vii) $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$.

Example 6. Suppose $|\mathbf{r}(t)| = c$ where c is a constant. Then $\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$ is a constant. Hence, by property (4),

$$0 = \frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \mathbf{r}'(t)\cdot\mathbf{r}(t)+\mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\mathbf{r}'(t)\cdot\mathbf{r}(t).$$

Thus, $\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$, so $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

Integrals of vector functions can be defined analogously. Suppose $\mathbf{r}(t)$ is a continuos vector function. Let $[a, b]$ be an interval in the domain of $\mathbf{r}(t)$ and consider a (regular) partition of $[a, b]$ into n subintervals, each of length Δt. In each subinterval, choose a representative point t_i^*. The definite integral of $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is then defined as

$$\begin{aligned}\int_a^b \mathbf{r}(t)\ \mathrm{dt} &= \lim_{n\to\infty}\sum_{i=1}^n \mathbf{r}(t_i^*)\Delta t\\ &= \lim_{n\to\infty}\left[\left(\sum_{i=1}^n f(t_i^*)\Delta t\right)\mathbf{i} + \left(\sum_{i=1}^n g(t_i^*)\Delta t\right)\mathbf{j} + \left(\sum_{i=1}^n h(t_i^*)\Delta t\right)\mathbf{k}\right].\end{aligned}$$

Hence,

$$\int_a^b \mathbf{r}(t)\ \mathrm{dt} = \left(\int_a^b f(t)\ \mathrm{dt}\right)\mathbf{i} + \left(\int_a^b g(t)\ \mathrm{dt}\right)\mathbf{i} + \left(\int_a^b h(t)\ \mathrm{dt}\right)\mathbf{i}.$$

As with real-valued functions, we use $\int \mathbf{r}(t)\ \mathrm{dt}$ for the indefinite integral. Hence, $\int \mathbf{r}(t)\ \mathrm{dt} = \mathbf{R}(t) + \mathbf{C}$ where $\mathbf{R}(t)$ is a vector function such that $\mathbf{R}'(t) = \mathbf{r}(t)$ (an antiderivative of $\mathbf{r}(t)$) and $\mathbf{C}$ is a constant vector.

Example 7. Using standard integration techniques, we have

$$\int te^{2t}\mathbf{i} + \frac{1}{t+1}\mathbf{j} + t\cos(\pi t)\mathbf{k}\ \mathrm{dt} = \frac{1}{4}(2t-1)e^{2t}\mathbf{i} + \ln|t+1|\mathbf{j} + \frac{1}{\pi^2}(\cos(\pi t) + t\pi\sin(\pi t))\mathbf{k} + \mathbf{C}.$$

Fundamental Theorem of Calculus for vector functions If $\mathbf{r}(t)$ be a continuous vector function and $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$$\int_a^b \mathbf{r}(t)\ \mathrm{dt} = [\mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

The formula for arc length of a space curve is analogous to that of a real valued function. If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a vector equation of a space curve on the interval $[a, b]$ with $f'(t), g'(t), h'(t)$ continuous, and if the curve is traversed exactly once as t increases from a to b, then the length of the curve is

$$L = \int_a^b |\mathbf{r}'(t)| \text{ dt} = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \text{ dt}.$$

Example 8. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \ln\cos t\mathbf{k}$ be the equation of the space curve C on the interval $[0, \pi/4]$. Then the length of C is

$$\begin{aligned} L &= \int_0^{\pi/4} |\mathbf{r}'(t)| \text{ dt} = \int_0^{\pi/4} \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \text{ dt} \\ &= \int_0^{\pi/4} \sqrt{[-\sin t]^2 + [\cos t]^2 + \left[\frac{-\sin t}{\cos t}\right]^2} \text{ dt} \\ &= \int_0^{\pi/4} \sqrt{1 + \tan^2 t} \text{ dt} = \int_0^{\pi/4} \sqrt{\sec^2 t} \text{ dt} = \int_0^{\pi/4} \sec t \text{ dt} \\ &= [\ln|\sec t + \tan t|]_0^{\pi/4} = \left(\ln(\sqrt{2} + 1)\right) - (\ln(1 + 0)) = \ln(\sqrt{2} + 1). \end{aligned}$$

One comment about this argument. We should really have put $|\sec t|$ when we took the integral. However, because $\sec t$ is positive on this interval.

Note: A curve may have multiple parameterizations, but all parameterizations will give the same arc length. Another way of saying this is that arc length is *independent* of parameterization of the curve.

If a curve C is given by a vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ on the interval $[a, b]$ where $\mathbf{r}'(t)$ is continuous and C is traversed exactly once as t increases from a to b. The arc length function s is defined as

$$s(t) = \int_a^t |\mathbf{r}'(u)| \text{ du}.$$

Note that we have replaced the variable of integration by u since t is being used by the bound of integration. Differentiating both sides of this equation give

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

Example 9. Consider the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$. We choose the point $(1, 0, 0)$ (corresponding to $t = 0$) as our initial point from which to measure arc length and we move in the direction of increasing t. The arc length function is then

$$s(t) = \int_0^t |(-\sin t)^2 + \cos^2 t + 1| \text{ du} = \int_0^t \sqrt{2} \text{ du} = \sqrt{2}t.$$

We can now reparameterize our curve by setting $s = s(t)$ and solving for t, so $t = s/\sqrt{2}$, and

$$\mathbf{r}(t(s)) = \cos(s/\sqrt{2})\mathbf{i} + \sin(s/\sqrt{2})\mathbf{j} + (s/\sqrt{2})\mathbf{k}.$$

This is called parameterization of the curve with respect to arc length. If we set $s = 1$, the $\mathbf{r}(t(1))$ is the positive vector of the point 1 unit from $(1, 0, 0)$.

A parameterization $\mathbf{r}(t)$ is called smooth on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq 0$ on I. A curve C is called smooth if it has a smooth parameterization $\mathbf{r}(t)$. For such a curve, the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Recall this indicates the direction of the curve but it can also tell us how "sharp" C is as it bends. This is the idea behind curvature. That is, curvature is the measure of how quickly the curve changes direction at a point. Formally, curvature is defined as the magnitude of the rate of change of the unit tangent vector with respect to arc length, so

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent and s is arc length. By the chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

and so

$$\kappa(t) = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Example 10. Consider the curve with vector equation $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, t^2 \rangle$. Note that $\mathbf{r}'(t) = \langle 1, t, 2t \rangle$. The unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 1, t, 2t \rangle}{\sqrt{1 + 5t^2}}.$$

By the product rule,

$$\begin{aligned}\mathbf{T}'(t) &= -\frac{5t}{(1+5t^2)^{3/2}}(\langle 1, t, 2t \rangle) + \frac{\langle 0, 1, 2 \rangle}{\sqrt{1+5t^2}} \\ &= \frac{1}{(1+5t^2)^{3/2}} \left(-5\langle t, t^2, 2t^2 \rangle + \langle 0, 1+5t^2, 2+10t^2 \rangle \right) = \frac{\langle -5t, 1, 2 \rangle}{(1+5t^2)^{3/2}}.\end{aligned}$$

Thus,

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\langle -5t, 1, 2\rangle|/(1+5t^2)^{3/2}}{\sqrt{1+5t^2}} = \frac{\sqrt{25t^2+5}}{(1+5t^2)^2} = \frac{\sqrt{5}\sqrt{1+5t^2}}{(1+5t^2)^2} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}.$$

Thus, at the point $(2,1,4)$, corresponding to $t=2$, the curvature is $\sqrt{5}/(21)^{3/2}$.

Theorem 11. The curvature of a curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Example 12. Consider the previous example. We have $\mathbf{r}''(t) = \langle 0, 1, 2\rangle$ and a computation shows that

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = 0\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The magnitude of this vector is $\sqrt{5}$. Hence, by the theorem,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{5}}{(1+5t^2)^{3/2}}.$$

A common (and useful) technique in Calc I/II is linear approximation. In this, we construct a line that approximates a curve near a point using tangents. In 3d space, given a curve (say a helix), we can ask what *plane* best approximates the curve near a point. We do this using tangent vectors.

Let $\mathbf{r}(t)$ be a smooth space curve with unit tangent vector $\mathbf{T}(t)$. Since $|\mathbf{T}(t)| = 1$ for all t, then $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$. Hence, $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$, but it need not be a unit vector. At any point where $\kappa \neq 0$, we define the (principal) unit normal vector as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

This vector tells us the direction the curve is turning at each point. The binormal vector is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ and it is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

The vectors $\mathbf{B}$ and $\mathbf{N}$ at a point P on a curve C determine a plane, called the normal plane of C at P. It consists of all lines that are orthogonal to the tangent vector $\mathbf{T}$.

Example 13. Consider the circular helix with equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$. We will determine the normal plane of C at the point $P(1, 0, 0)$. Standard computations show

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -\sin t, \cos t, 1\rangle}{\sqrt{2}} \\
\mathbf{T}'(t) &= \frac{1}{\sqrt{2}}\langle -\cos t, \sin t, 0\rangle \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{\sqrt{2}}\frac{\langle -\cos t, \sin t, 0\rangle}{1/\sqrt{2}} = \langle -\cos t, \sin t, 0\rangle \\
\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\langle \sin t, \cos t, 1\rangle.
\end{aligned}$$

Now the normal plane to C at P has normal vector $\mathbf{r}'(0) = \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle$. Hence, an equation of the plane is

$$0(x-1) + 1(y-0) + 1(z-0) = 0,$$

or just $y + z = 0$.

The vectors $\mathbf{T}$ and $\mathbf{N}$ also determine a plane, called the osculating plane of C at P. It is the plane that comes closes to containing the part of the curve near P. Hence if the curve lies in a plane (is a planar curve) then this is just the plane that contains the curve.

An easy computation shows that the curvature κ of a circle of radius a is $1/a$ (see Example 3 on page 864). This implies that the radius of a circle is $1/\kappa$. The circe that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\mathbf{N}$ points), and has radius $\rho = 1/\kappa$ is called the osculating circle of C at P.

4. Motion in space

Recall that we can think of a vector function $\mathbf{r}(t)$ as tracing the path (or position) of a particle moving along a space curve. With that interpretation, it follows that $\mathbf{r}'(t)$ represents the velocity vector $\mathbf{v}(t)$. The speed of the particle is the magnitude of the velocity vector, so $|\mathbf{v}(t)| = |\mathbf{r}'(t)|$. The acceleration is $\mathbf{a}(t) = \mathbf{r}''(t)$.

Example 14. Suppose a particle's acceleration on a curve is given by the vector function $\mathbf{a}(t) = \sin t\mathbf{i} + 2\cos t\mathbf{j} + 6t\mathbf{k}$. Further suppose that $\mathbf{v}(0) = -\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{j} - 4\mathbf{k}$ (that is, these are the velocity and position vectors at time 0). We wish to find the functions for velocity and position. To do this, we use integration:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ \mathrm{dt} = -\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t^2\mathbf{k} + \mathbf{C}.$$

Thus, $-\mathbf{k} = \mathbf{v}(0) = -\mathbf{i} + \mathbf{C}$, so $\mathbf{C} = \mathbf{i} - \mathbf{k}$ and hence,

$$\mathbf{v}(t) = (1 - \cos t)\mathbf{i} + 2\sin t\mathbf{j} + (3t^2 - 1)\mathbf{k}.$$

Now

$$\mathbf{r}(t) = \int \mathbf{v}(t)\ \mathrm{dt} = (1 - \sin t)\mathbf{i} - 2\cos t\mathbf{j} + (t^3 - t)\mathbf{k} + \mathbf{C}.$$

Thus,

$$\mathbf{j} - 4\mathbf{k} = \mathbf{r}(0) = \mathbf{i} - 2\mathbf{j} + \mathbf{D}.$$

Thus, $\mathbf{D} = -\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and so

$$\mathbf{r}(t) = (-\sin t)\mathbf{i} + (3 - 2\cos t)\mathbf{j} + (t^3 - t - 4)\mathbf{k}.$$

Newton's Second Law of Motion: If, at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m produces an acceleration $\mathbf{a}(t)$, then $\mathbf{F}(t) = m\mathbf{a}(t)$.

Example 15. Suppose an object with mass m moves in a circular path with constant angular speed ω and has position vector $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t\mathbf{j}$. The velocity and acceleration are then given by

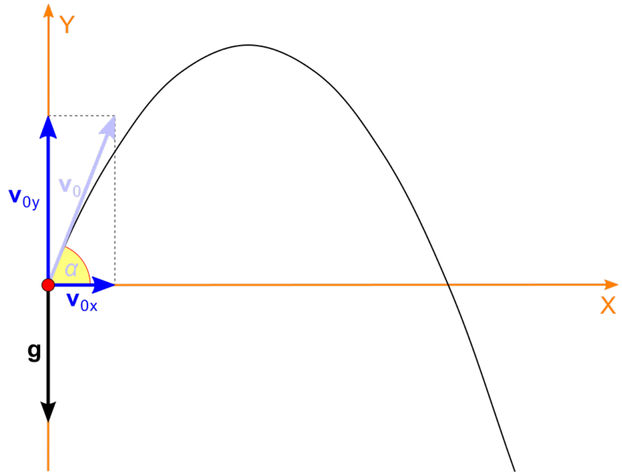
$$\begin{aligned}\mathbf{v}(t) &= -a\omega\sin\omega t\mathbf{i} + a\omega\cos\omega t\mathbf{j}\\ \mathbf{a}(t) &= -a\omega^2\cos\omega t\mathbf{i} - a\omega^2\sin\omega t\mathbf{j}.\end{aligned}$$

Thus, Newton's Second Law gives,

$$\mathbf{F}(t) = m\mathbf{a}(t) = -m\omega^2\mathbf{r}(t).$$

Hence, the foce acts in the direction opposite to the radius vector $\mathbf{r}(t)$ and so points in the direction of the origin (centripetal force).

Suppose a projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$.



There is no loss in assuming that the projectile starts at the origin. The force of gravity acts downward, and hence,

$$\mathbf{F} = m\mathbf{a} = -mg\mathbf{j}$$

where $g = |\mathbf{a}| \approx 9.8m/s^2$. Thus, $\mathbf{a} = -g\mathbf{j}$ and since $\mathbf{v}'(t) = \mathbf{a}$, we have

$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{C}$$

where $\mathbf{C} = \mathbf{v}(0) = \mathbf{v}_0$. Hence, $\mathbf{r}'(t) = \mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$. Integrating again we find

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{D}$$

and $\mathbf{D} = \mathbf{r}(0) = \mathbf{0}$. Hence, $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0$. Write $|\mathbf{v}_0| = v_0$ for initial speed of the projectile, then by the Pythagorean Theorem,

$$\mathbf{v}_0 = v_0 \cos\alpha\mathbf{i} + v_0 \sin\alpha\mathbf{j}$$

and so

$$\mathbf{r}(t) = (v_0 \cos\alpha)t\mathbf{i} + [(v_0 \sin\alpha)t - \frac{1}{2}gt^2]\mathbf{j}.$$

The parametric equations of trajectory are then

$$x = (v_0 \cos\alpha)t \qquad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2.$$

Example 16. A batter hits a baseball 3 ft above the ground toward the center field fence, which is 10 ft high and 400 ft from home plate. The ball leaves the bat with speed 115 ft/s at an angle 50° above the horizontal. Is it a home run? (Does the ball clear the fence?)

We denote by x the distance from home plate and by y the height above the ground. Thus, $\mathbf{r}_0 = 3\mathbf{j}$. Initial velocity is determined by the above,

$$\mathbf{v}(0) = \mathbf{v}_0 = v_0 \cos \alpha \mathbf{i} + v_0 \sin \alpha \mathbf{j} = 115 \cos 50^\circ \mathbf{i} + 115 \sin 50^\circ \mathbf{j}.$$

Acceleration is $\mathbf{a} = -g\mathbf{j}$ where $g \approx 32.174$ ft/s^2. Using the above, but adjusting for $\mathbf{r}_0$, we have

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{r}_0 - \frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 \\
&= 3\mathbf{j} - \frac{1}{2}gt^2\mathbf{j} + t(115 \cos 50^\circ \mathbf{i} + 115 \sin 50^\circ \mathbf{j}) \\
&= 115 \cos 50^\circ t\mathbf{i} + (3 - \frac{1}{2}gt^2 + 115 \sin 50^\circ t)\mathbf{j}.
\end{aligned}$$

Hence,

$$x = 115 \cos 50^\circ t \quad \text{and} \quad y = 3 - \frac{1}{2}gt^2 + 115 \sin 50^\circ t.$$

The ball reaches the fence when $x = 400$. To find the time that this happens, we set

$$400 = 115 \cos 50^\circ t \Rightarrow t \approx 5.4.$$

Hence, the ball reaches the fence 5.4 seconds after it leaves the bat. At this time, $y(500) \approx 8.654$ ft, which isn't high enough to clear the fence. Hence, it's not a homerun[1].

Wrote $v = |\mathbf{v}|$ for the speed of a particle, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}.$$

Hence, $\mathbf{v} = v\mathbf{T}$. Differentiating both sides gives

$$\mathbf{a} = \mathbf{v}' = v'\mathbf{T} + v\mathbf{T}'.$$

Recall that curvature is given by

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v}.$$

Hence, $|\mathbf{T}'| = \kappa v$. The unit normal is $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$ so $\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N}$ and hence,

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}.$$

We set the coefficients of $\mathbf{T}$ and $\mathbf{N}$ as $a_T = v'$ and $a_N = \kappa v^2$, respectively, and call these the tangential component and normal component of acceleration.

Note that

$$\mathbf{v} \cdot \mathbf{a} = vT \cdot (v'\mathbf{T} + \kappa v^2\mathbf{N}) = vv'\mathbf{T} \cdot \mathbf{T} + \kappa v^3\mathbf{T} \cdot \mathbf{N} = vv'.$$

[1] Unless a fan reaches over and grabs it and the umpire doesn't see it.

Some comment s regarding this computation are in order. Since $|\mathbf{T}| = 1$, then $\mathbf{T}\cdot\mathbf{T} = |\mathbf{T}|^2 = 1$. Moreover, as $\mathbf{T}$ is a vector function with constant magnitude, then an earlier computation (Example 6) shows that $\mathbf{T}\cdot\mathbf{N} = 0$. Hence,

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}.$$

Using Theorem 11, we also have

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

Calculus III
Chapter 14 - Partial derivatives

1. Functions of several variables

In the previous section, we studied curves, which were represented by functions of one variable. Now we being to study surfaces, which are represented by functions of two variables. For curves, it is easy to define and interpret tangents, though we had to do a little work to put everything in vector notation. For a surface, the analogy is not quite so easy. What is meant by a tangent at a point? This is the question we will consider in the coming sections.

A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range (or codomain) is the set of values that f takes on, that is, $\{f(x, y) : (x, y) \in D\}$. We write $z = f(x, y)$ and refer to x and y as the independent variables, and z as the dependent variable.

Example 1. (1) Consider the function $f(x, y) = \sqrt[4]{x - 3y}$. The domain of f consists of all $(x, y) \in \mathbb{R}^2$ such that $x - 3y \geq 0$. Equivalently, $y \leq \frac{1}{3}x$. This is the region below and including the line $y = \frac{1}{3}x$.

(2) Consider the function $f(x, y) = \ln(x^2 + y^2 - 4)$. The domain of f consists of all $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 - 4 > 0$. This consists of all points outside, but not including, the circle $x^2 + y^2 = 4$.

More generally, we define a function f of n variables as a rule that assigns to each ordered n-tuple $(x_1, \ldots, x_n)$ in a set $D \subset \mathbb{R}^n$ a unique real number denoted by $f(x_1, \ldots, x_n)$. Here D is the domain and the codomain is $\{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in D\}$.

If f is a function of two variables with domain D, then the graph of f is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$. We call such a graph a surface.

Example 2. Consider the function $f(x, y) = 6 - 3x - 2y$. The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$. This is the equation of a plane. To graph, we plot the intercepts. If $y = z = 0$, then $x = 2$, so the x-intercept is $(2, 0, 0)$. Similarly, if $x = z = 0$, then $y = 3$, so the y-intercept is $(0, 3, 0)$. The z-intercept is $(0, 0, 6)$. A plane is determined by three points so this is enough to draw the plane.

Example 3. Consider the function $f(x, y) = \sqrt{9 - x^2 - y^2}$. The graph of f has the equation $z = \sqrt{9 - x^2 - y^2}$, or $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$. This is the equation of a sphere centered at the origin with radius 3. However, since $z \geq 0$, then the graph of f is just the top half of the sphere.

We recognize some graphs from our study of quadric surfaces. Others can be determined using the following method, which should be reminiscent of topographic maps.

The level curves of a function f of two variables are the curves with equation $f(x, y) = k$, where k is a constant in the range of f. One can think of a level curve as the (two-dimensional projection of) the graph of f at height k.

Example 4. Consider again the function $f(x, y) = \sqrt{9 - x^2 - y^2}$. A level curve of f as equation $k = \sqrt{9 - x^2 - y^2}$, or $x^2 + y^2 = 9 - k^2$. Thus, the level curves are concentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$.

Example 5. Consider the function $f(x, y) = \ln(x^2 + 4y^2)$. The level curves of f have equation $k = \ln(x^2 + 4y^2)$, so $e^k = x^2 + 4y^2$. Thus, the level curves are concentric ellipses.

All of this generalizes to functions of three variables (and beyond) in a natural way.

2. Limits and continuity

Limits in multivariable calculus have the same general definition as in single variable, but there is somewhat more nuance to the study. Generally, we think of a limit of a function $f(x, y)$ at a point (a, b) as describing the *behavior* of f near that point.

Here is a somewhat imprecise definition of the limit, that we will soon make more precise. Suppose the domain of f is D and that D includes points near[1] (a, b). The domain need not contain (a, b) itself. If we let (x, y) vary, then we "define" the limit of f at (a, b) to be the value as (x, y) approaches (a, b).

For example, using Table 1 (on page 904) in the text, one can reasonably conjecture that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1.$$

Now we will be more precise. Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). We say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L, and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

Translated, this says that for any possible $\epsilon > 0$ (think of this as the acceptable error) then there is some circle of radius $\delta > 0$ such that all of the points (x, y) (strictly) within the circle satisfy that the distance from $f(x, y)$ to L is at most ϵ.

In single-variable calculus, points x can approach a from one of two directions (left or right), and the limit exists if and only if both sided limits exist and are equal. In multivariable calculus, there are an infinite number of directions[2]. That is, points (x, y) can approach (a, b) along any of the infinite number of curves (paths) that pass through (a, b) and lie on the surface. For a limit to exist, then, it must be shown that the limit exists for each path and agrees on *every* direction. This idea is often more useful for showing that a limit *does not* exist, rather than showing that it does.

Path test If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

[1]A reasonable question here is, 'What does near mean? What we really mean is that there is some circular region (perhaps quite small) around (a, b) contained in D.

[2]That escalated quickly!

Example 6. Consider the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$. Note that the domain of f consists of all $(x,y) \in \mathbb{R}^2$ such that $(x,y) \neq (0,0)$. We claim that the limit as $(x,y) \to (0,0)$ does not exist. Let's start by approaching $(0,0)$ along the x-axis. Then $y = 0$ and so we have $f(x,0) = x^2/x^2 = 1$ (for all $x \neq 0$). Thus, $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the x-axis. Now we approach $(0,0)$ along the y-axis, so $x = 0$ and so $f(0,y) = -y^2/y^2 = -1$ (for all $y \neq 0$). Thus, $f(x,y) \to -1$ as $(x,y) \to (0,0)$ along the y-axis. Since these two limits are different, the limit does not exist.

Example 7. Consider the function $f(x,y) = \frac{xy-y}{(x-1)^2+y^2}$. Note that the domain of f consists of all $(x,y) \in \mathbb{R}^2$ such that $(x,y) \neq (1,0)$. We claim that the limit as $(x,y) \to (1,0)$ does not exist. Let's start by approaching $(1,0)$ along the line $y = 0$, so $f(x,0) = 0/(x-1)^2 = 0$ (for all $x \neq 1$). Thus, $f(x,y) \to 0$ as $(x,y) \to (1,0)$ along the line $y = 0$. Now we approach $(1,0)$ along the line $x = 1$, so $f(1,y) = 0/y^2 = 0$ (for all $y \neq 0$). However, this does not guarantee that the limit exists. The Path Test above only shows *nonexistence of the limit* We approach $(1,0)$ along the line $y = x-1$, so $f(1/2, y) = \frac{x(x-1)-1}{2(x-1)^2}$. Thus, $f(x,y) \to 1/2$ as $(x,y) \to (1,0)$ along the line $y = x-1$. Thus, the limit does not exist.

All of the limit laws you learned in Calc I still hold (the sum of a limit is the limit of a sum, etc.) In particular, the Squeeze Theorem still holds.

Example 8. Consider the function $f(x,y) = \frac{3x^2y}{x^2+y^2}$. Since $x^2+y^2 \geq x^2$, then

$$0 \leq \left| \frac{3x^2y}{x^2+y^2} \right| \leq \frac{3x^2|y|}{x^2} = 3|y|.$$

Since $3|y| \to 0$ as $(x,y) \to (0,0)$, then $f(x,y) \to 0$ as $(x,y) \to (0,0)$ by the Squeeze Theorem.

A function f of two variables is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say f is continuous on D if f is continuous at every point $(a,b) \in D$.

This definition is much the same as for single variable functions. Intuitively, we think that the function has no holes or breaks. Polynomial functions (in two variables), which are (finite) sums of terms of the form cx^my^n, are continuous functions. Rational functions, those of the form $\frac{p(x,y)}{q(x,y)}$ where p and q are polynomials, are not necessarily continuous.

Example 9. The function $f(x,y)$ in Example 8 is not continuous because, though the limit exists at $(0,0)$, the function is not defined at $(0,0)$. However, it is continuous at every other point in its domain ($\mathbb{R}^2 \backslash \{(0,0\})$. If we instead defined the function as

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

then $f(x,y)$ is continuous.

Again, rules for continuity follow in much the same way as for single variable functions. For example, the sum/difference/product of continuous functions is continuous, and the composition of continuous functions is continuous.

Example 10. One last example, just to not leave things dangling. Consider again the limit

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

The trick here is to switch to polar coordinates. Set $x = r\cos\theta$ and $y = r\sin\theta$. Then $x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$, so

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{x^2+y^2\to 0^+} \frac{\sin(r^2)}{r^2} = \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2}.$$

We can evaluate the last limit using l'Hospital's Rule (since the limit is of the form 0/0),

$$\lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \overset{\text{LR}}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = \cos(0) = 1.$$

3. Partial Derivatives

As noted in the introduction, the idea of a tangent to a surface is not a straight analog of that for single-variable functions. Later, we will frame this in the idea of a tangent plane to a surface, which will perhaps seem like the more natural analog of a tangent line to a curve. For now, we will do our best to adapt old methods and see what information we can glean from them.

Let $f(x, y)$ be a function of two variables and let S be the graph of f. If we fix $y = b$ for some constant b, then we can set $g(x) = f(x, b)$ and then $g(x)$ is the function of a curve C_1 that lives on the surface S. Thus, we can take the derivative of g and this defines the slope of the tangent line to C_1 at a point (x, b). If $g'(x)$ exists at a point $x = a$, then we call $g'(a)$ the partial derivative of f with respect to x at (a, b). We generally denote $g'(a)$ by $f_x(a, b)$. Thus,

$$f_x(a, b) = g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h} = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

Similarly, if we fix $x = a$ and define $G(y) = f(a, y)$, then $G'(a)$ (if it is defined) is the partial derivative of f with respect to y at (a, b), denoted $f_y(a, b)$, and

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

Here $G(y)$ is the function of a curve C_2 that also lives on S.

A reasonable question is what makes the curves C_1 and C_2 special, other than convenience. The short answer is: nothing. *However*, we will see in the next section that the tangent vector to a curve C that lies on S at a point (a, b) lies in the plane determined by the tangent vectors to C_1 and C_2 at that point. In linear algebraic terms, the tangent vectors for C_1 and C_2 *span* the tangent plane.

In general we let x and y vary. If f is a function of two variables, is partial derivatives are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \quad \text{and} \quad f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

These may be denoted in any number of ways. If $z = f(x, y)$, then

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$
$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f.$$

The discussion above explains to us how to compute f_x and f_y for $z = f(x, y)$. For f_x, we regard (hold) y constant and compute the derivative with respect to x. For f_y, we regard (hold) x constant and compute the derivative with respect to y.

Example 11. (1) Let $f(x, y) = x^2y - 3y^4$. Then $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2 - 12y^3$.

(2) Let $z = x \sin(xy)$. Then using the product rule and chain rule we have

$$\frac{\partial z}{\partial x} = \sin(xy) + xy\cos(xy) \quad \text{and} \quad \frac{\partial z}{\partial x} = x^2 \cos(xy).$$

(3) Consider $x^2 + y^2 + z^2 - 2z = 4$. To find $\frac{\partial z}{\partial x}$, we *could* solve for z, or we could use implicit differentiation. Holding y constant and regarding z as a function of x, we have

$$2x + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0.$$

Thus,

$$\frac{\partial z}{\partial x} = \frac{-x}{z-1}.$$

Similarly

$$\frac{\partial z}{\partial y} = \frac{-y}{z-1}.$$

Since f_x is again a function of two variables, then we can consider *its* partial derivatives with respect to x or y. Similarly we can do this for f_y. Set $z = f(x, y)$, then we can define the second partial derivatives of f as

$$\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$

One can define higher derivatives similarly (e.g., f_{xxx} or f_{xyx}).

Example 12. Let $f(x, y) = x^2y - 3y^4$. By Example 11 (1), $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2 - 12y^3$. Then, $f_{xx} = 2y$, $f_{xy} = 2x$, $f_{yx} = 2x$, and $f_{yy} = 36y^2$.

The fact that $f_{xy} = f_{yx}$ in the previous example is not a coincidence.

Theorem 13 (Clairaut's Theorem)**.** Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

4. Tangent Planes

Just as we approximate curves with lines, we can approximate surfaces with planes. The idea is that, when we look at a surface locally (think: zoom in), the surface appears to be flat (or, at least, flatter than if look from further away).

Let S be a surface with equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 be the surface obtained by intersection S with the plane $y = y_0$. Let C_2 be the surface obtained by intersection S with the plane $x = x_0$. Let T_1 and T_2 be the tangent lines to C_1 and C_2, respectively, at the point P. Then tangent plane to S at the point P is the plane determined by (containing[3]) T_1 and T_2. This plane contains *all* tangent lines to curves on S that pass through P.

The equation of a plane is

$$\begin{aligned} 0 &= A(x - x_0) + B(y - y_0) + C(z - z_0) \\ z - z_0 &= -\frac{A}{C}(x - x_0) - \frac{B}{C}(y - y_0). \end{aligned}$$

Set $a = -A/C$ and $b = -B/C$. Then we have

$$z - z_0 = a(x - x_0) + b(y - y_0).$$

This is the equation of the tangent plane to S at P, and it only remains to find a and b. If we intersect the tangent plane at $y = y_0$, then this is the tangent line T_1. Setting $y = y_0$ above gives $z - z_0 = a(x - x_0)$. Thus, as the equation to the tangent line, we must have $a = f_x(x_0, y_0)$. Similarly, $b = f_y(x_0, y_0)$. It follows that the equation of the tangent plane to the surface S at P is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 14. Consider the surface S with equation $z = 3x^2 + xy - 4y^2$ and the point $P(1, -1, -2)$. Setting $f(x, y) = 3x^2 + xy - 4y^2$ gives

$$\begin{aligned} f_x(x, y) &= 6x + y & \qquad f_y(x, y) &= x - 8y \\ f_x(1, -1) &= 5 & \qquad f_y(1, -1) &= 9. \end{aligned}$$

Thus, the equation of the tangent plane at $(1, -1, -2)$ is $z + 2 = 5(x - 1) + 9(y + 1)$, or $z = 5x + 9y + 2$.

[3]For those with some linear algebra: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the tangent vectors to C_1 and C_2 at P. Then the tangent plane is the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Tangent lines approximate a curve close to a given point. The idea is that lines are simpler than curves and so can allow for significantly faster computations, if we are willing to deal with a small amount of error[4]. Tangent planes serve the same purpose but for surfaces.

Consider the previous example. We found the tangent plane to the graph of the equation $f(x, y) = 3x^2 + xy - 4y^2$ at the point $P(1, -1, -2)$. We can view this plane as a linear function of two variables,

$$L(x, y) = 5x + 9y + 2.$$

We call this function the *linearization of f at P* and it is a good approximation to $f(x, y) = z$ when (x, y) is near $(1, -1)$. For example, $L(.95, -1) = -2.25$ while the actual (retail) value is $f(1, -1) = -2.2425$.

In general, the linearization of f at (a, b) is the function,

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The linear (tangent plane) approximation of f at (a, b) is

$$f(x, y) \approx L(x, y).$$

What does it mean for a function to be differentiable? It is heavily tied up with the above discussion. In general, a function is differentiable at a point P if its linear approximation is a *good* approximation of the function near that point. (See the text for an example of a function where the linear approximation is not so good.)

Consider a function $z = f(x, y)$ of two variables and let (a, b) be a point in the domain of f. Suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$ You should think of Δx and Δy being small values, so that $(a + \Delta x, b + \Delta y)$ is *close* to (a, b). The increment of z is defined as

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

That is, Δz is the difference of the function values between the two points. We say f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1, \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Theorem 15. If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

[4]The size of the error depends on many factors, chief among them are how crazy the curve is and how far we are from the given point.

Example 16. Consider the function $f(x, y) = \sqrt{xy}$ at the point $(1, 4)$. The first partial derivatives are given by

$$f_x(x, y) = \frac{1}{2}y(xy)^{-1/2}, \qquad f_y(x, y) = \frac{1}{2}x(xy)^{-1/2}.$$

Both of these functions exist and are continuous at $(1, 4)$. Since $f_x(1, 4) = 1$ and $f_y(1, 4) = \frac{1}{4}$, then the linearization of f at $(1, 4)$ is $L(x, y) = 2 + (x - 1) + \frac{1}{4}(y - 4)$.

For a differentiable function $z = f(x, y)$, the differentials dx and dy are independent variables. The (total) differential dz is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Setting $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$, then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Hence, we can write the linear approxiattion as

$$f(x, y) \approx f(a, b) + dz.$$

As (a, b) changes to $(a + \Delta x, b + \Delta y)$, Δz represents the change in height of the surface, while dz represents the change in height of the tangent plane.

Example 17. Suppose metal encloses a cylindrical can that is 10 cm high and 4 cm in diameter. The metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick. We will use differentials to estimate the amount of metal used.

The volume of a cylinder is $V = \pi r^2 h$ where $r = 2$ (radius) and $h = 10$ (height). The differential is

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = 2\pi rh\ dr + \pi r^2\ dh = 40\pi\ dr + 4\pi\ dh.$$

The amount of metal the change in volume, which is approximated by dV. So

$$\Delta V \approx dV = 40\pi(0.05) + 4\pi(0.2) = 2.8\pi\ \text{cm}^3.$$

5. The Chain Rule

First we recall the chain rule from single-variable calculus. Suppose $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions. Then y is a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

This extends in a natural way to a chain rule for two-variable functions $z = f(x, y)$. However, there are two versions, depending on whether x and y are functions of a single variable, or of two variables themselves.

The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example 18. Let $z = \frac{x-y}{x+2y}$, $x = e^{\pi t}$, $y = e^{-\pi t}$. Then

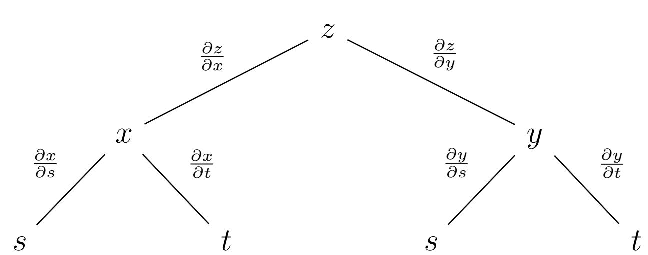
$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \left(\frac{3y}{(x+2y)^2}\right)\left(\pi e^{\pi t}\right) + \left(\frac{-3x}{(x+2y)^2}\right)\left(-\pi e^{-\pi t}\right) \\
&= \frac{3\pi}{(x+2y)^2}\left(y e^{\pi t} + x e^{-\pi t}\right).
\end{aligned}$$

Next we consider the case where x and y are functions of two variables.

The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

One way to visualize (and remember) this rule is by the following tree diagram:



Now $\frac{\partial z}{\partial s}$ is the sum of products of terms that end at s. This strategy works for functions of any number of variables, which are then functions of any number of variables.

Example 19. Suppose $z = \sqrt{x}e^{xy}$ where $x = 1 + st$ and $y = s^2 - t^2$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \left(\frac{e^{xy}}{2\sqrt{x}} + \sqrt{x}ye^{xy}\right)(t) + \left(x^{3/2}e^{xy}\right)(2s)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{e^{xy}}{2\sqrt{x}} + \sqrt{x}ye^{xy}\right)(s) + \left(x^{3/2}e^{xy}\right)(2t)$$

The chain rule allows us to think differently about implicit differentiation. Suppose $F(x, y) = 0$ defines y implicitly as a differentiable function of x, so $y = f(x)$ where $F(x, f(x)) = 0$ for all x in the domain of f. If F is differentiable, then the Chain Rule implies that

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0.$$

If $\partial F/\partial y \neq 0$, then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}.$$

Example 20. Suppose $y \cos x = x^2 + y^2$. Write $F(x, y) = x^2 + y^2 - y \cos x$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x + y \sin x}{2y - \cos x}.$$

We can repeat this for functions of two variables. Suppose $z = f(x, y)$ and $F(x, y, z) = 0$, so $F(x, y, f(x, y)) = 0$. If F and f are differentiable, then we can use the Chain Rule to find

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, then so long as $\frac{\partial F}{\partial z} \neq 0$, we have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Example 21. Suppose $x^2 - y^2 + z^2 - 2z = 4$. Set $F(x, y, z) = x^2 - y^2 + z^2 - 2z - 4$. Then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2x}{2z - 2} = \frac{-x}{z - 1}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{-2y}{2z - 2} = \frac{y}{z - 1}.$$

6. Directional derivatives and the gradient vector

Let $z = f(x, y)$. The partial derivatives f_x and f_y represent the change of z in the x- and y-directions, or in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$. It is reasonable, then, to consider the change of z in the direction of an arbitrary vector $\mathbf{u} = \langle a, b\rangle$.

Let S be the surface with equation $z = f(x, y)$ and let $z_0 = f(x_0, y_0)$, so $P(x_0, y_0, z_0)$ lies on S. There is a vertical plane that passes through P in the direction of $\mathbf{u} = \langle a, b\rangle$ intersects S in a curve C. The slope of the tangent line T to C at P is the rate of change of z in the direction of $\mathbf{u}$. Let $Q(x, y, z)$ be another point on C, and let P' and Q' be the projections of P and Q onto the xy-plane. Then $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$, so $\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb\rangle$ for some scalar h. Thus, $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$ and $y = y_0 + hb$ and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

We take the limit as $h \to 0$ and obtain the rate of change of z (with respect to distance) in the direction of $\mathbf{u}$.

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists. Note that $D_{\mathbf{i}} = f_x$ and $D_{\mathbf{j}} = f_y$.

In general, the directional derivative can be difficult to compute. The next theorem gives way to compute for unit vectors.

Theorem 22. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Proof. Define $g(h) = f(x_0 + ha, y_0 + hb)$. Then

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_{\mathbf{u}} f(x_0, y_0).$$

On the other hand, $g(h) = f(x, y)$ where $x = x_0 + ha$ and $y = y_0 + hb$, so by the Chain Rule,

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$

setting $h = 0$ gives $x = x_0$, $y = y_0$, and

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b. \qquad \square$$

Example 23. Let $f(x,y) = xy^3 - x^2$. We will find the directional derivative of f at the point $(1,2)$ in the direction of the angle $\theta = \pi/3$. This angle corresponds to the unit vector $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle = \langle 1/2, \sqrt{3}/2 \rangle$. Using the previous theorem,

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b = (y^3 - 2x) \cdot \frac{\sqrt{3}}{2} + 3xy^2 \cdot \frac{1}{2}$$

By Theorem 22, if $\mathbf{u}$ is a unit vector, then $D_{\mathbf{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \mathbf{u}$. We call the vector $\langle f_x(x,y), f_y(x,y) \rangle$ the gradient of f, denoted $\operatorname{grad} f$ or ∇f. Thus, assuming the hypotheses of Theorem 22, the conclusion can be written $D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$.

Example 24. Let $f(x,y) = \frac{x}{x^2+y^2}$ and $\mathbf{v} = \langle 3, 5 \rangle$. The vector $\mathbf{v}$ is not a unit vector, but

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{24}} \langle 3, 5 \rangle$$

is a unit vector in the direction of $\mathbf{v}$. The gradient of f is

$$\nabla f = \left\langle \frac{y^2 - x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2} \right\rangle.$$

Hence, the direction derivative of f in the direction of $\mathbf{v}$ is

$$\begin{aligned} D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u} &= \left\langle \frac{y^2 - x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2} \right\rangle \cdot \frac{1}{\sqrt{24}} \langle 3, 5 \rangle \\ &= \frac{1}{\sqrt{24}} \left(\frac{-3x^2 - 10xy + 3y^2}{(x^2+y^2)^2} \right). \end{aligned}$$

Hence,

$$D_{\mathbf{u}} f(1,2) = \frac{1}{\sqrt{24}} \left(\frac{-11}{25} \right).$$

If one is only interested in $D_{\mathbf{u}} f(1,2)$, then one can compute $\nabla f(1,2)$ before taking the dot product.

As usual, all of this generalizes easily to functions of three variables. Given a function of three variables $f(x,y,z)$, the gradient is defined as $\nabla f = \langle f_x, f_y, f_z \rangle$. If $\mathbf{u}$ is a unit vector, then we have $D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$.

The next theorem tells us which direction gives the greatest rate of change of a function f.

Theorem 25. Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Proof. We have

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Hence, the maximum value of $D_{\mathbf{u}}$ is $|\nabla f|$ and it occurs when $\theta = 0$, so when $\mathbf{u}$ has the same direction as ∇f. $\square$

Example 26. Consider $f(x, y) = xe^{xy}$. The gradient is given by

$$\nabla f(x, y) = \langle e^{xy} + xye^{xy}, x^2e^{xy}\rangle.$$

At the point $(0, 2)$, $\nabla f(0, 2) = \langle 1, 0\rangle$. Thus, the maximum value of $D_{\mathbf{u}}$ is 1 and it occurs when $\mathbf{u} = \mathbf{i}$.

Suppose S is a surface with equation $F(x, y, z) = k$ (so S is a level surface of a function F of three variables), and let $P(x_0, y_0, z_0)$ be a point on S. Let C be a curve that lies on S and passes through P, with equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, so that $\mathbf{r}(t_0)$ is the position vector of P. Since C lies on S, $F(x(t), y(t), z(t)) = k$. By the Chain Rule,

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0.$$

Equivalently, $\nabla F \cdot \mathbf{r}'(t) = 0$, so for $t = t_0$,

(1) $$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0.$$

If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, we can define the tangent plane to the level surface $F(x, y, z) = k$ at P is the plane through P with normal vector $\nabla F(x_0, y_0, z_0)$. That is, the plane with equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line to S at P is the line passing through P and perpendicular to the tangent plane. Its symmetric equations are given by

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

Example 27. Consider the surface with equation $x = y^2 + z^2 + 1$. Write $F(x, y, z) = y^2 + z^2 - x + 1$. The gradient of F is $\nabla F = \langle -1, 2y, 2z\rangle$. At the point $P(3, 1, -1)$, we have $\nabla F(3, 1, -1) = \langle -1, 2, -2\rangle$. Hence, at this point, the equation of the tangent plane through P is

$$-1(x - 3) + 2(y - 1) - 2(z + 1) = 0,$$

or $-x + 2y - 2z = 3$. The normal line at P is

$$\frac{x - 3}{-1} = \frac{y - 1}{2} = \frac{z + 1}{-2}.$$

7. Maximum and minimum values

Here we consider the two variable problem of locating (local/absolute) maxima and minima.

A function $f(x, y)$ of two variables has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near[5] (a, b). The number $f(a, b)$ is called a local maximum value. Similarly, if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b) and $f(a, b)$ is a local minimum value.

Replacing the condition *when* (x, y) *is near* (a, b) with *for all* (x, y) we get the definitions of absolute maximum and absolute minimum.

Theorem 28. If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (equivalently, $\nabla f = \mathbf{0}$).

Proof. Let $g(x) = f(x, b)$. If f has a local maximum or minimum at (a, b), then g has a local maximum or minimum at (a, b), so $g'(a) = 0$. But $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$. Similarly, $f_y(a, b) = 0$. $\square$

We call a point (a, b) such that $\nabla f(a, b) = \mathbf{0}$ a critical point. At such a point, the equation of the tangent plane is $z = z_0$, so we think that at local maxima, the tangent plane must be horizontal. By the previous theorem, all local maxima are critical points. However, not every critical point gives rise to a local maximum or minimum.

Example 29. Let $f(x, y) = xy - 2x - 2y - x^2 - y^2$. Then $f_x(x, y) = y - 2 - 2x$ and $f_y(x, y) = x - 2 - 2y$. Thus, if $f_x(x, y) = 0$ then $y = 2 + 2x$ and if $f_y(x, y) = 0$ then $x = 2 + 2y$. Putting these together, we have $y = 2 + 2(2 + 2y) = 6 + 4y$, or $y = -2$ and $x = -2$. Hence, $(-2, -2)$ is the only critical point of f. Without graphing software, or more algebraic manipulation, it is difficult to decide whether it is a local maximum or minimum.

Let $f(x, y)$ be a function of two variables with second-order partial derivatives. Define the Hessian of f at (a, b) as the determinant of the matrix

$$D = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}.$$

Hence, by Clairaut's Theorem, $D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

[5]Formally, what we mean is that there is some disk with center (a, b) and all of the points in this disk satisfy the condition.

Second Derivative Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that (a, b) is a critical point of f. Let D be the Hessian of f.

(1) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(2) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(3) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum (in this case, (a, b) is called a saddle point of f.

Example 30. In the previous example,

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

Since $f_{xx}(x, y) = 2 > 0$, then $(-2, -2)$ is a local minimum.

Example 31. Let $f(x, y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$. Then

$$\nabla f = \langle 3x^2 - 6x - 9, 3y^2 - 6y \rangle = 3\langle (x-3)(x+1), y(y-2) \rangle.$$

The critical points occur when $\nabla f = \mathbf{0}$, so they are $(3, 0)$, $(-1, 0)$, $(3, 2)$, $(-1, 2)$. We have

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = \begin{vmatrix} 6(x-1) & 0 \\ 0 & 6(y-1) \end{vmatrix} = 36(x-1)(y-1).$$

We have $D(3, 0) < 0$, $D(-1, 0) < 0$, $D(3, 2) > 0$, and $D(-1, 2) < 0$. Thus, all points are saddle points except $(3, 2)$. Since $f_{xx}(3, 2) > 0$, then $(3, 2)$ is a local minimum.

Example 32. Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible.

Let x, y, and z represent the three numbers. Then $x + y + z = 12$. Consider the function $F(x, y, z) = x^2 + y^2 + z^2$, Solving the first equation for z and plugging into the second gives,

$$\begin{aligned} f(x, y) &= F(x, y, 12 - x - y) = x^2 + y^2 + (12 - x - y)^2 \\ &= 2(x^2 + y^2 + xy - 12x - 12y + 72). \end{aligned}$$

We seek to minimize f. We have $\nabla f = 2\langle 2x + y - 12, 2y + x - 12 \rangle$. Setting $\nabla f = \mathbf{0}$ gives a critical value of $(4, 4)$. A check as above shows that this is a local minimum. Since f is an ellipsoid and may not take on negative values in the domain, then it follows that this is an absolute minimum. Thus, the three numbers are all 4.

A function need not have an absolute minimum or maximum. However, on a closed set, this is guaranteed. This is the point of the Extreme Value Theorem from Calc I. There is an analogous multivariable theorem.

Let $D \subset \mathbb{R}^2$. A point (a, b) is a boundary point if every disk with center (a, b) contains points in D and point not in D. Note that a boundary point need not be an element of D. A closed set in $\mathbb{R}^2$ is one that contains all of its boundary points. A bounded set in $\mathbb{R}^2$ is one that is contained in some disk.

Example 33. The disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$ is a closed set because it contains all of the points on or inside the circle $x^2 + y^2 = 1$. It is trivially a bounded set.

Theorem 34 (Extreme Value Theorem)**.** If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(1) Find the values of f at the critical points of f in D.
(2) Find the extreme values of f on the boundary of D.
(3) The largest of the values above is the absolute maximum and the smallest is the absolute minimum.

Example 35. Let $f(x, y) = x + y - xy$, and let D be the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$. We aim to find the absolute extrema of f.

First we look for the critical points of f. We have $\nabla f = \langle 1-y, 1-x\rangle$ Thus, the only critical point is $(1, 1)$ and $f(1, 1,) = 1$.

Now we consider f on the boundary of D. One side of the triangle (from $(0, 0)$ to $(4, 0)$) is just the x-axis, so $y = 0$ and $0 \leq x \leq 4$. Here, $f(x, 0) = x$. The maximum value is $f(4, 0) = 4$ and the minimum value is $f(0, 0) = 0$. Another side of the triangle (from $(0, 0)$ to $(0, 2)$) is the y-axis, so $x = 0$ and $0 \leq y \leq 2$. Here $f(0, y) = y$. The maximum value is $f(0, 2) = 2$ and the minimum value is $f(0, 0) = 0$.

The final side of the triangle (from $(0, 2)$ to $(4, 0)$) is the line $y = -\frac{1}{2}x + 2$, with $0 \leq x \leq 4$. Here,

$$g(x) = f\left(x, -\frac{1}{2}x + 2\right) = \frac{1}{2}x^2 - \frac{3}{2}x + 2 = \frac{1}{2}(x^2 - 3x + 4).$$

This is a parabola whose vertex is at $x = 3/2$ and $g(3/2) = 7/8$. Otherwise, we must check the endpoints of the interval $[0, 4]$, $g(0) = 2$ and $g(4) = 4$.

We conclude that the absolute maximum value is 4, and this occurs at $(4, 0)$. The absolute minimum value is 0, and this occurs at $(0, 0)$.

8. Lagrange Multipliers

Recall our problem from Example 32: Find three positive numbers whose sum is 12 and the sum of whose squares is as small as possible. We solved by turning our function of three variables into a function of two variables and then finding a (local) minimum. In this section, we'll see another strategy to solve such a problem.

Let $f(x, y)$ be a function of two variables. We aim to find the extreme values of f subject to the constrains $g(x, y) = k$. In other works, we want the largest (or smallest) value of c such that the level curve $f(x, y) = c$ intersects the level curve $g(x, y) = k$. This occurs when the curves have a common tangent line. If this happens at a point (x_0, y_0), then the normal lines of the curves at this point are identical, so the gradient vectors are parallel. This means that there exists a scalar λ such that

$$\nabla f(x_0, y_0) = \lambda g(x_0, y_0).$$

Example 36. Consider $f(x, y) = x + 3y + 2$ subject to the constraint $x^2 + y^2 = 10$. Set $g(x, y) = x^2 + y^2 - 10$. We are looking for maximum values of f that lie on the circle centered at the origin of radius $\sqrt{10}$. We compute $\nabla f = \langle 1, 3\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. We need to find the value of λ such that $\nabla f = \lambda \nabla g$. We have $2\lambda x = 1$. Since $x \neq 0$, then $\lambda = \frac{1}{2x}$. Similarly, $2\lambda y = 3$, so $\lambda = \frac{3}{2y}$. Setting these two values of λ equal and solving gives $y = 3x$. Substituting this into the constraint g gives $0 = g(x, 3x) = 10x^2 - 10$, so $x = \pm 1$. Thus, the possible solutions are $(1, 3)$ and $(-1, -3)$. Plugging these into f we get $f(1, 3) = 12$ and $f(-1, -3) = -8$, giving the maximum and minimum values, respectively.

Example 37. A company manufactures x units of one item and y units of another. The total cost in dollars, C, of producing these two items is approximated by the function

$$C(x, y) = 5x^2 + 2xy + 3y^2 + 800.$$

The combined production quota is 39, which means that the constraint is $g(x, y) = x+y-39$. We wish to find the minimum production cost. We have $\nabla f = \lambda \nabla g$, so

$$(10x + 2y)\mathbf{i} + (2x + 6y)\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j}).$$

Clearly, $x, y \neq 0$. Thus, we have $8x = 4y$, so $y = 2x$. Substituting this into the constraint, we get $x + 2x = 39$, so $x = 13$ and $y = 26$. Thus, the minimum cost is \$4349.

Suppose $f(x, y, z)$ is subject to the constraint $g(x, y, z) = k$, and that f has an extreme value at $P(x_0, y_0, z_0)$ on the surface S. Let C be a curve on S that passes through P with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Suppose t_0 is the parameter value such that

$\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Set $h(t) = f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$. Since f has an extreme value at t_0, so does h and thus $h'(t_0) = 0$. By the chain rule,

$$\begin{aligned} 0 = h'(t_0) &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0). \end{aligned}$$

Hence, $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ for every such curve C. From our discussion of tangent planes to level surfaces, we know that $\nabla g(x_0, y_0, z_0)$ is also orthogonal to $\mathbf{r}'(t_0)$ (see (1)). Hence, $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel, so if $\nabla g(x_0, y_0, z_0) \neq 0$, then there exists a scalar λ, called a Lagrange multiplier, satisfying

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

Example 38. An open rectangular box has volume 32 cm^3. What are the lengths of the edges giving the minimum surface area?

Strategy 1: Let $V = xyz$ where x, y, z represent the dimensions of the box. Then $xyz = 32$. Since $x, y, z \neq 0$, then $z = \frac{32}{xy}$. The function for surface area is $S = 2xy + 2xz + yz$ (because there is no top). We want to minimize S. We rewrite as

$$S = 2xy + 2xz + yz = 2xy + \frac{64}{y} + \frac{32}{x}.$$

Then $\nabla S = \langle 2y - \frac{32}{x^2}, 2x - \frac{64}{y^2} \rangle$. Setting each component equal to zero gives $x^4 - 8x = 0$. Thus, $x = 0$ or $x = 2$. Since we assume $x \neq 0$, then we must have $x = 2$ and in this case, $y = 4$ and $z = 4$. By the second derivative test, this is a local minimum. One can then check that as x and y are increased, or decreased, the surface area increases. Thus, $(2, 4)$ is a global minimum and the value of S at this point is $S = 48$.

Lagrange Multipliers: Let f the surface area function $f(x, y, z) = 2xy + 2xz + yz$, and let g be the volume function (constraint) $g(x, y, z) = xyz - 32$. Then $\nabla f = \lambda \nabla g$, so

$$(2y + 2z)\mathbf{i} + (2x + z)\mathbf{j} + (2x + y)\mathbf{k} = \lambda(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}).$$

We know that none of x, y, or z is zero. The second two equations give us

$$x\lambda = \frac{2x + z}{z} = \frac{2x + y}{y}.$$

Simplifying the right-hand equality gives $y = z$. The equation $2y + 2z = \lambda yz$ now gives $\lambda = \frac{4}{y}$. Substituting this into the equation $2x + y = \lambda xy$, we get $y = 2x$. Thus, using the constraint, we have $32 = xyz = x(2x)(2x)$, so $x = 2$. Thus, $x = 2$, $y = 4$, $z = 4$ is the global minimum subject to the constraint.

Calculus III

Chapter 15 - Multiple integrals

1. Double integrals over rectangles

Suppose S is a surface with graph $z = f(x, y)$ is defined on the rectangular region

$$R = [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}.$$

Suppose, for the time being, that $z \geq 0$ for all $(x, y) \in R$. We now confront the problem of approximating, and eventually finding exactly, the volume under S (between S and the xy-plane).

First, we partition $[a, b]$ into regular subintervals $[x_{i-1}, x_i]$ of length Δx. Similarly, we partition $[c, d]$ into regular subintervals $[y_{i-1}, y_i]$ of length Δy (we need not have $\Delta x = \Delta y$). This in turn partitions R into regular subintervals

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) : x_{i-1} \leq x \leq x_i, y_{i-1} \leq y \leq y_i\}$$

of area $\Delta A = \Delta x \Delta y$. Choose a sample point (x^*_{ij}, y^*_{ij}) in each R_{ij}. Then the box with volume $f(x^*_{ij}, y^*_{ij})\Delta A$ approximates the volume under S on the region R_{ij} (it may be an over or under estimate). We sum over all such rectangles to get an estimate of volume under S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A.$$

(These are called double Riemann sums).

As the area of the R_{ij} become smaller, the approximation improves. Thus,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A.$$

This leads directly to the definition of the double integral.

The double integral of f over the rectangle R is

$$\iint_R f(x, y)\ \mathrm{dA} = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A$$

if this limit exists. We say that f is integrable if for every $\epsilon > 0$, there exists an integer N such that

$$\left| \iint_R f(x,y)\ \mathrm{dA} - \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \right| < \epsilon$$

for every choice of sample points and for all integers $m, n > N$.

Example 1. Let $R = [0,4] \times [-1,2]$. We will approximate $\iint_R (1 - xy^2)\ \mathrm{dA}$ using Riemann sums with $m = 2$ and $n = 3$. Thus, we partition R into 6 intervals:

$$\begin{aligned} R_{11} &= [0,2] \times [-1,0] \qquad & R_{12} &= [0,2] \times [0,1] \qquad & R_{13} &= [0,2] \times [1,2] \\ R_{21} &= [2,4] \times [-1,0] \qquad & R_{22} &= [2,4] \times [0,1] \qquad & R_{23} &= [2,4] \times [1,2]. \end{aligned}$$

Note that every R_{ij} has area $\Delta A = 2$. In each region, we choose our sample point to be the lower right corner. Thus, our points are

$$\begin{aligned} (x_{11}*, y_{11}^*) &= (2,-1) \qquad & (x_{12}*, y_{12}^*) &= (2,0) \qquad & (x_{13}*, y_{13}^*) &= (2,1) \\ (x_{21}*, y_{21}^*) &= (4,-1) \qquad & (x_{22}*, y_{22}^*) &= (4,0) \qquad & (x_{23}*, y_{23}^*) &= (4,1). \end{aligned}$$

Let $f(x,y) = 1 - xy^2$. Then

$$\iint_R (1 - xy^2)\ \mathrm{dA} \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(x_{ij}^*, y_{ij}^*)\Delta A = -12.$$

This tells us that more of the surface lives under the xy-plane.

Keeping the above setup, we now see how to integrate formally using the Fundamental Theorem of Calculus. By $\int_c^d f(x,y)\ \mathrm{dy}$, we mean that x is held fixed and we integrate with respect to y. This is called *partial integration with respect to y*. This defines a function of x,

$$A(x) = \int_c^d f(x,y)\ \mathrm{dy}.$$

Now we integrate A with respect to x, so

$$\int_a^b A(x)\ \mathrm{dx} = \int_a^b \left[\int_c^d f(x,y)\ \mathrm{dy} \right]\ \mathrm{dx} = \int_a^b \int_c^d f(x,y)\ \mathrm{dy}\ \mathrm{dx}.$$

We call this an iterated integral. Note that we could have done all this the other way (holding y constant originally) to get the iterated integral,

$$\int_c^d \int_a^b f(x,y)\ \mathrm{dx}\ \mathrm{dy} = \int_c^d \left[\int_a^b f(x,y)\ \mathrm{dx} \right]\ \mathrm{dy}.$$

Fubini's Theorem If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)\ \mathrm{dA} = \int_a^b \int_c^d f(x,y)\ \mathrm{dy}\ \mathrm{dx} = \int_c^d \int_a^b f(x,y)\ \mathrm{dx}\ \mathrm{dy}.$$

Fubini's Theorem says that, so long is f is continuous, we can integrate in either order. However, it will often be the case that one order is much easier than another. Consider the following problem from the reading assignment.

Example 2. Consider $\iint_R xe^{xy}\ \mathrm{dA}$ on $R = [-1, 2] \times [0, 1]$. By Fubini's Theorem,

$$\iint_R xe^{xy}\ \mathrm{dA} = \int_0^1 \int_{-1}^2 xe^{xy}\ \mathrm{dy}\ \mathrm{dx}.$$

Set $u = xy$, so $du = x$ dy. Then

$$\begin{aligned}\int_{-1}^2 \int_0^1 xe^{xy}\ \mathrm{dy}\ \mathrm{dx} &= \int_{-1}^2 \int_0^x e^u\ \mathrm{du}\ \mathrm{dx} = \int_{-1}^2 [e^u]_0^x\ \mathrm{dx} \\ &= \int_{-1}^2 e^x - 1\ \mathrm{dx} = [e^x - x]_{-1}^2 \\ &= (e^2 - 2) - (e^{-1} + 1) = e^2 - e^{-1} - 3.\end{aligned}$$

Example 3. Evaluate the integral,

$$\int_1^3 \int_1^5 \frac{\ln y}{xy}\ \mathrm{dy}\ \mathrm{dx}.$$

The function $\frac{\ln y}{xy}$ can be factored into a function of x and y. That is, $\frac{\ln y}{xy} = \frac{1}{x}\frac{\ln y}{y}$. Hence, Fubini's Theorem gives,

$$\int_1^3 \int_1^5 \frac{\ln y}{xy}\ \mathrm{dy}\ \mathrm{dx} = \left(\int_1^3 \frac{1}{x}\ \mathrm{dx}\right)\left(\int_1^5 \frac{\ln y}{xy}\ \mathrm{dy}\right).$$

Set $u = \ln y$, so $\mathrm{du} = \frac{1}{y}$ dy. Then

$$\left(\int_1^3 \frac{1}{x}\ \mathrm{dx}\right)\left(\int_1^5 \frac{\ln y}{y}\ \mathrm{dy}\right) = \left([\ln x]_1^3\right)\left(\int_0^{\ln 5} u\ \mathrm{dy}\right) = (\ln 3)\left(\left[\frac{1}{2}u^2\right]_0^{\ln 5}\right) = \frac{(\ln 5)^2(\ln 3)}{2}.$$

2. Double integrals over general regions

In this section, we consider how to evaluate $f(x,y)$ over a more general region D (that is, D need not be a rectangle). Suppose D is bounded, so that we can enclose it in a rectangular region R. Define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \text{ but } (x,y) \notin D. \end{cases}$$

If F is integrable over R, then we define the double integral of f over D as

$$\iint_D f(x,y)\ \mathrm{dA} = \iint_R F(x,y)\ \mathrm{dA}.$$

In practice, this cannot be evaluated in general. We will consider a few special cases when it is possible to compute.

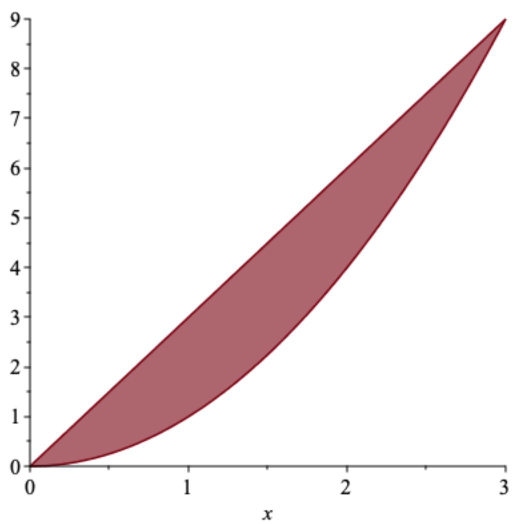
A plane region D is said to be of type I if it lies between the graphs of two continuous functions of x, so

$$D = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}.$$

If f is continuous on D, then

$$\iint_D f(x,y)\ \mathrm{dA} = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ \mathrm{dy}\ \mathrm{dx}.$$

Example 4. We will evaluate $\iint_D xy\ \mathrm{dA}$ where D is enclosed by the curves $y = x^2$ and $y = 3x$. This is a type I region. The x-bounds are 0 and 3 (these are the points of intersection of the two curves).



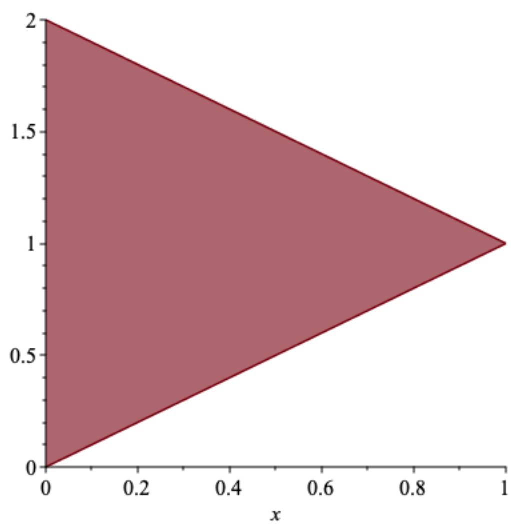
$$\begin{aligned} \iint_D xy\ \mathrm{dA} &= \int_0^3 \int_{x^2}^{3x} xy\ \mathrm{dy}\ \mathrm{dx} \\ &= \int_0^3 \left[\frac{1}{2}xy^2\right]_{x^2}^{3x} \mathrm{dy}\ \mathrm{dx} \\ &= \frac{1}{2}\int_0^3 9x^3 - x^5\ \mathrm{dx} \\ &= \frac{1}{2}\left[\frac{9}{4}x^4 - \frac{1}{6}x^6\right]_0^3 = \frac{243}{8}. \end{aligned}$$

Example 5. Consider the previous example, but we regard D as the region enclosed by $x = \sqrt{y}$ and $x = \frac{1}{3}y$. This is a type II region. The y-bounds are 0 and 9. Then

$$\iint_D xy \text{ dA} = \int_0^9 \int_{\frac{1}{3}y}^{\sqrt{y}} xy \text{ dx dy} = \int_0^9 \left[\frac{1}{2}x^2 y\right]_{\frac{1}{3}y}^{\sqrt{y}} \text{ dy} = \frac{1}{2}\int_0^9 y^2 - \frac{1}{9}y^3 \text{ dy}$$
$$= \frac{1}{2}\left[\frac{1}{3}y^3 - \frac{1}{36}y^4\right]_0^9 = \frac{1}{2}\left[\frac{1}{3}9^3 - \frac{1}{36}9^4\right] = \frac{243}{8}.$$

Example 6. We will find the volume of the region bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$.

Write $f(x, y) = x$. Note that, if $x < 0$, then $z < 0$ (because we are bounded by $z = x$). Similar logic now implies that $y > 0$, so we are in the first octant. Thus, we integrate f over the region D bounded by $x + y = 2$, $y = x$, and $x = 0$ in the xy-plane. This is the triangular region below.



$$\begin{aligned} V &= \int_0^1 \int_x^{2-x} x \text{ dy dx} \\ &= \int_0^1 [xy]_x^{2-x} \text{ dx} \\ &= \int_0^1 (x(2-x) - x(x)) \text{ dx} \\ &= 2\int_0^1 x - x^2 \text{ dx} \\ &= 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}. \end{aligned}$$

Properties of Double Integrals Let D be a region, suppose f and g are integrable over D, and let c be a constant. Then

$$\iint_D (f(x,y) + g(x,y)) \text{ dA} = \iint_D f(x,y) \text{ dA} + \iint_D g(x,y) \text{ dA}$$
$$\iint_D cf(x,y) \text{ dA} = c\iint f(x,y) \text{ dA}.$$

If $f(x, y) \geq g(x, y)$ for all $(x, y) \in D$, then

$$\iint_D f(x,y) \text{ dA} \geq \iint_D g(x,y) \text{ dA}.$$

If $D = D_1 \cup D_2$, where $D_1 \cap D_2$ do not overlap except perhaps on the boundary. Then

$$\iint f(x,y)\ \mathrm{dA} = \iint_{D_1} f(x,y)\ \mathrm{dA} + \iint_{D_2} f(x,y)\ \mathrm{dA}.$$

Integrating the constant function $f(x,y) = 1$ gives the area of D,

$$\iint_D 1\ \mathrm{dA} = A(D).$$

It follows that if $m \leq f(x,y) \leq M$ for all $(x,y) \in D$, where m and M are constants, then

$$mA(D) \leq \iint_D f(x,y)\ \mathrm{dA} \leq MA(D).$$

3. Double integrals in polar coordinates

Certain regions (like circles) are easier to integrate in polar coordinates. Recall that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad , y = r\sin\theta.$$

We will not go through the formality of setting up integration in polar coordinates as it is merely a conglomeration of what you learned in Calc II and our previous discussions on double integration. We will only remark that any time we make a change of variable, there is a price to be paid in the integration (think u-substitution).

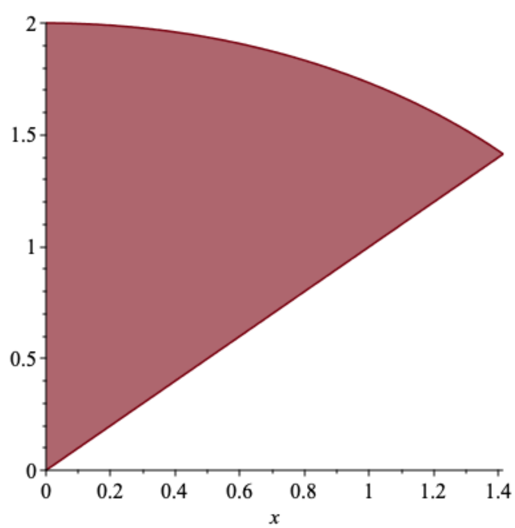
If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$ and $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\ \mathrm{dA} = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\ \mathrm{dr}\ \mathrm{d}\theta.$$

Notice the extra factor of r that has appeared. Rather than spend too much time discussing why the r is here, we will move on to examples and come back to a much more general explanation in a later section (15.9).

Example 7. We will evaluate the integral $\iint_R (2x - y)\ \mathrm{dA}$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

The region R is shown below. In polar coordinates, this is the region bounded by $0 \le r \le 2$ and $\pi/4 \le \theta \le \pi/2$



$$\begin{aligned}
&\iint_R (2x - y)\ \mathrm{dA} \\
&= \int_{\pi/4}^{\pi/2} \int_0^2 (2r\cos\theta - r\sin\theta) r\ \mathrm{dr}\ \mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/2} \int_0^2 r^2(2\cos\theta - \sin\theta)\ \mathrm{dr}\ \mathrm{d}\theta \\
&= \int_{\pi/4}^{\pi/2} (2\cos\theta - \sin\theta) \left[\frac{1}{3}r^3\right]_0^2\ \mathrm{d}\theta \\
&= \frac{8}{3}\int_{\pi/4}^{\pi/2} 2\cos\theta - \sin\theta\ \mathrm{d}\theta \\
&= \frac{8}{3}\left[2\sin\theta + \cos\theta\right]_{\pi/4}^{\pi/2} \\
&= \frac{8}{3}\left[(2+0) - (\sqrt{2} + \sqrt{2}/2)\right) = \frac{1}{3}\left(16 - 12\sqrt{2}\right).
\end{aligned}$$

There is also a notion of type I and type I polar regions. If f is continuous on a polar region of the form $D = \{(r, \theta) : \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$ then

$$\iint_D f(x, y) \text{ dA} = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r \text{ dr d}\theta.$$

The second type is defined similarly.

Example 8. We will find the area of the region enclosed by the cardiods

$$r = 1 + \cos\theta \quad \text{and} \quad r = 1 - \cos\theta.$$

When $\cos\theta \geq 0$, we have $1 + \cos\theta \geq 1 - \cos\theta$. The intersection of these two cardiods occur when $2\cos\theta = 0$, or when $\theta = \pm\pi/2$. By symmetry, the area is given by integrating the constant function $f(x, y) = 1$ as r changes from 0 to $1 - \cos\theta$ and $-\pi/2 \leq \theta \leq \pi/2$. Hence,

$$\begin{aligned}
A(D) &= \iint_D 1 \text{ dA} = \int_{-\pi/2}^{\pi/2} \int_0^{1-\cos\theta} r \text{ dr d}\theta \\
&= \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}r^2\right]_0^{1-\cos\theta} \text{d}\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 - \cos\theta)^2 \text{ d}\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 - 2\cos\theta + \cos^2\theta \text{ d}\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} 1 - 2\cos\theta + \frac{1}{2}(1 + \cos(2\theta)) \text{ d}\theta \\
&= \frac{1}{2}\left[\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{2}\left[((\pi/2) - 2 + \pi/4) - ((-\pi/2) + 2 - \pi/4)\right] = \frac{3\pi}{4} - 2.
\end{aligned}$$

Example 9. Consider the region R inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

Note that the sphere intersects the xy-plane in the circle $x^2 + y^2 = 16$ while the cylinder (obviously) intersects the xy-plane in the circle $x^2 + y^2 = 4$. We integrate the sphere over the region bounded between the circles. By symmetry, we have

$$\begin{aligned}
V &= 2\iint_{4 \leq x^2 + y^2 \leq 16} \sqrt{16 - x^2 - y^2} \text{ dA} \\
&= 2\int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\, r \text{ dr d}\theta = 2\int_0^{2\pi} \text{d}\theta \int_2^4 \sqrt{16 - r^2} r \text{ dr} \\
&= 2\left[\theta\right]_0^{2\pi} \left[-\frac{1}{3}(16 - r^2)^{3/2}\right]_2^4 = -\frac{2}{3}(2\pi)\left(0 - 12^{3/2}\right) = 32\pi\sqrt{3}.
\end{aligned}$$

Suppose a lamina (i.e., a thin plate) occupies a region D of the xy-plane and that its density (in units of mass per unit area0 at a point $(x, y) \in D$ is given by $\rho(x, y)$, where ρ is a continuous function on D. Consider a small rectangle of D, containing the point (x, y), and take Δm to be the mass of the rectangle and ΔA to be the area of the rectangle. Then

$$\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$$

where the limit is taken as the dimensions of the rectangle approach 0.

Then to find the mass of the rectangle we divide D into small regular rectangular regions R_{ij}. We choose a sample point (x_{ij}^*, y_{ij}^*) in each region and take $\rho(x_{ij}^*, y_{ij}^*)$ to be zero when the sample point lies outside of D. Thus, the mass of R_{ij} can be approximated by $\rho(x_{ij}^*, y_{ij}^*)\Delta A$, while the mass of the lamina can be approximated by

$$m \approx \sum_{i=1}^{k} \sum_{j=1}^{\ell} \rho(x_{ij}^*, y_{ij}^*)\Delta A.$$

Taking limits (assuming they exist) gives

$$m = \lim_{k,\ell \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{\ell} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x, y) \text{ dA}.$$

Example 10. Let D be the triangular region enclosed by the lines $y = 0$, $y = 2x$, and $x + 2y = 1$. Consider a lamina that occupies the region D with density function $\rho(x, y) = x$. We will find the mass of the lamina. The region is best described as the intersection of the lines $x = \frac{1}{2}y$ and $x = 1 - 2y$. Note that these intersect when $y = 2/5$. Hence,

$$\begin{aligned} m &= \int_0^{2/5} \int_{\frac{1}{2}y}^{1-2y} x \text{ dx dy} \\ &= \int_0^{2/5} \int_{\frac{1}{2}y}^{1-2y} \left[\frac{1}{2}x^2\right]_{\frac{1}{2}y}^{1-2y} \text{ dx dy} \\ &= \int_0^{2/5} \frac{7}{2}y^2 - 4y + 1 \text{ dy} = \frac{2}{25}. \end{aligned}$$

Continuing with the above, we can approximate the moment of the rectangle R_{ij} with respect to the x-axis by $\left[\rho(x_{ij}^*, y_{ij}^*)\Delta A\right] y_{ij}^*$. Hence, the moment of the lamina about the x-axis is

$$M_x = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\rho(x, y) \text{ dA}.$$

Similarly, the moment of the lamina about the y-axis is

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^*\rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x,y)\ \mathrm{dA}.$$

The center of mass $(\overline{x}, \overline{y})$ where $m\overline{x} = M_y$ and $m\overline{y} = M_x$.

Example 11. Keep the setup of the previous example. Then we have

$$M_x = \int_0^{2/5}\int_{\frac{1}{2}y}^{1-2y} yx\ \mathrm{dx}\ \mathrm{dy} = \frac{7}{750}, \qquad M_y = \int_0^{2/5}\int_{\frac{1}{2}y}^{1-2y} x^2\ \mathrm{dx}\ \mathrm{dy} = \frac{31}{750}.$$

Hence, the center of mass is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1}{m}M_y = \frac{7}{60} \quad \text{and} \quad \overline{y} = \frac{1}{m}M_x = \frac{31}{60}.$$

In a similar way to how we computed the length of a curve using single integrals, so too can we compute the surface area of a surface using double integrals. The area of the surface with equation $z = f(x,y)$, $(x,y) \in D$, where f_x and f_y are continuous is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ \mathrm{dA}.$$

Example 12. Consider the sphere $x^2 + y^2 + z^2 = 4$. We will find the surface area of the sphere that lies above the plane $z = 1$.

These surfaces intersect on the curve $x^2 + y^2 + 1 = 4$, or $x^2 + y^2 = 3$. That is, on the circle centered at the origin of radius $\sqrt{3}$. The sphere itself is described by the equation $z = f(x,y) = \sqrt{4 - x^2 - y^2}$ (since we only care about part of the upper hemisphere). We will integrate by converting to polar coordinates,

$$\begin{aligned}
A(S) &= \iint_{0\le x^2+y^2\le\sqrt{3}} \sqrt{1 + (f_x)^2 + (f_y)^2}\ \mathrm{dA} \\
&= \iint_{0\le x^2+y^2\le\sqrt{3}} \sqrt{1 + \left[(-x)\left(4 - x^2 - y^2\right)^{-1/2}\right]^2 + \left[(-y)\left(4 - x^2 - y^2\right)^{-1/2}\right]^2}\ \mathrm{dA} \\
&= \iint_{0\le x^2+y^2\le\sqrt{3}} \sqrt{1 + \left(x^2 + y^2\right)\left(4 - x^2 - y^2\right)^{-1}}\ \mathrm{dA} \\
&= \iint_{0\le x^2+y^2\le\sqrt{3}} \sqrt{\frac{4 - x^2 - y^2 + (x^2 + y^2)}{4 - x^2 - y^2}}\ \mathrm{dA} = \int_0^{\sqrt{3}}\int_0^{2\pi} \frac{2r}{\sqrt{4 - r^2}}\ \mathrm{d}\theta\ \mathrm{dr} \\
&= 2\pi\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}}\ \mathrm{dr} \quad \text{let } u = 4 - r^2 \text{ so du} = -2r\ \mathrm{dr} \\
&= -2\pi\int_4^1 u^{-1/2}\ \mathrm{du} = -2\pi\left[2u^{1/2}\right]_4^1 = -4\pi(1 - 2) = 4\pi.
\end{aligned}$$

6. Triple Integrals

We will approach triple integrals primarily through analogy with single and double integrals. Let f be a function of three variables defined (for now) on the set

$$B = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}.$$

That is, B is a rectangular box. We partition the x-interval into ℓ subintervals of length Δx, the y-interval into m subintervals of length Δy, and the z-interval into n subintervals of length Δz. This divides B into lmn subboxes of Volume $\Delta v = \Delta x \Delta y \Delta z$, with

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k].$$

Choosing sample points in each interval gives a triple Riemann sum whose limit defines the triple integral over B,

$$\iiint_B f(x, y, z)\ \mathrm{dV} = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V.$$

If f is continuous, then the three-variable version of Fubini's Theorem says,

$$\iiint_B f(x, y, z)\ \mathrm{dV} = \int_r^s \int_c^d \int_a^b f(x, y, z)\ \mathrm{dx\ dy\ dz},$$

where the integration may be taken in any order.

Example 13. Let $E = \{(x, y, z) : 0 \leq x \leq 2, 0 \leq y \leq 1, 0 \leq z \leq 3\}$. Then

$$\begin{aligned}
\iiint_E (xy + z^2)\ \mathrm{dV} &= \int_0^2 \int_0^1 \int_0^3 (xy + z^2)\ \mathrm{dz\ dy\ dx} = \int_0^2 \int_0^1 \left[xyz + \frac{1}{3}z^3\right]_0^3\ \mathrm{dy\ dx} \\
&= \int_0^2 \int_0^1 3xy + 9\ \mathrm{dy\ dx} = 3\int_0^2 \left[\frac{1}{2}xy^2 + 3y\right]_0^1\ \mathrm{dx} \\
&= 3\int_0^2 \frac{1}{2}x + 3\ \mathrm{dx} = 3\left[\frac{1}{4}x^2 + 3x\right]_0^2 = 21.
\end{aligned}$$

We also have a notion of the more general regions. A solid region E is type 1 if it lies between the graph of two continuous functions of x and y:

$$E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Then,

$$\iiint_E f(x, y, z)\ \mathrm{dV} = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ \mathrm{dz}\right]\ \mathrm{dA}.$$

Now suppose that the projection of D onto the xy-plane is a type I plane region, that is,

$$E = \{(x, y, z) : (x, y) \in D, a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Then,

$$\iiint_E f(x,y,z)\ \mathrm{dV} = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ \mathrm{dz\ dy\ dx}.$$

A similar result holds if the projection of D is a type II plane region. We have now seen two types, Type 1-I and Type 1-II. There are a total of six such types.

Example 14. Suppose E lies below the plane $z = x$ and above the triangular region with vertices $(0,0,0)$, $(\pi,0,0)$, and $(0,\pi,0)$. We will attempt to evaluate $\iiint_E \sin y\ \mathrm{dV}$.

Let D be the projection of the given triangular region in the xy-plane. Then D is type I with $0 \le x \le \pi$ and $0 \le y \le \pi - x$. As z is bounded between the xy-plane and $z = x$, then E is type 1-I. Thus,

$$\begin{aligned}
\iiint_E \sin y\ \mathrm{dV} &= \int_0^\pi \int_0^{\pi-x} \int_0^x \sin y\ \mathrm{dz\ dy\ dx} = \int_0^\pi \int_0^{\pi-x} [z\sin y]_0^x\ \mathrm{dy\ dx} \\
&= \int_0^\pi \int_0^{\pi-x} x\sin y\ \mathrm{dy\ dx} = \int_0^\pi [-x\cos y]_0^{\pi-x}\ \mathrm{dx} \\
&= \int_0^\pi -x\left(\cos(\pi-x) - 1\right)\ \mathrm{dx} \\
&= \int_0^\pi x(1+\cos(x))\ \mathrm{dx} \quad (\text{trig id: } \cos(\pi-\theta) = -\cos(\theta)) \\
&= \left[\frac{1}{2}x^2 + x\sin x + \cos x\right]_0^\pi \quad (\text{using integration-by-parts}) \\
&= \frac{\pi^2}{2} - 2.
\end{aligned}$$

Recall that $\int_a^b 1\ \mathrm{dx} = b - a$, which is the length of the interval $[a,b]$. Similarly, we have seen that $\iint_D 1\ \mathrm{dA} = A(D)$, the area of the region D. The same idea applies here. If $E \subset \mathbb{R}^3$, then $\iiint_E 1\ \mathrm{dV} = V(E)$, the volume of the region E.

Example 15. Find the volume of the solid E enclosed by the paraboloids $y = x^2 + z^2$ and $y = 8 - x^2 - z^2$.

When these paraboloids intersect, we have $8 - x^2 - z^2 = x^2 + z^2$, or $x^2 + z^2 = 4$. Thus, the projection of their intersection in the xz-plane is a circle D of radius 2 centered at the origin. We make a change of variable to polar coordinates, setting $x = r\cos\theta$ and $z = r\sin\theta$. Thus,

$$\begin{aligned}
\iiint_E 1\ \mathrm{dV} &= \iint_D \int_{x^2+z^2}^{8-x^2-z^2} 1\ \mathrm{dy\ dA} = \iint_D 8 - 2(x^2+z^2)\ \mathrm{dA} \\
&= \int_0^2 \int_0^{2\pi} (8-2r^2)r\ \mathrm{d\theta\ dr} = 4\pi \int_0^2 4r - r^3\ \mathrm{dr} = 4\pi\left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 16\pi.
\end{aligned}$$

As we saw in the Example 15, it is often advantageous to make a change of variable to evaluate a triple integral, just as it is for single and double integrals. In this section, we see two such changes that are both generalizations of polar coordinates. In the last section of this chapter, we investigate more general coordinate changes.

In the cylindrical coordinate system, we envision each point in three-space sitting on a cylinder centered at the origin. That is, each point P is represented by the ordered triple (r, θ, z), where (r, θ) are the polar coordinates of the projection of P onto the xy-plane. To convert to rectangular coordinates, we use the formulas

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

and to convert from rectangular coordinates to cylindrical coordinates we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z.$$

We see that working with cylindrical coordinates is not significantly different from working with polar coordinates.

Suppose E is a type 1 region whose projection D onto the xy-plane is described in polar coordinates. That is, $D = \{(r, \theta) : \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ and

$$E = \{(x, y, z) : (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

Then

$$\iiint_E f(x, y, z)\ \mathrm{dV} = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(x,y)}^{u_2(x,y)} f(r\cos\theta, r\sin\theta, z) r\ \mathrm{dz}\ \mathrm{dy}\ \mathrm{dx}.$$

Pay attention to the factor of r, which appears for the same reason as when working with polar coordinates.

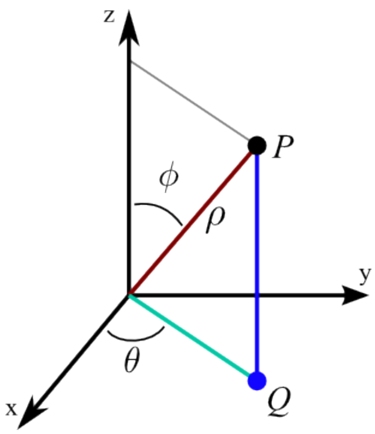
Example 16. Let E be the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$. We will find the volume of E.

In cylindrical coordinates, E is the cylinder $r = 2$ bounded above by the sphere $r^2 + z^2 = 4$. Hence, $E = \{(r, \theta, z) : 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4 - r^2} \le z \le \sqrt{4 - r^2}\}$. Then,

$$\iiint_E \mathrm{dV} = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\ \mathrm{dz}\ \mathrm{dr}\ \mathrm{d}\theta = \int_0^{2\pi} \int_0^1 2r\sqrt{4 - r^2}\ \mathrm{dr}\ \mathrm{d}\theta$$

$$= 2\pi\left[-\frac{2}{3}(4 - r^2)^{3/2}\right]_0^1 = \frac{4\pi}{3}\left(8 - 3^{3/2}\right).$$

In other cases, it is useful to think of points in three-dimensional space as sitting on the surface of a sphere. In some ways, this is the true analogue of polar coordinates in three dimensions. Here, a point is described by a radius, ρ and two angles, θ and ϕ.



(Image: Nykamp DQ, "Spherical coordinates." From Math Insight. `http://mathinsight.org/image/spherical_coordinates`)

To convert from rectangular to spherical coordinates, we use the equations

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi.$$

By the distance formula,

$$\rho^2 = x^2 + y^2 + z^2.$$

If E is a spherical wedge given by

$$E = \{(x, y, z) : a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\},$$

then the formula for triple integration in spherical coordinates is

$$\iiint_E f(x, y, z) \text{ dV} = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \text{ d}\rho \text{ d}\theta \text{ d}\phi.$$

Here, the factor of $\rho^2 \sin\phi$ is the price to be paid for change of variable. In the next section we will see the general argument for factors such as these.

Example 17. Let E be the region that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 1$. We will evaluate $\iiint_E y^2 z^2 \text{ dV}$.

Here E is the region

$$E = \{(x, y, z) : 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/3\}.$$

Hence, making a change to spherical coordinates gives

$$
\begin{aligned}
&\iiint_E y^2 z^2 \ \mathrm{dV} \\
&\qquad = \int_0^{\pi/3} \int_0^{2\pi} \int_0^1 (\rho \sin\phi \sin\theta)^2 (\rho\cos\phi)^2 \rho^2 \sin\phi \ \mathrm{d}\rho \ \mathrm{d}\theta \ \mathrm{d}\phi \\
&\qquad = \int_0^{\pi/3} \sin^3\phi \cos^2\phi \ \mathrm{d}\phi \int_0^{2\pi} \sin^2\theta \ \mathrm{d}\theta \int_0^1 \rho^6 \ \mathrm{d}\rho \\
&\qquad = \left(\int_0^{\pi/3} \sin\phi (1-\cos^2\phi)\cos^2\phi \ \mathrm{d}\phi\right) \left(\int_0^{2\pi} \frac{1}{2}(1-\cos(2\theta)) \ \mathrm{d}\theta\right) \left(\int_0^1 \rho^6 \ \mathrm{d}\rho\right) \\
&\qquad = \left[\frac{1}{5}\cos^5\phi - \frac{1}{3}\cos^3\phi\right]_0^{\pi/3} \left[\frac{1}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\right]_0^{2\pi} \left[\frac{1}{7}\rho^7\right]_0^1 \\
&\qquad = \frac{47}{480} \cdot \frac{1}{7} \cdot \pi = \frac{47\pi}{3360}
\end{aligned}
$$

9. Change of variables in multiple integrals

We have seen several examples of how a change of variable(s) can be useful in simplifying integration. This includes u-substitution, polar coordinates, cylindrical coordinates, and spherical coordinates. In this section we study change of variables for multiple integrals in broader terms. We will also see precisely how to determine the extra factors that arise in this substitutions.

First, consider the integrable function $f(x)$ on $[a,b]$. If $x = g(u)$, $a = g(c)$, and $b = g(d)$, then the Substitution Rule (u-substitution) says that

$$\int_a^b f(x)\ \mathrm{dx} = \int_c^d f(g(u))g'(u)\ \mathrm{du} = \int_c^d f(x(u))\frac{\mathrm{dx}}{\mathrm{du}}\ \mathrm{du}.$$

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function from the uv-plane to the xy-plane. The uv-plane isn't really any different from the xy-plane, but it is useful to use a different pair of variables. Then, through T, x and y are related to u and v by the equations

$$x = g(u,v) \qquad y = h(u,v).$$

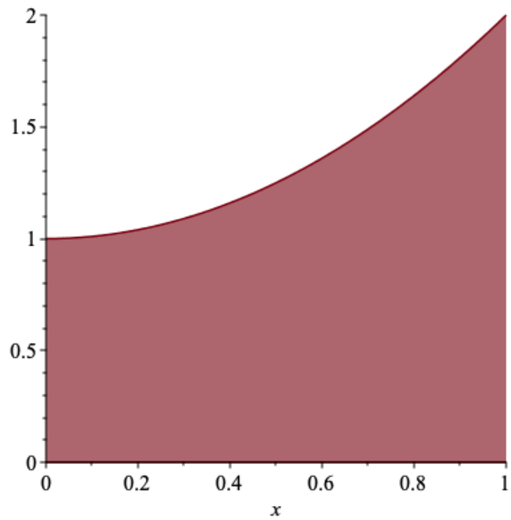
Sometimes we suppress the g, h notation and just write $x = x(u,v)$, and $y = y(u,v)$. We will usually assume that g and h have continuous first-order partial derivatives (that is, T is a C^1 transformation). The transformation T is said to be one-to-one (or injective) if $T(u_1,v_1) \neq T(u_2,v_2)$ whenever $(u_1,v_1) \neq (u_2,v_2)$ (that is, different preimages have different images). When T is one-to-one, it has an inverse transformation $T^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T^{-1}(T(u,v)) = (u,v)$.

Example 18. Let S be the region bounded by the lines $u = 0$, $u = 1$, $v = 0$, and $v = 1$. Let T be the transformation determined by $x = v$ and $y = u(1 + v^2)$.

Here, S is the rectangular region $S = \{(u,v) : 0 \leq u \leq 1, 0 \leq v \leq 1\}$. The transformation T carries the boundary of S to the boundary of the image $T(S)$. Thus, we record what T does to the four boundary lines of S:

$$\begin{aligned}
\ell_1 &: (v = 0, 0 \leq u \leq 1) \longrightarrow (x = 0, y = u) \\
\ell_2 &: (u = 0, 0 \leq v \leq 1) \longrightarrow (x = v, y = 0) \\
\ell_3 &: (v = 1, 0 \leq u \leq 1) \longrightarrow (x = 1, y = 2u) \\
\ell_4 &: (u = 1, 0 \leq v \leq 1) \longrightarrow (x = v, y = 1 + v^2).
\end{aligned}$$

The lines ℓ_1, ℓ_2, and ℓ_3 are all mapped to lines in the xy-plane. However, ℓ_4 is mapped to the parabola $y = 1 + x^2$ as x ranges from 0 to 1.



Let S be a (small) rectangular region in the uv-plane and let (u_0, v_0) be the lower left corner of S. Denote the dimensions of S by Δu and Δv. Let T be a transformation and let R be the image of S under T. While S may be rectangular, the region R need not be. We want to evaluate $\iint_R f(x,y)\ \mathrm{dA}$ by making a change of variable. That is, we want to evaluate in the following way,

$$\iint_R f(x,y)\ \mathrm{dA} \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\Delta x\Delta y \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i,v_j),h(u_i,v_j))\,(???)\,\Delta u\Delta v.$$

So, the question is: how does $\Delta x\Delta y$ compare to $\Delta u\Delta v$. This will turn out to be the *Jacobian*. Hence, **the Jacobian tells us how ΔA in the xy-plane compares to ΔA in the uv-plane**.

To integrate over R we need to approximate with rectangles and we can do this by finding tangent vectors. Set $(x_0, y_0) = T(u_0, v_0)$ (this is necessarily a boundary point of R). Then $\mathbf{r}(u,v) = g(u,v)\mathbf{i} + h(u,v)\mathbf{j}$ is the position vector of the image of the point (u, v). The lower boundary of S is $v = v_0$ and its image is $\mathbf{r}(u, v_0)$ with tangent vector (at (x_0, y_0))

$$\mathbf{r}_u = g_u(u_0,v_0)\mathbf{i} + h_u(u_0,v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}.$$

Similarly, starting with the left-hand side of S gives,

$$\mathbf{r}_v = g_v(u_0,v_0)\mathbf{i} + h_v(u_0,v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}.$$

The image region R now can be approximated by the parallelogram with secant vectors

$$\mathbf{a} = \mathbf{r}(u_0+\Delta u, v_0) - \mathbf{r}(u_0,v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v_0+\Delta v) - \mathbf{r}(u_0,v_0).$$

Since

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u},$$

then

$$\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u$$

and similarly,

$$\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v.$$

Thus, the area of R is approximated by

$$|(\Delta u \mathbf{r}_u) \times (\Delta v \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v.$$

The cross product gives

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}.$$

We call this remaining determinant the Jacobian of the transformation T, denoted by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Thus,

$$\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

and so

$$\iint_R f(x,y)\ \mathrm{dA} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta A \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v.$$

Change of variables in a double integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is contious on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\ \mathrm{dA} = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|\ \mathrm{du\ dv}.$$

It might seem a little strange that there is an absolute value on the Jacobian, but we don't include it when we do u-substitution. This has to do with orientation. When doing u-substitution in single-variable calculus, we maintain the order of the limits which reverses the orientation.

Example 19. Consider the change to polar coordinates, where

$$x = g(r,\theta) = r\cos\theta \qquad y = h(r,\theta) = r\sin\theta.$$

The Jacobian in this case is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus,

$$\iint_R f(x,y)\ \mathrm{dA} = \iint_S f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \ \mathrm{dr}\ \mathrm{d}\theta = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\ \mathrm{dr}\ \mathrm{d}\theta.$$

Example 20. Suppose R is the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$. We will evaluate $\iint_R (4x+8y)\ \mathrm{dA}$ with the transformation $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$. First, consider the image of the boundary. The boundary lines are

$$\ell_1 : y = -3x \mapsto v = 0 \qquad \ell_3 : y = -3x+8 \mapsto v = 8$$

$$\ell_2 : y = x-4 \mapsto u = 4 \qquad \ell_4 : y = x+4 \mapsto u = -4.$$

Thus, $S = \{(u,v) : -4 \le u \le 4, 0 \le v \le 8\}$. The Jacobian of the tranformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = 1/4.$$

Thus,

$$\begin{aligned} \iint_R (4x+8y)\ \mathrm{dA} &= \iint_S 4\left(\frac{1}{4}(u+v)\right) + 8\left(\frac{1}{4}(v-3u)\right)\frac{1}{4}\ \mathrm{du}\ \mathrm{dv} \\ &= \frac{1}{4}\int_0^8 \int_{-4}^4 3v - 5u\ \mathrm{du}\ \mathrm{dv} = \frac{1}{4}\int_0^8 \left[3uv - \frac{5}{2}u^2\right]_{-4}^4 \ \mathrm{dv} \\ &= \int_0^8 6v\ \mathrm{dv} = \left[3v^2\right]_0^8 = 192. \end{aligned}$$

The idea for change of variables for triple integrals is similar. Suppose T is a transformation that maps a region S in uvw-space onto a region R in xyz-space is given by

$$x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$$

The Jacobian of T is the determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Under hypotheses similar to the previous theorem,

$$\iiint_R f(x,y,z)\ \mathrm{dV} = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\ \mathrm{du\ dv\ dw}.$$

Example 21. Consider the change of variable to spherical coordinates

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi.$$

The Jacobian is

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} &= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\
&= \cos\phi\,((-\rho\sin\phi\sin\theta)(\rho\cos\phi\sin\theta) - (\rho\sin\phi\cos\theta)(\rho\cos\phi\cos\theta)) \\
&\qquad - \rho\sin\phi\,((\sin\phi\cos\theta)(\rho\sin\phi\cos\theta) - (-\rho\sin\phi\sin\theta)(\sin\phi\sin\theta)) \\
&= -\rho^2\cos^2\phi\left(\sin\phi\sin^2\theta + \sin\phi\cos^2\theta\right) \\
&\qquad - \rho^2\sin^2\phi\left(\sin\phi\cos^2\theta + \sin\phi\sin^2\theta\right) \\
&= -\rho^2\left(\cos^2\phi\sin\phi + \sin^2\phi\sin\phi\right) \\
&= -\rho^2\sin\phi.
\end{aligned}$$

This gives the formula for integration with spherical coordinates.

Calculus III

Chapter 16 - Vector Calculus

1. Vector Fields

Previously we have studied vector valued functions. Recall that we defined functions $f : \mathbb{R} \to V_2$ (resp. V_3) and these defined space curves in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$). Now we consider generalizations of this concept: vector fields. Vector fields can be used to model a wide assortment of physical phenomenon. In particular, energy and heat flux, gravitational force field, and forces acting on a charge at a certain position.

Let D be a set in $\mathbb{R}^2$. A vector field on $\mathbb{R}^2$ is a function $\mathbf{F} : D \to V_2$. Similarly, if E is a subset of $\mathbb{R}^3$, then a vector field on $\mathbb{R}^3$ is a function $\mathbf{F} : E \to V_3$.

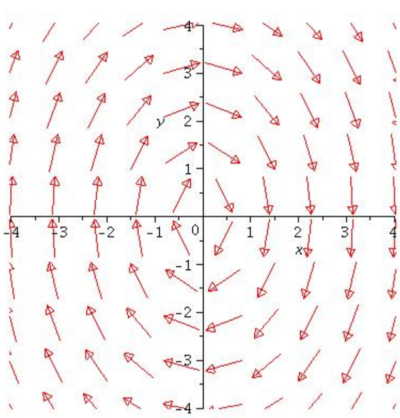
Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^2$ with domain D. Then for every $(x, y) \in D$,

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P(x, y), Q(x, y)\rangle.$$

We will sometimes simply write $P = P(x, y)$ and $Q = Q(x, y)$. We call P and Q the component functions of $\mathbf{F}$. When P and Q are scalars, $\mathbf{F}$ my be called a scalar field. As with our discussion in Chapter 13, $\mathbf{F}$ is continuous if and only if P and Q are continuous. All of this extends to vector fields in $\mathbb{R}^3$ where we write

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P(x, y), Q(x, y)\rangle.$$

Example 1. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$. We sketch some vectors for $\mathbf{F}$ below.



The vectors seem to be moving in an elliptic path. Let $\mathbf{x} = 2x\mathbf{i} + y\mathbf{j}$ be a position vector, then

$$\mathbf{x} \cdot \mathbf{F}(x, y) = 2xy - 2yx = 0.$$

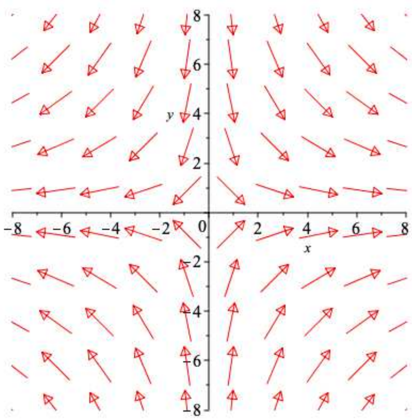
Hence, $\mathbf{F}(x, y)$ is perpendicular to the position vector $\langle 2x, y\rangle$ and so it is tangent to an ellipse with center the origin and radius $|\mathbf{x}| = \sqrt{4x^2 + y^2}$. Moreover,

$$|\mathbf{F}(x, y)| = \sqrt{y^2 + (-2x)^2} = |\mathbf{x}|,$$

so the magnitude of $\mathbf{F}(x, y)$ is equal to the distance from the origin to that point on the ellipse.

One example of a vector field we have already seen is the gradient vector field ∇.

Example 2. Define $f(x, y) = \frac{1}{2}(x^2 - y^2)$. Then the gradient of f is $\nabla f = \langle x, -y\rangle$.



The gradient vectors are perpendicular to the level curves of f (as discussed in Chapter 14). The length of the vectors actually depends on how close together the level curves are. This follows because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph.

Given a vector field $\mathbf{F}$, one can ask whether there exists a scalar function f, called a potential function for $\mathbf{F}$, such that $\mathbf{F} = \nabla f$. If such a f exists, we say that $\mathbf{F}$ is a conservative vector field. It is not clear yet why we would care about such vector fields.

Example 3. Newton's Law of Gravitation states that the magnitude of the gravitational force between two objects with masses m and M is

$$|\mathbf{F}| = \frac{mMG}{r^2}$$

where r is the distance between the objects and G is the gravitational constant. Assume the object with mass M is located at the origin in $\mathbb{R}^3$. Let the position vector of the object with mass m be $\mathbf{x} = \langle x, y, z\rangle$. Then $r = |\mathbf{x}|$, so $r^2 = |\mathbf{x}|^2$. The gravitational force exerted on this second object acts toward the origin, and the unit vector in this direction is

$$-\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Therefore the gravitational force acting on the object at $\mathbf{x} = \langle x, y, z\rangle$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}.$$

This is a vector field, called a gravitational field, and it associates a vector with every point $\mathbf{x}$ in space.

2. Line integrals

In this section, we consider the problem of integrating over a curve C (as opposed to an interval). Imagine having a long winding bar in space, then dropping a curtain from that. The area of the curtain is the line integral.

Let C be a smooth curve in $\mathbb{R}^2$ with parametric equations $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ where $a \leq t \leq b$. Recall that C smooth is equivalent to $\mathbf{r}'$ continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. We divide the interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of length Δs_i (because s is typically used for arc length). Choose a sample point $P_i(x_i^*, y_i^*)$ in each interval. Now let f be a function whose domain includes C. Then the line integral of f along C is

$$\int_C f(x, y)\ ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i.$$

Recall that the length of the curve C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Let $s(t)$ be the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{or} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

Thus, we can reparametrize the line integral to get

$$\int_C f(x, y)\ ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt.$$

The formula for a line integral in three variables can be obtained similarly to get

$$\int_C f(x, y, z)\ ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt.$$

In both cases, representing C by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, gives the generic formula

$$\int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)|\ dt.$$

Taking $f(x, y, z) = 1$ recovers the formula for arc length.

Example 4. Consider the curve $C : x = t, y = \cos 2t, z = \sin 2t$, $0 \leq t \leq 2\pi$, and the line integral $\int_C (x^2 + y^2 + z^2)\ ds$. Now $x(t)^2 + y(t)^2 + z(t)^2 = t^2 + 1$ and furthermore,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{1 + (-2 \sin 2t)^2 + (2 \cos 2t)^2} = \sqrt{5}.$$

Then

$$\int_C (x^2+y^2+z^2)\ ds = \int_0^{2\pi} (t^2+1)\sqrt{5}\ dt = \sqrt{5}\left[t^3+t\right]_0^{2\pi} = \sqrt{5}(8\pi^3+2\pi).$$

The definition of a line integral required that C be a *smooth* curve. However, this is not strictly necessary. If C is the union of a finite number of smooth curves $C_1,\ldots,C_n$, then we say that C is a piecewise-smooth curve and

$$\int_C f(x,y)\ ds = \int_{C_1} f(x,y)\ \mathrm{ds} + \cdots + \int_{C_n} f(x,y)\ ds.$$

The line integral we have seen so far is sometimes called the line integral with respect to arc length. We can also consider the line integral with respect to the x-axis:

$$\int_C f(x,y)\ dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*,y_i^*)\Delta x_i$$

and the line integral with respect to the y-axis:

$$\int_C f(x,y)\ dy = \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*,y_i^*)\Delta y_i.$$

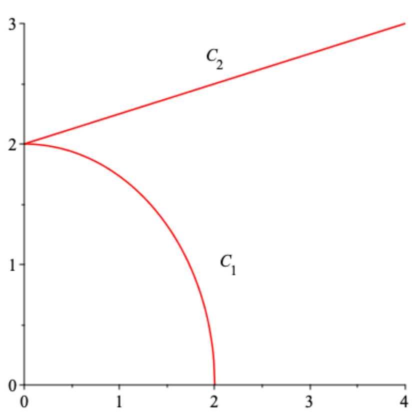
In essence, we are only considering change with respect to x or y, respectively. If $x = x(t)$ and $y = y(t)$ on the interval $a \le t \le b$, then $dx = x'(t)\ dt$ and $dy = y'(t)\ dt$ and so the above formulas become,

$$\int_C f(x,y)\ dx = \int_a^b f(x(t),y(t))x'(t)\ dt \quad \text{and} \quad \int_C f(x,y)\ dy = \int_a^b f(x(t),y(t))y'(t)\ dt.$$

We will frequent abbreviate the combined line integral with respect to x and y as

$$\int_C P(x,y)\ dx + \int_C Q(x,y)\ dy = \int_C P(x,y)\ dx + Q(x,y)\ dy.$$

Example 5. Let C be the arc of the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$, followed by the line segment from $(0,2)$ to $(4,3)$. We will evaluate the line integral $\int_C x^2\ dx + y^2\ dy$.



The curve C_1 can be parameterized by $x = 2\cos(t)$, $y = 2\sin(t)$, where $0 \le t \le \pi/2$.

The curve C_2 can be parameterized using our standard vector representation of a line segment:

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1,$$

for $0 \le t \le 1$. Here $\mathbf{r}_0 = \langle 0,2\rangle$ and $\mathbf{r}_1 = \langle 4,3\rangle$, so

$$\mathbf{r}(t) = \langle x(t),y(t)\rangle = (1-t)\langle 0,2\rangle + t\langle 4,3\rangle = \langle 4t, 2+t\rangle.$$

Now we evaluate the two line integral separately. First we evaluate along C_1,

$$\int_{C_1} x^2\ dx + y^2\ dy = \int_0^{\pi/2} 4\cos^2 t(-2\sin t)\ dt + \int_0^{\pi/2} 4\sin^2 t(2\cos t)\ dt$$
$$= 8\left[\frac{1}{3}\cos^3 t\right]_0^{\pi/2} + 8\left[\frac{1}{3}\sin^3 t\right]_0^{\pi/2} = -\frac{8}{3} + \frac{8}{3} = 0.$$

Now we evaluate along C_2,

$$\int_{C_2} x^2\ dx + y^2\ dy = \int_0^1 (4t)^2(4)\ dt + \int_0^1 (2+t)^2(1)\ dt$$
$$= 64\int_0^1 t^2\ dt + \int_0^1 (4+4t+t^2)\ dt$$
$$= \frac{64}{3} + \left(4 + 2 + \frac{1}{3}\right) = \frac{64+12+6+1}{3} = \frac{83}{3}.$$

Hence,

$$\int_C x^2\ dx + y^2\ dy = \int_{C_1} x^2\ dx + y^2\ dy + \int_{C_2} x^2\ dx + y^2\ dy = -\frac{2}{3} + \frac{83}{3} = 27.$$

Suppose in the previous example we instead considered C_1 as the circle from $(0,2)$ to $(2,0)$. This reverses the orientation and we obtain $\int_{C_1} x^2\ dx + y^2\ dy = \frac{2}{3}$. In general, we have

$$\int_{-C} f(x,y)\ dx = -\int_C f(x,y)\ dx \quad \text{and} \quad \int_{-C} f(x,y)\ dy = -\int_C f(x,y)\ dy.$$

However, arc length does not depend on orientation and so

$$\int_{-C} f(x,y)\ ds = \int_C f(x,y)\ ds.$$

If $f(x)$ is a variable force moving a particle from a to b along the x-axis, then the work done by the force is $W = \int_a^b f(x)\ dx$. On the other hand, if $\mathbf{F}$ is a constant force moving an object from point P to point Q in space, then the work done by the force is $W = \mathbf{F}\cdot\mathbf{D}$, where $\mathbf{D} = \overrightarrow{PQ}$.

Here we consider a continuous force field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ on $\mathbb{R}^3$. Suppose $\mathbf{F}$ moves a particle along a smooth curve C, described by the parameter t, $a \leq t \leq b$. If we divide $[a,b]$ into regular subintervals, then this in turn divides C into subarcs $P_{i-1}P_i$ with lengths Δs_i. Select a sample point $P_i(x_i^*, y_i^*, z_i^*)$ on the ith subinterval corresponding to t_i^*. Assuming Δs_i is small, then it moves from P_{i-1} to P_i (approximately) in the direction of $\mathbf{T}(t_i^*)$. Thus, the work done by $\mathbf{F}$ in moving the particle from P_{i-1} to P_i is approximately,

$$\mathbf{F}(x_i^*, y_i^*, z_i^*)\cdot[\Delta s_i \mathbf{T}(t_i^*)] = [\mathbf{F}(x_i^*, y_i^*, z_i^*)\cdot\mathbf{T}(t_i^*)]\,\Delta s_i.$$

It follows that the work W done by the force field F is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\mathbf{F}(x_i^*, y_i^*, z_i^*) \cdot \mathbf{T}(t_i^*)\right] \Delta s_i = \int_C \mathbf{F}(x,y,z) \cdot \mathbf{T}(x,y,z)\ ds = \int_C \mathbf{F} \cdot \mathbf{T}\ ds.$$

We call this last integral the line integral of $\mathbf{F}$ along C. If C is described by the vector equation $\mathbf{r}(t)$, $a \leq t \leq b$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ and so

$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right] |\mathbf{r}'(t)|\ dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt.$$

Hence, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt = \int_C \mathbf{F} \cdot \mathbf{T}\ ds.$$

Example 6. Let $\mathbf{F}(x,y,z) = (x+y^2)\mathbf{i} + xz\mathbf{j} + (y+z)\mathbf{k}$. and let $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} - 2t\mathbf{k}$, $0 \leq t \leq 2$. Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^2 \big((t^2+t^6)\mathbf{i} + (-2t^3)\mathbf{j} + (t^3-2t)\big) \cdot (2t\mathbf{i} + 3t^2\mathbf{j} - 2\mathbf{k})\ dt \\
&= \int_0^2 2t^7 - 6t^5 + 4t\ dt = \left[\frac{1}{4}t^8 - t^6 + 2t^2\right]_0^2 = 8.
\end{aligned}$$

3. The Fundamental Theorem for Line Integrals

Just as the integral acts as a sort of inverse operation to differentiation, so too does the line integral act as a sort of inverse operation to taking gradients.

Theorem 7. Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. The proof of this theorem is striaghtforward after a change of variable. Recall from the definition of a line integral along a curve,

$$\begin{aligned} \int_C \nabla f \cdot \, d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) \quad \text{(by the Chain Rule)} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \end{aligned}$$

The last step follows from the (usual) FTC. □

By the theorem, we now know that the line integral of ∇f tells us the net change of f.

Example 8. Find the work done by the gravitational field

$$\mathbf{F}(x) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point $(1, 1, 1)$ to the point $(2, 3, 0)$.

The vector field $\mathbf{F}$ is conservative since $\mathbf{F} = \nabla f$ where

$$f = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}.$$

Hence, by the FTC for line integrals,

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \, d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1,1,1) - f(2,3,0) \\ &= \frac{mMG}{\sqrt{1^2+1^2+1^2}} - \frac{mMG}{\sqrt{2^2+3^2+0^2}} = mMG\left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{13}}\right). \end{aligned}$$

Let $\mathbf{F}$ be a continuous vector field with domain D. We say that $\int_C \mathbf{F} \cdot \, d\mathbf{r}$ is independent of path (ion D) if $\int_{C_1} \nabla \mathbf{F} \cdot \, d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \, d\mathbf{r}$ for any two paths C_1 and C_2 in D with the same initial and terminal points. One consequence of the FTC is that line integrals of conservative vector fields are independent of path. That is, if C_1 and C_2 are two piecewise-smooth curves from A to B, then

$$\int_{C_1} \nabla f \cdot \, d\mathbf{r} = \int_{C_2} \nabla f \cdot \, d\mathbf{r}.$$

In general, there is no reason that we should expect this to hold.

A curve C is closed if its terminal point coincides with its initial point.

Theorem 9. Let $\mathbf{F}$ be a continuous vector field with domain D. Then $\int_C \mathbf{F} \cdot \, d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot \, d\mathbf{r} = 0$ for every closed path C in D.

Proof. Suppose $\int_C \mathbf{F} \cdot \, d\mathbf{r} = 0$ for every closed path C in D. Let C_1 and C_2 be paths in D from point A to point B and let C be the curve C_1 followed by $-C_2$. Then

$$0 = \int_C \mathbf{F} \cdot \, d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \, d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot \, d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \, d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot \, d\mathbf{r}.$$

It follows that $\int_{C_1} \mathbf{F} \cdot \, d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \, d\mathbf{r}$. The other direction is similar. $\square$

A consequence of this theorem is that if $\mathbf{F}$ is conservative, then $\int_C \mathbf{F} \cdot \, d\mathbf{r} = 0$ for every closed path C. By the next theorem, conservative vector fields are the only vector fields that are independent of path.

A region $D \subset \mathbb{R}^2$ is open if for every point $P \in D$, there isa. disk with center P that lies entirely in D (i.e., D contains no boundary points). The region D is connected if any two points in D can be joined by a path *that lies in* D.

Theorem 10. Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot \, d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D.

We've seen benefits and alternative characterizations of conservative vector field. The next question is whether it is possible to determine whether a field is conservative.

Theorem 11. If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Proof. Since $\mathbf{F}$ is conservative, there exists a function f such that $\nabla f = \mathbf{F}$. Then

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial Q}{\partial y}.$$

Now by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}. \qquad \square$$

This theorem doesn't tell us when a vector field is conservative, but it does tell us when it *is not* conservative.

Example 12. The vector field $\mathbf{F}(x, y) = (xy)\mathbf{i} + (x^2)\mathbf{j}$ is not conservative since $\frac{\partial P}{\partial y} = x$ while $\frac{\partial Q}{\partial x} = 2x$. (Unless D is restricted to $\{0\}$, which isn't a particularly interesting region.)

The next theorem is a partial converse to the previous one. First, we need some additional terminology. A curve C with equation $\mathbf{r}(t)$ on $a \le t \le b$ is simple if $\mathbf{r}(t_1) \ne \mathbf{r}(t_2)$ whenever $a < t_1 < t_2 < b$ (no intersections other than the endpoints). A region D is simply connected if every simple closed curve in D encloses points only in D.

Theorem 13. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Then $\mathbf{F}$ is conservative.

Example 14. Suppose that $\mathbf{F} = (3 + 2xy^2)\mathbf{i} + (2x^2y)\mathbf{j}$. The domain of F is all of $\mathbb{R}^2$, which is simply connected. Now $P = 3 + 2xy^2$ and $Q = 2x^2y$, and

$$\frac{\partial P}{\partial y} = 4xy = \frac{\partial Q}{\partial x}.$$

Thus, $\mathbf{F}$ is conservative.

Here we explain the procedure for finding f such that $\nabla f = \mathbf{F}$. We know such an f exists, then $f_x(x, y) = 3 + 2xy^2$ and $f_y(x, y) = 2x^2y$. We integrate $f_x(x, y)$ with respect to x to get

$$f(x, y) = \int f_x(x, y)\ dx = \int 3 + 2xy^2\ dx = 3x + x^2y^2 + g(y).$$

Here, the constant of integration is a function only in terms of y. Differentiating $f(x, y)$ with respect to y gives

$$2x^2y = f_y(x, y) = 2x^2y + g'(y).$$

It follows that $g(y) = K$ for some constant. Thus, $f(x, y) = 3x + x^2y^2 + K$.

Now suppose that C is the arc of the hyperbola $y = 1/x$ from $(1, 1)$ to $(4, \frac{1}{4})$. Using the above, we have that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(4, 1/4) - f(1, 1) = (12 + 1) - (3 + 1) = 9.$$

4. Green's Theorem

We now come to the second "big" theorem of this chapter. Green's Theorem connects certain line integrals to double integrals.

Let D be a region bounded by a simple closed curve. We choose *counterclockwise* to be the positive orientation of C (this is our convention but it is important that we stick to it).

Green's Theorem. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA.$$

We will sometimes replace $\int_C$ with $\oint_C$ to emphasize that C is closed and positively oriented. In other cases, we may write $\int_{\delta D}$ in place if $\int_C$ to emphasize that we are working over the boundary of D (again, positively oriented).

We say that D is a simple region if it is both type I and type II.

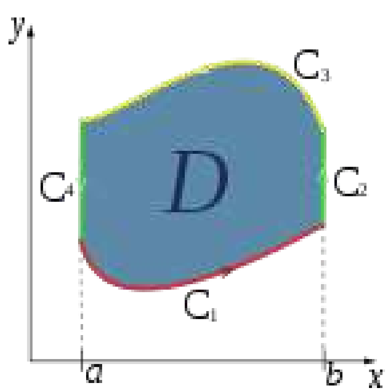
Proof of Green's Theorem in the simple case. Assume D is a simple region. Then it is type I and so there are continuous functions $g_1(x)$ and $g_2(x)$ such that

$$D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}.$$

Thus, by the FTC,

$$\iint_D \frac{\partial P}{\partial y}\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\ dy\ dx = \int_a^b P(x, g_2(x)) - P(x, g_1(x))\ dx.$$

Now we break δD into the union of four curves as shown below,



The curve C_1 is the graph of $y = g_1(x)$. Similarly, $-C_3$ is the graph of $y = g_2(x)$ (because of our orientation). Hence,

$$\int_{C_1} P(x, y)\ dx = \int_a^b P(x, g_1(x))\ dx$$

$$\int_{C_3} P(x, y)\ dx = -\int_{-C_3} P(x, y)\ dx = -\int_a^b P(x, g_2(x))\ dx.$$

On the other hand, at C_2 and C_4, x is constant and so

$$\int_{C_2} P(x,y)\ dx = 0 = \int_{C_4} P(x,y)\ dx.$$

Putting this all together we get

$$\begin{aligned}
\int_C P(x,y)\ dx &= \int_{C_1} P(x,y)\ dx + \int_{C_2} P(x,y)\ dx + \int_{C_33} P(x,y)\ dx + \int_{C_4} P(x,y)\ dx \\
&= \int_a^b P(x,g_1(x))\ dx - \int_a^b P(x,g_2(x))\ dx \\
&= \int_a^b P(x,g_2(x)) - P(x,g_1(x))\ dx = \iint_D \frac{\partial P}{\partial y}\ dA.
\end{aligned}$$

A similar computation (but expressing D as a type II region) shows that

$$\int_C Q\ dy = \iint_D \frac{\partial Q}{\partial x}\ dA.$$

Combining these two gives Green's Theorem for D a simple region. $\square$

Example 15. Let C be the ellipse $x^2 + 2y^2 = 2$. We will evaluate $\int_C y^4\ dx + 2xy^3\ dy$. First observe that, by Green's Theorem,

$$\int_C y^4\ dx + 2xy^3\ dy = \iint_D (2y^3) - (4y^3)\ dA = \iint_D -2y^3\ dA.$$

To evaluate this integral, we make a change of coordinates $x = \sqrt{2}u$ and $y = v$. Then our new region, D', has boundary $2 = (\sqrt{2}u)^2 + 2(v)^2 = 2(u^2+v^2)$, or equivalently $u^2 + v^2 = 1$. The Jacobian of this transformation is $\sqrt{2}$ and so we subsequently make a change to polar coordinates to get,

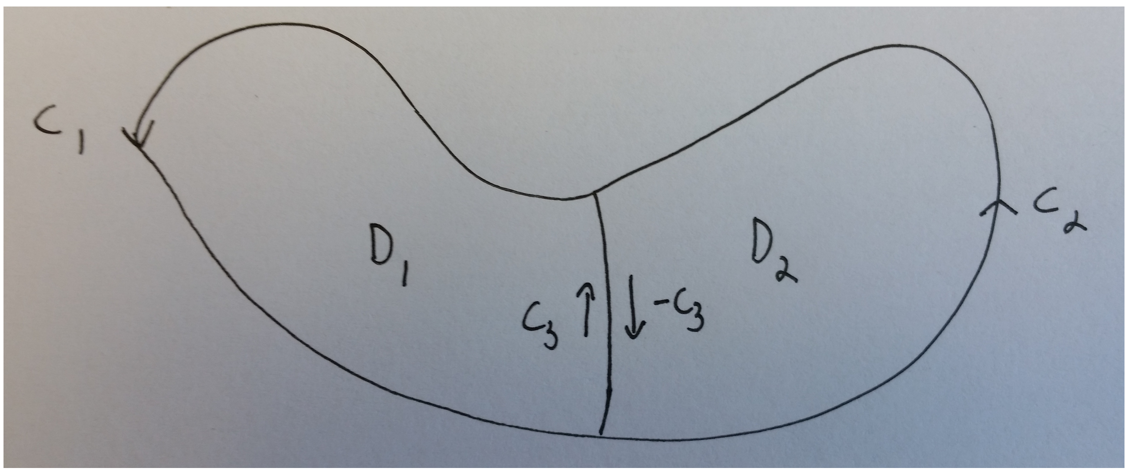
$$\begin{aligned}
\int_C y^4\ dx + 2xy^3\ dy &= \iint_D -2y^3\ dA = \iint_{D'} -2v^3\sqrt{2}\ du\ dv \\
&= \int_0^1 \int_0^{2\pi} -2\sqrt{2}(r\cos\theta)^3 r\ d\theta\ dr \\
&= -2\sqrt{2} \int_0^1 \int_0^{2\pi} r^4 \cos\theta(1-\sin^2\theta)\ d\theta\ dr \\
&= -2\sqrt{2} \int_0^1 r^4 \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^{2\pi}\ dr = 0.
\end{aligned}$$

Example 16. Let C be the triangle from $(0,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$ (remember: orientation matters!). Let $\mathbf{F} = \langle \sqrt{x^2+1}, \arctan x \rangle$. Note that $\mathbf{F}$ is not conservative. However, we can evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.

The region D enclosed by C is type I and can be written, $D = \{(x,y) : 0 \le x \le 1, x \le y \le 1\}$. Hence,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \sqrt{x^2+1}\ dx + \arctan x\ dy = \iint_D \left(\frac{\partial}{\partial x}(\arctan x) - \frac{\partial}{\partial y}(\sqrt{x^2+1}) \right)\ dA \\
&= \int_0^1 \int_x^1 \left(\frac{1}{x^2+1} - 0 \right)\ dy\ dx = \int_0^1 \left[\frac{y}{x^2+1} \right]_x^1\ dx = \int_0^1 \frac{1}{x^2+1} - \frac{x}{x^2+1}\ dx \\
&= \left[\arctan x - \frac{1}{2} \ln|x^2+1| \right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln(2).
\end{aligned}$$

Green's Theorem can easily be extended to the case where D is a finite union of simple regions. Suppose $D = D_1 \cup D_2$ where both D_1 and D_2 are simple (as below).



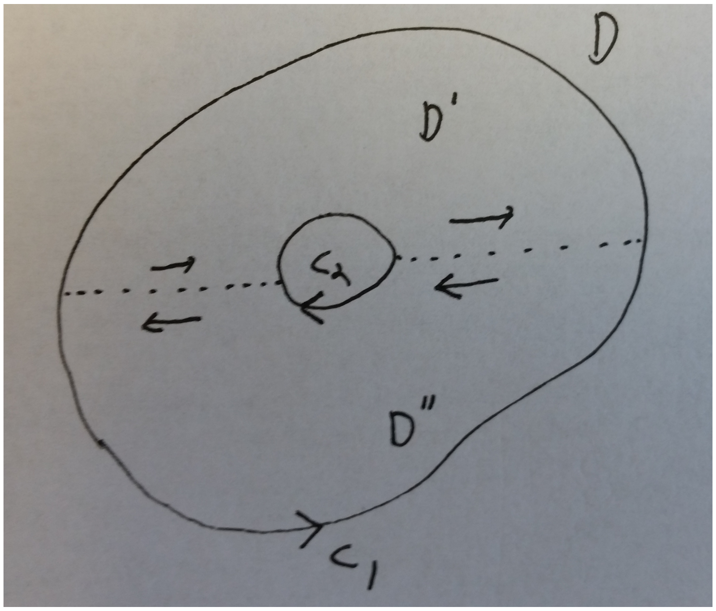
The boundary of D_1 is $C_1 \cup C_3$ while the boundary of D_2 is $C_2 \cup (-C_3)$. Hence, applying Green's Theorem to both regions independently gives,

$$\begin{aligned}
\int_{C_1 \cup C_3} P\ dx + Q\ dy &= \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA \\
\int_{C_2 \cup (-C_3)} P\ dx + Q\ dy &= \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA
\end{aligned}$$

Now,

$$\begin{aligned}
\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA &= \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA \\
&= \int_{C_1 \cup C_3} P\ dx + Q\ dy + \int_{C_2 \cup (-C_3)} P\ dx + Q\ dy \\
&= \int_{C_1} P\ dx + Q\ dy + \int_{C_2} P\ dx + Q\ dy = \int_C P\ dx + Q\ dy.
\end{aligned}$$

Green's Theorem can also be extended to regions with finitely many holds. Suppose D has one hole. Let C_1 be the outer boundary of D and C_2 the boundary for the hole as below.



We divide into $D' \cup D''$. The curves on the boundary between D' and D'' cancel (as in the above argument) and so

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, dA \\ &= \int_C P \, dx + Q \, dy.\end{aligned}$$

5. Curl and Divergence

Suppose we have fluid moving through a pipe occupying a region E in three-dimensional space. We let $\mathbf{V}(x, y, z)$ denote the velocity of the fluid at a point $(x, y, z) \in E$. Then $\mathbf{V}(x, y, z)$ is a vector field. At the same time, the fluid at point (x, y, z) may rotate around some axis, denoted $\operatorname{curl} \mathbf{V}(x, y, z)$ and the length of the curl denotes how quickly the fluid rotates around the axis. When there is no rotation, then $\operatorname{curl} \mathbf{V} = \mathbf{0}$.

We denote by ∇ (pronounced "del") the operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

(Yes yes yes, this is the same symbol for gradient.) Given a scalar function f, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Let $= \mathbf{P}\mathbf{i} + \mathbf{Q}\mathbf{j} + \mathbf{R}\mathbf{k}$ be a vector field on $\mathbb{R}^3$. Suppose the partial derivatives of P, Q, and R all exist. The curl of $\mathbf{F}$ is defined as

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

Example 17. Let $\mathbf{F} = x^3yz^2\mathbf{j} + y^4z^3\mathbf{k}$. Then

$$\begin{aligned} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^3yz^2 & y^4z^3 \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3yz^2 & y^4z^3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 0 & y^4z^3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & x^3yz^2 \end{vmatrix}\mathbf{k} \\ &= (4y^3z^3 - 2x^3yz)\mathbf{i} + 0\mathbf{j} + 3x^2yz^2\mathbf{k}. \end{aligned}$$

Theorem 18. If f is a function of three variables that has continuous second-order partial derivatives, then $\operatorname{curl}(\nabla f) = \mathbf{0}$. Hence, if $\mathbf{F}$ is conservative, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

Proof. We have

$$\operatorname{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix}\mathbf{k} = 0$$

by Clairaut's Theorem. $\square$

One can use the above theorem to show that $\mathbf{F}$ is *not* conservative, but it does not guarantee that $\mathbf{F}$ is conservative. The next theorem is a partial converse when $\mathbf{F}$ is defined on all of $\mathbb{R}^3$.

Theorem 19. If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Back to our velocity field example, $\mathbf{V}$. One can measure from a point (x, y, z) the net rate of change (with respect to time) of the mass of fluid flowing from the point (x, y, z) per unit of volume. This is known as the *divergence* and is denoted $\operatorname{div}\mathbf{V}(x, y, z)$.

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, $\partial R/\partial z$ all exist, then the divergence of $\mathbf{F}$ is defined as

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Example 20. Let $\mathbf{F} = x^3yz^2\mathbf{j} + y^4z^3\mathbf{k}$ as in Example 17. Then

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = 0 + (x^3 + y^2) + (3y^4z^2).$$

The next theorem is a straightforward computation. It follows for essentially the same reason that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

Theorem 21. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then $\operatorname{div}\operatorname{curl} = \mathbf{0}$.

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Then

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

It follows that we can write Green's Theorem in vector form as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl}\mathbf{F}) \cdot \mathbf{k}\; dA.$$

One can also show (see Stewart) that if $\mathbf{n}$ is the outward unit normal vector to C, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\; ds = \iint_D \operatorname{div}\mathbf{F}(x, y)\; dA.$$

6. Parametric surfaces and their areas

Much of this section will be done via analogy with parametric curves. Suppose that

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. The parametric surface S is the set of all points (x,y,z) such that $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$ for all $(u,v) \in D$. The three equations defining S are called the parametric equations of S.

In some sense, we've seen this already with spherical coordinates.

Example 22. The sphere $x^2 + y^2 + z^2 = a^2$ is represented by the parametric functions

$$x = a\sin\phi\cos\theta \quad y = a\sin\phi\sin\theta \quad z = a\cos\phi,$$

with $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Example 23. Suppose $\mathbf{r}(u,v) = \langle u^2, u\cos v, u\sin v\rangle$. Since $y^2 + z^2 = u^2$ (a cylinder), then this surface resembles a helix winding around the x-axis. This is known as a helicoid.

Now suppose that $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ defines a smooth surface S. Let P_0 be a point on S with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant, putting $u = u_0$, then $r(u_0, v)$ becomes a vector function of v and defines a curve C_1 (called a grid curve). The tangent vector to C_1 at P_0 is then

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k}.$$

Similarly,

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k}.$$

Since S is smooth, $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ and gives the normal vector for the tangent plane to S at P_0.

Let R be a small rectangle of D (the domain of $\mathbf{r}(u,v)$) with lower left corner (u_i^*, v_j^*). Let $\mathbf{r}_u^*$ and $\mathbf{r}_v^*$ be the tangent vectors at that point. Let Δu and Δv denote the dimension of R. We can use the tangent plane to approximate the area of the image of R on S as

$$|(\Delta u \mathbf{r}_u^*) \times (\Delta v \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*|\Delta u \Delta v.$$

It follows that

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} |\mathbf{r}_u^* \times \mathbf{r}_v^*|\Delta u\Delta v = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

Example 24. Consider the helicoid $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$. We have $\mathbf{r}_u = \cos v\mathbf{i} + \sin v\mathbf{j}$ and $\mathbf{r}_v = -u\sin v\mathbf{i} + u\cos v\mathbf{j} + \mathbf{k}$. Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \sin v\mathbf{i} - \cos v\mathbf{j} + u\mathbf{k}.$$

Hence,

$$\begin{aligned} A(S) &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \; dA = \int_0^1 \int_0^\pi \sqrt{1+u^2} \; dv \; du \\ &= \pi \int_0^1 \sqrt{1+u^2} \; du \quad \text{let } u = \tan\theta \text{ so } du = \sec^2\theta \; d\theta \\ &= \pi \int_0^{\pi/4} \sec^3\theta \; d\theta = \pi \left[\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|) \right]_0^{\pi/4} \\ &= \frac{\pi}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1) \right). \end{aligned}$$

If a surface is represented by the equation $z = f(x,y)$, then we can write this in parametric form as

$$x = x \quad y = y \quad z = f(x,y).$$

We have $\mathbf{r}_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}$. From this we recover our definition of surface area of a surface

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \; dA.$$

7. Surface integrals

Previously we studied line integrals in which (initially) we took the integral with respect to arc length. We extend this to study *surface integrals* with respect to surface area.

Suppose that a surface S has vector equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k} \qquad (u,v) \in D.$$

Assume that D is a rectangle (the general case is similar). We divide D into regular rectangles R_{ij} of area $\Delta A = \Delta u \Delta v$. This divides S into *patches* $\mathbf{r}(R_{ij}) = S_{ij}$. Given a function f, defined on S, we choose a point P_{ij}^* in each patch. The surface integral of f over the surface S is defined as

$$\iint_S f(x,y,z)\ dS = \lim_{m,n\to\infty} f(P_{ij}^*)\Delta S_{ij},$$

where ΔS_{ij} is the surface area of the patch S_{ij}. To actually evaluate this, we need to reparameterize in terms of u and v (just as we did with t for line integrals). We approximate each ΔS_{ij} by the tangent plane, so

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v.$$

Hence,

$$\iint_S f(x,y,z)\ dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\ dA.$$

Note that,

$$\iint 1\ dA = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\ dA = A(D).$$

Example 25. Let S be the cone with parametric equations

$$x = u\cos v, y = u \sin v, z = u, \qquad 0 \le u \le 1, 0 \le v \le \pi/2.$$

We will evaluate the surface integral $\iint_S xyz\ dS$. First, we compute $\mathbf{r}_u \times \mathbf{r}_v$,

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} &= -u\cos v\mathbf{i} - u\sin v\mathbf{j} + (u\cos^2 v + u\sin^2 v)\mathbf{k} \\ &= -u\cos v\mathbf{i} - u\sin v\mathbf{j} + u\mathbf{k}.\end{aligned}$$

Hence, $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(-u\cos v)^2 + (-u\sin v)^2 + u^2} = u\sqrt{2}$. Now we have

$$\begin{aligned}\iint_S xyz\ dS = \iint_D (u\cos v)(u\sin v)(u)(u\sqrt{2})\ dA &= \sqrt{2}\int_0^{\pi/2}\int_0^1 u^4 \cos v \sin v\ du\ dv \\ &= \frac{\sqrt{2}}{5}\int_0^{\pi/2} \cos v \sin v\ dv = \frac{\sqrt{2}}{5}\left[\frac{1}{2}\sin^2 v\right]_0^{\pi/2} = \frac{1}{5\sqrt{2}}.\end{aligned}$$

If S is a piecewise-smooth surface, so $S = S_1 \cup \cdots S_n$ and intersections are only along the boudary, then we can decompose the surface integral as

$$\iint_S f(x,y,z)\ dS = \iint_{S_1} f(x,y,z)\ dS + \cdots + \iint_{S_n} f(x,y,z)\ dS.$$

In our discussion of curves, it was useful to define *direction* and this played a role in certain line integrals. We want a similar notion for curves, as it will be useful in defining surface integrals over vector fields. A classic example of an *oriented surface* is a sphere. We can define a *continuous* function $f(x,y,z) \to V_3$ that at each point gives the unit normal vector to the sphere at the point (x,y,z). A surface is oriented if such a function exists, but not all surfaces are orientable in this way. Consider the *Möbius strip*, which really only has one side.

Example 26. Let $z = g(x,y)$ be the graph of a surface S. Then S can be parameterized by

$$x = x, \quad y = y, \quad z = g(x,y).$$

Hence, $\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$, so

$$\mathbf{r}_x \times \mathbf{r}_y = -\left(\frac{\partial g}{\partial x}\right)\mathbf{i} - \left(\frac{\partial g}{\partial y}\right)\mathbf{j} + \mathbf{k}$$

and so the surface integral formula becomes,

$$\iint_S f(x,y,z)\ dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA.$$

Now we also have that **a unit normal** is

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{-\left(\frac{\partial g}{\partial x}\right)\mathbf{i} - \left(\frac{\partial g}{\partial y}\right)\mathbf{j} + \mathbf{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}.$$

In some sense, this is the *natural orientation* because the $\mathbf{k}$-component is positive. Thus, this gives the upward orientation of the surface.

In general, if S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u,v)$, we associate to it an orientation by setting

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}.$$

An opposite orientation is given by $-\mathbf{n}$.

Example 27. Consider the parameterization of the sphere by spherical coordinates,

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta \mathbf{i} + a \sin\phi \sin\theta \mathbf{j} + a \cos\phi \mathbf{k}.$$

A computation shows that $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$ (this is just the Jacobian). Hence,

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \frac{1}{a}\mathbf{r}(\phi, \theta).$$

This tells us that the normal vector points in the same direction as the position vector (so outward from the sphere).

For a *closed surface* E (boundary is a solid region) the positive orientation is the one in which the normal vectors point outward from E. (Like many things, this is just convention but it is pretty standard).

Now we're in a place to define surface integrals over vector fields, much as we did over curves.

Let S be an oriented surface with unit normal vector $\mathbf{n}$ and suppose fluid flows through S with density $\rho(x, y, z)$ and velocity $\mathbf{v}(x, y, z)$. The rate of (fluid) flow (mass per unit time) per unit area if $\rho\mathbf{v}$. Hence, dividing S into small patches S_{ij} we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of $\mathbf{n}$ by

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij}).$$

Hence, the rate of flow through S is given by

$$\iint_S \rho\mathbf{v} \cdot \mathbf{n}\, dS.$$

Hence, if $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S (or the flux of $\mathbf{F}$ across S) is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

In the above example, $\mathbf{F} = \rho\mathbf{v}$.

If S is given by $\mathbf{r}(u, v)$, then we can obtain $\mathbf{n}$ as before to get

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS \\ &= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA \\ &= \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.\end{aligned}$$

Example 28. Let S be the hemisphere $x^2+y^2+z^2=4$, $z \geq 0$, oriented downward, and let $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 2z\mathbf{k}$. We will compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

We parameterize S by usng spherical coordinates

$$\mathbf{r}(\phi, \theta) = 2\sin\phi\cos\theta\mathbf{i} + 2\sin\phi\sin\theta\mathbf{j} + 2\cos\phi\mathbf{k}.$$

Hence,

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = 2\sin\phi\sin\theta\mathbf{i} - 2\sin\phi\cos\theta + 4\cos\phi\mathbf{k}.$$

We know from before that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 4\sin^2\phi\cos\theta\mathbf{i} + 4\sin^2\phi\sin\theta\mathbf{j} + 4\sin\phi\cos\phi\mathbf{k}.$$

Now

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 8(\sin^3\phi\sin\theta\cos\theta - \sin^3\phi\sin\theta\cos\theta + 2\sin\phi\cos^2\phi) = 16\sin\phi\cos^2\phi.$$

The domain D is just the projection of the hemisphere into the xy-plane, so it's a disk of radius 2 centered at the origin. But we need to be careful. Because our sphere is oriented downward and this is the *negative* orientation of the sphere, we pick up an extra negative sign. Hence,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= -\iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA = -\int_0^{\pi/2}\int_0^{2\pi} 16\sin\phi\cos^2\phi\ d\theta\ dr \\ &= -32\pi\int_0^{\pi/2} \sin\phi\cos^2\phi\ dr = -32\pi\left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi/2} = -\frac{32\pi}{3}.\end{aligned}$$

If $z = g(x, y)$ is the graph of a surface, then we can parameterize in the usual way and get that

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + r\mathbf{k}) \cdot \left(-\left(\frac{\partial g}{\partial x}\right)\mathbf{i} - \left(\frac{\partial g}{\partial y}\right)\mathbf{j} + \mathbf{k}\right)$$

so that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\left(\frac{\partial g}{\partial x}\right) - Q\left(\frac{\partial g}{\partial y}\right) + R\right)\ dA.$$

As an exercise, repeat the previous example by taking $z = \sqrt{4 - x^2 - y^2}$

8. Stokes' Theorem

Stokes' Theorem. Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components hav continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Stokes' Theorem connects line integrals to surface integrals, essentially what Green's Theorem does with (ordinary) double integrals. If S lies in the xy-plane with upward orientation, then the unit normal is just $\mathbf{k}$. Hence, $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \; dA$, which is just Green's Theorem. Hence, Green's Theorem is a special case of Stokes' Theorem. However, the proof of Stokes' Theorem that we give uses Green's Theorem so it's not enough to just prove Stokes' Theorem to get Green's Theorem.

Proof of a special case. Suppose S has equation $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. We assume the orientation of S is upward, in which case the positive orientation of C corresponds to the positive orientation of C_1. Suppose C_1 is parameterized by $x = x(t)$ and $y = y(t)$, $a \leq t \leq b$. Then C is parameterized by $x = x(t)$, $y = y(t)$, and $z = (g(x(t), y(t))$, $a \leq t \leq b$. By the Chain Rule,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) \, dt = \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right] \, dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] \, dt \\
&= \int_{C_1} \left(P + R\frac{\partial z}{\partial x} \right) \, dx + \left(Q + R\frac{\partial z}{\partial y} \right) \, dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x} \right) \right] \, dA \quad \text{by Green's Theorem} \\
&= \iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x \partial y} \right) \right. \\
&\qquad \left. - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y \partial x} \right) \right] \, dA \\
&= \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] \, dA \\
&= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.
\end{aligned}$$

□

Example 29. Suppose S is the cone $x = \sqrt{y^2 + z^2}$, $0 \le x \le 2$, oriented in the direction of the positive x-axis. Let $\mathbf{F} = \arctan(x^2yz^2)\mathbf{i} + x^2y\mathbf{j} + x^2z^2\mathbf{k}$. We will evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ using Stoke's Theorem.

The boundary curve C is $y^2 + z^2 = 4$, in the counterclockwise direction when viewed from the front. Hence, C is parameterized by $\mathbf{r}(t) = 2\mathbf{i} + 2\cos t\mathbf{j} + 2\sin t\mathbf{k} \qquad 0 \le t \le 2\pi$. Now

$$\mathbf{F}(\mathbf{r}(t)) = \arctan(32\cos t\sin^2 t)\mathbf{i} + 8\cos t\mathbf{i} + 16\sin^2 t \qquad \mathbf{r}'(t) = -2\sin t\mathbf{j} + 2\cos t\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -16\cos t\sin t + 32\cos t\sin^2 t$$

Using Stokes' Theorem,

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\ &= 16\int_0^{2\pi} 2\cos t\sin^2 t - \cos t\sin t\ dt = 16\left[\frac{2}{3}\sin^3 t\ dt - \frac{1}{2}\sin^2 t\right]_0^{2\pi} = 0\end{aligned}$$

Now we'll use Stokes' Theorem the other way. Recall that when S is a surface given by $z = g(x, y)$, then we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\left(\frac{\partial g}{\partial x}\right) - Q\left(\frac{\partial g}{\partial y}\right) + R\right)\ dA.$$

Example 30. Let C be the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant. Let $\mathbf{F} = \mathbf{i} + (x + yz)\mathbf{j} + (xy - \sqrt{z})\mathbf{k}$. We'll use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

A straightforward computation shows that $\operatorname{curl} \mathbf{F} = (x - y)\mathbf{i} - y\mathbf{j} + \mathbf{k}$. The region S is the portion of the plane $2x + 2y + z = 1$ over $D = \left\{(x, y) : 0 \le x \le \frac{1}{3}, 0 \le y \le \frac{1}{2}(1 - 3x)\right\}$. We orient S upward and since S is given by $z = 1 - 3x - 2y$, then by Stokes' Theorem,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-(x - y)(-3) - (-y)(-2) + (1)]\ dA \\ &= \int_0^{1/3}\int_0^{(1-3x)/2} (3x - 5y + 1)\ dA = \int_0^{1/3}\left[(1 + 3x)y - \frac{5}{2}y^2\right]_0^{(1-3x)/2}\ dy\ dx \\ &= \int_0^{1/3}\left[\frac{1}{2}(1 + 3x)(1 - 3x) - \frac{5}{8}(1 - 3x)^2\right]\ dx \\ &= -\frac{1}{8}\int_0^{1/3} 81x^2 - 30x + 1\ dx = -\frac{1}{8}\left[9x^3 - 15x^2 + x\right]_0^{1/3} = -\frac{1}{8} \cdot -\frac{1}{3} = \frac{1}{24}.\end{aligned}$$

Suppose S_1 and S_2 are two oriented surfaces with the same oriented boundary curve C, both satisfying the hypotheses of Stokes' Theorem. Then

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

9. The Divergence Theorem

In vector form, one can rewrite Green's Theorem as

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \mathrm{div}\mathbf{F}(x, y)\, dA.$$

The Divergence Theorem is a higher-dimensional analog of this statement. It tells us that, under the right conditions, the flux of a vector field $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E. We give the statement for *simple solid regions*, i.e., those that are simultaneously type 1, 2, and 3.

The Divergence Theorem. Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F}\, dV.$$

Proof. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\iiint_E \mathrm{div}\mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial P}{\partial z}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial P}{\partial z}\, dV$$

and

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \mathbf{n}\, dS = \iint_S P\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\mathbf{k} \cdot \mathbf{n}\, dS.$$

We will prove $\iiint_E \frac{\partial Q}{\partial y}\, dV = \iint_S Q\mathbf{j} \cdot \mathbf{n}\, dS$. The statement $\iiint_E \frac{\partial P}{\partial z}\, dV = \iint_S R\mathbf{k} \cdot \mathbf{n}\, dS$ is in the text, and the statement $\iiint_E \frac{\partial P}{\partial x}\, dV = \iint_S P\mathbf{i} \cdot \mathbf{n}\, dS$ is part of the final reading assignment. It is clear that the Divergence Theorem follows from these statements.

Since E is simple, it is a type 3 region:

$$E = \{(x, y, z) : (x, z) \in D : u_1(x, z) \le y \le u_2(x, z)\}$$

The boundary surface S consists of three pieces: a left surface S_1, a right surface S_2, and a horizontal surface S_3. The equation of S_2 is $y = u_2(x, z)$ and the outward normal $\mathbf{n}$ points in the positive y-direction. Conversely, on S_1, the equation is $y = u_1(x, z)$ and the normal points in the negative y-direction. Finally, on S_3, $\mathbf{j} \cdot \mathbf{n} = 0$ (because $\mathbf{j}$ is horizontal and $\mathbf{n}$ is vertical). Hence, $\iint_{S_3} Q\mathbf{j}\cdot\mathbf{n}\; dS = 0$, while

$$\iint_{S_2} Q\mathbf{j}\cdot\mathbf{n}\; dS = \iint_D Q(x,u_2(x,z),z)\; dA,$$
$$\iint_{S_1} Q\mathbf{j}\cdot\mathbf{n}\; dS = -\iint_D Q(x,u_1(x,z),z)\; dA.$$

Thus,

$$\begin{aligned}\iint_S Q\mathbf{j}\cdot\mathbf{n}\; dS &= \iint_{S_1} Q\mathbf{j}\cdot\mathbf{n}\; dS + \iint_{S_2} Q\mathbf{j}\cdot\mathbf{n}\; dS + \iint_{S_2} Q\mathbf{j}\cdot\mathbf{n}\; dS \\ &= \iint_D \left[Q(x,u_2(x,z),z - \iint_D Q(x,u_1(x,z),z)\; dA\right]\; dA \\ &= \iiint_E \frac{\partial Q}{\partial y}\; dV \quad \text{by the FTC.}\end{aligned}$$

□

Example 31. Let S be the sphere with center the origin and radius 2 and let

$$\mathbf{F} = (x^3+y^2)\mathbf{i} + (y^3+z^3)\mathbf{j} + (x^3+y^3)\mathbf{k}.$$

We will calculate the flux of $\mathbf{F}$ across S. That is, we will compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

Note that div$\mathbf{F} = 3(x^2+y^2+z^2)$ and the region E is just the solid sphere of radius 2 centered at the origin. We make a change to spherical coordinates, so div$\mathbf{F} = 3\rho^2$. Then by the Divergence Theorem.

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F}\; dV &= \int_0^2 \int_0^\pi \int_0^{2\pi} (3\rho^2)(\rho^2\sin\phi)\; d\theta\; d\phi\; d\rho \\ &= 6\pi \int_0^2 \int_0^\pi \rho^4 \sin\phi\; d\phi\; d\rho = 6\pi \int_0^2 \left[\rho^4(-\cos\phi)\right]_0^\pi\; d\rho \\ &= 12\pi \int_0^2 \rho^4\; d\rho = 12\pi \left[\frac{1}{5}\rho^5\right]_0^2 = \frac{384\pi}{5}.\end{aligned}$$

Example 32. Let S be the surface of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $z=y-2$ and $z=0$. Let $\mathbf{F} = (xy+2xz)\mathbf{i} + (x^2+y^2)\mathbf{j} + (xy-z^2)\mathbf{k}$. We will compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

We have div$\mathbf{F} = (y+2z)+(2y)-2z = 3y$. Let D be the circle of radius 2 in the xy-plane centered at the origin. Then the region E bounded by S (in cylindrical coordinates), is

$$E = \{(r,\theta,z) : 0 \le r \le 2, 0 \le \theta \le 2\pi, r\sin\theta - 2 \le z \le 0\}.$$

In cylindrical coordinates, $\text{div}\mathbf{F} = 3r\sin\theta$. Now by the Divergence Theorem,

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F}\ dV &= \int_0^{2\pi}\int_0^2\int_0^{r\sin\theta-2}(3r\sin\theta)(r)\ dz\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 (3r^2\sin\theta)(r\sin\theta-2)\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 3r^3\sin^2\theta - 6r^2\sin\theta\ dr\ d\theta \\
&= \int_0^{2\pi}\left[\frac{3}{4}r^4\sin^2\theta - 2r^3\sin\theta\right]_0^2\ d\theta \\
&= \int_0^{2\pi} 12\sin^2\theta - 16\sin\theta\ d\theta \\
&= \int_0^{2\pi} 6(1-\cos(2\theta)) - 16\sin\theta\ d\theta \\
&= \left[6\left(\theta - \frac{1}{2}\sin(2\theta)\right) + 16\cos\theta\right]_0^{2\pi} = 12\pi.
\end{aligned}$$

10. Summary

We'll summarize the important methods from this chapter. As the book notes, all of our major theorems are in fact just generalizations of the Fundamental Theorem of Calculus:

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

In this section, we do not focus on the strict hypotheses of these methods, though those are important, as one can refer back for the details. Instead, we will focus on *choosing* the right method for solving a problem.

Line integrals Given a (scalar) function $f(x, y, z)$ defined on a curve C with parameterization $\mathbf{r}(t)$, $a \leq t \leq b$, the line integral of f along C is

$$\int_C f(x, y, z)\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\ dt.$$

If $\mathbf{F}$ is a vector field defined along C, then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt.$$

If $\mathbf{F}$ is conservative ($\mathbf{F} = \nabla f$), then the Fundamental Theorem for Line Integrals gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Example 33. Let $\mathbf{F}(x, y, z) = (y^2 z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2 z)\mathbf{k}$ and let C be the curve parameterized by $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t + 1)\mathbf{j} + t^2\mathbf{k}$, $0 \leq t \leq 1$.

First, let's try to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the definition. We have

$$\mathbf{F}(\mathbf{r}(t)) = \left[(t(t+1))^2 + 2t^3\right]\mathbf{i} + 2(\sqrt{t}(t+1)t^2)\mathbf{j} + (\sqrt{t}(t+1)^2 + 2t^3)\mathbf{k}$$

We can already tell that $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ is going to be ridiculously complicated.

As an alternative, note that $\operatorname{curl} \mathbf{F} = 0$. Since $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ and its components have continuous partial derivatives, this means that $\mathbf{F}$ is conservative. So, we look for a potential function. Since $\mathbf{F} = \nabla f$ for some scalar function f, then

$$f(x, y, z) = \int f_x(x, y, z)\ dx = \int y^2 z + 2xz^2\ dx = xy^2 z + x^2 z^2 + g(y, z)$$

for some function $g(y, z)$. Since $2xyz = f_y(x, y, z) = 2xyz + g'(y, z)$, then $g(y, z) = h(z)$, a function in z. Thus, $xy^2 + 2x^2 z = f_z(x, y, z) = xy^2 + 2x^2 z + h'(z)$, so $h(z) = K$, a constant. Thus, we take $f = xy^2 z + x^2 z^2$. Note that $\mathbf{r}(1) = \langle 1, 2, 1\rangle$ and $\mathbf{r}0 = \langle 0, 1, 0\rangle$. Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, 0) = (4 + 1) - 0 = 5.$$

Another way to evaluate line integrals of vector fields in $\mathbb{R}^2$ is using Green's Theorem:

$$\int_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA,$$

where $C = \partial D$, positively oriented (counterclockwise). This is especially useful when C isn't a smooth curve but D is a "nice" region.

Example 34. Let C be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$, oriented counterclockwise. We'll evaluate $\int_C e^{2x+y}\ dx + e^{-y}\ dy$.

This notation is just another way to write $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle e^{2x+y}\ dx, e^{-y}\rangle$. The region D enclosed by C is $D = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$. Hence, by Green's Theorem,

$$\begin{aligned}
\int_C e^{2x+y}\ dx + e^{-y}\ dy &= \iint_D \left(0 - e^{2x+y}\right)\ dA = -\int_0^1 \int_0^{1-x} e^{2x+y}\ dy\ dx \\
&= -\int_0^1 \left[e^{2x+y}\right]_0^{1-x}\ dx = -\int_0^1 e^{x+1} - e^{2x}\ dx \\
&= -\left[e^{x+1} - \frac{1}{2}e^{2x}\right]_0^1 = e - \frac{1}{2} - \frac{1}{2}e^2.
\end{aligned}$$

Surface integrals Given a (scalar) function $f(x,y,z)$ defined on a surface s with parameterization $\mathbf{r}(u,v)$, $(u,v) \in D$, the surface integral of f over S is

$$\iint_S f(x,y,z)\ dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\ dA.$$

If $\mathbf{F}$ is a vector field defined on S, then the surface integral of $\mathbf{F}$ over C (flux of $\mathbf{F}$ across S) is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot \mathbf{n}\ dS = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA.$$

Surface integrals also give another way to evaluate line integrals.

Example 35. Let S be the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward, and let $\mathbf{F} = -2yz\mathbf{i} + y\mathbf{j} + 3x\mathbf{k}$. We will evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

An easy computation shows that $\operatorname{curl} \mathbf{F} = -(3+2y)\mathbf{j} + 2z\mathbf{k}$. One could use the special case of surface integrals when S is the graph of the function $z = g(x,y)$, or we could just parameterize with

$$\mathbf{r}(u,v) = \langle u, v, 5 - u^2 - v^2\rangle \qquad (u,v) \in D.$$

Hence, $\mathbf{r}_u \times \mathbf{r}_v = \langle 2u, 2v, 1\rangle$ and so

$$\operatorname{curl} \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \langle 0, -(3+2v), 2(5-u^2-v^2)\rangle \cdot \langle 2u, 2v, 1\rangle = 10 - 2u^2 - 6v^2 - 6v.$$

We evaluate the integral by switching to cylindrical coordinates,

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D 10-2u^2-6v^2-6v\ dA \\ &= \int_0^2\int_0^{2\pi}\big(10-2r\cos^2(x)-6r\sin^2(x)-6r\sin(x)\big)\,r\ d\theta\ dr = 8\pi\end{aligned}$$

On the other hand, if $\mathbf{F}$ is a vector field defined on a "nice" oriented surface S and C its boundary curve, then by Stokes' Theorem:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}.$$

Example 36. Keeping the setup of the previous example. The boundary curve C is a circle of radius 2, centered at the origin, in the plane $z=1$. We parameterize as $\mathbf{r}(t) = \{2\cos t, 2\sin t, 1\}$, $0\le t\le 2\pi$. Then,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\ dt = \int_0^{2\pi} 8\sin^2 t + 4\sin t\cos t\ dt = 8\pi.$$

Suppose in the previous example we wanted to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$. We could parameterize the surface using cylindrical coordinates:

$$\mathbf{r}(r,\theta) = \{r\cos\theta, r\sin\theta, 5-r^2\}, \qquad 0\le r\le 2, 0\le\theta\le 2\pi.$$

Now, $\mathbf{r}_r\times\mathbf{r}_\theta = 2r^2\cos\theta\mathbf{i} + 2r^2\sin\theta\mathbf{j} + r\mathbf{k}$ and so

$$\mathbf{F}(\mathbf{r}(r,\theta))\cdot(\mathbf{r}_r\times\mathbf{r}_\theta) = (\text{it's not pretty}).$$

As an alternative, we can evaluate using the Divergence Theorem. Let E be a simple solid region and S the boundary region of E, then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\ dV.$$

Example 37. With the setup of the previous example, we have $\operatorname{div}\mathbf{F} = 1$. Then,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\ dV = \iint_D\int_1^{5-x^2-y^2} 1\ dz\ dA = \iint_D 4-x^2-y^2\ dA.$$

The region D can be expressed in polar coordinates $D = \{(r,\theta) : 0\le r\le 2, 0\le\theta\le 2\pi\}$. Hence,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D 4-x^2-y^2\ dA = \int_0^2\int_0^{2\pi}(4-r^2)r\ d\theta\ dr \\ &= 2\pi\int_0^2 4r-r^3\ dr = 2\pi\left[2r^2-\frac{1}{4}r^4\right]_0^2 = 8\pi.\end{aligned}$$