Quivers supporting graded twisted Calabi-Yau algebras

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Calabi-Yau algebras

Let A be an algebra over an algebraically closed, characteristic zero field $\Bbbk$. Let $A^e = A \otimes_{\Bbbk} A^{op}$ be the **enveloping algebra** of A.

The algebra A is said to be **twisted Calabi-Yau (CY)** of (global) dimension d if A has a finite resolution by finitely-generated projectives in $\mathrm{A^e}$-MOD (*homologically smooth*) and

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & i \neq d \\ A^{\mu_A} & i = d \end{cases}$$

where μ_A is the Nakayama automorphism of A.

A is Calabi-Yau if $\mu_A = \mathrm{id}_A$.

CY algebras in nature

- A connected ($\mathbb{N}$-)graded algebra is twisted CY if and only if it is Artin-Schelter regular [Reyes-Rogalski-Zhang].
- Ore extensions of twisted CY algebras are twisted CY [Liu-Wang-Wu].
- Normal extensions of CY algebras are CY [Chirvasitu-Kanda-Smith].

CY algebras in nature

- Skew group algebras $A\#G$ where $A = \Bbbk[x_1, \ldots, x_n]$ and $G \subset SL_n(A)$ finite are CY [Iyama-Reiten]. Additional results for A p-Koszul CY [Wu-Zhu] and/or H an involutory CY Hopf [Liu-Wu-Zhu].

- Mutations (in the sense of Fomin-Zelevinsky) of CY algebras are CY [Vitoria, Zeng].

- Others: dimer models, NCCRs,...

Vacualgebras

Let $Q = (Q_0, Q_1)$ be a finite quiver and $\Bbbk Q$ its path algebra. For a path $p \in \Bbbk Q$, denote by $s(p)$ the **source** of p and by $t(p)$ the **target** of p.

Given an automorphism σ of Q_0, we say $\omega = \sum c_k p_k \in \Bbbk Q$ is a σ-**twisted superpotential** if ω is closed under cyclic permutation and $s(p_k) = \sigma(t(p_k))$ for all k with $c_k \neq 0$. When $\sigma = \mathrm{id}|_{Q_0}$, ω is said to be a **superpotential**.

For each $a \in Q_1$ we define the derivation operator δ_a on paths p via the rule $\delta_a p = a^{-1}p$ where $a^{-1}p = 0$ if $s(p) \neq a$.

Vacualgebras

An algebra of the form $\Bbbk Q/(\partial_a \omega : a \in Q_1)$ where ω is a (twisted) superpotential is said to be a **vacualgebra**.

Theorem (Bocklandt, Reyes-Rogalski)

If A is graded twisted CY of global dimension 3, then A is a vacualgebra on a strongly connected quiver with homogeneous superpotential.

Question

Which quivers support such algebras?

CY algebras of finite growth

Let $A = \Bbbk Q/(\partial_a \omega : a \in Q_1)$ be graded twisted CY of global dimension 3 with $|Q_0| = n$ and $\deg(\omega) = s$.

The **adjacency matrix** of a quiver Q is

$$(M_Q)_{ij} = \# \text{ of arrows from vertex } i \text{ to } j.$$

The **matrix polynomial of** Q is defined to be

$$p_Q(t) = I_n - M_Q t + M_Q^T P t^{s-1} - P t^s$$

where P is the permutation matrix (of the vertices) determined by the Nakayama automorphism μ_A.

The **matrix-valued Hilbert-series** of A is $H_A(t) = 1/p_Q(t)$.

CY algebras of finite growth

Theorem (Reyes-Rogalski)

Suppose $A = \Bbbk Q/(\partial_a \omega : a \in Q_1)$ is graded twisted CY of global dimension 3 and finite GK dimension. Let P be the permutation matrix corresponding to the Nakayama automorphism μ_A.

- *Every coefficient matrix in $H_A(t)$ is nonnegative.*
- *Every root of $p_Q(t)$ is a root of unity.*
- *M_Q and P commute.*
- *If $P = I$ and M_Q is normal, then either*

  $\deg(\omega) = 3$ *and* $\rho(M_Q) = 3$ *or*

  $\deg(\omega) = 4$ *and* $\rho(M_Q) = 2$.

Moreover, $\mathrm{GKdim}(A) =$ the multiplicity of 1 as a root of $p_Q(t)$.

Questions

Question

Which quivers have adjacency matrices with a good Hilbert series?

Question

Of the quivers in the last question, which quivers have superpotentials that produce CY algebras of global dimension 3 and finite GK dimension?

2-vertex quivers ($P = I$)

Suppose $A = \Bbbk Q/(\partial_a \omega : |Q_0| = 2, a \in Q_1)$ is graded twisted CY of global dimension 3 and finite GK dimension. Define

$$\theta(M_Q) := \left. \frac{d}{dt} \det\left(p_Q(t)\right) \right|_{t=1}.$$

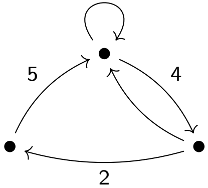
Then $\theta(M_Q) = 0$ when $\mathrm{GKdim}(A) < \infty$. Write $M_Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\theta(M_Q) = (b-c)^2$, then $b = c$ and so M_Q is symmetric.

If $\deg(\omega) = 3$, then $M_Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

If $\deg(\omega) = 4$, then $M_Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

Bad example

Let Q be the quiver.



The Hilbert series corresponding to Q suggests polynomial growth for a degree 3 superpotential.

However, a Gröbner basis argument shows that Q has no such superpotential.

3-vertex quivers

Proposition (G-Rogalski)

Let $A \cong \Bbbk Q/(\partial_a \omega : a \in Q_1)$ *where* A *is graded twisted Calabi-Yau of global dimension 3 and finite GK dimension with* $\deg(\omega) = 3$ *or* 4. *The number of loops at each vertex of* Q *does not exceed* $6 - \deg(\omega)$.

To prove this, we use the **Golod-Shafarevich inequality:** If B is a quadratic algebra with n generators and d relations, then

$$H_B(t) \geq |(1 - nt + dt^2)^{-1}|.$$

3-vertex quivers ($P = I$)

Suppose $A = \Bbbk Q/(\partial_a \omega : |Q_0| = 3, a \in Q_1, \deg(\omega) = 3)$ is graded twisted CY of global dimension 3 and finite GK dimension.

The matrix polynomial of Q,

$$p_Q(t) = I_3 - M_Q t + M_Q^T t^2 - I_3 t^3,$$

is antipalindromic. Hence, roots appear in inverse pairs.

By the hypothesis on GK dimension, 1 is a root of $p_Q(t)$ of multiplicity at least 3.

Hence, $\det(p_Q(t)) = (1-t)^3 r(t)$ where

$$r(t) = \prod_{i=1}^{3} (1 - k_i t + t^2), k_i \in \mathbb{R}, |k_i| \leq 2.$$

3-vertex quivers ($P = I$)

Write $M_Q = [m_{ij}]_{i,j=0\ldots3}$ and set

$$\lambda = \sum m_{ii}, \quad \beta = \sum_{i>j} m_{ii}m_{jj}, \quad \gamma = \sum_{i>j} m_{ij}m_{ji}.$$

Setting $\det(p_Q(t)) = (1-t)^3 r(t)$ we find,

$$\begin{aligned} \lambda &= k_1 + k_2 + k_3 + 3, \\ \gamma &= \beta - (k_1k_2 + k_1k_3 + k_2k_3) - 2\lambda + 3. \end{aligned}$$

For every choice of m_{ii}, we can find the maximum possible value m of γ. Then solve for the k_i for every choice of γ, $0 \leq \gamma \leq m$.

3-vertex quivers ($P = I$) - Example

Suppose Q has three loops at each of at least two vertices. This determines β and λ. The maximum value of γ is 0. Thus,

$$M_Q = \begin{pmatrix} 3 & a & 0 \\ 0 & 3 & b \\ c & 0 & \alpha \end{pmatrix}, \quad \alpha \leq 3.$$

Then $\theta(M_Q) = 0$ implies $a = 0$ or $a(\alpha - 3) + 3bc = 0$. Hence $\alpha < 3$.

We can now check that the Hilbert series has negative entries unless $\alpha = 0$, $a = b = c = 1$. This matrix can be checked directly to find that it has roots not lying on the unit circle.

3-vertex quivers ($P = I$) - Example

Suppose Q has no loops. Then the maximum value of γ is 3. One can show as before that no good matrices exist when $\gamma > 0$. Write,

$$M_Q = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}.$$

The Hilbert series corresponding to Q is good if and only if $(a, b, c) = (3x, 3y, 3z)$ where (x, y, z) is a solution to Markov's equation: $x^2 + y^2 + z^2 = 3xyz$.

Question

Do all such quivers have good superpotentials?

YES!

Mutations

Let Q be a quiver without loops or oriented 2-cycles and fix a vertex v.

Steps:

1. For every pair of arrows $a, b \in Q_1$ with $t(a) = v$ and $s(b) = v$, create a new arrow $[ba] : s(a) \to t(b)$.
2. Reverse each arrow with source or target at v.
3. Remove any maximal disjoint collection of oriented 2-cycles.

The resulting quiver $\widetilde{Q}$, is called a **mutation** of Q. There is also a method to mutate the corresponding superpotential.

By a result of Vitoria, two vacualgebras related by mutation are derived equivalent. Thus, if one is the CY, so is the other by a recent result of Zeng.

Mutations

Theorem (G-Rogalski)

The triple $(3x, 3y, 3z)$ is a solution to Markov's equation if and only if the corresponding quiver can be obtained from the quiver with adjacency matrix

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix}$$

by successive mutations.

These mutations provide the only examples of non-normal matrices whose quivers support a good superpotential (in the 3-vertex case).

3-vertex quivers ($P = I$)

Theorem (G-Rogalski)

Let $A = \Bbbk Q/(\partial_a \omega : a \in Q_1, |Q_0| = 3)$ *be graded twisted CY of global and GK dimension* 3 *with* $\deg(\omega) = 3$ *or* 4. *If* $P = I$, *then* M_Q *is one the matrices below.*

$$\deg(\omega) = 3: \quad \begin{pmatrix} 2&1&1\\1&0&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 2&1&1\\1&1&0\\1&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1\\1&1&1\\1&1&1 \end{pmatrix}, \begin{pmatrix} 1&1&1\\1&2&0\\1&0&2 \end{pmatrix}, \begin{pmatrix} 0&a&0\\0&0&b\\c&0&0 \end{pmatrix},$$

$$\deg(\omega) = 4: \quad \begin{pmatrix} 1&1&0\\0&1&1\\1&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&1\\1&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&1\\1&0&1\\1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&1\\1&1&0 \end{pmatrix}.$$

The indeterminates a, b, c *are positive integers satisfying the equation* $a^2 + b^2 + c^2 = abc$.

3-vertex quivers ($P = I$)

All quivers in the above theorem except for

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

are McKay quivers for some group action on an AS regular algebra in $6 - \deg(\omega)$ variables.

Question

Are there (twisted) CY algebras on the other quivers?

3-vertex quivers ($P = I$) - Remaining Quivers

Let Q be the quiver with adjacency matrix

$$M_Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Let Q' be the quiver with adjacency matrix $M_Q - I$. Then Q' supports a twisted CY algebra B of global and GK dimension 2 [Reyes-Rogalski].

It follows that $A = B[t]$ is a twisted graded CY algebra supported on Q with global and GK dimension 3, and $\deg(\omega) = 3$.

3-vertex quivers ($P = I$) - Remaining Quivers

Let Q be the quiver below.

$$\begin{tikzcd}
1 \arrow[loop left, "x_1"] \arrow[r, bend left, "x_2"] & 3 \arrow[l, bend left, "x_5"] \arrow[r, bend right, "x_6"'] & 2 \arrow[l, bend right, "x_4"'] \arrow[loop right, "x_3"]
\end{tikzcd}$$

Set $A = \Bbbk Q/(\partial_a \omega : a \in Q_1)$ with

$$\omega = \frac{1}{2}x_1^4 + x_1^2 x_2 x_5 + x_2 x_5 x_2 x_5 + x_2 x_6 x_4 x_5 + x_4 x_6 x_4 x_6 + x_3^2 x_4 x_6 + \frac{1}{2} x_3^4.$$

Then $\Omega = x_1^2 + x_2 x_5 + x_5 x_2 + x_4 x_6 + x_6 x_4 + x_3^2 +$ is a normal element in A and $B = A/(\Omega)$ is a twisted CY algebra of global and GK dimension 2 [Reyes-Rogalski].

Thus, A is a graded twisted CY algebra supported on Q with global and GK dimension 3, and $\deg(\omega) = 4$.

2-vertex quivers ($P \neq I$)

Proposition (G-Rogalski)

Let $A \cong \Bbbk Q/(\partial_a \omega : a \in Q_1, |Q_0| = 2)$ *be graded twisted CY of global and GK dimension* 3 *with* $\deg(\omega) = 3$ *or* 4*. If* $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *is the matrix corresponding to the Nakayama automorphism of* A*, then* M_Q *is one of the following:*

$$\deg(\omega) = 3 : \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right),$$
$$\deg(\omega) = 4 : \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right).$$

3-vertex quivers ($P \neq I$)

Proposition (G-Rogalski)

Let $A \cong \Bbbk Q/(\partial_a \omega : a \in Q_1, |Q_0| = 3)$ *be graded twisted CY of global dimension 3 and finite GK dimension with* $\deg(\omega) = 3$ *or* 4*. If*

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is the matrix corresponding to the Nakayama automorphism of A*, then* M_Q *is one of the following:*

$$\deg(\omega) = 3 : \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$
$$\deg(\omega) = 4 : \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix}.$$

3-vertex quivers ($P \neq I$)

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Let $A \cong \Bbbk Q/(\partial_a \omega : a \in Q_1, |Q_0| = 3)$ be graded twisted CY of global dimension 3 and finite GK dimension with $\deg(\omega) = 3$ *or* 4*. If*

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Thank You!