Pointed Hopf actions on quantum generalized Weyl algebras

Jason Gaddis

Workshop on Noncommutative Geometry and Noncommutative Invariant Theory
Banff International Research Station

September 26, 2022

(Joint work with Robert Won)

Hopf actions on $\mathbb{Z}$-graded algebras

Let $\Bbbk$ be a field. All algebras are $\Bbbk$-algebras.

Goal

Study Hopf actions in the setting of $\mathbb{Z}$-graded algebras.

The Weyl algebra

$$A_1(\Bbbk) = \Bbbk\langle u, v : uv - vu - 1\rangle$$

is $\mathbb{Z}$-graded (set $\deg(u) = 1$ and $\deg(v) = -1$) but exhibits no finite-dimensional quantum symmetry (Cuadra-Etingof-Walton).

Our interest is in actions on generalized Weyl algebras (GWAs) over a polynomial ring in one variable. These algebras are known to be twisted Calabi-Yau (Liu).

So, this is a natural extension of the problem of studying Hopf actions on connected $\mathbb{N}$-graded twisted Calabi-Yau algebras (i.e., Artin-Schelter regular algebras).

Quantum GWAs

Definition

Let $q \in \mathbb{k}^{\times}$ and let $h(t) \in \mathbb{k}[t]$ be non-constant. The corresponding *quantum generalized Weyl algebra* is

$$\mathbb{k}[t](u, v, q, h) = \mathbb{k}\langle u, v, t \mid ut - qtu, vt - q^{-1}tv, vu = h(t), uv = h(qt)\rangle.$$

Throughout we will assume q is a root of unity, $q \neq 1$.

A quantum GWA is $\mathbb{Z}$-graded (set $\deg(u) = 1$, $\deg(v) = -1$, and $\deg(t) = 0$).

Example

- Setting $h = t$, we obtain the *quantum planes:*

$$\mathbb{k}_q[u, v] = \mathbb{k}\langle u, v \mid uv - qvu\rangle$$

- Setting $h = t - 1$, we obtain the *quantum Weyl algebras:*

$$A_1^q(\mathbb{k}) = \mathbb{k}\langle u, v \mid uv - qvu - 1\rangle$$

(Generalized) Taft algebras

Definition

Let $m, n \in \mathbb{N}$ such that $m > 1$ and $m \mid n$, and let $\lambda \in \Bbbk$ be a primitive m^{th} root of unity. The *generalized Taft algebra* corresponding to this data is

$$T_n(\lambda, m) := \Bbbk\langle x, g \mid g^n - 1, x^m, gx - \lambda xg \rangle.$$

• (G, Won, Yee) Classified linear actions of Taft algebras on quantum planes and quantum Weyl algebras.

• (Cline, G) Extended the above to linear actions on quantum affine spaces and quantum matrix algebras. Studied actions of generalized Taft algebras, as well as their higher-dimensional analogues.

The actions we consider here are generally distinct from those studied above.

Weakly $\mathbb{Z}$-graded actions

Definition

Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded algebra and let H be a Hopf algebra that acts on A.

- We say the action of H on A is *$\mathbb{Z}$-graded* if A_i is an H-module for each $i \in \mathbb{Z}$.

- We say that the action of H on A is *weakly $\mathbb{Z}$-graded* if A_0 and $A_{-i} \oplus A_i$ are H-modules for every $i \in \mathbb{N}$.

The weakly $\mathbb{Z}$-graded setting captures group actions that preserve the $\mathbb{Z}$-grading of A up to the automorphism of $\mathbb{Z}$ which sends 1 to -1.

Quantum thickenings

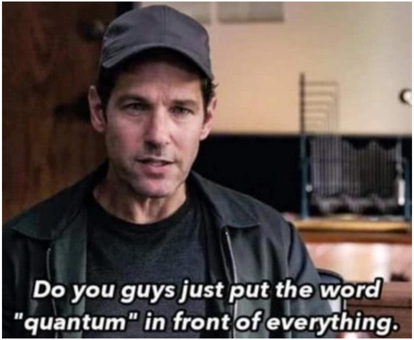
Classic problem

Determine the groups that act faithfully on a quantum GWA A.

Quantum problem

Study which cyclic subgroups G of $\mathrm{Aut}(A)$ are restrictions to the group of group-like elements of a generalized Taft algebra T which acts inner-faithfully on A.

Such a T-action can be viewed as a quantum thickening of the action of G.



Automorphisms of quantum GWAs (Suárez-Alvarez and Vivas)

Let $A = \mathbb{k}[t](u, v, q, h)$. Write $h = \sum h_i t^i$ and let $\ell = \gcd\{i - j \mid h_i h_j \neq 0\}$. Set

$$C_\ell = \begin{cases} \mathbb{k}^\times & \text{if } h \text{ is a monomial} \\ \{\ell^{\text{th}} \text{ roots of unity}\} & \text{otherwise.} \end{cases}$$

For $(\gamma, \mu) \in C_\ell \times \mathbb{k}^\times$, define $\eta_{\gamma,\mu} \in \mathrm{Aut}(A)$ by

$$\eta_{\gamma,\mu}(t) = \gamma t, \quad \eta_{\gamma,\mu}(v) = \mu v, \quad \eta_{\gamma,\mu}(u) = \mu^{-1}\gamma^{\deg_t(h)} u.$$

When $q \neq -1$, then $\mathrm{Aut}(A) = \{\eta_{\gamma,\mu} \mid (\gamma, \mu) \in C_\ell \times \mathbb{k}^\times\}$.

If $q = -1$, then there is an order 2 automorphism Ω defined by

$$\Omega(t) = -t, \quad \Omega(v) = u, \quad \Omega(u) = v.$$

In this case, every automorphism of A is either some $\eta_{\gamma,\mu}$ or else $\Omega \circ \eta_{\gamma,\mu}$.

So, every automorphism of a quantum GWA is weakly $\mathbb{Z}$-graded. But, when $q \neq -1$, every automorphism is actually $\mathbb{Z}$-graded.

Actions on the polynomial ring in one variable

For our main result, it was necessary to first study actions of generalized Taft actions on the polynomial base ring $\Bbbk[t]$.

For $f = \sum f_i t^i \in \Bbbk[t]$, set $\mathrm{supp}(f) = \{i \mid f_i \neq 0\} \subset \mathbb{Z}$.

Proposition

Let $T = T_n(\lambda, m)$. Let $\gamma \in \Bbbk \backslash \{0, 1\}$ and $0 \neq \phi \in \Bbbk[t]$ with $\deg_t(\phi) = d$.

I. If $\Bbbk[t]$ is a T-module algebra with $g(t) = \gamma t$ and $x(t) = \phi$, then

(1) *γ is a primitive m^{th} root of unity,*

(2) *$\lambda = \gamma^{d-1}$ and $\gcd(d-1, m) = 1$, and*

(3) *$\mathrm{supp}(\phi) \subseteq \{d, d-m, d-2m, \ldots\}$.*

Furthermore, the action is inner-faithful if and only if $m = n$.

II. Conversely, if γ and ϕ satisfy the conditions (1)—(3), then there is a unique T-module algebra structure on $\Bbbk[t]$ such that $g(t) = \gamma t$ and $x(t) = \phi$.

Main results

Theorem

Let $A = \Bbbk[t](u, v, q, h)$ with $q^2 \neq 1$ and let $T = T_n(\lambda, m)$.

(A) There is an inner-faithful weakly $\mathbb{Z}$-graded T-module algebra structure on A if and only if

1. $\mathrm{supp}(h)$ *is contained in a single congruence class modulo m, and*
2. *there exists an integer k coprime to m such that $\mathrm{lcm}(m, \mathrm{ord}(q^k)) = n$.*

(B) Assuming the conditions in (A) are satisfied, the inner-faithful weakly $\mathbb{Z}$-graded T-module algebra structures on A are parametrized by $\gamma, \mu \in \Bbbk^\times$ and $\phi(t) \in \Bbbk[t]$ of degree d such that

1. $\mathrm{ord}(\gamma) = m$ *and* $\lambda = \gamma^{d-1}$,
2. $\mathrm{lcm}(m, \mathrm{ord}(\mu)) = n$,
3. $\mathrm{supp}(\phi)$ *is contained in a single congruence class modulo m, and*
4. μq^{1-d} *is an m^{th} root of unity.*

These conditions guarantee an action even in the case of $q = -1$. However, when $q = -1$, there may be additional T-actions.

Main results

We can also frame our results in terms of quantum thickenings.

Theorem

Let $A = \Bbbk[t](u, v, q, h)$ with $q^2 \neq 1$. Let $G = \langle \eta_{\gamma,\mu} \rangle$ be a cyclic subgroup of $\mathrm{Aut}(A)$ *of order n. Let $m = \mathrm{ord}(\gamma)$ so $m \mid n$.*

(A) The action of G is the restriction of the action to the group of group-likes of an inner-faithful weakly $\mathbb{Z}$-graded $T_n(\lambda, m)$-module algebra action if and only if there exists an integer k coprime to m such that μq^k is an m^{th} root of unity.

(B) The actions of each $T_n(\lambda, m)$ whose group-like elements restrict to the action of G are parameterized by nonzero polynomials $\phi(t) \in \Bbbk[t]$ of degree d such that

1. $\gcd(d - 1, m) = 1$, *and*
2. $\mathrm{supp}(\phi)$ *is contained in a single congruence class modulo m.*

Invariants

For a Hopf algebra H and an H-module algebra A, the *fixed ring of A by H* is

$$A^H = \{a \in H \mid h(a) = \epsilon(h)a \text{ for all } h \in H\}.$$

Theorem

Let $A = \Bbbk[t](u, v, q, h)$ and let $T = T_n(\lambda, m)$. Suppose that A is an inner-faithful weakly $\mathbb{Z}$-graded T-module algebra where g acts as $\eta_{\gamma,\mu} \in \mathrm{Aut}(A)$ with $\gamma \neq 1$. Then for some polynomial $H \in \Bbbk[Z]$,

$$A^T \cong \Bbbk[U, V, Z]/(UV - H).$$

Thank You!