Auslander's Theorem for Dihedral Actions on Preprojective Algebras over Extended Dynkin Quivers

Jacob Barahona Kamsvaag

Miami University

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Introduction

$\Bbbk$ a field, G a group acting on a $\Bbbk$-algebra R.

McKay Correspondence: provides bijections between

- irreducible $\Bbbk$-representations of G
- indecomposable R^G-direct summands of R
- certain module categories over $\mathrm{End}_{R^G}(R)$
- certain module categories over the skew group ring $R\#G$.

Auslander's Theorem: provides conditions for when the $\Bbbk$-linear map $\eta : R\#G \to \mathrm{End}_{R^G}(R)$ given by

$$\eta(r\#g)(a) = rg(a)$$

is a $\Bbbk$-algebra isomorphism.

Graded Algebras

Definition

A $\Bbbk$-algebra R is called $\mathbb{N}$-*graded* if there exists a collection $\{R_i\}_{i \in \mathbb{N}}$ of $\Bbbk$-subspaces of R such that

- $R = \bigoplus_{i \in \mathbb{N}} R_i$
- $R_i R_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$.

If $R_0 \cong \Bbbk$, we say R is *connected*. If $\dim_{\Bbbk}(R_i) < \infty$ for all $i \in \mathbb{N}$ we say R is *locally finite*.

Example: $R = \Bbbk[x_1, \ldots, x_n]$,
$R_i = \{\text{homogeneous polynomials of degree } i\}$
$R_0 = \Bbbk$

Algebras will be $\Bbbk$-algebras, graded will mean $\mathbb{N}$-graded.

Gelfand-Kirillov Dimension

Definition

Let R be a graded locally finite algebra. The *Gelfand-Kirrilov dimension* of R, denoted $\mathrm{GKdim}(R)$, is

$$\mathrm{GKdim}(R) = \overline{\lim_{n\to\infty}} \log_n(R_{\leq n})$$

This is a standard invariant of R (does not depend on which grading we take) which measures the growth of the graded pieces.

Different settings

Classical: $R = \Bbbk[x_1, \ldots, x_n]$, G (finite) acts by linear changes of variables. $\text{GKdim}(R) = n$.

Connected Graded Noncommutative: R is Artin-Schelter regular, G (finite) acts homogeneously on R. Example, R is the quantum plane $\Bbbk_q[x, y]$: similar to $\Bbbk[x, y]$ except $xy = qyx$. $\text{GKdim}(R) = 2$.

Nonconnected Graded Noncommutative: R is twisted Calabi-Yau, G (finite) acts homogeneously on R. Example: R is the preprojective algebra over an extended Dynkin quiver of type ADE. $\text{GKdim}(R) = 2$.

Quivers

Definition

A *quiver* Q is a tuple (Q_0, Q_1, s, t) where

- Q_0 is a set of *vertices* $\{e_0, \ldots, e_n\}$
- Q_1 is a set of *arrows* $\{\alpha_0, \ldots, \alpha_m\}$
- $s, t : Q_1 \to Q_0$ are functions assigning to each arrow α_i a *source* vertex $s(\alpha_i)$ and *target* vertex $t(\alpha_i)$.

A *path of length* ℓ in Q:
$p = \alpha_{i_1} \cdots \alpha_{i_\ell}$ where $t(\alpha_{i_j}) = s(\alpha_{i_{j+1}})$ for all $j = 1, \ldots, \ell - 1$.

$Q_\ell = \{\text{paths of length } \ell\}$

Example: $\widetilde{A_2}$

$Q = \widetilde{A_2}$

$$\begin{tikzcd}
& e_0 \arrow[dl, "\alpha_0"'] & \\
e_1 \arrow[rr, "\alpha_1"'] & & e_2 \arrow[ul, "\alpha_2"']
\end{tikzcd}$$

$Q_0 = \{e_0, e_1, e_2\}$
$Q_1 = \{\alpha_0, \alpha_1, \alpha_2\}$
$Q_2 = \{\alpha_0\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_0\}$

The Double Quiver $\overline{Q}$

Definition

Let $Q = (Q_0, Q_1, s, t)$ be a quiver. The *double* of Q, denoted $\overline{Q} = (\overline{Q_0}, \overline{Q_1}, \overline{s}, \overline{t})$, is the quiver obtained by taking

- $\overline{Q_0} = Q_0$
- $\overline{Q_1}$ is Q_1 together with the set of symbols $\{\alpha_i^* : \alpha_i \in Q_1\}$
- $\overline{s} = s$ and $\overline{t} = t$ on Q_1, and for each $\alpha_i \in Q_1$,

$$\overline{s}(\alpha_i^*) = t(\alpha_i) \quad \text{and} \quad \overline{t}(\alpha_i^*) = s(\alpha_i).$$

Q_1 =*nonstar arrows*
$Q_1^* := \overline{Q}_1 \setminus Q_1$ =*star arrows*

We drop the bar notation and just write $s(\alpha_i^*)$ and $t(\alpha_i^*)$

The Double of $\widetilde{A_2}$

$Q = \widetilde{A_2}$, $\overline{Q}$:

$$\begin{tikzcd}
& e_0 \arrow[dl, shift left, "\alpha_0"] \arrow[dr, shift left, "\alpha_2^*"] & \\
e_1 \arrow[ur, shift left, "\alpha_0^*"] \arrow[rr, shift left, "\alpha_1"] & & e_2 \arrow[ul, shift left, "\alpha_2"] \arrow[ll, shift left, "\alpha_1^*"]
\end{tikzcd}$$

$$\overline{Q}_2 = \{\alpha_0\alpha_0^*, \alpha_0^*\alpha_0, \alpha_1\alpha_1^*, \alpha_1^*\alpha_1, \alpha_2\alpha_2^*, \alpha_2^*\alpha_2, \\ \alpha_0\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_0, \alpha_0^*\alpha_2^*, \alpha_2^*\alpha_1^*, \alpha_1^*\alpha_0^*\}$$

$\overline{Q}_n$ contains $3 \cdot 2^n$ many paths.

The Path Algebra

Definition

Let Q be a quiver. The *path algebra* $\Bbbk Q$ is defined as follows:

- Vector space: $\Bbbk$-linear combinations of paths.
- If $p = \alpha_1 \cdots \alpha_\ell$ and $q = \beta_1 \cdots \beta_k$ then $p \cdot q$ is concatentation:

$$pq = \begin{cases} \alpha_1 \cdots \alpha_\ell \beta_1 \cdots \beta_k & t(p) = s(q) \\ 0 & \text{otherwise.} \end{cases}$$

- Trivial paths e

$$ep = \begin{cases} p & s(p) = e \\ 0 & s(p) \neq e \end{cases} \qquad pe = \begin{cases} p & t(p) = e \\ 0 & t(p) \neq e. \end{cases}$$

The Preprojective Algebra

Definition

Let Q be a quiver. The *preprojective algebra* over Q, denoted Π_Q, is:

$$\Pi_Q = \Bbbk\overline{Q}/\left(\sum_{\alpha\in Q_1} \alpha\alpha^* - \alpha^*\alpha\right)$$

$\Sigma = \sum_{\alpha\in Q_1} \alpha\alpha^* - \alpha^*\alpha$ is called the *preprojective relation*.

Example: $\Pi_{\widetilde{A_2}}$

$$\begin{tikzcd}
 & e_0 \arrow[dl, shift left, "\alpha_0"] \arrow[dr, shift left, "\alpha_2^*"] & \\
e_1 \arrow[ur, shift left, "\alpha_0^*"] \arrow[rr, shift left, "\alpha_1"] & & e_2 \arrow[ul, shift left, "\alpha_2"] \arrow[ll, shift left, "\alpha_1^*"]
\end{tikzcd}$$

$$\Sigma = \alpha_0\alpha_0^* - \alpha_0^*\alpha_0 + \alpha_1\alpha_1^* - \alpha_1^*\alpha_1 + \alpha_2\alpha_2^* - \alpha_2^* - \alpha_2$$

Obtain new relations using idempotents:

$$\begin{aligned}
e_0\Sigma e_0 &= \alpha_0\alpha_0^* - \alpha_2^*\alpha_2, \\
e_1\Sigma e_1 &= \alpha_1\alpha_1^* - \alpha_0^*\alpha_0, \\
e_2\Sigma e_2 &= \alpha_2\alpha_2^* - \alpha_1^*\alpha_1.
\end{aligned}$$

Then in $\Pi_{\widetilde{A_2}}$, $\alpha_i^*\alpha_i = \alpha_{i+1}\alpha_{i+1}^*$, where the index is taken mod 3.

$\Pi_{\widetilde{A_{n-1}}}$

$$
\begin{tikzcd}
 & & e_0 \arrow[dll, shift left] \arrow[drr, shift left] & & \\
e_1 \arrow[urr, shift left] \arrow[r, shift left] & e_2 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & e_{n-2} \arrow[l, shift left] \arrow[r, shift left] & e_{n-1} \arrow[l, shift left] \arrow[ull, shift left]
\end{tikzcd}
$$

- α_i has source e_i and target e_{i+1}
- α_i^* has source e_{i+1} and target e_i
- $\alpha_i^*\alpha_i = \alpha_{i+1}\alpha_{i+1}^*$

where the indices are taken mod n.

The canonical form of an element

The preprojective relation allows us to move star arrows to the left:

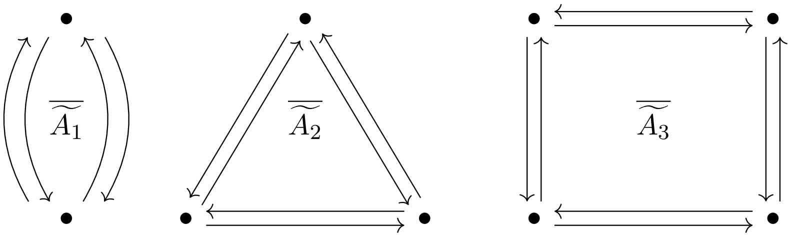
$$\begin{aligned}\alpha_j^*\alpha_j\alpha_{j+1}\cdots\alpha_{j+k} &= \alpha_{j+1}\alpha_{j+1}^*\alpha_{j+1}\alpha_{j+2}\cdots\alpha_{j+k}\\ &= \alpha_{j+1}\alpha_{j+2}\alpha_{j+2}^*\alpha_{j+2}\alpha_{j+3}\cdots\alpha_{j+k}\\ &\vdots\\ &= \alpha_{j+1}\cdots\alpha_{j+k}\alpha_{j+k}^*\alpha_{j+k}\\ &= \alpha_{j+1}\cdots\alpha_{j+k+1}\alpha_{j+k+1}^*.\end{aligned}$$

Every path can be written in the form pq where

- p is trivial or only contains nonstar arrows
- q is trivial or only contains star arrows

This is the *canonical form* of a general element.

The first few $\widetilde{A_{n-1}}$'s...



Quiver automorphisms

Definition

Let Q be a quiver. A *quiver automorphism* is a pair of maps $\sigma = \sigma_0 \cup \sigma_1$, $\sigma_0 : Q_0 \to Q_0$, $\sigma_1 : Q_1 \to Q_1$ such that for all $\alpha \in Q_1$

$$s(\sigma_1(\alpha)) = \sigma_0(s(\alpha)) \quad \text{and} \quad t(\sigma_1(\alpha)) = \sigma_0(t(\alpha))$$

Equivalently, a quiver automorphism is a pair of bijections $\sigma_0 : Q_0 \to Q_0$ and $\sigma_1 : Q_1 \to Q_1$ that form a commuting square with the source and target functions:

$$\begin{tikzcd} Q_1 \arrow[r, "\sigma_1"] \arrow[d, "s"'] & Q_1 \arrow[d, "t"] \\ Q_0 \arrow[r, "\sigma_0"] & Q_0 \end{tikzcd}$$

Extending quiver automorphisms

Let Q be a quiver and σ and quiver automorphism of Q.

- Q is *Schurian* if no two arrows share the same source and target.
- If σ is a quiver automorphism of Q and Q is Schurian, then σ is completely determined by σ_0.
- Extend σ to $\Bbbk Q$ linearly and multiplicatively. Then $\sigma \in \mathrm{Aut}_{\mathrm{gr}}(\Bbbk Q)$.

Quiver automorphisms

Theorem (BK)

Let Q be a quiver such that $\overline{Q}$ is schurian, and let $\sigma \in \mathrm{Aut}_{\mathrm{gr}}(\Bbbk Q)$ be induced from a quiver automorphism of $\overline{Q}$. Suppose either of the following hold:

- $\sigma(Q_1) = Q_1$ *and* $\sigma(Q_1^*) = Q_1^*$
- $\sigma(Q_1) = Q_1^*$ *and* $\sigma(Q_1^*) = Q_1$

Then $\sigma \in \mathrm{Aut}_{\mathrm{gr}}(\Pi_Q)$.

Dihedral automorphisms

Theorem (BK)

Let $Q = \widetilde{A_{n-1}}$, $n \geq 3$. The quiver automorphism group of $\overline{Q}$ contains a subgroup isomorphic to D_n which extends to a subgroup of $\mathrm{Aut}_{\mathrm{gr}}(\Pi_Q)$, also isomorphic to D_n.

- Define $\rho, r_0 : \overline{Q} \to \overline{Q}$ by $\rho(e_i) = e_{i+1}$ and $r_0(e_i) = e_{-i}$.
- Then

$$r_0(\rho(e_i)) = r_0(e_{i+1}) = e_{-i-1} = \rho^{-1}(e_{-i}) = \rho^{-1}(r_0(e_i))$$

- Therefore $\langle \rho, r_0 \rangle \cong D_n$.

Note: for each $e_i \in Q_0$, there exists a reflection $r_i \in D_n$ such that $r_i(e_i) = e_i$.

The Skew Group Ring

Definition

Let R be an algebra and G a subgroup of $\mathrm{Aut}(R)$. Let $R\#G$ be the set of formal sums

$$\left\{\sum a_g\#g : a_g \in R, g \in G\right\}$$

and define a multiplication on $R\#G$ by

$$(r_1\#g_1)(r_2\#g_2) = r_1g_1(r_2)\#g_1g_2,$$

and extending linearly.

Example: $\Pi_{\widetilde{A_2}}\#\langle\rho\rangle$

Example: $R = \Pi_{\widetilde{A_2}}$, $G = \langle p \rangle$. Let

$$1\#f = 1\#1 + 1\#\rho + 1\#\rho^2.$$

Then

$$\begin{aligned}(e_0\#1)(1\#f)(e_0\#1) &= (e_0\#1 + e_0\#\rho + e_0\#\rho^2)(e_0\#1) \\ &= e_0e_0\#1 + e_0\rho(e_0)\#1 + e_0\rho^2(e_0)\#\rho^2 \\ &= e_0\#1 + e_0e_1\#\rho + e_0e_2\#\rho^2 \\ &= e_0\#1.\end{aligned}$$

Pertinency

Definition

Let R be a graded algebra and G a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(R)$. Let

$$1\#f_G := \sum_{g \in G} 1\#g.$$

The *pertinency* of the G-action on R is defined to be

$$\mathfrak{p}(R, G) = \operatorname{GKdim} R - \operatorname{GKdim}(R\#G)/(1\#f_G)$$

When the group G is clear, we surpress the subscript in f_G and write f.

Auslander-Pertinency Theorem

Theorem (Bao-He-Zhang, 2019)

Let R be a Noetherian locally-finite graded algebra and G a finite group subgroup of $\mathrm{Aut}_{\mathrm{gr}}(R)$. Assume further that R is CM of global dimension 2 with $\operatorname{GKdim} R \geq 2$. Then $\eta_{R,G}$ is an isomorphism if and only if $\mathfrak{p}(R,G) \geq 2$.

For $R = \Pi_{\widetilde{A_{n-1}}}$, $\operatorname{GKdim} R = 2$, so

$$\begin{aligned} \mathfrak{p}(R,G) \geq 2 &\Leftrightarrow \operatorname{GKdim}(R\#G)/(1\#f_G) = 0 \\ &\Leftrightarrow \dim_{\Bbbk}(R\#G)/(1\#f_G) < \infty \\ &\Leftrightarrow \dim_{\Bbbk} R' < \infty, \end{aligned}$$

where R' is the *identity component of* $R\#G$, i.e. R' is the image in the composition

$$R \hookrightarrow R\#G \to (R\#G)/(1\#f)$$

Sufficient Condition

Theorem (BK)

Let $R = \Pi_{\widetilde{A_{n-1}}}$ and let G be a subgroup of D_n such that $r_i \notin G$ for some $i = 0, \dots, n-1$. Then $\dim_{\Bbbk} R' < \infty$.

- $r_i \notin G \Rightarrow \mathrm{stab}_G(e_i) = 1$. Then $e_i g(e_i) = 0$ for all $g \neq 1$.
- So
$$(e_i\#1)(1\#f)(e_i\#1) = \sum_{g \in G} e_i g(e_i)\#g = e_i\#1$$
- For any $p \in Q_{\geq 2n+1}$, p contains at least $n+1$ (WLOG) nonstar arrows.
- p passes through e_i, i.e. p can be written in the form $p = p'e_ip''$
- $e_i\#1 \in (1\#f)$ so $p\#1 = (p'\#1)(e_i\#1)(p''\#1) \in (1\#f)$

W_n and D_n

- If G doesn't satisfy the condition of the theorem, then G contains the group

$$W_n := \langle r_i \in D_n : i = 0, \dots, n-1 \rangle$$

- If n is odd, then $W_n = D_n$. If n is even, then

$$W_n = \left\langle r_0, \rho^2 \right\rangle,$$

which is of index 2 in D_n.

- The theorem applies to every proper subgroup of D_n except for W_n when n is even.

The Path Mirroring Map

Theorem (BK)

Let $R = \Pi_{\widetilde{A_{n-1}}}$ and $G = W_n$. Then Auslander's map $\eta_{R,G}$ is not surjective.

Definition

Let $R = \Pi_{\widetilde{A_{n-1}}}$ and $G = W_n$ and consider R as a right R^G-module. Define the *path mirroring map* $\phi : R \to R$ $\mathbb{k}$-linearly by

$$\phi(p) = r_i(p) \quad \text{where } e_i = s(p).$$

The Path Mirroring Map

Lemma (BK)

The path mirroring map ϕ is R^G-linear.

Must show $\phi(ax) = \phi(a)x$ for all $a \in R$ and $x \in R^G$.

- $\eta_{R,G}$ is an algebra homomorphism so $\eta(1\#r_i) = r_i$ is R^G-linear for all $i = 0, \ldots, n-1$.
- $a \in R$ can be written $a = a_0 + \cdots + a_{n-1}$ where the summands in a_i all have source e_i.

$$\begin{aligned}
\phi(ax) &= \phi(a_0x) + \cdots + \phi(a_{n-1}x) \\
&= r_0(a_0x) + \cdots + r_{n-1}(a_{n-1}x) \\
&= r_0(a_0)x + \cdots + r_{n-1}(a_{n-1})x \\
&= \phi(a_0)x + \cdots + \phi(a_{n-1})x \\
&= \phi(a)x.
\end{aligned}$$

Degree 0 maps

Definition

Let R be a graded algebra. A $\Bbbk$-linear map $\psi : R \to R$ is a *degree* 0 *map* if $\psi(R_i) \subseteq R_i$ for all $i \in \mathbb{N}$.

- The path mirroring map ϕ is a degree 0 map.
- If $z = \sum_{g \in G} a_g \# g \in R\#G$ and $\eta(z)$ is a degree 0 map, then $a_g \in R_0$ for all $g \in G$.

The Path Mirroring Map

Lemma (BK)

The path mirroring map ϕ is not in the image of $\eta_{R,G}$.

- Suppose $\eta(\sum a_g \# g) = \phi$.
- ϕ is degree 0, so each $a_g \in R_0$
- For all $\alpha_i \in Q_1$, $\mathrm{stab}_{D_n}(\alpha_i) = 1$ so $g(\alpha_i)$ is distinct for all $g \in G$.
- We have
$$\alpha^*_{i-1} = \phi(\alpha_i) = \sum a_g g(\alpha_i),$$
and $a_g g(\alpha_i) \in \mathrm{span}_{\Bbbk}(g(\alpha_i))$.
- By linear independence $a_g g(\alpha_i) = 0$ for all $g \neq r_i$, and $a_{r_i} = 1$. This holds for all i, a contradiction.

Auslander's map is not iso only for W_n and D_n

We have $\mathfrak{p}(R, W_n) < 2$.

Theorem (GKMW, 2019)

Let G be a group acting on an algebra R. Then for any subgroup H of G,

$$\mathfrak{p}(R, G) \leq \mathfrak{p}(R, H)$$

Hence $\mathfrak{p}(R, D_n) < 2$, so η_{R,D_n} is not an isomorphism.

Scalar Automorphisms

Weispfenning [13] shows that $\mathrm{Aut}_{\mathrm{gr}}(R) \cong D_n \ltimes N$ where

$$N = \{g \in \mathrm{Aut}_{gr}(R) : g(e_i) = e_i \text{ for } i = 0, \ldots, n-1\}$$

For each $g \in N$, for every arrow β, there exists $\xi_\beta \in \mathbb{k}^*$ such that $g(\beta) = \xi_\beta \beta$.
N is a group of *scalar automorphisms*.

Other quivers

Preprojective algebras over extended Dynkin diagrams of type D ($\widetilde{D}_n, n \geq 5$) and E ($\widetilde{E}_n$, $n = 6, 7, 8$) are of interest:

That's it!

Thank you for coming!

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