Quantum Generalized Weyl Algebras

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Motivation

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Rescaling gives the normalized version of the Fundamental Equation of Quantum Mechanics:

$$PQ - QP = I$$

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We are interested in studying this equation and its generalizations from the viewpoint of algebra.

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An **automorphism** of a $\mathbb{C}$-algebra A is a bijective, structure-preserving function from A to itself. We denote the group of automorphisms of A by $\mathrm{Aut}(A)$.

Generalized Weyl Algebra

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Let D be a commutative algebra, $\sigma \in \mathrm{Aut}(D)$, and $a \in D$, $a \neq 0$. The **generalized Weyl algebra** (GWA) $D(\sigma, a)$ is generated over D by x and y subject to the relations

$$xd = \sigma(d)x, \quad yx = \sigma^{-1}(d)y, \quad yx = a, \quad xy = \sigma(a).$$

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This type of algebras has been studied by many authors before Bavula. Notably in the work of Hodges, Jordan, Joseph, Smith, and Stafford.

Classical Weyl Algebra

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For example, let $D = \mathbb{C}[h]$, $a = h$, $\sigma(h) = h - 1$.

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xh &= \sigma(h)x = (h-1)x \\
yh &= \sigma(h)^{-1}y = (h+1)y \\
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This is just the first Weyl algebra $A_1(\mathbb{C})$

Quantum Generalized Weyl Algebra

Let $R = D(\sigma, a)$ be a GWA. We say R is *quantum* if $D = \mathbb{C}[h]$ or $\mathbb{C}[h^{\pm 1}]$ and $\sigma(h) = qh$, $q \in \mathbb{C} \backslash \{0, 1\}$.

$$xh = qhx, yh = q^{-1}hy, yx = a, xy = a(qh)$$

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There are many important families of algebras that may be GWAs: quantum Weyl algebras and primitive quotients of $U_q(\mathfrak{sl}_2)$.

Shephard, Todd, Chevalley Theorem

If A is a $\mathbb{C}$-algebra and G a subgroup of $\mathrm{Aut}(A)$, then $A^G = \{a \in A : g(a) = a \text{ for all } g \in G\}$.

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Theorem (Shephard, Todd, Chevalley)

Let $A = \mathbb{C}[x_1, \ldots, x_n]$ and let G be a finite group of linear automorphisms of A. Then the fixed ring A^G is again a polynomial ring if and only if G is generated by reflections.

Example: $A = \mathbb{C}[x, y]$ and $g \in \mathrm{Aut}\, A$ given by $g(x) = y$ and $g(y) = x$. Then $A^G = \mathbb{C}[x + y, xy]$.

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We want to work towards developing a STC-like theorem for (quantum) GWAs. Others have contributed to this previously, including Jordan-Wells, Kirkman-Kuzmanovich, and Gaddis-Won.

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Theorem (-, Gaddis)

Let D be an integral domain, let $R = D(\sigma, a)$ be a GWA, and let $\phi \in \mathrm{Aut}(R)$ with $|\phi| < \infty$. Suppose $\phi|_D$ restricts to an automorphism of D, $\phi(x) = \mu^{-1}x$, and $\phi(y) = \mu y$ for $\mu \in \mathbb{C}^\times$. Set $n = |\phi|_D|$ and $m = |\mu|$. If $\gcd(n, m) = 1$, then $R^{\langle\phi\rangle} = D^{\langle\phi\rangle}(\sigma^m, A)$ with $A = \prod_{i=0}^{m-1} \sigma^{-i}(a)$.

Example

Let $R = \mathbb{C}[h^{\pm 1}](\sigma, a)$ be a quantum GWA with $a = (h^2 - 1)(h^2 - 4)$ and $q = 1/2$. Let $\eta \in \mathrm{Aut}(R)$ be given by

$$\eta(h) = \gamma h, \quad \eta(x) = \mu^{-1} x, \quad \eta(y) = \mu y$$

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Then $R^{\langle \eta \rangle}$ is generated by $X = x^3$, $Y = y^3$, and $H = h^2$. The defining polynomial is

$$\begin{aligned} A(H) &= (H-1)(H-4)(4H-1)(4H-4)(16H-4)(16H-1) \\ &= 4(H-1)^2(H-4)(4H-1)(16H-1)(16H-4) \\ &= 4^6(H-1)^2(H-4)\left(H - \frac{1}{4}\right)^2\left(H - \frac{1}{16}\right). \end{aligned}$$

Thank You for Your Listening!