Dihedral actions on preprojective algebras of type A

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The McKay correspondence

The classical McKay correspondence provides connections between

- finite subgroups of $\mathrm{SL}_2(\mathbb{C})$,
- simple Lie algebras of type ADE,
- Kleinian singularities,
- preprojective algebras of reduced McKay quivers.

Definition (The Auslander Map)

Let V be a finite dimensional vector space and G a finite group acting linearly on $R = \mathbb{C}[V]$. The *Auslander map* $\eta_{R,G} : \underline{R\#G} \to \underline{\mathrm{End}_{R^G} R}$ is defined by

$$a\#g \longmapsto \left(\begin{array}{ccc} R & \longrightarrow & R \\ b & \longmapsto & ag(b) \end{array} \right)$$

The McKay correspondence

Theorem (Auslander's Theorem)

V f.d. vsp G acts w/o reflections on V (G small)

then $\eta_{R,G}$ is an iso.

Suppose G is small, so that $\eta_{R,G} : R\#G \to \mathrm{End}_{R^G} R$ is an isomorphism. Then there are bijective correspondences between isomorphism classes of the following:

- indecomposable direct summands of $\overset{R}{\cancel{A}}$ as left $\underset{R}{\cancel{A}}^G$-modules,
- indecomposable finitely generated, projective, initial left $\mathrm{End}_{\underset{R}{\cancel{A}}^G}(\overset{R}{\cancel{A}})$-modules,
- indecomposable finitely generated, projective, initial $\overset{R}{\cancel{A}}\#G$-modules, and
- simple left G-modules.

Pertinency

Let R be an algebra and G a finite group acting on R. Set

$$f_G = \sum_{g \in G} 1 \# g \in R \# G$$

The *pertinency* of the G-action on R is defined as

$$p(R,G) = \mathrm{GKdim}((R \# G)) - \mathrm{GKdim}((R \# G)/(f_G))$$

Theorem (Bao, He, Zhang)

Let R be a Noetherian locally-finite graded algebra and G a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(R)$. Assume further that R is GKdim-CM of global dimension 2 with $\mathrm{GKdim}\, R \geq 2$. Then $\eta_{R,G}$ is a graded algebra isomorphism if and only if $\mathrm{p}(R,G) \geq 2$.

Examples of Auslander's Theorem

The Auslander map is an isomorphism in each of the following situations:

(1) The algebra A is noetherian and AS regular of (global) dimension two and H is a semisimple Hopf algebra acting linearly and inner faithfully on A with trivial homological determinant. (Chan, Kirkman, Walton, Zhang)

(2) The algebra is a two-dimensional Artin-Schelter regular algebra and G is a small group (in the noncommutative sense). (Crawford)

(3) The algebra is $\mathbb{C}_{-1}[x_1, \ldots, x_n]$ and G is any subgroup of $\mathcal{S}_n$ acting linearly as permutations of the generators (i.e., $\sigma(x_i) = x_{\sigma(i)}$). (G, Kirkman, Moore, Won)

(4) The algebra is a (generic) graded down-up algebra and H is a finite group coacting homogeneously and inner faithfully. (Chen, Kirkman, Zhang)

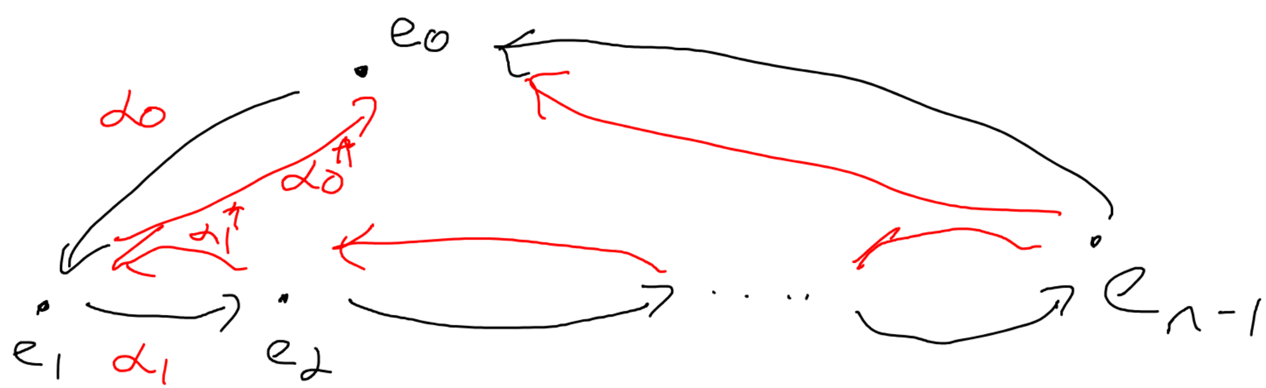
We are interested in cases where the the algebra is $\mathbb{N}$-graded (as in all examples above) but non-connected.

Preprojective algebras

Definition (Double quiver)

Let Q be a quiver. For each $\alpha \in Q_1$ with $s(\alpha) = e_i$ and $t(\alpha) = e_j$, set α^* to be an arrow $s(\alpha^*) = e_j$ and $t(\alpha^*) = e_i$, and let Q_1^* be the set of such arrows. Define the *double* of Q to be the quiver $\overline{Q}$ with $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \cup Q_1^*$.

$\widetilde{A}_{n-1}$



$\Bbbk = \mathbb{C}$

Definition (Preprojective algebra)

$$\Bbbk\overline{Q} \Big/ \Big(\sum_{a \in \overline{Q}_1} a a^* - a^* a \Big)$$

The preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$

For the remainder, we fix $Q = \widetilde{A_{n-1}}$, $n \geq 3$, and $R = \Pi_Q$.

- If P is a nonconstant monomial in R, then there exist nonstar arrows $\beta_1, \dots, \beta_\ell$ and star arrows $\gamma_1, \dots, \gamma_m$ such that
$$p = \underbrace{\beta_1 \cdots \beta_\ell}\underbrace{\gamma_1 \cdots \gamma_m}.$$
- R is a locally finite graded noetherian algebra of global and GK dimension 2.
- R is GKdim-CM.
- The matrix-valued Hilbert series of R is
$$H_R(t) = \frac{1}{(1 - M_{\overline{Q}} t)^2}.$$
- Let G be a finite subgroup of $\mathrm{Aut}_{\mathrm{gr}}(R)$ and let R' be the image of R in the composition
$$R \hookrightarrow R\#G \twoheadrightarrow (R\#G)/(f_G).$$
Then $\mathrm{GKdim}(R') = \underline{\mathrm{GKdim}((R\#G)/(f_G))}$. Hence, $\eta_{R,G}$ is a graded algebra isomorphism if and only if $\dim_{\Bbbk}(R') < \infty$.

The preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$ - Scalar automorphisms

Theorem

Let

$$F = \{\sigma \in \mathrm{Aut}_{\mathrm{gr}}(R) : |\sigma| < \infty \text{ and } \underline{\sigma(e_i) = e_i} \text{ for all } i = 0, \ldots, n-1\}.$$

Let $\sigma \in F$ with $m = |\sigma|$, $1 < m < \infty$, and let $G = \langle \sigma \rangle$. As above, write $\sigma(\alpha_i) = \xi_i \alpha_i$ and $\sigma(\alpha_i^*) = \xi_i^* \alpha_i^*$, $i = 0, \ldots, n-1$, with $\xi_1 \xi_1^* = 1$. In each of the following cases, $\eta_{R,G}$ is an isomorphism.

1. *There is some primitive mth root of unity ζ such that $\underline{\xi_i = \zeta}$ for $i = 0, \ldots, n-1$.*
2. *There is some primitive mth root of unity ζ such that $\xi_0 \xi_1 \cdots \xi_{n-1} = \zeta$.*
3. *For all $i, j = 0, \ldots, n-1$ with $i \neq j$, we have $\gcd(|\xi_i|, |\xi_j|) = 1$.*

$\rho = \underline{\alpha_0 \alpha_1 \cdots \alpha_{m-1}}$ indices mod n

$f_\sigma \rho + \alpha_0 f_\sigma \alpha_1 \alpha_2 \cdots \alpha_{m-1} + \cdots + \rho f_\sigma = m \rho \neq 1$

The preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$ - Dihedral actions

We identify two quiver automorphisms of $\overline{Q}$ which extend to automorphisms of R.

1. Define $\rho : \overline{Q} \to \overline{Q}$ by $\rho(e_i) = e_{i+1}$, where the index is taken mod n. Hence,

$$\rho(\alpha_i) = \alpha_{i+1} \quad \text{and} \quad \rho(\alpha_i^*) = \alpha_{i+1}^*.$$

2. Define $r_0 : \overline{Q} \to \overline{Q}$ by $r_0(e_i) = e_{n-i}$. Hence,

$$r_0(\alpha_i) = \alpha_{n-i-1}^* \quad \text{and} \quad r_0(\alpha_i^*) = \alpha_{n-i-1}.$$

The subgroup $\langle \rho, r_0 \rangle$ of $\mathrm{Aut}_{\mathrm{gr}}(R)$ is isomorphic to D_n and so we identify D_n with this group acting on R by graded automorphisms.

The preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$ - Dihedral actions

Theorem

Let G be a subgroup of D_n. Suppose there is some reflection through a vertex not contained in G. Then $\dim_{\mathbb{k}}(R') < \infty$ and so $\eta_{R,G}$ is an isomorphism.

$$\mathrm{stab}_G(e_i) = \{1\}$$

$$e_i f_G e_i = e_i \# 1$$

$$\ell(p) \geq 2n \quad \Rightarrow \quad p \in (f_G)$$

The preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$ - The case $G = D_n$

For $\ell \geq k \geq 0$, set

$$B_{\ell,k} = Q_\ell Q_k^* \cup Q_k Q_\ell^*.$$

For any $p \in B_{\ell,k}$, $\mathcal{O}(p) = B_{\ell,k}$.

The corresponding orbit sums $\mathbb{O}(\ell, k) = \sum_{p \in B_{\ell,k}} p$ form a $\Bbbk$-basis for R^{D_n}, so that

$$H_{R^{D_n}}(t) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + \cdots = \frac{1}{(1-t)(1-t^2)}.$$

The orbit sums $\mathbb{O}(\ell, k)$ satisfy the following relations:

$$\mathbb{O}(1,0)\mathbb{O}(\ell,k) = \begin{cases} \mathbb{O}(\ell+1,k) + \mathbb{O}(\ell,k+1) & \text{if } \ell > k \\ \mathbb{O}(\ell+1,k) & \text{if } \ell = k \end{cases}$$

$$\mathbb{O}(1,1)^m = \mathbb{O}(m,m).$$

The preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$ - The case $G = D_n$

Set $s_0 = \mathbb{O}(0,0) = 1$, $s_1 = \mathbb{O}(1,0)$, and $s_2 = \mathbb{O}(2,0)$. Then one can show that

1. s_1 and s_2 commute,
2. for all $\ell \geq k \geq 0$, $\mathbb{O}(\ell, k) \in \mathbb{k}[s_1, s_2]$,
3. $\mathbb{k}[s_1, s_2]$ and R^{D_n} have the same Hilbert series.

It follows that $R^{D_n} = \underline{\mathbb{k}[\overset{1}{s_1}, \overset{2}{s_2}]}$. Putting this together gives the following result.

Theorem

The Auslander map is not an isomorphism for the pair (R, D_n).

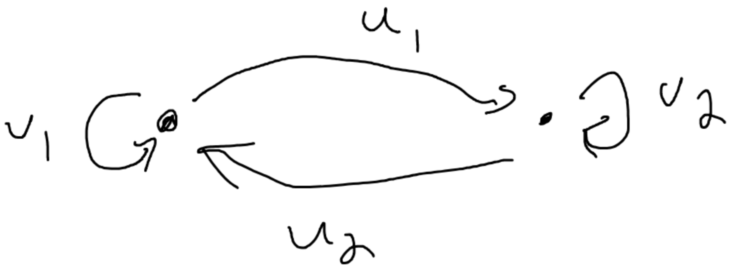
R free over R^{D_n}

$\mathrm{End}_{R^G}(R)$ contains neg degree maps

$\eta_{R,G}$ graded hom

The preprojective algebra $\Pi_{\widetilde{A_{n-1}}}$ - The case $G = W_n$

Let W_n be the subgroup of G generated by reflections through a vertex. If n is odd, then $W_n = D_n$, but if n is even, then W_n is a proper subgroup of index 2 in D_n. Assuming n is even, we can follow a similar strategy as above. The key difference is that the invariant ring is no longer connected graded.



(relns)

2-CY algebras

Theorem

The Auslander map is not an isomorphism for the pair (R, W_n).

Corollary

Let $G = D_n$ or $G = W_n$. Then $\mathrm{p}(R, G) = 1$.

Auslander's Theorem for $\Pi_{\widetilde{A_{n-1}}}$

Theorem

Let G be a subgroup of D_n. Then $\eta_{R,G}$ is an isomorphism if and only if there is some reflection through a vertex not contained in G.

Thank you!