Auslander's Theorem for permutation actions on noncommutative algebras

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Introduction

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Invariant Theory

Formally, symmetry is a property of invariance (or stability) with respect to some transformation. This may mean a geometric transformation (reflection, rotation, scaling), or it may relate to change with respect to the passage of time, change in temperature, etc.

Emily Noether proved that there is a correspondence between symmetries of physical systems and conservation laws in physics. Studying symmetry may lead to answers to fundamental questions regarding the shape of the universe.

Invariant Theory

In classical (algebraic) invariant theory, one generally studies actions of groups on polynomial rings.

Throughout, we will consider polynomials over an algebraically closed field $\Bbbk$ of characteristic zero.

Let G be a group acting (as linear automorphisms) on $A = \Bbbk[x_1, \dots, x_n]$. Our primary object of interest will be A^G, the subring of invariants.

Note: We regard A as an algebra (ring + vector space over $\Bbbk$), so a linear automorphism really means a $\Bbbk$-linear automorphism.

Invariant Theory - Examples

Formally, $A^G = \{p \in A : g(p) = p \text{ for all } g \in G\}$.

Let $A = \Bbbk[x, y]$.

- If $G = \langle \sigma \rangle$ where $\sigma(x) = x$ and $\sigma(y) = -y$, then $A^G = \Bbbk[x, y^2]$.
- If $G = \langle \sigma \rangle$ where $\sigma(x) = y$ and $\sigma(y) = x$, then $A^G = \Bbbk[xy, x + y]$.
- If $G = \langle \sigma \rangle$ where $\sigma(x) = -x$ and $\sigma(y) = -y$, then $A^G = \Bbbk[x^2, y^2, xy]$.

Note that in the first two examples, A^G is again a polynomial ring. However, in the last example $A^G \cong \Bbbk[a, b, c]/(ab - c^2)$.

Shephard-Todd-Chevalley

An element $g \in G$ is a reflection if g fixes a codimension 1 subspace of $\Bbbk[x_1, \ldots, x_n]$. In the case of a group acting linearly on a polynomial ring, we can represent $g \in G$ as a matrix M. Then g is a reflection if all but one of the eigenvalues of M are 1.

Theorem (Shephard-Todd 1954, Chevalley 1955)

Let G be a finite group acting on $A = \Bbbk[x_1, \ldots, x_n]$. Then $A^G \cong A$ if and only if G is generated by reflections.

The group G is small if it contains no reflections.

Invariant Theory - Example

The following result is typically attributed to my great[8] (mathematical) grandfather Carl Gauss.

Theorem

Let $A = \Bbbk[x_1, \ldots, x_n]$ and $G = \mathcal{S}_n$ acting on A by permutations (so $\sigma(x_i) = x_{\sigma(i)}$). Then A^G is a polynomial ring generated by the elementary symmetric functions in the x_i.

A "dual" to STC

The skew group algebra $A\#G$ has basis $A \otimes G$ and multiplication

$$(a \otimes g)(b \otimes h) = (ag(b)) \otimes gh \quad \text{for all } a, b \in A, g, h \in G.$$

Example

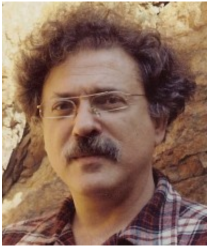
Let $A = \Bbbk[x, y]$ and let $G = \langle \sigma \rangle$ where $\sigma(x) = y$ and $\sigma(y) = x$. Then in $A\#G$ we have,

$$x\sigma \cdot y\sigma = (x\sigma(y))\sigma^2 = x^2 e.$$

A "dual" to STC

Define the Auslander map $\gamma_{A,G} : A\#G \mapsto \mathrm{End}_{A^G}(A)$ by

$$a\#g \mapsto (b \mapsto ag(b)).$$



Theorem (Auslander 1962)

If G is small, then $\gamma_{A,G}$ is an isomorphism.

Auslander's theorem first appeared in his 1962 paper, "On the purity of the branch locus". It is not actually stated as a theorem at all! The result relates finitely generated projective modules over $\Bbbk[x,y]\#G$ to maximal Cohen-Macaulay modules over $\Bbbk[x,y]^G$.

There are two (standard) ways to extend classical invariant theory to the noncommutative setting:

- One could replace A with a noncommutative algebra.

Example

Let $\Bbbk_{-1}[x_1, \dots, x_n] = \Bbbk\langle x_1, \dots, x_n : x_i x_j + x_j x_i = 0 \text{ for } i \neq j \rangle$ *denote the* (-1)*-skew polynomial ring in* n *variables.*

- One could replace G with a Hopf algebra.

Example

The Sweedler algebra $H_2 = \Bbbk\langle g, x : g^2 = 1, x^2 = 0, gx + xg = 0 \rangle$ *acts on* $\Bbbk[u, v]$ *by* $g(u) = u$, $g(v) = -v$, $x(u) = 0$, $x(v) = u$.

AS regular algebras

An algebra A is connected $\mathbb{N}$-graded if it has a vector space decomposition

$$A = \bigoplus_{i=0}^{\infty} A_i$$

such that $A_i A_j \subset A_{i+j}$ and $A_0 = \Bbbk$. Throughout, a graded algebra is understood to be connected $\mathbb{N}$-graded.

Definition

A graded algebra A is Artin-Schelter (AS) regular if

- $\operatorname{gldim}(A) = d < \infty$;
- $\operatorname{GKdim}(A) < \infty$;
- $\operatorname{Ext}^i_A(\Bbbk) = \delta_{id}\Bbbk(\ell)$.

The constant ℓ is the Gorenstein parameter of A.

AS regular algebras

Examples of AS regular algebras include

- The (-1)-skew polynomial ring $\Bbbk_{-1}[x_1, \ldots, x_n]$.

- More generally, for $q \in \Bbbk^\times$, the *q-skew polynomial ring* is defined as $\Bbbk_q[x_1, \ldots, x_n] = \Bbbk\langle x_1, \ldots, x_n : x_i x_j - q x_j x_i = 0 \text{ for } i < j\rangle$.

- The *three-dimensional Sklyanin algebra* with parameters $(a, b, c) \in \mathbb{P}^2$ is generated by x_0, x_1, x_2 with three relations

$$0 = a x_i x_{i+1} + b x_{i+1} x_i + c x_{i+2}^2 \quad \text{indices mod } i.$$

Question

In what context do the Shephard-Todd-Chevalley Theorem and Auslander's theorem hold when A is replaced by an AS regular algebra?

Reflections

But first...

Question

What do we mean by a reflection in the noncommutative setting?

In general, it is too much to ask that $A^G \cong A$.

Example

Let $A = \Bbbk_{-1}[x, y]$ *and* $G = \langle g \rangle$ *where* $g(x) = y$ *and* $g(y) = -x$. *Then* $A^G = \Bbbk[x^2 + y^2, xy]$. *Note that* g *is not a reflection in the classical sense.*

Definition

A finite group G acting linearly on an AS regular algebra A is a reflection group if A^G is AS regular.

Reflections

Let A be a graded algebra. The trace function of a graded automorphism g acting on A is defined to be the formal power series

$$\mathrm{Tr}_A(g,t) = \sum_{j=0}^{\infty} \mathrm{tr}\left(g|_{A_j} \right) t^j.$$

Definition (Kirkman, Kuzmanovich, and Zhang 2008)

Let A be a graded algebra of GK dimension n. Then $g \in G$ is a quasi-reflection if its trace series is of the form

$$\mathrm{Tr}_A(g,t) := \frac{1}{(1-t)^{n-1}q(t)}, \quad q(1) \neq 0.$$

Conjecture (NC Shephard-Todd-Chevalley)

Let A be an AS regular algebra and G a finite group of graded automorphisms of A. Then A^G is AS regular if and only if G is generated by quasi-reflections of A.

Theorem (Kirkman, Kuzmanovich, and Zhang 2008)

The conjecture holds when A is a noetherian AS regular algebra A with Hilbert series $H_A(t) = (1-t)^{-n}$ (a so-called quantum polynomial ring) and G is a finite abelian group of graded automorphisms of A.

The Auslander Map

Conjecture (NC Auslander's Theorem)

If A is an AS regular algebra and G a finite group acting linearly on A without quasi-reflections, then the Auslander map is an isomorphism.

One can try to adapt the original proof to the noncommutative setting.

In general, with the right initial conditions, injectivity is not a problem. On the other hand, to prove surjectivity, the original proof requires that every minimal prime ideal in A is unramified over A^G.

We have run into quantum rigidity, that is, noncommutative algebras have few prime ideals.

Some progress

Theorem (Mori, Ueyama 2015)

Let A be a noetherian AS-regular algebra of dimension $d \geq 2$ and $G \subset \operatorname{GrAut} A$ a finite ample subgroup. Then $A \# G \cong \operatorname{End}_{A^G}(A)$ as graded algebras.

One problem with the ampleness condition, apart from being hard to check, is that it implies something stronger than Auslander's theorem. It actually implies that A^G is a graded isolated singularity.

Pertinency

Let A be a graded algebra and write $A = \bigoplus_n A_n$. The Gelfand-Kirillov (GK) dimension of A is

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \log_n(\dim_k A_n).$$

Definition (Bao, He, Zhang 2016)

Let A be an graded algebra with $\mathrm{GKdim}\, A < \infty$ and G a finite group acting linearly on A. The pertinency of the action of G on A is defined to be

$$\mathrm{p}(A, G) = \mathrm{GKdim}\, A - \mathrm{GKdim}(A\#G)/(f_G)$$

where (f_G) is the two-sided ideal generated by $f_G = \sum_{g\in G} 1\#g$.

Theorem (BHZ)

Given the above setup, $A\#G \cong \mathrm{End}_{A^G}(A)$ *if and only if* $\mathrm{p}(A, G) \geq 2$.

Pertinency

Example (BHZ)

Consider $W = \langle \sigma \rangle$, $|\sigma| = n \geq 2$, *acting on* $\mathbb{k}_{-1}[x_1, \ldots, x_n]$ *by* $\sigma : x_i \mapsto x_{i+1}, x_n \mapsto x_1$ *for* $1 \leq i \leq n-1$. *If* $n = 2^d$, $d \geq 2$, *then*

$$p(\mathbb{k}_{-1}[x_1, \ldots, x_n], W) = n \geq 2.$$

Hence, the Auslander map is an isomorphism in this case.

Question

Is the Auslander map an isomorphism for $\mathbb{k}_{-1}[x_1, \ldots, x_n]$ *and any subgroup of* $\mathcal{S}_n$*?*

Strategery

We need to understand the ideal

$$(f_G) = \left(\sum_{g \in G} 1 \# g \right).$$

Theorem (BHZ)

Let A be finitely generated over a central subalgebra T. Let A' be the image of the map

$$A \hookrightarrow A\#G \to (A\#G)/(f_G)$$

and let $T' \subseteq A'$ be the image of T. Then

$$\operatorname{GKdim} T' = \operatorname{GKdim} A' = \operatorname{GKdim}(A\#G)/(f_G).$$

So we need only understand $(f_G) \cap A$ or even $(f_G) \cap T$.

Strategery

Let $J \subset (f_G) \cap T$ be an ideal.

Assuming we can show GKdim $T/J \leq n-2$ we have

$$\begin{aligned} p(A, G) &= \operatorname{GKdim} A - \operatorname{GKdim}(A\#G)/(f_G) \\ &\geq \operatorname{GKdim} A - \operatorname{GKdim} T/J \\ &\geq n - (n-2) \\ &= 2. \end{aligned}$$

Thus, under this assumption, the Auslander map is an isomorphism for A and G.

Producing elements

Let R be a commutative algebra and G a finite group acting on R.

For $g \in G$, let $I(g)$ be the ideal generated by $\{r - g.r : r \in R\}$.

Lemma (Brown, Lorenz 1994)

$$\prod_{\substack{g \in G \\ g \neq e}} I(g) \subset (f_G) \cap R$$

There are two problems with applying/adapting this lemma:

- The proof of this lemma is highly commutative.
- It produces elements of degree $|G| - 1 = n! - 1$, often much higher than lowest degree element in (f_G).

However, the idea can be adapted for algebras with large centers.

Example: $\mathbb{k}_{-1}[x_1, x_2, x_3]$ and $\mathcal{S}_3$

Let $T = \mathbb{k}[x_1^2, x_2^2, x_3^2] \subset C(V_3)$ and $f = \sum_{\sigma \in \mathcal{S}_3} 1\#\sigma$. Define

$$\begin{aligned}
f_1 &= x_1^2 f - f x_2^2 \\
&= (x_1^2 - x_2^2)\#(1) + (x_1^2 - x_2^2)\#(13) \\
&\qquad + (x_1^2 - x_3^2)\#(23) + (x_1^2 - x_3^2)\#(123) \\
f_2 &= x_1^2 f_1 - f_1 x_3^2 \\
&= (x_1^2 - x_2^2)(x_1^2 - x_3^2)\#(1) + (x_1^2 - x_3^2)(x_1^2 - x_2^2)\#(23) \\
f_3 &= x_2^2 f_2 - f_2 x_3^2 \\
&= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2)\#(1) \in (f) \cap C(A).
\end{aligned}$$

This provides only one of the elements we need. We must use noncommutativity to obtain the second element.

Example: $\Bbbk_{-1}[x_1, x_2, x_3]$ and $\mathcal{S}_3$

Recall

$$f_2 = (x_1^2 - x_2^2)(x_1^2 - x_3^2)\#(1) + (x_1^2 - x_3^2)(x_1^2 - x_2^2)\#(23).$$

Now

$$\begin{aligned} g_{23} &= (x_2 f_2 - f_2 x_3)(x_2 - x_3) \\ &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2 - x_3)^2\#(1) \\ &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 + x_3^2)\#(1) \in (f) \cap C(A). \end{aligned}$$

We can similarly construct g_{12} and g_{13}. Set $g = g_{12} + g_{13} + g_{23}$.

Example: $\Bbbk_{-1}[x_1, x_2, x_3]$ and $\mathcal{S}_3$

The elements

$$\begin{aligned} f_3 &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2) \\ g &= (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 + x_3^2) \\ &\qquad + (x_1^2 - x_2^2)(x_1^2 + x_3^2)(x_2^2 - x_3^2) \\ &\qquad + (x_1^2 + x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2) \end{aligned}$$

are relatively prime in $T = \Bbbk[x_1^2, x_2^2, x_3^2]$ and GKdim $T/(f_3, g) \leq 1$.

Theorem (G-Kirkman-Moore-Won)

Let G be any subgroup of $\mathcal{S}_n$ acting on $V_n = \Bbbk_{-1}[x_1, \dots x_n]$ as permutations. Then $\mathrm{p}(V_n, G) \geq 2$ so the Auslander map is an isomorphism.

All the Auslanders!

Theorem (G-Kirkman-Moore-Won)

The Auslander map is an isomorphism for the following:

- *subgroups of* $\mathcal{S}_n$ *acting on* $\Bbbk_{-1}[x_1, \ldots, x_n]$*,*
- *subgroups of* $\mathcal{S}_n$ *acting on the* (-1)*-quantum Weyl algebra,*
- *subgroups of* $\mathcal{S}_3$ *acting on the three-dimensional Sklyanin algebra* $S(1, 1, -1)$*,*
- *the cyclic group* $\langle(1\ 2\ 3)\rangle$ *acting on a generic three-dimensional Sklyanin algebra* $S(a, b, c)$*,*
- *subgroups of weighted permutations acting on the down-up algebra* $A(2, -1)$*,*
- $\langle -I_n, (1\ 3)(2\ 4)\rangle$ *acting on* $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$.

Graded isolated singularities

Definition (Ueyama 2013)

A^G is a graded isolated singularity if $\operatorname{gldim} \operatorname{tails} A^G < \infty$.

Theorem (Ueyama 2016)

If A^G is a graded isolated singularity, then

- *A^G is an AS-Gorenstein algebra of dimension $d \geq 2$,*
- *$A \in \mathrm{CM}^{\mathrm{gr}}(A^G)$ is a $(d-1)$-cluster tilting module, and*
- *$\mathrm{Ext}^1_{A^G}(A, M)$ and $\mathrm{Ext}^1_{A^G}(M, A)$ are f.d. for $M \in \mathrm{CM}^{\mathrm{gr}}(A^G)$.*

Theorem (Mori and Ueyama 2016)

If $\operatorname{GKdim} A \geq 2$, A^G is a graded isolated singularity if and only if $\dim_{\Bbbk} A\#G/(f_G) < \infty$ if and only if $\mathrm{p}(A, G) = n$.

Graded isolated singularities

Theorem (BHZ)

Let $A = \Bbbk_{-1}[x_1, \ldots, x_{2^n}]$ *and* $G = \langle (1\ 2\ \cdots\ 2^n) \rangle$. *Then* $\mathrm{p}(A, G) = 2^n$ *so* A^G *is a graded isolated singularity.*

Theorem (G-Kirkman-Moore-Won)

For the following, A^G *is a graded isolated singularity:*

- $\langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$ *acting on* $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$,
- $\langle (1\ 2)(3\ 4) \cdots (2n-1\ \ 2n) \rangle$ *acting on* $\Bbbk_{-1}[x_1, \ldots, x_{2n}]$,
- $\langle (1\ 2\ 3) \rangle$ *acting on a generic Sklyanin algebra* $S(a, b, c)$,
- $\langle -I_n, (1\ 3)(2\ 4) \rangle$ *acting on* $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$.

Whither the upper bounds?

Constructing elements of (f_G) gives lower bounds for $\mathrm{p}(A, G)$. Can we find upper bounds as well?

Theorem (G-Kirkman-Moore-Won)

If $G' \leq G$ then $\mathrm{p}(A, G) \leq \mathrm{p}(A, G')$.

This resolves a conjecture of Bao-He-Zhang for the group case.

Corollary (G-Kirkman-Moore-Won)

Let A be a noetherian connected graded algebra and suppose G contains a quasi-reflection g. If A and $A^{\langle g \rangle}$ have finite global dimension, then the Auslander map $\gamma_{A,G}$ is not an isomorphism.

Computing pertinency exactly

Lower bounds: constructing elements of (f_G)

Upper bounds: subgroup theorem

Subgroups of $\mathcal{S}_3$ acting on $\mathbb{k}_{-1}[x_1, x_2, x_3]$:

conjugacy class	p(A, G)
$\langle(12)\rangle$	2
$\langle(123)\rangle$	2 or 3
$\langle(12), (23)\rangle$	2

Computing pertinency exactly

Subgroups of $\mathcal{S}_4$ acting on $\Bbbk_{-1}[x_1, x_2, x_3, x_4]$:

conjugacy class	p(A, G)
$\langle(12)\rangle$	2
$\langle(12)(34)\rangle$	4
$\langle(123)\rangle$	2 or 3
$\langle(1234)\rangle$	4
$\langle(12),(34)\rangle$	2
$\langle(12)(34),(13)(24)\rangle$	4
$\langle(1234),(24)\rangle$	2
$\langle(123),(124)\rangle$	2 or 3
$\langle(123),(12)\rangle$	2
$\langle(1234),(12)\rangle$	2

Other questions

- What is the pertinency of $\langle(1\ 2\ 3)\rangle$ acting on $\mathbb{k}_{-1}[x_1, x_2, x_3]$ as permutations (computational evidence suggests it is 2)? In general, need more methods for constructing upper bounds.
- Direct connections between Tr and pertinency?
- Weighted permutations? Have some partial results in this direction.
- Hopf actions! A theorem of Chan, Kirkman, Walton, and Zhang (2016) shows that the Auslander map is an isomorphism for A an AS regular algebra of dimension 2 and H a semisimple Hopf algebra.

Thank You!