Taft Algebra Actions on Quantum Heisenberg Algebras

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The Big Idea

We classify actions of (generalized) Taft algebras on quantum Heisenberg algebras. This gives an alternative way to realize actions on graded down-up algebras.

Background

Heisenberg's *Uncertainty Principle* is an expression of the relationship between measuring position and momentum of a particle in motion [Hei27]. Quantum Heisenberg algebras are algebraic manifestations of this. Mathematically speaking, they are important examples of noncommutative algebras with "good" ring-theoretic properties, and they give an alternate presentation for the graded down-up algebras, which are important in representation theory and noncommutative projective algebraic geometry.

Throughout, we assume that $\Bbbk$ is an algebraically closed field of characteristic zero.

Definition

For $p, q \in \Bbbk^\times$, the (p, q)-Heisenberg algebra is defined as

$$H_{p,q} = \Bbbk\langle d, u, z : zd - pdz, zu - p^{-1}uz, du - qud - z\rangle.$$

In particular, a $\Bbbk$-basis of $H_{p,q}$ is $\{u^i d^j z^k : i, j, k \in \mathbb{N}\}$.

The algebra $H_{1,1}$ is the *classical* Heisenberg algebra. The quantum Heisenberg algebras have been studied in [KS93, Gad16].

Noncommutative algebras express few classical symmetries due to the notion of *quantum rigidity*. In this project, we are looking for quantum symmetries, and Taft Algebras are "quantum-thickenings" of cyclic automorphisms of an algebra.

Definition

Let $m, n \in \mathbb{N}$ such that $m > 1$ and $m \mid n$, and let $\lambda \in \Bbbk$ be a primitive mth root of unity. The *generalized Taft algebra* corresponding to this data is

$$T_n(\lambda, m) := \Bbbk\langle x, g : g^n - 1, x^m, xg - \lambda gx\rangle.$$

A generalized Taft algebra $T_n(\lambda, m)$ has Hopf algebra structure given by

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1 \qquad S(g) = g^{-1}$$
$$\Delta(x) = x \otimes g + 1 \otimes x \qquad \epsilon(x) = 0 \qquad S(x) = -xg^{-1}.$$

The Taft algebras are examples of noncommutative, non-cocommutative, non-semisimple Hopf algebras. Their actions have been studied on a variety of families of algebras [GWY19].

Results

We will consider actions on $H_{p,q}$ that respect the graded structure.

Definition

We say an action of $T_n(\lambda, m)$ on $H_{p,q}$ is *linear* if $h(u), h(d), h(z) \in \Bbbk u + \Bbbk d + \Bbbk z$ for all $h \in T_n(\lambda, m)$. This is equivalent to $h = g$ and $h = x$ satisfying this condition.

First we consider automorphisms of $H_{p,q}$. This shows that, generically, all such automorphisms are diagonal, as the next lemma shows.

Lemma

Assume $p, q \neq 1$. Let g be a linear automorphism of $H_{p,q}$. Then the following hold:

1. If $p \neq -1$ or $q \neq -1$, then there exist $\mu, \nu \in \Bbbk^\times$ such that
$$g(u) = \mu u, \quad g(d) = \nu d, \quad g(z) = \mu\nu z.$$
2. If $p = q = -1$, then then there exist $\mu, \nu \in \Bbbk^\times$ such that
$$g(u) = \mu u, \quad g(d) = \nu d, \quad g(z) = \mu\nu z, \quad \text{or}$$
$$g(u) = \nu d, \quad g(d) = \mu u, \quad g(z) = \mu\nu z$$

Suppose that $H_{p,q}$ is a $T = T_n(\lambda, m)$-module algebra for some choice of n and λ and that the action is linear. Let us consider the case that the action of g is diagonal. By the previous lemma, this is forced when $p, q \neq -1$. So, the action is given as below:

$$g(u) = \mu u \qquad g(d) = \nu d \qquad g(z) = \mu\nu z$$
$$x(u) = \eta_1 u + \gamma_1 d + \delta_1 z \qquad x(d) = \eta_2 u + \gamma_2 d + \delta_2 z \qquad x(z) = \eta_3 u + \gamma_3 d + \delta_3 z.$$

Theorem

Suppose that $H_{p,q}$ is an inner-faithful $T = T_n(\lambda, m)$-module algebra with action given by the above lemma. Then (at least) one of the following cases hold:

1. If $\gamma_1 \neq 0$, then $\mu = p^2 q$, $\nu = q^{-1}$, and $\lambda = p^2 q^2$.
2. If $\delta_1 \neq 0$, then $\mu = pq$, $\nu = q^{-1}$, and $\lambda = q$.
3. If $\eta_2 \neq 0$, then $\mu = q$, $\nu = p^{-2} q^{-1}$, and $\lambda = p^{-2} q^{-2}$.
4. If $\delta_2 \neq 0$, then $\mu = q$, $\nu = p^{-1} q^{-1}$, and $\lambda = q^{-1}$.

Moreover, the only cases that are compatible with each other are (1) with (3), (1) with (4), and (2) with (4).

Connection To Down-up Algebras

Definition

Let $\alpha, \beta \in \Bbbk$. The *(graded) down-up algebra* $A(\alpha, \beta)$ is defined as

$$\Bbbk\langle d, u : d^2 u - \alpha dud - \beta ud^2, du^2 - \alpha udu - \beta u^2 d\rangle.$$

The families $A(\alpha, \beta)$ and $H_{p,q}$ are related by the following isomorphism:

$$H_{p,q} \cong A(q + p^{-1}, -p^{-1} q)$$

Future Work

Regardless of p and q, there are diagonal actions of g, so the Theorem classifies certain Taft algebra actions in all cases. When $p = q = -1$, there are more actions to be explored. Currently, we are looking into these actions to see what progress can be made. When $p = q = -1$, actions may have the following additional form:

$$g(u) = \nu d \qquad g(d) = \mu u \qquad g(z) = \mu\nu z$$
$$x(u) = \eta_1 u + \gamma_1 d + \delta_1 z \qquad x(d) = \eta_2 u + \gamma_2 d + \delta_2 z \qquad x(z) = \eta_3 u + \gamma_3 d + \delta_3 z.$$

Another route we can also take would be to extend these ideas to other related families of Hopf Algebras, such as Drinfeld doubles of the Taft Algebras.

References

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