Isomorphism problems in noncommutative algebra

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Introduction

Obligatory first line: Let $\mathbb{k}$ be an algebraically closed field of characteristic zero.

Given a family of algebras (commutative or noncommutative), the *isomorphism problem* asks under what conditions two members are isomorphic.

A related (but generally harder) problem is to determine all such isomorphisms between two algebras or within a family of algebras.

Examples

Isomorphism problems have a long history in commutative and noncommutative algebra.

- The Zariski Cancellation Problem (ZCP): Does an isomorphism $B[x] \cong \Bbbk[x_1, \ldots, x_{n+1}]$ imply $B \cong \Bbbk[x_1, \ldots, x_n]$? ($n = 1$ Abhyankar-Eakin-Heinzer (1972), $n = 2$ Fujita (1979) and Miyanishi-Sugie (1980))
- Isomorphisms of certain generalized Weyl algebras (Bavula-Jordan 2000).
- $\mathcal{U}_q^{\pm}(\mathfrak{g}_1) \cong \mathcal{U}_q^{\pm}(\mathfrak{g}_2) \Leftrightarrow \mathfrak{g}_1 \cong \mathfrak{g}_2$ for $q \in \Bbbk^{\times}$ not a root of unity and $\mathfrak{g}_1, \mathfrak{g}_2$ simple Lie algebras (Yakimov 2014).
- Certain families of quantum Weyl algebras (G (2014), Goodearl-Hartwig (2015), Levitt-Yakimov (2016)).

The Zariski Cancellation Problem

One formulation of the Zariski Cancellation Problem (ZCP) asks whether an isomorphism of algebras $A[x] \cong B[x]$ implies $A \cong B$. If A is an algebra such that this holds for some algebra B, then A is said to be **cancellative**.

Theorem (Bell-Zhang '16)

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Let A be a finitely generated domain of Gelfand-Kirillov dimension two. If A is not commutative, then A is cancellative.

This result uses a combination of tools, including the *Makar-Limanov invariant* and the *discriminant*.

Isomorphisms of graded algebras

Theorem (Bell-Zhang '16)

Let A and B be two connected graded algebras finitely generated in degree one. If $A \cong B$ as ungraded algebras, then $A \cong B$ as graded algebras.

This result has far reaching effects for the study of isomorphism problems of quantum algebras.

Quantum Affine Space

Let $\mathcal{A}_n = \{\mathbf{p} \in \mathcal{M}_n(\mathbb{k}^\times) : p_{ii} = 1, p_{ij} = p_{ji}^{-1} \text{ for all } i \neq j\}$ (**multiplicatively antisymmetric**).

A matrix $\mathbf{p} \in \mathcal{A}_n$ is a permutation of $\mathbf{q} \in \mathcal{A}_n$ if there exists a permutation $\sigma \in \mathcal{S}_n$ such that $q_{ij} = p_{\sigma(i)\sigma(j)}$ for all i, j.

For $\mathbf{p} \in \mathcal{A}_n$, the **(multiparameter) quantum affine *n*-space** $\mathcal{O}_{\mathbf{p}}(\mathbb{k}^n)$ is defined as the algebra with basis $\{x_i\}$, $1 \leq i \leq n$, subject to the relations $x_i x_j = p_{ij} x_j x_i$ for all $1 \leq i, j \leq n$.

Theorem (Mori ($n = 3$) '06, Vitoria '11, G '17)

$\mathcal{O}_{\mathbf{p}}(\mathbb{k}^n) \cong \mathcal{O}_{\mathbf{q}}(\mathbb{k}^m)$ iff $n = m$ and $\mathbf{p}$ is a permutation of $\mathbf{q}$.

Quantum matrix algebras

Fix parameters $\lambda \in \mathbb{k}^\times$, $\lambda \neq \pm 1$, and $\mathbf{p} \in \mathcal{A}_n$. The **(multiparameter) quantum ($n \times n$) matrix algebra**, $\mathcal{O}_{\lambda,\mathbf{p}}(\mathcal{M}_n(\mathbb{k}))$, is the algebra with basis $\{X_{ij}\}$, $1 \leq i, j \leq n$, subject to the relations

$$X_{lm}X_{ij} = \begin{cases} p_{li}p_{jm}X_{ij}X_{lm} + (\lambda - 1)p_{li}X_{im}X_{lj} & l > i, m > j \\ \lambda p_{li}p_{jm}X_{ij}X_{lm} & l > i, m \leq j \\ p_{jm}X_{ij}X_{lm} & l = i, m > j. \end{cases}$$

Question

Under what conditions do we have an isomorphism

$$\mathcal{O}_{\lambda,\mathbf{p}}(\mathcal{M}_n(\mathbb{k})) \cong \mathcal{O}_{\mu,\mathbf{q}}(\mathcal{M}_m(\mathbb{k}))?$$

By the Bell-Zhang theorem, we may restrict to only discussing *graded* isomorphisms.

Quantum matrix algebras

When $\lambda = q^{-2}$ and $p_{ij} = q$ for all $i > j$ we obtain a single parameter quantum matrix algebra $\mathcal{O}_q(M_2(\Bbbk))$.

Theorem (G '14)

$\mathcal{O}_p(M_2(\Bbbk)) \cong \mathcal{O}_q(M_2(\Bbbk))$ *if and only if* $p = q^{\pm 1}$.

This result comes entirely from considering the defining relations on the two algebras.

Question

If G *is an algebraic group and* $\mathcal{O}_q(G)$ *its quantized coordinate ring, is it true that* $\mathcal{O}_p(G) \cong \mathcal{O}_q(G)$ *if and only if* $p = q^{\pm 1}$*? In particular, does this hold for* $G = \mathrm{GL}_n(\Bbbk)$ *or* $G = \mathrm{SL}_n(\Bbbk)$*?*

Quantum matrix algebras

Let $\{X_{ij}\}$ and $\{Y_{ij}\}$ be the standard generators for $\mathcal{O}_{\lambda,\mathbf{p}}(\mathcal{M}_n(\mathbb{k}))$ and $\mathcal{O}_{\mu,\mathbf{q}}(\mathcal{M}_n(\mathbb{k}))$, respectively.

Some isomorphisms $\Phi : \mathcal{O}_{\lambda,\mathbf{p}}(\mathcal{M}_n(\mathbb{k})) \to \mathcal{O}_{\mu,\mathbf{q}}(\mathcal{M}_n(\mathbb{k}))$.

- $\lambda = \mu$ and $p_{ij} = q_{ij}$, $\Phi(X_{ij}) = Y_{ij}$.
- $\lambda = \mu$ and $p_{ij} = \lambda^{-1} q_{ji}$ for $i > j$, $\Phi(X_{ij}) = Y_{ji}$.
- $\lambda = \mu^{-1}$ and $p_{ij} = q_{n+1-i,n+1-j}$, $\Phi(X_{ij}) = Y_{n+1-i,n+1-j}$.
- The composition of the previous two isomorphisms.

We will show that these conditions on the generators are the only ones under which there is an isomorphism.

Normal elements

Let A, A' be connected graded algebras generated in degree one.

We define ideals $I_0 \subset I_1 \subset \cdots \subset I_n$ of A where I_k/I_{k-1} is generated by all (homogeneous) degree one normal elements in A/I_{k-1}. We set $I_k = 0$ for $k < 0$.

If $\Phi : A \to A'$ is an isomorphism of connected graded algebras then $A/I_k \cong A'/\Phi(I_k)$ for all $k \geq 0$.

Quantum matrix algebras

We can visualize the generators of I_k/I_{k-1} in $\mathcal{O}_{\lambda,\mathbf{p}}(\mathcal{M}_n(\mathbb{k}))$.

X_{11}	X_{12}	X_{13}	$\cdots$	$X_{1(n-2)}$	$X_{1(n-1)}$	X_{1n}
X_{21}	X_{22}	X_{23}	$\cdots$	$X_{2(n-2)}$	$X_{2(n-1)}$	X_{2n}
X_{31}	X_{32}	X_{33}	$\cdots$	$X_{3(n-2)}$	$X_{3(n-1)}$	X_{3n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$X_{(n-2)1}$	$X_{(n-2)2}$	$X_{(n-2)3}$	$\cdots$	$X_{(n-2)(n-2)}$	$X_{(n-2)(n-1)}$	$X_{(n-2)n}$
$X_{(n-1)1}$	$X_{(n-1)2}$	$X_{(n-1)3}$	$\cdots$	$X_{(n-1)(n-2)}$	$X_{(n-1)(n-1)}$	$X_{(n-1)n}$
X_{n1}	X_{n2}	X_{n3}	$\cdots$	$X_{n(n-1)}$	$X_{n(n-1)}$	X_{nn}

$I_0 = \langle X_{n1}, X_{1n} \rangle$. $I_1/I_0 = \langle X_{(n-1)1}, X_{n2}, X_{1(n-1)}, X_{2n} \rangle$.
$I_2/I_1 = \langle X_{(n-2)1}, X_{(n-1)2}, X_{n3}, X_{1(n-2)}, X_{(n-1)2}, X_{3n} \rangle$.

Quantum matrix algebras

Theorem (G '17)

$\mathcal{O}_{\lambda,\mathbf{p}}(\mathcal{M}_n(\mathbb{k})) \cong \mathcal{O}_{\mu,\mathbf{q}}(\mathcal{M}_m(\mathbb{k}))$ *if and only if* $n = m$ *and one of the following cases holds:*

- $\lambda = \mu$ *and* $\mathbf{p} = \mathbf{q}$;
- $\lambda = \mu$ *and* $p_{ij} = \lambda^{-1} q_{ji}$ *for all* i, j;
- $\lambda = \mu^{-1}$ *and* $p_{ij} = q_{n+1-i,n+1-j}$ *for all* i, j;
- $\lambda = \mu^{-1}$ *and* $p_{ij} = \lambda^{-1} q_{n+1-j,n+1-i}$ *for all* i, j.

Question

What is the corresponding result for multiparameter quantum $m \times n$ *matrices?*

Quantized Weyl algebras

Fix $\mathbf{p} \in \mathcal{A}_n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{k}^\times)^n$. The **(multiparameter) quantized Weyl algebras** $A_n^{\mathbf{p},\gamma}(\mathbb{k})$ is the algebra with basis $\{x_i, y_i\}$, $1 \leq i \leq n$, subject to the relations

$$\begin{aligned}
y_i y_j &= p_{ij} y_j y_i && \text{(all } i,j) \\
x_i x_j &= \gamma_i p_{ij} x_j x_i && (i < j) \\
x_i y_j &= p_{ji} y_j x_i && (i < j) \\
x_i y_j &= \gamma_j p_{ji} y_j x_i && (i > j) \\
x_j y_j &= 1 + \gamma_j y_j x_j + \sum_{l<j} (\gamma_l - 1) y_l x_l && \text{(all } j).
\end{aligned}$$

Question

Under what conditions do we have an isomorphism

$$A_m^{\mathbf{q},\mu}(\mathbb{k}) \cong A_n^{\mathbf{p},\gamma}(\mathbb{k})?$$

Quantized Weyl algebras

If $n = 1$, then we write $A_1^q(\mathbb{k}) = \mathbb{k}\langle x, y : xy - qyx = 1\rangle$.

Theorem (G '14)

$A_1^p(\mathbb{k}) \cong A_1^q(\mathbb{k})$ *if and only if* $p = q^{\pm 1}$

This result adapted work by Alev and Dumas on the automorphism group of $A_1^q(\mathbb{k})$.

However, one can use the discriminant to simplify the proof in the case that p, q are roots of unity.

Quantized Weyl algebras

Theorem (Goodearl-Hartwig '15)

Assume no γ_i, μ_i is a root of unity.

$A_n^{\mathbf{p},\gamma}(\mathbb{k}) \cong A_m^{\mathbf{q},\mu}(\mathbb{k})$ *if and only if* $n = m$, *there exists* $\varepsilon \in \{\pm 1\}^n$ *such that*

$$\mu_i = \gamma_i^{\varepsilon_i},$$

and $\mathbf{p}, \mathbf{q}$ *satisfy*

$$q_{ij} = \begin{cases} p_{ij} & \textit{if } (\varepsilon_i, \varepsilon_j) = (1, 1), \\ p_{ji} & \textit{if } (\varepsilon_i, \varepsilon_j) = (-1, 1), \\ \gamma_i^{-1} p_{ji} & \textit{if } (\varepsilon_i, \varepsilon_j) = (1, -1), \\ \gamma_i p_{ij} & \textit{if } (\varepsilon_i, \varepsilon_j) = (-1, -1). \end{cases}$$

Quantized Weyl algebras

Theorem (Levitt-Yakimov '16)

Assume $A_n^{\mathbf{p},\gamma}(\Bbbk), A_m^{\mathbf{q},\mu}(\Bbbk)$ *are f.g. free over their centers.*

$A_n^{\mathbf{p},\gamma}(\Bbbk) \cong A_m^{\mathbf{q},\mu}(\Bbbk)$ *if and only if* $n = m$, *there exists* $\varepsilon \in \{\pm 1\}^n$ *such that*

$$\mu_i = \gamma_i^{\varepsilon_i},$$

and $\mathbf{p}, \mathbf{q}$ *satisfy*

$$q_{ij} = \begin{cases} p_{ij} & \text{if } (\varepsilon_i, \varepsilon_j) = (1,1), \\ p_{ji} & \text{if } (\varepsilon_i, \varepsilon_j) = (-1,1), \\ \gamma_i^{-1} p_{ji} & \text{if } (\varepsilon_i, \varepsilon_j) = (1,-1), \\ \gamma_i p_{ij} & \text{if } (\varepsilon_i, \varepsilon_j) = (-1,-1). \end{cases}$$

Homogenized quantized Weyl algebras

Fix $\mathbf{p} \in \mathcal{A}_n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\Bbbk^\times)^n$. The homogenized **(multiparameter) quantized Weyl algebras** $H_n^{\mathbf{p},\gamma}$ is the algebra with basis $\{x_i, y_i, z\}$, $1 \leq i \leq n$, z central, subject to the relations

$$
\begin{aligned}
y_i y_j &= p_{ij} y_j y_i && \text{(all } i, j) \\
x_i x_j &= \gamma_i p_{ij} x_j x_i && (i < j) \\
x_i y_j &= p_{ji} y_j x_i && (i < j) \\
x_i y_j &= \gamma_j p_{ji} y_j x_i && (i > j) \\
x_j y_j &= z^2 + \gamma_j y_j x_j + \sum_{l<j} (\gamma_l - 1) y_l x_l && \text{(all } j).
\end{aligned}
$$

Quantized Weyl algebras

In $H_n^{\mathbf{p},\gamma}$, $I_0 = \langle z \rangle$, and $I_k / I_{k-1} = \langle x_k, y_k \rangle$ for $0 < k \leq n$.

Theorem (G' 17)

$H_n^{\mathbf{p},\gamma} \cong H_m^{\mathbf{q},\mu}$ if and only if $n = m$, there exists $\varepsilon \in \{\pm 1\}^n$ such that

$$\mu_i = \gamma_i^{\varepsilon_i},$$

and $\mathbf{p}, \mathbf{q}$ satisfy

$$q_{ij} = \begin{cases} p_{ij} & \text{if } (\varepsilon_i, \varepsilon_j) = (1, 1), \\ p_{ji} & \text{if } (\varepsilon_i, \varepsilon_j) = (-1, 1), \\ \gamma_i^{-1} p_{ji} & \text{if } (\varepsilon_i, \varepsilon_j) = (1, -1), \\ \gamma_i p_{ij} & \text{if } (\varepsilon_i, \varepsilon_j) = (-1, -1). \end{cases}$$

Thank You!