The Center of the Quantum Matrix Algebras at Odd Roots of Unity

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The Big Idea

We determine a presentation of the center of the Quantum Matrix Algebras (QMAs) for q a nontrivial odd root of unity, and discuss its application in studying the automorphism group of $M_q(2)$. We also show that the automorphism group of $M_q(n)$, for all $n \geq 2$, contains a free group on two generators.

Background

The quantum matrix algebra $M_q(n)$ arises as the coordinate ring of the quantum analogue of $M_n(\mathbb{C})$. It is the $\mathbb{C}$-algebra generated by n^2 elements $Z_{i,j}, i, j = 1, 2 \dots, n$, subject to the relations

$$\begin{aligned} Z_{i,j}Z_{i,k} &= qZ_{i,k}Z_{i,j}, \quad \text{if } j < k \\ Z_{i,j}Z_{k,j} &= qZ_{k,j}Z_{i,j}, \quad \text{if } i < k \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j}, \quad \text{if } i < s, t < j \\ Z_{i,j}Z_{s,t} &= Z_{s,t}Z_{i,j} + (q - q^{-1})Z_{i,t}Z_{s,j}, \quad \text{if } i < s, j < t, \end{aligned}$$

where $q \in \mathbb{C}^*$ is the quantum parameter. When q is not a root of unity, it was shown in [NYM93] that the center of $M_q(n)$ is generated by an element called the *quantum determinant*:

$$\det{}_q = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} Z_{1,\sigma(1)} Z_{2,\sigma(2)} \cdots Z_{n,\sigma(n)},$$

where $\ell(\sigma)$ is the number of inversions in σ. On the other hand, the situation when q is a root of unity is significantly more complicated as shown by Jakobsen and Zhang in [JZ97]. In particular, suppose q is a primitive mth root of unity for an odd integer $m \geq 3$. Then the center $Z(M_q(n))$ is generated by

$$\begin{aligned} \{Z_{i,j}^m, \det{}_q, (\det{}_q(t))^r \tau((\det{}_q(n-t))^{m-r}) \mid \\ i, j, t = 1, \dots, n \text{ and } r = 0, \dots, m\}, \end{aligned}$$

where $\det_q(t)$ is the quantum determinant of the quantum matrix subalgebra generated by the $Z_{i,j}$ such that $1 \leq i \leq t, n-t+1 \leq j \leq n$, and where τ is the transpose automorphism $Z_{i,j} \mapsto Z_{j,i}$ [JZ97, Theorem 6.2].

The Ideal of Relations

Denote $X_{i,j} = Z_{i,j}^m, D = \det_q$, and $Y_{t,r} = (\det_q(t))^r \tau((\det_q(n-t))^{m-r})$.

Lemma

Let $1 \leq t, t' \leq n-1$. Then

1. $\det_q(t)\tau(\det_q(t')) = \tau(\det_q(t'))\det_q(t)$
2. $\det_q(t)\det_q(t') = \det_q(t')\det_q(t)$
3. $\tau(\det_q(t))\tau(\det_q(t')) = \tau(\det_q(t'))\tau(\det_q(t))$

In addition, [PJZ99] shows that $D^m = \det(Z_{i,j})$. Combining these two results, we obtain the following families of relations:

- $Y_{t,i}Y_{t,j} = Y_{t,k}Y_{t,\ell}$, for $i + j = k + \ell$ and $1 \leq t \leq n-1$
- $Y_{t,1}Y_{t,m-1} = \det(A_t)\det(B_{n-t})$ for $1 \leq t \leq n-1$
- $Y_{t,1}Y_{t,k-1} = \det(B_{n-t})Y_{t,k}$, for $2 \leq k \leq m-1$ and $1 \leq t \leq n-1$
- $Y_{t,m-1}Y_{t,\ell+1} = \det(A_t)Y_{t,\ell}$ for $1 \leq \ell \leq m-2$ and $1 \leq t \leq n-1$

where $A_t = (X_{i,j})_{1 \leq i \leq t, n-t+1 \leq j \leq n}$ and $B_t = (X_{i,j})_{n-t+1 \leq i \leq n, 1 \leq j \leq t}$. Let I be the ideal generated by these families of relations together with $D^m - \det(Z_{i,j})$.

Theorem

Let q be a primitive mth root of unity, $m \geq 3$ an odd integer. Then I is the ideal of relations for $Z = Z(M_q(n))$. In other words,

$$Z \cong \frac{\mathbb{C}[X_{i,j}, Y_{t,r}, D]}{I},$$

where $1 \leq i, j \leq n$, $1 \leq t \leq n-1$, and $1 \leq r \leq m-1$.

Proof (sketch). Denote $R = \mathbb{C}[X_{i,j}, Y_{t,r}, D]/I$.

Step 1: Show that the generators of I form a Gröbner basis with respect to the lex monomial order

$$X_{1,1} > \cdots > X_{1,n} > X_{2,1} > \cdots > X_{n,n} > D > Y_{1,1} > \cdots > Y_{1,m-1} > Y_{2,1} > \cdots > Y_{n-1,m-1}$$

using Bergman's diamond lemma [Ber78, Theorem 1.2].

Step 2: Show that $\dim(Z) = \dim(R)$, where dim denotes the Krull dimension. This is done by computing the Hilbert series of R and using $\dim(Z) = n^2$ [AST91, Theorem 1].

Step 3: Prove that each $Y_{t,1}$ is not a zero divisor in R by showing the ideal quotient $(I : (Y_{t,1}))$ is I using the algorithm in [GGVI10]. From the relations of the form $Y_{t,i}Y_{t,j} = Y_{t,k}Y_{t,\ell}$ for $i + j = k + \ell$, we obtain that all $Y_{t,r}$ are not zero divisors in R. Then from the relations of the form $Y_{t,m-1}Y_{t,\ell+1} = \det(A_t)Y_{t,\ell}$, we obtain that all $\det(A_t)$ are not zero divisors in the quotient.

Step 4: Let S be the multiplicative subset of R generated by the $\det(A_t)$. Compute the localization

$$R_S = \left(\frac{\mathbb{C}[X_{i,j}, Y_{t,m-1}, D]}{(D^m - \det(X_{i,j}), Y_{t,m-1}^m - \det(A_t)^{m-1}\det(B_{n-t}))} \right)_S.$$

Using the characterization of prime ideals of polynomial rings proven in [Fer97], prove that the latter ring is an integral domain. Thus, since each $\det(A_t)$ is not a zero divisor, it follows that R is an integral domain.

Step 5: Since both R and Z are integral domains of Krull dimension 4, and since there is an obvious surjection from R onto Z, conclude that $R \cong Z$. □

Discriminants and Automorphisms

Noncommutative discriminants are useful tools for solving invariant theoretic problems such as the automorphism problem for noncommutative algebras, the Zariski cancellation problem, and the Morita cancellation problem. In an attempt to understand the automorphism groups of the QMAs at odd roots of unity, we considered a slight variant of the discriminant introduced in [LWZ20]. Namely, while the terminology and definitions are essentially identical, we study localizations at central maximal ideals rather than quotients of central maximal ideals:

Definition

Let A be an algebra and $Z = Z(A)$ its center. Let P be a property defined for $\mathbb{C}$-algebras which is invariant under algebra isomorphism.

1. The *P-locus* of A is
$$L_P(A) = \{\mathfrak{m} \in \text{MaxSpec}(Z) \mid A_{\mathfrak{m}} \text{ has property } P\}.$$
2. The *P-discriminant set* of A is
$$D_P(A) = \text{MaxSpec}(Z) \setminus L_P(A).$$
3. The *P-discriminant ideal* of A is
$$I_P(A) = \bigcap_{\mathfrak{p} \in D_P(A)} \mathfrak{p} \subseteq Z.$$
4. If $I_P(A)$ is a principal ideal of Z generated by an element d, then d is called the *P-discriminant* of A and is denoted $d = d_P(A)$, or just $d(A)$ if the property P is clear. Note that $d(A)$ is unique up to a unit of Z.

Note that if σ is an automorphism of A, then it restricts to an automorphism of Z. Denote this restricted automorphism by σ as well and suppose it sends a maximal ideal $\mathfrak{m}_1$ to another maximal ideal $\mathfrak{m}_2$. Then there is an induced isomorphism between the localizations $Z_{\mathfrak{m}_1}$ and $Z_{\mathfrak{m}_2}$ defined by $\frac{r}{s} \mapsto \frac{\sigma(r)}{\sigma(s)}$. Therefore, for any property P, the P-discriminant ideal and the P-discriminant, if it exists, are preserved by σ.

Specializing, let $A = M_q(2)$ and let $Z = Z(A) = \mathbb{C}[X_{1,1}, X_{1,2}, X_{2,1}, X_{2,2}, Y_{1,1}, \dots, Y_{1,m-1}, D]/I$ where I is the ideal of relations as above. Note that Z is the coordinate ring of an irreducible affine algebraic variety X of dimension 4. Let $P \in X$ correspond to the maximal ideal $\mathfrak{p}$ of Z, and let J_P denote the Jacobian matrix of the generators of I evaluated at P. That is, if we denote $I = (f_1, \dots, f_N)$, then

$$J_P = \begin{pmatrix} \frac{\partial f_1}{\partial X_{1,1}}(P) & \cdots & \frac{\partial f_1}{\partial D}(P) \\ \cdots & \ddots & \cdots \\ \frac{\partial f_N}{\partial X_{1,1}}(P) & \cdots & \frac{\partial f_N}{\partial D}(P) \end{pmatrix}.$$

By a theorem in algebraic geometry known as the Jacobian criterion, see e.g. [GBL+07, Theorem 5.6.12, Corollary 5.6.14],

$$\text{rk } J_P = (m + 4) - \dim \mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m}$ is the maximal ideal of the localization $Z_{\mathfrak{p}}$. Note that as a corollary, $Z_{\mathfrak{p}}$ is regular local if and only if $\text{rk } J_P = m$.

With this theorem in hand, we now let P be the property of being local with $\dim \mathfrak{m}/\mathfrak{m}^2 = 1$ for $\mathfrak{m}$ the unique maximal ideal. It is easy to see P is indeed invariant under algebra isomorphisms. Upon computing the rank of the Jacobian at each point P in the variety X, we obtain the following result:

Theorem

Let σ be an automorphism of $M_q(2)$. Then σ fixes the ideal $(X_{1,2}, X_{2,1}, Y_{1,1}, Y_{1,2}, \dots, Y_{1,m-1})$ of A. Consequently, since $(Z_{1,2}, Z_{2,1})$ is a completely prime ideal and $((Z_{1,2}, Z_{2,1}))^m = (X_{1,2}, X_{2,1}, Y_{1,1}, Y_{1,2}, \dots, Y_{1,m-1})$, σ also fixes the ideal $(Z_{1,2}, Z_{2,1})$.

We were unable to compute a discriminant for $n = 2$, or indeed for any n, such that the discriminant ideal was principal. The following result shows why this might be expected:

Theorem

Let D_1 (resp. D_2) be the quantum determinant of the quantum matrix subalgebra generated by the $Z_{i,j}$ with $1 \leq i, j \leq n-1$ (resp. $2 \leq i, j \leq n-1$). Then the following maps are automorphisms of $M_q(n)$:

$$\phi : Z_{i,j} \mapsto \begin{cases} Z_{i,j}, & (i,j) \neq (n,n) \\ Z_{n,n} + D_1^{m-1}, & (i,j) = (n,n) \end{cases}$$

$$\psi : Z_{i,j} \mapsto \begin{cases} Z_{i,j}, & (i,j) \neq (1,1) \\ Z_{1,1} + D_2^{m-1}, & (i,j) = (1,1) \end{cases}.$$

Moreover, the free group $\langle \phi \rangle * \langle \psi \rangle$ is a proper subgroup of $\text{Aut}(M_q(n))$.

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References

[AST91] Michael Artin, William Schelter, and John Tate. Quantum deformations of $\mathrm{GL_n}$. *Communications on Pure and Applied Mathematics*, 44(8-9):879–895, 1991.

[Ber78] George M Bergman. The diamond lemma for ring theory. *Advances in Mathematics*, 29(2):178–218, 1978.

[Fer97] Miguel Ferrero. Prime ideals in polynomial rings in several indeterminates. *Proceedings of the American Mathematical Society*, 125(1):67–74, 1997.

[GBL+07] G.M. Greuel, O. Bachmann, C. Lossen, G. Pfister, and H. Schönemann. *A Singular Introduction to Commutative Algebra.* Springer Berlin Heidelberg, 2007.

[GGVI10] Shuhong Gao, Yinhua Guan, and Frank Volny IV. A new incremental algorithm for computing Gröbner bases. In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, pages 13–19, 2010.

[JZ97] Hans Plesner Jakobsen and Hechun Zhang. The center of the quantized matrix algebra. *Journal of Algebra*, 196(2):458–474, 1997.

[LWZ20] D-M Lu, Q-S Wu, and JJ Zhang. A Morita cancellation problem. *Canadian Journal of Mathematics*, 72(3):708–731, 2020.

[NYM93] Masatoshi Noumi, Hirofumi Yamada, and Katsuhisa Mimachi. Finite dimensional representations of the quantum group $\mathrm{GL}_q(n;c)$ and the zonal spherical functions on $\mathrm{U}_q(n-1)\backslash\mathrm{U}_q(n)$. *Japanese Journal of Mathematics. New series*, 19(1):31–80, 1993.

[PJZ99] Hans Plesner Jakobsen and Hechun Zhang. Cyclic representations of the quantum matrix algebras. *Communications in Algebra*, 27(2):493–510, 1999.