

QUIVERS AND CALABI-YAU ALGEBRAS

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ABSTRACT. These lectures are meant as a gentle introduction to the study of (twisted) Calabi-Yau algebras, which are related to Calabi-Yau manifolds, or the shapes that appear in the extra dimensions allowed by string theory. This theory is closely related to the theory of quivers and (super)potentials. Thus, the first lecture will focus primarily on necessary background material on quivers and quivers with relations. The second lecture will introduce examples of Calabi-Yau algebras. The third lecture will be dedicated to work of mine with Daniel Rogalski related to the classification of quivers supporting Calabi-Yau algebras under certain conditions.

0. INTRO

Ginzburg [10] motivates the study of Calabi-Yau geometry in the following way. Let (X, X') be a pair of mirror dual Calabi-Yau varieties. Mirror symmetry predicts a natural correspondence between smooth Calabi-Yau deformations of X and smooth crepant resolutions of X' .

My real interest here is in the study of (twisted) Calabi-Yau algebras, which act as noncommutative homogeneous rings of Calabi-Yau varieties. These are in a very precise way a generalization of the types of algebras arising in noncommutative (projective) algebraic geometry (see Section 2). Eventually I will state the definition of these algebras but, as it is somewhat technical, I'll begin by setting the stage with some preliminary work.

The study of Calabi-Yau algebras is tied heavily with the theory of quivers and (super)potentials. Thus, the first lecture will focus primarily on necessary background material on quivers and quivers with relations. The second lecture will introduce examples of Calabi-Yau algebras. The third lecture will be dedicated to work of mine with Daniel Rogalski related to the classification of quivers supporting Calabi-Yau algebras under certain conditions.

1. QUIVERS AND PATH ALGEBRAS

Much of the background in this section comes from Michael Wemyss' MSRI notes [15].

A **quiver** is a directed graph consisting of a (finite) number of vertices and arrows. Here's a simple one:

$$1 \bullet \xrightarrow{a} 2 \bullet \xrightarrow{b} 3 \bullet \tag{1}$$

It can also contain loops and be disconnected, like the example below.

$$\begin{array}{ccc}
 1 \bullet & \xrightarrow{a} & \bullet 2 \\
 & & \bullet 3 \xrightarrow{b} \bullet 3
 \end{array} \tag{2}$$

But I am already being a little dishonest about one thing. At each vertex i there should be a **trivial loop**, sometimes denoted e_i , so the second example really looks like

$$\begin{array}{ccc}
 1 \bullet & \xrightarrow{a} & \bullet 2 \\
 \curvearrowleft & & \curvearrowleft \\
 e_1 & & e_2
 \end{array}
 \quad
 \begin{array}{c}
 \bullet 3 \xrightarrow{b} \bullet 3 \\
 \curvearrowleft \\
 e_3
 \end{array}$$

By convention, we do not draw the trivial loops and they may simply be identified with the vertices themselves. We'll see soon that they play an important role in constructing an algebraic object associated to the quiver.

Formally, a quiver Q is an ordered pair (Q_0, Q_1) where Q_0 is the set of vertices and Q_1 is the set of arrows. As stated previously, we will restrict to the case that both $|Q_0|, |Q_1| < \infty$. For each arrow $a \in Q_1$, let $s(a) \in Q_0$ denote the **source** and $t(a) \in Q_0$ the **target** of a (that is, the starting and ending vertex, respectively)¹. In the quiver 1, $s(a) = 1$ and $t(a) = 2$.

A **path** p in a quiver $Q = (Q_0, Q_1)$ is a sequence (a_1, a_2, \dots, a_n) in Q_1 such that $t(a_i) = s(a_{i+1})$ for $i = 1, \dots, n-1$. We will typically write $p = a_1 a_2 \dots a_n$ ². and we extend the notions of source and target above to paths, so $s(p) = s(a_1)$ and $t(p) = t(a_n)$. In Example 1, there is a path ab .

To a quiver Q we can attach an algebraic object, called a **path algebra**. (Recall that an algebra is a ring that is also a vector space over a field.) The elements of the path algebra are formal (finite) sums of paths with coefficients in a field \mathbb{k} . We define multiplication of paths by concatenation (if it makes sense). That is, given paths p, q ,

$$p \cdot q = \begin{cases} pq & \text{if } s(q) = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if Q is 1, then in the corresponding path algebra $a \cdot b = ab$ and $b \cdot a = 0$.

The trivial paths act like the identity at each vertex, so

$$e_i \cdot p = \begin{cases} p & \text{if } s(p) = i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad p \cdot e_j = \begin{cases} p & \text{if } t(p) = j \\ 0 & \text{otherwise.} \end{cases}$$

¹**Convention warning:** As far as I can tell, there is no standard notation for these terms. Other sources (including [15]) call the source the *tail* and the target the *head*.

²**Convention warning:** Some think that quivers should be treated like functions and therefore a path should read right-to-left. Wemyss calls these people 'savage barbarians'.

Thus, the trivial loops form a set of orthogonal idempotents. We extend the multiplication operation linearly. The identity element is the sum of the trivial loops, so $1 = \sum_{i \in Q_0} e_i$. We denote the path algebra of a quiver Q over a field \mathbb{k} by $\mathbb{k}Q$.

Example 1.1. Consider the quiver Q with one vertex and no loops (except the trivial loop). It is not difficult to see that $\mathbb{k}Q \cong \mathbb{k}$.

Example 1.2. Consider the quiver Q with one vertex and one loop (and the trivial loop). A basis for $\mathbb{k}Q$ is $\{e, a\}$, where e denotes the trivial loop at the vertex and a the (non)trivial loop. Then $\mathbb{k}Q \cong \mathbb{k}[x]$, the algebra of polynomials in one variable over \mathbb{k} .

Example 1.3. Consider the quiver Q below.

$$1 \bullet \xrightarrow{a} \bullet_2 \quad (3)$$

A basis for $\mathbb{k}Q$ is $\{e_1, e_2, a\}$. Hence, an arbitrary element of $\mathbb{k}Q$ has the form $c_1 e_1 + c_2 e_2 + c_3 a$. One can show with little effort that $\mathbb{k}Q \cong U_2(\mathbb{k})$, the algebra of 2×2 upper-triangular matrices with entries in \mathbb{k} . The map is given by $e_1 \mapsto E_{11}$, $a \mapsto E_{12}$, and $e_2 \mapsto E_{22}$ extended linearly.

To any quiver Q we can attach *relations* R . By this we mean that R is an ideal in $\mathbb{k}Q$ and we study the factor ring $\mathbb{k}Q/R$.

Example 1.4. Consider the quiver Q with one vertex and two loops, x and y . Then $\mathbb{k}Q \cong \mathbb{k}\langle x, y \rangle$, the free algebra on two generators. Let $R = (xy - yx)$. Then $\mathbb{k}Q/R \cong \mathbb{k}[x, y]$, the polynomial ring in two variables.

Example 1.5. Consider the quiver Q below.

$$1 \bullet \begin{array}{c} \xleftarrow{a} \\ \xrightarrow{b} \end{array} \bullet_2 \quad (4)$$

Let R be the ideal of $\mathbb{k}Q$ generated by $\{ab - e_2, ba - e_1\}$. Then $\mathbb{k}Q/R \cong \mathcal{M}_2(\mathbb{k})$, the ring of 2×2 matrices over R . The map is given by $e_1 \mapsto E_{11}$, $b \mapsto E_{12}$, $e_2 \mapsto E_{22}$, and $a \mapsto E_{21}$ extended linearly.

There is at least one highly nontrivial result that we can state at this point, though we won't prove it. It concerns basic finite-dimensional \mathbb{k} -algebras.

By **basic** we mean $A/\text{rad}(A)$ is isomorphic to n copies of the field \mathbb{k} . Let Q_+ denote the **arrow ideal**, that is, the ideal of $\mathbb{k}Q$ generated by the elements of Q_1 . An ideal I of $\mathbb{k}Q$ is **admissible** if $Q_+^m \subset I \subset Q_+^2$ for some $m \geq 2$.

Theorem 1.6. Let A be a basic finite-dimensional \mathbb{k} -algebra and $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of A . The quiver associated to A , denoted Q_A , has vertices labeled by the idempotents e_1, \dots, e_n and $\dim_{\mathbb{k}} e_i(\text{rad } A / \text{rad}^2 A)e_j$ arrows between the vertices labeled by e_i and e_j . Moreover, $A \cong \mathbb{k}Q_A/I$ for some (admissible) ideal I .

2. THE CALABI-YAU CONDITION AND EXAMPLES

Definition 2.1. Let A be an algebra (over \mathbb{k}).

- The enveloping algebra of A is defined as $A^e = A \otimes_{\mathbb{k}} A^{\text{op}}$.
- A is said to be homologically smooth if it has a finite length resolution by finitely generated projectives in the category of $A^e - \text{Mod}$.
- A is said to be Calabi-Yau of dimension d if it is homologically smooth and there are isomorphisms

$$\text{Ext}_{A^e}^i \cong \begin{cases} 0 & i \neq d \\ A & i = d, \end{cases}$$

as A -bimodules.

We'll now look at some examples of Calabi-Yau algebras.

Artin-Schelter regular algebras. We say an algebra A is \mathbb{N} -graded if it has a vector space decomposition $A = \bigoplus_{n \geq 0} A_n$ such that $A_i \cdot A_j \subset A_{i+j}$. Furthermore, an \mathbb{N} -graded algebra A is **connected** if $A_0 = \mathbb{k}$.

Definition 2.2. Let \mathbb{k} be an algebraically closed, characteristic zero field. A connected \mathbb{N} -graded algebra A is said to be **Artin-Schelter regular** if $\text{GKdim } A < \infty$, $\text{gldim } A = d < \infty$, and $\text{Ext}_A^i(\mathbb{k}, A) \cong \delta_{id} \mathbb{k}(\ell)$ for some $\ell \in \mathbb{Z}$. Here \mathbb{k} is the trivial module $A/A_{\geq 1}$.

An Artin-Schelter regular ring is commutative if and only if it is a polynomial ring. In (global) dimension 2, the Artin-Schelter regular rings that are generated in degree one are the quantum planes

$$\mathcal{O}_q(\mathbb{k}^2) = \mathbb{k}\langle x, y : xy - qyx \rangle, \quad q \in \mathbb{k}^\times,$$

and the Jordan plane,

$$\mathcal{J} = \mathbb{k}\langle x, y : xy - yx - y^2 \rangle.$$

The classification in dimension 3 is known due to the work of Artin and Schelter [2], and that of Artin, Tate, and Van den Bergh [1]. This includes more exotic algebras, like the **Sklyanin algebras**. These are generated by three elements x_0, x_1, x_2 with parameters $a, b, c \in \mathbb{k}$ subject to the three relations

$$ax_i x_{i+1} + bx_{i+1} x_i + cx_{i+2}^2 \quad \text{indices taken mod 3.}$$

The classification in dimension 4 is still open.

The next theorem illustrates the close connection between Calabi-Yau algebras and Artin-Schelter regular algebras. However, the statement of this theorem requires a generalization of the Calabi-Yau condition, which we will discuss in Section 3.

Theorem 2.3 ([11]). A connected \mathbb{N} -graded algebra is twisted Calabi-Yau if and only if it is Artin-Schelter regular.

Preprojective algebras. We begin with some background courtesy of Thibault's PhD thesis [13]. The Auslander-Reiten translate of a left Λ -module M is given by $\tau(M) = D \operatorname{Ext}_{\Lambda}^1(M, \Lambda)$. Its inverse is defined as $\tau^{-1}(M) = \operatorname{Ext}_{\Lambda}^1(D\Lambda, M)$. A Λ module M is **preprojective** if $M = \tau^{-m}(P)$ for some $m \geq 0$ and indecomposable projective module P . Classically, the **preprojective algebra** of a finite-dimensional hereditary algebra Λ , denoted $\Pi(\Lambda)$, is the direct sum over representatives of isomorphism classes of preprojective modules.

Definition 2.4. Let Q be a finite quiver. We define a new quiver \overline{Q} , the double quiver of Q , by attaching for each arrow $a \in Q_1$ a new arrow $a^* : t(a) \rightarrow s(a)$. The **preprojective algebra** of Q is defined to be

$$\Pi_Q := \mathbb{k}\overline{Q} / \left(\sum_{a \in Q_1} [a, a^*] \right).$$

The relations defining Π_Q are called **mesh relations**.

Example 2.5. Let Q be the quiver with one vertex and one loop a . Then \overline{Q} is the quiver with one vertex and two loops a, a^* . The preprojective algebra Π_Q is then $\mathbb{k}\overline{Q}/(aa^* - a^*a) \cong \mathbb{k}[x, y]$.

Example 2.6. Let Q be the quiver (1). The preprojective algebra Π_Q is $\mathbb{k}\overline{Q}/(aa^* - a^*a) \cong \mathbb{k}[x, y]/(x^2, y^2)$.

Example 2.7. Let Q be the quiver in 4. Set $b^* = c$ and $a^* = d$. Then the preprojective algebra Π_Q is $\mathbb{k}\overline{Q}/(ad - da + bc - cb)$

Theorem 2.8 (Crawley-Boevey, Holland [7]). If A is the preprojective algebra of a non-Dynkin quiver, then A is Calabi-Yau of dimension 2.

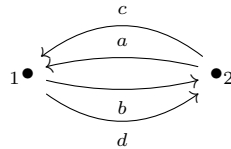
In some sense, the converse of Theorem 2.8 also holds.

Skew group algebras. Let G be a group acting on an algebra A . The skew group ring $A \# G$ is defined with basis that of $A \otimes \mathbb{k}G$ and multiplication given by

$$(a \otimes g)(b \otimes h) = ag(b) \otimes gh.$$

Example 2.9. Let $A = \mathbb{k}[x, y]$ and $G = \langle g \rangle \cong C_2$ acting on A by $g(x) = -x$ and $g(y) = -y$.

Let Q be the quiver below.



If $I = (ad - cb, da - bc) \subset \mathbb{k}Q$, then $\mathbb{k}Q/I \cong A \# G$. The isomorphism is given by $\psi : \mathbb{k}Q/I \rightarrow A \# G$ with

$$\begin{aligned} \psi(e_u) &= \frac{1}{2}(1\#e + 1\#g), & \psi(a) &= x\#e + x\#g, & \psi(c) &= y\#e + y\#g, \\ \psi(e_v) &= \frac{1}{2}(1\#e - 1\#g), & \psi(b) &= x\#e - x\#g, & \psi(d) &= y\#e - y\#g. \end{aligned}$$

Here is something interesting to notice that is totally not a coincidence. The relations defining I may be obtained by taking the mesh relations from Example 2.7 and splitting them by vertex. That is, $da - bc$ is based at vertex 1 while $ad - bc$ is based at vertex 2.

The quiver Q that shows up in this example is special.

Definition 2.10. Let G be a finite group, V be a representation of G , and V_0, \dots, V_n be (distinct isomorphism classes of) the irreducible representations of G . The **McKay quiver** of G with respect to V is the quiver with n vertices and $\dim_{\mathbb{k}} \text{Hom}_{\mathbb{k}G}(V_i, V_j \otimes V)$ arrows from vertex i to vertex j .

Note that $\dim_{\mathbb{k}} \text{Hom}_{\mathbb{k}G}(V_i, V_j \otimes V)$ is just the number of times that V_i appears as a summand in the decomposition of $V_j \otimes V$ into a sum of irreducibles. This is completely combinatorial and can be computed entirely in terms of characters.

Example 2.11. Let A and G be as in Example 2.9. As G is abelian, it has two indecomposable representations, V_0 and V_1 , with V_0 denoting the trivial representation. The character table is

	e	g
χ_0	1	1
χ_1	1	-1

As a G -module, $A \cong 2V_1 =: V$. Let χ be the representation corresponding to V . Then

$$\chi_0 \cdot \chi = 2(\chi_0 \cdot \chi_1) = 2\chi_1.$$

Thus, $\dim_{\mathbb{k}} \text{Hom}(V_0, V_0 \otimes V) = 0$ while $\dim_{\mathbb{k}} \text{Hom}(V_1, V_0 \otimes V) = 2$. Similarly,

$$\chi_1 \cdot \chi = 2(\chi_1 \cdot \chi_1) = 2\chi_0.$$

Thus, $\dim_{\mathbb{k}} \text{Hom}(V_0, V_1 \otimes V) = 2$ while $\dim_{\mathbb{k}} \text{Hom}(V_1, V_1 \otimes V) = 0$. It follows that the McKay quiver corresponding to (A, G) is the quiver from Example 2.9.

Theorem 2.12 (Craw, MacLagan, Thomas [6]). Let V be a finite-dimensional vector space over \mathbb{C} and $G \subset \text{GL}(V) \cong \text{GL}_n(\mathbb{C})$ finite abelian. Then $\mathbb{C}[V] \# G$ is isomorphic to a quotient of $\mathbb{k}Q$ where Q is the McKay quiver corresponding to (V, G) .

Theorem 2.13 (Bocklandt, Schedler, Wemyss [5]). Let V be a finite-dimensional vector space over \mathbb{C} and $G \subset \text{SL}(V) \cong \text{SL}_n(\mathbb{C})$ finite. Then $\mathbb{C}[V] \# G$ is Calabi-Yau of dimension n .

See the appendix for additional examples of skew group algebras as path algebras of quivers.

3. VACUALGEBRAS AND GRADED CALABI-YAU ALGEBRAS OF GLOBAL DIMENSION 3

Vacualgebras. Some of the algebras I will be most interested in arise as path algebras with relations in a very special way.

Definition 3.1. A superpotential Ω on a quiver Q is a (finite) sum (with coefficients in \mathbb{k}) of cyclic paths that is closed under cyclic permutation of the vertices. Given any arrow $a \in Q_1$, one may define a derivation operator δ_a on $\mathbb{k}Q$ where, for any path p , $\delta_a(p) = a^{-1}p$ if $p = aq$ and it is zero otherwise. This map is then extended linearly. A vacualgebra (or Jacobi algebra) is a path algebra with relations on a quiver Q of the form $\mathbb{k}Q/(\delta_a\Omega : a \in Q_1)$, where Ω is a superpotential on Q .

Remark 3.2. Because superpotentials are closed under cyclic permutation, it is often convenient to write them *up to permutation*. That is, we write it as an element of $\mathbb{k}Q/[\mathbb{k}Q, \mathbb{k}Q]$. For example, the element $ab + ba$ is a superpotential on the quiver in (4), but we will typically write just ab .

Example 3.3. Consider the quiver Q with one vertex and three loops: x, y, z . Consider the superpotential $\Omega = xyz - yxz$. Then the derivation operators on Ω give

$$\delta_x(\Omega) = yz - zy, \quad \delta_y(\Omega) = zx - xz, \quad \delta_z(\Omega) = xy - yx.$$

Thus, the corresponding vacualgebra $\mathbb{k}Q/(\delta_a\Omega : a \in Q_1)$ is isomorphic to the polynomial ring $\mathbb{k}[x, y, z]$.

See the appendix for a more elaborate example, one connecting vacualgebras to skew group algebras. Our primary reason for considering vacualgebras is due to the following claim of Ginzburg [10]:

Any Calabi-Yau algebra of dimension 3 ‘arising in nature’ is defined by a (super)potential.

Theorem 3.4 (Bocklandt [4]). An \mathbb{N} -graded Calabi-Yau algebra of global dimension 3 is isomorphic to a vacualgebra.

For higher-dimension Calabi-Yau algebras, one must take *higher order* derivations. On the other hand, for dimension 2 Calabi-Yau algebras, the rule is that one takes *no* derivations, but instead splits the relations according to vertices. For a proof, see [4].

Classification of quivers supporting graded Calabi-Yau algebras. A reasonable question at this point is what quivers can appear in the classification above. Unfortunately, this is simply too big of a question to answer given current technology, so we restrict it in a reasonable way. In the theory of Artin-Schelter regular algebras (i.e., connected graded Calabi-Yau algebras) one often restricts to the case of finite GK dimension. Thus, we add the same hypothesis and achieve a full classification for quivers with two or three vertices.

Theorem 3.5 (G, Rogalski [9]). Let $A \cong \mathbb{k}Q/(\partial_a\omega : a \in Q_1, |Q_0| = 2)$ be graded (twisted) Calabi-Yau of global dimension 3 and finite GK dimension. Then M_Q is one of the following:

$$\deg(\omega) = 3 : \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \quad \deg(\omega) = 4 : \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Moreover, each quiver above supports such an algebra.

All of the quivers appearing in the above theorem are McKay quivers.

Theorem 3.6 (G, Rogalski [9]). Let $A \cong \mathbb{k}Q/(\partial_a \omega : a \in Q_1, |Q_0| = 3)$ be graded (twisted) Calabi-Yau of global and GK dimension 3 with $\deg(\omega) = 3$ or 4. Then M_Q is one the following. The indeterminates a, b, c are positive integers satisfying the equation $a^2 + b^2 + c^2 = abc$.

$$\begin{aligned} \deg(\omega) = 3 : & \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \\ \deg(\omega) = 4 : & \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Moreover, each quiver above supports such an algebra.

Here is a brief overview of the proof. Let A be a graded (twisted) Calabi-Yau algebra of global dimension 3 with a degree 3 superpotential³. Denote by M the adjacency matrix of the associated quiver. The matrix polynomial of M is⁴ $p(t) = I - Mt + M^T t^2 - It^3$. A root of $p(t)$ is some $\alpha \in \mathbb{k}$ such that $p(\alpha)$ is singular. Equivalently, α is a root of the polynomial $\det p(t)$. The matrix-valued Hilbert series associated to A is $h_A(t) = 1/p(t)$.

The first step in our proof is to bound the number of loops at each vertex. This is accomplished by using a version of Golod-Schafarevich bound. We prove that the number of loops at each vertex cannot exceed 3⁵. Also, extending a result in the connected graded case, the GK dimension of A is the multiplicity of 1 as a root of $p(t)$ (some obfuscation here).

Suppose Q has three vertices⁶. If A has finite GK dimension, then every root of $p(t)$ is a root of unity [12]. Thus, $\det p(t) = (1 - t)^3 r(t)$ where $r(t)$ has degree 6 and is the product of cyclotomic polynomials. It follows from the definition that $\det p(t)$ is anti-palindromic. This, $r(t)$ is palindromic and so its roots appear in inverse pairs. It follows that

$$r(t) = \prod_{i=1}^3 (1 - k_i t + t^2), k_i \in \mathbb{R}, |k_i| \leq 2.$$

Write $M_Q = [m_{ij}]_{i,j=0\dots 3}$ and set

$$\lambda = \sum_{i=1}^3 m_{ii}, \quad \beta = \sum_{1 \leq i < j \leq 3} m_{ii} m_{jj}, \quad \gamma = \sum_{1 \leq i < j \leq 3} m_{ij} m_{ji}.$$

Note that both λ and β are determined by the number of loops at the vertices (so λ is bounded by 9). Thus, once we fix a particular number of loops at each vertex, only γ is unknown.

³We assume in our paper that the superpotential is degree 3 or degree 4 and conjecture that these are the only two possibilities.

⁴I am actually making an additional assumption that the *Nakayama automorphism* acts trivially on the vertices. This is automatic if A is Calabi-Yau but not necessarily in the twisted case.

⁵This number is 2 when the degree of the superpotential is 4.

⁶This is the hard case. The easy case is when $|Q_0| = 2$.

The coefficient of t^8 in $\det p(t)$ is λ and the coefficient of t^7 is $\gamma - \beta - \lambda$. Computing the matrix polynomial of an arbitrary 3×3 matrix and comparing to $\det p(t)$ gives the following equations

$$\sum_{i=1}^3 k_i = \lambda - 3, \quad \gamma = \beta - 2\lambda + 3 - \sum_{i \neq j} k_i k_j.$$

We can now use Mathematica to determine the maximum value of gamma corresponding to each set of diagonal entries (the largest we get is 5). This allows us to construct a finite number of families of matrices that *could* support a Calabi-Yau algebra. We then use various criteria to rule out most of the families and check that the remaining quivers do support a Calabi-Yau algebra.

Quiver mutation. Most of the quivers appearing in the above theorem are McKay quivers. There are a few exceptions. The quiver $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ is a polynomial extension of a Calabi-Yau algebra of global dimension 2. The quiver $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ is a *normal extension* of a Calabi-Yau algebra of global dimension 2.

The most interesting case are the family of quivers $\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}$ satisfying $a^2 + b^2 + c^2 = abc$.

Let Q be a quiver without loops or oriented 2-cycles and fix a vertex v . For an arrow $a \in Q_1$ denote by $s(a)$ the source and by $t(a)$ the target of a .

Step 1 For every pair of arrows $a, b \in Q_1$ with $t(a) = v$ and $s(b) = v$, create a new arrow $[ba] : s(a) \rightarrow t(b)$.

Step 2 Reverse each arrow with source or target at v .

Step 3 Remove any maximal disjoint collection of oriented 2-cycles.

The resulting quiver \tilde{Q} , is called a *mutation* of Q .

The quiver with adjacency matrix $\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix}$ supports a Calabi-Yau skew group ring of global dimension 3. The remaining members of the family are mutations of this quiver.

Let A' be an algebra obtained from A by successive mutations. By Vitória, A and A' are derived equivalent [14, Corollary 3.8]. A recent result of Zeng proves that the Calabi-Yau property is preserved under derived equivalence [16, Theorem 1.1].

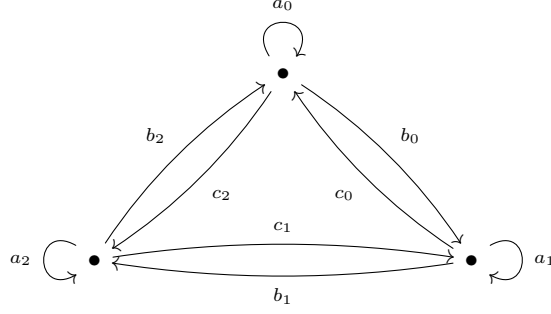
Isomorphisms of graded path algebras. We say a nonzero element $r \in \mathbb{k}Q$ is *homogeneous* if all summands of r have the same (path) length, source, and target. An ideal $I \subset \mathbb{k}Q$ is *homogeneous* if it is generated by homogeneous elements. Let $A = \mathbb{k}Q/I$ and $B = \mathbb{k}Q'/J'$ with I, J homogeneous ideals. Set $(A_n)_{uv}$ to be the paths (modulo relations in I) from vertex u to vertex v . Thus, we record $\dim A_n$ as a $(|Q_0| \times |Q_0|)$ -matrix where the values of $(\dim A_n)_{uv}$ are $\dim((A_n)_{uv})$ (and similarly for B). We say A and B are *isomorphic as graded path algebras* if $M_Q = PM'_Q$ for some permutation matrix P corresponding to $\sigma \in \mathcal{S}_{|Q_0|}$ and there exists an algebra isomorphism $\phi : A \rightarrow B$ such that $\phi((A_n)_{uv}) = (B_n)_{\sigma(u)\sigma(v)}$.

The next theorem generalizes a result of Bell and Zhang when $|Q_0| = 1$ [3, Theorem 0.1].

Theorem 3.7 (G [8]). If two graded path algebras $A = \mathbb{k}Q/I$ and $B = \mathbb{k}Q/J$ are isomorphic as (ungraded) algebras, then they are isomorphic as graded path algebras.

APPENDIX A. ADDITIONAL EXAMPLES OF SKEW GROUP ALGEBRAS

Example A.1. Consider the quiver Q below



Define a superpotential $\Omega = a_0 b_0 c_0 + a_0 c_2 b_2 + a_1 c_0 b_0 + a_1 b_1 c_1 + a_2 b_2 c_2 + a_2 c_1 b_1 \in \mathbb{k}Q$. The relations obtained by taking cyclic partial derivatives with respect to the arrows of Q are

$$\begin{aligned} 0 &= b_0 c_0 + c_2 b_2 = c_0 b_0 + b_1 c_1 = c_1 b_1 + b_2 c_2 \\ &= c_0 a_0 + a_1 c_0 = c_1 a_1 + a_2 c_1 = c_2 a_1 + a_0 c_2 \\ &= a_0 b_0 + b_0 a_1 = a_1 b_1 + b_1 a_2 = a_2 b_2 + b_2 a_0 \end{aligned}$$

This algebra may not actually be Calabi-Yau, but it is twisted Calabi-Yau of global and GK dimension 3 (see Section 3).

Now consider $R = \mathbb{k}_{-1}[x, y, z]$, the (-1) -skew polynomial ring in three variables. Let ω denote a primitive third root of unity. Define an action of $C_3 = \langle \rho \rangle$ on R by

$$\rho(x) = x, \quad \rho(y) = \omega y, \quad \rho(z) = \omega^2 y.$$

The skew group ring $R \# C_3$ is isomorphic to the vacualgebra $A = \mathbb{k}Q / (\partial_a \Omega : a \in Q_1)$ by the map Φ given below

$$\begin{aligned} e_0 &= \Phi \left(\frac{1}{3} (1 \# e + 1 \# \rho + 1 \# \rho^2) \right) & a_0 &= \Phi \left(\frac{1}{3} (x \# e + x \# \rho + x \# \rho^2) \right) \\ e_1 &= \Phi \left(\frac{1}{3} (1 \# e + \omega \# \rho + \omega^2 \# \rho^2) \right) & a_1 &= \Phi \left(\frac{1}{3} (x \# e + \omega x \# \rho + \omega^2 x \# \rho^2) \right) \\ e_2 &= \Phi \left(\frac{1}{3} (1 \# e + \omega^2 \# \rho + \omega \# \rho^2) \right) & a_2 &= \Phi \left(\frac{1}{3} (x \# e + \omega^2 x \# \rho + \omega x \# \rho^2) \right) \\ b_0 &= \Phi \left(\frac{1}{3} (y \# e + \omega y \# \rho + \omega^2 y \# \rho^2) \right) & c_0 &= \Phi \left(\frac{1}{3} (z \# e + z \# \rho + z \# \rho^2) \right) \\ b_1 &= \Phi \left(\frac{1}{3} (y \# e + \omega^2 y \# \rho + \omega y \# \rho^2) \right) & c_1 &= \Phi \left(\frac{1}{3} (z \# e + \omega z \# \rho + \omega^2 z \# \rho^2) \right) \\ b_2 &= \Phi \left(\frac{1}{3} (y \# e + y \# \rho + y \# \rho^2) \right) & c_2 &= \Phi \left(\frac{1}{3} (z \# e + \omega^2 z \# \rho + \omega z \# \rho^2) \right) \end{aligned}$$

To verify that Φ is actually an automorphism, one checks that the images vanish on the relations of A and then use GK dimension.

Example A.2. Let Q be the quiver in Example A.1. Suppose we take $R = \mathbb{k}_{-1}[x, y, z]$ as before and $G = C_3 = \langle \rho \rangle$ but with the action

$$\rho(x) = y, \quad \rho(y) = z, \quad \rho(z) = x.$$

Define a new superpotential on Q by

$$\Omega_1 = a_0 b_0 c_0 + a_0 c_2 b_2 + a_1 b_1 c_1 + a_1 c_0 b_0 + a_2 b_2 c_2 + a_2 c_1 b_1 - \frac{2}{3}(a_0^3 + a_1^3 + a_2^3) - 2(b_0 b_1 b_2 + c_2 c_1 c_0)$$

The relations obtained by taking cyclic partial derivatives with respect to the arrows of Q are

$$\begin{aligned} 0 &= b_0 c_0 + c_2 b_2 - 2a_0^2 = c_0 a_0 + a_1 c_0 - 2b_1 b_2 = a_0 b_0 + b_0 a_1 - 2c_2 c_1 \\ &= b_1 c_1 + c_0 b_0 - 2a_1^2 = c_1 a_1 + a_2 c_1 - 2b_2 b_0 = a_1 b_1 + b_1 a_2 - 2c_0 c_1 \\ &= b_2 c_2 + c_1 b_1 - 2a_2^2 = c_2 a_2 + a_0 c_2 - 2b_1 b_2 = a_2 b_2 + b_2 a_0 - 2c_1 c_0. \end{aligned}$$

The algebra is twisted Calabi-Yau of global and GK dimension 3. One can again make the isomorphism explicit, simply use the isomorphism in Example A.1 but replace x, y, z in Φ with $r = x + y + z$, $s = x + \omega^2 y + \omega z$, and $t = x + \omega y + \omega^2 z$.

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