



## 1 DEFINITIONS

1. The pmf for a Bernoulli r.v.  $X_i$  is given by

$$p_{X_i}(k) = \begin{cases} p_0 & k = 1 \\ 1 - p_0 & k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.1)$$

2. The pmf for a Binomial Random variable

$$X = \sum_{i=1}^n X_i \quad (1.2)$$

is given by

$$p_X(X = k) = \binom{n}{k} p_0^k (1 - p_0)^{n-k}, \quad 0 \leq k \leq n \quad (1.3)$$

where  $X_i$  are i.i.d.

3. Let  $Z \sim \mathcal{N}(0, 1)$ . Then the  $Q$  function is defined as,

$$Q(z) = \Pr(Z > z), \quad z \geq 0 \quad (1.4)$$

4. The MGF of  $Y$  is defined as

$$M_Y(s) = E(e^{-sY}) \quad (1.5)$$

5. The MGF of  $Z$  is

$$M_Z(s) = \frac{1}{2} e^{\frac{1}{2}s^2} \quad (1.6)$$

- 6.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad (1.7)$$

## 2 PROBLEMS

1. Show that the mean and variance of  $X_i$  are  $\mu = p_0$  and  $\sigma^2 = p(1 - p)$ .
2. Show that the mean and variance of

$$Y_i = \frac{X_i - \mu}{\sigma} \quad (2.1)$$

are 0 and 1 respectively.

3. Show that the mean and variance of

$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^n Y_i \quad (2.2)$$

are 0 and 1 respectively.

4. Show that

$$M_Y(s) = \left[ E \left( e^{-\frac{sY_i}{\sqrt{n}}} \right) \right]^n \quad (2.3)$$

5. Show that

$$E \left( e^{-\frac{sY_i}{\sqrt{n}}} \right) = 1 + \frac{1}{2n} + \frac{1}{n} R(s, n) \quad (2.4)$$

where  $R(s, n)$  is an infinite series.

6. Show that

$$\lim_{n \rightarrow \infty} Y = Z \quad (2.5)$$

This is known as the *Central Limit Theorem*.

7. Let

$$p = \frac{1}{n} \sum_{i=1}^n X_i \quad (2.6)$$

Show that

$$E(p) = \mu = p_0 \quad (2.7)$$

$$\text{var}(p) = \frac{\sigma^2}{n} = \frac{p_0(1-p_0)}{n} \quad (2.8)$$

8. Show that

$$\lim_{n \rightarrow \infty} \frac{p - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = Z \quad (2.9)$$

9. Find  $z$  if

$$\Pr(|Z| < z) = 1 - \alpha \quad (2.10)$$

10. Show that

$$p_0 - \sqrt{\frac{p_0(1-p_0)}{n}} Q^{-1} \left( \frac{\alpha}{2} \right) < p < p_0 + \sqrt{\frac{p_0(1-p_0)}{n}} Q^{-1} \left( \frac{\alpha}{2} \right) \quad (2.11)$$

11. Among 4000 newborns, 2080 are male. Find the  $1 - \alpha = 0.99$  confidence interval of the probability that a male child is born.