



## 1 AXIOMS OF PROBABILITY

### 1.1 Definitions

1. For any event  $A$ ,  $0 \leq \Pr(A) \leq 1$ .
2.  $A \cup B \triangleq A + B$ .
3.  $A \cap B \triangleq AB$ .
4. The null and complete event are  $\phi = 0, S = 1$ .
5. If  $AB = 0$ ,  $\Pr(A + B) = \Pr(A) + \Pr(B)$ .
6.  $(A + B)' = A'B'$

### 1.2 Problems

Prove the following:

1.

$$A = AB + AB' \quad (1.2.1.1)$$

2.

$$\Pr(A) = \Pr(AB) + \Pr(AB') \quad (1.2.2.1)$$

3.

$$A + B = B + AB' \quad (1.2.3.1)$$

4.

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (1.2.4.1)$$

## 2 DISTRIBUTION OF THE SUM OF RANDOM VARIABLES

### 2.1 Definitions

1. The mean of  $X$  is defined as

$$E(X) = \sum_k k p_X(k) \quad (2.1.1.1)$$

2. The  $Z$  transform of  $X$  is defined as

$$M_X(z) = E(z^{-X}) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \quad (2.1.2.1)$$

3. There is a one to one relationship between the pmf and its Z transform.
4. If  $X_1$  and  $X_2$  are independent,

$$E[f(X_1)g(X_2)] = E[f(X_1)]E[g(X_2)] \quad (2.1.4.1)$$

5. For a Bernoulli random variable  $X$ , the pmf is

$$p_X(n) = \begin{cases} p & k = 1 \\ 1 - p & k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.1.5.1)$$

6.  $X_i$  are said to be i.i.d (independent and identically distributed) if they are independent and have the same pmf.

## 2.2 Problem

1. Find the Z-transform for  $X$ , given that  $X$  is a Bernoulli random variable with parameter  $p$ .
2. If  $X_1$  and  $X_2$  are independent, and

$$Y = X_1 + X_2, \quad (2.2.2.1)$$

show that

$$M_Y(z) = M_{X_1}(z)M_{X_2}(z) \quad (2.2.2.2)$$

3. Find the Z-transform of  $Y$ , given that  $X_i$  are i.i.d Bernoulli random variables with parameter  $p$ .
4. Find the pmf of  $Y$ .
5. Find the pmf of

$$Y = \sum_{i=1}^N X_i, \quad (2.2.5.1)$$

where  $X_i$  are i.i.d.

## 3 MOMENTS AND VARIANCE

### 3.1 Definitions

1. The variance of  $X$  is defined as:

$$\text{Var}(X) = E(X - E(X))^2 \quad (3.1.1.1)$$

2. The Z transform of  $X$  is defined as

$$M_X(z) = E(z^{-X}) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \quad (3.1.2.1)$$

3. Let  $X$  be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \leq k \leq 6 \\ 0 & \text{otherwise} \end{cases} \quad (3.1.3.1)$$

$X$  is said to be Discrete Uniform Random Variable

4. The  $n^{th}$  moment of  $X$  is defined as:

$$E(X^n) = \sum_{k=-\infty}^{\infty} k^n p_X(k) \quad (3.1.4.1)$$

### 3.2 Problems

1. Show that  $Var(X) = E(X^2) - [E(X)]^2$
2. Find  $M_X(z)$
3. Show that  $E(X) = \frac{d}{dz} M_X(z^{-1}) \big|_{z=1}$
4. Find  $E(X^2)$
5. Find  $Var(X)$ .

## 4 CONVOLUTION

### 4.1 Definitions

1. The Z transform of  $X$  is defined as

$$M_X(z) = E(z^{-X}) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \quad (4.1.1.1)$$

2. Let  $X$  be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \leq k \leq 6 \\ 0 & \text{otherwise} \end{cases} \quad (4.1.2.1)$$

$X$  is said to be Discrete Uniform Random Variable

3. Convolution of two sequences using Toeplitz matrices

$$\mathbf{y} = \mathbf{x} \otimes \mathbf{h} \quad (4.1.3.1)$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & \cdot & \cdot & \cdot & 0 \\ h_2 & h_1 & \cdot & \cdot & \cdot & 0 \\ h_3 & h_2 & h_3 & \cdot & \cdot & 0 \\ h_{m-1} & \cdot & \cdot & \cdot & h_2 & h_1 \\ h_m & h_{m-1} & \cdot & \cdot & \cdot & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \quad (4.1.3.2)$$

#### 4.2 Problems

1. If  $\mathbf{x} = \mathbf{h} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , find  $\mathbf{y}$ .
2. Find  $p_{X_1}(k) \otimes p_{X_2}(k)$  using toeplitz matrices.
3. Find  $M_Y(z)$ , such that  $Y = X_1 + X_2$
4. Find  $p_Y(k)$

### 5 Z-TRANSFORM APPLICATIONS

#### 5.1 Definitions

1.

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (5.1.1.1)$$

2. The Z transform of  $X$  is defined as

$$M_X(z) = E(z^{-X}) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \quad (5.1.2.1)$$

#### 5.2 Problems

1. If

$$p_Y(n) \xleftrightarrow{Z} M_Y(z), \quad (5.2.1.1)$$

show that

$$p_Y(n-k) \xleftrightarrow{Z} M_Y(z) z^{-k}, \quad (5.2.1.2)$$

2. Show that

$$u(n) \xleftrightarrow{z} \frac{1}{(1 - z^{-1})}, \quad |z| > 1 \quad (5.2.2.1)$$

3. Show that

$$nu(n) \xleftrightarrow{z} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \quad (5.2.3.1)$$

4. Let

$$M_Y(z) = \left\{ \frac{z^{-1}(1 - z^{-6})}{6(1 - z^{-1})} \right\}^2, \quad |z| > 1 \quad (5.2.4.1)$$

Show that

$$p_Y(n) = \frac{(n-1)u(n-1) - 2(n-7)u(n-7) + (n-13)u(n-13)}{36} \quad (5.2.4.2)$$

## 6 MARKOV CHAIN

### 6.1 Definitions

- Fig. 6.1.1.1 shows a Markovs chain with 5 states. Transition from one state to another happens over time.  $s_0$  and  $s_4$  are absorbing states.

$$p + q = 1 \quad (6.1.1.1)$$

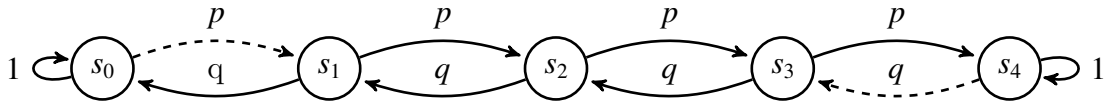


Fig. 6.1.1.1

- At time instant  $n$ ,

$$\mathbf{p}^{(n)} = \begin{pmatrix} P_0^{(n)} \\ P_1^{(n)} \\ P_2^{(n)} \\ P_3^{(n)} \\ P_4^{(n)} \end{pmatrix} \quad (6.1.2.1)$$

where  $P_i^{(n)}$  are defined to be the *stationary* probabilities.

- $P_{ij}$  is defined as the *transition* probability of going to state  $i$  from state  $j$ .

4. For a matrix  $\mathbf{A}$ , let

$$\mathbf{Ax} = \lambda \mathbf{x}. \quad (6.1.4.1)$$

Then  $\lambda$  is a scalar defined to be the eigenvalue of  $\mathbf{A}$  and  $\mathbf{x}$  is the corresponding eigenvector.

## 6.2 Problems

1. Let

$$P_0^{(n+1)} = P_0^{(n)} \quad (6.2.1.1)$$

$$P_1^{(n+1)} = pP_0^{(n)} + qP_2^{(n)} \quad (6.2.1.2)$$

$$P_2^{(n+1)} = pP_1^{(n)} + qP_3^{(n)} \quad (6.2.1.3)$$

$$P_3^{(n+1)} = pP_2^{(n)} + qP_4^{(n)} \quad (6.2.1.4)$$

$$P_4^{(n+1)} = P_4^{(n)} \quad (6.2.1.5)$$

Find the matrix  $\mathbf{P}$  such that  $\mathbf{p}^{(n+1)} = \mathbf{P}\mathbf{p}^{(n)}$

2. Show that 1 is an eigen value of  $\mathbf{P}$ .  
3. Show that

$$P_2^{(n+1)} = pP_1^{(n)} + qP_3^{(n)} \quad (6.2.3.1)$$

4. If  $P_0 = 1, P_N = 0$  and

$$P_i = pP_{i-1} + qP_{i+1} \quad (6.2.4.1)$$

show that

$$P_i = \frac{\left(\frac{p}{q}\right)^i - \left(\frac{p}{q}\right)^N}{1 - \left(\frac{p}{q}\right)^N}, 0 \leq i \leq N \quad (6.2.4.2)$$

## 7 GAUSSIAN DISTRIBUTION

### 7.1 Definitions

1. The CDF of  $X$  is defined as,

$$F_X(x) = \Pr(X \leq x) \quad (7.1.1.1)$$

2. The PDF of  $X$  is defined as,

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (7.1.2.1)$$

3. Let  $X \sim \mathcal{N}(0, 1)$ . Then the  $Q$  function is defined as,

$$Q(x) = \Pr(X > x), \quad x \geq 0 \quad (7.1.3.1)$$

## 7.2 Problems

1. Find

$$\Pr(|X - \mu| \leq k\sigma) \quad (7.2.1.1)$$

in terms of  $Q$  function.

2. Find

$$\Pr(X \leq x, |X - \mu| \leq k\sigma) \quad (7.2.2.1)$$

in terms of  $F_X(x)$

3. Find

$$F_X(x|X - \mu| \leq k\sigma) \quad (7.2.3.1)$$

4. Find

$$p_X(x|X - \mu| \leq k\sigma) \quad (7.2.4.1)$$

## 8 BIVARIATE GAUSSIAN

### 8.1 Definitions

1. Let

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{\mu} = E(\mathbf{X}) \quad (8.1.1.1)$$

$$\boldsymbol{\Sigma}_{\mathbf{x}} = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top] \quad (8.1.1.2)$$

Then  $\boldsymbol{\Sigma}_{\mathbf{x}}$  is defined to be the *covariance* matrix of  $\mathbf{x}$ .

2. For  $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{\mathbf{x}})$ ,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi \sqrt{|\boldsymbol{\Sigma}_{\mathbf{x}}|}} \exp -\frac{1}{2} \left( (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}_{\mathbf{x}}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right) \quad (8.1.2.1)$$

3. The *correlation coefficient* is defined as

$$\rho = \frac{E[(x_1 - \mu_1)(x_2 - \mu_2)]}{\sigma_1 \sigma_2} \quad (8.1.3.1)$$

where  $\mu_i, \sigma_i^2$  are the mean and variance of  $x_i$ .

## 8.2 Problems

1. Show that

$$E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top] = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \quad (8.2.1.1)$$

2. Prove that  $\boldsymbol{\Sigma}_{\mathbf{x}}$  is a diagonal matrix when  $x_1$  and  $x_2$  are independent.
3. Let

$$z_1 = x_1 + x_2 \quad (8.2.3.1)$$

$$z_2 = x_1 - x_2 \quad (8.2.3.2)$$

Find  $\mathbf{P}$  such that

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathbf{P}\mathbf{x} \quad (8.2.3.3)$$

4. Show that

$$\boldsymbol{\Sigma}_{\mathbf{z}} = \mathbf{P}\boldsymbol{\Sigma}_{\mathbf{x}}\mathbf{P}^\top \quad (8.2.4.1)$$

5. Check the independence of  $z_1$  and  $z_2$  given that  $\sigma_1 = \sigma_2$ .
6. Show that columns of  $\mathbf{P}$  are eigenvectors of  $\boldsymbol{\Sigma}_{\mathbf{z}}$ .
7. Show that the eigenvectors of  $\boldsymbol{\Sigma}_{\mathbf{z}}$  are orthogonal to each other.
8. Summarize your conclusion in one line.

## 9 TRANSFORMATION OF RANDOM VARIABLES

### 9.1 Definitions

1. The pdf of  $X \sim \mathcal{N}(\mu, \sigma^2)$  is defined as

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad -\infty < x < \infty \quad (9.1.1.1)$$

2. The Jacobian matrix transforming  $R, \Theta$  to  $X_1, X_2$  is defined as

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \quad (9.1.2.1)$$

- 3.

$$p_{R,\Theta}(r, \theta) = p_{X_1, X_2}(x_1, x_2) |\mathbf{J}| \quad (9.1.3.1)$$

4. The marginal distribution

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r, \theta) d\theta \quad (9.1.4.1)$$



5. The Laplace transform of  $p_Y(y)$  is given by

$$M_Y(s) = E(e^{-sY}) \quad (9.1.5.1)$$

6. The unit step function is defined as

$$u(y) = \begin{cases} 1 & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (9.1.6.1)$$

## 9.2 Problems

1. Let  $X_1, X_2 \in N \sim (0, 1)$  be i.i.d. Find  $p_{X_1, X_2}(x_1, x_2)$ .
2. Let

$$X_1 = R \cos \Theta \quad (9.2.2.1)$$

$$X_2 = R \sin \Theta \quad (9.2.2.2)$$

Find  $p_{R, \theta}(r, \theta)$ .

3. Find  $p_R(r)$ .
4. Find  $p_\Theta(\theta)$ .
5. Find the distribution of

$$Y = X_1^2 + X_2^2 \quad (9.2.5.1)$$

6. Find the Laplace transform of  $e^{-y}u(y)$
7. Find the Laplace transform of  $p_{X_1^2}(x_1)$ .
8. Find the Laplace transform of  $p_Y(y)$  using (9.2.5.1).
9. Find the distribution of  $Y$ .

## 10 ORDER STATISTICS

### 10.1 Definitions

1. The pdf of an *exponential* distribution is given by

$$p_X(x) = e^{-x}u(x) \quad (10.1.1.1)$$

where  $u(\cdot)$  is the unit step function.

### 10.2 Problems

1. Find  $F_X(x)$ .
2. Let  $X$  and  $Y$  be iid exponential. Find  $F_{XY}(z, z)$

3. Show that

$$\Pr(X \leq z, X > Y) = \int_{x=-\infty}^z \int_{y=-\infty}^x p_{X,Y}(x, y) dx dy \quad (10.2.3.1)$$

$$= \frac{e^{-2z}}{2} - e^{-z} \quad (10.2.3.2)$$

4. Find

$$\Pr(Y \leq z, X \leq Y) \quad (10.2.4.1)$$

5. Let

$$Z = \max(X, Y) \quad (10.2.5.1)$$

Show that

$$F_Z(z) = \Pr(X \leq z, X > Y) + \Pr(Y \leq z, X \leq Y) \quad (10.2.5.2)$$

6. Find the pdf of  $Z$ .
7. Find the pdf of  $W = \min(X, Y)$ .
8. Find  $F_W(vZ)$ , where  $v$  is a constant.
9. Find  $E[F_W(vZ)]$ .
10. Find the pdf of  $V = \frac{W}{Z}$ .

## 11 SOME DISTRIBUTIONS

### 11.1 Definitions

1. If  $Y = f(Z)$  be monotonic,

$$p_Y(y) dy = p_Z(z) dz \quad (11.1.1.1)$$

2. The pdf of the  $\chi^2(k)$  distribution is given by

$$p_X(x) = \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} u(x) \quad (11.1.2.1)$$

3. The Beta function is defined as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (11.1.3.1)$$

### 11.2 Problems

1. Let  $X_1 \sim \chi^2(m)$  and  $X_2 \sim \chi^2(n)$  be independent. For

$$Z = \frac{X_1/m}{X_2/n}, \quad (11.2.1.1)$$

show that

$$F_Z(z) = E \left[ F_{X_1} \left( \frac{mzX_2}{n} \right) \right] \quad (11.2.1.2)$$

2. Show that

$$p_Z(z) = E \left[ \frac{mX_2}{n} p_{X_1} \left( \frac{mzX_2}{n} \right) \right] \quad (11.2.2.1)$$

3. Show that

$$p_Z(z) = \frac{\left(\frac{m}{n}\right)^{m/2}}{B\left(\frac{m}{2}, \frac{n}{2}\right)} x^{\frac{m}{2}-1} \left(1 + \frac{m}{n}z\right)^{-\frac{m+n}{2}} u(z) \quad (11.2.3.1)$$

$Z$  has an  $F$  distribution with  $(m, n)$  degrees of freedom.

4. Show that  $Y = \frac{1}{Z}$  is monotonic.

5. Show that  $Y$  also has an  $F$  distribution with  $(n, m)$  degrees of freedom.

6. Find the pdf of  $\frac{mZ}{mZ+n}$ .

## 12 INEQUALITIES

### 12.1 Definitions

1. The mean of  $Y$  is defined as

$$E(Y) = \sum_k k p_Y(k) \quad (12.1.1.1)$$

2. The MGF of  $X$  is defined as

$$M_X(s) = E(e^{-sX}) \quad (12.1.2.1)$$

3. Let  $X \sim \mathcal{N}(0, 1)$ . Then the  $Q$  function is defined as,

$$Q(x) = \Pr(X > x), \quad x \geq 0 \quad (12.1.3.1)$$

### 12.2 Problems

1. Show that

$$E(Y) \geq \sum_{k=m}^{\infty} k p_Y(k), \quad m > 0 \quad (12.2.1.1)$$

2. Show that

$$\Pr(Y > m) \leq \frac{E(Y)}{m}, \quad m > 0 \quad (12.2.2.1)$$

3. Using (12.2.2.1), show that Show that

$$\Pr\left([Y - E(Y)]^2 > b^2\right) \leq \frac{\text{var}(Y)}{b^2}, \quad b > 0 \quad (12.2.3.1)$$

4. Show that

$$\Pr(|Y - E(Y)| > b) \leq \frac{\text{var}(Y)}{b^2}, \quad b > 0 \quad (12.2.4.1)$$

5. Show that

$$\Pr(X > a) = \Pr\left(e^{-sX} > e^{-sa}\right), \quad s < 0 \quad (12.2.5.1)$$

6. Using (12.2.2.1), show that

$$\Pr(X > a) \leq e^{as} M_X(s), \quad s < 0 \quad (12.2.6.1)$$

7. Show that the MGF of  $X$  is

$$M_X(s) = e^{\frac{1}{2}s^2} \quad (12.2.7.1)$$

8. Using (12.2.6.1) show that

$$Q(x) \leq e^{-\frac{x^2}{2}} \quad (12.2.8.1)$$