# 1 Axioms of Probability

## 1.1 Definitions

1. For any event $A, 0 \leq \Pr(A) \leq 1$.
2. $A \cup B \triangleq A + B$.
3. $A \cap B \triangleq AB$.
4. The null and complete event are $\phi = 0, S = 1$.
5. If $AB = 0, \Pr(A + B) = \Pr(A) + \Pr(B)$.
6. $(A + B)' = A'B'$

## 1.2 Problems

Prove the following:

1. $$A = AB + AB' \tag{1.2.1.1}$$
2. $$\Pr(A) = \Pr(AB) + \Pr(AB') \tag{1.2.2.1}$$
3. $$A + B = B + AB' \tag{1.2.3.1}$$
4. $$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.2.4.1}$$

# 2 Distribution of the sum of random variables

## 2.1 Definitions

1. The mean of $X$ is defined as
$$E(X) = \sum_k k p_X(k) \tag{2.1.1.1}$$
2. The $Z$ transform of $X$ is defined as
$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{2.1.2.1}$$

3. There is a one to one relationship betwwen the pmf and its Z transform.
4. If If $X_1$ and $X_2$ are independent,

$$E\left[f(X_1)\,g(X_2)\right] = E\left[f(X_1)\right]E\left[g(X_2)\right] \tag{2.1.4.1}$$

5. For a Bernoulli random variable $X$, the pmf is

$$p_X(n) = \begin{cases} p & k = 1 \\ 1 - p & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.5.1}$$

6. $X_i$ are said to be i.i.d (independent and identically distributed) if they are independent and have the same pmf.

### *2.2 Problem*

1. Find the Z-transform for $X$, given that $X$ is a Bernoulli random variable with parameter $p$.
2. If $X_1$ and $X_2$ are independent, and

$$Y = X_1 + X_2, \tag{2.2.2.1}$$

show that

$$M_Y(z) = M_{X_1}(z)M_{X_2}(z) \tag{2.2.2.2}$$

3. Find the Z-transform of $Y$, given that $X_i$ are i.i.d Bernoulli random variables with parameter $p$.
4. Find the pmf of $Y$.
5. Find the pmf of

$$Y = \sum_{i=1}^{N} X_i, \tag{2.2.5.1}$$

where $X_i$ are i.i.d.

## 3 Moments and variance

### *3.1 Definitions*

1. The variance of $X$ is defined as:

$$Var(X) = E(X - E(X))^2 \tag{3.1.1.1}$$

2. The $Z$ transform of $X$ is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{3.1.2.1}$$

3. Let $X$ be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \le k \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.3.1}$$

   $X$ is said to be Discrete Uniform Random Variable
4. The $n^{th}$ moment of $X$ is defined as:

$$E(X^n) = \sum_{k=-\infty}^{\infty} k^n p_X(k) \tag{3.1.4.1}$$

### *3.2 Problems*

1. Show that $Var(X) = E(X^2) - [E(X)]^2$
2. Find $M_X(z)$
3. Show that $E(X) = \frac{d}{dz} M_X(z^{-1}) \mid_{z=1}$
4. Find $E(X^2)$
5. Find $Var(X)$.

## 4 Convolution

### *4.1 Definitions*

1. The $Z$ transform of $X$ is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{4.1.5.1}$$

2. Let $X$ be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \le k \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{4.1.5.2}$$

   $X$ is said to be Discrete Uniform Random Variable

3. Convolution of two sequences using Toeplitz matrices

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{4.1.5.3}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_3 & . & . & 0 \\ h_{m-1} & . & . & . & h_2 & h_1 \\ h_m & h_{m-1} & . & . & . & h_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix} \tag{4.1.5.4}$$

### *4.2 Problems*

1. If $\mathbf{x} = \mathbf{h} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $\mathbf{y}$.
2. Find $p_{X_1}(k) \circledast p_{X_2}(k)$ using toeplitz matrices.
3. Find $M_Y(z)$, such that $Y = X_1 + X_2$
4. Find $p_Y(k)$

## 5 Z-Transform Applications

### *5.1 Definitions*

1. 

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.1.5.1}$$

2. The $Z$ transform of $X$ is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{5.1.5.2}$$

### *5.2 Problems*

1. If

$$p_Y(n) \overset{\mathcal{Z}}{\longleftrightarrow} M_Y(z), \tag{5.2.5.3}$$

show that

$$p_Y(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} M_Y(z) z^{-k}, \tag{5.2.5.4}$$

2. Show that

$$u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{(1 - z^{-1})}, \quad |z| > 1 \tag{5.2.5.5}$$

3. Show that

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{5.2.5.6}$$

4. Let

$$M_Y(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2, \quad |z| > 1 \tag{5.2.5.7}$$

Show that

$$p_Y(n) = \frac{(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)}{36} \tag{5.2.5.8}$$