1 Axioms of Probability

1.1 Definitions

1. For any event $A, 0 \leq \Pr(A) \leq 1$.
2. $A \cup B \triangleq A + B$.
3. $A \cap B \triangleq AB$.
4. The null and complete event are $\phi = 0, S = 1$.
5. If $AB = 0, \Pr(A + B) = \Pr(A) + \Pr(B)$.
6. $(A + B)' = A'B'$

1.2 Problems

Prove the following:

1. $$A = AB + AB' \tag{1.2.1.1}$$
2. $$\Pr(A) = \Pr(AB) + \Pr(AB') \tag{1.2.2.1}$$
3. $$A + B = B + AB' \tag{1.2.3.1}$$
4. $$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.2.4.1}$$

2 Distribution of the sum of random variables

2.1 Definitions

1. The mean of X is defined as
$$E(X) = \sum_k k p_X(k) \tag{2.1.1.1}$$
2. The Z transform of X is defined as
$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{2.1.2.1}$$

3. There is a one to one relationship betwwen the pmf and its Z transform.
4. If If X_1 and X_2 are independent,

$$E\left[f\left(X_1\right)g\left(X_2\right)\right] = E\left[f\left(X_1\right)\right]E\left[g\left(X_2\right)\right] \tag{2.1.4.1}$$

5. For a Bernoulli random variable X, the pmf is

$$p_X(n) = \begin{cases} p & k = 1 \\ 1-p & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.5.1}$$

6. X_i are said to be i.i.d (independent and identically distributed) if they are independent and have the same pmf.

2.2 *Problem*

1. Find the Z-transform for X, given that X is a Bernoulli random variable with parameter p.
2. If X_1 and X_2 are independent, and

$$Y = X_1 + X_2, \tag{2.2.2.1}$$

show that

$$M_Y(z) = M_{X_1}(z)M_{X_2}(z) \tag{2.2.2.2}$$

3. Find the Z-transform of Y, given that X_i are i.i.d Bernoulli random variables with parameter p.
4. Find the pmf of Y.
5. Find the pmf of

$$Y = \sum_{i=1}^{N} X_i, \tag{2.2.5.1}$$

where X_i are i.i.d.

3 Moments and variance

3.1 *Definitions*

1. The variance of X is defined as:

$$Var(X) = E(X - E(X))^2 \tag{3.1.1.1}$$

2. The Z transform of X is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{3.1.2.1}$$

3. Let X be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \le k \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.3.1}$$

   X is said to be Discrete Uniform Random Variable
4. The n^{th} moment of X is defined as:

$$E(X^n) = \sum_{k=-\infty}^{\infty} k^n p_X(k) \tag{3.1.4.1}$$

3.2 Problems

1. Show that $Var(X) = E(X^2) - [E(X)]^2$
2. Find $M_X(z)$
3. Show that $E(X) = \frac{d}{dz} M_X(z^{-1}) \mid_{z=1}$
4. Find $E(X^2)$
5. Find $Var(X)$.

4 Convolution

4.1 Definitions

1. The Z transform of X is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{4.1.1.1}$$

2. Let X be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \le k \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{4.1.2.1}$$

   X is said to be Discrete Uniform Random Variable

3. Convolution of two sequences using Toeplitz matrices

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{4.1.3.1}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_3 & . & . & 0 \\ h_{m-1} & . & . & . & h_2 & h_1 \\ h_m & h_{m-1} & . & . & . & h_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix} \tag{4.1.3.2}$$

4.2 Problems

1. If $\mathbf{x} = \mathbf{h} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $\mathbf{y}$.
2. Find $p_{X_1}(k) \circledast p_{X_2}(k)$ using toeplitz matrices.
3. Find $M_Y(z)$, such that $Y = X_1 + X_2$
4. Find $p_Y(k)$

5 Z-Transform Applications

5.1 Definitions

1.

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.1.1.1}$$

2. The Z transform of X is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{5.1.2.1}$$

5.2 Problems

1. If

$$p_Y(n) \overset{\mathcal{Z}}{\longleftrightarrow} M_Y(z), \tag{5.2.1.1}$$

show that

$$p_Y(n-k) \overset{\mathcal{Z}}{\longleftrightarrow} M_Y(z) z^{-k}, \tag{5.2.1.2}$$

2. Show that

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-z^{-1})}, \quad |z| > 1 \tag{5.2.2.1}$$

3. Show that

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \tag{5.2.3.1}$$

4. Let

$$M_Y(z) = \left\{ \frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)} \right\}^2, \quad |z| > 1 \tag{5.2.4.1}$$

   Show that

$$p_Y(n) = \frac{(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)}{36} \tag{5.2.4.2}$$

6 Markov Chain

6.1 Definitions

1. Fig. 6.1.1.1 shows a Markovs chain with 5 states. Transition from one state to another happens over time. s_0 and s_4 are absorbing states.

$$p + q = 1 \tag{6.1.1.1}$$

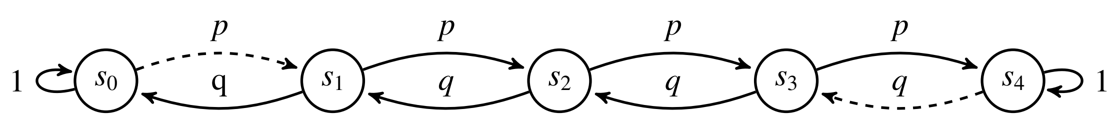

Fig. 6.1.1.1

2. At time instant n,

$$\mathbf{p}^{(n)} = \begin{pmatrix} P_0^{(n)} \\ P_1^{(n)} \\ P_2^{(n)} \\ P_3^{(n)} \\ P_4^{(n)} \end{pmatrix} \tag{6.1.2.1}$$

   where $P_i^{(n)}$ are defined to be the *stationary* probabilities.

3. $P_{i|j}$ is defined as the *transition* probability of going to state i from state j.

4. For a matrix $\mathbf{A}$, let

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}. \tag{6.1.4.1}$$

   Then λ is a scalar defined to be the eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is the corresponding eigenvector.

6.2 Problems

1. Let

$$P_0^{(n+1)} = P_0^{(n)} \tag{6.2.1.1}$$
$$P_1^{(n+1)} = pP_0^{(n)} + qP_2^{(n)} \tag{6.2.1.2}$$
$$P_2^{(n+1)} = pP_1^{(n)} + qP_3^{(n)} \tag{6.2.1.3}$$
$$P_3^{(n+1)} = pP_2^{(n)} + qP_4^{(n)} \tag{6.2.1.4}$$
$$P_4^{(n+1)} = P_4^{(n)} \tag{6.2.1.5}$$

   Find the matrix $\mathbf{P}$ such that $\mathbf{p^{(n+1)}} = \mathbf{Pp^{(n)}}$
2. Show that 1 is an eigen value of $\mathbf{P}$.
3. Show that

$$P_2^{(n+1)} = pP_1^{(n)} + qP_3^{(n)} \tag{6.2.3.1}$$

4. If $P_0 = 1, P_N = 0$ and

$$P_i = pP_{i-1} + qP_{i+1} \tag{6.2.4.1}$$

   show that

$$P_i = \frac{\left(\frac{p}{q}\right)^i - \left(\frac{p}{q}\right)^N}{1 - \left(\frac{p}{q}\right)^N}, 0 \le i \le N \tag{6.2.4.2}$$

7 Gaussian Distribution

7.1 Definitions

1. The CDF of X is defined as,

$$F_X(x) = \Pr\left(X \le x\right) \tag{7.1.1.1}$$

2. The PDF of X is defined as,

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{7.1.2.1}$$

3. Let $X \sim \mathcal{N}(0,1)$. Then the Q function is defined as,

$$Q(x) = \Pr\left(X > x\right), \quad x \geq 0 \tag{7.1.3.1}$$

7.2 Problems

1. Find

$$\Pr\left(|X-\mu| \leq k\sigma\right) \tag{7.2.1.1}$$

in terms of Q function.

2. Find

$$\Pr\left(X \leq x, |X-\mu| \leq k\sigma\right) \tag{7.2.2.1}$$

in terms of $F_X(x)$

3. Find

$$F_X\left(x| \, |X-\mu| \leq k\sigma\right) \tag{7.2.3.1}$$

4. Find

$$p_X\left(x| \, |X-\mu| \leq k\sigma\right) \tag{7.2.4.1}$$

8 Bivariate Gaussian

8.1 Definitions

1. Let

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{\mu} = E\left(\mathbf{X}\right) \tag{8.1.1.1}$$

$$\boldsymbol{\Sigma}_{\mathbf{x}} = E\left[\left(\mathbf{x}-\boldsymbol{\mu}\right)\left(\mathbf{x}-\boldsymbol{\mu}\right)^{\top}\right] \tag{8.1.1.2}$$

Then $\boldsymbol{\Sigma}_{\mathbf{x}}$ is defined to be the *covariance* matrix of $\mathbf{x}$.

2. For $\mathbf{x} \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}_{\mathbf{x}}\right)$,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\sqrt{|\boldsymbol{\Sigma}_{\mathbf{x}}|}} \exp -\frac{1}{2}\left(\left(\mathbf{x}-\boldsymbol{\mu}\right)^{\top} \boldsymbol{\Sigma}_{\mathbf{x}}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)\right) \tag{8.1.2.1}$$

3. The *correlation coefficient* is defined as

$$\rho = \frac{E\left[\left(x_1-\mu_1\right)\left(x_2-\mu_2\right)\right]}{\sigma_1\sigma_2} \tag{8.1.3.1}$$

where μ_i, σ_i^2 are the mean and variance of x_i.

8.2 Problems

1. Show that
$$E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^\top\right] = \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix} \tag{8.2.1.1}$$
2. Prove that $\mathbf{\Sigma_x}$ is a diagonal matrix when x_1 and x_2 are independent.
3. Let
$$z_1 = x_1 + x_2 \tag{8.2.3.1}$$
$$z_2 = x_1 - x_2 \tag{8.2.3.2}$$
Find $\mathbf{P}$ such that
$$\mathbf{z} = \begin{pmatrix}z_1 \\ z_2\end{pmatrix} = \mathbf{Px} \tag{8.2.3.3}$$
4. Show that
$$\mathbf{\Sigma_z} = \mathbf{P\Sigma_x P}^\top \tag{8.2.4.1}$$
5. Check the independence of z_1 and z_2 given that $\sigma_1 = \sigma_2$.
6. Show that columns of $\mathbf{P}$ are eigenvectors of $\mathbf{\Sigma_z}$.
7. Show that the eigenvectors of $\mathbf{\Sigma_z}$ are orthogonal to each other.
8. Summarize your conclusion in one line.

9 Transformation of Random Variables

9.1 Definitions

1. The pdf of $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ is defined as
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^- \frac{(x-\mu)^2}{2\sigma^2} \quad -\infty < x < \infty \tag{9.1.1.1}$$
2. The Jacobian matrix transforming R, Θ to X_1, X_2 is defined as
$$\mathbf{J} = \begin{pmatrix}\frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta}\end{pmatrix} \tag{9.1.2.1}$$
3.
$$p_{R,\Theta}(r,\theta) = p_{X_1,X_2}(x_1,x_2)\left|\mathbf{J}\right| \tag{9.1.3.1}$$

4. The marginal distribution

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}\left(r,\theta\right)\,d\theta \tag{9.1.4.1}$$

5. The Laplace transform of $p_Y(y)$ is given by

$$M_Y(s) = E\left(e^{-sY}\right) \tag{9.1.5.1}$$

6. The unit step function is defined as

$$u(y) = \begin{cases} 1 & y \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{9.1.6.1}$$

9.2 Problems

1. Let $X_1, X_2 \in N \sim (0,1)$ be i.i.d. Find $p_{X_1,X_2}\left(x_1,x_2\right)$.
2. Let

$$X_1 = R\cos\Theta \tag{9.2.2.1}$$
$$X_2 = R\sin\Theta \tag{9.2.2.2}$$

   Find $p_{R,\theta}\left(r,\theta\right)$.
3. Find $p_R(r)$.
4. Find $p_\Theta(\theta)$.
5. Find the distribution of

$$Y = X_1^2 + X_2^2 \tag{9.2.5.1}$$

6. Find the Laplace transform of $e^{-y}u(y)$
7. Find the Laplace transform of $p_{X_1^2}(x_1)$.
8. Find the Laplace transform of $p_Y(y)$ using (9.2.5.1).
9. Find the distribution of Y.

10 Order Statistics

10.1 Definitions

1. The pdf of an *exponential* distribution is given by

$$p_X(x) = e^{-x}u(x) \tag{10.1.1.1}$$

   where $u\left(\cdot\right)$ is the unit step function.

10.2 Problems

1. Find $F_X(x)$.
2. Let X and Y be iid exponential. Find $F_{XY}(z, z)$
3. Show that

$$\Pr(X \le z, X > Y) = \int_{x=-\infty}^{z} \int_{y=-\infty}^{x} p_{X,Y}(x, y)\, dx dy \tag{10.2.3.1}$$

$$= \frac{e^{-2z}}{2} - e^{-z} \tag{10.2.3.2}$$

4. Find

$$\Pr(Y \le z, X \le Y) \tag{10.2.4.1}$$

5. Let

$$Z = \max(X, Y) \tag{10.2.5.1}$$

Show that

$$F_Z(z) = \Pr(X \le z, X > Y) + \Pr(Y \le z, X \le Y) \tag{10.2.5.2}$$

6. Find the pdf of Z.
7. Find the pdf of $W = \min(X, Y)$.
8. Find $F_W(vZ)$, where v is a constant.
9. Find $E[F_W(vZ)]$.
10. Find the pdf of $V = \frac{W}{Z}$.

11 Some distributions

11.1 Definitions

1. If $Y = f(Z)$ be monotonic,

$$p_Y(y)\, dy = p_Z(z)\, dz \tag{11.1.1.1}$$

2. The pdf of the $\chi^2(k)$ distribution is given by

$$p_X(x) = \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} u(x) \tag{11.1.2.1}$$

3. The Beta function is defined as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{11.1.3.1}$$

11.2 Problems

1. Let $X_1 \sim \chi^2(m)$ and $X_2 \sim \chi^2(n)$ be independent. For
$$Z = \frac{X_1/m}{X_2/n}, \tag{11.2.1.1}$$
show that
$$F_Z(z) = E\left[F_{X_1}\left(\frac{mzX_2}{n}\right)\right] \tag{11.2.1.2}$$
2. Show that
$$p_Z(z) = E\left[\frac{mX_2}{n} p_{X_1}\left(\frac{mzX_2}{n}\right)\right] \tag{11.2.2.1}$$
3. Show that
$$p_Z(z) = \frac{\left(\frac{m}{n}\right)^{m/2}}{B\left(\frac{m}{2}, \frac{n}{2}\right)} x^{\frac{m}{2}-1}\left(1 + \frac{m}{n}z\right)^{-\frac{m+n}{2}} u(z) \tag{11.2.3.1}$$
Z has an F distribution with (m, n) degrees of freedom.
4. Show that $Y = \frac{1}{Z}$ is monotonic.
5. Show that Y also has an F distribution with (n, m) degrees of freedom.
6. Find the pdf of $\frac{mZ}{mZ+n}$.

12 Inequalities

12.1 Definitions

1. The mean of Y is defined as
$$E(Y) = \sum_k k p_Y(k) \tag{12.1.1.1}$$
2. The MGF of X is defined as
$$M_X(s) = E\left(e^{-sX}\right) \tag{12.1.2.1}$$
3. Let $X \sim \mathcal{N}(0, 1)$. Then the Q function is defined as,
$$Q(x) = \Pr(X > x), \quad x \geq 0 \tag{12.1.3.1}$$

12.2 Problems

1. Show that
$$E(Y) \geq \sum_{k=m}^{\infty} k p_Y(k), \quad m > 0 \tag{12.2.1.1}$$

2. Show that

$$\Pr(Y > m) \le \frac{E(Y)}{m}, \quad m > 0 \tag{12.2.2.1}$$

3. Using (12.2.2.1), show that Show that

$$\Pr\left([Y - E(Y)]^2 > b^2\right) \le \frac{var(Y)}{b^2}, \quad b > 0 \tag{12.2.3.1}$$

4. Show that

$$\Pr(|Y - E(Y)| > b) \le \frac{var(Y)}{b^2}, \quad b > 0 \tag{12.2.4.1}$$

5. Show that

$$\Pr(X > a) = \Pr\left(e^{-sX} > e^{-sa}\right), \quad s < 0 \tag{12.2.5.1}$$

6. Using (12.2.2.1), show that

$$\Pr(X > a) \le e^{as} M_X(s), \quad s < 0 \tag{12.2.6.1}$$

7. Show that the MGF of X is

$$M_X(s) = e^{\frac{1}{2}s^2} \tag{12.2.7.1}$$

8. Using (12.2.6.1) show that

$$Q(x) \le e^{-\frac{x^2}{2}} \tag{12.2.8.1}$$

13 Jensen's Inequality

13.1 Definitions

1. A function g is said to be convex if

$$g(\lambda x_1 + (1-\lambda)x_2) \le \lambda g(x_1) + (1-\lambda) g(x_2) \tag{13.1.1.1}$$

2. g is convex if and only if

$$g''(x) \ge 0 \tag{13.1.2.1}$$

3. The *information* associated with an event E is defined as

$$I(E) = -\log_2(p(E)) \tag{13.1.3.1}$$

4. The *entropy* of a random variable X is given by

$$H(X) = -E[\log_2(p(X))] \tag{13.1.4.1}$$

5. For a *bernoulli* random variable $Y \in \{0, 1\}$, the p.m.f is given by

$$p_Y(k) = \begin{cases} p & k = 0 \\ 1 - p & k = 1 \end{cases} \tag{13.1.5.1}$$

13.2 Problems

1. Show that $\log\left(\frac{1}{x}\right)$ is convex.
2. Show that

$$H(Y) = -p \log_2 p - (1 - p) \log_2 (1 - p) \tag{13.2.2.1}$$

and find the maximum value of $H(Y)$.

3. Let $X \in \{x_1, x_2\}$ and

$$\lambda = q = p_X(X = x_1) \tag{13.2.3.1}$$

in (13.1.1.1). Show that

$$E\left[g(X)\right] \geq g\left[E\left(X\right)\right]. \tag{13.2.3.2}$$

4. Using (13.2.3.2), show that

$$H(Y) \leq 1 \tag{13.2.4.1}$$

14 Maximum Likelihood Condition

14.1 Definitions

1. The pmf for a Binomial Random variable $Y \sim (n, p)$ is given by

$$p_Y\left(Y = k\right) = \binom{n}{k} p^k \left(1 - p\right)^{n-k}, \quad 0 \leq k \leq n \tag{14.1}$$

2. The pdf of an Exponential Random variable X is given by

$$p_X\left(x\right) = ce^{-cx} u\left(x\right) \tag{14.1}$$

14.2 Problems

1. Show that the solution of

$$\max_p p_Y\left(Y = k\right) \tag{14.2.1.1}$$

is

$$\hat{p} = \frac{k}{n} \tag{14.2.1.2}$$

2. Find

$$\Pr\left(X > T\right) \tag{14.2.2.1}$$

where T is a constant.

3. If X represents the lifetime of a bulb, show that the value of c that maximizes the probability of k out of n bulbs working after T hours is

$$c = \frac{1}{T}\log\left(\frac{n}{k}\right) \tag{14.2.3.1}$$

15 Confidence Intervals

15.1 Definitions

1. The pmf for a Bernoulli r.v. X_i is given by

$$p_{X_i}(k) = \begin{cases} p_0 & k = 1 \\ 1 - p_0 & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{15.1}$$

2. The pmf for a Binomial Random variable

$$X = \sum_{i=1}^{n} X_i \tag{15.1}$$

is given by

$$p_X\left(X = k\right) = \binom{n}{k} p_0^k \left(1 - p_0\right)^{n-k}, \quad 0 \le k \le n \tag{15.2}$$

where X_i are i.i.d.

3. Let $Z \sim \mathcal{N}(0, 1)$. Then the Q function is defined as,

$$Q(z) = \Pr\left(Z > z\right), \quad z \ge 0 \tag{15.1}$$

4. The MGF of Y is defined as

$$M_Y(s) = E\left(e^{-sY}\right) \tag{15.1}$$

5. The MGF of Z is

$$M_Z(s) = \frac{1}{2}e^{\frac{1}{2}s^2} \tag{15.1}$$

6.

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e \tag{15.1}$$

15.2 Problems

1. Show that the mean and variance of X_i are $\mu = p_0$ and $\sigma^2 = p\,(1-p)$.
2. Show that the mean and variance of
$$Y_i = \frac{X_i - \mu}{\sigma} \tag{15.1}$$
are 0 and 1 respectively.
3. Show that the mean and variance of
$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \tag{15.1}$$
are 0 and 1 respectively.
4. Show that
$$M_Y(s) = \left[E\left(e^{-\frac{sY_i}{\sqrt{n}}}\right)\right]^n \tag{15.1}$$
5. Show that
$$E\left(e^{-\frac{sY_i}{\sqrt{n}}}\right) = 1 + \frac{s^2}{2n} + \frac{1}{n}R(s,n) \tag{15.1}$$
where $R(s,n)$ is an infinite series.
6. Show that
$$\lim_{n\to\infty} Y = Z \tag{15.1}$$
This is known as the *Central Limit Theorem*.
7. Let
$$p = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{15.1}$$
Show that
$$E\,(p) = \mu = p_0 \tag{15.2}$$
$$var\,(p) = \frac{\sigma^2}{n} = \frac{p_0\,(1-p_0)}{n} \tag{15.3}$$
8. Show that
$$\lim_{n\to\infty} \frac{p - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = Z \tag{15.1}$$

9. Find z if

$$\Pr\left(|Z| < z\right) = 1 - \alpha \tag{15.1}$$

10. Show that

$$p_0 - \sqrt{\frac{p_0\left(1 - p_0\right)}{n}} Q^{-1}\left(\frac{\alpha}{2}\right) < p < p_0 + \sqrt{\frac{p_0\left(1 - p_0\right)}{n}} Q^{-1}\left(\frac{\alpha}{2}\right) \tag{15.1}$$

11. Among 4000 newborns, 2080 are male. Find the $1 - \alpha = 0.99$ confidence interval of the probability that a male child is born.

16 Autocorrelation and Power Spectral Density

16.1 Definitions

1. Let $X(t)$ be a random process. The *autocorrelation function* is then defined as

$$R_X\left(\tau\right) = E\left[X(t)X^*\left(t + \tau\right)\right] \tag{16.1}$$

2. The *Fourier* transform of $g(t)$ is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t}\, dt \tag{16.1}$$

3. The *power spectral density* of $X(t)$ is defined as

$$R_X(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_X(f) \tag{16.1}$$

4.

$$g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f) \tag{16.1}$$

$$\implies G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{16.2}$$

5.

$$g(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} G\left(\frac{f}{|a|}\right) \tag{16.1}$$

6.

$$\operatorname{rect}\left(t\right) = \begin{cases} 1 & t \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ 0 & \text{otherwise} \end{cases} \tag{16.1}$$

$$\operatorname{sinc}\left(f\right) = \frac{\sin\left(\pi f\right)}{\pi f} \tag{16.2}$$

7. If

$$Y(t) = X(t) * h(t), \tag{16.1}$$
$$S_Y(f) = |H(f)|^2 S_X(f) \tag{16.2}$$

8.

$$\triangle\left(\frac{t}{2}\right) = \begin{cases} 1 - |t| & t \in (-1, 1) \\ 0 & \text{otherwise} \end{cases} \tag{16.1}$$

9.

$$g_1(t) * g_2(t) \xleftrightarrow{\mathcal{F}} G_1(f) G_2(f) \tag{16.1}$$

10. The *Dirac delta* function is defined as

$$\delta(t) = 0, t \neq 0 \tag{16.1}$$
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{16.2}$$

16.2 Problems

1. Show that

$$\text{rect}\,(t) \xleftrightarrow{\mathcal{F}} \text{sinc}\,(f) \tag{16.1}$$

2. Show that

$$g(t - T) \xleftrightarrow{\mathcal{F}} G(f) e^{-\jmath 2\pi f T} \tag{16.1}$$

3. Let

$$Y = \int_0^T X(t) \tag{16.1}$$

Show that

$$Y = Y(t)|_{t=T} \tag{16.2}$$

where

$$Y(t) = X(t) * h(t) \tag{16.3}$$
$$h(t) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right) \tag{16.4}$$

4. Show that

$$\mathrm{rect}\,(t) * \mathrm{rect}\,(t) = \triangle\left(\frac{t}{2}\right) \tag{16.1}$$

5. Show that

$$\triangle\left(\frac{t}{2}\right) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}^2(f) \tag{16.1}$$

6. Show that

$$E\left(Y^2\right) = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df \tag{16.1}$$

7. Show that

$$E\left(Y^2\right) = T\int_{-T}^{T} \triangle\left(\frac{\tau}{T}\right) R_X(\tau)\,d\tau \tag{16.1}$$

8. Show that

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1 \tag{16.1}$$

9. Show that

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2 + (\jmath 2\pi f)^2} \tag{16.1}$$

10. For

$$T = 10, E\left[X(t)\right] = 8, R_X(\tau) = 64 + e^{-2|\tau|}, \tag{16.1}$$

find the mean and variance of Y.