1 Axioms of Probability

1.1 Definitions

1. For any event $A, 0 \le \Pr(A) \le 1$.
2. $A \cup B \triangleq A + B$.
3. $A \cap B \triangleq AB$.
4. The null and complete event are $\phi = 0, S = 1$.
5. If $AB = 0, \Pr(A + B) = \Pr(A) + \Pr(B)$.
6. $(A + B)' = A'B'$

1.2 Problems

Prove the following:

1. $$A = AB + AB' \tag{1.2.1.1}$$
2. $$\Pr(A) = \Pr(AB) + \Pr(AB') \tag{1.2.2.1}$$
3. $$A + B = B + AB' \tag{1.2.3.1}$$
4. $$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{1.2.4.1}$$

2 Distribution of the sum of random variables

2.1 Definitions

1. The mean of X is defined as
$$E(X) = \sum_k k p_X(k) \tag{2.1.1.1}$$
2. The Z transform of X is defined as
$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{2.1.2.1}$$

3. There is a one to one relationship betwwen the pmf and its Z transform.
4. If If X_1 and X_2 are independent,

$$E\left[f\left(X_1\right)g\left(X_2\right)\right] = E\left[f\left(X_1\right)\right]E\left[g\left(X_2\right)\right] \tag{2.1.4.1}$$

5. For a Bernoulli random variable X, the pmf is

$$p_X(n) = \begin{cases} p & k = 1 \\ 1-p & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.5.1}$$

6. X_i are said to be i.i.d (independent and identically distributed) if they are independent and have the same pmf.

2.2 Problem

1. Find the Z-transform for X, given that X is a Bernoulli random variable with parameter p.
2. If X_1 and X_2 are independent, and

$$Y = X_1 + X_2, \tag{2.2.2.1}$$

show that

$$M_Y(z) = M_{X_1}(z)M_{X_2}(z) \tag{2.2.2.2}$$

3. Find the Z-transform of Y, given that X_i are i.i.d Bernoulli random variables with parameter p.
4. Find the pmf of Y.
5. Find the pmf of

$$Y = \sum_{i=1}^{N} X_i, \tag{2.2.5.1}$$

where X_i are i.i.d.

3 Moments and variance

3.1 Definitions

1. The variance of X is defined as:

$$Var(X) = E(X - E(X))^2 \tag{3.1.1.1}$$

2. The Z transform of X is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{3.1.2.1}$$

3. Let X be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \le k \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{3.1.3.1}$$

   X is said to be Discrete Uniform Random Variable
4. The n^{th} moment of X is defined as:

$$E(X^n) = \sum_{k=-\infty}^{\infty} k^n p_X(k) \tag{3.1.4.1}$$

3.2 Problems

1. Show that $Var(X) = E(X^2) - [E(X)]^2$
2. Find $M_X(z)$
3. Show that $E(X) = \frac{d}{dz} M_X(z^{-1}) \mid_{z=1}$
4. Find $E(X^2)$
5. Find $Var(X)$.

4 Convolution

4.1 Definitions

1. The Z transform of X is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{4.1.1.1}$$

2. Let X be a random variable with pmf.

$$p_X(k) = \begin{cases} 1/6 & 1 \le k \le 6 \\ 0 & \text{otherwise} \end{cases} \tag{4.1.2.1}$$

   X is said to be Discrete Uniform Random Variable

3. Convolution of two sequences using Toeplitz matrices

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{4.1.3.1}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_3 & . & . & 0 \\ h_{m-1} & . & . & . & h_2 & h_1 \\ h_m & h_{m-1} & . & . & . & h_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & h_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_n \end{pmatrix} \tag{4.1.3.2}$$

4.2 *Problems*

1. If $\mathbf{x} = \mathbf{h} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find $\mathbf{y}$.
2. Find $p_{X_1}(k) \circledast p_{X_2}(k)$ using toeplitz matrices.
3. Find $M_Y(z)$, such that $Y = X_1 + X_2$
4. Find $p_Y(k)$

5 Z-Transform Applications

5.1 *Definitions*

1.

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.1.1.1}$$

2. The Z transform of X is defined as

$$M_X(z) = E\left(z^{-X}\right) = \sum_{k=-\infty}^{\infty} z^{-k} p_X(k) \tag{5.1.2.1}$$

5.2 *Problems*

1. If

$$p_Y(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} M_Y(z), \tag{5.2.1.1}$$

show that

$$p_Y(n-k) \stackrel{\mathcal{Z}}{\longleftrightarrow} M_Y(z) z^{-k}, \tag{5.2.1.2}$$

2. Show that

$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{5.2.2.1}$$

3. Show that

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{\left(1 - z^{-1}\right)^2}, \quad |z| > 1 \tag{5.2.3.1}$$

4. Let

$$M_Y(z) = \left\{ \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)} \right\}^2, \quad |z| > 1 \tag{5.2.4.1}$$

Show that

$$p_Y(n) = \frac{(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)}{36} \tag{5.2.4.2}$$

6 Markov Chain

6.1 Definitions

1. Fig. 6.1.1.1 shows a Markovs chain with 5 states. Transition from one state to another happens over time. s_0 and s_4 are absorbing states.

$$p + q = 1 \tag{6.1.1.1}$$

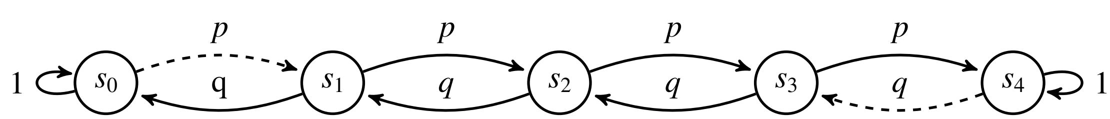

Fig. 6.1.1.1

2. At time instant n,

$$\mathbf{p}^{(n)} = \begin{pmatrix} P_0^{(n)} \\ P_1^{(n)} \\ P_2^{(n)} \\ P_3^{(n)} \\ P_4^{(n)} \end{pmatrix} \tag{6.1.2.1}$$

where $P_i^{(n)}$ are defined to be the *stationary* probabilities.

3. $P_{i|j}$ is defined as the *transition* probability of going to state i from state j.

4. For a matrix $\mathbf{A}$, let

$$\mathbf{Ax} = \lambda\mathbf{x}. \tag{6.1.4.1}$$

Then λ is a scalar defined to be the eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ is the corresponding eigenvector.

6.2 Problems

1. Let

$$P_0^{(n+1)} = P_0^{(n)} \tag{6.2.1.1}$$
$$P_1^{(n+1)} = pP_0^{(n)} + qP_2^{(n)} \tag{6.2.1.2}$$
$$P_2^{(n+1)} = pP_1^{(n)} + qP_3^{(n)} \tag{6.2.1.3}$$
$$P_3^{(n+1)} = pP_2^{(n)} + qP_4^{(n)} \tag{6.2.1.4}$$
$$P_4^{(n+1)} = P_4^{(n)} \tag{6.2.1.5}$$

Find the matrix $\mathbf{P}$ such that $\mathbf{p^{(n+1)}} = \mathbf{Pp^{(n)}}$

2. Show that 1 is an eigen value of $\mathbf{P}$.
3. Show that

$$P_2^{(n+1)} = pP_1^{(n)} + qP_3^{(n)} \tag{6.2.3.1}$$

4. If $P_0 = 1, P_N = 0$ and

$$P_i = pP_{i-1} + qP_{i+1} \tag{6.2.4.1}$$

show that

$$P_i = \frac{\left(\frac{p}{q}\right)^i - \left(\frac{p}{q}\right)^N}{1 - \left(\frac{p}{q}\right)^N}, 0 \le i \le N \tag{6.2.4.2}$$

7 Gaussian Distribution

7.1 Definitions

1. The CDF of X is defined as,

$$F_X(x) = \Pr\left(X \le x\right) \tag{7.1.1.1}$$

2. The PDF of X is defined as,

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{7.1.2.1}$$

3. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then the Q function is defined as,
$$Q(x) = \Pr(X > x), \quad x \geq 0 \tag{7.1.3.1}$$

7.2 Problems

1. Find
$$\Pr(|X - \mu| \leq k\sigma) \tag{7.2.1.1}$$
in terms of Q function.
2. Find
$$\Pr(X \leq x, |X - \mu| \leq k\sigma) \tag{7.2.2.1}$$
in terms of $F_X(x)$
3. Find
$$F_X(x \mid |X - \mu| \leq k\sigma) \tag{7.2.3.1}$$
4. Find
$$p_X(x \mid |X - \mu| \leq k\sigma) \tag{7.2.4.1}$$

8 Bivariate Gaussian

8.1 Definitions

1. Let
$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{\mu} = E(\mathbf{X}) \tag{8.1.1.1}$$
$$\boldsymbol{\Sigma}_\mathbf{x} = E\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top\right] \tag{8.1.1.2}$$
Then $\boldsymbol{\Sigma}_\mathbf{x}$ is defined to be the *covariance* matrix of $\mathbf{x}$.
2. For $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}_\mathbf{x})$,
$$f_\mathbf{X}(\mathbf{x}) = \frac{1}{2\pi\sqrt{|\boldsymbol{\Sigma}_\mathbf{x}|}} \exp -\frac{1}{2}\left((\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}_\mathbf{x}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right) \tag{8.1.2.1}$$
3. The *correlation coefficient* is defined as
$$\rho = \frac{E\left[(x_1 - \mu_1)(x_2 - \mu_2)\right]}{\sigma_1 \sigma_2} \tag{8.1.3.1}$$
where μ_i, σ_i^2 are the mean and variance of x_i.

8.2 Problems

1. Show that
$$E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^\top\right] = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{8.2.1.1}$$
2. Prove that $\mathbf{\Sigma_x}$ is a diagonal matrix when x_1 and x_2 are independent.
3. Let
$$z_1 = x_1 + x_2 \tag{8.2.3.1}$$
$$z_2 = x_1 - x_2 \tag{8.2.3.2}$$
Find $\mathbf{P}$ such that
$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathbf{Px} \tag{8.2.3.3}$$
4. Show that
$$\mathbf{\Sigma_z} = \mathbf{P\Sigma_x P}^\top \tag{8.2.4.1}$$
5. Check the independence of z_1 and z_2 given that $\sigma_1 = \sigma_2$.
6. Show that columns of $\mathbf{P}$ are eigenvectors of $\mathbf{\Sigma_z}$.
7. Show that the eigenvectors of $\mathbf{\Sigma_z}$ are orthogonal to each other.
8. Summarize your conclusion in one line.

9 Transformation of Random Variables

9.1 Definitions

1. The pdf of $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ is defined as
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^- \frac{(x-\mu)^2}{2\sigma^2} \quad -\infty < x < \infty \tag{9.1.1.1}$$
2. The Jacobian matrix transforming R, Θ to X_1, X_2 is defined as
$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_1}{\partial \Theta} \\ \frac{\partial X_2}{\partial R} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{9.1.2.1}$$
3.
$$p_{R,\Theta}(r, \theta) = p_{X_1,X_2}(x_1, x_2)\left|\mathbf{J}\right| \tag{9.1.3.1}$$

4. The marginal distribution

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta)\, d\theta \tag{9.1.4.1}$$

5. The Laplace transform of $p_Y(y)$ is given by

$$M_Y(s) = E\left(e^{-sY}\right) \tag{9.1.5.1}$$

6. The unit step function is defined as

$$u(y) = \begin{cases} 1 & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{9.1.6.1}$$

9.2 Problems

1. Let $X_1, X_2 \in N \sim (0,1)$ be i.i.d. Find $p_{X_1,X_2}(x_1, x_2)$.
2. Let

$$X_1 = R\cos\Theta \tag{9.2.2.1}$$
$$X_2 = R\sin\Theta \tag{9.2.2.2}$$

   Find $p_{R,\theta}(r,\theta)$.
3. Find $p_R(r)$.
4. Find $p_\Theta(\theta)$.
5. Find the distribution of

$$Y = X_1^2 + X_2^2 \tag{9.2.5.1}$$

6. Find the Laplace transform of $e^{-y}u(y)$
7. Find the Laplace transform of $p_{X_1^2}(x_1)$.
8. Find the Laplace transform of $p_Y(y)$ using (9.2.5.1).
9. Find the distribution of Y.