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Quiz 8 AI1110

1 Definitions

1. Let

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{\mu} = E(\mathbf{X}) \tag{1.1.1}$$

$$\mathbf{\Sigma_x} = E\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top\right] \tag{1.1.2}$$

Then $\mathbf{\Sigma_x}$ is defined to be the *covariance* matrix of $\mathbf{x}$.

2. For $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{\Sigma_x})$,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\sqrt{|\mathbf{\Sigma_x}|}} \exp -\frac{1}{2}\left((\mathbf{x} - \boldsymbol{\mu})^\top \mathbf{\Sigma_x}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right) \tag{1.2.1}$$

3. The *correlation coefficient* is defined as

$$\rho = \frac{E\left[(x_1 - \mu_1)(x_2 - \mu_2)\right]}{\sigma_1\sigma_2} \tag{1.3.1}$$

where μ_i, σ_i^2 are the mean and variance of x_i.

2 Problems

1. Show that

$$E\left[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^\top\right] = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{2.1.1}$$

2. Prove that $\mathbf{\Sigma_x}$ is a diagonal matrix when x_1 and x_2 are independent.
3. Let

$$z_1 = x_1 + x_2 \tag{2.3.1}$$

$$z_2 = x_1 - x_2 \tag{2.3.2}$$

Find $\mathbf{P}$ such that

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathbf{P}\mathbf{x} \tag{2.3.3}$$

4. Show that

$$\mathbf{\Sigma_z} = \mathbf{P}\mathbf{\Sigma_x}\mathbf{P}^\top \tag{2.4.1}$$

5. Check the independence of z_1 and z_2 given that $\sigma_{x_1} = \sigma_{x_2}$.
6. Show that columns of $\mathbf{P}$ are eigenvectors of $\mathbf{\Sigma_z}$.
7. Show that the eigenvectors of $\mathbf{P}$ are orthogonal to each other.
8. Summarize your conclusion in one line.