

# UGC Net - Problem 53

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Question 53) Suppose (X,Y) follows bivariate normal distribution with means  $\mu_1, \mu_2$ , standard deviations  $\sigma_1, \sigma_2$  and correlation coefficient  $\rho$ , where all parameters are un-known. Then, testing  $H_0: \sigma_1 = \sigma_2$  is equivalent to testing the independence of

- 1.) X and Y
- 2.) X and X-Y
- 3.) X+Y and Y
- 4.) X+Y and X-Y

Answer: 4, X+Y and X-Y

## 1 Solution:

### 1.1 Definition of Bivariate Gaussian and its independency

Bi-variate random variables are distribution of normal distribution to two coordinates. are said to be bivariate normal or jointly normal, if  $aX + bY$  has normal distribution  $\forall a, b \in R$ .

Random normal vector  $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$  is Bi-variate when it is jointly normal

Joint PDF of Z is given as

$$f_z(Z) = \frac{1}{2\pi\sqrt{\det\Sigma}} \exp \left\{ \frac{-1}{2} (z - m)^T \Sigma^{-1} (z - m) \right\}$$

Where,

$$m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \quad \Sigma = [ [Z - E(Z)][Z - E(Z)]^T ]$$

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix}$$

If X, Y, which are independent, then they are un-correlated or their co-variances are  $\rho\sigma_Y\sigma_X = 0$  then co-variance matrix becomes a diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}$$

## 1.2 Co-variance Matrix

$$\Sigma = [[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T]$$

When,  $\mathbf{Z}' = \mathbf{T}\mathbf{Z}$ , Where  $T$  is a transformation

$$\begin{aligned} \Sigma_{TZ} &= [[\mathbf{T}\mathbf{Z} - E(\mathbf{T}\mathbf{Z})][\mathbf{T}\mathbf{Z} - E(\mathbf{T}\mathbf{Z})]^T] \\ &= [[T\mathbf{Z} - TE(\mathbf{Z})][T\mathbf{Z} - TE(\mathbf{Z})]^T] \\ &= [T[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T T^T] \\ &= [T[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T T^T] \\ &= [T\Sigma T^T] \end{aligned}$$

$$\text{Where } [[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T] = \Sigma$$

$$\Sigma_{TZ} = [T\Sigma T^T] \quad (1)$$

## 1.3 Few observations:

1.  $T$  is diagonal matrix
2.  $\Sigma_{TZ}$  is diagonal matrix.
3.  $\sigma$  is symmetric.
4. columns of  $T$  are eigen vectors.

Now if condition 1 and 4 satisfies then we can say a transformation is independent.

## 1.4 Evaluating option 1

: Given  $\sigma_x = \sigma_y$

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_X \\ \rho\sigma_X\sigma_X & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X^2 \\ \rho\sigma_X^2 & \sigma_X^2 \end{bmatrix} = \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

$\Sigma$  is not a diagonal matrix so components of  $Z$  in option 1 are not independent

### 1.5 Evaluating option 2

X, X-Y can be written as  $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

Where  $T = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ ,  $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$

Co-variance matrix  $\Sigma$  for X, X-Y From Eq. 1  $\Sigma_{TZ} = [T\Sigma T^T]$

$$\begin{aligned} \Sigma_{TZ} &= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \sigma_X^2 \begin{bmatrix} 1 & \rho \\ 1-\rho & \rho-1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 1 & 1-\rho \\ 1-\rho & 2-\rho \end{bmatrix} \end{aligned}$$

$\Sigma_{TZ}$  is not a diagonal matrix, so components of TZ are not independent

### 1.6 Evaluating option 3

X+Y and Y can be written as  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

Where  $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ,  $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$

Co-variance matrix  $\Sigma$  for X+Y, Y From Eq. 1  $\Sigma_{TZ} = [T\Sigma T^T]$

$$\begin{aligned} \Sigma_{TZ} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \sigma_x^2 \begin{bmatrix} 2+2\rho & \rho+1 \\ \rho+1 & 1 \end{bmatrix} \end{aligned}$$

$\Sigma_{TZ}$  is not a diagonal matrix, so components of TZ are not independent

### 1.7 Evaluating option 4

X+Y and X-Y can be written as  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

From Eq. 1  $\Sigma_{TZ} = [T\Sigma T^T]$

$$\begin{aligned} \Sigma_{TZ} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \sigma_x^2 \begin{bmatrix} 1+\rho & \rho+1 \\ 1-\rho & \rho-1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2+2\rho & 0 \\ 0 & 2-2\rho \end{bmatrix} \end{aligned}$$

Hence option 4 is correct