Probability Questions

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1. There are 30 questions in a certain multiple choice examination paper. Each question has 4 options and exactly one is to be marked by the candidate. Three candidates A,B,C mark each of the 30 questions at random independently. The probability that all the 30 answers of the three students match each other perfectly is?
2. Consider a Markov Chain with state space {0,1,2,3,4} and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

3. Consider the function $f(x)$ defined as $f(x) = ce^{-x^4}, x \in \mathbb{R}$. For what value of c is f a probability density function?
   a) $\frac{2}{\Gamma(\frac{1}{4})}$
   b) $\frac{4}{\Gamma(\frac{1}{4})}$
   c) $\frac{3}{\Gamma(\frac{1}{3})}$
   d) $\frac{1}{4\Gamma(4)}$
4. A random sample of size 7 is drawn from a distribution with p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1+x^2}{3\theta(1+\theta^2)}, & -2\theta \leq x \leq \theta, \theta > 0 \\ 0, & otherwise \end{cases}$$

   and the observations are 12, -54, 26, -2, 24, 17, -39. What is the maximum likelihood estimation of θ.
   a) 12
   b) 24
   c) 26
   d) 27
5. Let X_1, X_2, X_3, X_4, X_5be i.i.d random variables having a continuous distribution function. Then P($X_1 > X_2 > X_3 > X_4 > X_5 | X_1 = \max(X_1, X_2, X_3, X_4, X_5)$) equals
   a) $\frac{1}{4}$
   b) $\frac{1}{5}$
   c) $\frac{1}{4!}$
   d) $\frac{1}{5!}$
6. Suppose (X,Y) follows bivariate normal distribution with means $\mu 1, \mu 2$, standard deviations $\sigma 1$, $\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
   a) X and Y
   b) X and X-Y
   c) X+Y and Y
   d) X+Y and X-Y

Question 53) Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

1 Question 54

(52) Suppose X∼ Cauchy(0,1). Then the distribution of $\frac{1-X}{1+X}$ is?

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2 Solution

A continuous random variable X follows **Cauchy distribution** with parameters μ and λ if its pdf is given by

$$f(x) = \begin{cases} \frac{\lambda}{\pi} \cdot \frac{1}{\lambda^2+(x-\mu)^2}, & -\infty < x < \infty; \\ & -\infty < \mu < \infty,\ \lambda > 0; \\ 0, & \text{Otherwise.} \end{cases}$$

The parameter μ and λ are location and scale parameters respectively.
When μ=0 and λ=1, then the distribution is called **Standard Cauchy Distribution**. The pdf of standard Cauchy distribution is

$$f(x) = \begin{cases} \frac{1}{\pi} \cdot \frac{1}{1+x^2}, & -\infty < x < \infty; \\ 0, & \text{Otherwise.} \end{cases}$$

Let,

$$Y = \frac{1-X}{1+X} \tag{1}$$

Then, cdf of Y is

$$F_Y(y) = P(Y \le y) \tag{2}$$

$$= P\left(\frac{1-X}{1+X} \le y\right) \tag{3}$$

$$= P\left((1-X) \le y.(1+X)\right) \tag{4}$$

$$= P\left((1-X) \le (y+y.X)\right) \tag{5}$$

$$= P\left((1-y) \le (X+y.X)\right) \tag{6}$$

$$= P\left(\frac{1-y}{1+y} \le X\right) \tag{7}$$

$$= 1 - P\left(X < \frac{1-y}{1+y}\right) \tag{8}$$

$$= 1 - \int_{-\infty}^{\frac{1-y}{1+y}} f(x)\,dx \tag{9}$$

$$= 1 - \int_{-\infty}^{\frac{1-y}{1+y}} \frac{1}{\pi} \cdot \frac{1}{1+x^2}\,dx \tag{10}$$

$$= 1 - \frac{1}{\pi} \cdot \left[tan^{-1}x\right]_{-\infty}^{\frac{1-y}{1+y}} \tag{11}$$

$$= 1 - \frac{1}{\pi} \cdot \left[\left[tan^{-1}x\right]_{-\infty}^{0} + \left[tan^{-1}x\right]_{0}^{\frac{1-y}{1+y}}\right] \tag{12}$$

$$= 1 - \frac{1}{\pi} \cdot \left[-\frac{\pi}{2} + \left[tan^{-1}x\right]_{0}^{\frac{1-y}{1+y}}\right] \tag{13}$$

$$F_Y(y) = 1 - \frac{1}{\pi} \cdot \left[-\frac{\pi}{2} + tan^{-1}\left(\frac{1-y}{1+y}\right)\right] \tag{14}$$

The pdf of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = -\frac{1}{\pi} \cdot \frac{d\left(tan^{-1}\left(\frac{1-y}{1+y}\right)\right)}{dy} \tag{15}$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1+\left(\frac{1-y}{1+y}\right)^2} \cdot \frac{d\left(\frac{1-y}{1+y}\right)}{dy} \tag{16}$$

$$= -\frac{1}{\pi} \cdot \frac{1}{1+\left(\frac{1-y}{1+y}\right)^2} \cdot \left(-\frac{2}{(y+1)^2}\right) \tag{17}$$

$$= \frac{1}{\pi} \cdot \frac{1}{1+y^2} \tag{18}$$

Hence, Y $\sim$ Cauchy(0,1).