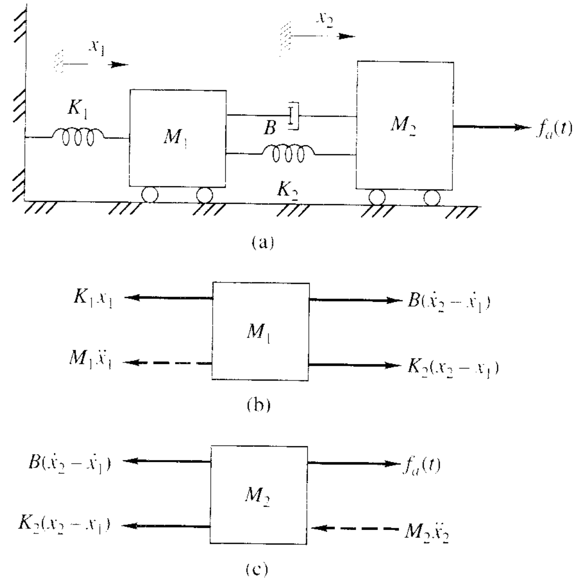


Figure 2.13 (a) Translational system for Example 2.2. (b), (c) Free-body diagrams.

EXAMPLE 2.2

Draw the free-body diagrams and use D'Alembert's law to write the two modeling equations for the two-mass system shown in Figure 2.13(a).

SOLUTION Because there are two masses that can move with different unknown velocities, a separate free-body diagram is drawn for each one. This is done in Figure 2.13(b) and Figure 2.13(c). In Figure 2.13(b), the forces K_1x_1 and $M_1\ddot{x}_1$ are similar to those in Example 2.1. As indicated in our earlier discussion of displacements, the net elongation of the spring and dashpot connecting the two masses is $x_2 - x_1$. Hence a positive value of $x_2 - x_1$ results in a reaction force by the spring to the right on M_1 and to the left on M_2, as indicated in Figure 2.13. Of course, the force on either free-body diagram could be labeled $K_2(x_1 - x_2)$, provided that the corresponding reference arrow were reversed. For a positive value of $\dot{x}_2 - \dot{x}_1$, the reaction force of the middle dashpot is to the right on M_1 and to the left on M_2. As always, the inertial forces $M_1\ddot{x}_1$ and $M_2\ddot{x}_2$ are opposite to the positive directions of the accelerations.

Summing the forces on each free-body diagram separately and taking into account the directions of the reference arrows give the following pair of differential equations:

$$B(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) - M_1\ddot{x}_1 - K_1x_1 = 0$$
$$f_a(t) - M_2\ddot{x}_2 - B(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) = 0$$

Rearranging, we have

$$M_1\ddot{x}_1 + B\dot{x}_1 + (K_1 + K_2)x_1 - B\dot{x}_2 - K_2x_2 = 0 \tag{16a}$$
$$-B\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 = f_a(t) \tag{16b}$$

Equations (16a) and (16b) constitute a pair of coupled second-order differential equations. In the next chapter, we shall discuss two alternative methods of presenting the information contained in such a set of equations.

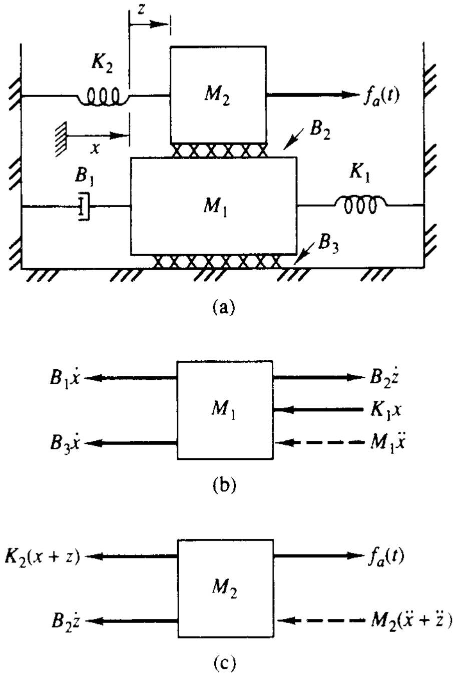


Figure 2.15 (a) Translational system for Example 2.4. (b), (c) Free-body diagrams.

The reader is encouraged to repeat this example when the displacement of each mass is expressed with respect to its own fixed reference position. If x_1 and x_2 denote the displacements of M_1 and M_2, respectively, with the positive senses to the right, we find that

$$\begin{aligned} M_1\ddot{x}_1 + (B_1 + B_2 + B_3)\dot{x}_1 + K_1x_1 - B_2\dot{x}_2 &= 0 \\ -B_2\dot{x}_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 &= f_a(t) \end{aligned} \tag{20}$$

When x_1 is replaced by x, and x_2 is replaced by $x + z$, this pair of equations reduces to those in (19). However, it is useful to be able to obtain (19) directly from the free-body diagrams in Figure 2.15.

Displacement variables must be expressed with respect to some reference position. These references are commonly chosen such that any springs are neither stretched nor compressed when the values of the displacement variables are zero. The following two examples show why this may not necessarily be the case for systems with vertical motion.

EXAMPLE 2.5

Draw the free-body diagram, including the effect of gravity, and find the differential equation describing the motion of the mass shown in Figure 2.16(a).

SOLUTION Assume that x is the displacement from the position corresponding to a spring that is neither stretched nor compressed. The gravitational force on the mass is Mg, and we include it in the free-body diagram shown in Figure 2.16(b) because the mass moves vertically. By summing the forces on the free-body diagram, we obtain

$$M\ddot{x} + B\dot{x} + Kx = f_a(t) + Mg \tag{21}$$

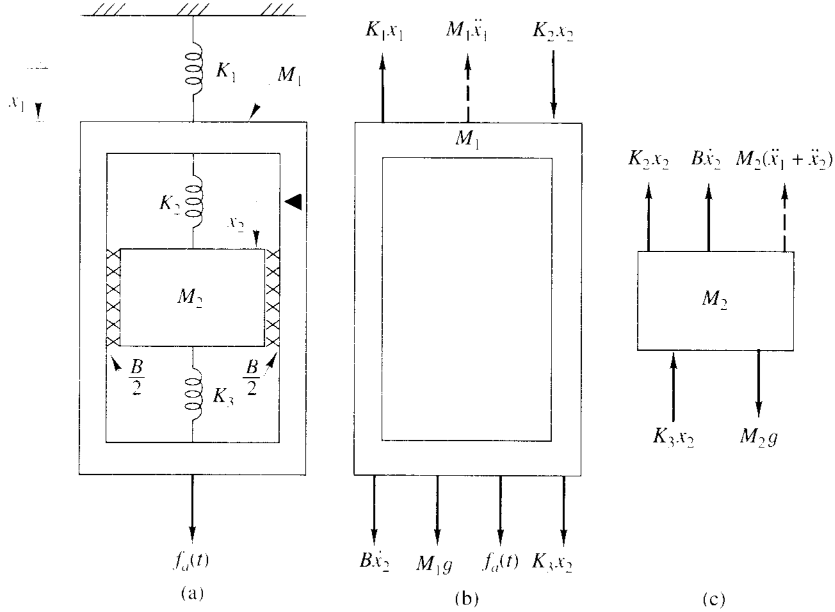


Figure 2.17 (a) Translational system for Example 2.6. (b), (c) Free-body diagrams.

EXAMPLE 2.6

For the system shown in Figure 2.17(a), x_1 and x_2 denote the elongations of K_1 and K_2, respectively. Note that x_1 is the displacement of mass M_1 with respect to a fixed reference but that x_2 is the relative displacement of M_2 with respect to M_1. When $x_1 = x_2 = 0$, all three springs shown in the figure are neither stretched nor compressed. Draw the free-body diagram for each mass, including the effect of gravity, and find the differential equations describing the system's behavior. Determine the values of x_1 and x_2 that correspond to the static-equilibrium position, when $f_a(t) = 0$ and when the masses are motionless.

SOLUTION The free-body diagrams are shown in parts (b) and (c) of the figure. Many of the comments made in Example 2.4 also apply to this problem. If x_1 and x_2 are positive, then K_1 and K_2 are stretched and K_3 is compressed. Under these circumstances, K_1 exerts an upward force on M_1, and K_2 and K_3 exert downward forces on M_1. The relative velocity of M_2 with respect to M_1 is $\dot{x}_2$, so frictional forces of $B\dot{x}_2$ are exerted downward on M_1 and upward on M_2. The inertial force on M_2 is proportional to its absolute acceleration, which is $\ddot{x}_1 + \ddot{x}_2$. Summing the forces on each of the free-body diagrams gives

$$\begin{aligned} M_1\ddot{x}_1 + K_1x_1 - B\dot{x}_2 - (K_2+K_3)x_2 &= M_1g + f_a(t) \\ M_2\ddot{x}_1 + M_2\ddot{x}_2 + B\dot{x}_2 + (K_2+K_3)x_2 &= M_2g \end{aligned} \tag{25}$$

This pair of coupled equations has two unknown variables. If the element values and $f_a(t)$ are known, and if the necessary initial conditions are given, then (25) can be solved for x_1 and x_2 as functions of time by the methods discussed in later chapters. Because x_2 is a relative displacement, the total displacement of M_2 is $x_1 + x_2$.

To find the displacements x_{1_0} and x_{2_0} that correspond to the static-equilibrium position, we replace $f_a(t)$ and all the displacement derivatives by zero. Then (25) reduces to

$$\begin{aligned} K_1 x_{1_0} - (K_2+K_3)x_{2_0} &= M_1 g \\ (K_2+K_3)x_{2_0} &= M_2 g \end{aligned} \tag{26}$$

from which

$$\begin{aligned} x_{1_0} &= \frac{(M_1+M_2)g}{K_1} \\ x_{2_0} &= \frac{M_2 g}{K_2+K_3} \end{aligned} \tag{27}$$

If we want the differential equations in terms of displacements z_1 and z_2 measured with respect to the equilibrium conditions given by (27), then we can write $x_1 = x_{1_0} + z_1$ and $x_2 = x_{2_0} + z_2$. Substituting these expressions into (25) and using (26), we find that

$$\begin{aligned} M_1\ddot{z}_1 + K_1 z_1 - B\dot{z}_2 - (K_2+K_3)z_2 &= f_a(t) \\ M_2\ddot{z}_1 + M_2\ddot{z}_2 + B\dot{z}_2 + (K_2+K_3)z_2 &= 0 \end{aligned}$$

As expected, these equations are similar to (25) except for the absence of the gravitational forces.

The last two examples illustrate the choice that we have when choosing references for linear systems with vertical motion. We will generally define displacements from the positions corresponding to unstressed springs. This means that we must show the Mg force on the free-body diagram for any mass that can move vertically. If, however, we were to define displacements from the static-equilibrium positions caused by the gravitational forces, then we would not show the Mg forces on our diagrams. The modeling equations will have the same form as for the first method, except that the Mg terms will be missing. The first method can be used even for nonlinear springs; however, the second cannot.

The Ideal Pulley

A pulley can be used to change the direction of motion in a translational mechanical system. Frequently, part of the system then moves horizontally and the rest moves vertically. The basic pulley consists of a cylinder that can rotate about its center and that has a cable resting on its surface. An ideal pulley has no mass and no friction. We assume that there is no slippage between the cable and the surface of the cylinder—that is, that they both move with the same velocity. We also assume that the cable is always in tension but that it cannot stretch. If we need to consider a cable that can stretch, we can approximate that effect by showing a separate spring leading to an ideal cable. If the pulley is not ideal, then its mass and any frictional effects must be considered, as will be discussed in Chapter 5. The action of an ideal pulley is illustrated in the following example.

EXAMPLE 2.7

Find and compare the equations describing the systems shown in Figure 2.18(a) and Figure 2.19(a). Let $x_1 = x_2 = 0$ correspond to the condition when the springs are neither stretched nor compressed.

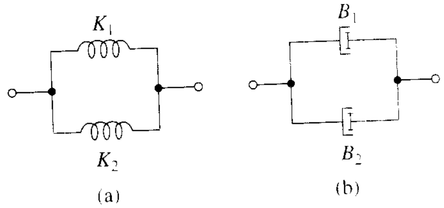


Figure 2.21 Parallel combinations. (a) $K_{eq} = K_1 + K_2$. (b) $B_{eq} = B_1 + B_2$.

Using (33) to cancel some of the terms in this equation, we obtain

$$M\ddot{z} + B\dot{z} + (K_1 + K_2)z = f_a(t)$$

which is identical to (30) except for the use of z in place of x. Thus (31) is valid for the equivalent spring constant even if the unstretched lengths of the springs are different, provided that x is interpreted as the displacement beyond the static-equilibrium position given by (34).

Two parallel springs or dashpots have their respective ends joined, as shown in Figure 2.21. From the last example, we see that for the parallel combination of two springs,

$$K_{eq} = K_1 + K_2 \tag{35}$$

Similarly, it can be shown that for two dashpots in parallel, as in part (b) of Figure 2.21,

$$B_{eq} = B_1 + B_2 \tag{36}$$

The formulas for parallel stiffness or friction elements can be extended to situations that may seem somewhat different from those in Figure 2.21. The key requirement for parallel elements is that respective ends move with the same displacement. The individual ends need not be tied directly together in the figure depicting the system. Although one pair of ends may sometimes be connected to a fixed surface, in other cases both pairs of ends may be free to move.

EXAMPLE 2.9

Find the equation describing the motion of the mass in the translational system shown in Figure 2.22(a). Show that the two springs can be replaced by a single equivalent spring, and the three friction elements by an equivalent element.

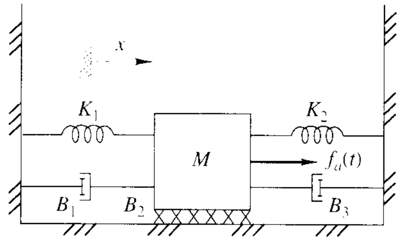
SOLUTION Note that each of the springs has one side attached to the mass and the other attached to the fixed surface on the perimeter of the diagram. When x is positive, K_1 is stretched and K_2 compressed. Thus the springs will exert forces of K_1x and K_2x on the mass to the left, as shown on the free-body diagram in Figure 2.22(b).

For each of the three friction elements, one side moves with the velocity of the mass and the other side is stationary. When $\dot{x}$ is positive, each of these elements exerts a retarding force on the mass to the left. Summing the forces shown on the free-body diagram, we have

$$M\ddot{x} + (B_1 + B_2 + B_3)\dot{x} + (K_1 + K_2)x = f_a(t)$$

With $K_{eq} = K_1 + K_2$ and $B_{eq} = B_1 + B_2 + B_3$, this equation becomes

$$M\ddot{x} + B_{eq}\dot{x} + K_{eq}x = f_a(t)$$



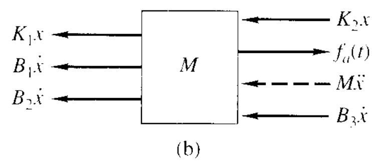


Figure 2.22 (a) Translational system with parallel stiffness and friction elements. (b) Free-body diagram.

When the parallel combinations of stiffness and friction elements in Figure 2.22(a) are replaced by equivalent elements, the system reduces to that shown in Figure 2.12(d), which is described by (15).

Series Combinations

Two springs or dashpots are said to be in **series** if they are joined at only one end of each element and if there is no other element connected to their common junction. The following example has a series combination of two springs and also illustrates the application of D'Alembert's law to a massless junction.

EXAMPLE 2.10

When $x_1 = x_2 = 0$, the two springs shown in Figure 2.23(a) are neither stretched nor compressed. Draw free-body diagrams for the mass M and for the massless junction A, and then write the equations describing the system. Show that the motion of point A is not independent of that of the mass M and that x_1 and x_2 are directly proportional to one another. Finally, find K_{eq} for a single spring that could replace the combination of K_1 and K_2.

SOLUTION The free-body diagrams are shown in parts (b) and (c) of the figure. Because there is no mass at point A, there is no inertial force in its free-body diagram. Summing the forces for each diagram gives

$$M\ddot{x}_1 + B\dot{x}_1 + K_1(x_1 - x_2) = f_a(t)$$
$$K_2x_2 = K_1(x_1 - x_2)$$

Solving the second equation for x_2 in terms of x_1 gives

$$x_2 = \left(\frac{K_1}{K_1 + K_2}\right)x_1$$

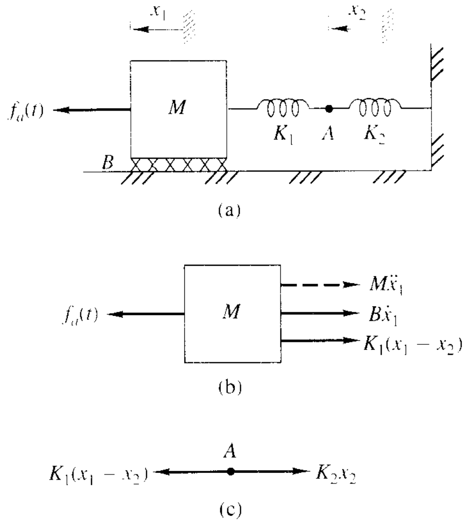


Figure 2.23 (a) Translational system with a massless junction. (b), (c) Free-body diagrams.

which shows that the two displacements are proportional to one another. Substituting this expression back into the first equation, we have

$$M\ddot{x}_1 + B\dot{x}_1 + K_1\left[1 - \frac{K_1}{K_1 + K_2}\right]x_1 = f_a(t)$$

from which

$$M\ddot{x}_1 + B\dot{x}_1 + \frac{K_1K_2}{K_1 + K_2}x_1 = f_a(t)$$

This equation describes the system formed when the two springs in Figure 2.23(a) are replaced by a single spring for which

$$K_{\text{eq}} = \frac{K_1K_2}{K_1 + K_2} \tag{37}$$

Series combinations of stiffness and friction elements are shown in Figure 2.24. It is assumed that no other element is connected to the common junctions. For the two springs in part (a) of the figure, the equivalent spring constant is given by (37). For two dashpots in series, as in part (b) of the figure, it can be shown that

$$B_{\text{eq}} = \frac{B_1B_2}{B_1 + B_2} \tag{38}$$

In order to reduce certain combinations of springs or dashpots to a single equivalent element, we may have to use the rules for both parallel and series combinations. The following example illustrates the procedure for two different combinations of dashpots.

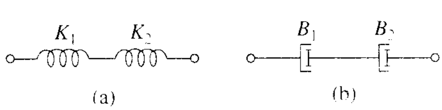


Figure 2.24 Series combinations. (a) $K_{\text{eq}} = K_1K_2/(K_1 + K_2)$. (b) $B_{\text{eq}} = B_1B_2/(B_1 + B_2)$.

*2.4. For the system shown in Figure P2.4, draw the free-body diagram for each mass and write the differential equations describing the system.

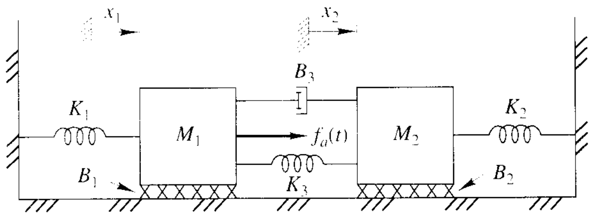


Figure P2.4

2.5. Repeat Problem 2.4 for the system shown in Figure P2.5.

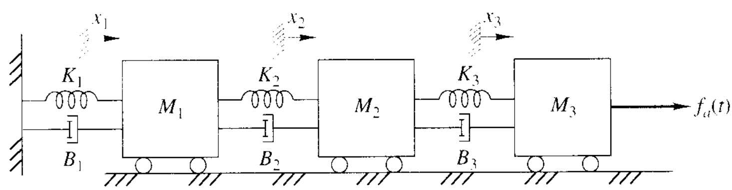


Figure P2.5

2.6. In the mechanical system shown in Figure P2.6, the spring forces are zero when $x_1 = x_2 = x_3 = 0$. Let the base be stationary so that $x_3(t) = 0$ for all values of t. Draw free-body diagrams and write a pair of coupled differential equations that govern the motion when the only input is $f_a(t)$.

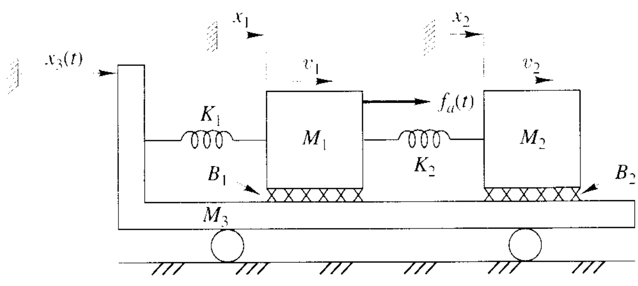


Figure P2.6

*2.7. For the system shown in Figure P2.7, the springs are undeflected when $x_1 = x_2 = 0$. The input is $x_2(t)$, the displacement of the left edge of M_2.

a. Write the equation governing the motion of M_1.

b. Write an expression for the force f_2, positive sense to the right, that must be applied to M_2 in order to achieve the displacement $x_2(t)$.

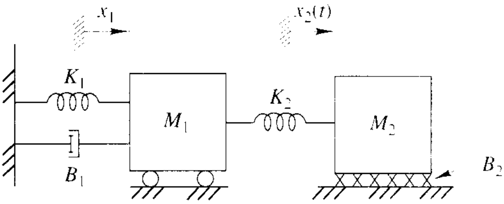


Figure P2.7

2.8. Repeat Problem 2.7 for the system shown in Figure P2.8.

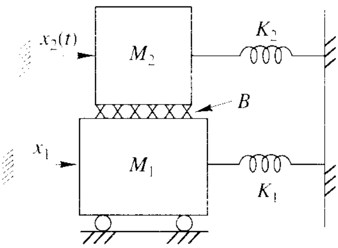


Figure P2.8

2.9. Repeat Problem 2.6 with the following changes:

a. The displacement $x_3(t)$ is the input and the applied force $f_a(t)$ is zero for all time.

b. Write an expression for the force f_3, positive sense to the right, that must be applied to M_3 in order to move M_3 with the specified displacement $x_3(t)$.

***2.10.** For the system shown in Figure P2.10, the distance between masses M_1 and M_2 is $A + x_2$, where A is a constant. The springs are undeflected when $x_1 = x_2 = 0$. Draw free-body diagrams and write the differential equations for the system.

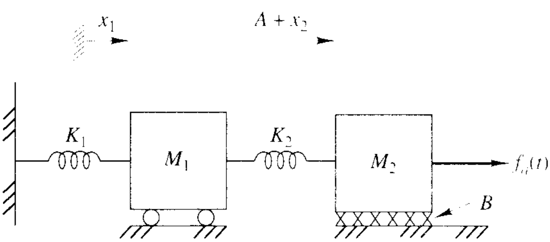


Figure P2.10

2.11. For the system shown in Figure P2.11, the input is the displacement $x_1(t)$. The springs are undeflected when $x_1 = x_2 = x_3 = 0$. The variable x_2 represents the displacement of M_2 with respect to M_1. Write the mathematical model to describe the motion of the masses as a set of coupled differential equations. Include appropriate free-body diagrams.

2.1 $M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1 + K_2)x_1 - K_2x_2 = f_a(t),$
$-K_2x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 = 0$

2.4 $M_1\ddot{x}_1 + (B_1 + B_3)\dot{x}_1 + (K_1 + K_3)x_1 - B_3\dot{x}_2 - K_3x_2 = f_a(t),$
$-B_3\dot{x}_1 - K_3x_1 + M_2\ddot{x}_2 + (B_2 + B_3)\dot{x}_2 + (K_2 + K_3)x_2 = 0$

2.7

a. $M_1\ddot{x}_1 + B_1\dot{x}_1 + (K_1 + K_2)x_1 = K_2x_2(t)$

b. $f_2 = -K_2x_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2(t)$

2.10 $M_1\ddot{x}_1 + K_1x_1 - K_2x_2 = 0$
$M_2\ddot{x}_1 + B\dot{x}_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 = f_a(t)$

2.12 $M_1\ddot{x}_1 + K_1x_1 + B\dot{x}_2 = K_1x_3(t)$
$-M_2\ddot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 = 0$

2.15

a. $M_1\ddot{x}_1 + B\dot{x}_1 + K_1x_1 - B\dot{x}_2 - K_1x_2 = M_1g$
$-B\dot{x}_1 - K_1x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + (K_1 + K_2)x_2 = M_2g + f_a(t)$

b. $x_{1_0} = [(M_1 + M_2)/K_2 + (M_1/K_1)]g$
$x_{2_0} = [(M_1 + M_2)/K_2]g$

c. $M_1\ddot{z}_1 + B\dot{z}_1 + K_1z_1 - B\dot{z}_2 - K_1z_2 = 0$
$-B\dot{z}_1 - K_1z_1 + M_2\ddot{z}_2 + B\dot{z}_2 + (K_1 + K_2)z_2 = f_a(t)$

2.18

a. $M_1\ddot{x}_1 + B\dot{x}_1 + 2Kx_1 - B\dot{x}_2 - Kx_2 = M_1g$
$-B\dot{x}_1 - Kx_1 + M_2\ddot{x}_2 + B\dot{x}_2 + 3Kx_2 - Kx_3 = M_2g$
$-Kx_2 + M_3\ddot{x}_3 + Kx_3 = M_3g - f_a(t)$

b. $x_{1_0} = (2M_1 + M_2 + M_3)g/(3K)$
$x_{2_0} = (M_1 + 2M_2 + 2M_3)g/(3K)$
$x_{2_0} - x_{1_0} = (-M_1 + M_2 + M_3)g/(3K)$
$x_{3_0} - x_{2_0} = M_3g/K$

EXAMPLE 5.10

Find the state-variable equations for the system shown in Figure 5.26(a), in which the pa
of gears couples two similar subsystems.

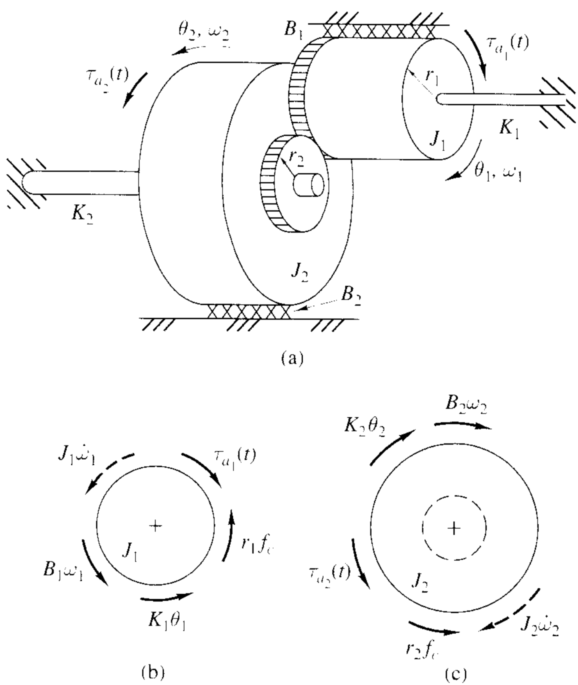


Figure 5.26 (a) System for Example 5.10. (b), (c) Free-body diagrams.

SOLUTION Because of the two moments of inertia and the two shafts, it might appear that we could choose ω_1, ω_2, θ_1, and θ_2 as state variables. However, θ_1 and θ_2 are related by the gear ratio, as are ω_1 and ω_2. Because the state variables must be independent, either θ_1 and ω_1 or θ_2 and ω_2 constitute a suitable set.

The free-body diagrams for each of the moments of inertia are shown in Figure 5.26(b) and Figure 5.26(c). As in Example 5.9, f_c represents the contact force between the two gears. Summing the torques on each of the free-body diagrams gives

$$J_1\dot{\omega}_1 + B_1\omega_1 + K_1\theta_1 + r_1 f_c = \tau_{a_1}(t) \tag{55a}$$

$$J_2\dot{\omega}_2 + B_2\omega_2 + K_2\theta_2 - r_2 f_c = \tau_{a_2}(t) \tag{55b}$$

By the geometry of the gears,

$$\begin{aligned} \theta_1 &= N\theta_2 \\ \omega_1 &= N\omega_2 \end{aligned} \tag{56}$$

where $N = r_2/r_1$.

Selecting θ_2 and ω_2 as the state variables, we can write $\dot{\theta}_2 = \omega_2$ as the first state-variable equation and combine (55) and (56) to obtain the required equation for $\dot{\omega}_2$ in terms of θ_2, ω_2, $\tau_{a_1}(t)$, and $\tau_{a_2}(t)$. We first solve (55b) for f_c and substitute that expression into (55a). Then, substituting (56) into the result gives

$$(J_2 + N^2J_1)\dot{\omega}_2 + (B_2 + N^2B_1)\omega_2 + (K_2 + N^2K_1)\theta_2 - N\tau_{a_1}(t) - \tau_{a_2}(t) = 0 \tag{57}$$

At this point, it is convenient to define the parameters

$$\begin{aligned} J_{2_{eq}} &= J_2 + N^2J_1 \\ B_{2_{eq}} &= B_2 + N^2B_1 \\ K_{2_{eq}} &= K_2 + N^2K_1 \end{aligned} \tag{58}$$

which can be viewed as the combined moment of inertia, damping coefficient, and stiffness constant, respectively, when the combined system is described in terms of the variables θ_2 and ω_2. For example, it is common to say that N^2J_1 is the equivalent inertia of disk 1 when that inertia is reflected to shaft 2. Similarly, N^2B_1 and N^2K_1 are the reflected viscous-friction coefficient and stiffness constant, respectively. Hence the parameters $J_{2_{eq}}$, $B_{2_{eq}}$, and $K_{2_{eq}}$ defined in (58) are the sums of the parameters associated with shaft 2 and the corresponding parameters reflected from shaft 1.

With the new notation, we can rewrite (57) as

$$J_{2_{eq}}\dot{\omega}_2 + B_{2_{eq}}\omega_2 + K_{2_{eq}}\theta_2 - N\tau_{a_1}(t) - \tau_{a_2}(t) = 0 \tag{59}$$

and the state-variable equations are

$$\begin{aligned} \dot{\theta}_2 &= \omega_2 \\ \dot{\omega}_2 &= \frac{1}{J_{2_{eq}}}\left[-K_{2_{eq}}\theta_2 - B_{2_{eq}}\omega_2 + N\tau_{a_1}(t) + \tau_{a_2}(t)\right] \end{aligned} \tag{60}$$

Note that the driving torque $\tau_{a_1}(t)$ applied to shaft 1 has the value $N\tau_{a_1}(t)$ when reflected to shaft 2.

If we wanted the system model in terms of θ_1 and ω_1, straightforward substitutions would lead to the equations

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 \\ \dot{\omega}_1 &= \frac{1}{J_{1_{eq}}}\left[-K_{1_{eq}}\theta_1 - B_{1_{eq}}\omega_1 + \tau_{a_1}(t) + \frac{1}{N}\tau_{a_2}(t)\right] \end{aligned}$$

where the combined parameters with the elements associated with shaft 2 reflected to shaft 1 are

$$J_{1_{eq}} = J_1 + \frac{1}{N^2}J_2$$

$$B_{1_{eq}} = B_1 + \frac{1}{N^2}B_2$$

$$K_{1_{eq}} = K_1 + \frac{1}{N^2}K_2$$

With the experience we have gained in deriving the mathematical models for separate translational and rotational systems, it is a straightforward matter to treat systems that combine both types of elements. The next example uses a rack and a pinion gear to convert rotational motion to translational motion. The final example combines translational and rotational systems via a cable attached to a disk.

EXAMPLE 5.11

Derive the state-variable model for the system shown in Figure 5.27. The moment of inertia J represents the rotor of a motor on which an applied torque $\tau_a(t)$ is exerted. The rotor is connected by a flexible shaft to a pinion gear of radius R that meshes with the linear rack. The rack is rigidly attached to the mass M, which might represent the bed of a milling machine. The outputs of interest are the displacement and velocity of the rack and the contact force between the rack and the pinion.

SOLUTION The free-body diagrams for the moment of inertia J, the pinion gear, and the mass M are shown in Figure 5.28. The contact force between the rack and the pinion is denoted by f_c. Forces and torques that will not appear in the equations of interest (such as the vertical force on the mass and the bearing forces on the rotor and pinion gear) have been omitted. Summing the torques in Figure 5.28(a) and Figure 5.28(b) and the forces in Figure 5.28(c) yields the three equations

$$J\dot{\omega} + B_1\omega + K(\theta - \theta_A) - \tau_a(t) = 0 \tag{61a}$$

$$Rf_c - K(\theta - \theta_A) = 0 \tag{61b}$$

$$M\dot{v} + B_2 v - f_c = 0 \tag{61c}$$

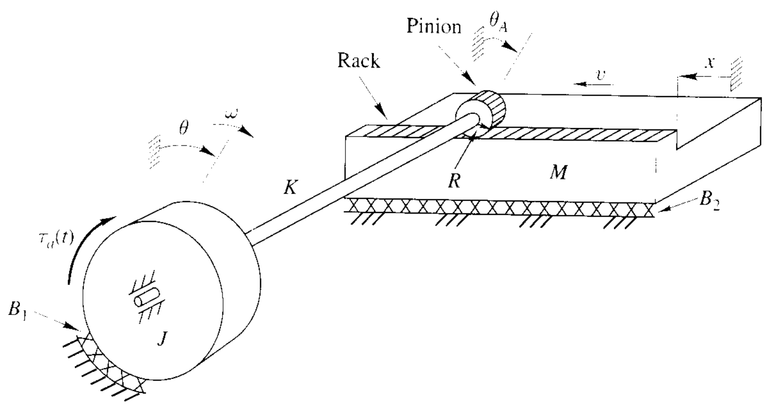


Figure 5.27 System for Example 5.11 with rack and pinion gear.

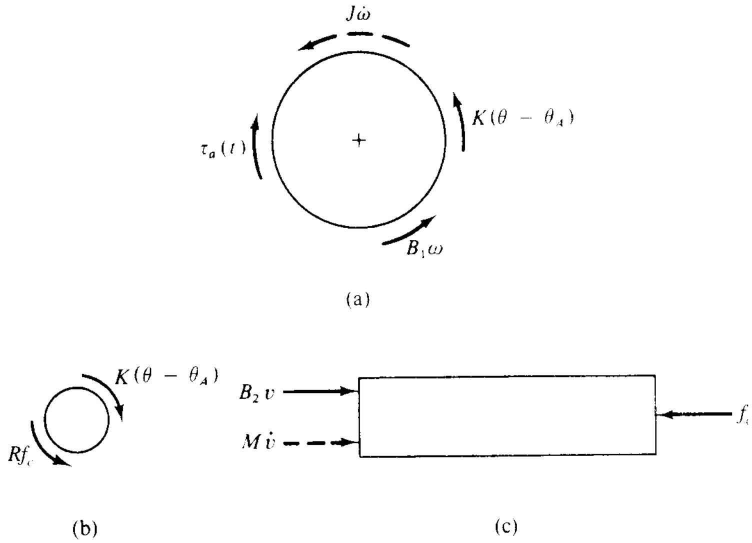


Figure 5.28 Free-body diagrams for Example 5.11. (a) Rotor. (b) Pinion gear. (c) Mass.

In addition, the geometric relationship

$$R\theta_A = x \tag{62}$$

must hold because of the contact between the rack and the pinion gear.

The fact that there are three energy-storing elements corresponding to the parameters J, M, and K suggests that the three variables ω, v, and $\theta_R = \theta - \theta_A$ might constitute a satisfactory set of state variables. However, the displacement x of the mass, which is usually of interest and which is one of the specified outputs, cannot be expressed as an algebraic function of ω, v, θ_R, and the input. Thus we need four state variables, which we choose to be θ, ω, x, and v. Using (62) to eliminate θ_A in (61a) gives

$$J\dot{\omega} + B_1\omega + K\theta - \frac{K}{R}x - \tau_a(t) = 0$$

and using (62) and (61b) to eliminate f_c in (61c) results in

$$M\dot{v} + B_2 v + \frac{K}{R^2}x - \frac{K}{R}\theta = 0$$

Thus the state-variable equations are

$$\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= \frac{1}{J}\left[-K\theta - B_1\omega + \frac{K}{R}x + \tau_a(t)\right] \\
\dot{x} &= v \\
\dot{v} &= \frac{1}{M}\left(\frac{K}{R}\theta - \frac{K}{R^2}x - B_2 v\right)
\end{aligned} \tag{63}$$

The outputs x and v are also state variables. The output equation for f_c can be found by substituting (62) into (61b). It is

$$f_c = \frac{K}{R}\left(\theta - \frac{x}{R}\right)$$

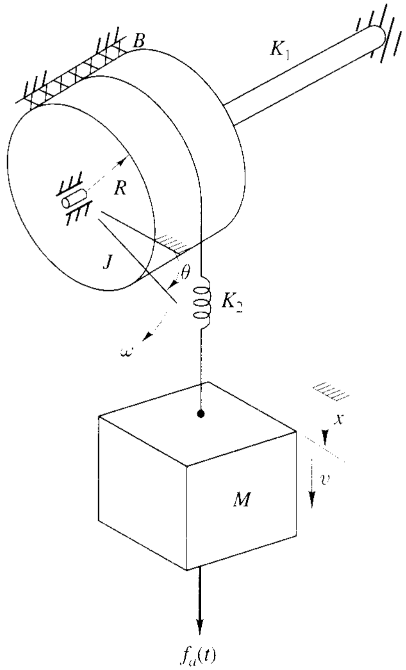


Figure 5.29 System for Example 5.12 with translational and rotational elements.

EXAMPLE 5.12

In the system shown in Figure 5.29, the mass and spring are connected to the disk by a flexible cable. Actually, the spring might be used to represent the stretching of the cable. The mass M is subjected to the external force $f_a(t)$ in addition to the gravitational force. Let θ and x be measured from references corresponding to the position where the shaft K_1 is not twisted and the spring K_2 is not stretched. Find the state-variable model, treating $f_a(t)$ and the weight of the mass as inputs and the angular displacement θ and the tensile force in the cable as outputs.

SOLUTION The free-body diagrams for the disk and the mass are shown in Figure 5.30, where f_2 denotes the force exerted by the spring. The downward displacement of the top end of the spring is $R\theta$, so

$$f_2 = K_2(x - R\theta) \tag{64}$$

Because of the four energy-storing elements corresponding to the parameters K_1, J, K_2, and M, we select θ, ω, x, and v as the state variables. From the free-body diagrams and with (64), we can write

$$J\dot{\omega} + B\omega + K_1\theta - RK_2(x - R\theta) = 0 \tag{65a}$$

$$M\dot{v} + K_2(x - R\theta) = f_a(t) + Mg \tag{65b}$$

Note that the reaction force $f_2 = K_2(x - R\theta)$ of the cable on the mass is not the same as the total external force $f_a(t) + Mg$ on the mass. As indicated by (65b), the difference is the inertial force $M\dot{v}$. Only if the mass were negligible would the external force be transmitted directly through the spring. From (65) and the identities $\dot{\theta} = \omega$ and $\dot{x} = v$, we can write the

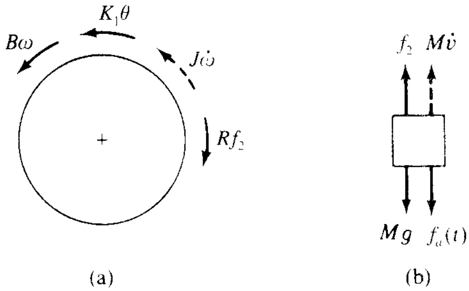


Figure 5.30 Free-body diagrams for Example 5.12. (a) Disk. (b) Mass.

state-variable equations

$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= \frac{1}{J}\left[-(K_1 + K_2R^2)\theta - B\omega + K_2Rx\right] \\
\dot{x} &= v \\
\dot{v} &= \frac{1}{M}[K_2R\theta - K_2x + f_a(t) + Mg]
\end{aligned}
\tag{66}
$$

The only output that is not a state variable is the tensile force in the cable, for which the output equation is given by (64).

In order to emphasize the effect of the weight Mg, suppose that $f_a(t) = 0$ and that the mass and disk are not moving. Let θ_0 denote the constant angular displacement of the disk and x_0 the constant displacement of the mass under these conditions. Then (65) becomes

$$
\begin{aligned}
K_1\theta_0 &= RK_2(x_0 - R\theta_0) \\
K_2(x_0 - R\theta_0) &= Mg
\end{aligned}
\tag{67}
$$

from which

$$
\begin{aligned}
\theta_0 &= \frac{RMg}{K_1} \\
x_0 &= \frac{Mg}{K_2} + \frac{R^2Mg}{K_1}
\end{aligned}
\tag{68}
$$

These expressions represent the constant displacements caused by the gravitational force Mg.

Now reconsider the case where $f_a(t)$ is nonzero and where the system is in motion. Let

$$
\begin{aligned}
\theta &= \theta_0 + \phi \\
x &= x_0 + z
\end{aligned}
\tag{69}
$$

so that ϕ and z represent the additional angular and vertical displacements caused by the input $f_a(t)$. Note that $\omega = \dot{\theta} = \dot{\phi}$ and $v = \dot{x} = \dot{z}$. Substituting (69) into (65) gives

$$
\begin{aligned}
J\dot{\omega} + B\omega + K_1(\theta_0 + \phi) - RK_2(x_0 + z - R\theta_0 - R\phi) &= 0 \\
M\dot{v} + K_2(x_0 + z - R\theta_0 - R\phi) &= f_a(t) + Mg
\end{aligned}
$$

Using (67) to cancel those terms involving θ_0, x_0, and Mg, we are left with

$$
\begin{aligned}
J\dot{\omega} + B\omega + K_1\phi - RK_2(z - R\phi) &= 0 \\
M\dot{v} + K_2(z - R\phi) &= f_a(t)
\end{aligned}
$$

so the corresponding state-variable equations are

$$
\begin{aligned}
\dot{\phi} &= \omega \\
\dot{\omega} &= \frac{1}{J}[-(K_1 + K_2R^2)\phi - B\omega + K_2Rz] \\
\dot{z} &= v \\
\dot{v} &= \frac{1}{M}[K_2R\phi - K_2z + f_a(t)]
\end{aligned} \tag{70}
$$

We see that (66) and (70) have the same form, except that in the latter case the term Mg is missing and θ and x have been replaced by ϕ and z. As long as the stiffness elements are linear, we can ignore the gravitational force Mg if we measure all displacements from the static-equilibrium positions corresponding to no inputs except gravity. This agrees with the conclusion reached in Examples 2.5 and 2.6. Note that if one of the desired outputs is the total tensile force in the cable, we must substitute (69) into (64) to get

$$
f_2 = K_2(x_0 - R\theta_0) + K_2(z - R\phi) \tag{71}
$$

where x_0 and θ_0 are given by (68). The first of the two terms in (71) is the constant tensile force resulting only from the weight of the mass. The second term is the additional tensile force caused by the input $f_a(t)$.

***2.4.** For the system shown in Figure P2.4, draw the free-body diagram for each mass and write the differential equations describing the system.

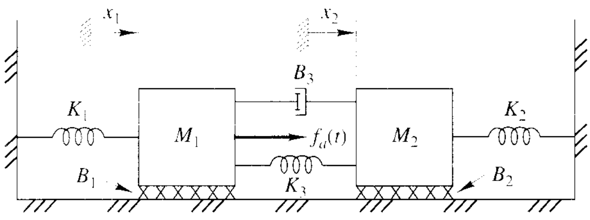


Figure P2.4

2.5. Repeat Problem 2.4 for the system shown in Figure P2.5.

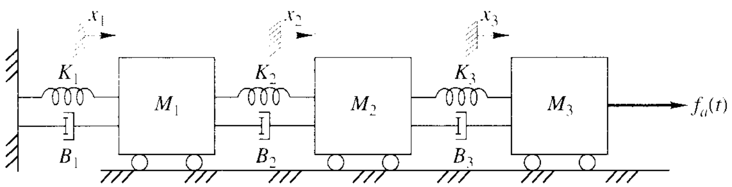


Figure P2.5

2.6. In the mechanical system shown in Figure P2.6, the spring forces are zero when $x_1 = x_2 = x_3 = 0$. Let the base be stationary so that $x_3(t) = 0$ for all values of t. Draw free-body diagrams and write a pair of coupled differential equations that govern the motion when the only input is $f_a(t)$.

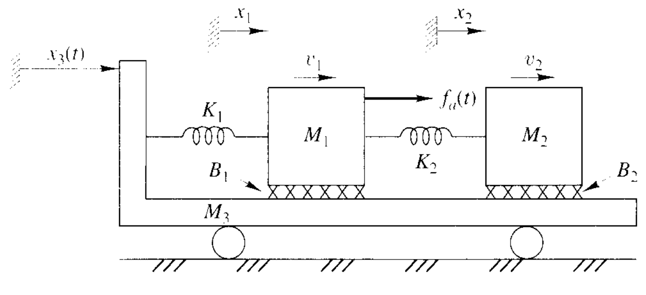


Figure P2.6

***2.7.** For the system shown in Figure P2.7, the springs are undeflected when $x_1 = x_2 = 0$. The input is $x_2(t)$, the displacement of the left edge of M_2.

a. Write the equation governing the motion of M_1.

b. Write an expression for the force f_2, positive sense to the right, that must be applied to M_2 in order to achieve the displacement $x_2(t)$.