MA 101 - Calculus I

Problem Sheet : Functional Series

1. Determine the exact intervals of convergence for the following:

   (i) $\sum n^2 x^n$ (ii) $\sum \dfrac{2^n}{n^2} x^n$ (iii) $\sum \dfrac{x^n}{n^n}$ (iv) $\sum \dfrac{1}{(n+1)^2 2^n} x^n$ (v) $\sum \dfrac{(-1)^n}{n^2 4^n} x^n$

   (vi) $\sum \sqrt{n} x^n$ (vii) $\sum \dfrac{3^n}{n.4^n} x^n$ (viii) $\sum \dfrac{n^3}{3^n} x^n$ (ix) $\sum \dfrac{3^n}{\sqrt{n}} x^{2n+1}$ (x) $\sum x^{n!}$

2. Consider a power series $\sum a_n x^n$ with radius of convergence R.

   (a) Prove that if all the coefficients a_n are integers and if infinitely many of them are non-zero, then $R \leqslant 1$.

   (b) If $|a_n|^{\frac{1}{n}} \to l$, then $R = \begin{cases} 0, & \text{if } l = \infty \\ \infty, & \text{if } l = 0 \\ \dfrac{1}{l}, & \text{if } 0 < l < \infty \end{cases}$.

   (c) If $a_n \neq 0$ for all large n and $\dfrac{|a_{n+1}|}{|a_n|} \to l$, the conclusion of (b) above still holds.

   (d) Verify the above with the following series whose co-efficients are given as:

   (i) $a_n = \dfrac{n^3}{3^n}$ (ii) $a_n = \dfrac{2^n}{n!}$ (iii) $a_{2n-1} = \dfrac{1}{4^n}; a_{2n} = \dfrac{1}{9^n}$.

3. Consider a power series $\sum a_n x^n$ with a finite radius of convergence R. Prove that if all the coefficients $a_n \geqslant 0$ for all n and if the series converges at R, then the series also converegs at $-R$.

4. For each $n \in \mathbb{N}$, let $f_n(x) = (\cos x)^n$. Show that

   (a) each f_n is continuous.

   (b) $\lim f_n(x) = 0$ unless x is a multiple of π.

   (c) $\lim f_n(x) = 1$ if x is an even multiple of π.

   (d) $\lim f_n(x)$ does not exist if x is an odd multiple of π.

5. For each $n \in \mathbb{N}$, let $f_n(x) = \frac{1}{n} \sin x$. Show that

   (a) each f_n is differentiable.

   (b) $\lim f_n(x) = 0$ for all $x \in \mathbb{R}$.

   (c) $\lim f_n'(x)$ need not exist (for instance at $x = \pi$).

6. For each $n \in \mathbb{N}$, let $f_n(x) = nx^n$ for $x \in [0, 1]$. Show that

   (a) $\lim f_n(x) = 0$ for all $x \in [0, 1)$.

   (b) $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1$.

7. For each $n \in \mathbb{N}$, let $f_n(x) = \left(x - \dfrac{1}{n}\right)^2$ for $x \in [0, 1]$.

   (a) Find $f(x) = \lim f_n(x)$.

   (b) Does (f_n) converge pointwise on $[0, 1]$?

   (c) Does it also converge uniformly ?

8. Obtain the Taylor series of the following functions about the indicated point a:

   (i) $\tan x$; $a = \frac{\pi}{4}$ (ii) $e^{\sin x}$; $a = 0$ (iii) $\ln(\cos x)$; $a = 0$ (iv) $\cos^2 x$; $a = 0$

   (v) $\cos^2 x$; $a = \frac{\pi}{4}$ (vi) $\dfrac{1}{x^3}$; $a = 7$ (vii) $\dfrac{1}{1+x^4}$; $a = 3$ (viii) $\tan^{-1} x$; $a = 0$

9. Suppose that $f(x)$ is differentiable on an interval I centered at $x = a$ and that

$$g(x) = b_0 + b_1(x-a) + \ldots + b_n(x-a)^n,$$

is a polynomial of degree n with constant coefficients $b_0, b_1, \ldots, b_n$. Let $E(x) = f(x) - g(x)$. Show that if

(a) $E(a) = 0$ (i.e., the approximation error is zero at $x = a$)

(b) $\lim\limits_{x \to a} \dfrac{E(x)}{(x-a)^n} = 0$ (i.e., the error is negligible when compared to $(x-a)^n$)

then $b_k = \dfrac{f'(a)}{k!}$, $k = 0, \ldots, n$. Thus the Taylor's polynomial is the only polynomial of degree less than or equal to n whose error is zero at $x = a$ and negligible when compared to $(x-a)^n$.

10. Find the Fourier Series of the following functions:

(i) $f(x) = x^3;\ -\pi \leqslant x \leqslant \pi$

(ii) $f(x) = x + |x|;\ -\pi \leqslant x \leqslant \pi$

(iii) $f(x) = \begin{cases} 1, & -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < x \leqslant \frac{3\pi}{2} \end{cases}$

(iv) $f(x) = \begin{cases} x, & -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x \leqslant \frac{3\pi}{2} \end{cases}$

(v) $f(x) = \begin{cases} 1, & -1 \leqslant x \leqslant 0 \\ -1, & 0 < x \leqslant 1 \end{cases}$

(vi) $f(x) = \begin{cases} x, & -2 \leqslant x < 0 \\ \pi - x, & 0 < x \leqslant 2 \end{cases}$

(vii) $f(x) = x^3;\ -2 \leqslant x \leqslant 2$

(viii) $f(x) = x + |x|;\ \frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

11. Expand the following functions such that we obtain (i) only a sine series and (ii) only a cosine series:

(i) $f(x) = x;\ 0 \leqslant x \leqslant 2\pi$

(ii) $f(x) = \pi - x;\ 0 \leqslant x \leqslant \pi$

(iii) $f(x) = \sin^2 x;\ 0 \leqslant x \leqslant \pi$

(iv) $f(x) = e^x;\ 0 \leqslant x \leqslant L$

(v) $f(x) = x^2;\ 0 \leqslant x \leqslant L$

(vi) $f(x) = 4 - x^2;\ 0 \leqslant x \leqslant L$

12. Find the Fourier series of $f(x) = \dfrac{(\pi - x)^2}{4}$ on $0 \leqslant x \leqslant 2\pi$ and hence show that

(a) $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots$

(b) $\dfrac{\pi^2}{12} = \dfrac{1}{1^2} - \dfrac{1}{3^2} + \dfrac{1}{5^2} - \ldots$

(c) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

13. Find the Fourier series of $f(x) = \sqrt{1 - \cos x}$ on $(0, 2\pi)$ and hence deduce that $\dfrac{1}{2} = \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{4n^2 - 1}$.

14. Let f be a periodic function with period 2π. Let f_n be the trignometric polynomial of order n given as follows

$$f_n(x) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx.$$

Show that if f_n minimizes the integral of the square of the error in approximating f, viz.,

$$\int_{-\pi}^{\pi} [f(x) - f_n(x)]^2 \, dx,$$

then the coefficients of f_n are given as Fourier coeffecients.