IIT Hyderabad
Department of Mathematics
MA 101 - Calculus I

25, November 2009 | **End Semester** | 180 mins | **50** marks

1. Say True or False, with a brief justification. (**5** marks)

   (a) The series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ is dominated on $0 \leqslant x \leqslant 1$.

   (b) A function f is Riemann integral over $[a, b]$ if and only if f is continuous over $[a, b]$.

   (c) $\int_0^1 x^m(1-x)^n \, dx = \int_0^1 x^n(1-x)^m \, dx$ for any $m > 0, n > 0$.

   (d) $e^{i\theta} = \sin\theta + i \ \cos\theta$.

   (e) $\int_{-\infty}^{+\infty} f(x) \, dx = \lim_{b\to\infty} \int_{-b}^{+b} f(x) \, dx$ for any real function f.

2. Give example of (**5** marks)

   (a) a power series whose radius of convergence is $(-3, 3)$.

   (b) a functional series that is pointwise convergent but not uniformly convergent.

   (c) a function f and an interval $[a, b]$ such that f is not Riemann integral over $[a, b]$.

   (d) an infinitely differentiable function whose Taylor's series does not converge to it.

   (e) two functions $\phi(x)$ and $\psi(x)$ and an interval $[a, b]$ such that $\int_a^b \phi(x) \cdot \psi(x) dx = 0$.

3. Find the radius of convergence of the power series $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$. (**2** marks)

4. Discuss the Maclaurin's series expansion of $f(x) = (1 + x)^m$ for any $m \in \mathbb{R}$ and hence find a series expansion of $\sin^{-1} x$. (**4** marks)

5. Expand $\frac{1}{1 + x^2}$ in powers of x and hence find a power series expansion of $\tan^{-1} x$. (**4** marks)

6. Comment on the convergence of the series $\sum a_n$ where $a_n = \begin{cases} \frac{n}{2^n} & \text{, if } n \text{ is odd} \\ \frac{1}{2^n} & \text{, if } n \text{ is even} \end{cases}$. (**3** marks)

7. Discuss the Gamma function as an improper integral with respect to its convergence and show that $\Gamma(n + 1) = n!$. (**5** marks)

8. Expand in Fourier series the function $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in the interval $[-\pi, \pi]$ and hence show that $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$. (**4** marks)

9. Comment on the convergence of the following integrals: (**4** marks)

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\displaystyle\int_0^\infty x\sin x\ dx$.

10. Consider the function $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$. (**2**+**3** marks)

- Show that $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}(x,y)$ where $(x,y) \neq (0,0)$.
- Is $\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = \dfrac{\partial^2 f}{\partial y \partial x}(0,0)$? Substantiate.

11. Evaluate $\displaystyle\int_a^b e^x\ dx$ by calculating it as the limit of Riemann sum. (**3** marks)

12. Find the *total* area of a figure bounded by $y = x$, $y = 2x$ and the curve $y = x^3$. (**2** marks)

13. Consider the cycloid $x = r(t - \sin t)$; $y = r(1 - \cos t)$. (**1**+**3** marks)

    (a) Find the arc length of one arch $(0 \leqslant t \leqslant 2\pi)$.

    (b) Find the surface area of the solid generated by rotating this arch about the x-axis.

ALL THE BEST