Application Assignment: Filter #114

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1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for filter number 114. This is a bandpass filter whose specifications are available below.

2 Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

2.1 The Digital Filter

1. *Tolerances:* The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

2. *Passband:* The passband of filter number j, j going from 109 to 135 is from $\{3 + 0.6(\text{j-109})\}$kHz to $\{3 + 0.6(\text{j-107})\}$kHz. Since our filter number is 114, substituting $j = 114$ gives the passband range for our bandpass filter as 6 kHz - 7.2 kHz. Hence, the un-normalized discrete time filter passband frequencies are $F_{p1} = 7.2$ kHz and $F_{p2} = 6.0$ kHz. The corresponding normalized digital filter passband frequencies are $\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.3\pi$ and $\omega_{p2} = 2\pi\frac{F_{p2}}{F_s} = 0.25\pi$ kHz. The centre frequency is then given by $\omega_c = \frac{\omega_{p1}+\omega_{p2}}{2} = 0.275\pi$.

3. *Stopband:* The *transition band* for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband. Hence, the un-normalized *stopband* frequencies are $F_{s1} = 7.2 + 0.3 = 7.5$ kHz and $F_{s2} = 6.0 - 0.3 = 5.7$ kHz. The corresponding normalized frequencies are $\omega_{s1} = 0.3125\pi$ and $\omega_{s2} = 0.2375\pi$.

2.2 The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) as $\Omega = \tan \frac{\omega}{2}$. Using this relation, we obtain the analog passband and stopband frequencies as $\Omega_{p1} = 0.5095$, $\Omega_{p2} = 0.4142$ and $\Omega_{s1} = 0.5345$, $\Omega_{s2} = 0.3914$ respectively.

3 The IIR Filter Design

Filter Type: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the *Chebyschev approximation* to design our bandpass IIR filter.

3.1 The Analog Filter

1. *Low Pass Filter Specifications:* If $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{1}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.4594$ and $B = \Omega_{p1} - \Omega_{p2} = 0.0953$. The low pass filter has the passband edge at $\Omega_{Lp} = 1$ and stopband edges at $\Omega_{Ls_1} = 1.4653$ and $\Omega_{Ls_2} = -1.5511$. We choose the stopband edge of the analog low pass filter as $\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.4653$.

2. *The Low Pass Chebyschev Filter Paramters:* The magnitude squared of the Chebyschev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{2}$$

where $c_N(x) = \cosh(N \cosh^{-1} x)$ and the integer N, which is the order of the filter, and ϵ are design paramters. Since $\Omega_{Lp} = 1$, (2) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{3}$$

Also, the design paramters have the following constraints

$$\begin{aligned} \frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \le \epsilon \le \sqrt{D_1}, \\ N \ge \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \end{aligned} \tag{4}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$. After appropriate substitutions, we obtain $N \ge 4$ and $0.3184 \le \epsilon \le 0.6197$. In Figure 1, we plot $|H(j\Omega)|$

for a range of values of ϵ, for $N = 4$. We find that for larger values of ϵ, $|H(j\Omega)|$ decreases in the transition band. We choose $\epsilon = 0.4$ for our IIR filter design.

3. *The Low Pass Chebyschev Filter:* Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \tag{5}$$

where

$$c_4(x) = 8x^4 + 8x^2 + 1. \tag{6}$$

The poles of the frequency response in (2) lying in the left half plane are in general obtained as $r_1 \cos\phi_k + jr_2 \sin\phi_k$, where

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, k = 0, 1, \ldots, N-1$$
$$r_1 = \frac{\beta^2 - 1}{2\beta}, r_2 = \frac{\beta^2 + 1}{2\beta}, \beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{\frac{1}{N}} \tag{7}$$

Thus, for N even, the low-pass stable Chebyschev filter, with a gain G has the form

$$H_{a,LP}(s_L) = \frac{G_{LP}}{\prod_{k=1}^{\frac{N}{2}-1}(s_L^2 - 2r_1\cos\phi_k s_L + r_1^2\cos^2\phi_k + r_2^2\sin^2\phi_k)} \tag{8}$$

Substituting $N = 4$, $\epsilon = 0.5$ and $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$, from (7) and (8), we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \tag{9}$$

In Figure 2 we plot $|H(j\Omega)|$ using (5) and (9), thereby verifying that our low-pass Chebyschev filter design meets the specifications.

4. *The Band Pass Chebyschev Filter:* The analog bandpass filter is obtained from (9) by substituting $s_L = \frac{s^2+\Omega_0^2}{Bs}$. Hence

$$H_{a,BP}(s) = G_{BP}H_{a,LP}(s_L)\Big|_{s_L=\frac{s^2+\Omega_0^2}{Bs}}, \tag{10}$$

where G_{BP} is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that $H_{a,BP}(j\Omega_{p1}) = 1$, we obtain

$$H_{a,BP}(s) = \frac{2.7776 \times 10^{-5}s^4}{s^8 + 0.1055s^7 + 0.8589s^6 + 0.0676s^5 + 0.2735s^4 + 0.0143s^3 + 0.0383s^2 + 0.001s + 0.002} \tag{11}$$

In Figure 3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

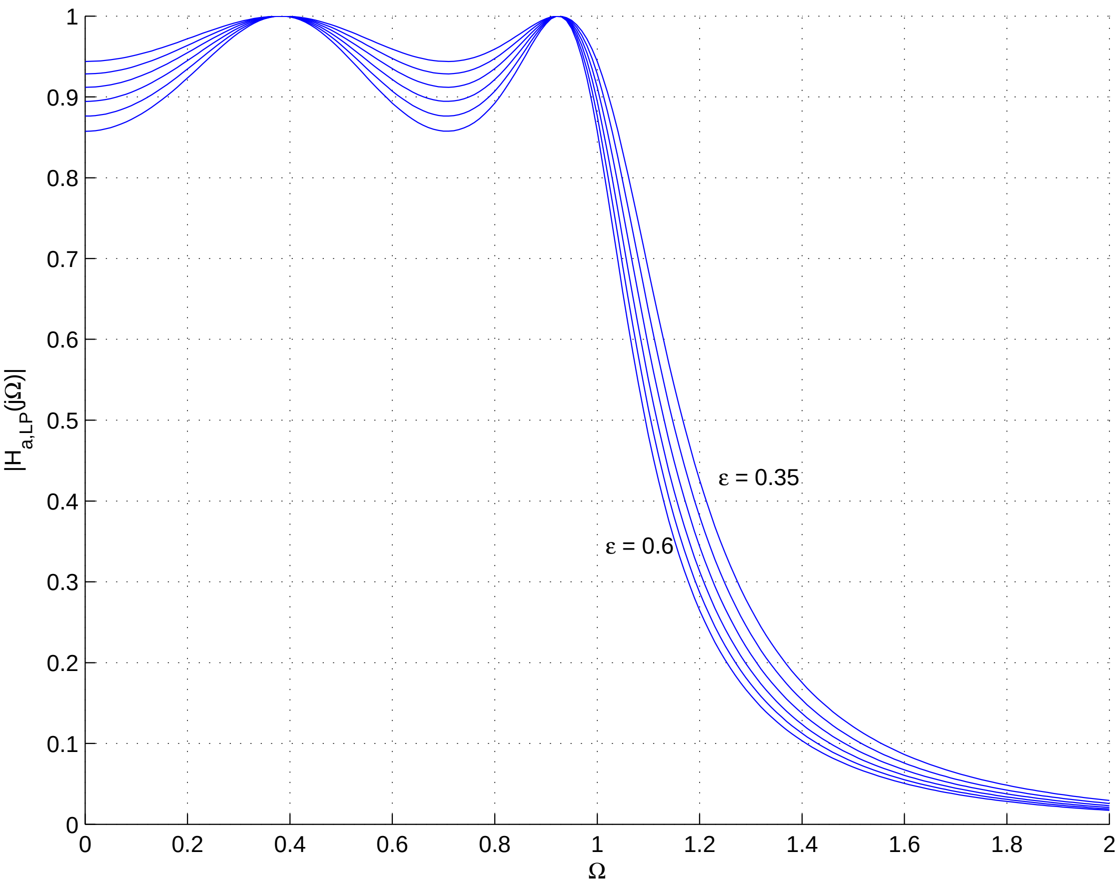

Figure 1: The Analog Low-Pass Frequency Response for $0.35 \leq \epsilon \leq 0.6$

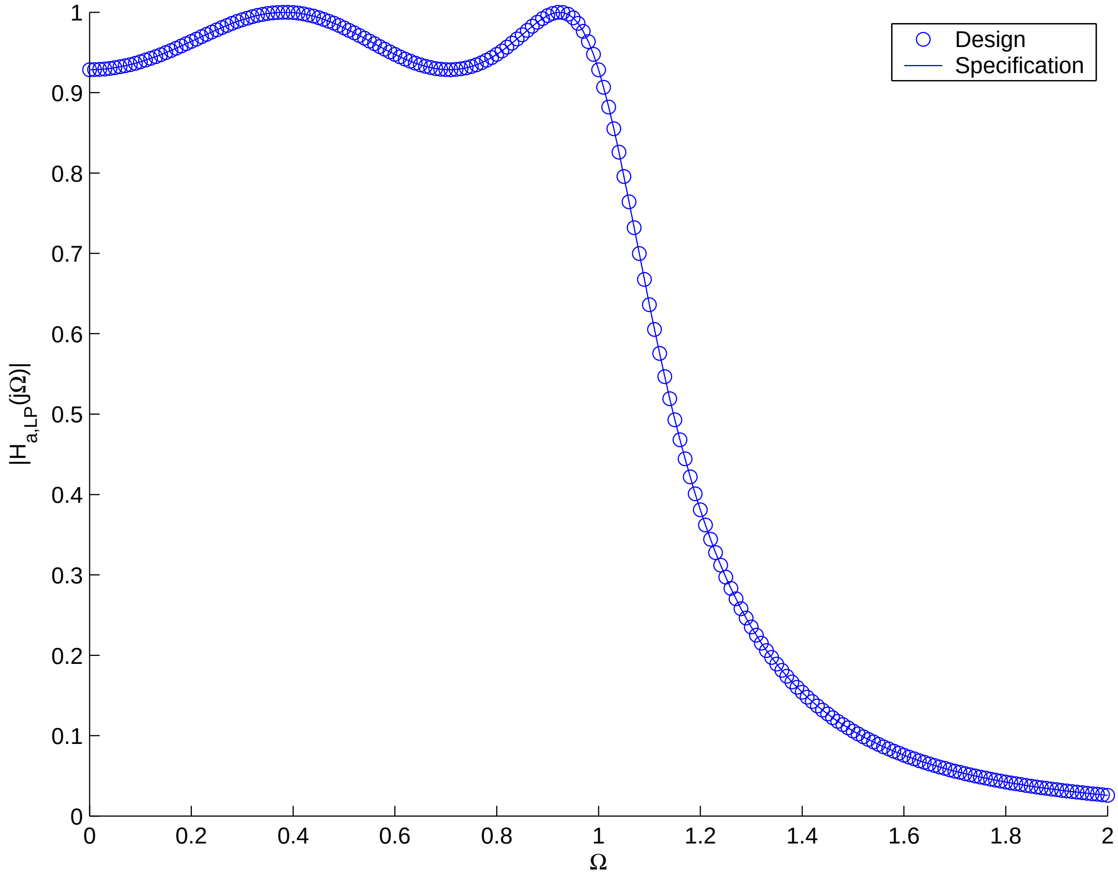

Figure 2: The magnitude response plots from the specifications in Equation 5 and the design in Equation 9

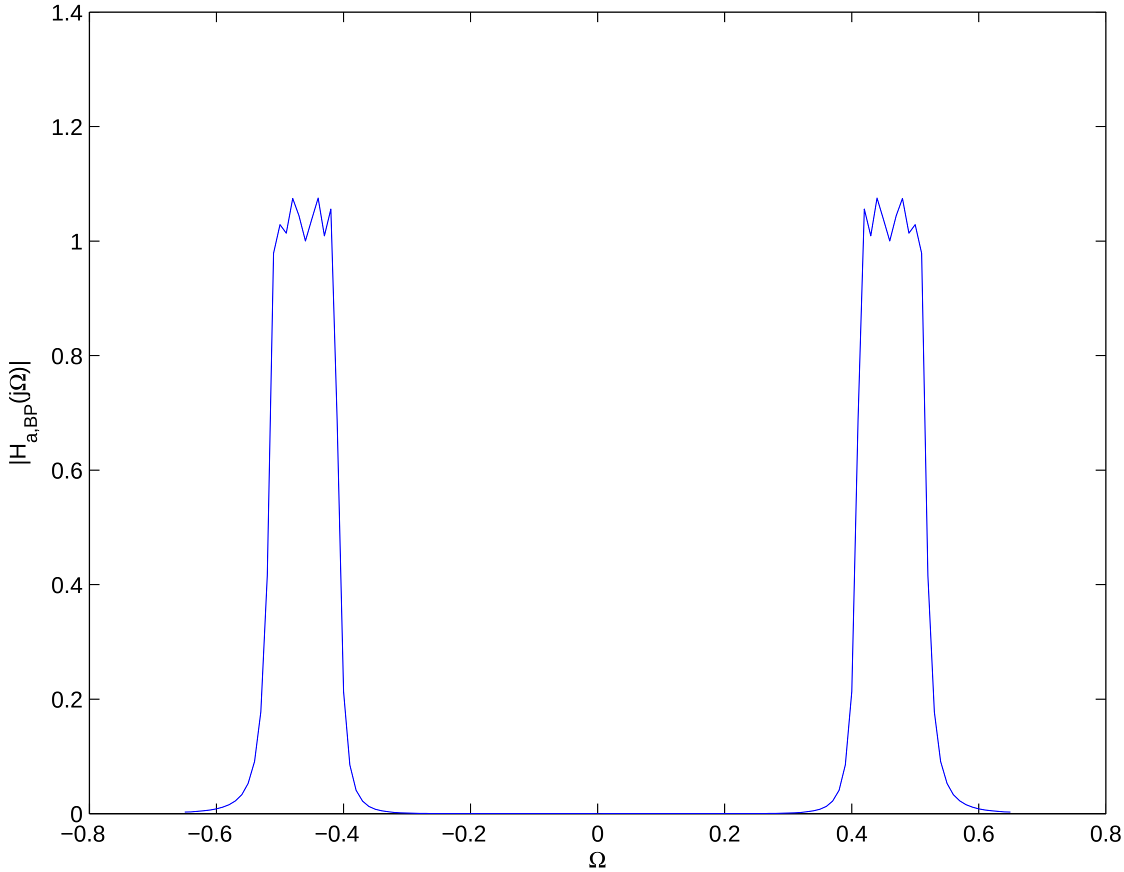

Figure 3: The analog bandpass magnitude response plot from Equation 11

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{12}$$

where G is the gain of the digital filter. From (11) and (12), we obtain

$$H_{d,BP}(z) = G\frac{N(z)}{D(z)} \tag{13}$$

where $G = 2.7776 \times 10^{-5}$,

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \tag{14}$$

and

$$\begin{aligned} D(z) = 2.3609 - 12.0002z^{-1} + 31.8772z^{-2} - 53.7495z^{-3} + 62.8086z^{-4} \\ -51.4634z^{-5} + 29.2231z^{-6} - 10.5329z^{-7} + 1.9842z^{-8} \end{aligned} \tag{15}$$

The plot of $|H_{d,BP}(z)|$ with respect to the normalized angular freqency (normalizing factor π) is available in Figure 4. Again we find that the passband and stopband frequencies meet the specifications well enough.

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is δ.

1. The *passband frequency* ω_l is defined as $\omega_l = \frac{\omega_{p1}-\omega_{p2}}{2}$. Substituting the values of ω_{p1} and ω_{p2} from section 2.1, we obtain $\omega_l = 0.025\pi$.

2. *The impulse response* $h_{lp}(n)$ of the desired lowpass filter with cutoff frequency ω_l is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi}w(n), \tag{16}$$

where $w(n)$ is the Kaiser window obtained from the design specifications.

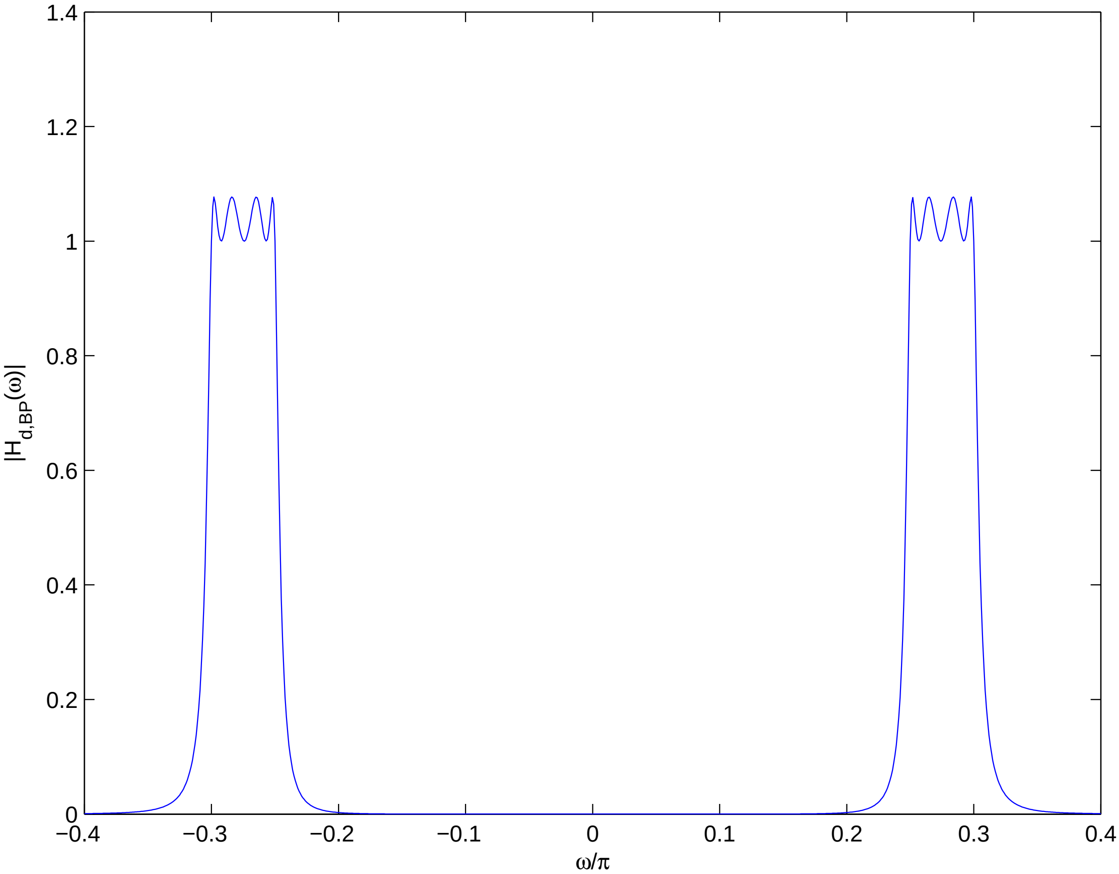

Figure 4: The magnitude response of the bandpass digital filter designed to meet the given specifications

4.2 The Kaiser Window

The Kaiser window is defined as

$$
\begin{aligned}
w(n) &= \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, \qquad -N \leq n \leq N, \quad \beta > 0 \\
&= 0 \qquad\qquad \text{otherwise,}
\end{aligned} \tag{17}
$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero in x and β and N are the window shaping factors. In the following, we find β and N using the design parameters in section 2.1.

1. N is chosen according to
$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{18}$$
where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2. β is chosen according to
$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{19}$$
In our design, we have $A = 16.4782 < 21$. Hence, from (19) we obtain $\beta = 0$ and since $N = 48$, substituting these in (17) gives us the rectangular window
$$
\begin{aligned}
w(n) &= 1, \ -48 \leq n \leq 48 \\
&= 0 \quad \text{otherwise}
\end{aligned} \tag{20}
$$

From (16) and (20), we obtain the desired lowpass filter impulse response

$$
\begin{aligned}
h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \qquad -48 \leq n \leq 48 \\
&= 0, \qquad \text{otherwise}
\end{aligned} \tag{21}
$$

The Frequency response of the filter in (21) is shown in Figure 5.