

CONTENTS

1 Phase Margin 1

Abstract—This manual is an introduction to control systems based on GATE problems. Links to sample Python codes are available in the text.

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
control/codes
```

1 PHASE MARGIN

1.1. Find the Phase Margin of $G(s)$ in degrees where

$$G(s) = \frac{2}{(s+1)(s+2)} \quad (1.1.1)$$

Solution: Phase Margin: It is the difference between phase of the system and -180° at the gain crossover frequency, (the gain crossover frequency being the frequency at which the open-loop gain first reaches 1).

Phase Margin is given by,

$$P.M = \phi - \angle G(j\omega)|_{\omega=\omega_{pc}} = \phi + 180^\circ \quad (1.1.2)$$

where,

$$\phi = \angle G(j\omega)|_{\omega=\omega_{gc}} \quad (1.1.3)$$

ω_{pc} is the Phase crossover frequency (The frequency at which the phase of open-loop transfer function reaches -180°).

ω_{gc} is the Gain crossover frequency (The frequency at which the gain of the open-loop transfer function reaches 1).

Given,

$$G(s) = \frac{2}{(s+1)(s+2)} \quad (1.1.4)$$

$$G(j\omega) = \frac{1}{(j\omega+1)(j\omega+2)} \quad (1.1.5)$$

We can find magnitude and phase as

$$|G(j\omega)| = \frac{2}{(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})} \quad (1.1.6)$$

$$\angle G(j\omega) = -\tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right) \quad (1.1.7)$$

We know that, Gain in dB = 0 at $\omega = \omega_{gc}$

$$20\log_{10}|G(j\omega_{gc})| = 0 \quad (1.1.8)$$

$$|G(j\omega_{gc})| = 1 \quad (1.1.9)$$

$$\frac{2}{(\sqrt{\omega_{gc}^2+1})(\sqrt{\omega_{gc}^2+4})} = 1 \quad (1.1.10)$$

Solving we get,

$$\omega_{gc}^2(\omega_{gc}^2+5) = 0 \quad (1.1.11)$$

$$\Rightarrow \omega_{gc} = 0, +j\sqrt{5}, -j\sqrt{5} \quad (1.1.12)$$

As frequency is a real quantity

Hence, $\omega_{gc} \neq \text{Imaginary}$

$$\therefore \omega_{gc} = 0 \quad (1.1.13)$$

From (1.1.3) and (1.1.7)

$$\phi = \angle G(j\omega_{gc}) = -\tan^{-1}(\omega_{gc}) - \tan^{-1}\left(\frac{\omega_{gc}}{2}\right) \quad (1.1.14)$$

$$\Rightarrow \phi = 0^\circ \quad (1.1.15)$$

$$\therefore P.M = 180^\circ + 0^\circ = 180^\circ \quad (1.1.16)$$

1.2. We can verify the above result using phase plot. The following code plots Fig(1.3)

```
codes/ee18btech11017.py
```

1.3. The Phase plot is as shown, We can observe

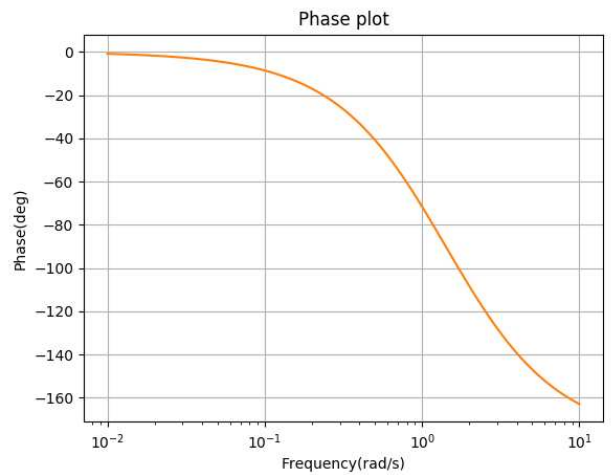


Fig. 1.3

that at $\omega_{gc} = 0$, $\phi = 0^\circ$

$$\therefore P.M = 180^\circ \quad (1.3.1)$$

1.4. **Application:** Phase margin is measure of stability in closed-loop, dynamic-control systems.(i.e, For stability of a system both gain margin and phase margin should be positive.)

1.5. **Example:**Phase Margin is a measure of stability.

Consider a unity negative feedback system with open loop gain ,

$$G(s) = \frac{10000}{s(s+10)^2} \quad (1.5.1)$$

We can find mgnitude and phase as

$$|G(j\omega)| = \frac{10^4}{\omega \sqrt{(\omega^2 + 10^2)^2}} \quad (1.5.2)$$

$$\angle G(j\omega) = -90^\circ - 2\tan^{-1} \frac{\omega}{10} \quad (1.5.3)$$

At the gain cross over frequency $\omega_{gc}, |G(j\omega)| = 1$

$$\frac{10^4}{\omega_{gc} \sqrt{(\omega_{gc}^2 + 10^2)^2}} = 1 \quad (1.5.4)$$

$$(1.5.5)$$

Simplifying we get,

$$\omega_{gc}^3 + 10^2 \omega_{gc} - 10^4 = 0 \quad (1.5.6)$$

$$\Rightarrow \omega_{gc} = 20, -10 + 20j, -10 - 20j \quad (1.5.7)$$

As frequency is a real quantity

Hence, $\omega_{gc} \neq \text{Imaginary}$

$$\therefore \omega_{gc} = 20 \quad (1.5.8)$$

From (1.5.3) and (1.5.8) we get,

$$P.M = 180^\circ - 90^\circ - 2\tan^{-1} \left(\frac{\omega_{gc}}{10} \right) \quad (1.5.9)$$

$$\Rightarrow P.M = -36.9^\circ \quad (1.5.10)$$

We can verify the above result using phase plot.The following code plots Fig(1.5)

```
codes/ee18btech11017.py
```

The Phase plot is as shown, We can observe that at $\omega_{gc} = 20$, $\phi = -216.86^\circ$

$$\therefore P.M = 180^\circ + -216.86^\circ = -36.9^\circ \quad (1.5.11)$$

$$(1.5.12)$$

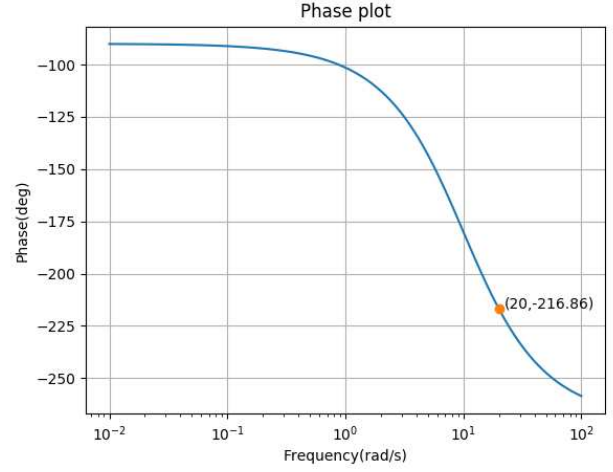


Fig. 1.5

As Phase Margin is negative . Therefore the closed loop system is unstable . We can verify closed loop stability using Routh-Hurwitz criterion.

1.6. Verifying the above example using Routh-Hurwitz criterion.

Let $T(s)$ be Closed loop transfer function ,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)}{1 + G(s)} \quad (1.6.1)$$

The characteristic equation is

$$D(s) = 0 \quad (1.6.2)$$

$$1 + G(s) = 0 \quad (1.6.3)$$

$$\Rightarrow s^3 + 20s^2 + 100s + 10000 = 0 \quad (1.6.4)$$

Constructing routh array for (1.6.4)...

$$\begin{array}{c|ccc} s^3 & 1 & 100 & 0 \\ s^2 & 20 & 10000 & 0 \\ s & -400 & 0 & 0 \end{array} \quad (1.6.5)$$

$$\begin{array}{c|ccc} s^3 & 1 & 100 & 0 \\ s^2 & 20 & 10000 & 0 \\ s & -400 & 0 & 0 \\ s^0 & 10000 & 0 & 0 \end{array} \quad (1.6.6)$$

There are two sign changes in the first column

of the routh array. So, two poles lie on right half of s-plane.

Therefore, the system is unstable.

The following code generates routh array.

codes/RouthHurwitz.py

- 1.7. Now consider $G(s)$ from (1.5.1), in above example and multiply it with $H(s) = 10^{-1}$. So, now open loop gain becomes $G(s)H(s)$.

$$G(s)H(s) = \frac{1000}{s(s+10)^2} \quad (1.7.1)$$

We can find magnitude and phase as

$$|G(j\omega)H(j\omega)| = \frac{10^3}{\omega \sqrt{(\omega^2 + 10^2)^2}} \quad (1.7.2)$$

$$\angle G(j\omega)H(j\omega) = -90^\circ - 2\tan^{-1} \frac{\omega}{10} \quad (1.7.3)$$

At the gain cross over frequency ω_{gc} , $|G(j\omega)H(j\omega)| = 1$

$$\frac{10^3}{\omega_{gc} \sqrt{(\omega_{gc}^2 + 10^2)^2}} = 1 \quad (1.7.4)$$

$$(1.7.5)$$

Simplifying we get,

$$\omega_{gc}^3 + 10^2 \omega_{gc} - 10^3 = 0 \quad (1.7.6)$$

$$\Rightarrow \omega_{gc} = 6.82, -3.41 + 11.6j, -3.41 - 11.6j \quad (1.7.7)$$

As frequency is a real quantity

Hence, $\omega_{gc} \neq \text{Imaginary}$

$$\therefore \omega_{gc} = 6.82 \quad (1.7.8)$$

From (1.7.3) and (1.7.8) we get,

$$P.M = 180^\circ - 90^\circ - 2\tan^{-1} \left(\frac{\omega_{gc}}{10} \right) \quad (1.7.9)$$

$$\Rightarrow P.M = 21.42^\circ \quad (1.7.10)$$

We can verify the above result using phase plot. The following code plots Fig(1.7)

codes/ee18btech11017.py

The Phase plot is as shown,

We can observe that at $\omega_{gc} = 6.82$, $\phi = -158.58^\circ$

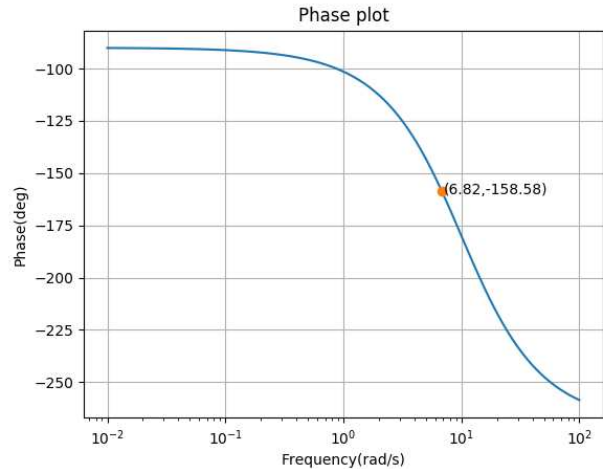


Fig. 1.7

$$\therefore P.M = 180^\circ + -158.58^\circ = 21.42^\circ \quad (1.7.11)$$

$$(1.7.12)$$

As Phase Margin is positive. Therefore the closed loop system is stable. We can verify closed loop stability using Routh-Hurwitz criterion.

Verification:

Verifying the above example using Routh-Hurwitz criterion.

Let $T(s)$ be Closed loop transfer function,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)H(s)}{1 + H(s)G(s)} \quad (1.7.13)$$

The characteristic equation is

$$D(s) = 0 \quad (1.7.14)$$

$$1 + G(s)H(s) = 0 \quad (1.7.15)$$

$$\Rightarrow s^3 + 20s^2 + 100s + 1000 = 0 \quad (1.7.16)$$

Constructing routh array for (1.7.16)...

$$\begin{array}{c|ccc} s^3 & 1 & 100 & 0 \\ s^2 & 20 & 1000 & 0 \\ s & 50 & 0 & 0 \end{array} \quad (1.7.17)$$

$$\left| \begin{array}{c|ccc} s^3 & 1 & 100 & 0 \\ s^2 & 20 & 1000 & 0 \\ s & 50 & 0 & 0 \\ s^0 & 1000 & 0 & 0 \end{array} \right| \quad (1.7.18)$$

There are no sign changes in the first column of the routh array. So, all the poles lie on left half of s-plane.

Therefore, the system is stable.

The following code generates routh array.

```
codes/RouthHurwitz.py
```

- 1.8. Consider $G(s)$ in Equation(1.1.1) and multiply it with $H(s) = \frac{100}{s+1}$. So, now the open loop gain becomes $G(s)H(s)$.

$$G(s)H(s) = \frac{100}{(s+1)^2(s+2)} \quad (1.8.1)$$

We can find magnitude and phase as

$$|G(j\omega)H(j\omega)| = \frac{10^2}{\sqrt{(\omega^2 + 1)^2} \sqrt{\omega^2 + 4}} \quad (1.8.2)$$

$$\angle G(j\omega)H(j\omega) = -\tan^{-1} \frac{\omega}{2} - 2\tan^{-1}(\omega) \quad (1.8.3)$$

At the gain crossover frequency ω_{gc} , $|G(j\omega)H(j\omega)| = 1$

$$\frac{10^2}{\sqrt{\omega_{gc}^2 + 4} \sqrt{(\omega_{gc}^2 + 10^2)^2}} = 1 \quad (1.8.4)$$

$$(1.8.5)$$

Simplifying we get,

$$\omega_{gc}^6 + 6\omega_{gc}^4 + 9\omega_{gc}^2 - 9996 = 0 \quad (1.8.6)$$

Values of ω_{gc} are

$$\Rightarrow \omega_{gc} = -4.42, 4.42, -2.21 + 4.2j, \quad (1.8.7)$$

$$-2.21 - 4.2j, 2.21 + 4.2j, 2.21 - 4.2j \quad (1.8.8)$$

As frequency is a real quantity

Hence, $\omega_{gc} \neq \text{Imaginary, Negative}$

$$\therefore \omega_{gc} = 4.42 \quad (1.8.9)$$

From (1.8.3) and (1.8.9) we get,

$$P.M = 180^\circ - 2\tan^{-1}(\omega_{gc}) - \tan^{-1}\left(\frac{\omega_{gc}}{2}\right) \quad (1.8.10)$$

$$\Rightarrow P.M = -40.15^\circ \quad (1.8.11)$$

We can verify the above result using phase plot. The following code plots Fig(1.8)

```
codes/ee18btech11017.py
```

The Phase plot is as shown,

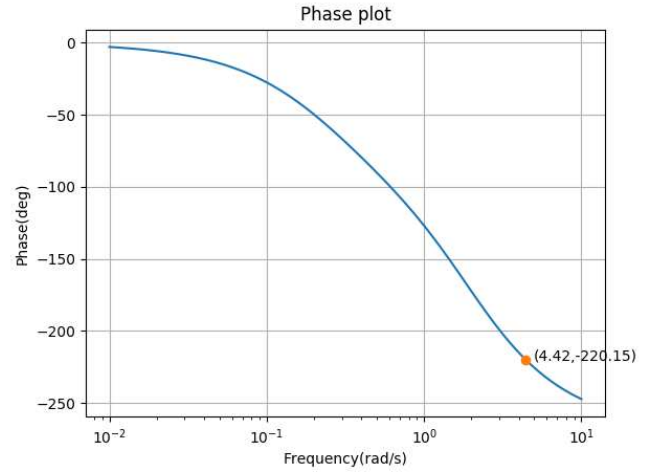


Fig. 1.8

We can observe that at $\omega_{gc} = 4.42$, $\phi = -220.15^\circ$

$$\therefore P.M = 180^\circ + -220.15^\circ = -40.15^\circ \quad (1.8.12)$$

As Phase Margin is negative. Therefore the closed loop system is unstable. We can verify closed loop stability using Routh-Hurwitz criterion.

Verification:

Verifying the above problem using Routh-Hurwitz criterion.

Let $T(s)$ be Closed loop transfer function,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)H(s)}{1 + H(s)G(s)} \quad (1.8.13)$$

The characteristic equation is

$$D(s) = 0 \quad (1.8.14)$$

$$1 + G(s)H(s) = 0 \quad (1.8.15)$$

$$\Rightarrow s^3 + 4s^2 + 5s + 102 = 0 \quad (1.8.16)$$

Constructing routh array for (1.8.16)...

$$\begin{array}{c|ccc} s^3 & 1 & 5 & 0 \\ s^2 & 4 & 102 & 0 \\ s & -20.5 & 0 & 0 \end{array} \quad (1.8.17)$$

$$\begin{array}{c|ccc} s^3 & 1 & 5 & 0 \\ s^2 & 4 & 102 & 0 \\ s & -20.5 & 0 & 0 \\ s^0 & 102 & 0 & 0 \end{array} \quad (1.8.18)$$

There are two sign changes in the first column of the routh array. So, two poles lie on right half of s-plane.

Therefore, the system is unstable.

The following code generates routh array.

```
codes/RouthHurwitz.py
```