CONTENTS

Abstract—**This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

1 PHASE MARGIN

1.1. Find the Phase Margin of $G(s)$ in degrees where

$$G(s) = \frac{2}{(s+1)(s+2)} \quad (1.1.1)$$

Solution: Phase Margin:It is the difference between phase of the system and $-180°$ at the gain crossover frequency,(the gain crossover frequency being the frequency at which the open-loop gain first reaches 1).
Phase Margin is given by,

$$P.M = \phi - \angle G(j\omega)|_{\omega=\omega_{pc}} = \phi + 180° \quad (1.1.2)$$

where,

$$\phi = \angle G(j\omega)|_{\omega=\omega_{gc}} \quad (1.1.3)$$

ω_{pc} is the Phase crossover frequency (The frequency at which the phase of open-loop transfer function reaches -180°).
ω_{gc} is the Gain crossover frequency (The frequency at which the gain of the open-loop transfer fuction reaches 1).
Given,

$$G(s) = \frac{2}{(s+1)(s+2)} \quad (1.1.4)$$

$$G(j\omega) = \frac{1}{(j\omega+1)(j\omega+2)} \quad (1.1.5)$$

We can find magnitude and phase as

$$|G(j\omega)| = \frac{2}{(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})} \quad (1.1.6)$$

$$\angle G(j\omega) = -tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{2}) \quad (1.1.7)$$

We know that, Gain in dB = 0 at $\omega = \omega_{gc}$

$$20log_{10}|G(j\omega_{gc})| = 0 \quad (1.1.8)$$

$$|G(j\omega_{gc})| = 1 \quad (1.1.9)$$

$$\frac{2}{(\sqrt{\omega_{gc}^2+1})(\sqrt{\omega_{gc}^2+4})} = 1 \quad (1.1.10)$$

Solving we get,

$$\omega_{gc}^2(\omega_{gc}^2+5) = 0 \quad (1.1.11)$$

$$=> \omega_{gc} = 0, +j\sqrt{5}, -j\sqrt{5} \quad (1.1.12)$$

As frequency is a real quantity
Hence, $\omega_{gc} \neq$ Imaginary

$$\therefore \omega_{gc} = 0 \quad (1.1.13)$$

From (1.1.3) and (1.1.7)

$$\phi = \angle G(j\omega_{gc}) = -tan^{-1}(\omega_{gc}) - tan^{-1}(\frac{\omega_{gc}}{2}) \quad (1.1.14)$$

$$=> \phi = 0° \quad (1.1.15)$$

$$\therefore P.M = 180° + 0° = 180° \quad (1.1.16)$$

1.2. We can verify the above result using phase plot.The following code plots Fig(1.3)

codes/ee18btech11017.py

1.3. The Phase plot is as shown, We can observe

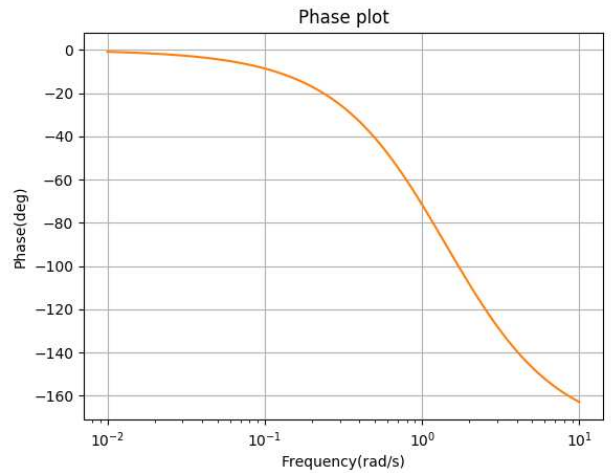Fig. 1.3

that at $\omega_{gc} = 0$, $\phi = 0°$

$$\therefore P.M = 180° \quad (1.3.1)$$

1.4. **Application:** Phase margin is measure of stability in closed-loop, dynamic-control systems.(i.e, For stability of a system both gain margin and phase margin should be positive.)

1.5. **Example:**Phase Margin is a measure of stability.

Consider a unity negative feedback system with open loop gain ,

$$G(s) = \frac{10000}{s(s + 10)^2} \qquad (1.5.1)$$

We can find mgnitude and phase as

$$|G(j\omega)| = \frac{10^4}{\omega \sqrt{(\omega^2 + 10^2)^2}} \qquad (1.5.2)$$

$$\angle G(j\omega) = -90° - 2tan^{-1}\frac{\omega}{10} \qquad (1.5.3)$$

At the gain cross over frequency $\omega_{gc}, |G(j\omega)| = 1$

$$\frac{10^4}{\omega_{gc} \sqrt{(\omega_{gc}^2 + 10^2)^2}} = 1 \qquad (1.5.4)$$

$$(1.5.5)$$

Simplifying we get,

$$\omega_{gc}^3 + 10^2\omega_{gc} - 10^4 = 0 \quad (1.5.6)$$

$$=> \omega_{gc} = 20, -10 + 20j, -10 - 20j \quad (1.5.7)$$

As frequency is a real quantity
Hence, $\omega_{gc} \neq$ Imaginary

$$\therefore \omega_{gc} = 20 \qquad (1.5.8)$$

From (1.5.3) and (1.5.8) we get,

$$P.M = 180° - 90° - 2tan^{-1}(\frac{\omega_{gc}}{10}) \qquad (1.5.9)$$

$$=> P.M = -36.9° \qquad (1.5.10)$$

We can verify the above result using phase plot.The following code plots Fig(1.5)

codes/ee18btech11017.py

The Phase plot is as shown, We can observe that at $\omega_{gc} = 20$, $\phi = -216.86°$

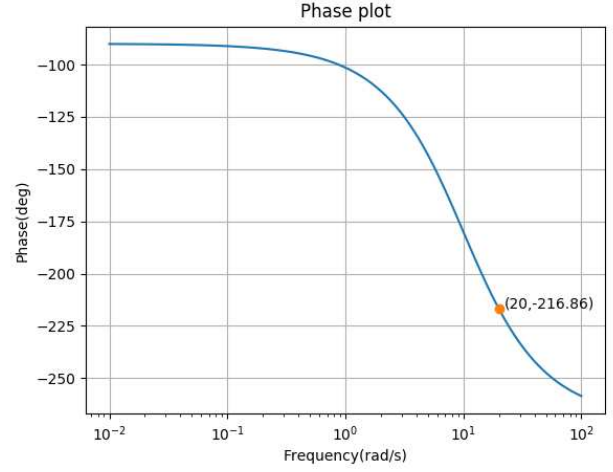$$\therefore P.M = 180° + -216.86° = -36.9° \quad (1.5.11)$$

$$(1.5.12)$$

Fig. 1.5

As Phase Margin is negative . Therefore the closed loop system is unstable . We can verify closed loop stability using Routh-Hurwitz criterion.

1.6. Verifying the above example using Routh-Hurwitz criterion.

Let $T(s)$ be Closed loop transfer function ,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)}{1 + G(s)} \qquad (1.6.1)$$

The characteristic equation is

$$D(s) = 0 \qquad (1.6.2)$$

$$1 + G(s) = 0 \qquad (1.6.3)$$

$$=> s^3 + 20s^2 + 100s + 10000 = 0 \quad (1.6.4)$$

Constructing routh array for (1.6.4)..,

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix} \begin{vmatrix} 1 & 100 & 0 \\ 20 & 10000 & 0 \\ -400 & 0 & 0 \end{vmatrix} \qquad (1.6.5)$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 100 & 0 \\ 20 & 10000 & 0 \\ -400 & 0 & 0 \\ 10000 & 0 & 0 \end{vmatrix} \qquad (1.6.6)$$

There are two sign changes in the first column

of the routh array. So, two poles lie on right half of s-plane.

Therefore,the system is unstable.

The following code generates routh array.

codes/RouthHurwitz.py

1.7. Now consider $G(s)$ from (1.5.1),in above example and multiply it with $H(s) = 10^{-1}$. So, now open loop gain becomes $G(s)H(s)$.

$$G(s)H(s) = \frac{1000}{s(s + 10)^2} \quad (1.7.1)$$

We can find mgnitude and phase as

$$|G(j\omega)H(j\omega)| = \frac{10^3}{\omega \sqrt{(\omega^2 + 10^2)^2}} \quad (1.7.2)$$

$$\angle G(j\omega)H(j\omega) = -90° - 2tan^{-1}\frac{\omega}{10} \quad (1.7.3)$$

At the gain cross over frequency $\omega_{gc}, |G(j\omega)H(j\omega)| = 1$

$$\frac{10^3}{\omega_{gc} \sqrt{(\omega_{gc}^2 + 10^2)^2}} = 1 \quad (1.7.4)$$

$$(1.7.5)$$

Simplifying we get,

$$\omega_{gc}^3 + 10^2\omega_{gc} - 10^3 = 0 \quad (1.7.6)$$

$$=> \omega_{gc} = 6.82, -3.41 + 11.6j, -3.41 - 11.6j \quad (1.7.7)$$

As frequency is a real quantity
Hence, $\omega_{gc} \neq$ Imaginary

$$\therefore \omega_{gc} = 6.82 \quad (1.7.8)$$

From (1.7.3) and (1.7.8) we get,

$$P.M = 180° - 90° - 2tan^{-1}(\frac{\omega_{gc}}{10}) \quad (1.7.9)$$

$$=> P.M = 21.42° \quad (1.7.10)$$

We can verify the above result using phase plot.The following code plots Fig(1.7)

codes/ee18btech11017.py

The Phase plot is as shown,
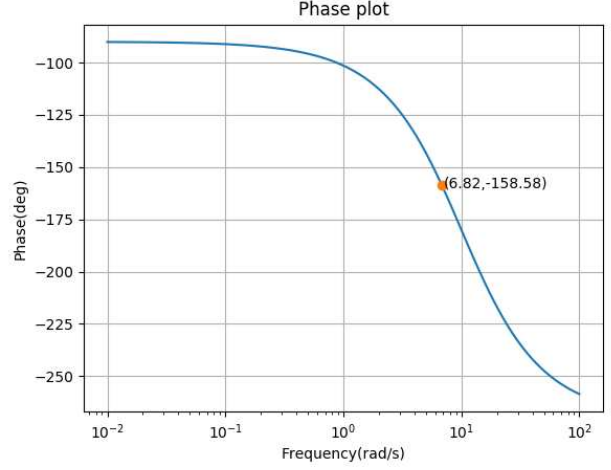We can observe that at $\omega_{gc} = 6.82$, $\phi = -158.58°$

Fig. 1.7

$$\therefore P.M = 180° + -158.58° = 21.42° \quad (1.7.11)$$

$$(1.7.12)$$

As Phase Margin is positive. Therefore the closed loop system is stable . We can verify closed loop stability using Routh-Hurwitz criterion.

Verification:

Verifying the above example using Routh-Hurwitz criterion.
Let $T(s)$ be Closed loop transfer function ,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)H(s)}{1 + H(s)G(s)} \quad (1.7.13)$$

The characteristic equation is

$$D(s) = 0 \quad (1.7.14)$$

$$1 + G(s)H(s) = 0 \quad (1.7.15)$$

$$=> s^3 + 20s^2 + 100s + 1000 = 0 \quad (1.7.16)$$

Constructing routh array for (1.7.16)..,

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix} \begin{vmatrix} 1 & 100 & 0 \\ 20 & 1000 & 0 \\ 50 & 0 & 0 \end{vmatrix} \quad (1.7.17)$$

$$\begin{array}{c|ccc} s^3 & 1 & 100 & 0 \\ s^2 & 20 & 1000 & 0 \\ s & 50 & 0 & 0 \\ s^0 & 1000 & 0 & 0 \end{array} \tag{1.7.18}$$

There are no sign changes in the first column of the routh array. So, all the poles lie on left half of s-plane.

Therefore,the system is stable.

The following code generates routh array.

codes/RouthHurwitz.py