Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Phase Margin

1.1. Find the Phase Margin of $G(s)$ in degrees where

$$G(s) = \frac{2}{(s+1)(s+2)} \tag{1.1.1}$$

Solution: Phase Margin:It is the difference between phase of the system and $-180°$ at the gain crossover frequency,(the gain crossover frequency being the frequency at which the open-loop gain first reaches 1).
Phase Margin is given by,

$$P.M = \phi - \angle G(j\omega)|_{\omega=\omega_{pc}} = \phi + 180° \tag{1.1.2}$$

where,

$$\phi = \angle G(j\omega)|_{\omega=\omega_{gc}} \tag{1.1.3}$$

ω_{pc} is the Phase crossover frequency (The frequency at which the phase of open-loop transfer function reaches -180°).
ω_{gc} is the Gain crossover frequency (The frequency at which the gain of the open-loop transfer fuction reaches 1).
Given,

$$G(s) = \frac{2}{(s+1)(s+2)} \tag{1.1.4}$$

$$G(j\omega) = \frac{1}{(j\omega+1)(j\omega+2)} \tag{1.1.5}$$

We can find magnitude and phase as

$$|G(j\omega)| = \frac{2}{(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})} \tag{1.1.6}$$

$$\angle G(j\omega) = -tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{2}) \tag{1.1.7}$$

We know that, Gain in dB = 0 at $\omega = \omega_{gc}$

$$20log_{10}|G(j\omega_{gc})| = 0 \tag{1.1.8}$$

$$|G(j\omega_{gc})| = 1 \tag{1.1.9}$$

$$\frac{2}{(\sqrt{\omega_{gc}^2+1})(\sqrt{\omega_{gc}^2+4})} = 1 \tag{1.1.10}$$

Solving we get,

$$\omega_{gc}^2(\omega_{gc}^2+5) = 0 \tag{1.1.11}$$

$$=> \omega_{gc} = 0, +j\sqrt{5}, -j\sqrt{5} \tag{1.1.12}$$

As frequency is a real quantity
Hence, $\omega_{gc} \neq$ Imaginary

$$\therefore \omega_{gc} = 0 \tag{1.1.13}$$

From (1.1.3) and (1.1.7)

$$\phi = \angle G(j\omega_{gc}) = -tan^{-1}(\omega_{gc}) - tan^{-1}(\frac{\omega_{gc}}{2}) \tag{1.1.14}$$

$$=> \phi = 0° \tag{1.1.15}$$

$$\therefore P.M = 180° + 0° = 180° \tag{1.1.16}$$

1.2. We can verify the above result using phase plot.The following code plots Fig(1.3)

```
codes/ee18btech11017.py
```

1.3. The Phase plot is as shown, We can observe

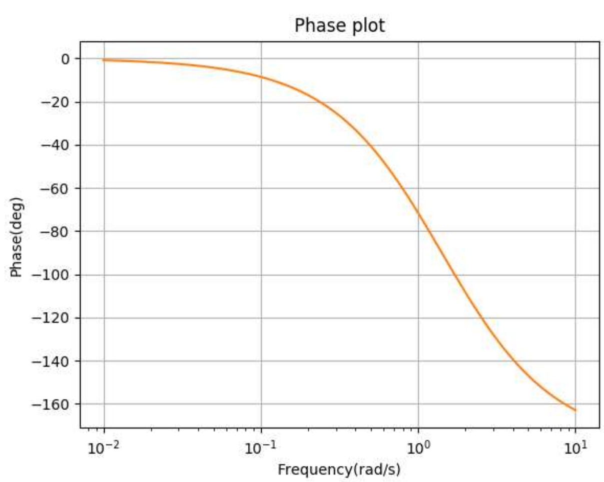

Fig. 1.3

that at $\omega_{gc} = 0$, $\phi = 0°$

$$\therefore P.M = 180° \tag{1.3.1}$$

1.4. **Application:** Phase margin is measure of stability in closed-loop, dynamic-control systems.(i.e, For stability of a system both gain margin and phase margin should be positive.)

1.5. **Example:**Phase Margin is a measure of stability.
Consider a unity negative feedback system with open loop gain ,

$$G(s) = \frac{10000}{s(s+10)^2} \tag{1.5.1}$$

We can find mgnitude and phase as

$$|G(j\omega)| = \frac{10^4}{\omega\sqrt{(\omega^2+10^2)^2}} \tag{1.5.2}$$

$$\angle G(j\omega) = -90^\circ - 2tan^{-1}\frac{\omega}{10} \tag{1.5.3}$$

At the gain cross over frequency ω_{gc},$|G(j\omega)| = 1$

$$\frac{10^4}{\omega_{gc}\sqrt{(\omega_{gc}^2+10^2)^2}} = 1 \tag{1.5.4}$$

$$\tag{1.5.5}$$

Simplifying we get,

$$\omega_{gc}^3 + 10^2\omega_{gc} - 10^4 = 0 \tag{1.5.6}$$

$$=> \omega_{gc} = 20, -10+20j, -10-20j \tag{1.5.7}$$

As frequency is a real quantity
Hence, $\omega_{gc} \neq$ Imaginary

$$\therefore \omega_{gc} = 20 \tag{1.5.8}$$

From (1.5.3) and (1.5.8) we get,

$$P.M = 180^\circ - 90^\circ - 2tan^{-1}(\frac{\omega_{gc}}{10}) \tag{1.5.9}$$

$$=> P.M = -36.9^\circ \tag{1.5.10}$$

We can verify the above result using phase plot.The following code plots Fig(1.5)

```
codes/ee18btech11017.py
```

The Phase plot is as shown, We can observe that at $\omega_{gc} = 20$, $\phi = -216.86^\circ$

$$\therefore P.M = 180^\circ + -216.86^\circ = -36.9^\circ \tag{1.5.11}$$

$$\tag{1.5.12}$$

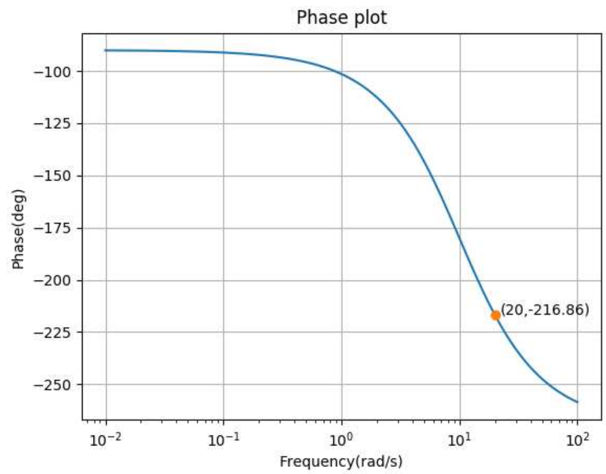

Fig. 1.5

As Phase Margin is negative . Therefore the closed loop system is unstable . We can verify closed loop stability using Routh-Hurwitz criterion.

1.6. Verifying the above example using Routh-Hurwitz criterion.
Let $T(s)$ be Closed loop transfer function ,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)}{1+G(s)} \tag{1.6.1}$$

The characteristic equation is

$$D(s) = 0 \tag{1.6.2}$$

$$1 + G(s) = 0 \tag{1.6.3}$$

$$=> s^3 + 20s^2 + 100s + 10000 = 0 \tag{1.6.4}$$

Constructing routh array for (1.6.4)..,

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix} \begin{vmatrix} 1 & 100 & 0 \\ 20 & 10000 & 0 \\ -400 & 0 & 0 \end{vmatrix} \tag{1.6.5}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 100 & 0 \\ 20 & 10000 & 0 \\ -400 & 0 & 0 \\ 10000 & 0 & 0 \end{vmatrix} \tag{1.6.6}$$

There are two sign changes in the first column

of the routh array. So, two poles lie on right half of s-plane.
Therefore,the system is unstable.
The following code generates routh array.

```
codes/RouthHurwitz.py
```

1.7. Now consider $G(s)$ from (1.5.1),in above example and multiply it with $H(s) = 10^{-1}$. So, now open loop gain becomes $G(s)H(s)$.

$$G(s)H(s) = \frac{1000}{s(s+10)^2} \tag{1.7.1}$$

We can find mgnitude and phase as

$$|G(j\omega)H(j\omega)| = \frac{10^3}{\omega\sqrt{(\omega^2+10^2)^2}} \tag{1.7.2}$$

$$\angle G(j\omega)H(j\omega) = -90^\circ - 2tan^{-1}\frac{\omega}{10} \tag{1.7.3}$$

At the gain cross over frequency ω_{gc},$|G(j\omega)H(j\omega)| = 1$

$$\frac{10^3}{\omega_{gc}\sqrt{(\omega_{gc}^2+10^2)^2}} = 1 \tag{1.7.4}$$

$$\tag{1.7.5}$$

Simplifying we get,

$$\omega_{gc}^3 + 10^2\omega_{gc} - 10^3 = 0 \tag{1.7.6}$$

$$=> \omega_{gc} = 6.82, -3.41 + 11.6j, -3.41 - 11.6j \tag{1.7.7}$$

As frequency is a real quantity
Hence, $\omega_{gc} \neq$ Imaginary

$$\therefore \omega_{gc} = 6.82 \tag{1.7.8}$$

From (1.7.3) and (1.7.8) we get,

$$P.M = 180^\circ - 90^\circ - 2tan^{-1}(\frac{\omega_{gc}}{10}) \tag{1.7.9}$$

$$=> P.M = 21.42^\circ \tag{1.7.10}$$

We can verify the above result using phase plot.The following code plots Fig(1.7)

```
codes/ee18btech11017.py
```

The Phase plot is as shown,
We can observe that at $\omega_{gc} = 6.82$, $\phi = -158.58^\circ$

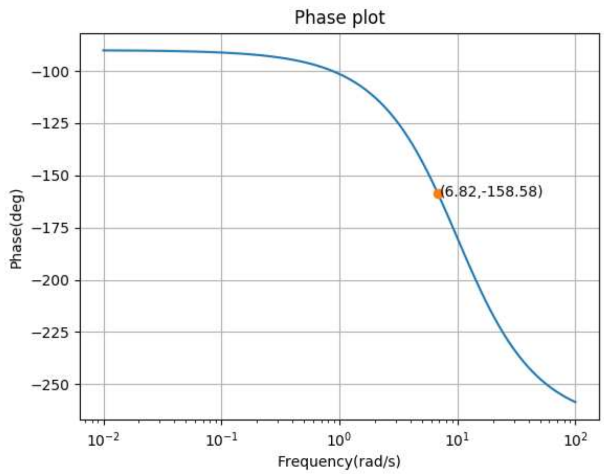

Fig. 1.7

$$\therefore P.M = 180^\circ + -158.58^\circ = 21.42^\circ \tag{1.7.11}$$

$$\tag{1.7.12}$$

As Phase Margin is positive. Therefore the closed loop system is stable . We can verify closed loop stability using Routh-Hurwitz criterion.
Verification:

Verifying the above example using Routh-Hurwitz criterion.
Let $T(s)$ be Closed loop transfer function ,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)H(s)}{1+H(s)G(s)} \tag{1.7.13}$$

The characteristic equation is

$$D(s) = 0 \tag{1.7.14}$$

$$1 + G(s)H(s) = 0 \tag{1.7.15}$$

$$=> s^3 + 20s^2 + 100s + 1000 = 0 \tag{1.7.16}$$

Constructing routh array for (1.7.16)..,

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix} \begin{vmatrix} 1 & 100 & 0 \\ 20 & 1000 & 0 \\ 50 & 0 & 0 \end{vmatrix} \tag{1.7.17}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 100 & 0 \\ 20 & 1000 & 0 \\ 50 & 0 & 0 \\ 1000 & 0 & 0 \end{vmatrix} \tag{1.7.18}$$

There are no sign changes in the first column of the routh array. So, all the poles lie on left half of s-plane.
Therefore,the system is stable.
The following code generates routh array.

```
codes/RouthHurwitz.py
```

1.8. Consider $G(s)$ in Equation(1.1.1) and mulitply it with $H(s) = \frac{50}{s+1}$. So, now the open loop gain becomes $G(s)H(s)$.

$$G(s)H(s) = \frac{100}{(s+1)^2(s+2)} \tag{1.8.1}$$

We can find mgnitude and phase as

$$|G(j\omega)H(j\omega)| = \frac{10^2}{\sqrt{(\omega^2+1)^2}\sqrt{\omega^2+4}} \tag{1.8.2}$$

$$\angle G(j\omega)H(j\omega) = -tan^{-1}\frac{\omega}{2} - 2tan^{-1}(\omega) \tag{1.8.3}$$

At the gain crossover frequency ω_{gc},$|G(j\omega)H(j\omega)| = 1$

$$\frac{10^2}{\sqrt{\omega_{gc}^2+4}\sqrt{(\omega_{gc}^2+10^2)^2}} = 1 \tag{1.8.4}$$

$$\tag{1.8.5}$$

Simplifying we get,

$$\omega_{gc}^6 + 6\omega_{gc}^4 + 9\omega_{gc}^2 - 9996 = 0 \tag{1.8.6}$$

Values of ω_{gc} are

$$=> \omega_{gc} = -4.42, 4.42, -2.21 + 4.2j, \tag{1.8.7}$$

$$-2.21 - 4.2j, 2.21 + 4.2j, 2.21 - 4.2j \tag{1.8.8}$$

As frequency is a real quantity
Hence, $\omega_{gc} \neq$ Imaginary , Negative

$$\therefore \omega_{gc} = 4.42 \tag{1.8.9}$$

From (1.8.3) and (1.8.9) we get,

$$P.M = 180^\circ - 2tan^{-1}(\omega_{gc}) - tan^{-1}(\frac{\omega_{gc}}{2}) \tag{1.8.10}$$

$$=> P.M = -40.15^\circ \tag{1.8.11}$$

We can verify the above result using phase plot.The following code plots Fig(1.8)

```
codes/ee18btech11017.py
```

The Phase plot is as shown,

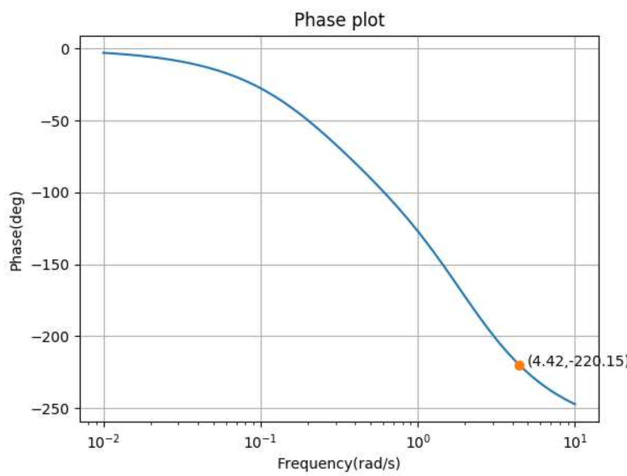

Fig. 1.8

We can observe that at $\omega_{gc} = 4.42$, $\phi = -220.15^\circ$

$$\therefore P.M = 180^\circ + -220.15^\circ = -40.15^\circ \tag{1.8.12}$$

As Phase Margin is negative. Therefore the closed loop system is unstable . We can verify closed loop stability using Routh-Hurwitz criterion.
Verification:

Verifying the above problem using Routh-Hurwitz criterion.
Let $T(s)$ be Closed loop transfer function ,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)H(s)}{1 + H(s)G(s)} \tag{1.8.13}$$

The characteristic equation is

$$D(s) = 0 \tag{1.8.14}$$
$$1 + G(s)H(s) = 0 \tag{1.8.15}$$

$$=> s^3 + 4s^2 + 5s + 102 = 0 \tag{1.8.16}$$

Constructing routh array for (1.8.16)..,

$$\begin{vmatrix} s^3 \\ s^2 \\ s \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \end{vmatrix} \tag{1.8.17}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 5 & 0 \\ 4 & 102 & 0 \\ -20.5 & 0 & 0 \\ 102 & 0 & 0 \end{vmatrix} \tag{1.8.18}$$

There are two sign changes in the first column of the routh array. So, two poles lie on right half of s-plane.
Therefore,the system is unstable.
The following code generates routh array.

```
codes/RouthHurwitz.py
```

1.9. From Equation (1.1.16) we can observe that Phase Margin is positive. So,the closed loop system is stable.
Let $T(s)$ be Closed loop transfer function ,

$$T(s) = \frac{N(s)}{D(s)} = \frac{G(s)}{1 + G(s)} \tag{1.9.1}$$

The characteristic equation is

$$D(s) = 0 \tag{1.9.2}$$
$$1 + G(s) = 0 \tag{1.9.3}$$

$$s^2 + 3s + 4 = 0 \tag{1.9.4}$$
$$=> s = -1.5 + 1.3j, -1.5 - 1.3j \tag{1.9.5}$$

The above poles lie on left half of s-plane.So the closed loop system with unity negative feedback is stable.