On the Concavity of the Sum-Rate Function in OFDM Systems

Jeonghee Chi, Jieun Kim, Youjung Ha, and Hyang-Won Lee, *Member, IEEE*

***Abstract*—We study the power allocation problem for maximizing the sum rate in orthogonal frequency-division multiplexing systems, where multiple point-to-point links share a set of subcarriers. This problem is hard to solve, in general, due to its nonconvexity. In this letter, we characterize the conditions for the problem to be convex, in which case the problem can be solved efficiently using known algorithms such as gradient projection method. We validate our analysis through simulations.**

***Index Terms*—OFDM, sum-rate function, concavity conditions, non-convex optimization.**

I. Introduction

WITH the explosive growth of the number of mobile devices, it is often the case that several devices attempt to transmit data simultaneously in an uncoordinated manner. For instance, smartphone users in the vicinity may want to exchange data through Bluetooth, in which case multiple point-to-point links would transmit over the same frequency. These uncoordinated communications may disrupt each other by generating severe interference. The problem of managing interference will thus become more important as the number of mobile devices increases. In this paper, we focus on understanding the fundamental properties of sum-rate maximization problems in OFDM systems.

The sum-rate maximization problem is non-convex and thus most of the previous works focus on developing heuristic algorithms (for maximizing the sum-rate) under some simplifying assumptions. In [1], the so-called iterative water-filling algorithm is developed. Although the method is simple and can be readily implemented in a distributed manner, the obtained solution may significantly deviates from the optimum under severe interference. In [2], the sum-rate maximization problem is convexified and solved under the assumption that the Signal-to-Interference-plus-Noise-Ratio (SINR) of each user is high. In [3] and [4], the problem is solved in the low interference regime. The work in [5] develops the (time-sharing) condition for nonconvex spectrum management problem to have zero duality, and algorithms for solving the problem. However, it is hard to characterize when the time-sharing condition holds.

Recently, it was shown in [6] that the problem is NP-Hard in general. Furthermore, the authors characterize the conditions that the optimal solution is Frequency Division Multiple Access (FDMA), i.e., each subcarrier is used by at most one user. Such a case arises when interference channel gains are high and thus users generate severe interference to each other.

In contrast to [6], our primary focus in this paper is on the low interference regime where interference channel gains are small. We analyze the convexity property of the sum-rate maximization problem in OFDM systems. We show that the problem is convex when interference channel gains are all small enough, by characterizing the channel gain conditions for the sum-rate function to be concave. Our results in this paper thus provide a specific region where the sum-rate maximization problem can be solved efficiently.

Manuscript received January 2, 2014; revised June 1, 2014; accepted June 22, 2014. Date of publication July 1, 2014; date of current version September 8, 2014. This work was supported by the Faculty Research Fund of Konkuk University in 2012. The associate editor coordinating the review of this paper and approving it for publication was Y. Li. *(Corresponding author: H.-W. Lee.)*

The authors are with the Department of Internet and Multimedia Engineering, Konkuk University, Seoul 143-701, Korea (e-mail: jhchi@konkuk.ac.kr; kjeun56@konkuk.ac.kr; ujlub@konkuk.ac.kr; leehw@konkuk.ac.kr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/LCOMM.2014.2334323

II. System Model and Problem Description

Consider an OFDM system where there are K peer-to-peer wireless communication links or users (We will use link and user interchangeably throughout the paper). Each user consists of a transmitter and a receiver, and the users share N subcarriers. Denote by g_{lk}^n the channel gain from the transmitter of user l to the receiver of user k. Let p_k^n be the transmit power of user k over subcarrier n. Suppose that the subcarriers are orthogonal, that is, there is no interference between different subcarriers. Let η_k^n be the thermal noise that user k experiences over subcarrier n, and $\sum_{l\neq k} g_{lk}^n p_l^n$ be the interference from other users to user k. In this letter, we study the following problem of maximizing the sum rate.

$$\max \quad \sum_{k=1}^{K}\sum_{n=1}^{N}\log\left(1+\frac{g_{kk}^n p_k^n}{\eta_k^n+\sum\limits_{l\neq k} g_{lk}^n p_l^n}\right) \tag{1}$$

$$\text{subject to} \quad \sum_{n=1}^{N} p_k^n \leq P_k^{\max},\ \forall k=1,\ldots,K \tag{2}$$

$$p_k^n \geq 0,\ k=1,\ldots,K,\ n=1,\ldots,N. \tag{3}$$

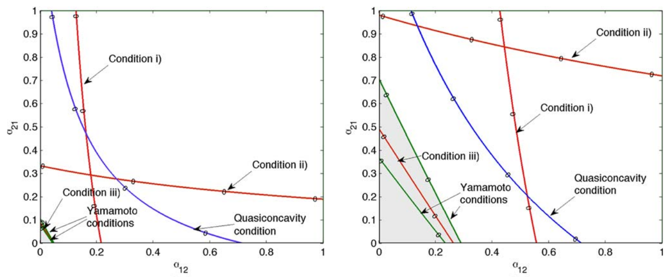
The constraint of (2) indicates that the total transmit power of user k cannot exceed the peak power $P_k^{\max}$. It is well known that the above problem is difficult to solve in general due to its non-convexity. Our goal in this letter is to characterize the conditions for the objective function in (1) to be concave. Note that if the interference channel gain $g_{lk}^n = 0, \forall l \neq k, \forall n$, then the objective function is concave. It is thus conceivable that if the interference channel gains are all sufficiently small, the objective function is concave as well. In the following, we quantify how small the channel gains should be for the concavity to hold, and validate the analysis through simulations.

III. Channel Gain Conditions for Concavity

A. The Case of 2 Users and 1 Subcarrier

We start with the case of $K=2$ and $N=1$, i.e., a subcarrier is shared by 2 users. Let $\alpha_{12}=g_{12}/g_{22}$, $\alpha_{21}=g_{21}/g_{11}$, $\sigma_1=\eta_1/g_{11}$, and $\sigma_2=\eta_2/g_{22}$. The objective function is then

$$R(p)=\log\left(1+\frac{p_1}{\sigma_1+\alpha_{21}p_2}\right)+\log\left(1+\frac{p_2}{\sigma_2+\alpha_{12}p_1}\right). \tag{4}$$



(a) (b)

Fig. 1. Comparison with conditions in [8] and quasi-concavity condition (5). (a) $P_1^{\max} = P_2^{\max} = 5, \sigma_1 = 2.1, \sigma_2 = 1.5$. (b) $P_1^{\max} = P_2^{\max} = 1, \sigma_1 = 2.1, \sigma_2 = 1.5$.

For brevity, we have omitted the superscript n of subcarriers. The following three conditions are sufficient for the concavity of $R(p)$:

i) $-a_1 + (\alpha_{12})^2(b_2 - a_2) < 0$
ii) $-a_2 + (\alpha_{21})^2(b_1 - a_1) < 0$
iii) $(\alpha_{21}a_1 + \alpha_{12}a_2)^2 - \{-a_1 + (\alpha_{12})^2(b_2 - a_2)\}\{-a_2 + \alpha_{21}^2(b_1 - a_1)\} \leq 0$,

for every possible values of a_1, b_1, a_2, and b_2 such that

$$\frac{2}{(\sigma_1 + \alpha_{21}P_2^{\max} + P_1^{\max})^2} \leq a_1 \leq \frac{2}{(\sigma_1)^2}$$
$$\frac{2}{(\sigma_1 + \alpha_{21}P_2^{\max})^2} \leq b_1 \leq \frac{2}{(\sigma_1)^2}$$
$$\frac{2}{(\sigma_2 + \alpha_{12}P_1^{\max} + P_2^{\max})^2} \leq a_2 \leq \frac{2}{(\sigma_2)^2}$$
$$\frac{2}{(\sigma_2 + \alpha_{12}P_1^{\max})^2} \leq b_2 \leq \frac{2}{(\sigma_2)^2}.$$

Theorem 1: If the channel gains α_{12} and α_{21} meet the conditions i), ii) and iii), then $R(p)$ is concave.

Proof: See Appendix A. ■

The variables a_1, a_2, b_1, and b_2 (precisely defined in Appendix A) are in fact coupled. For instance, if a_1 is close to its maximum value, all the other variables would be close to their maximum values as well. This clearly gives more room for the channel gains to satisfy conditions i), ii) and iii). Consequently, our conditions (derived assuming that a_1, a_2, b_1, and b_2 are uncorrelated) are only sufficient and may be loose. We compare our conditions with the following condition

$$\alpha_{12}\alpha_{21}\left(\frac{P_1^{\max}}{\sigma_1} + \frac{P_2^{\max}}{\sigma_2}\right) + \alpha_{12}\frac{\sigma_1}{\sigma_2} + \alpha_{21}\frac{\sigma_2}{\sigma_1} < 1. \quad (5)$$

The condition (5) is a sufficient condition for the sum-rate function to be quasi-concave, which is an immediate consequence of the proof of Theorem 3.2 in [6]. Furthermore, it is also necessary for quasi-concavity under some conditions (see Theorem 5 in [7]). Since a concave function is always quasi-concave, the condition (5) can provide a necessary condition for concavity in some cases, although the necessity is not always guaranteed.

Fig. 1 compares our conditions, the quasi-concavity condition, and the conditions developed in [8]. Our conditions and Yamamoto's conditions give about the same region for concavity. As seen in Fig. 1(a), our conditions (and Yamamoto's conditions) can possibly be very loose. Note, however, that when $P_1^{\max}$ and $P_2^{\max}$ are small, our conditions and Yamamoto's are relatively closer to the quasi-concavity condition (Fig. 1(b)).

B. The Case of K Users and N Subcarriers

For the general case of K users and N subcarriers, we can similarly derive the conditions for the objective function to be concave. Define $\alpha^n_{lk} = g^n_{lk}/g^n_{kk}$ and $\sigma^n_k = \eta^n_k/g^n_{kk}$. Let $\mathcal{K} = \{1, 2, \ldots, K\}$ and $\mathcal{N} = \{1, 2, \ldots, N\}$. The indices k, l, m are reserved for users and n for subcarriers. Consider the following extended version of conditions i), ii) and iii):

iv) $-a^n_l + \sum_{k\neq l}(\alpha^n_{lk})^2(b^n_k - a^n_k) < 0, \forall l \in \mathcal{K}, n \in \mathcal{N}$
v) $(K-1)^2\{-\alpha^n_{ml}a^n_l - \alpha^n_{lm}a^n_m + \sum_{k\neq m,l}\alpha^n_{lk}\alpha^n_{mk}(b^n_k - a^n_k)\}^2 - \{-a^n_l + \sum_{k\neq l}(\alpha^n_{lk})^2(b^n_k - a^n_k)\}\{-a^n_m + \sum_{k\neq m}(\alpha^n_{mk})^2 (b^n_k - a^n_k)\} \leq 0, \forall m > l, l \in \mathcal{K}, n \in \mathcal{N}$,

for every $a^n_k, b^n_k, k \in \mathcal{K}, n \in \mathcal{N}$ such that

$$\frac{2}{(\sigma^n_k + \sum_l \alpha^n_{lk}P_l^{\max})^2} \leq a^n_k \leq \frac{2}{(\sigma^n_k)^2}$$
$$\frac{2}{\left(\sigma^n_k + \sum_{l\neq k}\alpha^n_{lk}P_l^{\max}\right)^2} \leq b^n_k \leq \frac{2}{(\sigma^n_k)^2}.$$

Note that a^n_k and b^n_k correspond to a_1 and b_1 (or a_2 and b_2) in the case of $K = 2$ and $N = 1$.

Theorem 2: The objective function is concave if the channel gains α^n_{lk} satisfy the conditions iv) and v).

Proof: The proof is a straightforward extension of the proof of Theorem 1, and can be found in Appendix B.[1] ■

Note that the conditions i), ii) and iii) in Theorem 1 are a special case of the conditions iv) and v). As shown in Theorems 1 and 2, if the interference channel gains are small, the objective function is concave, in which case the problem is a convex optimization and thus can be solved using the standard method such as primal-dual gradient projection method.

In [8], Yamamoto *et al.* analyzes the convexity property of the sum-rate maximization problem. A sufficient condition for convexity is derived based on the fact that a function is concave if and only if its Hessian is negative semidefinite, and the fact that a symmetric matrix with nonnegative diagonal entries is positive semidefinite if it is diagonally dominant. Our approach works directly on the definition of concave function, and uses the fact that a continuous function is convex if and only if it is midpoint convex. As a result, our conditions are different from the conditions developed in [8]. We remark that our research has been conducted independently without the knowledge that Yamamoto *et al.* investigated the same problem.

IV. Simulation Results

We validate our analysis in the previous section by comparing the following well-known algorithms:

- Gradient Projection Method (GPM) that iteratively updates the transmit power vector via primal-dual gradient projection. GPM converges to an optimal solution with sufficiently small step size if the problem is convex [9].
- FDMA in [6] that finds a power vector such that no subcarriers are shared by two or more users. This algorithm performs well in the high interference regime.
- IWFA in [1] that iteratively updates the power vector by regarding interference as a constant and applying water-filling algorithm. This algorithm performs well in the low interference regime.

[1]The full version of this paper is available online at http://home.konkuk.ac.kr/~leehw/concavityfull.pdf.

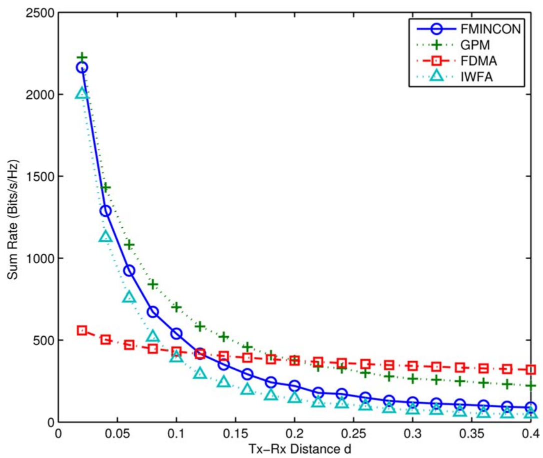


Fig. 2. Comparison of sum-rate performance: $K = 16$, $N = 20$.

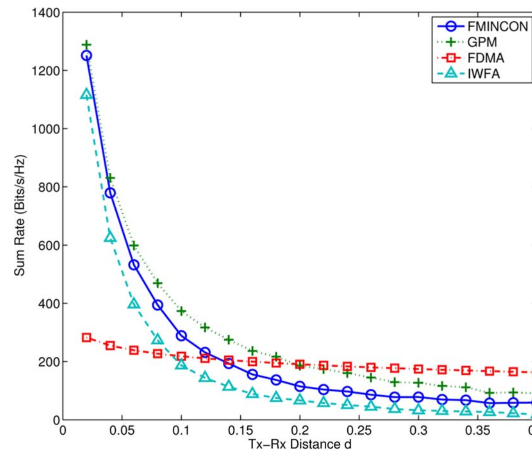


Fig. 3. Comparison of sum-rate performance: $K = 20$, $N = 10$.

- FMINCON in MATLAB that solves general optimization problems including convex and non-convex. We chose the interior point algorithm in FMINCON, as recommended by MATLAB.

For simulations, the distance between the transmitter and receiver of each user is equally set to d. For a given value of d, we place each transmitter in the unit square uniformly at random, pick an angle from 0° to 360° uniformly at random, and then place its receiver d distance apart from the transmitter (possibly outside the unit square); so that the angle between the horizontal axis and the transmitter-receiver line is equal to the random angle. The channel gain g_{lk}^n is generated as $g_{lk}^n = |cg_{lk}^n d_{lk}^{-2}|^2$ where cg_{lk}^n is the complex standard normal random variable and d_{lk} is the distance between transmitter l and receiver k. The thermal noise η_k^n is -40 dB. The peak power $P_k^{\max}$ is 5 dB for all k. There are 20 subcarriers shared by 16 users randomly placed in the unit square. The value of d is varied from 0.02 to 0.4 at 0.02 intervals.

Fig. 2[2] shows that when d (Tx-Rx Distance) is small, the sum-rates of FMINCON and GPM are higher than that of FDMA. For small values of d, the channel gains α_{lk}^n are small as well. As shown in Theorem 1, 2, in this case, the problem might be convex. This is why GPM (that exactly solves convex problems) outperforms FDMA for small values of d. On the contrary, for large values of d, the problem may no longer be convex, and hence, GPM cannot guarantee an optimal solution. Fig. 2 clearly shows that in this case, FDMA yields the highest sum-rate. This result is consistent with the result in [6] that FDMA is optimal in the case of severe interference. Note further that IWFA performs relatively well in the low interference regime, however its performance is degraded significantly in the high interference regime.

We could observe similar results for other values of K and N, but present only the case of $K = 20$ and $N = 10$ in Fig. 3 due to the page limit. These results show that there is no algorithm that dominates others over the whole interference regime, and hence, different algorithms should be used depending on the level of interference.

[2]Each point in Figs. 2 and 3 is the average of valid results out of 100 different scenarios. We say a result is valid if GPM gives a feasible power allocation after 30000 iterations (Other algorithms terminate relatively quickly and always guarantee a feasible solution). Hence, for a simulation scenario, if GPM solution is not feasible after termination, all the results obtained by GPM, FMINCON, FDMA and IWFA under that scenario are discarded. Otherwise, the results are accounted for the calculation of average performance.

V. Conclusion and Future Work

In this letter we studied the power allocation problem for sum-rate in OFDM systems. This problem is hard to solve in general due to its non-convexity. We characterized the conditions for the problem to be convex, and showed that the problem becomes convex when interference channel gains are sufficiently small. It would be an interesting future study to investigate whether non-FDMA solutions are optimal under the conditions found in this work.

Appendix A
Proof of Theorem 1

By definition, $R(p)$ is concave if

$$\lambda R(p) + (1-\lambda) R(q) - R\left(\lambda p + (1-\lambda) q\right) \le 0 \tag{6}$$

for every feasible power vector p, q, and every $\lambda \in [0, 1]$. Because $R(p)$ is a continuous function, if (6) is satisfied for $\lambda = 1/2$, it is satisfied for every $\lambda \in [0, 1]$ [10]. Consequently, it suffices to show the inequality (6) for $\lambda = 1/2$.

Let $f_1(p) = \log(\sigma_1 + \alpha_{21} p_2 + p_1)$ and $g_1(p) = \log(\sigma_1 + \alpha_{21} p_2)$. Define $f_2(p)$ and $g_2(p)$ similarly. The rate function $R(p)$ can then be written as $R(p) = f_1(p) - g_1(p) + f_2(p) - g_2(p)$. It follows that

$$\begin{aligned}
&\lambda R(p) + (1-\lambda) R(q) - R\left(\lambda p + (1-\lambda) q\right) \\
&= \lambda \left(f_1(p) - f_1\left(\lambda p + (1-\lambda) q\right)\right) \\
&\quad + (1-\lambda)\left(f_1(q) - f_1\left(\lambda p + (1-\lambda) q\right)\right) \\
&\quad - \lambda \left(g_1(p) - g_1\left(\lambda p + (1-\lambda) q\right)\right) \\
&\quad - (1-\lambda)\left(g_1(q) - g_1\left(\lambda p + (1-\lambda) q\right)\right) \\
&\quad + \lambda \left(f_2(p) - f_2\left(\lambda p + (1-\lambda) q\right)\right) \\
&\quad + (1-\lambda)\left(f_2(q) - f_2\left(\lambda p + (1-\lambda) q\right)\right) \\
&\quad - \lambda \left(g_2(p) - g_2\left(\lambda p + (1-\lambda) q\right)\right) \\
&\quad - (1-\lambda)\left(g_2(q) - g_2\left(\lambda p + (1-\lambda) q\right)\right).
\end{aligned} \tag{7}$$

We use the following second order form of Taylor's theorem (also known as second order mean value theorem).

Theorem 3 ([9]): Let S be a nonempty open convex set in $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be twice differentiable. Then, for every $\mathbf{x}_1$ and $\mathbf{x}_2$ in S,

$$f(\mathbf{x}_2) = f(\mathbf{x}_1) + \nabla f(\mathbf{x}_1)^T(\mathbf{x}_2 - \mathbf{x}_1) + \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_1)^T\nabla^2 f(\mathbf{x})(\mathbf{x}_2 - \mathbf{x}_1),$$

where $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for some $\lambda \in [0, 1]$.

By Taylor's theorem, the first in (7) term can be written as

$$\begin{aligned}\lambda\{f_1(p) - f_1(\lambda p + (1-\lambda)q)\} &= \lambda\Big\{\nabla f_1(\lambda p + (1-\lambda)q)^T(1-\lambda)(p-q) \\ &\quad + \frac{1}{2}(1-\lambda)(p-q)^T\nabla^2 f_1\left(r^{(1)}\right)(1-\lambda)(p-q)\Big\}, \quad (8)\end{aligned}$$

where $r^{(1)} = \left[r_1^{(1)}\ r_2^{(1)}\right]^T$ is a convex combination of p and $\lambda p + (1-\lambda)q$. Similarly, applying the Taylor's Theorem to the second term in (7) yields

$$\begin{aligned}(1-\lambda)\{f_1(q) - f_1(\lambda p + (1-\lambda)q)\} &= (1-\lambda)\Big\{\nabla f_1(\lambda p + (1-\lambda)q)^T(-\lambda)(p-q) \\ &\quad + \frac{1}{2}(-\lambda)(p-q)^T\nabla^2 f_1\left(r^{(2)}\right)(-\lambda)(p-q)\Big\}, \quad (9)\end{aligned}$$

where $r^{(2)}$ is a convex combination of q and $\lambda p + (1-\lambda)q$. Letting $\lambda = 1/2$ and adding (8) and (9) yield

$$\begin{aligned}&\frac{1}{4}\nabla f_1\left(\frac{p}{2}+\frac{q}{2}\right)^T(p-q) + \frac{1}{16}(p-q)^T\nabla^2 f_1\left(r^{(1)}\right)(p-q) \\ &- \frac{1}{4}\nabla f_1\left(\frac{p}{2}+\frac{q}{2}\right)^T(p-q) + \frac{1}{16}(p-q)^T\nabla^2 f_1\left(r^{(2)}\right)(p-q) \\ &= -\frac{1}{16}(p_1 - q_1 + \alpha_{21}(p_2-q_2))^2 a_1, \quad (10)\end{aligned}$$

where $a_1 = (1/(\sigma_1 + \alpha_{21}r_2^{(1)} + r_1^{(1)})^2) + (1/(\sigma_1 + \alpha_{21}r_2^{(2)} + r_1^{(2)})^2)$. Similarly, the third and fourth terms in (7) can be written as

$$\frac{1}{16}(\alpha_{21})^2(p_2-q_2)^2 b_1$$

where $b_1 = (1/(\sigma_1 + \alpha_{21}r_2^{(3)})^2) + (1/(\sigma_1 + \alpha_{21}r_2^{(4)})^2)$ and $r^{(3)}$ (respectively, $r^{(4)}$) is a convex combination of p and $\lambda p + (1-\lambda)q$ (respectively, q and $\lambda p + (1-\lambda)q$). Hence, the first four terms in (7) are

$$-\frac{1}{16}a_1(p_1 - q_1 + \alpha_{21}(p_2-q_2))^2 + \frac{1}{16}b_1(\alpha_{21})^2(p_2-q_2)^2.$$

By symmetry, the last four terms in (7) are

$$-\frac{1}{16}a_2(p_2 - q_2 + \alpha_{12}(p_1-q_1))^2 + \frac{1}{16}b_2(\alpha_{12})^2(p_1-q_1)^2,$$

where $a_2 = (1/(\sigma_2 + \alpha_{12}r_1^{(5)} + r_2^{(5)})^2) + (1/(\sigma_2+\alpha_{12}r_1^{(6)} + r_2^{(6)})^2)$, $b_2 = (1/(\sigma_2 + \alpha_{12}r_1^{(7)})^2) + (1/(\sigma_2 + \alpha_{12}r_1^{(8)})^2)$, and $r^{(5)}$, $r^{(6)}$, $r^{(7)}$, $r^{(8)}$ are defined similarly to $r^{(1)}$, $r^{(2)}$, $r^{(3)}$, $r^{(4)}$. Consequently, for $\lambda = 1/2$, the left hand side of (6) can be written as

$$\begin{aligned}&\lambda R(p) + (1-\lambda)R(q) - R(\lambda p + (1-\lambda)q) \\ &= -\frac{1}{16}a_1(p_1 - q_1 + \alpha_{21}(p_2-q_2))^2 + \frac{1}{16}b_1\alpha_{21}^2(p_2-q_2)^2 \\ &\quad - \frac{1}{16}a_2(p_2 - q_2 + \alpha_{12}(p_1-q_1))^2 + \frac{1}{16}b_2\alpha_{12}^2(p_1-q_1)^2 \\ &= \frac{1}{16}\left(-a_1 + \alpha_{12}^2(b_2-a_2)\right)(p_1-q_1)^2 \\ &\quad + \frac{1}{16}\left(-a_2 + \alpha_{21}^2(b_1-a_1)\right)(p_2-q_2)^2 \\ &\quad - \frac{1}{8}(\alpha_{21}a_1 + \alpha_{12}a_2)(p_1-q_1)(p_2-q_2). \quad (11)\end{aligned}$$

Note that the ranges of a_1, b_1, a_2, and b_2 are

$$\begin{aligned}\frac{2}{(\sigma_1 + \alpha_{21}P_2^{\max} + P_1^{\max})^2} &\le a_1 \le \frac{2}{(\sigma_1)^2} \\ \frac{2}{(\sigma_1 + \alpha_{21}P_2^{\max})^2} &\le b_1 \le \frac{2}{(\sigma_1)^2} \\ \frac{2}{(\sigma_2 + \alpha_{12}P_1^{\max} + P_2^{\max})^2} &\le a_2 \le \frac{2}{(\sigma_2)^2} \\ \frac{2}{(\sigma_2 + \alpha_{12}P_1^{\max})^2} &\le b_2 \le \frac{2}{(\sigma_2)^2}.\end{aligned}$$

Therefore, the expression in (11) is non-positive if the conditions i), ii) and iii) are satisfied. This completes the proof.

References

[1] W. Yu, G. Ginis, and J. M. Cioffi, "Distributed multiuser power control for digital subscriber lines," *IEEE J. Sel. Areas Commun.*, vol. 20, no. 5, pp. 1105–1115, Jun. 2002.

[2] M. Chiang, "Balancing transport and physical layers in wireless multihop networks: Jointly optimal congestion control and power control," *IEEE J. Sel. Areas Commun.*, vol. 23, no. 1, pp. 104–116, Jan. 2005.

[3] B. Radunovic and J. Le Boudec, "Optimal power control, scheduling, routing in UWB networks," *IEEE J. Sel. Areas Commun.*, vol. 22, no. 7, pp. 1252–1270, Sep. 2004.

[4] R. Cruz and A. Santhanam, "Optimal routing, link scheduling and power control in multihop wireless networks," in *Proc. IEEE INFOCOM*, San Francisco, CA, USA, Mar. 30–Apr. 3, 2003, pp. 702–711.

[5] W. Yu and R. Lui, "Dual methods for nonconvex spectrum optimization of multicarrier systems," *IEEE Trans. Commun.*, vol. 54, no. 7, pp. 1310–1322, Jul. 2006.

[6] S. Hayashi and Z.-Q. Luo, "Spectrum management for interference-limited multiuser communication systems," *IEEE Trans. Inf. Theory*, vol. 55, no. 3, pp. 1153–1175, Mar. 2009.

[7] W. E. Diewert, M. Avriel, and I. Zang, "Nine kinds of quasiconcavity and concavity," *J. Econ. Theory*, vol. 25, no. 3, pp. 379–420, Dec. 1981.

[8] D. Yamamoto, *Convexity analysis and splitting algorithm for the sum-rate maximization problem*. Kyoto, Japan: Graduate School of Informatics, Kyoto University, Feb. 2010.

[9] D. Bertsekas, *Nonlinear Programming*. Belmont, MA, USA: Athena Scientific, 1999.

[10] C. Niculescu and L.-E. Persson, *Convex Functions and Their Applications: A Contemporary Approach*. New York, NY, USA: Springer-Verlag, 2006.