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1 CONVEX FUNCTIONS

A single variable function f is said to be convex if

$$f[\lambda x + (1 - \lambda)y] \leq \lambda f(x) + (1 - \lambda)f(y), \quad (1.1)$$

for $0 < \lambda < 1$.

Problem 1.1. Download and execute the following python script. Is $\ln x$ convex or concave?

```
wget https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/1.1.py
```

Problem 1.2. Modify the above python script as follows to plot the parabola $f(x) = x^2$. Is it convex or concave?

```
wget https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/1.2.py
```

Problem 1.3. Execute the following script to obtain Fig. 1.3. Comment.

```
wget https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/1.3.py
```

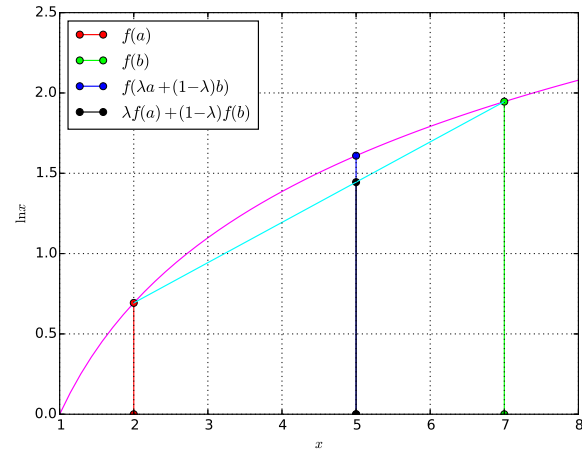


Fig. 1.1: $\ln x$ versus x

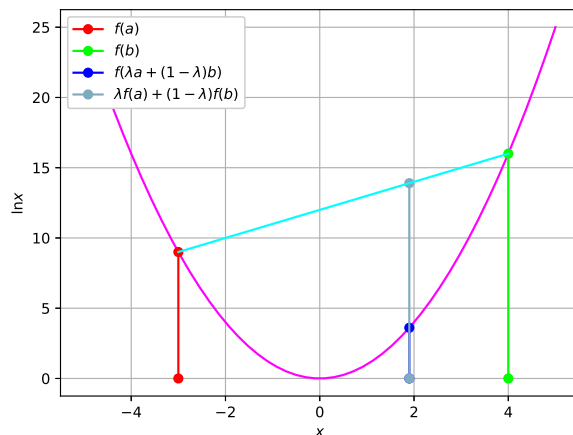


Fig. 1.2: x^2 versus x

Problem 1.4. Modify the script in the previous problem for $f(x) = x^2$. What can you conclude?

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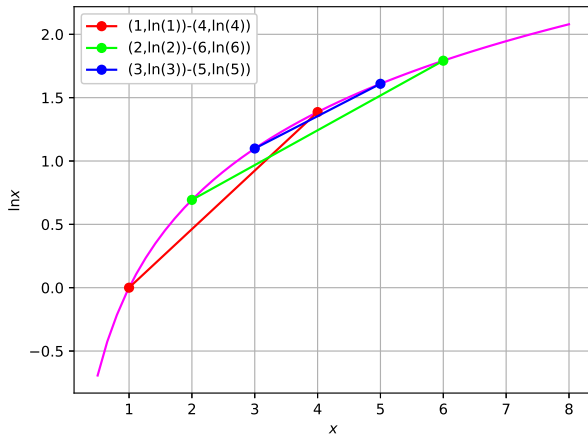
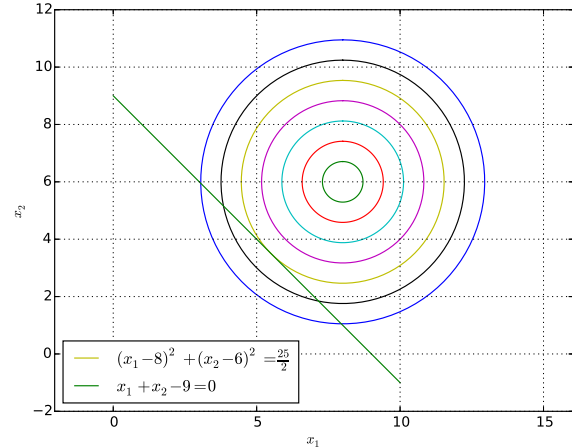


Fig. 1.3: Segments are below the curve

Fig. 2.1: Finding $\min_{\mathbf{x}} f(\mathbf{x})$

Problem 1.5. Let

$$f(\mathbf{x}) = x_1 x_2, \quad \mathbf{x} \in \mathbf{R}^2 \quad (1.2)$$

Sketch $f(\mathbf{x})$ and deduce whether it is convex. Can you theoretically explain your observation using (1.1)?

2 CONVEX OPTIMIZATION

2.1 Karush Kuhn-Tucker Conditions

Problem 2.1. Plot the circles

$$f(\mathbf{x}) = (x_1 - 8)^2 + (x_2 - 6)^2 = r^2 \quad (2.1)$$

$\mathbf{x} = (x_1, x_2)^T$, for different values of r along with the line

$$g(\mathbf{x}) = x_1 + x_2 - 9 = 0 \quad (2.2)$$

using the following program. From the graph, find

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad (2.3)$$

$$g(\mathbf{x}) = x_1 + x_2 - 9 = 0 \quad (2.4)$$

wget <https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/2.1.py>

Problem 2.2. Obtain a theoretical solution for problem 2.1 using coordinate geometry.

Solution: From (2.2) and (2.1),

$$r^2 = (x_1 - 8)^2 + (3 - x_1)^2 \quad (2.5)$$

$$= 2x_1^2 - 22x_1 + 73 \quad (2.6)$$

$$\Rightarrow r^2 = \frac{(2x_1 - 11)^2 + 5^2}{2} \quad (2.7)$$

which is minimum when $x_1 = \frac{11}{2}, x_2 = \frac{7}{2}$. The minimum value is $\frac{25}{2}$ and the radius $r = \frac{5}{\sqrt{2}}$.

Problem 2.3. Define

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \quad (2.8)$$

and

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial \lambda} \end{pmatrix} \quad (2.9)$$

Solve the equations

$$\nabla L(\mathbf{x}, \lambda) = 0. \quad (2.10)$$

How is this related to problem 2.1? What is the sign of λ ? L is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

Solution: From (2.2) and (2.1),

$$L(\mathbf{x}, \lambda) = (x_1 - 8)^2 + (x_2 - 6)^2 - \lambda(x_1 + x_2 - 9) \quad (2.11)$$

$$\Rightarrow \nabla L(\mathbf{x}, \lambda) = \begin{pmatrix} 2x_1 - 16 - \lambda \\ 2x_2 - 12 - \lambda \\ x_1 + x_2 - 9 \end{pmatrix} \quad (2.12)$$

$$= \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 16 \\ 12 \\ 9 \end{pmatrix} = 0 \quad (2.13)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ \frac{7}{2} \\ -5 \end{pmatrix} \quad (2.14)$$

using the following python script. Note that this method yields the same result as the previous exercises. Thus, λ is negative.

wget <https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/2.3.py>

Problem 2.4. Modify the code in problem 2.1 to find a graphical solution for minimising

$$f(\mathbf{x}) = (x_1 - 8)^2 + (x_2 - 6)^2 \quad (2.15)$$

with constraint

$$g(\mathbf{x}) = x_1 + x_2 - 9 \geq 0 \quad (2.16)$$

Solution: This problem reduces to finding the radius of the smallest circle in the shaded area in Fig. 2.4. It is clear that this radius is 0.

wget <https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/2.4.py>

Problem 2.5. Now use the method of Lagrange multipliers to solve problem 2.4 and compare with the graphical solution. Comment.

Solution: Using the method of Lagrange multipliers, the solution is the same as the one obtained in problem 2.4, which is different from the graphical solution. This means that the Lagrange multipliers method cannot be applied blindly.

Problem 2.6. Repeat problem 2.5 by keeping $\lambda = 0$. Comment.

Solution: Keeping $\lambda = 0$ results in $x_1 = 8, x_2 = 6$, which is the correct solution. The minimum value

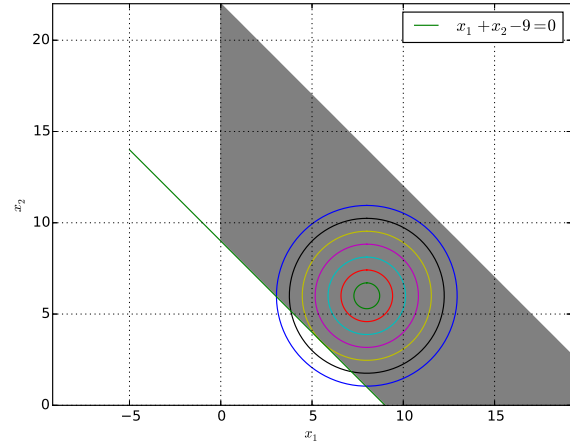


Fig. 2.4: Smallest circle in the shaded region is a point.

of $f(\mathbf{x})$ without any constraints lies in the region $g(\mathbf{x}) = 0$. In this case, $\lambda = 0$.

Problem 2.7. Find a graphical solution for minimising

$$f(\mathbf{x}) = (x_1 - 8)^2 + (x_2 - 6)^2 \quad (2.17)$$

with constraint

$$g(\mathbf{x}) = x_1 + x_2 - 9 \leq 0. \quad (2.18)$$

Summarize your observations.

Solution: In Fig. 2.7, the shaded region represents the constraint. Thus, the solution is the same as the one in problem 2.4. This implies that the method of Lagrange multipliers can be used to solve the optimization problem with this inequality constraint as well. Table 2.7 summarizes the conditions for this based on the observations so far.

wget <https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/2.7.py>

TABLE 2.7: Summary of conditions.

Cost	Constraint	λ
$f(\mathbf{x})$	$g(\mathbf{x}) = 0$	< 0
	$g(\mathbf{x}) \geq 0$	0
	$g(\mathbf{x}) \leq 0$	< 0

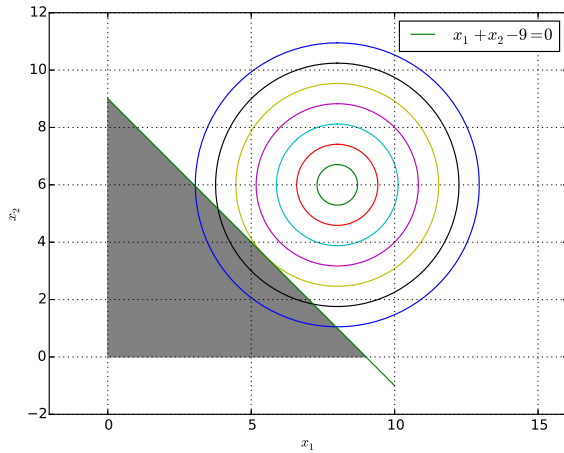


Fig. 2.7: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

Problem 2.8. Find a graphical solution for

$$\min_{\mathbf{x}} f(\mathbf{x}) = (x_1 - 8)^2 + (x_2 - 6)^2 \quad (2.19)$$

with constraint

$$g(\mathbf{x}) = x_1 + x_2 - 18 = 0 \quad (2.20)$$

Solution:

wget <https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/2.8.py>

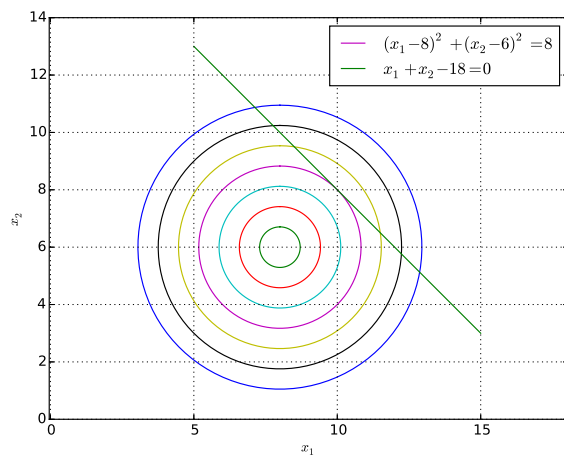


Fig. 2.8: Finding $\min_{\mathbf{x}} f(\mathbf{x})$.

Problem 2.9. Repeat problem 2.8 using the method of Lagrange multipliers. What is the sign of λ ?

Solution: From (2.25) and (2.25),

$$L(\mathbf{x}, \lambda) = (x_1 - 8)^2 + (x_2 - 6)^2 - \lambda(x_1 + x_2 - 18) \quad (2.21)$$

$$\Rightarrow \nabla L(\mathbf{x}, \lambda) = \begin{pmatrix} 2x_1 - 16 - \lambda \\ 2x_2 - 12 - \lambda \\ x_1 + x_2 - 18 \end{pmatrix} \quad (2.22)$$

$$= \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 16 \\ 12 \\ 18 \end{pmatrix} = 0 \quad (2.23)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \lambda \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 4 \end{pmatrix} \quad (2.24)$$

using the following python script. Thus, λ is positive and the minimum value of f is 8.

wget <https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/2.9.py>

Problem 2.10. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = (x_1 - 8)^2 + (x_2 - 6)^2 \quad (2.25)$$

with constraint

$$g(\mathbf{x}) = x_1 + x_2 - 18 \geq 0 \quad (2.26)$$

Solution: Since the unconstrained solution is outside the region $g(\mathbf{x}) \geq 0$, the solution is the same as the one in problem 2.8.

Problem 2.11. Based on the problems so far, generalise the Lagrange multipliers method for

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \geq 0 \quad (2.27)$$

Solution: Considering $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x})$, for $g(\mathbf{x}) = x_1 + x_2 - 18 \geq 0$ we found $\lambda > 0$ and for $g(\mathbf{x}) = x_1 + x_2 - 9 \leq 0$, $\lambda < 0$. A single condition can be obtained by framing the optimization problem as

$$\min_{\mathbf{x}} f(\mathbf{x}), \quad g(\mathbf{x}) \leq 0 \quad (2.28)$$

with the Lagrangian

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g(\mathbf{x}), \quad (2.29)$$

provided

$$\nabla L(\mathbf{x}, \lambda) = 0 \Rightarrow \lambda > 0 \quad (2.30)$$

else, $\lambda = 0$.

Problem 2.12. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = 4x_1^2 + 2x_2^2 \quad (2.31)$$

with constraints

$$g_1(\mathbf{x}) = 3x_1 + x_2 - 8 = 0 \quad (2.32)$$

$$g_2(\mathbf{x}) = 15 - 2x_1 - 4x_2 \geq 0 \quad (2.33)$$

Solution: Considering the Lagrangian

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda g_1(\mathbf{x}) - \mu g_2(\mathbf{x}) \quad (2.34)$$

$$= 4x_1^2 + 2x_2^2 + \lambda(3x_1 + x_2 - 8) - \mu(15 - 2x_1 - 4x_2), \quad (2.35)$$

$$\nabla L(\mathbf{x}, \lambda) = \begin{pmatrix} 8x_1 + 3\lambda + 2\mu \\ 4x_2 + \lambda + 4\mu \\ 3x_1 + x_2 - 8 \\ -2x_1 - 4x_2 + 15 \end{pmatrix} = 0 \quad (2.36)$$

resulting in the matrix equation

$$\Rightarrow \begin{pmatrix} 8 & 0 & 3 & 2 \\ 0 & 4 & 1 & 4 \\ 3 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \\ 15 \end{pmatrix} \quad (2.37)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 1.7 \\ 2.9 \\ -3.12 \\ -2.12 \end{pmatrix} \quad (2.38)$$

using the following python script. The (incorrect) graphical solution is available in Fig. 2.12

```
wget https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/2.12.py
```

Note that $\mu < 0$, contradicting the necessary condition in (2.30).

Problem 2.13. Obtain the correct solution to the previous problem by considering $\mu = 0$.

Problem 2.14. Solve

$$\min_{\mathbf{x}} f(\mathbf{x}) = 4x_1^2 + 2x_2^2 \quad (2.39)$$

with constraints

$$g_1(\mathbf{x}) = 3x_1 + x_2 - 8 = 0 \quad (2.40)$$

$$g_2(\mathbf{x}) = 15 - 2x_1 - 4x_2 \leq 0 \quad (2.41)$$

Problem 2.15. Based on whatever you have done so far, list the steps that you would use in general

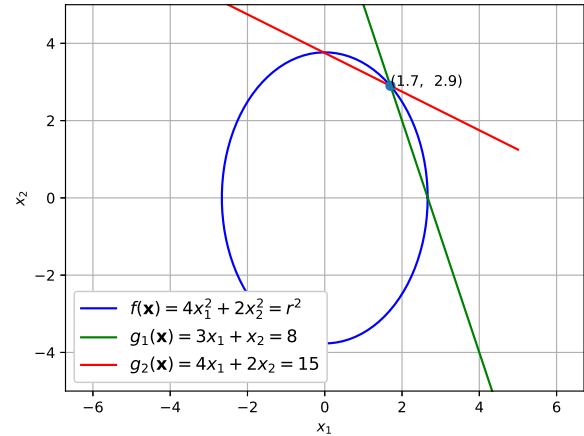


Fig. 2.12: Incorrect solution is at intersection of all curves $r = 5.33$

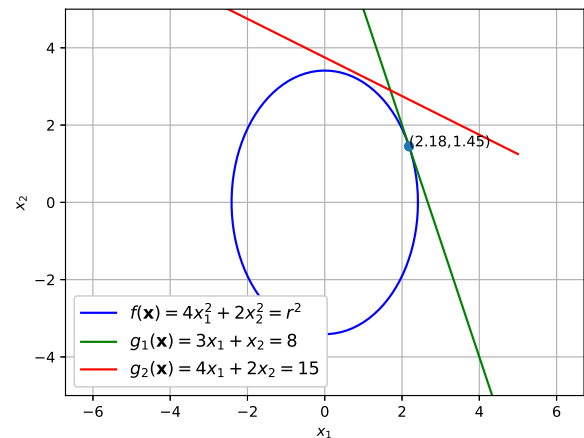


Fig. 2.13: Optimal solution is where $g_1(x)$ touches the curve $r = 4.82$

for solving a convex optimization problem like (2.39) using Lagrange Multipliers. These are called Karush-Kuhn-Tucker(KKT) conditions.

Solution: For a problem defined by

$$\mathbf{x}^* = \min_{\mathbf{x}} f(\mathbf{x}) \quad (2.42)$$

$$\text{subject to } h_i(\mathbf{x}) = 0, \forall i = 1, \dots, m \quad (2.43)$$

$$\text{subject to } g_i(\mathbf{x}) \leq 0, \forall i = 1, \dots, n \quad (2.44)$$

the optimal solution is obtained through

$$\mathbf{x}^* = \min_{\mathbf{x}} L(\mathbf{x}, \lambda, \mu) \quad (2.45)$$

$$= \min_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^m \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^n \mu_i g_i(\mathbf{x}), \quad (2.46)$$

using the KKT conditions

$$\Rightarrow \nabla_{\mathbf{x}} f(\mathbf{x}) + \sum_{i=1}^m \nabla_{\mathbf{x}} \lambda_i h_i(\mathbf{x}) + \sum_{i=1}^n \mu_i \nabla_{\mathbf{x}} g_i(\mathbf{x}) = 0 \quad (2.47)$$

$$\text{subject to } \mu_i g_i(\mathbf{x}) = 0, \forall i = 1, \dots, n \quad (2.48)$$

$$\text{and } \mu_i \geq 0, \forall i = 1, \dots, n \quad (2.49)$$

Problem 2.16. Maximize

$$f(\mathbf{x}) = \sqrt{x_1 x_2} \quad (2.50)$$

with the constraints

$$x_1^2 + x_2^2 \leq 5 \quad (2.51)$$

$$x_1 \geq 0, x_2 \geq 0 \quad (2.52)$$

Problem 2.17. Solve

$$\min_{\mathbf{x}} x_1 + x_2 \quad (2.53)$$

with the constraints

$$x_1^2 - x_1 + x_2^2 \leq 0 \quad (2.54)$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Solution: Using the method of Lagrange multipliers,

$$\nabla \{f(\mathbf{x}) + \mu g(\mathbf{x})\} = 0, \quad \mu \geq 0 \quad (2.55)$$

resulting in the equations

$$2x_1\mu - \mu + 1 = 0 \quad (2.56)$$

$$2x_2\mu + 1 = 0 \quad (2.57)$$

$$x_1^2 - x_1 + x_2^2 = 0 \quad (2.58)$$

which can be simplified to obtain

$$\left(\frac{1-\mu}{2\mu}\right)^2 + \left(\frac{1}{2\mu}\right)^2 + \frac{1-\mu}{2\mu} = 0 \quad (2.59)$$

$$\Rightarrow 1 + \mu^2 - 2\mu + 1 + 2\mu(1-\mu) = 0 \quad (2.60)$$

$$\Rightarrow \mu^2 = 2, \text{ or } \mu = \pm \sqrt{2} \quad (2.61)$$

From (2.39), $\mu \geq 0 \Rightarrow \mu = \sqrt{2}$. The desired solution is

$$\mathbf{x} = \begin{pmatrix} \frac{\sqrt{2}-1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{pmatrix} \quad (2.62)$$

Graphical solution: The constraint can be expressed as

$$x_1^2 - x_1 + x_2^2 \leq 0 \quad (2.63)$$

$$\Rightarrow \left(x_1 - \frac{1}{2}\right)^2 + x_2^2 \leq \left(\frac{1}{2}\right)^2 \quad (2.64)$$

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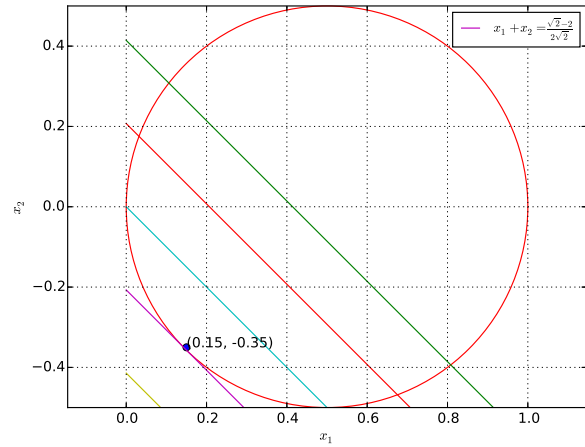


Fig. 2.17: Optimal solution is the lower tangent to the circle

3 SEMI-DEFINITE PROGRAMMING

Problem 3.1. The problem

$$\min_{\mathbf{x}} x_{11} + x_{12} \quad (3.1)$$

with constraints

$$x_{11} + x_{22} = 1 \quad (3.2)$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \geq 0 \quad (\geq \text{ means positive definite}) \quad (3.3)$$

is known as a semi-definite program. Find a numerical solution to this problem. Compare with the solution in problem 2.17.

Solution: The *cvxopt* solver needs to be used in order to find a numerical solution. For this, the given problem has to be reformulated as

$$\min_{\mathbf{x}} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} \quad \text{s.t.} \quad (3.4)$$

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{22} \end{pmatrix} = 1 \quad (3.5)$$

$$x_{11} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + x_{12} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + x_{22} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.6)$$

The following script provides the solution to this problem.

```
wget https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/3.1.py
```

Problem 3.2. Repeat the previous exercise using *cvxpy*

Solution:

```
wget https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/3.1-cvx.py
```

Problem 3.3. Minimize

$$-x_{11} - 2x_{12} - 5x_{22} \quad (3.7)$$

subject to

$$2x_{11} + 3x_{12} + x_{22} = 7 \quad (3.8)$$

$$x_{11} + x_{12} \geq 1 \quad (3.9)$$

$$x_{11}, x_{12}, x_{22} \geq 0 \quad (3.10)$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \geq 0 \quad (3.11)$$

using *cvxopt*.

Problem 3.4. Repeat the above exercise by converting the problem into a convex optimization problem in two variables and using graphical plots.

Problem 3.5. Solve the above problem using the KKT conditions. Comment.

4 LINEAR PROGRAMMING

Problem 4.1. Graphically obtain a solution to the following

$$\max_{\mathbf{x}} 6x_1 + 5x_2 \quad (4.1)$$

with constraints

$$x_1 + x_2 \leq 5 \quad (4.2)$$

$$3x_1 + 2x_2 \leq 12 \quad (4.3)$$

$$\text{where } x_1, x_2 \geq 0 \quad (4.4)$$

Solution: The following program plots the solution in Fig. 4.1

```
wget https://raw.githubusercontent.com/gadepall/EE2250/master/manual/codes/4.1.py
```

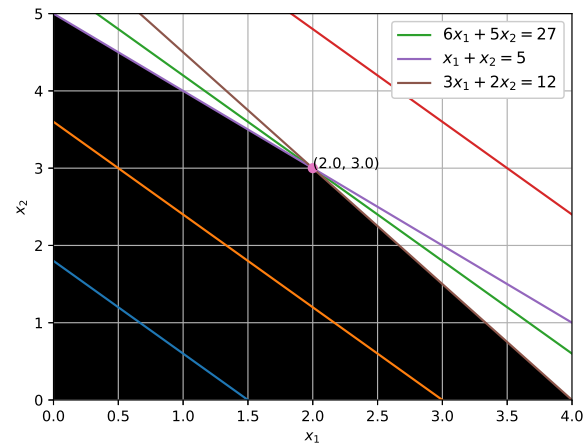


Fig. 4.1: The cost function intersects with the two constraints at $\mathbf{x} = (2, 3)$.

Problem 4.2. Now use *cvxopt* to obtain a solution to problem 4.1.

Solution: The given problem is expressed as follows

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad \text{s.t.} \quad (4.5)$$

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \quad (4.6)$$

where

$$\mathbf{c} = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 0 \\ 0 \end{pmatrix} \quad (4.7)$$

The desired solution is then obtained using the following program.

```
wget https://raw.githubusercontent.com/gadepall/
EE2250/master/manual/codes/4.2.py
```

Problem 4.3. Repeat the previous exercise using *cvxpy*

Solution:

```
wget https://raw.githubusercontent.com/gadepall/
EE2250/master/manual/codes/4.2-cvx.py
```

Problem 4.4. Verify your solution to the above problem using the method of Lagrange multipliers.

Problem 4.5. Maximise $5x_1 + 3x_2$ w.r.t the constraints

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ 5x_1 + 2x_2 &\leq 10 \\ 3x_1 + 8x_2 &\leq 12 \\ \text{where } x_1, x_2 &\geq 0 \end{aligned}$$