JEE Problems in Optimization

***Abstract*—A collection of problems from JEE papers related to 2D coordinate geometry are available in this document. These problems should be solved using Optimization techniques. Verify using *cvxpy*.**

1. Let $\mathbf{P}$ be the point on the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 8 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{1}$$
which is at a minimum distance from the centre $\mathbf{C}$ of the circle
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 0 & 12 \end{pmatrix} \mathbf{x} = 1 \tag{2}$$
Find the equation of the circle passing through $\mathbf{C}$ and having its centre at $\mathbf{P}$.
2. Let $\mathbf{P}$ be a point on the parabola
$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 & 4 \end{pmatrix} \mathbf{x} = 0 \tag{3}$$
Given that the distance of $\mathbf{P}$ from the centre of the circle
$$\mathbf{x}^T \mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{x} + 8 = 0 \tag{4}$$
is minimum. Find the equation of the tangent to the parabola at $\mathbf{P}$.
3. Find the eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half the distance between its foci.
4. $\mathbf{P}$ and $\mathbf{Q}$ are two distinct points on the parabola
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 4 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{5}$$
with parameters t and t_1 respectively. If the normal at $\mathbf{P}$ passes through $\mathbf{Q}$, then find the minimum value of t_1^2 using a descent algorithm.
5. A tangent at a point on the ellipse
$$\mathbf{x}^T V \mathbf{x} = 51 \tag{6}$$
where
$$V = \begin{pmatrix} 3 & 0 \\ 0 & 27 \end{pmatrix} \tag{7}$$
meets the coordinate axes at $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{O}$ be the origin, find the minimum area of $\triangle OAB$.
6. Find the shortest distance between the line
$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{8}$$
and the curve
$$\mathbf{x}^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} + 2 = 0 \tag{9}$$
7. Let S be the set of all complex numbers z satisfying $\left|z - 2 + \jmath\right| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, find the principal argument of
$$\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2\jmath} \tag{10}$$
8. Let $\omega \neq 1$ be a cube root of unity. Find the minimum of the set
$$\left|a + b\omega + c\omega^2\right|, \tag{11}$$
where a, b, c are distinct nonzero integers.
9. Let
$$\mathbf{M} = \begin{pmatrix} \sin^4\theta & -1 - \sin^2\theta \\ 1 + \cos^2\theta & \cos^4\theta \end{pmatrix} = \alpha\mathbf{I} + \beta\mathbf{M}^{-1} \tag{12}$$
where α, β are real functions of θ and $\mathbf{I}$ is the identity matrix. If
$$\alpha^* = \min_\theta \alpha(\theta) \tag{13}$$
$$\beta^* = \min_\theta \beta(\theta), \tag{14}$$
find $\alpha^* + \beta^*$.
10. Find the minimum value of
$$\cos(P + Q)\cos(Q + R)\cos(R + P) \tag{15}$$
in $\triangle PQR$.
11. Find the minimum value of α for which
$$4\alpha x^2 + \frac{1}{x} \geq 1, x > 0. \tag{16}$$

12. Let

$$S = S_1 \cap S_2 \cap S_3, \tag{17}$$

where

$$S_1 = \{z \in C : |z| < 4\} \tag{18}$$

$$S_2 = \left\{ z \in C : \Im \left[\frac{z - 1 + \jmath\sqrt{3}}{1 - \jmath\sqrt{3}} \right] \right\}, \tag{19}$$

$$S_3 = \{z \in C : \Re(z) > 0\} \tag{20}$$

Find

$$\min_{z \in S} \left|1 - 3\jmath - z\right| \tag{21}$$

13. A line

$$L : \left(m - 1\right)\mathbf{x} = -3 \tag{22}$$

passes through

$$\mathbf{E} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \tag{23}$$

and

$$\mathbf{x}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 16 & 0 \end{pmatrix}\mathbf{x} = 0, 0 \le \begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} \le 6 \tag{24}$$

at the point **F**. Find m such that the area of $\triangle EFG$ is maximum.

14. If $|z - 3 - 2j| \le 2$, find

$$\min_{z} \left|2z - 6 + 5\jmath\right| \tag{25}$$

15. Find

$$\max_{z} \left| Arg\left(\frac{1}{1 - z}\right) \right| \tag{26}$$

$$s.t \quad |z| = 1, z \ne 1. \tag{27}$$

16. Find the maximum value of the function

$$f(x) = 2x^3 - 15x^2 + 36x - 48 \tag{28}$$

on the set

$$A = \left\{x : x^2 + 20 \le 9x\right\} \tag{29}$$

17. Find the minimum distance of a point on the curve

$$\mathbf{x}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 - 1 \end{pmatrix}\mathbf{x} = 4 \tag{30}$$

from the origin.

18. Find

$$\min_{z} \left|z + \frac{1}{2}\right| \tag{31}$$

$$s.t \quad |z| \ge 2 \tag{32}$$

19. Find the minimum value of

$$\tan A + \tan B \tag{33}$$

$$s.t \quad A + B = 6, \tag{34}$$

$$A, B \ge 0 \tag{35}$$

20. Show that

$$\sin A_1 + \sin A_2 + \cdots + \sin A_n \le n \sin\left(\frac{A_1 + A_2 + \cdots + A_n}{n}\right)$$

$$0 < A_i < \pi, i = 1, 2, \ldots, n, n \ge 1 \tag{36}$$

21. Let $\mathbf{F}_1, \mathbf{F}_2$ be the foci of the standard ellipse with parameters a and b. If **P** be any point on the ellipse, find the maximum area of $\triangle PF_1F_2$.
22. A circle $\|\mathbf{x}\| = 1$ intersects the $X-$ axis at **P** and **Q**. Another circle with centre **Q** intersects this circle above the $X-$ axis at **R** and the line segent PQ at **S**. Find the maximum area of $\triangle QSR$.
23. Let **M** be a fixed point in the first quadrant. A line through **M** intersects the positive axes at **P**, **Q** respectively. If **O** be the origin, find the minimum area of $\triangle OPQ$.