

Contour Integration

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Abstract—This manual provides a simple introduction to contour integrals.

1 THE LAPLACE DISTRIBUTION

Definition 1.1. The Laplace distribution is defined conditionally as

$$X \sim N(0, Y) \quad (1)$$

where $Y = \|h\|^2$ and $h \sim \text{CN}(0, 2)$ is complex circularly Gaussian.

Problem 1. Show that the conditional characteristic function of X is

$$\phi_{X/Y}(j\omega) = e^{-\frac{1}{2}Y\omega^2} \quad (2)$$

Problem 2. Given that

$$\phi_Y(j\omega) = \frac{1}{1 - 2j\omega}, \quad (3)$$

show that

$$\phi_X(j\omega) = E[\phi_{X/Y}(j\omega)] = \frac{1}{1 + \omega^2} \quad (4)$$

2 CONTOUR INTEGRATION

In Fig. 3, let $z = x + jy$, $C = C_1 + C_2$. It is obvious that the line integral in the anti-clockwise direction

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \int_{C_1} \frac{e^{-jzt}}{1 + z^2} dz + \int_{C_2} \frac{e^{-jzt}}{1 + z^2} dz \quad (5)$$

The symbol on the integral on the LHS shows that the integration is over a closed path in the anti-clockwise direction.

Problem 3. Show that

$$\int_{C_1} \frac{e^{-jzt}}{1 + z^2} dx = \int_{-R}^R \frac{e^{-jxt}}{1 + x^2} dx \quad (6)$$

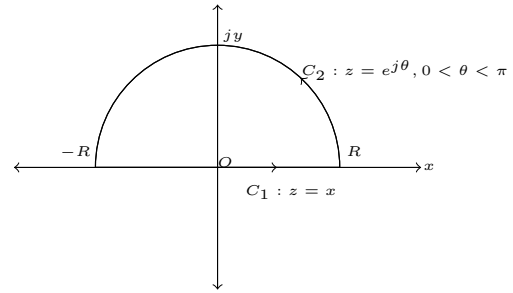


Fig. 3

Problem 4. Show that

$$\lim_{R \rightarrow \infty} \left| \frac{e^{-jzt}}{1 + z^2} \right| = 0, \quad t < 0 \quad (7)$$

by substituting $z = Re^{j\theta}$, $0 < \theta < \pi$.

Problem 5. Let C be a closed curve within the curve T . Then given that

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \oint_T \frac{e^{-jzt}}{1 + z^2} dz, \quad (8)$$

show that

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \int_{-\infty}^{\infty} \frac{e^{-jxt}}{1 + x^2} dx, \quad t < 0 \quad (9)$$

Problem 6. Given that

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi j f(z_0), \quad (10)$$

where z_0 is within C , show that

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \pi e^t \quad t < 0 \quad (11)$$

Problem 7. Now find

$$\int_{-\infty}^{\infty} \frac{e^{-jxt}}{1 + x^2} dx, \quad t > 0 \quad (12)$$

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Problem 8. Obtain an expression for the probability density function (PDF) of X .

3 CAUCHY'S INTEGRAL FORMULA

Problem 9. Show that

$$\oint_C \frac{f(z)}{z - z_0} dz = f(z_0) \oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0} dz \quad (13)$$

Problem 10. For $C : z = z_0 + \rho e^{j\theta}, 0 < \theta < 2\pi$, show that

$$\oint_C \frac{dz}{z - z_0} = 2\pi j \quad (14)$$

Problem 11. Let

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - K \right| < \delta \implies |z - z_0| < \epsilon \quad (15)$$

for some $\epsilon, \delta > 0$.

1) Show that

$$|f(z) - Kz - f(z_0) + Kz_0| < \epsilon\delta \quad (16)$$

2) Let

$$f_1(z) = f(z) - Kz \quad (17)$$

$$f_2(z) = Kz \quad (18)$$

Show that

$$\begin{aligned} |f_1(z) + f_2(z) - f_1(z_0) - f_2(z_0)| \\ = |f(z) - f(z_0)| < \lambda \end{aligned} \quad (19)$$

3) Let

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \quad (20)$$

Show that

$$|z - z_0| = \rho \implies |g(z)| < \frac{\lambda}{\rho} \quad (21)$$

Problem 12. Let

$$|g(z)| < \frac{\lambda}{\rho} \quad (22)$$

Show that, if $C : z = \rho e^{j\theta}, 0 < \theta < 2\pi$ and $z_i \in C, \Delta z_m = z_m - z_{m-1}, \zeta_m \in (z_{m-1}, z_m)$,

$$\lim_{\substack{\Delta z_m \rightarrow 0 \\ n \rightarrow \infty}} \left| \sum_{m=1}^n g(\zeta_m) \Delta z_m \right| < 2\pi\lambda \quad (23)$$

Definition 3.1.

$$\int_C g(z) dz = \lim_{\substack{\Delta z_m \rightarrow 0 \\ n \rightarrow \infty}} \left| \sum_{m=1}^n g(\zeta_m) \Delta z_m \right| \quad (24)$$

Problem 13. Show that

$$\left| \int_C \frac{f(z) - f(z_0)}{z - z_0} dz \right| < 2\pi\lambda \quad (25)$$

Problem 14. Show that

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi j f(z_0) \quad (26)$$

4 CAUCHY'S INTEGRAL THEOREM

Definition 4.1. The derivative of $f(z)$ is defined as

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad (27)$$

$f(z)$ is said to be *analytic* if its derivative exists.

Problem 15. Let $z = x + jy, \Delta z = \Delta x + j\Delta y, f(z) = u(x, y) + jv(x, y)$.

1) By letting $\Delta y \rightarrow 0$ followed by $\Delta x \rightarrow 0$, show that

$$f'(z) = \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} \quad (28)$$

2) By letting $\Delta x \rightarrow 0$ followed by $\Delta y \rightarrow 0$, show that

$$f'(z) = \frac{\partial v}{\partial y} - j \frac{\partial u}{\partial y} \quad (29)$$

Problem 16. Show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (30)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (31)$$

These are known as the *Cauchy-Riemann* equations.

Problem 17. Using (24), show that

$$\begin{aligned} \oint_C f(z) dz &= \oint_C (u dx - v dy) \\ &\quad + j \oint_C (u dy + v dx) \end{aligned} \quad (32)$$

Problem 18. According to *Green's Theorem*, for a closed curve C ,

$$\oint_C (u dx - v dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy \quad (33)$$

where R is bounded by C . Show that

$$\oint_C f(z) dz = 0. \quad (34)$$

assuming that $f(z)$ is analytic everywhere inside C .

Problem 19. In Fig. 19, show that

$$\oint_C f(z) dz = \oint_T f(z) dz \quad (35)$$

given that $f(z)$ is analytic in the region between $C - T$.

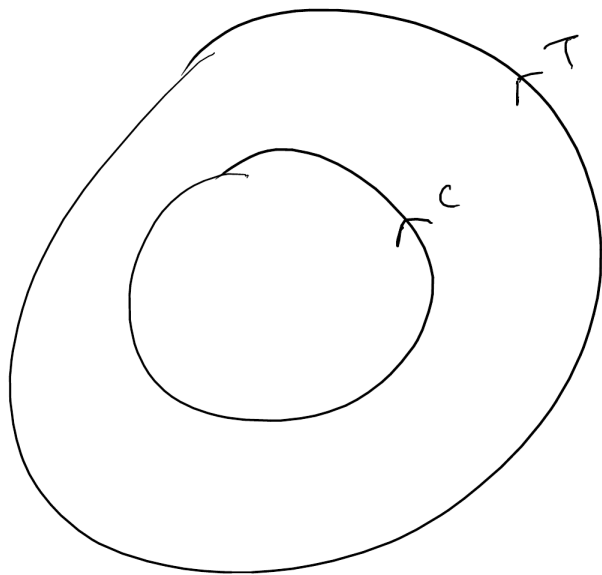


Fig. 19