GATE problems in Complex Analysis

Abstract—**This manual has problems on Complex Analysis taken from GATE papers in Mathematics and Electronics and Communication Engineering.**

1. The contour C given below is on the complex plane $z = x + \jmath y$. Find the value of the integral $\frac{1}{\pi\jmath}\oint_C \frac{dz}{z^2-1}$.

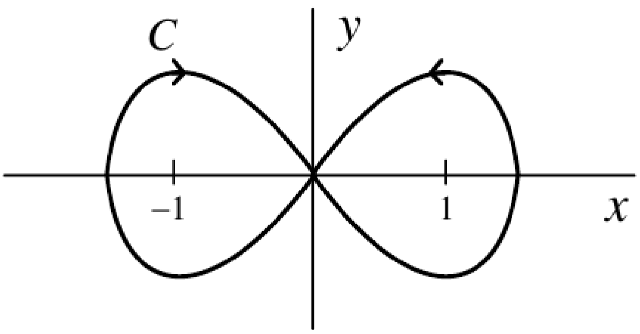

Fig. 1

2. An integral I is given by
$$I = \oint_C \frac{z^2-1}{z^2+1} e^z \, dz. \tag{1}$$
If $C : |z| = 3$, find the value of I.
3. Consider contour integration performed over $C : |z| = 1$ in the anticlockwise direction. Which of the following is NOT true?
   a) The residue of $\frac{z}{z^2-1}$ at $z = 1$ is $\frac{1}{2}$.
   b) $\oint_C z^2 \, dz = 0$
   c) $\frac{1}{2\pi\jmath}\oint_C \frac{dz}{z} = 1$.
   d) $\bar{z}$ is an analytic function.
4. Find the value of
$$\oint_C \frac{z^2 - z + 4\jmath}{z + 2\jmath} dz, \quad C : |z| = 3. \tag{2}$$
5. Given
$$f(z) = \frac{1}{z+1} - \frac{2}{z+3}. \tag{3}$$
If $|z+1| = 1$, find $\frac{1}{2\pi\jmath}\oint_C f(z)\, dz$.
6. Find
$$\oint_C \frac{-3z+4}{z^2+4z+5} dz, \quad C : |z| = 1 \tag{4}$$
7. Find the residues of a complex function
$$X(z) = \frac{1-12z}{z(z-1)(z-2)} \tag{5}$$
at its poles.
8. If $f(z) = c_0 + c_1 z^{-1}$, find
$$\oint_C \frac{1+f(z)}{z} dz, \quad C : |z| = 1 \tag{6}$$
9. Find the residue of the function
$$f(z) = \frac{1}{(z+2)^2 (z-2)^2} \tag{7}$$
at $z = 2$.
10. If the semi-circular contour D of radius 2 is as shown in Fig. 10, then find
$$\oint_D \frac{dz}{z^2+4} \tag{8}$$

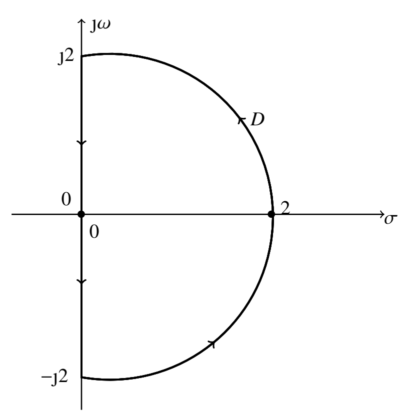

Fig. 10

11. Find
$$\oint_{|z-j|=2} \frac{1}{z^2-1}\, ds \tag{9}$$
12. Find $\operatorname*{Res}_{z=0} \frac{\sin z}{z^8}$
13. Let Γ denote the boundary of the square whose sides lies along $x = \pm 1$ and $y = \pm 1$, where Γ is described in the positive sense. Find
$$\oint_{\Gamma} \frac{z^2}{2z+3} dz \tag{10}$$
14. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$
15. For the function $f(z) = \frac{1-e^{-z}}{z}$, the point $z = 0$ is
    a) an essential singularity
    b) a pole of order zero
    c) a pole of order one
    d) a removable singularity
16. Find
$$\oint_{|z|=4} \frac{dz}{z^2-1} \tag{11}$$
17. Evaluate
$$\oint_{|z-\jmath|=\frac{7}{2}} \frac{e^{\frac{1}{z^2}}}{z^2+1} dz \tag{12}$$
18. Construct an analytic function $f(z)$ of which the real part is $u(x, y) = 2xy + \cos hx \sin y$, given that $f(0) = 0$.
19. The function $\sin z$ is analytic in
    a) $C \cup \{\infty\}$
    b) C except on the negative real axis.
    c) $C - \{0\}$
    d) C
20. If $f(z) = z^3$, then it
    (A) has an essential singularity at $z = \infty$
    (B) has a pole of order 3 at $z = \infty$
    (C) has a pole of order 3 at $z = 0$
    (D) is analytic at $z = \infty$
21. The function $f(z) = |\, z\,|^2$ is
    a) differentiable everywhere
    b) differentiable only at the origin
    c) not differentiable anywhere
    d) differentiable on real x-axis
22. Evaluate
$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5} dx$
using the method of residues.
23. Let T be any circle enclosing the origin and oriented counter-clockwise. Find $\int_{\Gamma} \frac{\cos z}{z^2} dz$
24. For the function $f(z) = \sin\frac{1}{z}, z = 0$ is a
    a) removable singularity
    b) simple pole
    c) branch point
    d) essential singularity
25. Evaluate the integral
$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}, a > 0,$
by the method of residue calculus.
26. Consider a function $f(z) = u + \jmath v$ defined on $\left|z - \jmath\right| < 1$ where u, v are real valued functions of x, y. Then $f(z)$ is analytic for u equals to
    a) $x^2 + y^2$
    b) $\ln(x^2 + y^2)$
    c) e^{xy}
    d) $e^{x^2-y^2}$
27. At $z = 0$, the function $f(z) = z^2\bar{z}$
    a) does not satisfy Cauchy-Reimann equations
    b) satisfies Cauchy-Reimann equations but is not differentiable
    c) is differentiable
    d) is analytic
28. Let γ be the curve : $r = 2 + 4\cos\theta, (0 \le \theta \le 2\pi)$. If $I_1 = \int_{\gamma} \frac{dz}{z-1}$ and $I_2 = \int_{\gamma} \frac{dz}{z-3}$ then
    a) $I_1 = 2I_2$
    b) $I_1 = I_2$
    c) $2I_1 = I_2$
    d) $I_1 = 0, I_2 \neq 0$
29. Let $f(z)$ be an analytic function with a simple pole at $z = 1$ and a double pole at $z = 2$ with residues 1 and -2 respectively. Further if $f(0) = 0$, $f(3) = -\frac{3}{4}$ and f is bounded as $z \to \infty$, then $f(z)$ must be
    a) $z(z-3) - \frac{1}{4} + \frac{1}{z-1} - \frac{2}{z-1} + \frac{1}{(z-2)^2}$
    b) $-\frac{1}{4} + \frac{1}{z-1} - \frac{2}{z-2} + \frac{1}{(z-2)^2}$
    c) $\frac{1}{z-1} - \frac{2}{z-2} + \frac{5}{(z-2)^2}$
    d) $\frac{15}{4} + \frac{1}{z-1} + \frac{2}{z-2} - \frac{7}{(z-2)^2}$
30. An example of a function with a non-isolated essential singularity at $z = 2$ is
    a) $\tan\frac{1}{z-2}$
    b) $\sin\frac{1}{z-2}$
    c) $e^{-(z-2)}$
    d) $\tan\frac{z-2}{z}$

31. Find $I = \int_c \frac{\cot(\pi z)}{(z-\jmath)^2} dz$, where C is the contour $4x^2 + y^2 = 2$ (counter clock-wise).
32. $\int_0^{2\pi} \frac{d\theta}{13-5\sin\theta} =$
33. For the positively oriented unit circle, find $\oint_{|z|=1} \frac{2\mathrm{Re}(z)}{z+2} dz$
34. The number of zeroes, counting multiplicities, of the polynomial $z^5 + 3z^3 + z^2 + 1$ inside the circle $|z| = 2$ is
   a) 0
   b) 2
   c) 3
   d) 5
35. Consider the functions $f(z) = x^2 + \jmath y^2$ and $g(z) = x^2 + y^2 + \jmath xy$. At $z = 0$,
   a) f is analytic but not g
   b) g is analytic but not f
   c) both f and g are analytic
   d) neither f nor g is analytic
36. Let γ be a simple closed curve in the complex.Then the set of all possible values of $\oint_\gamma \frac{dz}{z(1-z^2)}$ is
   a) $\{0, \pm\pi\jmath\}$
   b) $\{0, \pi\jmath, 2\pi\jmath\}$
   c) $\{0, \pm\pi\jmath, \pm2\pi\jmath\}$
   d) $\{0\}$
37. The principal value of the improper integral $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$ is
   a) $\frac{\pi}{e}$
   b) πe
   c) $\pi + e$
   d) $\pi - e$
38. The number of roots of the equation $z^5 - 12z^2 + 14 = 0$ that lie in the region $\{z \in C : 2 \leq |z| < \frac{5}{2}\}$ is
   a) 2
   b) 3
   c) 4
   d) 5
39. Find the value of $\int_0^{2\pi} \exp(e^{\jmath\theta} - \jmath\theta)\, d\theta$
40. The sum of the residues at all the poles of $f(z) = \frac{\cot \pi z}{(z+a)^2}$, where a is a constant, $(a \neq 0, \pm1, \pm2, ...)$ is
   a) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi \mathrm{cosec}^2 \pi a$
   b) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi \mathrm{cosec}^2 \pi a$
   c) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi \mathrm{cosec}^2 \pi a$
   d) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi \mathrm{cosec}^2 \pi a$
41. Which of the following is not the real part of an analytic function?
   a) $x^2 - y^2$
   b) $\frac{1}{1+x^2+y^2}$
   c) $\cos x \cos hy$
   d) $x + \frac{x}{x^2+y^2}$
42. Let $f(z)$ be an analytic function. Then the value of $\int_0^{2\pi} f(e^{\jmath t}) \cos(t) dt$ equals
   a) 0
   b) $2\pi f(0)$
   c) $2\pi f'(0)$
   d) $\pi f'(0)$
43. Let $f(z) = 2z^2 - 1$. Then the maximum value of $|f(z)|$ on the unit disc $D = \{z \in C : |z| \leq 1\}$ equals
   a) 1
   b) 2
   c) 3
   d) 4
44. Let S be the positively oriented circle given by $\left|z - 3\jmath\right| = 2$. Find the value of $\int_S \frac{dz}{z^2+4}$
45. Let $f(z) = \cos z - \frac{\sin z}{z}$ for non-zero $z \in \mathbb{C}$ and $f(0) = 0$. Also,let $g(z) = \sin hz$ for $z \in \mathbb{C}$. Then $f(z)$ has a zero at $z = 0$ of order
   a) 0
   b) 1
   c) 2
   d) greater than 2
46. Refer to the previous question for $f(z)$ and $g(z)$. Then $\frac{g(z)}{zf(z)}$ has a pole at $z = 0$ of order
   a) 1
   b) 2
   c) 3
   d) greater than 3
47. For the function $f(z) = \sin\left(\frac{1}{\cos\left(\frac{1}{z}\right)}\right)$, the point $z = 0$ is
   a) a removable singularity
   b) a pole
   c) an essential singularity
   d) a non-isolated singularity
48. Consider the function $f(z) = \frac{e^{\jmath z}}{z(z^2+1)}$. The residue

of f at the isolated singular point in the upper half plane $\{z = x + \jmath y : y > 0\}$ is

a) $-\frac{1}{2e}$
b) $-\frac{1}{e}$
c) $\frac{e}{2}$
d) 2

49. The Cauchy Principal Value of the integral $\int_{-\infty}^{\infty} \frac{\sin x dx}{x(x^2+1)}$ is
a) $-2\pi(1 + 2e^{-1})$
b) $\pi(1 + e^{-1})$
c) $2\pi(1 + e)$
d) $-\pi(1 + e^{-1})$

50. Let $I = \int_C \frac{f(z)}{(z-1)(z-2)} dz$, where $f(z) = \sin\frac{\pi z}{2} + \cos\frac{\pi z}{2}$ and C is the curve $|z| = 3$ oriented anti-clockwise. Find the value of I

51. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic except for a simple pole at $z = 0$ an let $g : \mathbb{C} \to \mathbb{C}$ be analytic. Then, the value of $\frac{\operatorname{Res}_{z=0}\{f(z)g(z)\}}{\operatorname{Res}_{z=0} f(z)}$ is
a) $g(0)$
b) $g'(0)$
c) $\lim_{z\to 0} zf(z)$
d) $\lim_{z\to 0} zf(z)g(z)$

52. Let C be the contour $|z| = 2$ oriented in the anti-clockwise direction. The value of the integral $\oint_C ze^{\frac{3}{z}} dz$ is
a) $3\pi\jmath$
b) $5\pi\jmath$
c) $7\pi\jmath$
d) $9\pi\jmath$

53. The function $f(z) = |z|^2 + \jmath\bar{z} + 1$ is differentiable at
a) $\jmath$
b) 1
c) $-\jmath$
d) no point in $\mathbb{C}$

54. The inverse Laplace transform of $\frac{2s^2-4}{(s-3)(s^2-s-2)}$ is
a) $(1 + t)e^{-t} + \frac{7}{2}e^{-3t}$
b) $\frac{e^t}{3} + te^{-t} + 2t$
c) $\frac{7}{2}e^{3t} - \frac{e^{-t}}{6} - \frac{4}{3}e^{2t}$
d) $\frac{7}{2}e^{-3t} - \frac{e^t}{6} - \frac{4}{3}e^{-2t}$

55. The maximum modulus of e^{z^2} on the set $S = \{z \in \mathbb{C} : 0 \le \operatorname{Re}(z) \le 1, 0 \le \operatorname{Im}(z) \le 1\}$ is
a) $\frac{2}{e}$
b) e
c) $e + 1$
d) e^2

56. Let $\Omega = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and let C be a smooth curve lying in Ω with initial point $-1+2\jmath$ and final point $1 + 2\jmath$. The value of $\int_C \frac{1+2z}{1+z} dz$ is
a) $4 - \frac{1}{2}\ln 2 + \jmath\frac{\pi}{4}$
b) $-4 + \frac{1}{2}\ln 2 + \jmath\frac{\pi}{4}$
c) $4 + \frac{1}{2}\ln 2 - \jmath\frac{\pi}{4}$
d) $4 - \frac{1}{2}\ln 2 + \jmath\frac{\pi}{2}$

57. If $a \in \mathbb{C}$ with $|a| < 1$, then the value of
$$\frac{(1 - |a|^2)}{\pi} \int_\Gamma \frac{|dz|}{|z + a|^2},$$
where Γ is the simple closed curve $|z| = 1$ taken with the positive orientation, is ________

58. Let $C = \{z \in \mathbb{C} : |z - \jmath| = 2\}$. Then $\frac{1}{2\pi}\oint_C \frac{z^2-4}{z^2+4} dz$ is equal to ________

59. The value of $\frac{\jmath}{4-\pi}\int_{|z|=4} \frac{dz}{z\cos(z)}$ is equal to ________

60. Let $\gamma = \{z \in \mathbb{C} : |z| = 2\}$ be oriented in the counter-clockwise direction. Let
$$I = \frac{1}{2\pi\jmath}\oint_\gamma z^7 \cos\left(\frac{1}{z^2}\right) dz.$$
Then,the value of I is equal to ________

61. Let $f(z) = (x^2+y^2)+\jmath 2xy$ and $g(z) = 2xy+\jmath(y^2-x^2)$ for $z = x + \jmath y \in \mathbb{C}$. Then, in the complex plane $\mathbb{C}$.
a) f is analytic and g is NOT analytic.
b) f is NOT analytic and g is analytic.
c) neither f nor g is analytic.
d) both f and g are analytic.

62. Let C be the simple,positively oriented circle of radius 2 centered at the origin in the complex plane. Then find
$$\frac{2}{\pi\jmath}\int_C \left(ze^{\left(\frac{1}{z}\right)} + \tan\left(\frac{z}{2}\right) + \frac{1}{(z-1)(z-3)^2}\right) dz$$
.

63. Let Γ be the circle given by $z = 4e^{\jmath\theta}$, where θ varies from 0 to 2π. Find
$$\oint_\Gamma \frac{e^z}{z^2 - 2z} dz =$$

64. Let $f(z) = z^3 e^{z^2}$ for $z \in \mathbb{C}$ and let Γ be the circle

$z = e^{\jmath\theta}$, where θ varies from 0 to 4π. Then

$$\frac{1}{2\pi\jmath}\oint_{\Gamma}\frac{f'(z)}{f(z)}dz = \underline{\hspace{6cm}}.$$