Inverse Laplace transforms via residue theory

The Laplace transform:

For a function (signal) $f(t)$ which is zero for $t < 0$, the Laplace transform is

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Here we use $s = \sigma + j\omega$ in place of $z = x + jy$, so we have

$$F(s) = \int_0^\infty (f(t)e^{-\sigma t})e^{-j\omega t}dt$$

Problem: Given $F(s)$ how do we obtain $f(t)$?

The Fourier transform:

The Fourier transform of $f(t)$ is

$$\hat{f}(\omega) = \int_{-\infty}^\infty f(t)e^{-j\omega t}dt$$

and the inverse transform is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(\omega)e^{j\omega t}dt$$

In the Laplace transform let,

$$\begin{aligned}\phi(t) &= f(t)e^{-\sigma t}, \quad t \geq 0\\ &= 0, \quad t < 0\end{aligned}$$

where σ is a constant. Taking the Fourier transform of $\phi(t)$:

$$\hat{\phi}(\omega) = \int_0^\infty \phi(t)e^{-j\omega t}dt = \int_0^\infty f(t)e^{-\sigma t}e^{-j\omega t}dt = F(\sigma + j\omega)$$

So, taking the inverse Fourier transform:

$$\phi(t) = f(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{\omega=-\infty}^\infty F(\sigma + j\omega)e^{j\omega t}d\omega$$

Hence,

$$f(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^\infty F(\sigma + j\omega)e^{j\omega t}e^{\sigma t}d\omega$$

Let $s = \sigma + j\omega$, $ds = jd\omega$, then

$$\begin{aligned}f(t) &= \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds, \quad t \geq 0\\ &= 0, \quad t < 0\end{aligned}$$

The inverse Laplace transform

The formula for the inverse Laplace transform was obtained in the previous section as:

$$f(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

The relevant questions here are:

1. How do we choose the real parameter σ?
2. How do we evaluate the integral?

We already know that $f(t) = 0$, $t < 0$, and we shall see that this gives us an answer to (1).

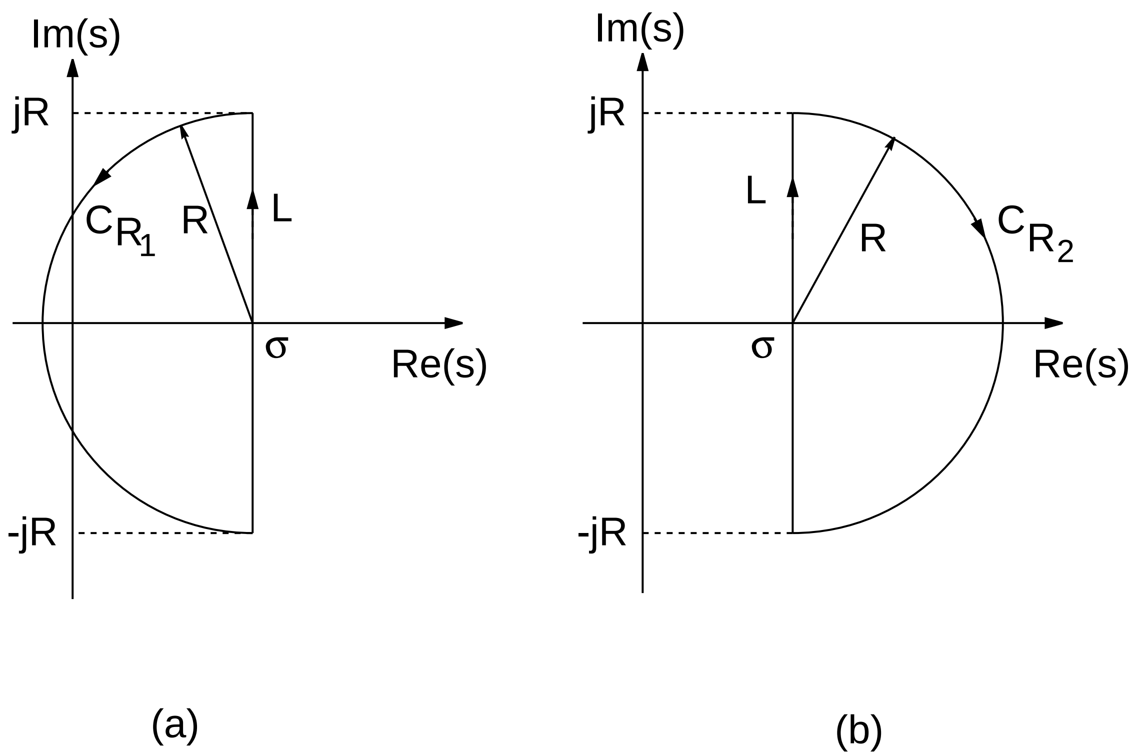

Figure 1: Integration contours

First, consider the closed contour $C_1 = C_{R_1} + L$ shown in Figure 1a. Then,

$$\begin{aligned}\oint_{C_1} F(s)e^{st}ds &= \int_L F(s)e^{st}ds + \int_{C_{R_1}} F(s)e^{st}ds \\ &= \int_{\sigma-jR}^{\sigma+jR} F(s)e^{st}ds + \int_{C_{R_1}} F(s)e^{st}ds \\ &= 2\pi j \sum_{\text{poles in } C_1} \text{Res}[F(s)e^{st}]\end{aligned}$$

Note that the residues are at the poles inside C_1. Now, as $R \to \infty$, $\int_{\sigma-jR}^{\sigma+jR} F(s)e^{st}ds$ becomes the integral we require, and we can show (Jordan's lemma, see appendix) that for $t > 0$, $\int_{C_{R_1}} F(s)e^{st}ds \to 0$

as $R \to \infty$. So, for $t > 0$,

$$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = \sum_{\text{poles left of } \sigma} \text{Res}[F(s)e^{st}]$$

Next consider the contour $C_2 = C_{R_2} + L$ shown in Figure 1b. As before,

$$\begin{aligned}
\oint_{C_2} F(s)e^{st}ds &= \int_L F(s)e^{st}ds + \int_{C_{R_2}} F(s)e^{st}ds \\
&= \int_{\sigma-jR}^{\sigma+jR} F(s)e^{st}ds + \int_{C_{R_2}} F(s)e^{st}ds \\
&= -2\pi j \sum_{\text{poles in } C_2} \text{Res}[F(s)e^{st}]
\end{aligned}$$

Note that the residues are at the poles inside C_2 and the minus sign is due to the fact that we are dealing with a clockwise contour. Again, using Jordan's lemma, we have that for $t < 0$, $\int_{C_{R_2}} F(s)e^{st}ds \to 0$ as $R \to \infty$. So, for $t < 0$,

$$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = -\sum_{\text{poles right of } \sigma} \text{Res}[F(s)e^{st}]$$

But we know that this must be zero, since $f(t) = 0$, $t < 0$. Hence σ must be chosen such that C_2 does not contain any poles of $F(s)e^{st}$ (as $R \to \infty$), and thus C_1 must contain *all* poles of $F(s)e^{st}$. This is the answer to question (1). Note, finally, that since e^{st} is analytic everywhere (i.e. has no poles), the poles of $F(s)e^{st}$ are the same as the poles of $F(s)$. This answers question (2) and we have

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = \sum_{\text{all poles of } F(s)} \text{Res}[F(s)e^{st}]$$

Example 1

Find the inverse transform of $F(s) = \frac{1}{s+a}$.
The function $\frac{e^{st}}{s+a}$ has a simple pole at $s = -a$. Hence

$$\text{Res}_{s=-a}[F(s)e^{st}] = \lim_{s \to -a}\left[(s+a)\frac{e^{st}}{s+a}\right] = e^{-at}$$

and so,

$$\begin{aligned}
L^{-1}\left[\frac{1}{s+a}\right] &= e^{-at}, \quad t \geq 0 \\
&= 0, \quad t < 0
\end{aligned}$$

Example 2

Find the inverse transform of $F(s) = \frac{1}{(s+a)^2}$.
In this case the function $\frac{e^{st}}{(s+a)^2}$ has a pole of order 2 at $s = -a$. Remember that if a function $f(z)$ has a pole of order n, then its residue at this pole is given by

$$c_1 = \frac{1}{(n-1)!}\lim_{z \to a}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$$

In our case the required residue is:

$$\frac{1}{(2-1)!}\lim_{s\to -a}\frac{d}{ds}\left[(s+a)^2\frac{e^{st}}{(s+a)^2}\right] = \lim_{s\to -a}[te^{st}] = te^{-at}$$

and so,

$$\begin{aligned} L^{-1}\left[\frac{1}{(s+a)^2}\right] &= te^{-at}, \quad t \geq 0 \\ &= 0, \quad t < 0 \end{aligned}$$

Example 3

Find the inverse transform of $F(s) = \frac{1}{(s+a)^2(s+b)}$.
In this case the function $\frac{e^{st}}{(s+a)^2(s+b)}$ has a pole of order 2 at $s=-a$ and a simple pole at $s=-b$. The residue at $s=-b$ is

$$\lim s \to -b\left[\frac{e^{st}}{(s+a)^2}\right] = \frac{e^{-bt}}{(a-b)^2}$$

The residue at $s=-a$ is

$$\frac{1}{(2-1)!}\lim\frac{d}{ds}\left[\frac{e^{st}}{s+b}\right] = \lim_{s\to -a}\left[\frac{te^{st}}{s+b} - \frac{e^{st}}{(s+b)^2}\right] = \frac{te^{-at}}{b-a} - \frac{e^{-at}}{(b-a)^2}$$

and so

$$L^{-1}\left[\frac{1}{(s+a)^2(s+b)}\right] = \frac{e^{-bt}}{(a-b)^2} + \frac{te^{-at}}{b-a} - \frac{e^{-at}}{(b-a)^2}, \quad t \geq 0$$

Appendix: Jordan's Lemma

In the theory for inverting Laplace transforms using residue theory, we ned the folowing result:

$$\lim_{R\to\infty}\int_{C_R} F(s)e^{st}ds = 0 \tag{1}$$

where $t > 0$ and C_R is the semicircular contour C_{R_1} shown in Figure 1a. Points on this contour are given by

$$s = \sigma + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$$

Looking at the table of standard transforms it can be seen that most statisfy the conditions $F(s) \to 0$ as $|s| \to \infty$ (e.g. $\frac{1}{s}$, $\frac{1}{s+a}$, etc). Therefore on C_R as $R \to \infty$, $F(s) \to 0$. This means that for any $\epsilon > 0$, an R can be found such that $|F(s)| = |F(\sigma + Re^{j\theta})| < \epsilon$. For this R we have

$$\left|\int_{C_R} F(s)e^{st}ds\right| \leq \epsilon\int_{C_R}|e^{st}|ds$$

On C_R,

$$|e^{st}| = |e^{\sigma + Re^{j\theta}}| = |e^{(\sigma + R\cos\theta + jR\sin\theta)t}| = e^{(\sigma + R\cos\theta)t}$$

and $ds = jRe^{j\theta}d\theta$. Therefore,

$$\epsilon\int_{C_R}|e^{st}|ds = \epsilon Re^{\sigma t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{Rt\cos\theta}d\theta = 2\epsilon Re^{\sigma t}\int_0^{\frac{\pi}{2}} e^{-Rt\sin\theta}d\theta$$

Plotting graphs of $y = \sin\theta$ and the straight line $y = \frac{2}{\pi}\theta$, we see that

$$\sin\theta \geq \frac{2}{\pi}\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

Hence the integral is less than

$$2R\epsilon e^{\sigma t}\int_0^{\frac{\pi}{2}} e^{-\frac{2Rt}{\pi}\theta}d\theta = \frac{\epsilon\pi e^{\sigma t}}{t}(1 - e^{-Rt})$$

For any $t > 0$ this last quantity goes to zero as $R \to \infty$ (because ϵ also goes to zero), and we have therefore proved (1).