

# Contour Integration

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**Abstract**—This manual provides a simple introduction to the application of complex analysis in finding the statistics of the Laplace distribution.

## 1 THE LAPLACE DISTRIBUTION

**Definition 1.1.** The Laplace distribution is defined conditionally as

$$X \sim N(0, Y) \quad (1)$$

where  $Y = \|h\|^2$  and  $h \sim \text{CN}(0, 2)$  is complex circularly Gaussian.

**Problem 1.** Show that the conditional characteristic function of  $X$  is

$$\phi_{X/Y}(j\omega) = e^{-\frac{1}{2}Y\omega^2} \quad (2)$$

**Problem 2.** Given that

$$\phi_Y(j\omega) = \frac{1}{1 - 2j\omega}, \quad (3)$$

show that

$$\phi_X(j\omega) = E[\phi_{X/Y}(j\omega)] = \frac{1}{1 + \omega^2} \quad (4)$$

## 2 CONTOUR INTEGRATION

In Fig. 3, let  $z = x + jy$ ,  $C = C_1 + C_2$ . It is obvious that the line integral in the anti-clockwise direction

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \int_{C_1} \frac{e^{-jzt}}{1 + z^2} dz + \int_{C_2} \frac{e^{-jzt}}{1 + z^2} dz \quad (5)$$

The symbol on the integral on the LHS shows that the integration is over a closed path in the anti-clockwise direction.

**Problem 3.** Show that

$$\int_{C_1} \frac{e^{-jzt}}{1 + z^2} dx = \int_{-R}^R \frac{e^{-jxt}}{1 + x^2} dx \quad (6)$$

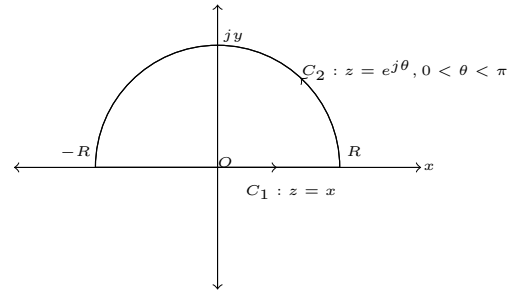


Fig. 3

**Problem 4.** Show that

$$\lim_{R \rightarrow \infty} \left| \frac{e^{-jzt}}{1 + z^2} \right| = 0, \quad t < 0 \quad (7)$$

by substituting  $z = Re^{j\theta}$ ,  $0 < \theta < \pi$ .

**Problem 5.** Let  $C$  be a closed curve within the curve  $T$ . Then given that

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \oint_T \frac{e^{-jzt}}{1 + z^2} dz, \quad (8)$$

show that

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \int_{-\infty}^{\infty} \frac{e^{-jxt}}{1 + x^2} dx, \quad t < 0 \quad (9)$$

**Problem 6.** Given that

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi j f(z_0), \quad (10)$$

where  $z_0$  is within  $C$ , show that

$$\oint_C \frac{e^{-jzt}}{1 + z^2} dz = \pi e^t \quad t < 0 \quad (11)$$

**Problem 7.** Now find

$$\int_{-\infty}^{\infty} \frac{e^{-jxt}}{1 + x^2} dx, \quad t > 0 \quad (12)$$

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**Problem 8.** Obtain an expression for the probability density function (PDF) of  $X$ .

### 3 CAUCHY'S INTEGRAL FORMULA

**Problem 9.** Show that

$$\oint_C \frac{f(z)}{z - z_0} dz = f(z_0) \oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0} dz \quad (13)$$

**Problem 10.** For  $C : z = z_0 + \rho e^{j\theta}, 0 < \theta < \pi$ , show that

$$\oint_C \frac{dz}{z - z_0} = 2\pi j \quad (14)$$

**Problem 11.** Let

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - K \right| < \delta \implies |z - z_0| < \epsilon \quad (15)$$

for some  $\epsilon, \delta > 0$ .

1) Show that

$$|f(z) - Kz - f(z_0) + Kz_0| < \epsilon\delta \quad (16)$$

2) Let

$$f_1(z) = f(z) - Kz \quad (17)$$

$$f_2(z) = Kz \quad (18)$$

Show that

$$\begin{aligned} |f_1(z) + f_2(z) - f_1(z_0) - f_2(z_0)| \\ = |f(z) - f(z_0)| < \kappa \end{aligned} \quad (19)$$

3) Let

$$g(z) = \left| \frac{f(z) - f(z_0)}{z - z_0} - K \right| \quad (20)$$

**Problem 12.** Let

$$|g(z)| < \frac{\epsilon}{\rho} \quad (21)$$

Show that, if  $C : z = \rho e^{j\theta}, 0 < \theta < 2\pi$  and  $z_i \in C, \Delta z_m = z_m - z_{m-1}, \zeta_m \in (z_{m-1}, z_m)$ ,

$$\left| \sum_{m=1}^n g(\zeta_m) \Delta z_m \right| < 2\pi\epsilon \quad (22)$$

**Definition 3.1.**

$$\int_C g(z) dz = \lim_{\substack{\Delta z_m \rightarrow 0 \\ n \rightarrow \infty}} \left| \sum_{m=1}^n g(\zeta_m) \Delta z_m \right| \quad (23)$$

**Problem 13.** Show that

$$\left| \int_C \frac{f(z) - f(z_0)}{z - z_0} dz \right| < 2\pi\epsilon \quad (24)$$

**Problem 14.** Show that

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi j f(z_0) \quad (25)$$

### 4 CAUCHY'S INTEGRAL THEOREM

**Definition 4.1.** The derivative of  $f(z)$  is defined as

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad (26)$$

$f(z)$  is said to be *analytic* if its derivative exists.

**Problem 15.** Let  $z = x + jy, \Delta z = \Delta x + \Delta y, f(z) = u(x, y) + jv(x, y)$ .

1) By letting  $\Delta y \rightarrow 0$  followed by  $\Delta x \rightarrow 0$ , show that

$$f'(z) = \frac{\partial u}{\partial x} + j \frac{\partial v}{\partial x} \quad (27)$$

2) By letting  $\Delta x \rightarrow 0$  followed by  $\Delta y \rightarrow 0$ , show that

$$f'(z) = \frac{\partial v}{\partial y} - j \frac{\partial u}{\partial y} \quad (28)$$

**Problem 16.** Show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (29)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (30)$$

These are known as the *Cauchy-Riemann* equations.

**Problem 17.** Using (23), show that

$$\begin{aligned} \oint_C f(z) dz &= \oint_C (u dx - v dy) \\ &\quad + j \oint_C (u dy + v dx) \end{aligned} \quad (31)$$

**Problem 18.** According to *Green's Theorem*, for a closed curve  $C$ ,

$$\oint_C (u dx - v dy) = \iint_R \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy \quad (32)$$

where  $R$  is bounded by  $C$ . Show that

$$\oint_C f(z) dz = 0. \quad (33)$$

assuming that  $f(z)$  is analytic everywhere inside  $C$ .

**Problem 19.** In Fig. 19, show that

$$\oint_C f(z) dz = \oint_T f(z) dz \quad (34)$$

given that  $f(z)$  is analytic in the region between  $C - T$ .

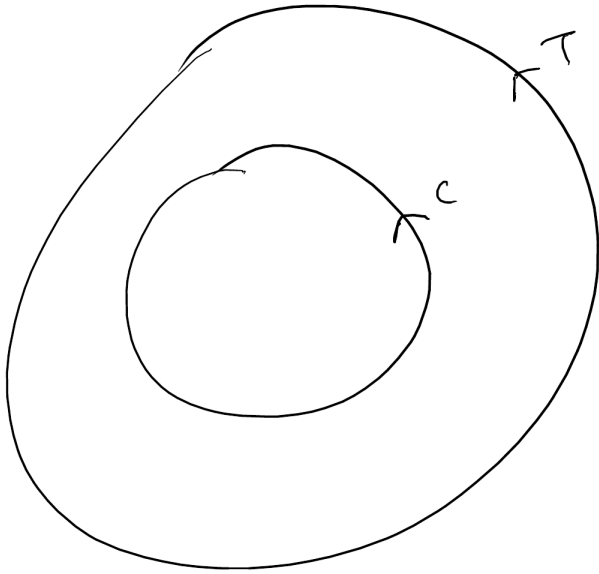


Fig. 19