GATE problems in Complex Analysis

Indian Institute of Technology
Hyderabad

Teaching Learning Center
IIT Hyderabad

Abstract—This manual has problems on Complex Analysis taken from GATE papers in Mathematics and Electronics and Communication Engineering.

1. The contour C given below is on the complex plane $z = x + \jmath y$. Find the value of the integral
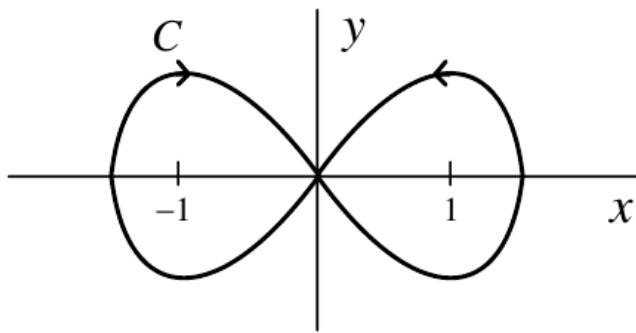$$\frac{1}{\pi \jmath} \oint_C \frac{dz}{z^2 - 1}.$$

Fig. 1

2. An integral I is given by
$$I = \oint_C \frac{z^2 - 1}{z^2 + 1} e^z \, dz. \qquad (1)$$

   If $C : |z| = 3$, find the value of I.

3. Consider contour integration performed over $C :$ $|z| = 1$ in the anticlockwise direction. Which of the following is NOT true?

   a) The residue of $\frac{z}{z^2-1}$ at $z = 1$ is $\frac{1}{2}$.

   b) $\oint_C z^2 \, dz = 0$

   c) $\frac{1}{2\pi\jmath} \oint_C \frac{dz}{z} = 1$.

   d) $\bar{z}$ is an analytic function.

4. Find the value of
$$\oint_C \frac{z^2 - z + 4\jmath}{z + 2\jmath} \, dz, \quad C : |z| = 3. \qquad (2)$$

5. Given
$$f(z) = \frac{1}{z + 1} - \frac{2}{z + 3}. \qquad (3)$$

   If $|z + 1| = 1$, find $\frac{1}{2\pi\jmath} \oint_C f(z) \, dz$.

6. Find
$$\oint_C \frac{-3z + 4}{z^2 + 4z + 5} \, dz, \quad C : |z| = 1 \qquad (4)$$

7. Find the residues of a complex function
$$X(z) = \frac{1 - 12z}{z \, (z - 1) \, (z - 2)} \qquad (5)$$

   at its poles.

8. If $f(z) = c + 0 + c_1 z^{-1}$, find
$$\oint_C \frac{1 + f(z)}{z} \, dz, \quad C : |z| = 1 \qquad (6)$$

9. Find the residue of the function
$$f(z) = \frac{1}{(z + 2)^2 \, (z - 2)^2} \qquad (7)$$

   at $z = 2$.

10. If the semi-circular contour D of radius 2 is as shown in Fig. 10, then find
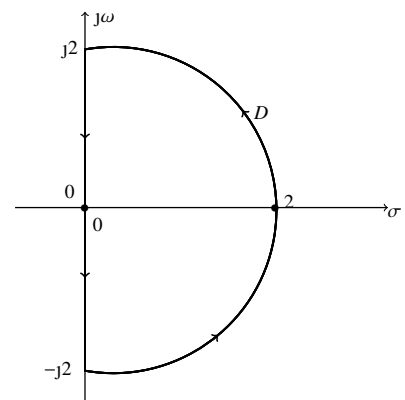$$\oint_D \frac{dz}{z^2 + 4} \qquad (8)$$

Fig. 10

11. Find
$$\oint_{|z-j|=2} \frac{1}{z^2-1}\,ds \qquad (9)$$

12. Find $\operatorname*{Res}_{z=0} \frac{\sin z}{z^8}$

13. Let Γ denote the boundary of the square whose sides lies along $x = \pm 1$ and $y = \pm 1$, where Γ is described in the positive sense. Find
$$\oint_{\Gamma} \frac{z^2}{2z+3}\,dz \qquad (10)$$

14. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

15. For the function $f(z) = \frac{1-e^{-z}}{z}$, the point $z = 0$ is
    a) an essential singularity
    b) a pole of order zero
    c) a pole of order one
    d) a removable singularity

16. Find
$$\oint_{|z|=4} \frac{dz}{z^2-1} \qquad (11)$$

17. Evaluate
$$\oint_{|z-j|=\frac{7}{2}} \frac{e^{\frac{1}{z^2}}}{z^2+1}\,dz \qquad (12)$$

18. Construct an analytic function $f(z)$ of which the real part is $u(x,y) = 2xy + \cos hx \sin y$, given that $f(0) = 0$.

19. The function $\sin z$ is analytic in
    a) $C \cup \{\infty\}$
    b) C except on the negative real axis.
    c) $C - \{0\}$
    d) C

20. If $f(z) = z^3$, then it
    (A) has an essential singularity at $z = \infty$
    (B) has a pole of order 3 at $z = \infty$
    (C) has a pole of order 3 at $z = 0$
    (D) is analytic at $z = \infty$

21. The function $f(z) = |z|^2$ is
    a) differentiable everywhere
    b) differentiable only at the origin
    c) not differentiable anywhere
    d) differentiable on real x-axis

22. Evaluate
$$\int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+2x+5}\,dx$$

using the method of residues.

23. Let T be any circle enclosing the origin and oriented counter-clockwise. Find $\int_\Gamma \frac{\cos z}{z^2}\,dz$

24. For the function $f(z) = \sin\frac{1}{z}, z = 0$ is a
    a) removable singularity
    b) simple pole
    c) branch point
    d) essential singularity

25. Evaluate the integral
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}, a > 0,$$
by the method of residue calculus.

26. Consider a function $f(z) = u + jv$ defined on $|z - j| < 1$ where u, v are real valued functions of x, y. Then $f(z)$ is analytic for u equals to
    a) $x^2 + y^2$
    b) $\ln(x^2 + y^2)$
    c) e^{xy}
    d) $e^{x^2-y^2}$

27. At $z = 0$, the function $f(z) = z^2\bar{z}$
    a) does not satisfy Cauchy-Reimann equations
    b) satisfies Cauchy-Reimann equations but is not differentiable
    c) is differentiable
    d) is analytic

28. Let γ be the curve : $r = 2 + 4\cos\theta, (0 \le \theta \le 2\pi)$. If $I_1 = \int_\gamma \frac{dz}{z-1}$ and $I_2 = \int_\gamma \frac{dz}{z-3}$ then
    a) $I_1 = 2I_2$
    b) $I_1 = I_2$
    c) $2I_1 = I_2$
    d) $I_1 = 0, I_2 \ne 0$

29. Let $f(z)$ be an analytic function with a simple pole at $z = 1$ and a double pole at $z = 2$ with residues 1 and -2 respectively. Further if $f(0) = 0$, $f(3) = -\frac{3}{4}$ and f is bounded as $z \to \infty$, then $f(z)$ must be
    a) $z(z-3) - \frac{1}{4} + \frac{1}{z-1} - \frac{2}{z-1} + \frac{1}{(z-2)^2}$
    b) $-\frac{1}{4} + \frac{1}{z-1} - \frac{2}{z-2} + \frac{1}{(z-2)^2}$
    c) $\frac{1}{z-1} - \frac{2}{z-2} + \frac{5}{(z-2)^2}$
    d) $\frac{15}{4} + \frac{1}{z-1} + \frac{2}{z-2} - \frac{7}{(z-2)^2}$

30. An example of a function with a non-isolated essential singularity at $z = 2$ is
    a) $\tan\frac{1}{z-2}$
    b) $\sin\frac{1}{z-2}$
    c) $e^{-(z-2)}$
    d) $\tan\frac{z-2}{z}$

31. Find $I = \int_C \frac{\cot(\pi z)}{(z-\jmath)^2} dz$, where C is the contour $4x^2 + y^2 = 2$ (counter clock-wise).

32. $\int_0^{2\pi} \frac{d\theta}{13-5\sin\theta} =$

33. For the positively oriented unit circle, find $\oint_{|z|=1} \frac{2\text{Re}(z)}{z+2} dz$

34. The number of zeroes, counting multiplicities, of the polynomial $z^5 + 3z^3 + z^2 + 1$ inside the circle $|z| = 2$ is

    a) 0
    b) 2
    c) 3
    d) 5

35. Consider the functions $f(z) = x^2 + \jmath y^2$ and $g(z) = x^2 + y^2 + \jmath xy$. At $z = 0$,

    a) f is analytic but not g
    b) g is analytic but not f
    c) both f and g are analytic
    d) neither f nor g is analytic

36. Let γ be a simple closed curve in the complex.Then the set of all possible values of $\oint_\gamma \frac{dz}{z(1-z^2)}$ is

    a) $\{0, \pm\pi\jmath\}$
    b) $\{0, \pi\jmath, 2\pi\jmath\}$
    c) $\{0, \pm\pi\jmath, \pm2\pi\jmath\}$
    d) $\{0\}$

37. The principal value of the improper integral $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx$ is

    a) $\frac{\pi}{e}$
    b) πe
    c) $\pi + e$
    d) $\pi - e$

38. The number of roots of the equation $z^5 - 12z^2 + 14 = 0$ that lie in the region $\{z \in C : 2 \le |z| < \frac{5}{2}\}$ is

    a) 2
    b) 3
    c) 4
    d) 5

39. Find the value of $\int_0^{2\pi} \exp(e^{\jmath\theta} - \jmath\theta)\, d\theta$

40. The sum of the residues at all the poles of $f(z) = \frac{\cot \pi z}{(z+a)^2}$, where a is a constant, $(a \ne 0, \pm1, \pm2, ...)$ is

    a) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi\text{cosec}^2\pi a$

    b) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi\text{cosec}^2\pi a$

    c) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi\text{cosec}^2\pi a$

    d) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi\text{cosec}^2\pi a$

41. Which of the following is not the real part of an analytic function?

    a) $x^2 - y^2$
    b) $\frac{1}{1+x^2+y^2}$
    c) $\cos x \cos hy$
    d) $x + \frac{x}{x^2+y^2}$

42. Let $f(z)$ be an analytic function. Then the value of $\int_0^{2\pi} f(e^{\jmath t}) \cos(t) dt$ equals

    a) 0
    b) $2\pi f(0)$
    c) $2\pi f'(0)$
    d) $\pi f'(0)$

43. Let $f(z) = 2z^2 - 1$. Then the maximum value of $|f(z)|$ on the unit disc $D = \{z \in C : |z| \le 1\}$ equals

    a) 1
    b) 2
    c) 3
    d) 4

44. Let S be the positively oriented circle given by $|z - 3\jmath| = 2$. Find the value of $\int_S \frac{dz}{z^2+4}$

45. Let $f(z) = \cos z - \frac{\sin z}{z}$ for non-zero $z \in C$ and $f(0) = 0$. Also,let $g(z) = \sin hz$ for $z \in C$. Then $f(z)$ has a zero at $z = 0$ of order

    a) 0
    b) 1
    c) 2
    d) greater than 2

46. Refer to the previous question for $f(z)$ and $g(z)$. Then $\frac{g(z)}{zf(z)}$ has a pole at $z = 0$ of order

    a) 1
    b) 2
    c) 3
    d) greater than 3

47. For the function $f(z) = \sin\left(\frac{1}{\cos(\frac{1}{z})}\right)$, the point $z = 0$ is

    a) a removable singularity
    b) a pole
    c) an essential singularity
    d) a non-isolated singularity

48. Consider the function $f(z) = \frac{e^{\jmath z}}{z(z^2+1)}$. The residue

of f at the isolated singular point in the upper half plane $\{z = x + \jmath y : y > 0\}$ is

a) $-\frac{1}{2e}$

b) $-\frac{1}{e}$

c) $\frac{e}{2}$

d) 2

49. The Cauchy Principal Value of the integral $\int\limits_{-\infty}^{\infty} \frac{\sin x dx}{x(x^2+1)}$ is

a) $-2\pi(1 + 2e^{-1})$

b) $\pi(1 + e^{-1})$

c) $2\pi(1 + e)$

d) $-\pi(1 + e^{-1})$

50. Let $I = \int\limits_C \frac{f(z)}{(z-1)(z-2)}dz$, where $f(z) = \sin\frac{\pi z}{2} + \cos\frac{\pi z}{2}$ and C is the curve $|z| = 3$ oriented anti-clockwise. Find the value of I

51. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic except for a simple pole at $z = 0$ an let $g : \mathbb{C} \to \mathbb{C}$ be analytic. Then, the value of $\dfrac{\underset{z=0}{\text{Res}}\{f(z)g(z)\}}{\underset{z=0}{\text{Res}} f(z)}$ is

a) $g(0)$

b) $g'(0)$

c) $\lim\limits_{z\to 0} zf(z)$

d) $\lim\limits_{z\to 0} zf(z)g(z)$

52. Let C be the contour $|z| = 2$ oriented in the anti-clockwise direction. The value of the integral $\oint\limits_C ze^{\frac{3}{z}}dz$ is

a) $3\pi\jmath$

b) $5\pi\jmath$

c) $7\pi\jmath$

d) $9\pi\jmath$

53. The function $f(z) = |z|^2 + \jmath\bar{z} + 1$ is differentiable at

a) $\jmath$

b) 1

c) $-\jmath$

d) no point in $\mathbb{C}$

54. The inverse Laplace transform of $\frac{2s^2-4}{(s-3)(s^2-s-2)}$ is

a) $(1 + t)e^{-t} + \frac{7}{2}e^{-3t}$

b) $\frac{e^t}{3} + te^{-t} + 2t$

c) $\frac{7}{2}e^{3t} - \frac{e^{-t}}{6} - \frac{4}{3}e^{2t}$

d) $\frac{7}{2}e^{-3t} - \frac{e^t}{6} - \frac{4}{3}e^{-2t}$

55. The maximum modulus of e^{z^2} on the set $S = \{z \in \mathbb{C} : 0 \le \text{Re}(z) \le 1, 0 \le \text{Im}(z) \le 1\}$ is

a) $\frac{2}{e}$

b) e

c) $e + 1$

d) e^2

56. Let $\Omega = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ and let C be a smooth curve lying in Ω with initial point $-1+2\jmath$ and final point $1 + 2\jmath$. The value of $\int\limits_C \frac{1+2z}{1+z}dz$ is

a) $4 - \frac{1}{2}\ln 2 + \jmath\frac{\pi}{4}$

b) $-4 + \frac{1}{2}\ln 2 + \jmath\frac{\pi}{4}$

c) $4 + \frac{1}{2}\ln 2 - \jmath\frac{\pi}{4}$

d) $4 - \frac{1}{2}\ln 2 + \jmath\frac{\pi}{2}$

57. If $a \in \mathbb{C}$ with $|a| < 1$, then the value of

$$\frac{(1 - |a|^2)}{\pi} \int\limits_\Gamma \frac{|dz|}{|z + a|^2},$$

where Γ is the simple closed curve $|z| = 1$ taken with the positive orientation, is _____

58. Let $C = \{z \in \mathbb{C} : |z - \jmath| = 2\}$. Then $\frac{1}{2\pi}\oint\limits_C \frac{z^2-4}{z^2+4}dz$ is equal to _____

59. The value of $\frac{1}{4-\pi}\int\limits_{|z|=4} \frac{dz}{z\cos(z)}$ is equal to _____

60. Let $\gamma = \{z \in \mathbb{C} : |z| = 2\}$ be oriented in the counter-clockwise direction. Let

$$I = \frac{1}{2\pi\jmath}\oint\limits_\gamma z^7 \cos\left(\frac{1}{z^2}\right)dz.$$

Then, the value of I is equal to _____

61. Let $f(z) = (x^2+y^2)+\jmath 2xy$ and $g(z) = 2xy+\jmath(y^2-x^2)$ for $z = x + \jmath y \in \mathbb{C}$. Then, in the complex plane $\mathbb{C}$.

a) f is analytic and g is NOT analytic.

b) f is NOT analytic and g is analytic.

c) neither f nor g is analytic.

d) both f and g are analytic.

62. Let C be the simple, positively oriented circle of radius 2 centered at the origin in the complex plane. Then find

$$\frac{2}{\pi\jmath} \int\limits_C \left(ze^{\left(\frac{1}{z}\right)} + \tan\left(\frac{z}{2}\right) + \frac{1}{(z-1)(z-3)^2}\right)dz$$

.

63. Let Γ be the circle given by $z = 4e^{\jmath\theta}$, where θ varies from 0 to 2π. Find

$$\oint\limits_\Gamma \frac{e^z}{z^2 - 2z}dz =$$

64. Let $f(z) = z^3 e^{z^2}$ for $z \in \mathbb{C}$ and let Γ be the circle

$z = e^{\jmath\theta}$, where θ varies from 0 to 4π. Then

$$\frac{1}{2\pi\jmath} \oint_\Gamma \frac{f'(z)}{f(z)} dz = \underline{\hspace{3cm}}.$$