A Contour Integral Approach to Evaluate the Cumulative Distribution Function of a Random Variable from its Characteristic Function

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***Abstract*— The Gil-Pelaez theorem has found remarkable application in communication theory to simplify outage and diversity analysis in mobile communication systems. The theorem allows for the cumulative distribution function (c.d.f.) of a random variable to be expressed in terms of a real improper integral involving its characteristic function (c.f.). Most of the methods using the theorem require the c.d.f. to be evaluated at zero. A significant feature of the integrand is a built in singularity at the origin, which needs to be accounted for when the real integral is sought to be transformed to a complex contour integral. Unfortunately, this has not been done in the numerous applications of the theorem in the available literature, and the mathematical validity of these needs to be proved. In this paper, based on the theory of generalized functions, an alternative derivation of the Gil-Pelaez theorem has been provided along with a mathematical basis for the contour integral approach to evaluate the c.d.f.**

***Index Terms*— Gil-Pelaez theorem, Cauchy principal value, generalized functions**

I. Introduction

Recent advances in wireless communications have resulted in a corresponding complexity in the performance analysis of modern communication systems, and researchers in communication theory are exploring different branches of mathematics to find simpler solutions to their problems. One result, that simplifies the process of finding the outage probability in a fading environment with multiple interferers [1] and symbol error probabilty in systems using diversity techniques [2] is the Gil-Pelaez theorem [3]. The most appealing aspect of the theorem is the expression of the c.d.f. of a random variable (r.v.) in terms of a real integral involving its c.f. Though a version of this result was earlier proved by Gurland [4], the proof is rather abstract involving the Lebesgue integral whereas the Reimann integral is used in [3]. Shephard [5] provided a new approach to the proof of the Gil-Pelaez theorem and used it to derive a multivariate inversion theorem.

Under some conditions, it is possible to express the c.d.f. of the r.v. in terms of a contour integral. This contour integral can then be easily evaluated by using the residue theorem [6], resulting in a simple method to find the c.d.f. of a r.v., given its c.f. In general, the method of transforming a real integral to a contour integral requires that the integrand have no singularity on the interval of integration. Else, the real integral would not exist and we obtain the Cauchy principal value (c.p.v.) of the integral instead. Fortunately, this is not a problem, because the expression of the c.d.f. in terms of the c.p.v. of the improper integral in the Gil-Pelaez formula is implied in the derivation [3]. But most of the works that refer to this theorem do not seem to have taken note of this fact.

A c.f. approach has been suggested by Zhang [1] to compute the outage probability in a cellular network with multiple Nakagami interferers having arbitrary fading parameters. By using the Gil-Pelaez theorem, this approach expresses the outage probability in terms of a contour integral, which was evaluated using the method of residues. In the process, the c.p.v. of the real integral was erroneously defined as the integral itself , that allowed the c.f. approach to be used, even if the integral did not exist. Because of this, given the singularity of the integrand at the origin, a justification for the contour integral approach needs to be provided.

In this paper, using a slighltly different approach from [1], we provide a proof of the Gil-Pelaez theorem through the theory of generalized functions [7]. Then, for random variables having well defined c.fs., a contour integral approach is used to evaluate the c.d.f. Our approach is rigorous, and accounts for the singularity of the integrand at the origin, while transforming the real improper integral to a contour integral.

The rest of the paper is organized as follows. In Section II, we prove the Gil-Pelaez theorem using generalized functions. Then the process of transforming the real integral to the corresponding contour integral to obtain the c.d.f. of a r.v. from its c.f. is described in Section III. Conclusions are available in Section IV.

II. Proof of the Gil-Pelaez Theorem

Before proceeding with the proof of the theorem, we introduce the concept of the c.p.v., that is available in standard texts on mathematical analysis.

Definition: The Cauchy principal value (c.p.v.) of a definite integral

$$\int_A^B f(t)dt \tag{1}$$

whose integrand becomes infinite at a point a in the interval of integration, i.e.

$$\lim_{t\to a} |f(t)| = \infty \tag{2}$$

is defined as [8]

$$c.p.v. \int_A^B f(t)dt = \lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(t)dt + \int_{a-\epsilon}^B f(t)dt \right]. \quad (3)$$

The integral itself is defined as

$$\int_A^B f(t)dt = \lim_{\epsilon \to 0} \int_A^{a-\epsilon} f(t)dt + \lim_{\eta \to 0} \int_{a-\eta}^B f(t)dt, \quad (4)$$

where both ϵ and η approach zero through positive values. It may so happen that neither of the two limits in (4) exist, i.e. the integral itself has no meaning but the c.p.v. defined by (3) exists.

The c.d.f. $F_\gamma(x)$ of a r.v. γ can be expressed in the form

$$F_\gamma(x) = \int_{-\infty}^x dF_\gamma(y). \quad (5)$$

If $\phi_\gamma(t)$ be the c.f. of γ,

$$F_\gamma(x) = \frac{1}{2\pi} \int_{-\infty}^x \int_{-\infty}^\infty \phi_\gamma(t) e^{-jyt} dt\, dy \quad (6)$$

from the inverse fourier transform relationship. With an appropriate change of variables in the outer integral, we obtain

$$\begin{aligned} F_\gamma(x) &= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty \phi_\gamma(t) e^{-j(x-y)t} dt\, dy \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty dt\, \phi_\gamma(t) e^{-jxt} \int_0^\infty e^{jyt} dy, \quad (7) \end{aligned}$$

when $\phi_\gamma(t)$ is continuous over the interval of integration. From [7, Eqn. 23, p.151],

$$\int_0^\infty e^{jyt} dy = \pi\delta(t) + j\; Pf\left(\frac{1}{t}\right), \quad (8)$$

where $Pf\left(\frac{1}{t}\right)$ is a *pseudofunction*[1] defined by

$$\int_{-\infty}^\infty Pf\left(\frac{1}{t}\right) \psi(t) dt = c.p.v. \int_{-\infty}^\infty \frac{\psi(t)}{t} dt, \quad (9)$$

when the c.p.v. of the integral on the right hand side of the above equation exists for the continuous function $\psi(t)$. Substituting (8) in (7),

$$F_\gamma(x) = \frac{1}{2}\phi_\gamma(0) + \frac{j}{2\pi} \int_{-\infty}^\infty Pf\left(\frac{1}{t}\right) \phi_\gamma(t) e^{-jxt} dt. \quad (10)$$

Since $\phi_\gamma(t)$ is a c.f., $\phi_\gamma(0) = 1$. From (9) and (10),

$$F_\gamma(x) = \frac{1}{2} - \frac{1}{2\pi j} \times c.p.v. \int_{-\infty}^\infty \frac{\phi_\gamma(t)}{t} e^{-jxt} dt. \quad (11)$$

Gil-Pelaez, while deriving the inversion formula, obtained[2] [3]

$$\lim_{\substack{\epsilon \to 0 \\ \lambda \to \infty}} \frac{1}{\pi} \int_\epsilon^\lambda \frac{e^{jtx}\phi_\gamma(-t) - e^{-jxt}\phi_\gamma(t)}{jt} dt = 2F_\gamma(x) - 1, \quad (12)$$

which is the same as (11). From [1], we obtain the probability

$$P(\gamma > 0) = \frac{1}{2} + \frac{1}{2\pi j} \int_\infty^\infty \frac{\phi_\gamma(t)}{t} dt. \quad (13)$$

A weaker expression for $F_\gamma(0)$ is obtained from the above as

$$\begin{aligned} F_\gamma(0) &= 1 - P(\gamma > 0) \\ &= \frac{1}{2} - \frac{1}{2\pi j} \int_\infty^\infty \frac{\phi_\gamma(t)}{t} dt. \quad (14) \end{aligned}$$

Evaluating the c.d.f. in (11) at zero yields a similar expression as (14), except that we obtain the c.p.v. of the integral in the expression. This is explained by the fact that classical texts on communication theory [9] express

$$\int_0^\infty e^{jyt} dy = \pi\delta(t) + \frac{j}{t}, \quad (15)$$

which was used to obtain (13).

We now provide an example to show the usefulness of (11) over (14).

Example 1: Let $\gamma = x_1 - x_2$, where x_1 and x_2 are exponential i.i.d. r.vs. with unit mean. Then the c.fs. of the r.vs. x_1 and x_2 are

$$\phi_{x_1}(t) = \phi_{x_2}(t) = \frac{1}{(1 - jt)}.$$

Also,

$$\begin{aligned} \phi_\gamma(t) &= \phi_{x_1}(t)\phi_{x_2}(-t) \\ &= \frac{1}{(1 + t^2)}. \quad (16) \end{aligned}$$

From (14) and (16), we obtain

$$F_\gamma(0) = \frac{1}{2} - \frac{1}{2\pi j} \int_\infty^\infty \frac{dt}{t(1+t^2)} dt. \quad (17)$$

Since

$$\frac{1}{t\,(1+t^2)} = \frac{1}{t} - \frac{t}{1+t^2}, \quad (18)$$

we obtain

$$\int_\infty^\infty \frac{dt}{t(1+t^2)} dt = \int_{-\infty}^\infty \frac{dt}{t} - \int_{-\infty}^\infty \frac{t}{1+t^2} dt. \quad (19)$$

From (4), we find that neither of the integrals on the right hand side of the above equation converge, and we cannot evaluate $F_\gamma(0)$ using (17). But

$$\begin{aligned} F_\gamma(0) &= P(\gamma \le 0) \\ &= P(x_1 \le x_2). \quad (20) \end{aligned}$$

By symmetry, since x_1 and x_2 are i.i.d., $P(x_1 \le x_2) = \frac{1}{2}$. Thus, we know that $F_\gamma(0) = \frac{1}{2}$, but there is no way to reach this result using (17). However, using (3), the c.p.v. of the integrals on the right hand side of (19) is found to be zero, since their respective integrands are odd functions, and we have

$$c.p.v. \int_{-\infty}^\infty \frac{dt}{t(1+t^2)} dt = 0. \quad (21)$$

From (11), (16) and (21), we obtain $F_\gamma(0) = \frac{1}{2}$.

The above discussion highlights the weakness in the approach suggested in [1] and provides a strong argument for using the general form of the Gil-Pelaez theorem derived in this section.

[1] Kanwal [7, Eqn. 17, p.25] has defined the *pseudofunction* through a linear functional. We have used the Reimann integral, only to reduce the level of abstraction.

[2] The final form of the formula where the integration limits are from 0 to ∞ holds only when the integral exists.

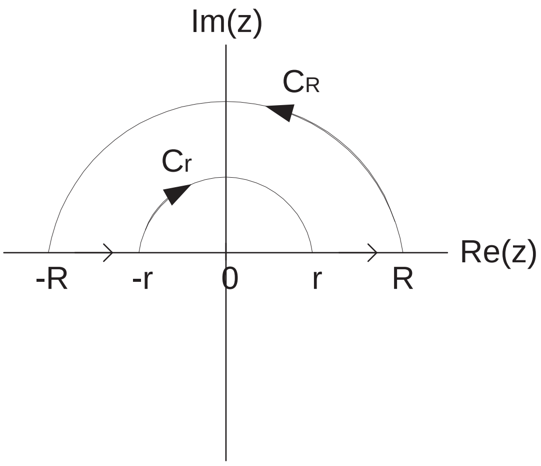

Fig. 1. The contour C.

III. The contour integral

The contour integral method suggested in [1] can be used to evaluate $F_\gamma(0)$, but since we are interested in evaluating $F_\gamma(x)$, we will have to rely on a more general approach. In the following, we have used readily available results in complex analysis to obtain a simple formula for the c.d.f. We state the following Lemma.

Lemma 3.1: Let ϕ_γ *be analytic on an open set containing the entire complex plane, except for a finite number of isolated singularities, none of which are on the real axis. Also, suppose that* $\phi_\gamma(z) \to 0$ *as* $z \to \infty$ *such that* $\forall\ \epsilon > 0$, *there exists* $R(\epsilon)$ *such that* $|\phi_\gamma(z)| < \epsilon$ *whenever* $|z| \geq R(\epsilon)$. *Then,*

$$F_\gamma(x) = \begin{cases} -\sum \left\{ \textit{residues of } \frac{\phi_\gamma(z)}{z} e^{-jxz} \textit{in } S \right\} & x \leq 0 \\ 1 + \sum \left\{ \textit{residues of } \frac{\phi_\gamma(z)}{z} e^{-jxz} \textit{in } S \right\} & x \geq 0, \end{cases} \tag{22}$$

where

$$S = \begin{cases} \{z \in \mathbf{C},\ |\ Im(z) > 0\} & x \leq 0 \\ \{z \in \mathbf{C},\ |\ Im(z) < 0\} & x \geq 0. \end{cases} \tag{23}$$

Note: For $x = 0$, *choose S such that it contains at least one pole of* $\phi_\gamma(z)$.

An outline of the proof of the above Lemma is provided below. More details are available in [6]. We consider the closed path C in the complex plane in Fig. 1. Let C_R and C_r be two semi-circular paths with radii R and r respectively. We have the contour integral

$$\int_C \frac{\phi_\gamma(z)}{z} e^{-jxz} dz = \int_{C_R} \frac{\phi_\gamma(z)}{z} e^{-jxz} dz + \int_{-R}^{-r} \frac{\phi_\gamma(t)}{t} e^{-jxt} dt + \int_{C_r} \frac{\phi_\gamma(z)}{z} e^{-jxz} dz + \int_r^R \frac{\phi_\gamma(t)}{t} e^{-jxt} dt. \tag{24}$$

For $x \leq 0$, choosing S to be the closed upper half plane, from (23) and [6, p.282]

$$\lim_{R\to\infty} \int_{C_R} \frac{\phi_\gamma(z)}{z} e^{-jxz} dz = 0. \tag{25}$$

Also [6, p.285],

$$\begin{aligned} \lim_{r\to 0} \int_{C_r} \frac{\phi_\gamma(z)}{z} e^{-jxz} dz &= -j\pi \left\{ \textit{residue of } \frac{\phi_\gamma(z)}{z} e^{-jxz}, z = 0 \right\} \\ &= -j\pi \phi_\gamma(0). \end{aligned} \tag{26}$$

From (24), (25) and (26), we get

$$\lim_{\substack{r\to 0 \\ R\to\infty}} \int_{-R}^{-r} \frac{\phi_\gamma(t)}{t} e^{-jxt} dt + \int_r^R \frac{\phi_\gamma(t)}{t} e^{-jxt} dt = j\pi + \int_C \frac{\phi_\gamma(z)}{z} e^{-jxz} \tag{27}$$

since $\phi_\gamma(0) = 1$. But the left hand side[3] of the above equation is the c.p.v. of $\int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t} e^{-jxt} dt$. Hence, using the residue theorem [6, p.286], we obtain

$$c.p.v. \int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t} e^{-jxt} dt = j\pi + \tag{28}$$
$$2\pi j \sum \left\{ \textit{residues of } \frac{\phi_\gamma(z)}{z} e^{-jxz} \right\} \textit{in } S.$$

Substituting the above in (11) gives us (22). A similar derivation holds for $x \geq 0$.

Example 2: In Example 1,

$$\phi_\gamma(t) = \frac{1}{1+t^2} \tag{29}$$

has poles at $\pm j$. From (22), we find that

$$F_\gamma(x) = \begin{cases} \frac{1}{2} e^x & x \leq 0 \\ 1 - \frac{1}{2} e^{-x} & x \geq 0 \end{cases} \tag{30}$$

and $F_\gamma(0) = \frac{1}{2}$, which was obtained earlier by using the c.p.v. of the real integral. Note that $F_\gamma(x)$ is continuous at 0, indicating the either of the poles can be used to evaluate $F_\gamma(0)$. In the next example, we find that this is not always the case.

Example 3: The c.f. of a central chi-square distribution γ with $2n$ degrees of freedom and parameter σ [10] is

$$\phi_\gamma(t) = \frac{1}{(1 - j2\sigma^2 t)^n}. \tag{31}$$

Since $\phi_\gamma(z)$ has multiple poles of order n at $z = -\frac{j}{2\sigma^2}$, and no poles in the upper half plane, from (22), we find that $F_\gamma(x) = 0, x < 0$. For $x \geq 0$, after finding the residue [6, p.250],

$$F_\gamma(x) = 1 + \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[\left(z - \frac{1}{j2\sigma^2} \right)^n \frac{\phi_\gamma(z) e^{-jxz}}{z} \right]_{z = -\frac{j}{2\sigma^2}}.$$

Substituting the expression for $\phi_\gamma(z)$ from (31),

$$\begin{aligned} F_\gamma(x) &= 1 + \frac{1}{(-j2\sigma^2)^n} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{e^{-jxz}}{z} \right]_{z = -\frac{j}{2\sigma^2}} \\ &= 1 + \left(\frac{x}{2\sigma^2} \right)^n \frac{1}{(n-1)!} \frac{d^{n-1}}{dt^{n-1}} \left[\frac{e^t}{t} \right]_{t = -\frac{x}{2\sigma^2}}. \end{aligned} \tag{32}$$

For any two functions $f(t)$ and $g(t)$, whose m^{th} order derivatives exist,

$$\frac{d^m}{dt^m} [f(t) g(t)] = \sum_{k=0}^{m} {}^m C_k \frac{d^{m-k} f(t)}{dt^{m-k}} \times \frac{d^k f(t)}{dt^k}. \tag{33}$$

[3]For $x = 0$, Zhang[1, Eqn. 54] has defined it as $\int_{-\infty}^{\infty} \frac{\phi_\gamma(t)}{t} dt$, which is not true in general. This has been demonstrated in Example 1.

For $f(t) = e^t$ and $g(t) = \frac{1}{t}$, from (32) and (33),

$$\begin{aligned} F_\gamma(x) &= 1 + \left(\frac{x}{2\sigma^2}\right)^n \frac{e^{-\frac{x}{2\sigma^2}}}{(n-1)!} \\ &\times \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-k-1)!k!}(-1)^k k! \left(-\frac{x}{2\sigma^2}\right)^{-(k+1)} \end{aligned} \quad (34)$$

After simplifying the above expression, we obtain

$$F_\gamma(x) = 1 - e^{-\frac{x}{2\sigma^2}} \sum_{k=0}^{n-1} \frac{1}{k!}\left(\frac{x}{2\sigma^2}\right)^k \quad x \geq 0 \quad (35)$$

which has been obtained in [10] by integrating the probability density function (p.d.f.)

In Example 3, when the number of degrees of freedom is odd, (say $2n+1$), the degree of the expression in the denominator of the c.f. is $n+\frac{1}{2}$. In such cases, we cannot apply the method of residues, because of the limitation imposed by Cauchy's integral formula [6]. In fact, a closed form expression for the c.d.f. in this case does not exist [10].

Despite these limitations, whenever the above approach holds, instead of integrating an expression to evaluate the c.d.f., it is sufficient to differentiate an expression involving the characteristic function. This is where this method scores over more general ones, like those based on the moment generating function [11] and the conditional error probability [12].

IV. Conclusions

Earlier attempts to evaluate probabilities from the c.f. using the contour integral approach have not paid sufficient attention to the singularity of the integrand at the origin, that results as a consequence of the Gil-Pelaez theorem. We have shown how this singularity can render such an approach mathematically invalid. To address this, we proved the general version of the Gil-Pelaez theorem through a generalized function based approach. It was then shown that this general form, in terms of the c.p.v., was implicit in the original proof of the theorem by Gil-Pelaez.

Using the general version of the theorem, for the first time, a rigorous method to evaluate the c.d.f. of a r.v. from its c.f., using contour integration, has been proposed. Through examples, we have shown how this method may be applied, and also discuss its limitations. We have also discovered two different expressions for the same integral in (8) and (15), which is the Fourier transform of the unit step function. This is a paradox that needs to be explained, since both expressions seem to result from valid concepts.

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