Contour Integration

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Abstract—**This manual provides a simple introduction to contour integrals.**

1 The Laplace Distribution

Definition 1.1. The Laplace distribution is defined *conditionally* as

$$X \sim \mathrm{N}(0, Y) \tag{1}$$

where $Y = \|h\|^2$ and $h \sim \mathrm{CN}(0, 2)$ is complex circularly Gaussian.

Problem 1. Show that the conditional characteristic function of X is

$$\phi_{X/Y}(\jmath\omega) = e^{-\frac{1}{2}Y\omega^2} \tag{2}$$

Problem 2. Given that

$$\phi_Y(\jmath\omega) = \frac{1}{1 - 2\jmath\omega}, \tag{3}$$

show that

$$\phi_X(\jmath\omega) = E\left[\phi_{X/Y}(\jmath\omega)\right] = \frac{1}{1 + \omega^2} \tag{4}$$

2 Contour Integration

In Fig. 3, let $z = x + \jmath y, C = C_1 + C_2$. It is obvious that the line integral in the anti-clockwise direction

$$\oint_C \frac{e^{-\jmath zt}}{1 + z^2}\, dz = \int_{C_1} \frac{e^{-\jmath zt}}{1 + z^2}\, dz + \int_{C_2} \frac{e^{-\jmath zt}}{1 + z^2}\, dz \tag{5}$$

The symbol on the integral on the LHS shows that the integration is over a closed path in the anti-clockwise direction.

Problem 3. Show that

$$\int_{C_1} \frac{e^{-\jmath zt}}{1 + z^2}\, dx = \int_{-R}^{R} \frac{e^{-\jmath xt}}{1 + x^2}\, dx \tag{6}$$

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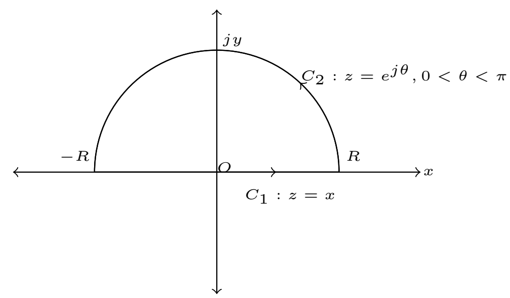

Fig. 3

Problem 4. Show that

$$\lim_{R\to\infty} \left| \frac{e^{-\jmath zt}}{1 + z^2} \right| = 0, \quad t < 0 \tag{7}$$

by substituting $z = Re^{\jmath\theta}, 0 < \theta < \pi$.

Problem 5. Let C be a closed curve within the curve T. Then given that

$$\oint_C \frac{e^{-\jmath zt}}{1 + z^2}\, dz = \oint_T \frac{e^{-\jmath zt}}{1 + z^2}\, dz, \tag{8}$$

show that

$$\oint_C \frac{e^{-\jmath zt}}{1 + z^2}\, dz = \int_{-\infty}^{\infty} \frac{e^{-\jmath xt}}{1 + x^2}\, dx, \quad t < 0 \tag{9}$$

Problem 6. Given that

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi\jmath f(z_0), \tag{10}$$

where z_0 is within C, show that

$$\oint_C \frac{e^{-\jmath zt}}{1 + z^2}\, dz = \pi e^t \quad t < 0 \tag{11}$$

Problem 7. Now find

$$\int_{-\infty}^{\infty} \frac{e^{-\jmath xt}}{1 + x^2}\, dx, \quad t > 0 \tag{12}$$

Problem 8. Obtain an expression for the probability density function (PDF) of X.

3 Cauchy's Integral Formula

Problem 9. Show that

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz \tag{13}$$

Problem 10. For $C : z = z_0 + \rho e^{\jmath\theta}, 0 < \theta < 2\pi$, show that

$$\oint_C \frac{dz}{z-z_0} = 2\pi\jmath \tag{14}$$

Problem 11. Let

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - K\right| < \delta \implies |z-z_0| < \epsilon \tag{15}$$

for some $\epsilon, \delta > 0$.

1) Show that

$$|f(z) - Kz - f(z_0) + Kz_0| < \epsilon\delta \tag{16}$$

2) Let

$$f_1(z) = f(z) - Kz \tag{17}$$

$$f_2(z) = Kz \tag{18}$$

Show that

$$\begin{aligned}&|f_1(z) + f_2(z) - f_1(z_0) - f_2(z_0)| \\ &\qquad = |f(z) - f(z_0)| < \lambda\end{aligned} \tag{19}$$

3) Let

$$g(z) = \frac{f(z)-f(z_0)}{z-z_0} \tag{20}$$

Show that

$$|z-z_0| = \rho \implies |g(z)| < \frac{\lambda}{\rho} \tag{21}$$

Problem 12. Let

$$|g(z)| < \frac{\lambda}{\rho} \tag{22}$$

Show that, if $C : z = \rho e^{\jmath\theta}, 0 < \theta < 2\pi$ and $z_i \in C, \Delta z_m = z_m - z_{m-1}, \zeta_m \in (z_{n-1}, z_n)$,

$$\lim_{\substack{\Delta z_m \to 0\\ n\to\infty}} \left|\sum_{m=1}^{n} g(\zeta_m)\Delta z_m\right| < 2\pi\lambda \tag{23}$$

Definition 3.1.

$$\int_C g(z)\,dz = \lim_{\substack{\Delta z_m \to 0\\ n\to\infty}} \sum_{m=1}^{n} g(\zeta_m)\Delta z_m \tag{24}$$

Problem 13. Show that

$$\left|\int_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < 2\pi\lambda \tag{25}$$

Problem 14. Show that

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi\jmath f(z_0) \tag{26}$$

Problem 15. By differentiating (26) with respect to z_0, show that

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi\jmath \frac{f^{(n)}(z_0)}{n!} \tag{27}$$

where $f^{(n)}(z)$ is the nth derivative of $f(z)$.

4 Cauchy's Integral Theorem

Definition 4.1. The derivative of $f(z)$ is defined as

$$\lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \tag{28}$$

$f(z)$ is said to be *analytic* if its derivative exists.

Problem 16. Let $z = x + \jmath y, \Delta z = \Delta x + \Delta y, f(z) = u(x,y) + \jmath v(x,y)$.

1) By letting $\Delta y \to 0$ followed by $\Delta x \to 0$, show that

$$f'(z) = \frac{\partial u}{\partial x} + \jmath\frac{\partial v}{\partial x} \tag{29}$$

2) By letting $\Delta x \to 0$ followed by $\Delta y \to 0$, show that

$$f'(z) = \frac{\partial v}{\partial y} - \jmath\frac{\partial u}{\partial y} \tag{30}$$

Problem 17. Show that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{31}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{32}$$

These are known as the *Cauchy-Riemann* equations.

Problem 18. Using (24), show that

$$\begin{aligned}\oint_C f(z)\,dz = &\oint_C (u\,dx - v\,dy) \\ &+ \jmath\oint_C (u\,dy + v\,dx)\end{aligned} \tag{33}$$

Problem 19. According to *Green's Theorem*, for a closed curve C,

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \tag{34}$$

where R is bounded by C. Show that

$$\oint_C f(z)\,dz = 0. \qquad (35)$$

assuming that $f(z)$ is analytic everywhere inside C.

Problem 20. In Fig. 20, show that

$$\oint_C f(z)\,dz = \oint_T f(z)\,dz \qquad (36)$$

given that $f(z)$ is analytic in the region between $C - T$.

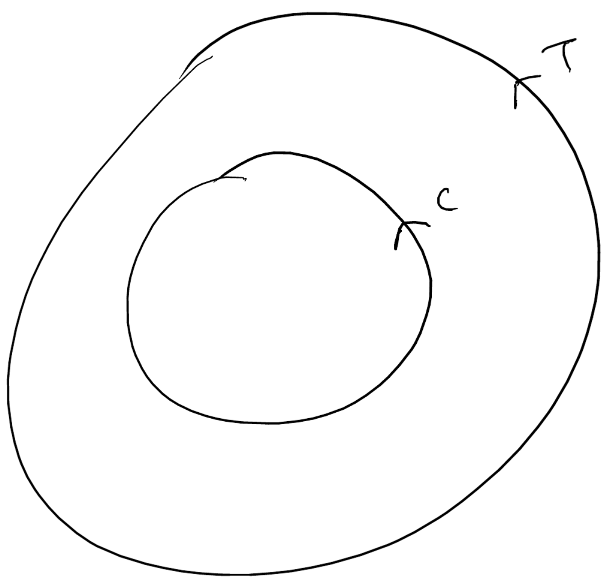

Fig. 20