Complex Analysis in Electrical Engineering

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***Abstract*—This manual provides applications of Complex Analysis in Electrical Engineering.**

1 The Inverse Z Transform

Problem 1. Show that z^n is analytic everywhere for $n \geq 0$.

Problem 2. Show that for $C : z = Re^{\jmath\theta}, 0 < \theta < 2\pi$,

$$\oint_C \frac{dz}{z^n} = \begin{cases} 2\pi\jmath & n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Definition 1.1. The Z transform of $x(n)$ is defined as

$$X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k} \tag{2}$$

Problem 3. Show that

$$\frac{1}{2\pi\jmath} \oint_C X(z) z^{n-1}\, dz = \sum_{k=-\infty}^{\infty} x(k) \oint_C z^{n-k-1}\, dz \tag{3}$$

$$= x(n) \tag{4}$$

Problem 4. The Z transform of $x(n)$ is given by

$$X(z) = \frac{z^{20}}{\left(z - \frac{1}{2}\right)(z-2)^5 \left(z + \frac{5}{2}\right)^2 (z+3)} \tag{5}$$

Also, it is known that $X(z)$ is analytic for $|z| = 1$. Find $x(-18)$.

2 The Laplace Transform

Problem 5. Let

$$\int_{-\infty}^{\infty} F(t) e^{-st}\, dt = \frac{1}{s\left(1 - s^2\right)} = \frac{M(s)}{s}, \quad s = \sigma + \jmath\omega \tag{6}$$

Using the inverse fourier transform relationship, show that

$$F(t) e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\jmath\omega t}}{s\left(1 - s^2\right)}\, d\omega \tag{7}$$

Problem 6. Show that

$$F(t) = \frac{1}{2\pi\jmath} \int_{\sigma - \jmath\infty}^{\sigma + \jmath\infty} \frac{e^{st}}{s\left(1 - s^2\right)}\, ds \tag{8}$$

Problem 7. Let

$$F(t) = \begin{cases} \frac{1}{2\pi\jmath} \oint_{C_1} \frac{e^{st}}{s\left(1-s^2\right)}\, ds & t > 0 \\ \frac{1}{2\pi\jmath} \oint_{C_2} \frac{e^{st}}{s\left(1-s^2\right)}\, ds & t < 0 \end{cases} \tag{9}$$

where C_1, C_2 are the closed contours on the left and right respectively as shown in Fig. 7. Find $F(t)$ given that the ROC of $\frac{M(s)}{s}$ is $0 < \text{Re}\,(s) < 1$.

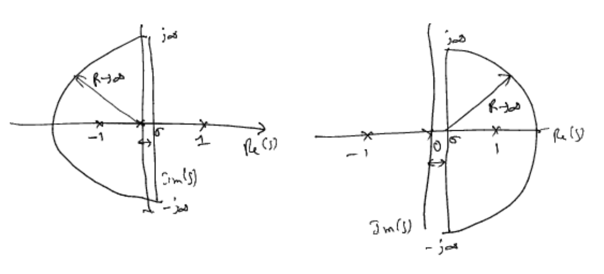

Fig. 7

3 The Gil-Pelaez Integral

Problem 8. Let

$$F(t) = \frac{1}{2} - \frac{1}{2\pi\jmath} \times \text{c.p.v.} \int_{-\infty}^{\infty} \frac{e^{-\jmath\omega t}}{\omega\left(1 + \omega^2\right)}\, d\omega, \tag{10}$$

where c.p.v. denotes the Cauchy Principal Value. Use the contour in Fig. 8 to evaluate $F(t), t < 0$ by showing that

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1)

$$\lim_{r\to 0}\int_{C_r}\frac{e^{-\jmath zt}}{z\,(1+z^2)}\,dz = -\jmath\pi,$$

$$C_r : z = re^{-\jmath\theta}, 0 < \theta < \pi \quad (11)$$

2)

$$\lim_{\substack{r\to 0\\ R\to\infty}}\int_{-R}^{-r}\frac{e^{-\jmath\omega t}}{\omega\,(1+\omega^2)}\,d\omega + \int_{r}^{R}\frac{e^{-\jmath\omega t}}{\omega\,(1+\omega^2)}\,d\omega$$

$$= \jmath\pi + \oint_{C}\frac{e^{-\jmath zt}}{z\,(1+z^2)}\,dz \quad (12)$$

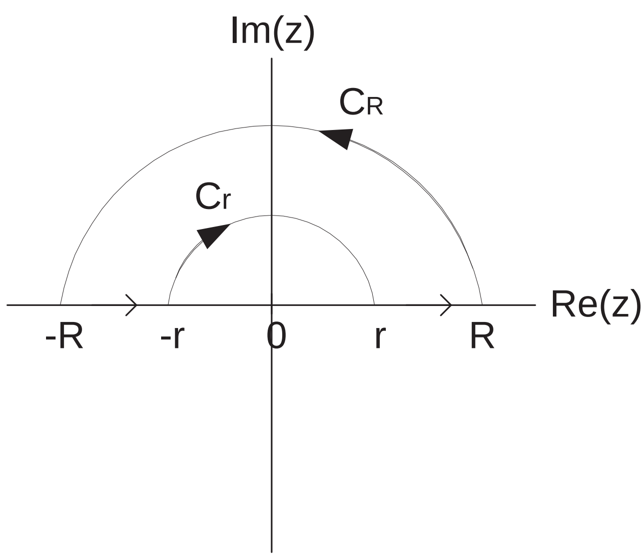

Fig. 8