Least Mean Square Algorithm

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Contents

***Abstract*—This manual provides an introduction to the LMS algorithm.**

1 Audio Source Files

1.1 Get the **audio_source**

```
svn checkout https://github.com/gadepall/
    EE5347/trunk/audio_source
cd audio_source
```

1.2 Play the **signal_noise.wav** and **noise.wav** file. Comment.

Solution: signal_noise.wav contains a human voice along with an instrument sound in the background. This instrument sound is captured in **noise.wav**.

2 Problem Formulation

Let the **signal_noise.wav** be $d(n)$, which contains a human voice along with an instrument sound in the background. This instrument sound is captured in **noise.wav** and denoted as $X(n)$. The goal is to extract the human voice by suppressing $X(n)$.

Known	d(n)	signal noise.wav
Unknown	X(n) e(n) W(n)	noise.wav

Let

$$d(n) = e(n) + W^T(n)X(n) \tag{2.1}$$

where $e(n)$ is an estimate of the human voice (desired signal)

$$X(n) = \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ .. \\ .. \\ x(n-M+1) \end{bmatrix}_{MX1} \tag{2.2}$$

$$W(n) = \begin{bmatrix} w_1(n) \\ w_2(n) \\ w_3(n) \\ .. \\ .. \\ w_{n-M+1}(n) \end{bmatrix}_{MX1} \tag{2.3}$$

and estimating $W(n)$. The human voice can be characterized as

$$e(n) = d(n) - W^T(n)X(n) \tag{2.4}$$

The goal is to find $W(n)$ that will allow $W^T(n)X(n)$ to mimic the instrument sound in $d(n)$. This is possible if $e(n)$ is minimum. This problem can be expressed as

$$\min_{W(n)} e^2(n) \tag{2.5}$$

3 LMS Algorithm

Problem 3.1. Show using (2.4) that

$$\nabla_{W(n)} e^2(n) = \frac{\partial e^2(n)}{\partial W(n)} \tag{3.1}$$

$$= -2X(n)d(n) + 2X(n)X^T(n)W(n) \tag{3.2}$$

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Problem 3.2. Use the gradient descent method to obtain an algorithm for solving

$$\min_{W(n)} e^2(n) \tag{3.3}$$

Solution: The desired algorithm can be expressed as

$$W(n+1) = W(n) - \bar{\mu}[\nabla_{W(n)} e^2(n)] \tag{3.4}$$
$$W(n+1) = W(n) + \mu X(n)e(n) \tag{3.5}$$

where $\mu = \bar{\mu}$.

Problem 3.3. Write a program to suppress $X(n)$ in $d(n)$.

Solution: Execute **LMS_NC_SPEECH.py**.

4 Wiener-Hopf Equation

Problem 4.1. Let

$$e(n) = d(n) - W^T(n)X(n) \tag{4.1}$$

Show that

$$E[e^2(n)] = r_{dd} - W^T(n)r_{xd} - r_{xd}^T W(n) + W^T(n)RW(n) \tag{4.2}$$

where

$$r_{dd} = E[d^2(n)] \tag{4.3}$$
$$r_{xd} = E[X(n)d(n)] \tag{4.4}$$
$$R = E[X(n)X^T(n)] \tag{4.5}$$

Problem 4.2. By computing

$$\frac{\partial J(n)}{\partial W(n)} = 0, \tag{4.6}$$

show that the optimal solution for

$$W^*(n) = \min_{W(n)} E\left[e^2(n)\right] = R^{-1} r_{xd} \tag{4.7}$$

This is the Wiener optimal solution.

5 Convergence of the LMS Algorithm

5.1 Convergence in the Mean

Problem 5.1. Show that R in (4.5) is symmetric as well as positive definite.

Let

$$\tilde{W}(n) = W(n) - W_* \tag{5.1}$$

where W_* is obtained in (4.7). Also, according to the LMS algorithm,

$$W(n+1) = W(n) + \mu X(n)e(n) \tag{5.2}$$
$$e(n) = d(n) - X^T(n)W(n) \tag{5.3}$$

Problem 5.2. Show that

$$E\left[\tilde{W}(n+1)\right] = [I - \mu R]E\left[\tilde{W}(n)\right] \tag{5.4}$$

Problem 5.3. Show that

$$R = U\Lambda U^T \tag{5.5}$$

for some U, Λ, such that Λ is a diagonal matrix and $U^T U = I$.

Problem 5.4. Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n+1)\right] = 0 \iff \lim_{n\to\infty}[I - \mu\Lambda]^n = 0 \tag{5.6}$$

Problem 5.5. Using (5.6), show that

$$0 < \mu < \frac{2}{\lambda_{\max}} \tag{5.7}$$

where $\lambda_{\max}$ is the largest entry of Λ.

5.2 Convergence in Mean-square sense

Let

$$X(n) = \begin{bmatrix} X_1(n) \\ X_2(n) \end{bmatrix} \tilde{W}(n) = \begin{bmatrix} \tilde{W}_1(n) \\ \tilde{W}_2(n) \end{bmatrix} \tag{5.8}$$

Problem 5.6. Show that

$$E[\tilde{W}^T(n)X(n)X^T(n)\tilde{W}(n)] = E[\tilde{W}^T(n)R\tilde{W}(n)] \tag{5.9}$$

for R defined in (4.5).

Problem 5.7. Show that

$$\begin{aligned} J(n) = E[e^2(n)] &= E[e_*^2(n)] \\ &+ E[\tilde{W}(n)X(n)X(n)^T\tilde{W}(n)^T] - E[\tilde{W}(n)X(n)e_*(n)] \\ &- E[e_*(n)X^T(n)\tilde{W}^T(n)] \end{aligned} \tag{5.10}$$

where

$$\tilde{W}(n) = W(n) - W_* \tag{5.11}$$
$$e_*(n) = d(n) - W_* X(n) \tag{5.12}$$

Problem 5.8. Show that

$$E\left[\tilde{W}(n)X(n)e_*(n)\right] = E\left[e_*(n)X^T(n)\tilde{W}^T(n)\right] = 0 \tag{5.13}$$

Problem 5.9. Show that

$$E\left[\tilde{W}^T(n)R\tilde{W}(n)\right] = \text{trace}\left(E\left[\tilde{W}^T(n)R\tilde{W}(n)\right]\right) \quad (5.14)$$
$$= \text{trace}\left(E\left[\tilde{W}(n)\tilde{W}^T(n)\right]R\right) \quad (5.15)$$

Problem 5.10. Using (5.11), (5.2) and (5.12), show that

$$\tilde{W}(n+1) = \left[I - \mu X(n)X^T(n)\right]\tilde{W}(n) + \mu X(n)e_*(n) \quad (5.16)$$

Problem 5.11. Let $\mu^2 \to 0$. Using (5.5) and (4.7), show that

$$E\left[\tilde{W}(n+1)\tilde{W}^T(n+1)\right] = (I - 2\mu R)\,E\left[\tilde{W}(n)\tilde{W}^n(n)\right] \quad (5.17)$$

Problem 5.12. Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n)\tilde{W}^T(n)\right] = 0 \iff 0 < \mu < \frac{1}{\lambda_{max}} \quad (5.18)$$

Problem 5.13. Find the value of the cost function at infinity i.e. $J(\infty)$

Problem 5.14. How can you choose the value of μ from the convergence of both in mean and mean-square sense?