Least Mean Square Algorithm

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***Abstract*—This manual provides an introduction to the LMS algorithm.**

1 Audio Source Files

1.1 Get the **audio_source**

```
svn checkout https://github.com/gadepall/
    EE5347/trunk/audio_source
cd audio_source
```

1.2 Play the **signal_noise.wav** and **noise.wav** file. Comment.
Solution: signal_noise.wav contains a human voice along with an instrument sound in the background. This instrument sound is captured in **noise.wav**.

2 Problem Formulation

2.1 See Table 2.1. The goal is to extract the human voice $e(n)$ from $d(n)$ by suppressing the component of $\mathbf{X}(n)$. Formulate an equation for this.
Solution: The maximum component of $\mathbf{X}(n)$ in

Signal	Label	Type	Filename
Known	d(n)	Human+Instrument	signal noise.wav
	X(n)	Instrument	noise.wav
Unknown	e(n)	Human estimate	
	W(n)	Weight Vector	

TABLE 2.1

$d(n)$ can be estimated as

$$\mathbf{W}^T(n)\mathbf{X}(n) \tag{2.1}$$

where

$$\mathbf{W}(n) = \begin{bmatrix} w_1(n) \\ w_2(n) \\ w_3(n) \\ .. \\ .. \\ w_{n-M+1}(n) \end{bmatrix}_{MX1} \tag{2.2}$$

Intuitively, the human voice $e(n)$ is obtained after removing as much of $\mathbf{X}(n)$ from $d(n)$ as possible. The first step in this direction is to estimate $\mathbf{W}$ in (2.1) using the metric

$$\min_{\mathbf{W}(n)} \|d(n) - \mathbf{W}^T(n)\mathbf{X}(n)\|^2 \tag{2.3}$$

The human voice can be then obtained as

$$e(n) = d(n) - \mathbf{W}^T(n)\mathbf{X}(n) \tag{2.4}$$

3 LMS Algorithm

3.1 Show using (2.4) that

$$\nabla_{\mathbf{W}(n)} e^2(n) = \frac{\partial e^2(n)}{\partial \mathbf{W}(n)} \tag{3.1}$$

$$= -2\mathbf{X}(n)d(n) + 2\mathbf{X}(n)X^T(n)\mathbf{W}(n) \tag{3.2}$$

3.2 Use the gradient descent method to obtain an algorithm for solving (2.3)

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Solution: The desired algorithm can be expressed as

$$\mathbf{W}(n+1) = \mathbf{W}(n) - \bar{\mu}[\nabla_{\mathbf{W}(n)} e^2(n)] \tag{3.3}$$
$$\mathbf{W}(n+1) = \mathbf{W}(n) + \mu\mathbf{X}(n)e(n) \tag{3.4}$$

where $\mu = \bar{\mu}$.

3.3 Write a program to suppress $\mathbf{X}(n)$ in $d(n)$.
Solution: Execute

```
wget https://raw.githubusercontent.com/
    gadepall/EE5347/master/lms/codes/
    LMS_NC_SPEECH.py
```

4 Wiener-Hopf Equation

4.1 Using (2.4), show that

$$E[e^2(n)] = r_{dd} - W^T(n) r_{xd} - r_{xd}^T \mathbf{W}(n) + W^T(n) R \mathbf{W}(n) \tag{4.1}$$

where

$$r_{dd} = E[d^2(n)] \tag{4.2}$$
$$r_{xd} = E[\mathbf{X}(n)d(n)] \tag{4.3}$$
$$R = E[\mathbf{X}(n)\mathbf{X}^T(n)] \tag{4.4}$$

4.2 By computing

$$\frac{\partial J(n)}{\partial \mathbf{W}(n)} = 0, \tag{4.5}$$

show that the optimal solution for

$$W^*(n) = \min_{\mathbf{W}(n)} E\left[e^2(n)\right] = R^{-1} r_{xd} \tag{4.6}$$

This is the Wiener optimal solution.

5 Convergence of the LMS Algorithm

5.1 Convergence in the Mean

5.1.1 Show that R in (4.4) is symmetric as well as positive definite.
Let

$$\tilde{W}(n) = \mathbf{W}(n) - W_* \tag{5.1}$$

where W_* is obtained in (4.6). Also, according to the LMS algorithm,

$$W(n+1) = \mathbf{W}(n) + \mu\mathbf{X}(n)e(n) \tag{5.2}$$
$$e(n) = d(n) - X^T(n)\mathbf{W}(n) \tag{5.3}$$

5.1.2 Show that

$$E\left[\tilde{W}(n+1)\right] = [I - \mu R] E\left[\tilde{W}(n)\right] \tag{5.4}$$

5.1.3 Show that

$$R = U\Lambda U^T \tag{5.5}$$

for some U, Λ, such that Λ is a diagonal matrix and $U^T U = I$.

5.1.4 Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n+1)\right] = 0 \iff \lim_{n\to\infty} [I - \mu\Lambda]^n = 0 \tag{5.6}$$

5.1.5 Using (5.6), show that

$$0 < \mu < \frac{2}{\lambda_{\max}} \tag{5.7}$$

where $\lambda_{\max}$ is the largest entry of Λ.

5.2 Convergence in Mean-square sense

Let

$$\mathbf{X}(n) = \begin{bmatrix} X_1(n) \\ X_2(n) \end{bmatrix} \tilde{W}(n) = \begin{bmatrix} \tilde{W}_1(n) \\ \tilde{W}_2(n) \end{bmatrix} \tag{5.8}$$

5.2.1 Show that

$$E[\tilde{W}^T(n)\mathbf{X}(n)X^T(n)\tilde{W}(n)] = E[\tilde{W}^T(n) R \tilde{W}(n)] \tag{5.9}$$

for R defined in (4.4).

5.2.2 Show that

$$J(n) = E[e^2(n)] = E[e_*^2(n)] + E[\tilde{W}(n)\mathbf{X}(n)\mathbf{X}(n)^T \tilde{W}(n)^T] - E[\tilde{W}(n)\mathbf{X}(n)e_*(n)] - E[e_*(n)X^T(n)\tilde{W}^T(n)] \tag{5.10}$$

where

$$\tilde{W}(n) = W(n) - W_* \tag{5.11}$$
$$e_*(n) = d(n) - W_*\mathbf{X}(n) \tag{5.12}$$

5.2.3 Show that

$$E\left[\tilde{W}(n)\mathbf{X}(n)e_*(n)\right] = E\left[e_*(n)X^T(n)\tilde{W}^T(n)\right] = 0 \tag{5.13}$$

5.2.4 Show that

$$E\left[\tilde{W}^T(n) R \tilde{W}(n)\right] = \operatorname{trace}\left(E\left[\tilde{W}^T(n) R \tilde{W}(n)\right]\right) \tag{5.14}$$
$$= \operatorname{trace}\left(E\left[\tilde{W}(n)\tilde{W}^T(n)\right] R\right) \tag{5.15}$$

5.2.5 Using (5.11), (5.2) and (5.12), show that

$$\tilde{W}(n+1) = \left[I - \mu\mathbf{X}(n)X^T(n)\right]\tilde{W}(n) + \mu\mathbf{X}(n)e_*(n) \tag{5.16}$$

5.2.6 Let $\mu^2 \to 0$. Using (5.5) and (4.6), show that

$$E\left[\tilde{W}(n+1)\tilde{W}^T(n+1)\right] = (I - 2\mu R)\, E\left[\tilde{W}(n)\tilde{W}^n(n)\right] \quad (5.17)$$

5.2.7 Show that

$$\lim_{n\to\infty} E\left[\tilde{W}(n)\tilde{W}^T(n)\right] = 0 \iff 0 < \mu < \frac{1}{\lambda_{max}} \quad (5.18)$$

5.2.8 Find the value of the cost function at infinity i.e. $J(\infty)$

5.2.9 How can you choose the value of μ from the convergence of both in mean and mean-square sense?