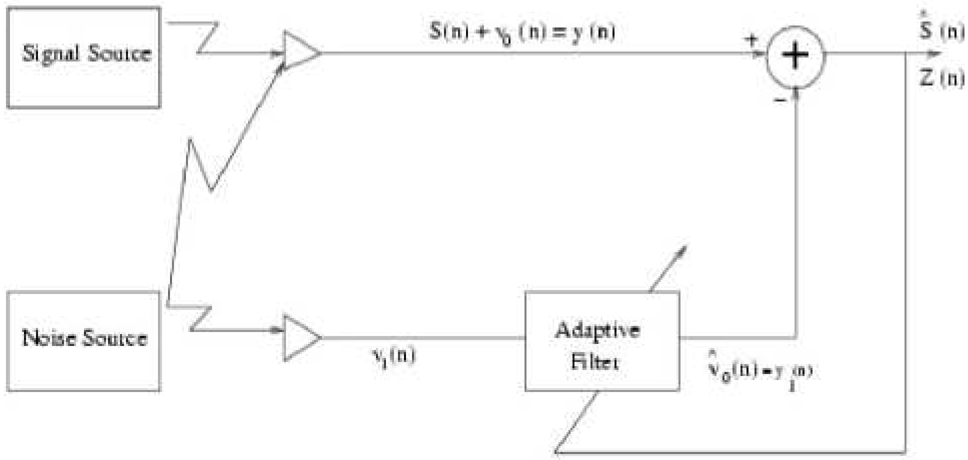
EE608 Adaptive Signal Processing

Problem Set 1 — Some Basic Problems in Estimation and Matrix Computation

1) Consider the following M-th order FIR adaptive structure for noise cancellation shown in class



Assume that $v_i(n)$ and s(n) are statistically independent.Let $e(n) = s(n) + v_0(n) - \hat{v}_0(n)$.Now suppose the adaptation rule is chosen such that $E[e^2(n)]$ is minimized.Then show that $E[e^2(n)] = E[(s(n) - \hat{s}(n))^2]$.Thus minimizing $E[e^2(n)]$ will indeed yield $\hat{s}(n)$ as the minimum mean sqaure estimate of s(n).

2) Let $\{y_k\}_{k=1}^N$ be N measurements of unknown constant c. The problem is to derive the expression for the l_2,l_1, and l_∞ estimate of the constant c using the measurements $\{y_k\}_{k=1}^N$
We model the measurements as sample values of a discrete time random variable Y having the density function

$$f_Y(y) = \sum_{k=1}^{N} \alpha_k \delta(y - y_k)$$

where α_k are such that $\sum_{k=1}^{N} \alpha_k = 1$.
Show that

(a) The l_2 estimate of c is given by

$$\hat{c} = \sum_{k=1}^{N} \alpha_k y_k$$

(b) The l_1 estimate of c is given by $\hat{c} = y_p$, for $1 \leq p \leq N$ such that

$$\sum_{y_k < \hat{c}} \alpha_k = \sum_{y_k > \hat{c}} \alpha_k$$

(c) The l_∞ estimate is given by

$$\hat{c} = \frac{min\{y_k\} + max\{y_k\}}{2}$$

3) Consider the problem of estimating the mean and variance of a Gaussian distributed random variable.

Let

$$X_k, k = 1, ...N$$

be N independent samples of a stationary Gaussian distribution.Thus

$$f_{X_k|M,V^{(x_k|m,\sigma^2)}} = \frac{1}{2\pi\sigma^2} e^{-(x_k-m)^2/(2\sigma^2)}$$

and letting $Z = \{X_1, ..., X_N\}, z = \{x_1, ..., x_n\}$

$$f_{Z|M,V^{(z|m,\sigma^2)}} = \prod_{k=1}^{N} f_{X_k|M,V^{(x_1|m,\sigma^2)}} = \frac{1}{(2\pi\sigma^2)^{N/2}} exp[- \quad \sum_{k=1}^{N} (x_k - m)^2/2\sigma^2]$$

(a) Assume σ^2 is known. Show that the MLE of m is given by

$$\hat{m}_{MLE} = \frac{1}{N} \quad \sum_{k=1}^{N} x_k$$

and that is an unbiased estimate.

(b) Assume m is known. Show that MLE of $\hat{\sigma^2}$ is given by

$$(\hat{\sigma^2})_{MLE} = \frac{1}{N} \quad \sum_{k=1}^{N} (x_k - m)^2$$

and that is unbiased.

(c) *Neither m nor σ^2 is known.* Show that

$$\hat{m}_{MLE} = \frac{1}{N} \quad \sum_{k=1}^{N} x_k, (\hat{\sigma^2})_{MLE} = \frac{1}{N} \quad \sum_{k=1}^{N} (x_k - \hat{m}_{MLE})^2$$

Furthermore show that, $\hat{m}_{MLE}$ is unbiased, while $(\hat{\sigma^2})_{MLE}$ is biased with

$$E[(\hat{\sigma^2})_{MLE}] = \frac{N-1}{N}\sigma^2$$

(d) Recursive computation of $\hat{m}_{MLE}$ and $(\hat{\sigma^2})_{MLE}$.Show that if $\hat{m}_{MLE} := \hat{m}_N$ and $(\hat{\sigma^2})_{MLE} := (\hat{\sigma^2})_N$ then

$$\hat{m}_N = \frac{1}{N}[(N-1)\hat{m}_{N-1} + x_N]$$

$$\hat{\sigma^2}_N = \frac{1}{N}[(N-1)\hat{\sigma^2}_{N-1} + \frac{N}{N-1}(x_N - m_N)^2]$$

4) Solve the following l_2 minimization problems.

(a) Let A be an m $\times$ n matrix, x an n $\times$ 1 vector, and y an m $\times$ 1 vector. Find x such that

$$\|Ax - y\|^2 = (A_x - y)^T(A_x - y)$$

is minimized.

(b) Let A be an m $\times$ n matrix, x an n $\times$ 1 vector, and y an m $\times$ 1 vector. Find x having minimum l_2 norm ($\|x\|^2 = x'x$) and satisfying $Ax = y$; ie.

$$\min_{subj.Ax=b} \|x\|^2$$

Note: These are two fundamentals finite dimensional l_2 approximation problems which can serve as prototypes for any other finite dimensional l_2 approximation problem.

5) **On QR Decomposition**

*Definition:*LetA be an m$\times$n matrix with m>n and rank p; then the QR decompositon of A is defined by

$$A = \begin{bmatrix} Q & N \end{bmatrix} \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} QR_1 & QR_2 \end{bmatrix} \stackrel{\text{def}}{=} QR$$

where

(a) the dimensions of various matrices are Q:m$\times$p, N:m$\times$$(m-p)$,$R_1 : p$$\times$p,$R_2 : p \times (n-p)$. $R = \begin{bmatrix} R_1 & R_2 \end{bmatrix}$ is p$\times$ n.

(b) $\begin{bmatrix} Q & N \end{bmatrix}$ is an orthogonal matrix,namely ,

$$\begin{bmatrix} Q & N \end{bmatrix} \begin{bmatrix} Q^T \\ N^T \end{bmatrix} = I_m = \begin{bmatrix} Q^T \\ N^T \end{bmatrix} \begin{bmatrix} Q & N \end{bmatrix}$$

Note the obvious $QQ^T \neq I$ and $NN^T \neq I$. But $Q^TQ = I_p$ and $N^TN = I_{m-p}$.

(c) R_1 is a p$\times$p upper triangular matrix.
Note, if p=n then $R_2 = 0$.

(a) Show that the columns of Q provide an orthogonal basis for the range space of A, while the columns of N provide an orthogonal basis for the null space of A^T.

(b) **Algorithm for obtaining the QR Decomposition**
Consider a 2$\times$1 column vector $\begin{bmatrix} x & y \end{bmatrix}^T$. Now it is shown earlier that the orthogonal Givens rotation

$$G = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, c = \frac{x}{\sqrt{x^2+y^2}}$$
$$s = \frac{y}{\sqrt{x^2+y^2}}$$

when applied to $\begin{bmatrix} x & y \end{bmatrix}^T$,yields

$$G \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{x^2+y^2} \\ 0 \end{bmatrix}$$

First figure out how the above method can be used to selectively zero elements of any column vector.Using this idea develop an algorithm for computing the QR decomposition of an m$\times$n $(m > n)$ matrix A. You can illustrate your algorithm by considering a full rank 4$\times$3 matrix A.

6) One possible norm for an m×n matrix A is defined as

$$\|A\| \stackrel{\text{def}}{=} \max_{x^T x=1, x \varepsilon \Re^n} (x^T A^T A x)$$

Now let the SVD of an m×n matrix A be A=UΣV^T.Show that $\|A\| = \sigma_{max}(A)$.

7) Let the SVD of an m×n matrix A be A=UΣV^T.Now define the pseudo inverse of A as

$$A^\dagger = V\Sigma^{-1}U^T$$

verify that the above $A^\dagger$ satisfies the following identities for the matrix to be a pseudo inverse.

(i) $(A^\dagger)^\dagger = A$, (iii) $A^\dagger A A^\dagger = A^\dagger$, (v) $(A^\dagger A)^T = A^\dagger A$, (vii) $A^\dagger = A^T(AA^T)^\dagger$.

(ii) $(A^T)^\dagger = (A^\dagger)^T$, (iv) $AA^\dagger A = A$, (vi) $A^\dagger = (A^T A)^\dagger A^T$,